20th International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems.
Lisbon, Portugal, July 22nd – 26th 2024

Session: Ontologies, Games, and Advanced Uncertainty Management

Fuzziness and Lie Algebras

Dr. Giuseppe Filippone

Joint work with **Drs. Gianmarco La Rosa, Manuel Mancini, Marco Elio Tabacchi**

2024 07 25

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Table of contents

History of fuzzification of basic concepts of algebra

Azriel Rosenfeld (1931–2004) and Lotfi Aliasker Zadeh (1921-2017)

History of fuzzification of basic concepts of algebra

- Chin-Liang Chang. “Fuzzy topological spaces”. In: Journal of Mathematical Analysis and Applications 24 (1968), pp. 182–190.
- Azriel Rosenfeld. “Fuzzy groups”. In: Journal of Mathematical Analysis and Applications 35.3 (1971), pp. 512–517.

The idea of trying to fuzzify algebra finally after several years later, after Lotfi’s student C.L. Chang had published his paper “Fuzzy topological spaces” (J. Math. Anal. Appl. 24 (1968) 182-190). After somewhat belatedly coming across that paper, I said to myself “If Chang can do it for topological spaces, I can do it for algebraic structures.” [John N. Mordeson and Davender S. Malik. Fuzzy Commutative Algebra. World Scientific Publishing Company, 1998, Foreword, p. X]

— *Aziel Rosenfeld*

History of fuzzification of basic concepts of algebra

❖ **According to the Scopus database, the paper "Fuzzy groups" by Azriel Rosenfeld has more than 1700 citations, while Chang’s work has just under 1700 citations!**

I took a copy of my own algebra book with me on a train ride from Washington to New York, with the intention of formulating natural fuzzifications of the basic concepts of algebra. Needless to say, I found a way to do this; in fact, by the end of the train ride I had written an essentially complete draft of my embarrassingly well-cited paper “Fuzzy groups” (J. Math. Anal. Appl. 35 (1971) 512-517). [John N. Mordeson and Davender S. Malik. Fuzzy Commutative Algebra. World Scientific Publishing Company, 1998, Foreword, p. X]

— Azriel Rosenfeld

History of fuzzification of basic concepts of algebra

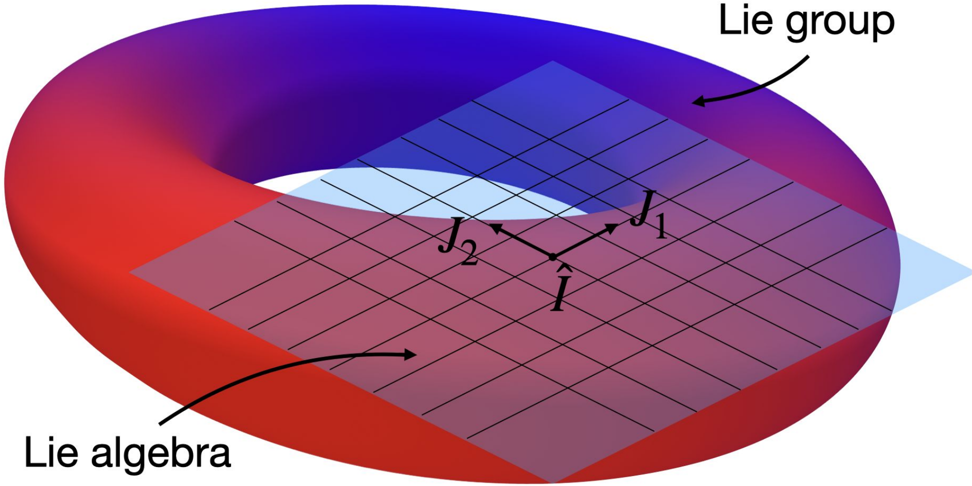
Let G be a groupoid. A fuzzy subset μ of G is called a fuzzy subgroupoid of G if, for all $x, y \in G$

$$\mu(xy) \geq \min(\mu(x), \mu(y))$$

Fuzzy Lie algebras

Fuzzy Lie algebras

Marius Sophus Lie (1842-1899)



Fuzzy Lie algebras

A Lie algebra $\mathbf{g}$, like any other algebraic structure, can be fuzzified in Rosenfeld style. A subalgebra of $\mathbf{g}$, which is a subspace, remains closed with respect to the *bracket operation*, i.e., an operation on Lie algebras.

➢ Samy El-Badawy Yehia. “Fuzzy ideals and fuzzy subalgebras of Lie algebras”, Fuzzy Sets and Systems 80.2 (1996), pp. 237–244.

The Transfer Principle

In the field of model theory, the transfer principle posits that the validity of statements within a given language for one structure implies their validity for another structure within the same language.

Formally, the transfer principle allows one to transfer assertions from one algebraic system to another.

The Transfer Principle

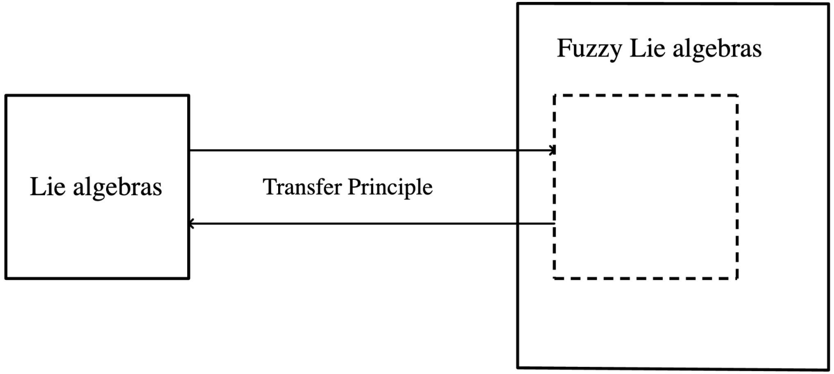
In any supposed transition, ending in any terminus, it is permissible to institute a general reasoning, in which the terminus may also be included. [Howard J. Keisler. Elementary Calculus: An Infinitesimal Approach, 1976, pp. 902-903]

— Gottfried W. Leibniz (1646-1716), the Law of Continuity

The Transfer Principle

- Michiro Kondo and Wieslaw Dudek. “On the transfer principle in fuzzy theory”. In: Mathware and Soft Computing 12.1 (2005), pp. 41–55.

Michiro Kondo and Wieslaw Dudek in their work applied the transfer principle in order to translate the results of crisp algebras to fuzzy algebras. Specifically, they divided the results of crisp algebraic structures into four types, and it was demonstrated that these types of propositions hold true for their fuzzy counterparts if and only if they hold true for their crisp versions.



The Transfer Principle

These types are the following:

- **type 0**: A subset A has a property P
- **type 1**: If a subset A has a property P, then it has a property Q
- **type 2**: Let $f : X \to Y$ be a homomorphism. If a subset B of Y has a property Q, then a subset $f^{-1}(B)$ of X has a property P
- **type 3**: Let $f : X \to Y$ be a surjective homomorphism. If a subset A of X has a property P , then a subset $f(A)$ of Y has a property Q

The Transfer Principle

To be more precise, such application of the transfer principle began at least in the work of Young Bae Jun and Michiro Kondo in 2004.

- Young Bae Jun and Michiro Kondo. “On transfer principle of fuzzy BCK/BCI-algebras”. In: Scientiae Mathematicae Japonicae, 2004.

The Transfer Principle

➢ Muhammad Akram, “Fuzzy Lie Structures”, Fuzzy Lie Algebras, 2018, Theorem 1.1, p. 8.

Theorem (crisp). Every ideal of a Lie algebra is a Lie subalgebra.

Theorem (fuzzy). Every fuzzy Lie ideal is a fuzzy Lie subalgebra.

Conclusions

- Fuzzy theory has influenced not only applied areas but also areas of abstract mathematics such as algebra.
- Rosenfeld work on fuzzy groups inspired many mathematicians to fuzzify different algebraic structures, e.g., Lie algebras.
- The Law of Continuity of Leibniz led to the transfer principle, which offers a highly effective method for constructing a fuzzy version of every proposition known from the classical theory, based on its crisp definition, and vice versa. Conversely, since the definition of fuzzy Lie algebras extends that of the classic theory, some results can be appreciated in a natural way only in the fuzzy context, whereas they are irrelevant to the matter in the crisp Lie algebras.

Thank you for your attention!