On the spectral problem of the quantum KdV hierarchy

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Abstract

The spectral problem for the quantum dispersionless Korteweg–de Vries (KdV) hierarchy, aka the quantum Hopf hierarchy, is solved by Dubrovin. In this article, following Dubrovin, we study Buryak–Rossi's quantum KdV hierarchy. In particular, we prove a symmetry property and a non-degeneracy property for the quantum KdV Hamiltonians. On the basis of this we construct a complete set of common eigenvectors. The analysis underlying this spectral problem implies certain vanishing identities for combinations of characters of the symmetric group. We also comment on the geometry of the spectral curves of the quantum KdV hierarchy and we give a representation of the quantum dispersionless KdV Hamiltonians in terms of multiplication operators in the class algebra of the symmetric group.

1 Introduction

Let $\Lambda := \mathbb{C}[q_1, q_2, q_3, \dots]$ denote the ring of polynomials of the indeterminates $q_1, q_2, q_3, \dots$ with complex coefficients. By a *quantum integrable hierarchy* we mean a sequence of pairwise-commuting linear operators $(H_m)_{m\in I}$ on $\Lambda \otimes R$ or on some completion of it, satisfying certain non-degeneracy property (cf. e.g. [1, 2, 3, 6, 11, 17]). Here I is an infinite index set and R is a certain ring.

To illustrate this subject and the context of this paper, let us start by recalling the definition of the classical KdV hierarchy and its Hamiltonian structures. Denote by $(\mathcal{A}_u, \partial_x)$ the differential polynomial ring of u, namely an element of $\mathcal{A}_u$ is a polynomial of $u_x, u_{xx}, u_{xxx}, \dots$ whose coefficients are formal power series in u with complex coefficients. Define a sequence of elements $h^{\mathsf{cl}}_{-1}, h^{\mathsf{cl}}_0, h^{\mathsf{cl}}_1, h^{\mathsf{cl}}_2, \dots$ in $\mathcal{A}_u[\epsilon^2]$ by the *Lenard-Magri recursion*:

$$(2m+3)\partial_x h^{\mathsf{cl}}_m = \left(2u\partial_x + u_x + \frac{\epsilon^2}{4}\partial_x^3\right) h^{\mathsf{cl}}_{m-1}, \quad m \ge 0, \tag{1.1}$$

$$h^{\mathsf{cl}}_{-1} = u, \quad h^{\mathsf{cl}}_m|_{u=u_x=u_{xx}=\dots=0} = 0, \tag{1.2}$$

as well as

$$\sum_{i\ge 1} i u_{ix} \frac{\partial h^{\mathsf{cl}}_m}{\partial u_{ix}} - \epsilon \frac{\partial h^{\mathsf{cl}}_m}{\partial \epsilon} = 0, \quad m \ge 0. \tag{1.3}$$

Here, $u_{ix} := \partial_x^i(u)$, $i \ge 0$. Explicitly,

$$h^{\mathsf{cl}}_0 = \frac{u^2}{2} + \frac{\epsilon^2}{12} u_{xx}\,, \quad h^{\mathsf{cl}}_1 = \frac{u^3}{6} + \epsilon^2\left(\frac{1}{24}u_x^2 + \frac{1}{12}uu_{xx}\right) + \frac{\epsilon^4}{240} u_{xxxx}\,, \tag{1.4}$$

etc. For an element $a \in \mathcal{A}_u[\epsilon]$, we denote by $\overline{a}$ the projection of a onto $\mathcal{A}_u[\epsilon]/\partial_x\mathcal{A}_u[\epsilon]$. Elements in $\mathcal{A}_u[\epsilon]/\partial_x\mathcal{A}_u[\epsilon]$ can be called *local functionals*. The *classical KdV hierarchy* is the following system of commuting Hamiltonian evolutionary PDEs:

$$\frac{\partial u}{\partial t_m} = \mathcal{P}_1\left(\frac{\delta \overline{h^{\mathsf{cl}}_m}}{\delta u(x)}\right) = \frac{1}{2m+1}\mathcal{P}_2\left(\frac{\delta \overline{h^{\mathsf{cl}}_{m-1}}}{\delta u(x)}\right), \quad m \ge 0, \tag{1.5}$$

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where $\frac{\delta}{\delta u(x)}$ denotes the variational derivative, and $\mathcal{P}_1, \mathcal{P}_2$ are two Hamiltonian operators given by

$$\mathcal{P}_1 = \partial_x, \quad \mathcal{P}_2 = 2u\partial_x + u_x + \frac{1}{4}\epsilon^2\partial_x^3, \tag{1.6}$$

respectively. The local functionals $\overline{h_m^{\mathsf{cl}}}$, $m \geq -1$ serve as the Hamiltonians. The Hamiltonian operators $\mathcal{P}_a$, $a = 1, 2$, define two Poisson structures on the space of local functionals $\mathcal{A}_u[\epsilon]/\partial_x\mathcal{A}_u[\epsilon]$ in the following way: for any $\overline{f}, \overline{g} \in \mathcal{A}_u[\epsilon]/\partial_x\mathcal{A}_u[\epsilon]$,

$$\left\{\overline{f}, \overline{g}\right\}_a := \overline{\frac{\delta \overline{f}}{\delta u(x)}\mathcal{P}_a\left(\frac{\delta \overline{g}}{\delta u(x)}\right)} \in \mathcal{A}_u[\epsilon]/\partial_x\mathcal{A}_u[\epsilon], \quad a = 1, 2. \tag{1.7}$$

The commutativity of KdV flows (1.5) can then be viewed equivalently from either

$$\left\{\overline{h_m^{\mathsf{cl}}}, \overline{h_n^{\mathsf{cl}}}\right\}_1 = 0, \quad \forall\, m, n \geq 0, \tag{1.8}$$

or

$$\left\{\overline{h_{m-1}^{\mathsf{cl}}}, \overline{h_{n-1}^{\mathsf{cl}}}\right\}_2 = 0, \quad \forall\, m, n \geq 0. \tag{1.9}$$

Quantization of the KdV hierarchy with respect to the first[1] Poisson structure $\{,\}_1$ was considered by Buryak and Rossi [11], Dubrovin [17] and Eliashberg [21]. Let us now give a brief review. Consider the coordinate functional $u(x)$, and write it using the Fourier modes representation:

$$u(x) = \sum_{k\in\mathbb{Z}} U_k \mathrm{e}^{\mathrm{i}kx}. \tag{1.10}$$

Here and below, we will consider x as a variable in S^1, and for any $f = f(u; u_x, u_{xx}, \ldots; \epsilon) \in \mathcal{A}_u[\epsilon]$, we regard $\overline{f}$ as $\int_{S^1} f(u; u_x, u_{xx}, \ldots; \epsilon)\, \mathrm{d}x$ (hereafter $\int_{S^1}$ denotes $\frac{1}{2\pi}\int_0^{2\pi}$, the operator extracting the zero Fourier mode). In terms of the Fourier modes, the first Poisson structure in (1.7) reads as follows:

$$\{U_k, U_l\}_1 = \mathrm{i}k\delta_{k+l,0}\,, \quad \forall\, k, l \in \mathbb{Z}. \tag{1.11}$$

The canonical quantizations with respect to $\{\cdot,\cdot\}_1$ for the space of local functionals $\mathcal{A}_u[\epsilon]/\partial_x\mathcal{A}_u[\epsilon]$ are defined as the following linear operators on $\Lambda[[\epsilon]]$:

$$\widehat{U}_k = \begin{cases} q_k, & k > 0, \\ U_0\,\mathrm{id}, & k = 0, \\ -\hbar k\frac{\partial}{\partial q_{-k}}, & k < 0, \end{cases} \tag{1.12}$$

$$\widehat{\overline{a}} = \,:\int_{S^1} a(u(x); u_x(x), u_{xx}(x), \ldots; \epsilon)\, \mathrm{d}x:, \quad \forall\, a \in \mathcal{A}_u[\epsilon], \tag{1.13}$$

where U_0 and $\hbar$ are parameters and $::$ denotes the normally ordered product. The normal ordering here means the following: we first move all U_k with $k > 0$ to the left of the U_k with $k < 0$ in each monomial of a, and then we replace each U_k by $\widehat{U}_k$. In such a canonical quantization, the Poisson bracket (1.11) is replaced by $\frac{1}{\mathrm{i}\hbar}[\cdot,\cdot]$, where $[\cdot,\cdot]$ denotes the commutator; indeed,

$$\left[\widehat{U}_k, \widehat{U}_l\right] = -\hbar k\delta_{k+l,0}, \qquad k, l \in \mathbb{Z}.$$

It turns out that the Poisson commutativity (1.8) does not hold after the quantization, namely $\left[\widehat{\overline{h_m^{\mathsf{cl}}}}, \widehat{\overline{h_n^{\mathsf{cl}}}}\right]$ do not vanish in general. The quantization problem, suggested by Dubrovin, is to find an $\hbar$-deformation $\overline{h_m}$ of $\overline{h_m^{\mathsf{cl}}}$, such that

$$\left[\widehat{\overline{h_m}}, \widehat{\overline{h_n}}\right] = 0, \quad \forall\, m, n \geq 0. \tag{1.14}$$

[1] Quantization of the KdV hierarchy with respect to the second Poisson structure $\{,\}_2$ was considered by Bazhanov, Lukyanov and Zamalodchikov [1, 2, 3].

For the case that the dispersive parameter ϵ vanishes, the existence of such an $\hbar$-deformation was obtained by Eliashberg [21], in the setting of Symplectic Field Theory [22], via an explicit generating series (cf. (2.37) below); for an arbitrary ϵ, the existence is obtained via a concrete construction by Buryak and Rossi [11] (cf. (2.33)-(2.34) below). Let us denote by h_m^{DR} the densities of the quantum Hamiltonians constructed by Buryak and Rossi, which of course satisfy $\overline{h_m^{\mathsf{DR}}}|_{\hbar=0} = \overline{h_m^{\mathsf{cl}}}$. Here we note that adding a total x-derivative in the density of a local functional does not affect the quantization of the functional. Below, for simplify we denote Buryak–Rossi's quantum KdV Hamiltonians $\widehat{\overline{h_m^{\mathsf{DR}}}}$ as H_m.

In this paper, we consider the eigenvalue problem for the operators H_m. For the reader's convenience, we list below the first few H_m:

$$H_{-1} = U_0, \tag{1.15}$$

$$H_0 = \hbar \sum_{k>0} k q_k \frac{\partial}{\partial q_k} - \frac{\hbar}{24} + \frac{U_0^2}{2}, \tag{1.16}$$

$$H_1 = \Delta + \hbar U_0 \sum_{k\geq 1} k q_k \frac{\partial}{\partial q_k} - \frac{\epsilon^2 \hbar}{12} \sum_{k\geq 1} k^3 q_k \frac{\partial}{\partial q_k} - \frac{\epsilon^2\hbar}{2880} - \frac{\hbar U_0}{24} + \frac{U_0^3}{6}, \tag{1.17}$$

where

$$\Delta := \frac{1}{2} \sum_{i,j\geq 1} \hbar (i+j) q_i q_j \frac{\partial}{\partial q_{i+j}} + \hbar^2 ij q_{i+j} \frac{\partial^2}{\partial q_i \partial q_j}. \tag{1.18}$$

Here we note that H_0 is a grading operator, and that Δ can be recognized as the celebrated *cut-and-join* operator (cf. [25] and Appendix A). In general, write

$$H_m = \sum_{g\geq 0} \epsilon^{2g} H_m^{[g]}, \tag{1.19}$$

where $H_m^{[g]}$ are independent of ϵ. It is helpful to notice that for every $m \geq -1$, the operator $K_m := H_m|_{U_0=0}/\sqrt{\hbar^{m+2}}$, after the rescaling

$$q \to q/\sqrt{\hbar} =: T, \tag{1.20}$$

depends only on the single parameter $\sigma = -\frac{\epsilon^2}{\sqrt{\hbar}}$ (cf. Remark 2.5). Here, $T = (T_1, T_2, T_3, \dots)$.

The eigenvalue problem for H_m with $\epsilon = 0$, i.e. for $H_m^{[0]}$, was solved by Dubrovin [17]. To state his result, let us introduce some notations.

Denote by $\mathscr{P}$ the set of partitions λ, namely $\lambda = (\lambda_1, \lambda_2, \dots) \in \mathscr{P}$ is a half-infinite non-increasing sequence of integers $\lambda_1 \geq \lambda_2 \geq \cdots$, called parts of λ, with only finitely many non-zero parts; the number $\ell(\lambda)$ of nonzero parts is called the length of the partition, and the number $|\lambda| := \sum_i \lambda_i$ is called the weight of the partition. Let us denote $\mathscr{P}_k$ the set of partitions λ having weight $|\lambda| = k$. Let $s_\lambda(T)$ denote the Schur polynomial associated to λ which can be defined by

$$s_\lambda(T) = \det(h_{\lambda_i - i + j}(T))_{1\leq i,j\leq \ell(\lambda)}. \tag{1.21}$$

Here $h_k(T)$, $k \in \mathbb{Z}$ are the complete homogeneous symmetric polynomials defined via

$$\sum_{k\geq 0} h_k(T) z^k := \exp\Bigg(\sum_{k\geq 1} \frac{T_k}{k} z^k\Bigg), \qquad h_k := 0, \text{ if } k < 0. \tag{1.22}$$

For example,

$$s_{(2)} = \frac{1}{2}(T_1^2 + T_2), \quad s_{(1,1)} = \frac{1}{2}(T_1^2 - T_2), \tag{1.23}$$

$$s_{(3)} = \frac{1}{6}(T_1^3 + 3T_1T_2 + 2T_3), \quad s_{(2,1)} = \frac{1}{3}(T_1^3 - T_3), \quad s_{(1,1,1)} = \frac{1}{6}(T_1^3 - 3T_1T_2 + 2T_3). \tag{1.24}$$

The collection of $s_\lambda(T)$ for all partitions form a basis of $\mathbb{C}[T_1, T_2, \dots] =: \widetilde{\Lambda}$ over $\mathbb{C}$ (see e.g. [30]). We also recall that the standard inner product $\langle\, ,\, \rangle : \widetilde{\Lambda} \times \widetilde{\Lambda} \to \mathbb{C}$ is defined by (see e.g. [30])

$$\langle s_\lambda(T), s_\mu(T)\rangle = \delta_{\lambda,\mu}. \tag{1.25}$$

Moreover, it is convenient to recall the following functions of partitions: for any $j \geq 0$ and any partition $\lambda \in \mathscr{P}$, define [4, 14, 17, 35]

$$P_j(\lambda) = \sum_{i=1}^{\ell(\lambda)} \left[\left(\lambda_i - i + \frac{1}{2}\right)^j - \left(-i + \frac{1}{2}\right)^j \right] \tag{1.26}$$

and a differently normalized set of functions Q_j, for $j \geq 0$, related to the P_j's by

$$Q_0(\lambda) = 1, \qquad Q_j(\lambda) = \frac{P_{j-1}(\lambda)}{(j-1)!} + \beta_j \quad \text{if } j \geq 1, \tag{1.27}$$

where β_j, for $j \geq 0$, are defined by the generating series

$$\frac{z/2}{\sinh(z/2)} = \sum_{j\geq 0} \beta_j z^j = 1 - \frac{z^2}{24} + \frac{7z^4}{5760} - \frac{31z^6}{967680} + \cdots , \tag{1.28}$$

or, equivalently, $\beta_j = \left(\frac{1}{2^{j-1}} - 1\right) B_j/j!$, B_j being the j_{th} Bernoulli numbers. The functions Q_j, for $j \geq 0$, given in (1.27) are the generators of the algebra of *shifted symmetric polynomials* [4]. They are building blocks of the Bloch–Okounkov theorem and have a famous relation to quasimodular forms [4, 14, 35] which we hope to come back to in a later publication.

Theorem 1.1 (Dubrovin [17])**.** *The following formulae hold true:*

$$H^{[0]}_m s_\lambda\left(\frac{q}{\sqrt{\hbar}}\right) = E^{[0]}_m(\lambda; \hbar, U_0) s_\lambda\left(\frac{q}{\sqrt{\hbar}}\right), \quad m \geq -1 \tag{1.29}$$

with the eigenvalues given explicitly by

$$\begin{aligned} E^{[0]}_m(\lambda; \hbar, U_0) &= \sum_{j=0}^{m+2} \frac{\sqrt{\hbar}^j \beta_j U_0^{m+2-j}}{(m+2-j)!} + \sqrt{\hbar} \sum_{i\geq 1} \frac{\left(U_0 + \sqrt{\hbar}\left(\lambda_i + \frac{1}{2} - i\right)\right)^{m+1} - \left(U_0 + \sqrt{\hbar}\left(\frac{1}{2} - i\right)\right)^{m+1}}{(m+1)!} \\ &= \sum_{j=0}^{m+2} \frac{\sqrt{\hbar}^j U_0^{m+2-j}}{(m+2-j)!} Q_j(\lambda), \end{aligned} \tag{1.30}$$

where $Q_j(\lambda)$ are defined in (1.27)*.*

For a general ϵ, to study the eigenvalue problem for H_m, let us make two observations. The first one is a symmetry property (cf. Corollary 2.4) of the quantum KdV Hamiltonians, and the second one is that the spectrum of the quantum dispersionless KdV Hamiltonians is *non-degenerate* (cf. Lemma 3.1). Based on these observations we have the following proposition.

Proposition 1.2. *There exists a unique collection $(r_\lambda(T;\sigma))_{\lambda\in\mathscr{P}}$ of elements in the free module $\widetilde{\Lambda} \otimes_{\mathbb{C}} \mathbb{C}[\![\sigma]\!]$ over $\mathbb{C}[\![\sigma]\!]$ such that*

$$\langle r_\lambda(T;\sigma), s_\lambda(T)\rangle = 1, \tag{1.31}$$

$$H_m r_\lambda\left(\frac{q}{\sqrt{\hbar}}; -\frac{\epsilon^2}{\sqrt{\hbar}}\right) = E_m\left(\lambda; \epsilon, \hbar, U_0\right) r_\lambda\left(\frac{q}{\sqrt{\hbar}}; -\frac{\epsilon^2}{\sqrt{\hbar}}\right) \tag{1.32}$$

for some $E_m(\lambda; \epsilon, \hbar, U_0) \in \mathbb{C}(\sqrt{\hbar}, U_0)[\![\epsilon^2]\!]$. Here H_m are Buryak–Rossi's quantum KdV Hamiltonians, cf. (2.33)*-*(2.34)*. Moreover, $(r_\lambda(T;\sigma))_{\lambda\in\mathscr{P}}$ is a basis of the free module $\widetilde{\Lambda} \otimes_{\mathbb{C}} \mathbb{C}[\![\sigma]\!]$, and*

$$r_\lambda(T; \sigma = 0) = s_\lambda(T), \qquad E_m(\lambda; \epsilon = 0, \hbar) = E^{[0]}_m(\lambda; \hbar), \tag{1.33}$$

where $E^{[0]}_m$ are defined in (1.30)*.*

The proof is given in Section 3.

We call $r_\lambda(T;\sigma)$ in the above proposition the *deformed Schur polynomials*. For example, we have

$$r_{(2)} = s_{(2)} + s_{(1,1)} \frac{4-\sqrt{16-\sigma^2}}{\sigma} = s_{(2)} + s_{(1,1)} \left(-\frac{\sigma}{8} + \frac{\sigma^3}{512} - \frac{\sigma^5}{16384} + \frac{5\sigma^7}{2097152} + \mathcal{O}(\sigma^9) \right). \tag{1.34}$$

More examples can be found in Section 3.2. It should be noted that $\langle r_\lambda(T;\sigma), r_\mu(T;\sigma)\rangle = 0$ for all $\lambda \neq \mu$ and that $r_{\lambda'}(T;\sigma) = (-1)^{|\lambda|} r_\lambda(-T;-\sigma)$ (see Proposition 3.3). Here λ' denotes the partition conjugate to λ. As an application of Proposition 1.2, we also find certain vanishing identities for combinations of characters in the symmetric group (see Section 3.3).

It is suggested by Dubrovin [18] that to understand geometry of the spectrum and the common eigenvectors of H_m one needs to study the corresponding *spectral curves*. To define these curves, introduce a gradation on $\widetilde{\Lambda}$ via the degree assignments $\deg T_k = k$, $k \geq 1$. We have $\widetilde{\Lambda} = \bigoplus_{k\geq 0} \widetilde{\Lambda}_k$, where elements in $\widetilde{\Lambda}_k$ are homogeneous of degree k with respect to deg. The commutativity (1.14) with $n = 0$ implies that each K_m $(m \geq -1)$ acts on the space $\widetilde{\Lambda}_k$ for any $k \geq 0$. The weight k spectral curves for the quantum KdV hierarchy are then defined by

$$\Sigma_{k,m} := \left\{ (\sigma,\rho) \in \mathbb{C}^2 : \ \det_{\widetilde{\Lambda}_k}(K_m(\sigma) - \rho) = 0 \right\}, \quad m \geq 1. \tag{1.35}$$

We will discuss the geometric meanings of these curves after the proof of Proposition 1.2 (see Section 3.2). Dubrovin [18] computes the geometric genus of the spectral curve $\Sigma_{k,1}$ up to $k \leq 7$. We continue his computation; below is a table of the geometric genus $g(\Sigma_{k,1})$.

k	0	1	2	3	4	5	6	7	8	9	10	$\cdots$
$g(\Sigma_{k,1})$	0	0	0	1	4	9	21	37	69	113	187	$\cdots$

(1.36)

We have the following conjectural statements.

Conjecture 1.1. *For any fixed $k \geq 1$, the curves $\Sigma_{k,m}$ $(m \geq 1)$ are irreducible with geometric genus $g(\Sigma_{k,m})$ independent of m and given explicitly by the expression*

$$g(\Sigma_{k,m}) = (k-1)\,|\mathscr{P}_k| + 1 - \sum_{\lambda \in \mathscr{P}_k} \ell(\lambda), \tag{1.37}$$

where $\mathscr{P}_k$ denotes the set of partitions of weight k.

It is straightforward to verify that the values of the right-hand side of (1.37) coincide with those of sequence A238641 in The On-Line Encyclopedia of Integer Sequences[2], introduced by Kimberling. The conjectural identity (1.37) was actually observed with the help of A238641.

Recall that the KdV hierarchy is a particular Dubrovin–Zhang (DZ) hierarchy [20], that corresponds to the trivial rank-1 Cohomological Field Theory (CohFT). For the definition of a CohFT see [28] or Section 2. For an arbitrary tautological CohFT (e.g. homogeneous CohFT, or Hodge case), following Dubrovin, one can consider the quantization of the first Poisson structure for the corresponding DZ hierarchy ([9, 10, 19, 20]). However, this Poisson structure is usually a deformation of $\eta^{\alpha\beta}\partial_x$, where $\eta^{\alpha\beta}$ are certain constants. To make the quantization procedure as simple as for the KdV case, one could perform a Miura-type transform reducing the Hamiltonian operator for the DZ hierarchy to $\eta^{\alpha\beta}\partial_x$ (cf. [15, 20, 24]) before doing the quantization. Considering the conjectural relationship between the DZ and the DR hierarchies associated to the same CohFT, it seems easier to directly quantize the DR hierarchy because the latter by definition possesses $\eta^{\alpha\beta}\partial_x$ as the first Hamiltonian operator. Such a quantization is successfully constructed by Buryak and Rossi [11] with explicit and recursive formulae for the associated quantum DR Hamiltonians, which we will review in Section 2. An analogous statement to Proposition 1.2 will also be given for the quantum DR hierarchy associated to a rank-1 CohFT.

[2] Available at https://oeis.org/A238641

Organization of the paper. In Section 2, we review the construction by Buryak and Rossi of the quantum DR hierarchy and prove a general symmetry property of the quantum DR Hamiltonians. In Section 3, we study the spectral problem of the quantum KdV hierarchy in details, in particular we prove Proposition 1.2. In Section 4 we give further remarks including an analogous statement to Proposition 1.2 for an arbitrary rank-1 CohFT. In Appendix A we give a representation of the quantum dispersionless KdV Hamiltonians in terms of multiplication operators in the class algebra of the symmetric group.

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2 Buryak–Rossi's quantum DR Hamiltonians and their symmetry property

In this section, we review the construction of classical and quantum double ramification (DR for short) Hamiltonians [7, 11, 12] for a rank l Cohomological Field Theory (CohFT). We will restrict to the rank-1 case in the later sections.

Let $\overline{\mathcal{M}}_{g,n}$ be the moduli space of stable algebraic curves over $\mathbb{C}$ of genus g with n distinct marked points, and let $c_{g,n} \in \mathrm{Hom}(V^{\otimes n}, H^{\mathsf{even}}(\overline{\mathcal{M}}_{g,n}, \mathbb{C}))$ be a CohFT [28]. This means that we are given an l-dimensional complex vector space V, endowed with a non-degenerate symmetric two-form $\eta \in \mathrm{Sym}^2(V^*)$ and a distinguished element $\mathbf{1} \in V$, along with a collection of even cohomology classes $c_{g,n}(v_1 \otimes \cdots \otimes v_n) \in H^{\mathsf{even}}(\overline{\mathcal{M}}_{g,n}, \mathbb{C})$, linearly depending on $v_1 \otimes \cdots \otimes v_n \in V^{\otimes n}$, which satisfy the following axioms. To state them, denote by $e_1 = \mathbf{1}, e_2, \ldots, e_l$ a basis of V, and let $\eta_{\alpha\beta} := \eta(e_\alpha, e_\beta)$ and $\eta^{\alpha\beta}$ the entries of η^{-1}. Here and below, free Greek indices take the integer values $1, \ldots, l$.

- $c_{g,n}$ is $\mathfrak{S}_n$-equivariant, with respect to the action of the symmetric group $\mathfrak{S}_n$ on $V^{\otimes n}$ permuting copies of V and on $\overline{\mathcal{M}}_{g,n}$ permuting marked points.
- Denoting $gl_1 : \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}$ the gluing morphism which identifies the last two marked points of a stable curve, we have
$$gl_1^* c_{g,n}(v_1 \otimes \cdots \otimes v_n) = \eta^{\alpha\beta} c_{g-1,n+2}(v_1 \otimes \cdots \otimes v_n \otimes e_\alpha \otimes e_\beta). \tag{2.1}$$
Here and in what follows, Einstein summation convention is assumed for contracted Greek indexes.
- Denoting $gl_2 : \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g,n}$, $g = g_1 + g_2$ and $n = n_1 + n_2$, the gluing morphism which identifies the last marked points of a pair of stable curves, we have
$$gl_2^* c_{g,n}(v_1 \otimes \cdots \otimes v_n) = \eta^{\alpha\beta} c_{g_1,n_1+1}(v_1 \otimes \cdots \otimes v_{n_1} \otimes e_\alpha) \otimes c_{g_2,n_2+1}(v_{n_1+1} \otimes \cdots \otimes v_n \otimes e_\beta), \tag{2.2}$$
where in the right side we sum over the contracted indexes α, β.
- Denoting $p : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ the map forgetting the last marked point of a stable curve, we have
$$p^* c_{g,n}(v_1 \otimes \cdots \otimes v_n) = c_{g,n+1}(v_1 \otimes \cdots \otimes v_n \otimes e_1). \tag{2.3}$$

- We have

$$c_{0,3}(e_\alpha \otimes e_\beta \otimes e_1) = \eta_{\alpha\beta}. \tag{2.4}$$

We shall assume in the following that we can reduce η to the form

$$\eta_{\alpha\beta} = \delta_{\alpha+\beta,l+1} \tag{2.5}$$

by a change of basis. This is always true for $l = 1$, and follows by the assumption $\eta(e_1, e_1) = 0$ for $l \geq 2$, cf. [16]. In what follows we shall also assume that the CohFT under consideration is *tautological* (see [11, Appendix B]).

Given a rank l CohFT $c_{g,n}$, Buryak [7] associates to it an *integrable* hierarchy of Hamiltonian PDEs in l components $u^1, \dots, u^l$, called the *DR hierarchy*. Denote $u = (u^1, \dots, u^l)$, and denote by $(\mathcal{A}_u, \partial_x)$ the differential polynomial ring of u, namely, an element of $\mathcal{A}_u$ is a polynomial of $u^\alpha_x, u^\alpha_{xx}, u^\alpha_{xxx}, \dots$ whose coefficients are formal power series in u^α with complex coefficients. The DR hierarchy has the form:

$$\frac{\partial u^\alpha}{\partial t^{\beta,m}} = \eta^{\alpha\rho}\partial_x\left(\frac{\delta \overline{h^{\mathsf{DR,cl}}_{\beta,m}}}{\delta u^\rho(x)}\right), \quad m \geq 0, \tag{2.6}$$

where $h^{\mathsf{DR,cl}}_{\beta,m}$ are certain elements in $\mathcal{A}_u[[\epsilon^2]]$, $m \geq 0$, defined below in (2.10). Here "integrable" means that the flows commute pairwise:

$$\frac{\partial}{\partial t^{\gamma,n}}\frac{\partial u^\alpha}{\partial t^{\beta,m}} = \frac{\partial}{\partial t^{\beta,m}}\frac{\partial u^\alpha}{\partial t^{\gamma,n}}. \tag{2.7}$$

Let us recall the construction of $h^{\mathsf{DR,cl}}_{\beta,m}$. Denote the Fourier expansion of u by

$$u^\alpha(x) = \sum_{k\in\mathbb{Z}} U^\alpha_k e^{ikx}. \tag{2.8}$$

For $m = -1$, they are defined by

$$h^{\mathsf{DR,cl}}_{\alpha,-1} = \sum_{k\in\mathbb{Z}} \eta_{\alpha\beta} U^\beta_k e^{ikx}, \tag{2.9}$$

and, for $m \geq 0$, by

$$h^{\mathsf{DR,cl}}_{\alpha,m} = \sum_{\substack{g,n\geq 0\\ 2g-2+n\geq 0\\ k_1,\dots,k_n\in\mathbb{Z}}} \frac{(-\epsilon^2)^g}{n!}\left(\int_{DR_g\left(-\sum_{i=1}^n k_i, k_1,\dots,k_n\right)} \psi_1^m \lambda_g c_{g,n+1}\left(e_\alpha \otimes \bigotimes_{i=1}^n e_{\alpha_i}\right)\right) U^{\alpha_1}_{k_1}\cdots U^{\alpha_n}_{k_n} e^{ix\sum_{i=1}^n k_i}, \tag{2.10}$$

where $DR_g(k_0, \dots, k_n) \in H_{2(2g-2+n)}(\overline{\mathcal{M}}_{g,n+1}, \mathbb{C})$ is the DR cycle [13, 23, 26], $\psi_1 \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{C})$ is the ψ-class, and $\lambda_g \in H^{2g}(\overline{\mathcal{M}}_{g,n}, \mathbb{C})$ is the g-th Chern class of the Hodge bundle. It is proved in [7] that one can view the above-defined $h^{\mathsf{DR,cl}}_{\alpha,m}$ as elements in $\mathcal{A}_u[[\epsilon^2]]$; for example, $h^{\mathsf{DR,cl}}_{\alpha,-1}$ is identified with the differential polynomial $\eta_{\alpha\beta}u^\beta$. The Hamiltonian operator $\eta^{\alpha\beta}\partial_x$ defines a Poisson structure on the space of local functionals $\mathcal{A}_u[[\epsilon^2]]/\partial_x\mathcal{A}_u[[\epsilon^2]]$ in the following way: for any $\overline{f}, \overline{g} \in \mathcal{A}_u[[\epsilon^2]]/\partial_x\mathcal{A}_u[[\epsilon^2]]$,

$$\{\overline{f}, \overline{g}\} := \overline{\frac{\delta \overline{f}}{\delta u^\alpha(x)}\eta^{\alpha\beta}\partial_x\left(\frac{\delta\overline{g}}{\delta u^\beta(x)}\right)} \in \mathcal{A}_u[[\epsilon^2]]/\partial_x\mathcal{A}_u[[\epsilon^2]]. \tag{2.11}$$

The commutativity (2.7) of the DR flows (2.6) then can also be interpreted as

$$\left\{\overline{h^{\mathsf{DR,cl}}_{\alpha,m}}, \overline{h^{\mathsf{DR,cl}}_{\beta,n}}\right\} = 0, \quad \forall\, m, n \geq 0. \tag{2.12}$$

In terms of the Fourier modes (2.8) the Poisson structure (2.11) reads

$$\{U_k^\alpha, U_l^\beta\} = \mathrm{i}k\eta^{\alpha\beta}\delta_{k+l,0}, \quad \forall\, k,l \in \mathbb{Z}. \tag{2.13}$$

Similarly to the rank-1 case considered in the Introduction, let us recall that the canonical quantization with respect to $\{\cdot,\cdot\}$ for the space of local functionals $\mathcal{A}_u[\epsilon]/\partial_x\mathcal{A}_u[\epsilon]$ is defined as follows:

$$\widehat{U}_k^\alpha = \begin{cases} q^{\alpha,k}, & k>0, \\ U_0^\alpha\,\mathrm{id}, & k=0, \\ -\hbar k\eta^{\alpha\beta}\frac{\partial}{\partial q^{\beta,-k}}, & k<0, \end{cases} \tag{2.14}$$

$$\widehat{\overline{a}} = :\int_{S^1} a(u(x);u_x(x),u_{xx}(x),\dots;\epsilon)\,\mathrm{d}x:, \quad \forall\, a\in\mathcal{A}_u[\epsilon], \tag{2.15}$$

where $::$ denotes the normally ordered product. As explained in the Introduction, this means that we first move all U_k^α with $k>0$ to the left of the U_k^α with $k<0$ in each monomial in a, and then we replace each U_k^α by the corresponding operator $\widehat{U}_k^\alpha$. The $\widehat{U}_k^\alpha$ with $k\in\mathbb{Z}$ and the $\widehat{\overline{a}}$ with $a\in\mathcal{A}_u[\epsilon^2]$ are considered as operators on Λ depending on parameters $\hbar,\epsilon$, i.e. formally as elements of $(\mathrm{End}\,\Lambda)\otimes\mathbb{C}[\epsilon,\hbar]$. In such a canonical quantization, the Poisson bracket is replaced by $\frac{1}{\mathrm{i}\hbar}[\cdot,\cdot]$; namely, we have

$$\left[\widehat{U}_k^\alpha, \widehat{U}_j^\beta\right] = -\hbar k\eta^{\alpha\beta}\delta_{k+j,0}, \qquad k,j\in\mathbb{Z}. \tag{2.16}$$

It turns out that the Poisson commutativity (2.12) does not hold after the quantization, namely,

$$\left[\widehat{\overline{h_{\alpha,m}^{\mathsf{DR,cl}}}}, \widehat{\overline{h_{\beta,n}^{\mathsf{DR,cl}}}}\right] \tag{2.17}$$

in general do not vanish. One needs to deform $h_{\alpha,m}^{\mathsf{DR,cl}}$ such that the quantizations of the deformed Hamiltonians commute. A successful and explicit deformation is obtained by Buryak and Rossi using the DR cycles in the moduli spaces of curves [11]. Indeed, introduce a family of elements $h_{\alpha,m}\in\mathcal{A}_u[[\epsilon^2,\hbar]]$, $m\geq -1$ by

$$h_{\alpha,-1}^{\mathsf{DR}} = \sum_{k\in\mathbb{Z}} \eta_{\alpha\beta}U_k^\beta \mathrm{e}^{\mathrm{i}kx} \tag{2.18}$$

and, for $m\geq 0$,

$$h_{\alpha,m}^{\mathsf{DR}} = \sum_{\substack{g,n\geq 0\\ 2g-2+n\geq 0\\ k_1,\dots,k_n\in\mathbb{Z}}} \frac{\hbar^g}{n!}\left(\int_{DR_g\left(-\sum_{i=1}^n k_i,k_1,\dots,k_n\right)} \psi_1^m\Lambda\left(-\frac{\epsilon^2}{\hbar}\right)c_{g,n+1}\left(e_\alpha\otimes\bigotimes_{i=1}^n e_{\alpha_i}\right)\right)U_{k_1}^{\alpha_1}\cdots U_{k_n}^{\alpha_n}\mathrm{e}^{\mathrm{i}x\sum_{i=1}^n k_i}, \tag{2.19}$$

where $\Lambda(\xi) = 1+\lambda_1\xi+\cdots+\lambda_g\xi^g$ is the Chern polynomial of the Hodge bundle, $\lambda_i\in H^{2i}(\overline{\mathcal{M}}_{g,n},\mathbb{C})$. Again, we can and should view $h_{\alpha,m}$ as elements in $\mathcal{A}_u[\epsilon,\hbar]$ via (2.8); for example, $h_{\alpha,-1}=\eta_{\alpha\beta}u^\beta$. For more details about this correspondence with differential polynomials see [7, 11]. Note also that $h_{\alpha,m}^{\mathsf{DR}}|_{\hbar=0} = h_{\alpha,m}^{\mathsf{DR,cl}}$ for all $m\geq -1$.

It is shown by Buryak and Rossi [11] that, denoting for convenience $H_{\alpha,m} = \widehat{\overline{h_{\alpha,m}^{\mathsf{DR}}}}$,

$$[H_{\alpha,m},H_{\beta,n}] = 0, \quad m,n\geq -1. \tag{2.20}$$

(We note that obtaining $H_{\alpha,m}$ is more direct from (2.19).) In particular $H_{1,0}$ has the expression [11, Lemma 3.6]

$$H_{1,0} = \frac{1}{2}\eta_{\alpha\beta}U_0^\alpha U_0^\beta + \hbar\sum_{\alpha=1}^d\sum_{k>0} kq^{\alpha,k}\frac{\partial}{\partial q^{\alpha,k}} + \mathrm{const}, \tag{2.21}$$

which is, up to constants, the grading operator with respect to the degree $\deg q^{\alpha,k} := k$. From this we know that all $H_{\alpha,m}$ preserve such degree.

Although one can obtain $H_{\alpha,m}$ from (2.19) directly, in practice it is actually more effective to compute them via the following recursion, found and proved by Buryak and Rossi [11]:

$$\frac{\partial}{\partial x}(D-1)\, h^{\mathsf{DR}}_{\alpha,m+1} = \frac{1}{\hbar}[h^{\mathsf{DR}}_{\alpha,m}, H_{1,1}], \qquad D := \epsilon\frac{\partial}{\partial\epsilon} + 2\hbar\frac{\partial}{\partial\hbar} + \sum_{\alpha=1}^{d}\sum_{k\geq 0} u^{\alpha}_{kx}\frac{\partial}{\partial u^{\alpha}_{kx}}, \tag{2.22}$$

$$\frac{\partial h^{\mathsf{DR}}_{\alpha,m+1}}{\partial u^1} = h^{\mathsf{DR}}_{\alpha,m}. \tag{2.23}$$

We now discuss a symmetry property of the quantum DR Hamiltonians. The space Λ has a basis given by the monomials

$$q^{\alpha_1,k_1}\cdots q^{\alpha_n,k_n}, \tag{2.24}$$

with $k_i > 0$ and $\alpha_i = 1,\ldots,l$. Introduce a sesquilinear form on Λ, with values in $\mathbb{C}[\hbar]$ defined on the basis elements (2.24) by

$$\begin{aligned}\left\langle q^{\alpha_1,k_1}\cdots q^{\alpha_n,k_n}, q^{\beta_1,j_1}\cdots q^{\beta_n,j_n}\right\rangle &= \hbar^n k_1\cdots k_n \sum_{\sigma\in\mathfrak{S}_n}\prod_{i=1}^{n}\delta_{k_i,j_{\sigma(i)}}\eta^{\alpha_i\beta_{\sigma(i)}} \\ &= \hbar^n k_1\cdots k_n \sum_{\sigma\in\mathfrak{S}_n}\prod_{i=1}^{n}\delta_{k_i,j_{\sigma(i)}}\delta_{\alpha_i+\beta_{\sigma(i)},l+1},\end{aligned} \tag{2.25}$$

cf. (2.5), setting $\langle q^{\alpha_1,k_1}\cdots q^{\alpha_n,k_n}, q^{\beta_1,j_1}\cdots q^{\beta_m,j_m}\rangle = 0$ whenever $n\neq m$. We shall denote by the same symbol $\langle,\rangle$ the natural sesquilinear extension to $\Lambda\otimes_{\mathbb{C}}\mathbb{C}[[\hbar,\epsilon]]$.

Remark 2.1. *For rank-1 CohFTs, the sesquilinear form (2.25) reduces, up to a rescaling in $\hbar$, to the standard inner product (1.25) on the space of symmetric polynomials [30] (cf. also Appendix A).*

Lemma 2.2. *For any $k\in\mathbb{Z}$, the operators $\widehat{U}^{\alpha}_{k}$ and $\widehat{U}^{\alpha}_{-k}$ satisfy the following relation:*

$$\left\langle \widehat{U}^{\alpha}_{k} f, g\right\rangle = \left\langle f, \widehat{U}^{\alpha}_{-k} g\right\rangle, \qquad f,g\in\Lambda. \tag{2.26}$$

Proof. Note that the statement is trivial for $k=0$ and it is symmetric with respect to $k\mapsto -k$. So we only need to consider the case that $k>0$. Take $k=k_1>0$. To show the statement it suffices to verify (2.26) when f,g are the monomials (2.24). The left-hand side of (2.26) can be nonzero only for

$$\begin{aligned}\left\langle \widehat{U}^{\alpha_1}_{k_1} q^{\alpha_2}_{k_2}\cdots q^{\alpha_{n+1}}_{k_{n+1}}, q^{\beta_1}_{j_1}\cdots q^{\beta_{n+1}}_{j_{n+1}}\right\rangle &= \left\langle q^{\alpha_1}_{k_1}\cdots q^{\alpha_{n+1}}_{k_{n+1}}, q^{\beta_1}_{j_1}\cdots q^{\beta_{n+1}}_{j_{n+1}}\right\rangle \\ &= \hbar^{n+1}k_1\cdots k_{n+1}\sum_{\sigma\in\mathfrak{S}_{n+1}}\prod_{i=1}^{n+1}\delta_{k_i,j_{\sigma(i)}}\eta^{\alpha_i\beta_{\sigma(i)}}.\end{aligned} \tag{2.27}$$

On the other hand, the right-hand side of (2.26) is nonzero only for

$$\begin{aligned}&\left\langle q^{\alpha_2,k_2}\cdots q^{\alpha_{n+1},k_{n+1}}, \widehat{U}^{\alpha_1}_{-k_1} q^{\beta_1,j_1}\cdots q^{\beta_{n+1},j_{n+1}}\right\rangle \\ &\quad = \hbar k_1\eta^{\alpha_1\beta}\left\langle q^{\alpha_2,k_2}\cdots q^{\alpha_{n+1},k_{n+1}}, \frac{\partial}{\partial q^{\beta,k_1}}\left(q^{\beta_1,j_1}\cdots q^{\beta_{n+1},j_{n+1}}\right)\right\rangle \\ &\quad = \sum_{i=1}^{n+1}\hbar k_1\eta^{\alpha_1\beta_i}\delta_{k_1,j_i}\left\langle q^{\alpha_2,k_2}\cdots q^{\alpha_{n+1},k_{n+1}}, q^{\beta_1,j_1}\cdots \widehat{q^{\beta_i,j_i}}\cdots q^{\beta_{n+1},j_{n+1}}\right\rangle\end{aligned} \tag{2.28}$$

where $\frown$ denotes omission of the corresponding term. By (2.25) we rewrite the last expression as

$$\hbar^{n+1}k_1\cdots k_{n+1}\sum_{i=1}^{n+1}\eta^{\alpha_1\beta_i}\delta_{k_1,j_i}\sum_{\pi}\delta_{k_2,j_{\pi(2)}}\eta^{\alpha_2\beta_{\pi(2)}}\cdots\delta_{k_{n+1},j_{\pi(n+1)}}\eta^{\alpha_{n+1}\beta_{\pi(n+1)}} \tag{2.29}$$

where the internal sum ranges over all bijections $\pi:\{2,\ldots,n+1\}\to\{1,\ldots,n+1\}\setminus\{i\}$. Changing summation indexes $(i,\pi)\to\rho\in\mathfrak{S}_{n+1}$ where ρ is defined by $\rho(1)=i, \rho(2)=\pi(2),\ldots,\rho(n+1)=\pi(n+1)$ we recognize that the last expression is the same as (2.27). The proof is complete. $\square$

Denote by $\mathcal{A}_u^{\mathbb{R}}$ the subalgebra of $\mathcal{A}_u$ consisting of differential polynomials with real coefficients.

Lemma 2.3. *For any real differential polynomial $a \in \mathcal{A}_u^{\mathbb{R}}$, the operator $A = \widehat{\widehat{a}}$ (cf. (2.15)) is (formally) self-adjoint with respect to (2.25):*

$$\langle Af, g\rangle = \langle f, Ag\rangle, \qquad f, g \in \Lambda. \tag{2.30}$$

Proof. It is enough to show the lemma when a is a monomial of the form $a = \partial_x^{j_1} u^{\alpha_1} \cdots \partial_x^{j_n} u^{\alpha_n}$, for some $n > 0$, $j_1, \dots, j_n \geq 0$, $\alpha_1, \dots, \alpha_n \in \{1, \dots, d\}$. In such case $A = \widehat{\widehat{a}}$ has the form

$$A = \sum_{\substack{k_1,\dots,k_n\in\mathbb{Z}\\ k_1+\dots+k_n=0}} (\mathrm{i}k_1)^{j_1}\cdots(\mathrm{i}k_n)^{j_n} :U_{k_1}^{\alpha_1}\cdots U_{k_n}^{\alpha_n}: \tag{2.31}$$

and so we can explicitly compute the adjoint operator A^*, satisfying $\langle Af, g\rangle = \langle f, A^* g\rangle$ using Lemma 2.2. This yields

$$A^* = \sum_{\substack{k_1,\dots,k_n\in\mathbb{Z}\\ k_1+\dots+k_n=0}} (-\mathrm{i}k_1)^{j_1}\cdots(-\mathrm{i}k_n)^{j_n} :U_{-k_n}^{\alpha_n}\cdots U_{-k_1}^{\alpha_1}: \tag{2.32}$$

and relabeling the indices of the sum by $k_i \mapsto -k_i$ we conclude that $A = A^*$. □

Corollary 2.4. *Assuming that the quantum DR Hamiltonians $H_{\alpha,m} = \widehat{\overline{h_{\alpha,m}^{\mathsf{DR}}}}$ (cf. (2.19) for the definition of $h_{\alpha,m}^{\mathsf{DR}}$) have real coefficients and assuming $U_0 \in \mathbb{R}$, then $H_{\alpha,m}$ are (formally) self-adjoint with respect to (2.25).*

Proof. The statement follows directly from Lemma 2.3. □

It follows from Corollary 2.4 that, under the same reality assumptions, the Buryak–Rossi's quantum Hamiltonians are *diagonalizable.*

Let us now consider the example of the trivial CohFT $\big(V = \mathbb{C},\ \eta = 1,\ c_{g,n} = 1\big)$. In this case, Buryak–Rossi's quantum Hamiltonian densities read as follows:

$$h_{-1}^{\mathsf{DR}} = u(x), \tag{2.33}$$

$$h_m^{\mathsf{DR}} = \sum_{\substack{g\geq 0,\ n\geq 0\\ 2g-2+n\geq 0}} \sum_{k_1,\dots,k_n\in\mathbb{Z}} \sum_{\ell=0}^{g} \frac{(-\epsilon^2)^\ell \hbar^{g-\ell}}{n!} \left(\int_{DR_g\left(-\sum_{i=1}^n k_i, k_1,\dots,k_n\right)} \psi_1^m \lambda_\ell\right) U_{k_1}\cdots U_{k_n}, \quad m \geq 0. \tag{2.34}$$

Here we omit the index α, as we now have $l = 1$. The corresponding quantum Hamiltonians $H_m = \widehat{\overline{h_m^{\mathsf{DR}}}}$ form Buryak–Rossi's quantum KdV hierarchy. Indeed, by comparing (2.33)–(2.34) with (2.10) (taking $c_{g,n} \equiv 1$) one sees that the classical $\hbar \to 0$ limit of $\overline{h_m^{\mathsf{DR}}}$ gives the classical DR Hamiltonians associated to the trivial CohFT, which coincides with the classical KdV Hamiltonians [7]. Recalling the expansion (1.19) and using the dimensional constraint it can be proved that $H_m^{[i]} = 0$ for $i > m + 2$; we observe that also $H_m^{[m+1]} = H_m^{[m+2]} = 0$.

Remark 2.5. *Noting that the intersection number on the DR cycle in (2.34) vanishes unless $m + \ell = 2g - 2 + n$, we find that h_m is homogeneous of degree $m + 2$ with respect to the grading*

$$\widetilde{\deg}\, u = 1, \qquad \widetilde{\deg}\, \epsilon = \frac{1}{2}, \qquad \widetilde{\deg}\, \hbar = 2. \tag{2.35}$$

Denote by

$$\mathcal{H}^{[0]}(z) := 1 + \sum_{m\geq -1} z^{m+2} H_m^{[0]} \tag{2.36}$$

the generating series of the quantum *dispersionless* KdV Hamiltonians, where $H_m^{[0]}$ are defined in (1.19). The following formula is given by Buryak and Rossi [11]:

$$\mathcal{H}^{[0]}(z) = \frac{1}{S(\sqrt{\hbar}z)} : \int e^{zS\left(i\sqrt{\hbar}z\frac{\partial}{\partial x}\right)u} dx :, \tag{2.37}$$

where the series $S(z)$ is defined as

$$S(z) = \frac{\sinh(z/2)}{z/2} = 1 + \frac{z^2}{24} + \frac{z^4}{1920} + \frac{z^6}{322560} + \cdots. \tag{2.38}$$

By comparing with Eliashberg's formula [21], one observes immediately that Buryak–Rossi's quantum *dispersionless* KdV Hamiltonians coincide with Eliashberg's ones.

3 The spectral problem of the quantum KdV hierarchy

In this section we study the spectral problem of the quantum KdV hierarchy and give applications.

3.1 Non-degeneracy property

The following lemma plays a crucial role in the proof of Prop. 1.2.

Lemma 3.1. *For any $w \geq 0$ there exists $m = m(w) \geq 0$ that $E_m^{[0]}(\lambda) \neq E_m^{[0]}(\mu)$ for all $\lambda \neq \mu$, $|\lambda| = w = |\mu|$. Here $E_m^{[0]}(\lambda)$ are the dispersionless eigenvalues, given in (1.30).*

Proof. We first claim that $P_j(\lambda) = P_j(\mu)$ for all $j \geq 0$ if and only if $\lambda = \mu$, where P_j are defined in (1.26). To prove this, we consider the generating series

$$\sum_{j\geq 1} P_j(\lambda)\frac{z^j}{j!} = \sum_{i\geq 1}\left[e^{z\left(\lambda_i - i + \frac{1}{2}\right)} - e^{z\left(-i+\frac{1}{2}\right)}\right] = \sum_{i\geq 1} e^{z\left(\lambda_i - i + \frac{1}{2}\right)} - \frac{1}{2\sinh(z/2)}, \tag{3.1}$$

provided that the complex variable z satisfies $\Re z > 0$. It follows that if $P_j(\lambda) = P_j(\mu)$ for all $j \geq 0$ we have

$$\sum_{i\geq 1} e^{z(\lambda_i - i)} = \sum_{i\geq 1} e^{z(\mu_i - i)} \tag{3.2}$$

which implies $\lambda_i = \mu_i$ for all $i \geq 1$, and so the partitions λ and μ must coincide, as claimed. It follows that for any pair of partitions $\lambda \neq \mu$, with $|\lambda| = |\mu|$, there exists $j = j(\lambda,\mu)$ such that $Q_j(\lambda) \neq Q_j(\mu)$, where Q_j are defined in (1.26). By looking at (1.30), namely

$$E_m(\lambda;\hbar,U_0) = \sum_{j=0}^{m+2} \frac{\sqrt{\hbar^j}U_0^{m+2-j}}{(m+2-j)!}Q_j(\lambda), \tag{3.3}$$

we conclude that as soon as $m > j(\lambda,\mu)$ for all λ,μ of the same weight $w = |\lambda| = |\mu|$, some coefficient of U_0 in $E_m^{[0]}(\lambda;\hbar,U_0)$ is different from the corresponding coefficient in $E_m^{[0]}(\mu;\hbar,U_0)$. $\square$

3.2 Proof of Proposition 1.2

Let us introduce

$$H_m = \sqrt{\hbar^{m+2}}K_m, \qquad U_k = \sqrt{\hbar}V_k, \qquad \sigma := -\frac{\epsilon^2}{\sqrt{\hbar}}, \tag{3.4}$$

so that (2.33) and (2.34) read as

$$K_{-1} = V_0, \tag{3.5}$$

$$K_m = \sum_{\substack{g\geq 0,\ n\geq 0\\ 2g-2+n\geq 0}} \sum_{\substack{k_1,\dots,k_n\in\mathbb{Z}\\ k_1+\cdots+k_n=0}} \frac{\sigma^{2g-2+n-m}}{n!}\left(\int_{DR_g(0,k_1,\dots,k_n)} \psi_1^m \lambda_{2g-2+n-m}\right) : \widehat{V}_{k_1}\cdots\widehat{V}_{k_n} : \quad (m \geq 0), \tag{3.6}$$

where we have used the dimensional constraint $m+\ell=2g-2+n$ in (2.34) (cf. Remark 2.5). Let us also introduce

$$q_k=\sqrt{\hbar}T_k,\quad U_0=\sqrt{\hbar}V_0. \tag{3.7}$$

Then the operators $\widehat{V}_k$ act on $\mathbb{C}[T_1,T_2,\dots]$. Explicitly,

$$\widehat{V}_k: f(T)\mapsto\begin{cases}T_kf(T), & k>0,\\ V_0f(T), & k=0,\\ -k\frac{\partial f(T)}{\partial T_{-k}}, & k<0.\end{cases} \tag{3.8}$$

In particular $H_m^{[0]}=\sqrt{\hbar^{m+2}}K_m^{[0]}$, where $K_m^{[0]}:=K_m|_{\sigma=0}$; therefore Dubrovin's result (cf. Theorem 1.1) states that

$$K_m^{[0]}s_\lambda(T)=F_m^{[0]}(\lambda;V_0)s_\lambda(T), \tag{3.9}$$

where $s_\lambda(T)$ are the Schur polynomials (1.21), and

$$F_m^{[0]}(\lambda;V_0)=\frac{1}{\sqrt{\hbar^{m+2}}}E_m^{[0]}(\lambda;\hbar,\sqrt{\hbar}V_0)=\sum_{j=0}^{m+2}\frac{V_0^{m+2-j}}{(m+2-j)!}Q_j(\lambda), \tag{3.10}$$

which is also independent of $\hbar$. The property (1.32) in the statement of Proposition 1.2 is then equivalent to

$$K_mr_\lambda(T;\sigma)=F_m(\lambda;\sigma,V_0)\,r_\lambda(T;\sigma), \tag{3.11}$$

for some $F_m(\lambda;\sigma,V_0)\in\mathbb{C}(V_0)[\![\sigma]\!]$; the relation with $E_m(\lambda;\hbar,\epsilon,U_0)$ in the statement of Proposition 1.2 is

$$E_m(\lambda;\hbar,\epsilon,U_0)=\sqrt{\hbar^{m+2}}F_m\left(\lambda;\sigma=-\frac{\epsilon^2}{\sqrt{\hbar}},V_0=\frac{U_0}{\sqrt{\hbar}}\right) \tag{3.12}$$

Therefore, let us first prove that (1.31) and (1.32) (the latter equivalently rewritten as (3.11)) imply (1.33). Indeed, looking at (3.11) for $\sigma=0$ we obtain that $r_\lambda(T;\sigma=0)$ is a basis on which $H_m^{[0]}$ are diagonal, and it follows from Theorem 1.1 that $r_\lambda(T;\sigma=0)$ must be a linear combination of the Schur polynomials $s_\mu(T)$ for which $E_m^{[0]}(\lambda)=E_m^{[0]}(\mu)$ for all $m\geq-1$; Lemma 3.1 implies then that $r_\lambda(T;\sigma=0)=c\,s_\lambda(T)$, for some $c\in\mathbb{C}$, and $E_m(\lambda;\sigma=0)=E_m^{[0]}(\lambda)$. Finally $c=1$ due to (1.31).

Let us proceed and prove the existence and the uniqueness of $r_\lambda(T;\sigma)$. Denote $r_\lambda(T;\sigma)=\sum_{k\geq0}r_\lambda^{[k]}(T)\sigma^k$ and $F_m(\lambda;\sigma)=\sum_{k\geq0}F_m^{[k]}(\lambda)\sigma^k$; we will show existence and uniqueness of the coefficients $r_\lambda^{[k]}(T)$ and $F_m^{[k]}$ for $k\geq0$, such that (1.31) and (1.32) hold true.

The case $k=0$, namely that $r_\lambda^{[0]}(T)=s_\lambda(T)$ and that $F_m^{[0]}(\lambda)$ is given by (3.10), follows from Dubrovin's Theorem 1.1, as explained above. Next, consider (3.11) at order σ^k for some $k\geq1$ and take the inner product, using (1.25), with $s_\mu(T)$ (for $|\mu|=|\lambda|$)

$$\sum_{j=0}^k\left\langle s_\mu(T),K_m^{[j]}r_\lambda^{[k-j]}(T)\right\rangle=\sum_{j=0}^kF_m^{[j]}(\lambda)\left\langle s_\mu(T),r_\lambda^{[k-j]}(T)\right\rangle. \tag{3.13}$$

Denote $\left\langle r_\lambda^{[\ell]}(T),s_\lambda(T)\right\rangle=r_{\lambda\mu}^{[\ell]}$; the normalization (1.31) implies that $r_{\lambda\lambda}^{[\ell]}=\delta_{\ell,0}$. Exploiting the symmetry property of the H_m (Corollary 2.4) we infer

$$\sum_{j=0}^k\left\langle K_m^{[j]}s_\mu(T),r_\lambda^{[k-j]}(T)\right\rangle=\sum_{j=0}^kF_m^{[j]}(\lambda)r_{\lambda\mu}^{[k-j]}, \tag{3.14}$$

and by separating terms corresponding to $j=0$ in the sums, we obtain

$$\left(F_m^{[0]}(\lambda)-F_m^{[0]}(\mu)\right)r_{\lambda\mu}^{[k]}=\sum_{j=1}^k\left\langle K_m^{[j]}s_\mu(T),r_\lambda^{[k-j]}(T)\right\rangle-\sum_{j=1}^kF_m^{[j]}(\lambda)r_{\lambda\mu}^{[k-j]}. \tag{3.15}$$

Write then the last relation in the case $\mu = \lambda$

$$F_m^{[k]}(\lambda) = \sum_{j=1}^{k} \sum_{\nu \in \mathscr{P}_{|\lambda|}} \left\langle K_m^{[j]} s_\lambda(T), s_\nu(T) \right\rangle r_{\lambda\nu}^{[k-j]}, \tag{3.16}$$

and in the case $\mu \neq \lambda$

$$\left(F_m^{[0]}(\lambda) - F_m^{[0]}(\mu) \right) r_{\lambda\mu}^{[k]} = \sum_{j=1}^{k} \left\langle K_m^{[j]} s_\mu(T), r_\lambda^{[k-j]}(T) \right\rangle - \sum_{j=1}^{k-1} F_m^{[j]}(\lambda) r_{\lambda\mu}^{[k-j]}. \tag{3.17}$$

For any $w \geq 0$, let us first consider $m_* \geq -1$ such that $F_{m_*}^{[0]}(\lambda) \neq F_{m_*}^{[0]}(\mu)$ whenever λ, μ are distinct partitions of w (cf. Lemma 3.1). Since we already know $F_{m_*}^{[0]}$ and $r_{\lambda\mu}^{[0]}$, we can use (3.16) by induction on k to obtain $F_{m_*}^{[k]}(\lambda)$ for all $k \geq 0$ and all λ partitions of w and to obtain $r_{\lambda\mu}^{[k]}$ for all $k \geq 0$ and all distinct partitions $\lambda \neq \mu$ of w. This proves uniqueness of $r_\lambda(T;\sigma)$ and $F_{m_*}(\lambda;\sigma)$; moreover, by construction, we conclude that (3.11) with $m = m_*$ holds true for all λ partitions of w. Next, for $\lambda \neq \mu$ being any distinct partitions of w, we have, exploiting symmetry of the Hamiltonians,

$$F_{m_*}(\mu) \langle r_\lambda, r_\mu \rangle = \langle r_\lambda, K_{m_*} r_\mu \rangle = \langle K_{m_*} r_\lambda, r_\mu \rangle = F_{m_*}(\lambda) \langle r_\lambda, r_\mu \rangle,$$

which proves, thanks to Lemma 3.1, that $\langle r_\lambda, r_\mu \rangle$ vanishes. Thus $r_\lambda(T;\sigma)$ form a basis of the free module $\widetilde{\Lambda} \otimes_{\mathbb{C}} \mathbb{C}[\![\sigma]\!]$. Finally, for any m, by commutativity and symmetry of the Hamiltonians,

$$F_{m_*}(\mu) \langle K_m r_\lambda, r_\mu \rangle = \langle K_m r_\lambda, K_{m_*} r_\mu \rangle = \langle K_{m_*} K_m r_\lambda, r_\mu \rangle = \langle K_m K_{m_*} r_\lambda, r_\mu \rangle = F_{m_*}(\lambda) \langle K_m r_\lambda, r_\mu \rangle,$$

from which, using again Lemma 3.1, we conclude that $\langle K_m r_\lambda, r_\mu \rangle$ vanishes for all $\lambda \neq \mu$. This implies, as $r_\lambda(T;\sigma)$ form a basis, that

$$K_m r_\lambda(T;\sigma) = F_m(\lambda;\sigma) r_\lambda(T;\sigma)$$

for some scalar-valued series $F_m(\lambda;\sigma) = \sum_{k\geq 0} F_m^{[k]}(\lambda)\sigma^k$. The proof is complete. □

In the above proof, the nondegeneracy property given in Lemma 3.1 is used. We note that a weaker form of the nondegeneracy property (namely that for a fixed $m \geq 1$, if $E_m^{[0]}(\lambda) = E_m^{[0]}(\mu)$ for some $\lambda \neq \mu$ then there exists $k > m$ such that $E_k^{[0]}(\lambda) \neq E_k^{[0]}(\mu)$) is actually sufficient for the proof, and that we can prove this weaker property for the quantum KdV Hamiltonians H_m even when H_m are restricted to $U_0 = 0$. This could also be helpful for other quantum integrable systems.

A natural question for the quantum KdV hierarchy is about the analyticity in σ of their spectrum and common eigenvectors. A way to answer this question is to look at the spectral curves (1.35) (see the Introduction). Indeed, the spectral curves are ramified coverings of the Riemann sphere $\sigma \in \mathbb{P}^1$ of degree $|\mathscr{P}_k|$. So the eigenvalues $F_m(\lambda;\sigma)$ above are the Taylor series at $\sigma = 0$ of the branches of the analytic functions ρ on $\Sigma_{k,m}$. Similarly, $r_{\lambda\mu}(\sigma)$ are meromorphic functions on the spectral curves $\Sigma_{k,m}$. The global geometry of the spectral curves is interesting. For example, it is possible to prove that at $\sigma = \infty$ the eigenvectors are the monomials $T_\lambda = T_{\lambda_1} \cdots T_{\lambda_{\ell(\lambda)}}$ and the corresponding branches of ρ behave as

$$\rho \sim \sigma^m \sum_{i=1}^{\ell(\lambda)} \lambda_i^{2m+1}, \quad \sigma \to \infty, \tag{3.18}$$

while, from Theorem 1.1 we know that at $\sigma = 0$ the eigenvectors are the Schur polynomials $s_\lambda(T)$; thus, such a deformation of Schur polynomials interpolates between the Schur and monomial bases of $\widetilde{\Lambda}$. Moreover, due to the symmetry property of the Hamiltonians, the branch points can only be complex conjugate points in the σ-plane; coincidence of eigenvalues for real values of σ yields instead singularities of the spectral curves. Further study of the geometry of spectral curves is deferred to future investigations.

Remark 3.2. *The initial value problem*

$$\hbar \frac{\partial}{\partial y_m} \Psi(\underline{y}) = H_m \Psi(\underline{y}), \quad m \geq 0, \tag{3.19}$$

$$\Psi|_{\underline{y}=0} = \sum_{\lambda \in \mathscr{P}} c_\lambda r_\lambda \left(\frac{q}{\sqrt{\hbar}}; -\frac{\epsilon^2}{\sqrt{\hbar}} \right), \tag{3.20}$$

where c_λ are arbitrarily given constants and $\underline{y} = (y_0, y_1, y_2, \dots)$, can be solved as

$$\Psi(\underline{y}) = \sum_{\lambda \in \mathscr{P}} c_\lambda \mathrm{e}^{\frac{1}{\hbar} \sum_{m \geq 0} y_m E_m(\lambda; \epsilon, \hbar, U_0)} r_\lambda \left(\frac{q}{\sqrt{\hbar}}; -\frac{\epsilon^2}{\sqrt{\hbar}} \right). \tag{3.21}$$

The classical property

$$s_{\lambda'}(T) = (-1)^{|\lambda|} s_\lambda(-T) \tag{3.22}$$

of the Schur polynomials, where λ' denotes the conjugate partition of λ (i.e. the diagram of λ' is obtained by flipping that of λ along its main diagonal), generalizes to a similar property of the deformed Schur polynomials.

Proposition 3.3. *For all $\lambda \in \mathscr{P}$ we have*

$$r_{\lambda'}(T; \sigma) = (-1)^{|\lambda|} r_\lambda(-T; -\sigma). \tag{3.23}$$

Proof. Let us denote $\widetilde{r}_\lambda(T; \sigma) = (-1)^{|\lambda|} r_{\lambda'}(-T; -\sigma)$; from (3.22) we obtain

$$\langle \widetilde{r}_\lambda(T; \sigma), s_\lambda(T) \rangle = \langle r_{\lambda'}(-T; -\sigma), s_{\lambda'}(-T) \rangle = 1, \tag{3.24}$$

where we also use the property (1.31) of deformed Schur polynomials. Next, it follows from Remark 2.5 that all quantum KdV Hamiltonians are homogeneous of degree $m + 2$ with respect to the degree assignment

$$\widetilde{\deg}\, q_k = 1, \qquad \widetilde{\deg}\, \epsilon = \frac{1}{2}, \qquad \widetilde{\deg}\, \hbar = 2. \tag{3.25}$$

Let us denote by $H_m = H_m(\epsilon, \hbar)$ the explicit dependence on the parameters and let us also introduce the involution $\Pi : \Lambda \mapsto \Lambda$ defined by $q_i \mapsto -q_i$ on the generators of the polynomial ring Λ. Thus, from the homogeneity property just mentioned (see (3.25)) we have

$$\Pi H_m(\mathrm{i}\epsilon, \hbar) \Pi = (-1)^m H_m(\epsilon, \hbar). \tag{3.26}$$

It follows then that $H_m(\mathrm{i}\epsilon, \hbar)$ is diagonal on $\Pi r_\lambda(q/\sqrt{\hbar}; \epsilon^2/\sqrt{\hbar}) = r_\lambda(-q/\sqrt{\hbar}; \epsilon^2/\sqrt{\hbar})$, hence it is diagonal on $\widetilde{r}_\lambda(q/\sqrt{\hbar}; -\epsilon^2/\sqrt{\hbar})$. Thus the elements $\widetilde{r}_\lambda(T; \sigma)$ satisfy the two defining properties (1.31) and (1.32) uniquely characterizing the deformed Schur polynomials (cf. Proposition 1.2) and so they must be equal to $r_\lambda(T; \sigma)$. $\square$

We give a few more examples of $r_\lambda(T; \sigma)$.

$$r_{(3)} = s_{(3)} + s_{(2,1)} \left(-\frac{2\sigma}{9} + \frac{\sigma^2}{324} + \frac{43\sigma^3}{5832} + \frac{193\sigma^4}{559872} + \mathcal{O}(\sigma^5) \right) + s_{(1,1,1)} \left(\frac{5\sigma}{72} + \frac{2\sigma^2}{81} - \frac{893\sigma^3}{373248} - \frac{115\sigma^4}{69984} + \mathcal{O}(\sigma^5) \right),$$

$$r_{(2,1)} = s_{(3)} \left(\frac{2\sigma}{9} + \frac{\sigma^2}{81} - \frac{2\sigma^3}{729} - \frac{\sigma^4}{729} + \mathcal{O}(\sigma^5) \right) + s_{(2,1)} + s_{(1,1,1)} \left(-\frac{2\sigma}{9} + \frac{\sigma^2}{81} + \frac{2\sigma^3}{729} - \frac{\sigma^4}{729} + \mathcal{O}(\sigma^5) \right),$$

$$
\begin{aligned}
r_{(4)} &= s_{(4)} + s_{(3,1)}\left(-\frac{5\sigma}{16} + \frac{\sigma^2}{192} + \frac{6055\sigma^3}{331776} + \mathcal{O}(\sigma^4)\right) + s_{(2,2)}\left(-\frac{5\sigma}{72} + \frac{59\sigma^2}{2592} + \frac{4715\sigma^3}{1492992} + \mathcal{O}(\sigma^4)\right) \\
&\quad + s_{(2,1,1)}\left(\frac{\sigma}{8} + \frac{37\sigma^2}{768} - \frac{727\sigma^3}{82944} + \mathcal{O}(\sigma^4)\right) + s_{(1,1,1,1)}\left(-\frac{7\sigma}{144} - \frac{95\sigma^2}{2592} - \frac{9119\sigma^3}{2985984} + \mathcal{O}(\sigma^4)\right), \\
r_{(3,1)} &= s_{(4)}\left(\frac{5\sigma}{16} + \frac{\sigma^2}{32} - \frac{7\sigma^3}{4096} + \mathcal{O}(\sigma^4)\right) + s_{(3,1)} + s_{(2,2)}\left(-\frac{\sigma}{8} - \frac{\sigma^2}{32} - \frac{13\sigma^3}{2048} + \mathcal{O}(\sigma^4)\right) \\
&\quad + s_{(2,1,1)}\left(-\frac{5\sigma}{16} + \frac{3\sigma^2}{64} + \frac{35\sigma^3}{4096} + \mathcal{O}(\sigma^4)\right) + s_{(1,1,1,1)}\left(\frac{\sigma}{8} + \frac{11\sigma^2}{256} - \frac{7\sigma^3}{1024} + \mathcal{O}(\sigma^4)\right), \\
r_{(2,2)} &= s_{(4)}\left(\frac{5\sigma}{72} + \frac{37\sigma^2}{1296} - \frac{133\sigma^3}{46656} + \mathcal{O}(\sigma^4)\right) + s_{(3,1)}\left(\frac{\sigma}{8} - \frac{\sigma^2}{48} - \frac{31\sigma^3}{5184} + \mathcal{O}(\sigma^4)\right) + s_{(2,2)} \\
&\quad + s_{(2,1,1)}\left(-\frac{\sigma}{8} - \frac{\sigma^2}{48} + \frac{31\sigma^3}{5184} + \mathcal{O}(\sigma^4)\right) + s_{(1,1,1,1)}\left(-\frac{5\sigma}{72} + \frac{37\sigma^2}{1296} + \frac{133\sigma^3}{46656} + \mathcal{O}(\sigma^4)\right).
\end{aligned}
$$

3.3 Vanishing identities

As a consequence of Proposition 1.2, we get some vanishing identities for certain combination of characters in the symmetric group. For convenience, let us denote by $\mathscr{P}_k$ the set of partitions λ of weight $|\lambda| = k$ and set

$$
\widetilde{H}_m^{[i]} := H_m^{[i]}\Big|_{U_0=0}, \qquad \widetilde{E}_m^{[0]}(\lambda;\hbar) = E_m^{[0]}(\lambda;\hbar,U_0)\Big|_{U_0=0}. \tag{3.27}
$$

By (1.30) we infer that $\widetilde{E}^{[0]}$ is given by

$$
\widetilde{E}_m^{[0]}(\lambda;\hbar) = \sqrt{\hbar^{m+2}}Q_{m+2}(\lambda), \tag{3.28}
$$

where Q_j are given in (1.27).

Lemma 3.4. *Suppose $\lambda,\mu \in \mathscr{P}_k$ are distinct partitions of the same weight, $\lambda \neq \mu$, satisfying $\widetilde{E}_m^{[0]}(\lambda;\hbar) = \widetilde{E}_m^{[0]}(\mu;\hbar)$. Then*

$$
\left\langle s_\lambda\left(\frac{q}{\sqrt{\hbar}}\right), \widetilde{H}_m^{[1]} s_\mu\left(\frac{q}{\sqrt{\hbar}}\right)\right\rangle = 0. \tag{3.29}
$$

Proof. Taking terms of order ϵ in the identity $[H_m, H_{m'}] = 0$ we have, after setting $U_0 = 0$,

$$
0 = [\widetilde{H}_m^{[0]}, \widetilde{H}_{m'}^{[1]}] + [\widetilde{H}_m^{[1]}, \widetilde{H}_{m'}^{[0]}]. \tag{3.30}
$$

Taking the matrix entry (λ,μ) of this relation, using the symmetry property of Corollary 2.4, and denoting $T = q/\sqrt{\hbar}$ we get

$$
\begin{aligned}
0 &= \left\langle s_\lambda(T), \left([\widetilde{H}_m^{[0]}, \widetilde{H}_{m'}^{[1]}] + [\widetilde{H}_m^{[1]}, \widetilde{H}_{m'}^{[0]}]\right) s_\mu(T)\right\rangle \\
&= \left\langle \widetilde{H}_m^{[0]} s_\lambda(T), \widetilde{H}_{m'}^{[1]} s_\mu(T)\right\rangle - \left\langle \widetilde{H}_{m'}^{[1]} s_\lambda(T), \widetilde{H}_m^{[0]} s_\mu(T)\right\rangle \\
&\quad + \left\langle \widetilde{H}_m^{[1]} s_\lambda(T), \widetilde{H}_{m'}^{[0]} s_\mu(T)\right\rangle - \left\langle \widetilde{H}_{m'}^{[0]} s_\lambda(T), \widetilde{H}_m^{[1]} s_\mu(T)\right\rangle \\
&= \left(\widetilde{E}_m^{[0]}(\lambda;\hbar) - \widetilde{E}_m^{[0]}(\mu;\hbar)\right)\left\langle s_\lambda(T), \widetilde{H}_{m'}^{[1]} s_\mu(T)\right\rangle - \left(\widetilde{E}_{m'}^{[0]}(\lambda;\hbar) - \widetilde{E}_{m'}^{[0]}(\mu;\hbar)\right)\left\langle s_\lambda(T), \widetilde{H}_m^{[1]} s_\mu(T)\right\rangle.
\end{aligned} \tag{3.31}
$$

Now assume λ,μ are as in the statement; using the first part in the proof of Lemma 3.1, we must have $\widetilde{E}_{m'}^{[0]}(\lambda;\hbar) \neq \widetilde{E}_{m'}^{[0]}(\mu;\hbar)$ for some m', hence we have $\left\langle s_\lambda(\widetilde{q}), \widetilde{H}_m^{[1]} s_\mu(\widetilde{q})\right\rangle = 0$. □

Rephrasing the statement of Lemma 3.4 for $m = 1$, we get the more explicit identity given in the following corollary.

Corollary 3.5. *Suppose $\lambda, \mu \in \mathscr{P}_k$ are distinct partitions, $\lambda \neq \mu$, satisfying $P_2(\lambda) = P_2(\mu)$, where P_2 is defined in (1.26). Then*

$$\sum_{\nu \in \mathscr{P}_k} |C_\nu| \chi_\lambda(C_\nu) \chi_\mu(C_\nu) \sum_{i=1}^{\ell(\nu)} \nu_i^3 = 0, \tag{3.32}$$

where $C_\nu \subset \mathfrak{S}_k$ is the conjugacy class of permutations with disjoint cycles of lengths $\nu_1, \dots, \nu_{\ell(\nu)}$ and χ_λ is the character of the irreducible representation of $\mathfrak{S}_k$ associated to λ.

Proof. By using the explicit formula (see e.g. [11, Section 4.1])

$$h_1|_{U_0=0} = \frac{u^3}{6} + \frac{\epsilon^2}{24} u u_{xx} - \frac{\hbar \epsilon^2}{2880}, \tag{3.33}$$

we obtain

$$H_1|_{U_0=0} = \Delta - \frac{\hbar \epsilon^2}{12} \left(\sum_{i \geq 1} i^3 q_i \frac{\partial}{\partial q_i} + \frac{1}{240} \right), \tag{3.34}$$

where Δ is the cut-and-join operator, see (1.18). Hence, since $\lambda \neq \mu$ are assumed to satisfy $P_2(\lambda) = P_2(\mu)$, we have $\widetilde{E}_1^{[0]}(\lambda; \hbar) = \widetilde{E}_1^{[0]}(\mu; \hbar)$, cf. (1.27) and (3.28), and so Lemma 3.4 yields

$$0 = \left\langle s_\lambda \left(\frac{q}{\sqrt{\hbar}} \right), \widetilde{H}_1^{[1]} s_\mu \left(\frac{q}{\sqrt{\hbar}} \right) \right\rangle = \left\langle s_\lambda \left(\frac{q}{\sqrt{\hbar}} \right), \sum_{i \geq 1} i^3 q_i \frac{\partial}{\partial q_i} s_\mu \left(\frac{q}{\sqrt{\hbar}} \right) \right\rangle. \tag{3.35}$$

By expanding the Schur polynomials on the basis of monomials with the help of

$$s_\lambda \left(\frac{q}{\sqrt{\hbar}} \right) = \sum_{\nu \in \mathscr{P}_k} \frac{|C_\nu| \, \chi_\lambda(C_\nu)}{k!} \frac{q_\nu}{\hbar^{\ell(\nu)/2}}, \tag{3.36}$$

(see [30]) and by using (A.6), the last equation can be rewritten as (3.32). □

4 Further remarks

This paper is a first step in the study of the spectral problems for the quantum KdV hierarchy following Dubrovin's suggestion [17, 18]. In particular, we have proved that the quantum KdV Hamiltonians are symmetric with respect to a natural inner product (this property is actually generalized and proved for any tautological CohFT; cf. Corollary 2.4) and that the quantum dispersionless KdV hierarchy possesses a non-degeneracy property. Moreover, we show that these two properties imply the existence of a complete set of common eigenvectors for the quantum KdV Hamiltonians. As an application, we obtain some vanishing identities for certain combinations of characters in the symmetric group; a simple example of this phenomenon is given in Corollary 3.5. More applications will be given in subsequent publications.

Let us note that Proposition 1.2 can be generalized to the quantum DR hierarchy associated to an arbitrary rank-1 CohFT (cf. [8, 11, 19, 34]). More precisely, recall that the quantum Hamiltonian densities for this hierarchy are

$$h_m^{\mathsf{Hodge}}(\underline{s}) = \sum_{\substack{g,n \geq 0 \\ 2g-2+n \geq 0 \\ k_1,\dots,k_n \in \mathbb{Z}}} \frac{\hbar^g}{n!} \left(\int_{DR_g\left(-\sum_{i=1}^n k_i, k_1, \dots, k_n\right)} \psi_1^m \Lambda\left(-\frac{\epsilon^2}{\hbar}\right) \mathrm{e}^{\sum_{j \geq 1} \mathrm{ch}_{2j-1} s_{2j-1}} \right) U_{k_1} \cdots U_{k_n} \mathrm{e}^{\mathrm{i}x \sum_{i=1}^n k_i}, \tag{4.1}$$

where $\underline{s} = (s_1, s_3, s_5, \dots)$ and ch_{2j-1} denotes the $(2j-1)$th component of the Chern character of the Hodge bundle on the moduli space of curves. We have the following proposition, where we denote $H_m^{\mathsf{Hodge}} = \widehat{h_m^{\mathsf{Hodge}}}$ the associated quantum DR Hamiltonians

Proposition 4.1. *There exists a unique collection* $(r_\lambda(T;\sigma,\underline{s}))_{\lambda\in\mathscr{P}}$ *of elements in the free module* $\mathbb{C}[T_1,T_2,\dots]\otimes_{\mathbb{C}}\mathbb{C}[\![\sigma,\underline{s}]\!]$ *over* $\mathbb{C}[\![\sigma,\underline{s}]\!]$ *such that*

$$\langle r_\lambda(T;\sigma,\underline{s}), s_\lambda(T)\rangle = 1, \tag{4.2}$$

$$H_m^{\mathsf{Hodge}} r_\lambda\left(\frac{q}{\sqrt{\hbar}};-\frac{\epsilon^2}{\sqrt{\hbar}},\underline{s}\right) = E_m\left(\lambda;\epsilon,\hbar,\underline{s}\right) r_\lambda\left(\frac{q}{\sqrt{\hbar}};-\frac{\epsilon^2}{\sqrt{\hbar}},\underline{s}\right), \tag{4.3}$$

for some $E_m(\lambda;\epsilon,\hbar,\underline{s})\in\mathbb{C}(\sqrt{\hbar})[\![\epsilon^2,\underline{s}]\!]$. *Moreover,* $(r_\lambda(T;\sigma,\underline{s}))_{\lambda\in\mathscr{P}}$ *is basis of the free module* $\mathbb{C}[T_1,T_2,\dots]\otimes_{\mathbb{C}}\mathbb{C}[\![\sigma,\underline{s}]\!]$, *and*

$$r_\lambda(T;\sigma=0,\underline{s}=0) = s_\lambda(T), \qquad E_m(\lambda;\epsilon=0,\hbar,\underline{s}=0) = E_m^{[0]}(\lambda;\hbar), \tag{4.4}$$

where $E_m^{[0]}$ *are defined in* (1.30).

The proof of this proposition is analogous to that of Proposition 1.2 and is therefore omitted.

We end this paper by giving an equivalent form of Buryak–Rossi's quantum DR recursion in the rank-1 case. Indeed, by using the dimensional constraint in (4.3), we infer that h_m^{Hodge} is homogeneous of degree $m+2$ with respect to the grading

$$\widetilde{\deg}\, u = 1, \qquad \widetilde{\deg}\,\epsilon = \frac{1}{2}, \qquad \widetilde{\deg}\,\hbar = 2, \qquad \widetilde{\deg}\, s_{2k-1} = -2k+1. \tag{4.5}$$

In particular, this implies the relation

$$\left(\frac{\epsilon}{2}\partial_\epsilon + 2\hbar\partial_\hbar - \sum_{k\ge1}(2k-1)s_{2k-1}\partial_{s_{2k-1}} + \sum_{k\ge0} u_{kx}\partial_{u_{kx}}\right)h_m^{\mathsf{Hodge}} = (m+2)h_m^{\mathsf{Hodge}}, \tag{4.6}$$

which can be combined with Buryak–Rossi's recursion (2.22) to give an alternative recursion

$$\left(m+1+\frac{\epsilon}{2}\frac{\partial}{\partial\epsilon} + \sum_{k\ge1}(2k-1)s_{2k-1}\partial_{s_{2k-1}}\right)\partial_x h_m^{\mathsf{Hodge}} = \frac{1}{\hbar}\left[h_{m-1}^{\mathsf{Hodge}}, H_1^{\mathsf{Hodge}}\right]. \tag{4.7}$$

In terms of the generating function $\mathcal{H}^{\mathsf{Hodge}}(z) := \sum_{m\ge-1} z^{m+1}h_m^{\mathsf{Hodge}}$ it gives

$$\mathscr{D}\,\mathcal{H}^{\mathsf{Hodge}} = 0, \qquad \mathscr{D} = z\frac{\partial}{\partial z} + \frac{\epsilon}{2}\frac{\partial}{\partial\epsilon} + \sum_{k\ge1}(2k-1)s_{2k-1}\partial_{s_{2k-1}} + \frac{z\partial_x^{-1}}{\hbar}\left[H_1^{\mathsf{Hodge}},\cdot\right]. \tag{4.8}$$

We note that a similar equation to (4.8) is also given in [8] (cf. Theorem 5.1 therein). Taking the $\hbar\to0$ limit in (4.8) we find

$$\mathscr{D}^{\mathsf{cl}}\,\mathcal{H}^{\mathsf{Hodge,cl}} = 0, \qquad \mathscr{D}^{\mathsf{cl}} = z\frac{\partial}{\partial z} + \frac{\epsilon}{2}\frac{\partial}{\partial\epsilon} + \sum_{k\ge1}(2k-1)s_{2k-1}\partial_{s_{2k-1}} + z\partial_x^{-1}\left\{H_1^{\mathsf{Hodge,cl}},\cdot\right\}. \tag{4.9}$$

A Quantum Hopf hierarchy and Young-Jucys-Murphy elements

As a consequence of Theorem 1.1 [17] we prove in this appendix a representation of the quantum dispersionless KdV Hamiltonians $H_m^{[0]}$ in terms of multiplication operators in the class algebra of the symmetric groups.

With respect to the gradation $\deg q_k = k$, consider the direct sum decomposition $\Lambda = \bigoplus_{k\ge0}\Lambda_k$ into homogeneous components. Denote by $\mathfrak{S}_k$ the symmetric group of permutations of $\{1,\dots,k\}$, by $\mathbb{C}[\mathfrak{S}_k]$ its group algebra, and introduce the *Frobenius map*[3]

$$\Phi : \mathbb{C}\left[\mathfrak{S}_k\right] \to \Lambda_k\left[1/\sqrt{\hbar}\right] \tag{A.1}$$

$$\sigma \mapsto \frac{q_\lambda}{\hbar^{\ell/2}k!}, \tag{A.2}$$

[3] It is convenient here to include the quantization parameter $\hbar$, though it may be absorbed by the scaling $T = q/\sqrt{\hbar}$.

defined here for permutations σ with ℓ disjoint cycles of lengths $\lambda_1 \ldots, \lambda_\ell$ and extended linearly, denoting by q_λ the monomial $q_\lambda := q_{\lambda_1} \cdots q_{\lambda_\ell}$.

It is well known [25] that $\Phi^{-1}\Delta\Phi = \Phi^{-1}H_1^{[0]}\big|_{U_0=0}\Phi$, see (1.18), is the operator in the class algebra $Z(\mathbb{C}[\mathfrak{S}_k])$ of the symmetric group (the center of the group algebra $\mathbb{C}[\mathfrak{S}_k]$) given by multiplication by the formal sum of all transpositions. Based on Dubrovin's result (Theorem 1.1) we can prove the following result, concerning an extension of this representation-theoretic interpretation to the entire sequence of the quantum dispersionless KdV Hamiltonians. To this end, let us introduce the *Young-Jucys-Murphy* (YJM) elements $\mathcal{J}_i \in \mathbb{C}[\mathfrak{S}_i]$ [27, 31, 32], defined as the sum of all transpositions in $\mathfrak{S}_i$ which are not in $\mathfrak{S}_{i-1}$. Explicitly:

$$\mathcal{J}_1 = 0, \qquad \mathcal{J}_i = (1,i) + (2,i) + \cdots + (i-1,i), \quad i > 1. \tag{A.3}$$

Proposition A.1. *For any $k \geq 0$, we have the identity*

$$1+\sum_{m\geq -1}\left(\frac{z}{\sqrt{\hbar}}\right)^{m+2} H_m^{[0]}\Big|_{\Lambda_k} = e^{zU_0/\sqrt{\hbar}}\left(\frac{z/2}{\sinh(z/2)} + 2z\sinh\left(\frac{z}{2}\right)\sum_{m\geq 0}\frac{z^m}{m!}\Phi\left(\mathcal{J}_1^m + \cdots + \mathcal{J}_k^m\right)\Phi^{-1}\right) \tag{A.4}$$

in $\mathrm{End}_{\mathbb{C}}(\Lambda_k) \otimes \mathbb{C}((\sqrt{\hbar}))[\![z]\!]$. *Here $\mathcal{J}_i$ are the YJM elements* (A.3) *and Φ the Frobenius map* (A.2).

Let us remark that Rossi [33, Theorem 5.3] gives an alternative representation of these operators in terms of free fermions.

Before the proof, we note that for rank-1 CohFTs, the sesquilinear form (2.25) reduces to

$$\langle q_\lambda, q_\mu\rangle = \hbar^{\ell(\lambda)} z_\lambda \delta_{\lambda,\mu}, \tag{A.5}$$

where $z_\lambda := \prod_{i\geq 1} m_i(\lambda)! i^{m_i(\lambda)}$, denoting $m_i(\lambda)$ the number of parts of λ which are equal to i, and $\ell(\lambda)$ the length of λ. Equivalently, denoting $C_\lambda \subset \mathfrak{S}_{|\lambda|}$ the conjugacy class of permutations with disjoint cycles of lengths $\lambda_1, \ldots, \lambda_{\ell(\lambda)}$, we have

$$\left\langle \frac{q_\lambda}{\hbar^{\ell(\lambda)/2}}, \frac{q_\mu}{\hbar^{\ell(\mu)/2}}\right\rangle = \frac{|\lambda|!}{|C_\lambda|}\delta_{\lambda,\mu}. \tag{A.6}$$

This is the standard inner product on the space of symmetric polynomials [30]; in particular the Schur basis is orthonormal

$$\left\langle s_\lambda\left(\frac{q}{\sqrt{\hbar}}\right), s_\mu\left(\frac{q}{\sqrt{\hbar}}\right)\right\rangle = \delta_{\lambda,\mu}. \tag{A.7}$$

Proof of Proposition A.1. Before entering the proof proper let us recall the following two fundamental properties of the YJM elements introduced in (A.3).

(i) The $\mathcal{J}_i$'s commute among themselves, though they are not in the class algebra $Z(\mathbb{C}[\mathfrak{S}_k])$ (the center of the group algebra); however the class algebra is generated by symmetric functions of the YJM elements.

(ii) Multiplication by a symmetric function $f(\mathcal{J}_1, \ldots, \mathcal{J}_k)$ of the YJM elements is diagonal on the character basis $\{\chi_\lambda\}_{\lambda\in\mathscr{P}_k}$ of the class algebra $Z(\mathbb{C}[\mathfrak{S}_k])$, with explicit eigenvalues

$$f(\mathcal{J}_1, \ldots, \mathcal{J}_k)\chi_\lambda = f(\{i-j\}_{(i,j)\in\lambda})\chi_\lambda, \qquad \lambda \in \mathscr{P}_k, \tag{A.8}$$

where $f(\{i-j\}_{(i,j)\in\lambda})$ is the evaluation of the symmetric function f at the *contents* $i-j$ of the partition λ. Here the partition λ is identified with the set of $(i,j) \in \mathbb{Z}^2$ satisfying $1 \leq i \leq \ell(\lambda)$, $1 \leq j \leq \lambda_i$.

Therefore to prove (A.4), let us note that, since $\Phi\chi_\lambda = s_\lambda(q/\sqrt{\hbar})$ [30], both sides act diagonally on the Schur basis, and so it suffices to prove that they have the same eigenvalues.

To this end it is convenient to employ the Frobenius notation for a partition $\lambda = (\lambda_1, \dots, \lambda_\ell)$, which consists in denoting

$$\lambda = (a_1, \dots, a_d | b_1, \dots, b_d), \tag{A.9}$$

where $d := \max\{i \geq 0 : \ \lambda_i \geq i\}$ is the length of the main diagonal in the Young diagram of λ, and for $i = 1, \dots, d$,

$$a_i := \lambda_i - i, \qquad b_i = \lambda'_i - i \tag{A.10}$$

are the number of cells to the right of the ith diagonal cell in the Young diagram of λ ($a_1 > \dots > a_d \geq 0$), and the number of cells below the ith diagonal cell in the Young diagram of λ ($b_1 > \dots > b_d \geq 0$), respectively; here, as before, we denoted λ' the conjugate partition.

Hereafter we denote $\lambda = (a_1, \dots, a_d | b_1, \dots, b_d)$ in Frobenius notation, and using (A.8) we compute

$$(\mathcal{J}_1^m + \dots + \mathcal{J}_k^m)\chi_\lambda = \left(\sum_{(i,j)\in\lambda} (i-j)^m \right) \chi_\lambda = \sum_{i=1}^d \left(\delta_{m,0} + \sum_{\ell=1}^{a_i} \ell^m + (-1)^m \sum_{\ell=1}^{b_i} \ell^m \right) \chi_\lambda \tag{A.11}$$

and using the *Faulhaber-Bernoulli formula* (for $m = 0, 1, 2, \dots$)

$$\sum_{\ell=1}^N \ell^m = \frac{F_{m+1}(N)}{m+1}, \qquad F_m(x) = \sum_{j=0}^{m-1} (-1)^j \binom{m}{j} B_j x^{m-j}, \tag{A.12}$$

we can write

$$(\mathcal{J}_1^m + \dots + \mathcal{J}_k^m)\chi_\lambda = \frac{1}{m+1} \sum_{i=1}^d \left(\delta_{m,0} + F_{m+1}(a_i) + (-1)^m F_{m+1}(b_i) \right) \chi_\lambda. \tag{A.13}$$

Multiplying by $z^m/m!$ and summing over $m \geq 0$ we obtain

$$\begin{aligned} \sum_{m\geq 0} \frac{z^m}{m!} (\mathcal{J}_1^m + \dots + \mathcal{J}_k^m)\chi_\lambda &= \sum_{m\geq 0} \frac{z^m}{(m+1)!} \sum_{i=1}^d \left(\delta_{m,0} + F_{m+1}(a_i) + (-1)^m F_{m+1}(b_i) \right) \chi_\lambda \\ &= \sum_{i=1}^d \left(1 + \frac{\mathrm{e}^{z a_i} - 1}{1 - \mathrm{e}^{-z}} + \frac{\mathrm{e}^{-z b_i} - 1}{1 - \mathrm{e}^{z}} \right) \chi_\lambda, \end{aligned} \tag{A.14}$$

where we have used the identity

$$\sum_{m\geq 0} \frac{F_{m+1}(x)}{(m+1)!} z^m = \frac{\mathrm{e}^{zx} - 1}{1 - \mathrm{e}^{-z}}. \tag{A.15}$$

Finally, from the results of [17], recalled in Theorem 1.1, we have

$$\mathcal{H}^{[0]}\left(\frac{z}{\sqrt{\hbar}}\right) s_\lambda\left(\frac{q}{\sqrt{\hbar}}\right) = \mathrm{e}^{zU_0/\sqrt{\hbar}} \left(\frac{1}{S(z)} + z \sum_{i=1}^d \left(\mathrm{e}^{z\left(a_i + \frac{1}{2}\right)} - \mathrm{e}^{-z\left(b_i + \frac{1}{2}\right)} \right) \right) s_\lambda\left(\frac{q}{\sqrt{\hbar}}\right), \tag{A.16}$$

where the generating function $\mathcal{H}^{[0]}$ is given in (2.37) and $S(z)$ is given in (2.38). Therefore both sides of (A.4) are diagonal on Schur polynomials $s_\lambda\left(q/\sqrt{\hbar}\right)$, and by the identity

$$\mathrm{e}^{z\left(a+\frac{1}{2}\right)} - \mathrm{e}^{-z\left(b+\frac{1}{2}\right)} = 2\sinh\left(\frac{z}{2}\right) \left(\frac{\mathrm{e}^{az} - 1}{1 - \mathrm{e}^{-z}} + \frac{\mathrm{e}^{-bz} - 1}{1 - \mathrm{e}^{z}} + 1 \right) \tag{A.17}$$

it is clear that the eigenvalues coincide, see (A.14) and (A.16), and the proof is complete. □

Remark A.2. *In view of the identification of Prop. A.1, Eliashberg's formula (2.37) matches with a formula of Lascoux and Thibon [29] expressing the action of multiplication in the class algebra by Newton polynomials of YJM elements in terms of differential operators.*

It remains an open question whether a similar description continues to hold for the quantum KdV hierarchy; in this respect it would be interesting to compare with the results of the recent preprint [5] where Hurwitz numbers appear in connection with the quantum Witten-Kontsevich series, a particular *quantum tau-function* for the quantum KdV hierarchy.

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