Metric operators, generalized hermiticity and partial inner product spaces

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Abstract

A quasi-Hermitian operator is an operator in a Hilbert space that is similar to its adjoint in some sense, via a metric operator, i.e., a strictly positive self-adjoint operator. Motivated by the recent developments of pseudo-Hermitian quantum mechanics, we analyze the structure of metric operators, bounded or unbounded. We introduce several generalizations of the notion of similarity between operators and explore to what extent they preserve spectral properties.

Next we consider canonical lattices of Hilbert spaces generated by unbounded metric operators. Since such lattices constitute the simplest case of a partial inner product space (PIP-space), we can exploit the technique of PIP-space operators. Thus we apply some of the previous results to operators on a particular PIP-space, namely, the scale of Hilbert spaces generated by a single metric operator. Finally, we reformulate the notion of pseudo-hermitian operators in the preceding formalism.

1 Introduction

Non-self-adjoint operators with real spectrum appear in different contexts : the so-called PT-symmetric quantum mechanics [9], pseudo-Hermitian quantum mechanics [18,19], Three-Hilbert-Space formulation of quantum mechanics [26], nonlinear pseudo-bosons [8], nonlinear supersymmetry, and so on. In addition, they appear under various names: pseudo-Hermitian, quasi-Hermitian, cryptohermitian operators.

The $\mathcal{PT}$-symmetric Hamiltonians are usually pseudo-Hermitian operators, a term introduced a long time ago by Dieudonné [13] (under the name 'quasi-Hermitian') for characterizing those bounded operators A which satisfy a relation of the form

$$GA = A^*G, \tag{1.1}$$

where G is a *metric operator*, i.e., a strictly positive self-adjoint operator. This operator G then defines a new metric (hence the name) and a new Hilbert space (sometimes called physical) in which A is symmetric and possesses a self-adjoint extension. For a systematic analysis of pseudo-Hermitian QM, we may refer to the review of Mostafazadeh [18] and the special issues [10, 11], which contain a variety of concrete applications in quantum physics.

According to (1.1), the generic structure of these operators is $A^* = GAG^{-1}$, i.e., A^* is *similar* to A, in some sense, via a metric operator G, i.e., a strictly positive self-adjoint operator $G > 0$, thus invertible, with (possibly unbounded) inverse G^{-1}. Now, in most of the literature, the metric operators are assumed to be bounded. In some recent works, however, unbounded metric operators are introduced [6–8, 19].

On the other hand, if G^{-1} is bounded, (1.1) implies that A is similar to a self-adjoint operator, thus it is a spectral operator of scalar type and real spectrum, in the sense of Dunford [14]. This is the case treated by Scholtz *et al.* [24] and Geyer *et al.* [16], who introduced the concept in the physics literature.

The aim of this talk is to study in a rigorous way the problem of *operator similarity* under a metric operator, bounded or unbounded. In particular, we will formulate the analysis in the framework of partial inner product spaces (PIP-spaces), since the latter appear naturally in this context. Most of the information contained here comes from our papers [3–5].

To conclude, we fix our notations. The framework in a separable Hilbert space $\mathcal{H}$, with inner product $\langle\cdot|\cdot\rangle$, linear in the first entry. Then, for any operator A in $\mathcal{H}$, we denote its domain by $D(A)$, its range by $R(A)$ and, if A is positive, its form domain by $Q(A) := D(A^{1/2})$.

2 Metric operators

By a *metric operator*, in a Hilbert space $\mathcal{H}$, we mean a strictly positive self-adjoint operator G, that is, $G > 0$ or $\langle G\xi|\xi\rangle \geqslant 0$ for every $\xi \in D(G)$ and $\langle G\xi|\xi\rangle = 0$ if and only if $\xi = 0$.

Of course, G is densely defined and invertible, but need not be bounded; its inverse G^{-1} is also a metric operator, bounded or not (in this case, in fact, 0 belongs to the continuous spectrum of G).

Let G, G_1, G_2 be metric operators. Then

(1) If G_1 and G_2 are both bounded, then $G_1 + G_2$ is a bounded metric operator;

(2) λG is a metric operator for every $\lambda > 0$;

(3) if G_1 and G_2 commute, their product G_1G_2 is also a metric operator;

(4) $G^{1/2}$ and, more generally, $G^{\alpha}(\alpha \in \mathbb{R})$, are metric operators.

Given a bounded metric operator G, define $\langle\xi|\eta\rangle_G := \langle G\xi|\eta\rangle$, $\xi, \eta \in \mathcal{H}$. This is a positive definite inner product on $\mathcal{H}$ with corresponding norm $\|\xi\|_G = \|G^{1/2}\xi\|$. We denote by $\mathcal{H}(G)$ the completion of $\mathcal{H}$ in this norm. Thus we get $\mathcal{H} \subseteq \mathcal{H}(G)$. If $G^{-1/2}$ is bounded, $\mathcal{H}$ and $\mathcal{H}(G)$ are the same as vector spaces and they carry different, but equivalent, norms.

Clearly, the conjugate dual space of $\mathcal{H}(G)^{\times}$ of $\mathcal{H}(G)$ is a subspace of $\mathcal{H}$ and $\mathcal{H}(G)^{\times} \equiv \mathcal{H}(G^{-1}) = D(G^{-1/2})$ with inner product $\langle\xi|\eta\rangle_{G^{-1}} = \langle G^{-1}\xi|\eta\rangle$. The upshot is a triplet of Hilbert spaces

$$\mathcal{H}(G^{-1}) \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}(G), \tag{2.1}$$

where $\hookrightarrow$ denotes a continuous embedding with dense range. If G^{-1} is bounded, $\mathcal{H}(G^{-1}) = \mathcal{H}(G) = \mathcal{H}$ with norms equivalent (but different) to the norm of $\mathcal{H}$. In the triplet (2.1), $G^{-1/2}$ is a unitary operator from $\mathcal{H}(G^{-1})$ onto $\mathcal{H}$ and from $\mathcal{H}$ onto $\mathcal{H}(G)$ In the same way, $G^{1/2}$ is a unitary operator from $\mathcal{H}$ onto $\mathcal{H}(G^{-1})$ and from $\mathcal{H}(G)$ onto $\mathcal{H}$.

Now, the triplet (2.1) is the central part of the infinite scale of Hilbert spaces built on the powers of $G^{-1/2}$, $V_I := \{\mathcal{H}_n, n \in \mathbb{Z}\}$, where $\mathcal{H}_n = D(G^{-n/2}), n \in \mathbb{N}$, with a norm equivalent to the graph norm, and $\mathcal{H}_{-n} = \mathcal{H}_n^{\times}$

$$\ldots \subset\ \mathcal{H}_2\ \subset\ \mathcal{H}_1\ \subset\ \mathcal{H}\ \subset\ \mathcal{H}_{-1} \subset\ \mathcal{H}_{-2} \subset\ \ldots \tag{2.2}$$

The obvious question is how to identify the end spaces of the scale:

$$\mathcal{H}_\infty(G^{-1/2}) := \bigcap_{n\in\mathbb{Z}} \mathcal{H}_n, \qquad \mathcal{H}_{-\infty}(G^{-1/2}) := \bigcup_{n\in\mathbb{Z}} \mathcal{H}_n. \tag{2.3}$$

By *quadratic interpolation* [12], one may build the continuous scale $\mathcal{H}_t, 0 \leqslant t \leqslant 1$, between $\mathcal{H}_1$ and $\mathcal{H}$, where $\mathcal{H}_t = D(G^{-t/2})$, with norm $\|\xi\|_t = \|G^{-t/2}\xi\|$. Next, defining $\mathcal{H}_{-t} = \mathcal{H}_t^\times$ and iterating, one obtains the full continuous scale $V_{\tilde{I}} := \{\mathcal{H}_t, t \in \mathbb{R}\}$, a simple example of a PIP-space [1]. Then, of course, one can replace $\mathbb{Z}$ by $\mathbb{R}$ in the definition (2.3) of the end spaces of the scale.

3 Similar and quasi-similar operators

Before proceeding to our main topic, we quote two easy properties. Let A be a linear operator in the Hilbert space $\mathcal{H}$, with domain $D(A)$. Then, (i) if $D(A)$ is dense in $\mathcal{H}$, it is also dense in $\mathcal{H}(G)$; (ii) if A is closed in $\mathcal{H}(G)$, it is also closed in $\mathcal{H}$.

Now we introduce the central definitions.

Definition 3.1

(1) Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, A and B densely defined linear operators in $\mathcal{H}$, resp. $\mathcal{K}$. A bounded operator $T : \mathcal{H} \to \mathcal{K}$ is called a *bounded intertwining operator* for A and B if
  - (io_1) $T : D(A) \to D(B)$;
  - (io_2) $BT\xi = TA\xi$, $\forall \xi \in D(A)$.

  If T is a bounded intertwining operator for A and B, then $T^* : \mathcal{K} \to \mathcal{H}$ is a bounded intertwining operator for B^* and A^*.

(2) A and B are *similar*, which is denoted $A \sim B$, if there exists a bounded intertwining operator T for A and B with bounded inverse $T^{-1} : \mathcal{K} \to \mathcal{H}$, which is intertwining for B and A. In addition, A and B are *metrically similar* if T is a metric operator.

(3) A and B are *unitarily equivalent* ($A \overset{u}{\sim} B$) if $A \sim B$ and $T : \mathcal{H} \to \mathcal{K}$ is unitary.

Obviously, $\sim$ and $\overset{u}{\sim}$ are equivalence relations.

The following properties are immediate. Let $A \sim B$. Then:

(i) $TD(A) = D(B)$.

(ii) A is closed iff B is closed.

(iii) $A \sim B$ iff $B^* \sim A^*$.

(iv) A^{-1} exists iff B^{-1} exists; in that case, $B^{-1} \sim A^{-1}$.

In the sequel, we will examine to what extent the spectral properties of operators behave under the similarity relation. In order to do that, it is worth recalling the basic definitions, especially since we are dealing with closed, non self-adjoint operators.

Given a closed operator A in $\mathcal{H}$, consider $A - \lambda I : D(A) \to \mathcal{H}$ and the resolvent $R_A(\lambda) := (A - \lambda I)^{-1}$. Then one defines:

- The resolvent set $\rho(A) := \{\lambda \in \mathbb{C} : A - \lambda I \text{ is one-to-one and } (A - \lambda I)^{-1} \text{ is bounded}\}$.
- The spectrum $\sigma(A) := \mathbb{C} \setminus \rho(A)$.

- The point spectrum $\sigma_p(A) := \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not one-to-one}\}$, that is, the set of eigenvalues of A.
- The continuous spectrum $\sigma_c(A) := \{\lambda \in \mathbb{C} : A - \lambda I$ is one-to-one and has dense range, different from $\mathcal{H}\}$, hence $(A - \lambda I)^{-1}$ is densely defined, but unbounded.
- The residual spectrum $\sigma_r(A) := \{\lambda \in \mathbb{C} : A - \lambda I$ is one-to-one, but its range is not dense$\}$, hence $(A - \lambda I)^{-1}$ is not densely defined.

With these definitions, the three sets $\sigma_p(A), \sigma_c(A), \sigma_r(A)$ are disjoint and

$$\sigma(A) = \sigma_p(A) \cup \sigma_c(A) \cup \sigma_r(A). \tag{3.1}$$

We note also that $\sigma_r(A) = \overline{\sigma_p(A^*)} = \{\overline{\lambda} : \lambda \in \sigma_p(A^*)\}$. Indeed, for any $\lambda \in \sigma_r(A)$, there exists $\eta \neq 0$ such that

$$0 = \langle (A - \lambda I)\xi | \eta \rangle = \langle \xi | (A^* - \overline{\lambda} I)\eta \rangle, \ \forall \, \xi \in D(A),$$

which implies $\overline{\lambda} \in \sigma_p(A^*)$. Also $\sigma_r(A) = \emptyset$ if A is self-adjoint.

Note that here we follow Dunford-Schwartz [15], but other authors give a different definition of the continuous spectrum, implying that it is no longer disjoint from the point spectrum, for instance, Reed-Simon [21] or Schmüdgen [23]. This alternative definition allows for eigenvalues embedded in the continuous spectrum, a situation common in many physical situations.

The answer to the question raised above is given in the following proposition.

Proposition 3.2 *Let A, B be closed operators such that $A \sim B$ with the bounded intertwining operator T. Then,*

(i) $\rho(A) = \rho(B)$.

(ii) $\sigma_p(A) = \sigma_p(B)$. *Moreover if $\xi \in D(A)$ is an eigenvector of A corresponding to the eigenvalue λ, then $T\xi$ is an eigenvector of B corresponding to the same eigenvalue. Conversely, if $\eta \in D(B)$ is an eigenvector of B corresponding to the eigenvalue λ, then $T^{-1}\eta$ is an eigenvector of A corresponding to the same eigenvalue. Moreover, the multiplicity of λ as eigenvalue of A is the same as its multiplicity as eigenvalue of B.*

(iii) $\sigma_c(A) = \sigma_c(B)$ *and* $\sigma_r(A) = \sigma_r(B)$.

(iv) If A is self-adjoint, then B has real spectrum and $\sigma_r(B) = \emptyset$.

The property (iv) means that B is then a spectral operator of scalar type with real spectrum, a notion introduced by Dunford [14].

In conclusion, similarity preserves the various parts of the spectra, but it does *not* preserve self-adjointness. This means we are on the good track, since we are seeking a form of similarity that transforms a non-self-adjoint operator into a self-adjoint one.

However, the notion of similarity just defined is too strong in many situations. A natural step is to drop the boundedness of T^{-1}.

Definition 3.3 We say that A is *quasi-similar* to B, and write $A \dashv B$, if there exists a bounded intertwining operator T for A and B which is invertible, with inverse T^{-1} densely defined (but not necessarily bounded).

Even if T^{-1} is bounded, A and B need not be similar, unless T^{-1} is also an intertwining operator. Indeed, T^{-1} does not necessarily map $D(B)$ into $D(A)$, unless of course if $TD(A) = D(B)$. Note that one can always suppose that T is a metric operator

Actually there is a great confusion in the literature about the terminology of (quasi-)similarity. We refer to our paper [5] for a detailed discussion.

We proceed now to show the stability of the different parts of the spectrum under the quasi-similarity relation $\dashv$, following mostly [3] and [5].

Proposition 3.4 *Let A and B be closed operators and assume that $A \dashv B$, with the bounded intertwining operator T. Then the following statements hold.*

(i) $\sigma_p(A) \subseteq \sigma_p(B)$ and for every $\lambda \in \sigma_p(A)$ one has $m_A(\lambda) \leqslant m_B(\lambda)$, where $m_A(\lambda)$, resp. $m_B(\lambda)$, denotes the multiplicity of λ as eigenvalue of the operator A, resp. B.

(ii) $\sigma_r(B) \subseteq \sigma_r(A)$.

(iii) If $TD(A) = D(B)$, then $\sigma_p(B) = \sigma_p(A)$.

(iv) If T^{-1} is bounded and $TD(A)$ is a core for B, then $\sigma_p(B) \subseteq \sigma(A)$.

(v) If T^{-1} everywhere defined and bounded, then

$$\rho(A) \setminus \sigma_p(B) \subseteq \rho(B) \quad \text{and} \quad \rho(B) \setminus \sigma_r(A) \subseteq \rho(A)$$

(vi) Assume that T^{-1} is everywhere defined and bounded and $TD(A)$ is a core for B. Then

$$\sigma_p(A) \subseteq \sigma_p(B) \subseteq \sigma(B) \subseteq \sigma(A).$$

The situation described in Proposition 3.4 (vi) is quite important for possible applications. Even if the spectra of A and B may be different, it gives a certain number of informations on $\sigma(B)$ once $\sigma(A)$ is known. For instance, if A has a pure point spectrum, then B is isospectral to A. More generally, if A is self-adjoint, then any operator B quasi-similar to A by means of an intertwining operator T with bounded inverse T^{-1}, has real spectrum.

We will illustrate the previous proposition by two examples, both taken from [3]. In the first one, $A \dashv B, A, B$ and T are all bounded, and the two spectra, which are pure point, coincide.

Example 3.5 In $\mathcal{H} = L^2(\mathbb{R}, \mathrm{d}x)$, take the operator Q of multiplication by x, on the dense domain

$$D(Q) = \left\{ f \in L^2(\mathbb{R}) : \int_{\mathbb{R}} x^2 |f(x)|^2 \, \mathrm{d}x < \infty \right\},$$

and define the two operators

- $P_\varphi := |\varphi\rangle\langle\varphi|$, for $\varphi \in L^2(\mathbb{R})$, with $\|\varphi\| = 1$,
- $A_\varphi f = \langle (I + Q^2) f | \varphi \rangle (I + Q^2)^{-1} \varphi, \quad \varphi \in D(A_\varphi)$.

Then

(i) $P_\varphi \dashv A_\varphi$ with the bounded intertwining operator $T = (I + Q^2)^{-1}$.

(ii) P_φ is everywhere defined and bounded, but the operator A_φ is closable iff $\varphi \in D(Q^2)$.

(iii If $\varphi \in D(Q^2)$, A_φ is bounded and everywhere defined, and $\sigma(A_\varphi) = \sigma(P_\varphi) = \{0, 1\}$.

In the second example, A and B are both unbounded. In that case, the two spectra coincide as a whole, but not their individual parts. In particular, A has a nonempty residual spectrum, whereas B does not

Example 3.6 In $\mathcal{H} = L^2(\mathbb{R}, \mathrm{d}x)$, define the two operators

- $(Af)(x) = f'(x) - \dfrac{2x}{1+x^2} f(x), \quad f \in D(A) = W^{1,2}(\mathbb{R})$
- $(Bf)(x) = f'(x), \quad f \in D(B) = W^{1,2}(\mathbb{R})$

Then

(i) $A \dashv B$ with the bounded intertwining operator $T = (I + Q^2)^{-1}$.

(ii) $\sigma(A) = \sigma(B)$.

(iii) $\sigma_p(A) = \emptyset$, $\sigma_r(A) = \{0\}$, but $\sigma(B) = \sigma_c(B) = i\mathbb{R}$.

It is easy to generalize the preceding analysis to the case of an unbounded intertwining operator, but we have to adapt the definition.

Definition 3.7 Let A, B two densely defined linear operators on the Hilbert spaces $\mathcal{H}, \mathcal{K}$, respectively. A closed (densely defined) operator $T : D(T) \subseteq \mathcal{H} \to \mathcal{K}$ is called an *intertwining operator* for A and B if

(io_0) $D(TA) = D(A) \subset D(T)$;

(io_1) $T : D(A) \to D(B)$;

(io_2) $BT\xi = TA\xi$, $\forall\, \xi \in D(A)$.

The first part of condition (io_0) means that $\xi \in D(A)$ implies $A\xi \in D(T)$.

Then we say again that A is *quasi-similar* to B, $A \dashv B$, if there exists a (possibly unbounded) intertwining operator T for A and B which is invertible, with inverse T^{-1} densely defined. Note that $A \dashv B$ does not imply $B^* \dashv A^*$, since (io_0) may fail for B^*. Furthermore, we say that A and B are *mutually quasi-similar* if we have both $A \dashv B$ and $B \dashv A$, which we denote by $A \dashv\vdash B$. Clearly, $\dashv\vdash$ is an equivalence relation and $A \dashv\vdash B$ implies $A^* \dashv\vdash B^*$.

We add that quasi-similarity with an unbounded intertwining operator may occur only under *singular*, even pathological, circumstances. For instance, one may note that, if $A \dashv B$ with the intertwining operator T and the resolvent set $\rho(A)$ is not empty, then T is necessarily bounded.

At this point, one may examine to what extent some parts of Proposition3.4 survive when the intertwining operator T is no longer bounded, and also what happens if $A \dashv\vdash B$. We refer to [5] for a thorough analysis.

$$\begin{tikzcd}
 & \mathcal{H}(G^{-1}) \arrow[dr] & \\
\mathcal{H}(R_{G^{-1}}) \arrow[ur] \arrow[dr] & & \mathcal{H}(R_G^{-1}) \\
 & \mathcal{H} \arrow[ur] \arrow[dr] & \\
\mathcal{H}(R_G) \arrow[ur] \arrow[dr] & & \mathcal{H}(R_{G^{-1}}^{-1}) \\
 & \mathcal{H}(G) \arrow[ur] &
\end{tikzcd}$$

Figure 1: The lattice of Hilbert spaces generated by a metric operator.

$$\begin{tikzcd}[column sep=small]
 & L^2(\mathbb{R}, x^{-2}\,\mathrm{d}x) \arrow[dr] & \\
L^2(\mathbb{R}, (1+x^{-2})\,\mathrm{d}x) \arrow[ur] \arrow[dr] & & L^2(\mathbb{R}, (1+x^{2})^{-1}\,\mathrm{d}x) \\
 & L^2(\mathbb{R}, \,\mathrm{d}x) \arrow[ur] \arrow[dr] & \\
L^2(\mathbb{R}, (1+x^{2})\,\mathrm{d}x) \arrow[ur] \arrow[dr] & & L^2(\mathbb{R}, (1+x^{-2})^{-1}\,\mathrm{d}x) \\
 & L^2(\mathbb{R}, x^{2}\,\mathrm{d}x) \arrow[ur] &
\end{tikzcd}$$

Figure 2: The lattice of Hilbert spaces generated by $G = x^2$.

4 The lattice generated by a single metric operator

Now we turn to the general case, where G and G^{-1} are both possibly unbounded.

Define $\mathcal{H}(R_G)$ as $D(G^{1/2})$ equipped with the graph norm $\|\xi\|^2_{R_G} := \|\xi\|^2 + \left\|G^{1/2}\xi\right\|^2$. Then define $\mathcal{H}(G)$ as the completion of $\mathcal{H}(R_G)$ in the norm $\|\xi\|^2_G := \left\|G^{1/2}\xi\right\|^2$. It follows that $\mathcal{H}(R_G) = \mathcal{H} \cap \mathcal{H}(G)$, with the projective norm [1, Sec. I.2.1].

Now, since $D(G^{1/2}) = Q(G)$, the form domain of G, we may write

$$\|\xi\|^2_{R_G} = \langle (1+G)\xi|\xi\rangle = \langle R_G\xi|\xi\rangle, \quad \|\xi\|^2_G = \langle G\xi|\xi\rangle, \ \text{ with } \ R_G = 1 + G,$$

which justifies the notation $\mathcal{H}(R_G)$.

Next, the conjugate dual $\mathcal{H}(R_G)^\times = \mathcal{H}(R_G^{-1})$, so that

$$\mathcal{H}(R_G) \ \subset \ \mathcal{H} \ \subset \ \mathcal{H}(R_G^{-1}) = \mathcal{H} + \mathcal{H}(G^{-1}),$$

with the inductive norm [1, Sec. I.2.1]. Putting everything together, we get the lattice shown in Fig. 1.

To give a concrete example, take $G = x^2$ in $L^2(\mathbb{R}, \mathrm{d}x)$, so that $R_G = 1 + x^2$. Then all the spaces in the diagram are weighted L^2 spaces, as shown in Fig. 2.

Actually one can go further, following a construction made in [2]. If G is unbounded, $R_G = 1 + G > 1$ and R_G^{-1} bounded, so that we have the triplet $\mathcal{H}(R_G) \subset \mathcal{H} \subset \mathcal{H}(R_G^{-1})$.

Iterating as before, we get the infinite Hilbert scale built on powers of $R_G^{1/2}$, $\mathcal{H}_n = D(R_G^{n/2}), n \in \mathbb{N}$, and $\mathcal{H}_{-n} = \mathcal{H}_n^\times$:

$$\ldots \subset \ \mathcal{H}_2 \ \subset \ \mathcal{H}_1 \ \subset \ \mathcal{H} \ \subset \ \mathcal{H}_{-1} \subset \ \mathcal{H}_{-2} \subset \ \ldots \tag{4.1}$$

Taking $\mathcal{H} = L^2(\mathbb{R}, \mathrm{d}x)$, we find familiar examples, namely,

- $G_x = (1 + x^2)^{1/2}$, so that $\mathcal{H}_\infty(G_x^{1/2})$ consists of fast decreasing L^2 functions.
- $G_p = (1 - \mathrm{d}^2/\mathrm{d}x^2)^{1/2} = \mathcal{F}G_x\mathcal{F}{-1}$, so that the scale consists of the Sobolev spaces $W^{n,2}(\mathbb{R})$.

5 Quasi-Hermitian operators

According to Dieudonné [13], a bounded operator A is called quasi-Hermitian if there exists a metric operator G such that $GA = A^*G$. However, this definition is too restrictive for applications, hence we generalize it in order to cover unbounded operators.

Definition 5.1 A closed operator A is called *quasi-Hermitian* if there exists a metric operator G such that $D(A) \subset D(G)$ and

$$\langle A\xi | G\eta \rangle = \langle G\xi | A\eta \rangle, \quad \xi, \eta \in D(A) \tag{5.1}$$

Let us consider first a bounded quasi-Hermitian operator A. If in addition, the metric operator G is bounded with bounded inverse, then (5.1) implies immediately that GA is self-adjoint in $\mathcal{H}$. Actually, there is more.

Proposition 5.2 *Let A be bounded. Then the following statements are equivalent.*

(i) A is quasi-Hermitian.

(ii) There exists a bounded metric operator G, with bounded inverse, such that GA $(= A^*G)$ is self-adjoint.

(iii) A is metrically similar to a self-adjoint operator K.

We turn now to unbounded quasi-Hermitian operators. The following results are easy.

Proposition 5.3 *Let A be an unbounded quasi-Hermitian operator and G a bounded metric operator. Then (i) A is quasi-Hermitian iff GA is symmetric in $\mathcal{H}$; (ii) If A is self-adjoint in $\mathcal{H}(G)$, then GA is symmetric in $\mathcal{H}$. If G^{-1} is also bounded, A is self-adjoint in $\mathcal{H}(G)$i ff GA is self-adjoint in $\mathcal{H}$.*

Now we turn the problem around. Namely, given the closed densely defined operator A, we seek whether there is a metric operator G that makes A quasi-Hermitian and self-adjoint in $\mathcal{H}(G)$. We first obtain a metric operator with bounded inverse.

Proposition 5.4 *Let A be closed and densely defined. Then the following statements are equivalent:*

(i) There exists a bounded metric operator G, with bounded inverse, such that A is self-adjoint in $\mathcal{H}(G)$.

(ii) There exists a bounded metric operator G, with bounded inverse, such that $GA = A^*G$, i.e., A is similar to its adjoint A^*, with intertwining operator G.

(iii) There exists a bounded metric operator G, with bounded inverse, such that $G^{1/2}AG^{-1/2}$ is self-adjoint.

(iv) A is a spectral operator of scalar type with real spectrum, i.e., $A = \int_{\mathbb{R}} \lambda \, \mathrm{d}X(\lambda)$, where $\{X(\lambda)\}$ is a spectral family (not necessarily self-adjoint).

Instead of requiring that A be similar to A^*, we may ask that they be only quasi-similar. The price to pay is that now G^{-1} is no longer bounded and, therefore, the equivalences stated in are no longer true. Then Proposition 5.4 is replaced by the following weaker result [4].

Proposition 5.5 *Let A be closed and densely defined. Consider the statements*

(i) There exists a bounded metric operator G such that $GD(A) = D(A^*)$, $A^*G\xi = GA\xi$, for every $\xi \in D(A)$, in particular, A is quasi-similar to its adjoint A^*, with intertwining operator G.

(ii) There exists a bounded metric operator G, such that $G^{1/2}AG^{-1/2}$ is self-adjoint.

(iii) There exists a bounded metric operator G such that A is self-adjoint in $\mathcal{H}(G)$; i.e., A is quasi-selfadjoint.

(iv) There exists a bounded metric operator G such that $GD(A) = D(G^{-1}A^*)$, $A^*G\xi = GA\xi$, for every $\xi \in D(A)$, in particular, A is quasi-similar to its adjoint A^*, with intertwining operator G.

Then, the following implications hold :

$$(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv).$$

If the range $R(A^*)$ of A^* is contained in $D(G^{-1})$, then the four conditions (i)-(iv) are equivalent.

When G is unbounded, we say that A is *strictly quasi-Hermitian* if it is quasi-Hermitian, in the sense of Definition 5.1, and $AD(A) \subset D(G)$ or, equivalently, $D(GA) = D(A)$. Therefore, A is strictly quasi-Hermitian iff $A \dashv A^*$.

More results may be obtained if one uses the PIP-*space formalism*, as we shall see below.

6 The LHS generated by metric operators

Denote by $\mathcal{M}(\mathcal{H})$ the set of all metric operators and by $\mathcal{M}_b(\mathcal{H})$ the set of bounded ones. There is a natural order on $\mathcal{M}(\mathcal{H})$

$$\begin{aligned} G_1 \preceq G_2 &\iff \exists\, \gamma > 0 \text{ such that } G_2 \leqslant \gamma G_1 \\ &\iff \mathcal{H}(G_1) \subset \mathcal{H}(G_2), \text{ where the identity is continuous and with dense range.} \end{aligned}$$

As a consequence, we have

$$\begin{gathered} G_2^{-1} \preceq G_1^{-1} \iff G_1 \preceq G_2 \ \text{ if } G_1, G_2 \in \mathcal{M}(\mathcal{H}) \\ G^{-1} \preceq I \preceq G,\ \forall\, G \in \mathcal{M}_b(\mathcal{H}) \end{gathered}$$

Thus, given $X, Y \in \mathcal{M}(\mathcal{H})$, one has $X \preceq Y \ \Leftrightarrow\ \mathcal{H}(X) \hookrightarrow \mathcal{H}(Y)$. We will show that the spaces $\{\mathcal{H}(X) : X \in \mathcal{M}(\mathcal{H})\}$ constitute a *lattice of Hilbert spaces (LHS).*

Let $\mathcal{O} \subset \mathcal{M}(\mathcal{H})$ be a family of metric operators, containing I and at least one unbounded element, and assume that

$$\mathcal{D} := \bigcap_{G \in \mathcal{O}} D(G^{1/2})$$

is a dense subspace of $\mathcal{H}$. Since every operator $G \in \mathcal{O}$ is self-adjoint and invertible, one can define on $\mathcal{D}$, the graph topology $\mathsf{t}_{\mathcal{O}}$ by means of the norms

$$\xi \in \mathcal{D} \mapsto \|G^{1/2}\xi\|, \quad G \in \mathcal{O}$$

Let $\mathcal{D}^\times :=$ denote the conjugate dual of $\mathcal{D}[\mathsf{t}_{\mathcal{O}}]$, with strong dual topology $\mathsf{t}_{\mathcal{O}}^\times$. Then the triplet

$$\mathcal{D}[\mathsf{t}_{\mathcal{O}}] \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{D}^\times[\mathsf{t}_{\mathcal{O}}^\times]$$

is the Rigged Hilbert Space associated to $\mathcal{O}$. We will show that $\mathcal{O}$ generates a canonical lattice of Hilbert spaces (LHS) interpolating between $\mathcal{D}$ and $\mathcal{D}^\times$.

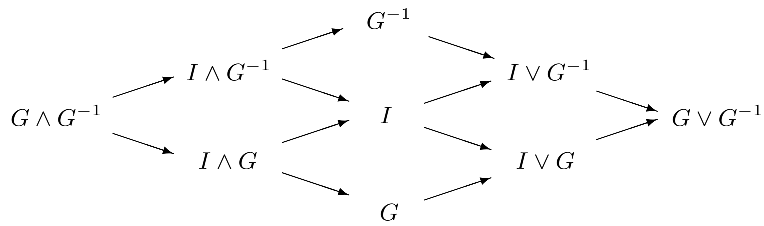

Figure 3: The lattice Σ generated by the metric operator G.

On the family $\{\mathcal{H}(X) : X \in \mathcal{O}^{-1}\}$ define the lattice operations as

$$\mathcal{H}(X \wedge Y) := \mathcal{H}(X) \cap \mathcal{H}(Y)$$
$$\mathcal{H}(X \vee Y) := \mathcal{H}(X) + \mathcal{H}(Y)$$

equipped, respectively, with the projective and the inductive norm, namely,

$$\|\xi\|^2_{X\wedge Y} = \|\xi\|^2_X + \|\xi\|^2_Y \,,$$
$$\|\xi\|^2_{X\vee Y} = \inf_{\xi=\eta+\zeta} \left(\|\eta\|^2_X + \|\zeta\|^2_Y\right),\ \eta \in \mathcal{H}(X), \zeta \in \mathcal{H}(Y)\,.$$

The corresponding operators read as

$$X \wedge Y := X \dotplus Y$$
$$X \vee Y := (X^{-1} \dotplus Y^{-1})^{-1}$$

Here $\dotplus$ stands for the *form sum* and $X, Y \in \mathcal{O}^{-1}$: given two positive operators, $T := T_1 \dotplus T_2$ is the positive operator associated to the quadratic form $\mathfrak{t} = \mathfrak{t}_1 + \mathfrak{t}_2$, where $\mathfrak{t}_1, \mathfrak{t}_2$ are the quadratic forms of T_1, T_2, respectively [17, §VI.2.5]. Note that both $X \vee Y$ and $X \wedge Y$ are inverses of a metric operator, but they need not belong to $\mathcal{O}^{-1}$. In particular, for $\mathcal{O} = \mathcal{M}(\mathcal{H})$, the corresponding family $\mathcal{M}(\mathcal{H})^{-1}$ is a lattice by itself, but the domain $\mathcal{D}$ usually fails to be dense.

Define $\mathcal{R} = \{G^{\pm 1/2}, G \in \mathcal{O}\}$ and the domain $\mathcal{D}_{\mathcal{R}} := \bigcap_{X\in\mathcal{R}} D(X)$. Let Σ be the minimal set of self-adjoint operators containing $\mathcal{O} \cup \mathcal{O}^{-1}$, stable under inversion and form sums, such that $\mathcal{D}_{\mathcal{R}}$ is dense in $H(Z)$, for every $Z \in \Sigma$.

Then $\mathcal{O}$ generates a lattice of Hilbert spaces $\mathcal{J}_\Sigma := \{\mathcal{H}(X) : X \in \Sigma\}$ and a PIP-space V_Σ with central Hilbert space $\mathcal{H} = \mathcal{H}(I)$. The total space is $V = \sum_{G\in\Sigma} \mathcal{H}(G)$ and the "smallest" space is $V^\# = \mathcal{D}_{\mathcal{R}}$. The compatibility and the partial inner product read, respectively, as

$$\xi \# \eta \iff \exists\, G \in \Sigma \ \text{ such that } \xi \in \mathcal{H}(G),\, \eta \in \mathcal{H}(G^{-1}),$$
$$\langle \xi | \eta \rangle_\Sigma = \langle G^{1/2} \xi | G^{-1/2} \eta \rangle_{\mathcal{H}}.$$

For simplicity, we write $\langle \xi | \eta \rangle_\Sigma = \langle \xi | \eta \rangle$.

For instance, for $\mathcal{O} = \{I, G\}$, the lattice Σ consists of the nine operators shown in Fig. 3.

7 (Quasi-)similarity for PIP-space operators

7.1 General PIP-space operators

Given the PIP-space V_Σ, an *operator* A on V_Σ is a map from a subset $\mathcal{D}(A) \subset V$ into V, such that

(i) $\mathcal{D}(A) = \bigcup_{X\in\mathsf{d}(A)} H(X)$, where $\mathsf{d}(A)$ is a nonempty subset of Σ;

(ii) For every $X \in \mathsf{d}(A)$, there exists $Y \in \Sigma$ such that the restriction of A to $H(X)$ is a continuous linear map into $H(Y)$ (we denote this restriction by A_{YX});

(iii) A has no proper extension satisfying (i) and (ii).

We denote by $\mathrm{Op}(V_\Sigma)$ the set of all operators on V_Σ. The continuous linear operator $A_{YX} : H(X) \to H(Y)$ is called a *representative* of A.

The properties of the operator A are encoded in the set $\mathsf{j}(A)$ of couples $(X, Y) \in \Sigma\times\Sigma$ such that $A : \mathcal{H}(X) \to \mathcal{H}(Y)$, continuously. Thus the operator A may be identified with the collection of its representatives, $A \simeq \{A_{YX} : (X, Y) \in \mathsf{j}(A)\}$. This is a *coherent* family that is, if $\mathcal{H}(W) \subset \mathcal{H}(X)$ and $\mathcal{H}(Y) \subset \mathcal{H}(Z)$, then one has $A_{ZW} = E_{ZY}A_{YX}E_{XW}$ ($E_{..} \simeq$ identity). More generally, $(X, Y) \in \mathsf{j}(A)$ if $Y^{1/2}AX^{-1/2}$ is bounded in $\mathcal{H}$.

Every operator has an *adjoint* $A^\times$ defined as follows: $(X, Y) \in \mathsf{j}(A)$ implies $(Y^{-1}, X^{-1}) \in \mathsf{j}(A^\times)$ and

$$\langle A^\times \eta|\xi\rangle = \langle \eta|A\xi\rangle, \text{ for } \xi \in \mathcal{H}(X), \eta \in \mathcal{H}(Y^{-1}).$$

In particular, $(X, X) \in \mathsf{j}(A)$ implies $(X^{-1}, X^{-1}) \in \mathsf{j}(A^\times)$.

An operator A is *symmetric* if $A = A^\times$. Therefore, $(X, X) \in \mathsf{j}(A)$ implies $(X^{-1}, X^{-1}) \in \mathsf{j}(A)$. Then, by interpolation, $(I, I) \in \mathsf{j}(A)$, that is, A has a bounded representative $A_{II} : \mathcal{H} \to \mathcal{H}$.

We will now examine the quasi-similarity properties of PIP-space operators.

(1) Let first $(G, G) \in \mathsf{j}(A)$, for some $G \in \mathcal{M}(\mathcal{H})$. Then the operator $\mathsf{B} = G^{1/2}A_{GG}G^{-1/2}$ is bounded on $\mathcal{H}$ and $A_{GG} \dashv \mathsf{B}$.

(2) Next, let $(G, G) \in \mathsf{j}(A)$, with G bounded and G^{-1} unbounded, so that

$$\mathcal{H}(G^{-1}) \subset \mathcal{H} \subset \mathcal{H}(G).$$

Consider the restriction A of A_{GG} to $\mathcal{H}$ and assume that $D(\mathsf{A}) = \{\xi \in \mathcal{H} : A\xi \in \mathcal{H}\}$ is dense in $\mathcal{H}$. Then $G^{1/2} : D(\mathsf{A}) \to D(\mathsf{B})$ and $\mathsf{B}\, G^{1/2}\eta = G^{1/2}\mathsf{A}\, \eta$, $\forall\, \eta \in D(\mathsf{A})$, i.e. $\mathsf{A} \dashv \mathsf{B}$, where the two operators act in $\mathcal{H}$.

Now we have $\mathsf{B}\, G^{1/2}\eta = G^{1/2}A\, \eta$, $\forall\, \eta \in \mathcal{H}(G)$ and $G^{1/2} : \mathcal{H}(G) \to \mathcal{H}$ is a unitary operator. Therefore, A and B are unitarily equivalent (but acting in different Hilbert spaces).

(3) Let finally $(G, G) \in \mathsf{j}(A)$ with G unbounded and G^{-1} bounded, so that

$$\mathcal{H}(G) \subset \mathcal{H} \subset \mathcal{H}(G^{-1}).$$

Then $A : \mathcal{H}(G) \to \mathcal{H}(G)$ is a densely defined operator in $\mathcal{H}$. Since $\mathsf{B} = G^{1/2}A_{GG}G^{-1/2}$ is bounded and everywhere defined on $\mathcal{H}$, one has $G^{-1/2}\mathsf{B}\xi = A_{GG}G^{-1/2}\xi$, $\forall\, \xi \in \mathcal{H}$, i.e., $\mathsf{B} \dashv A_{GG}$. However, by (1), $A_{GG} \dashv \mathsf{B}$, hence we have $A_{GG} \dashv\vdash B$. In addition $A_{GG} \overset{u}{\sim} B$, since $G^{\pm 1/2}$ are unitary between $\mathcal{H}$ and $\mathcal{H}(G)$.

7.2 The case of symmetric PIP-space operators

If $A \in \mathrm{Op}(V_\Sigma)$ is symmetric, $A = A^\times$ there is a possibility of self-adjoint *restrictions* to $\mathcal{H}$, that is, candidates for quantum observables.

However, if If $A = A^\times$, then $(G, G) \in \mathsf{j}(A)$ iff $(G^{-1}, G^{-1}) \in \mathsf{j}(A)$, which implies $(I, I) \in \mathsf{j}(A)$. Thus, every symmetric operator $A \in \mathrm{Op}(V_\Sigma)$ such that $(G, G) \in \mathsf{j}(A)$, with $G \in \mathcal{M}(\mathcal{H})$, has a *bounded* restriction A_{II} to $\mathcal{H}$.

Therefore, we conclude that the assumption $(G, G) \in \mathsf{j}(A)$ is too strong for applications ! Thus we will assume instead that $(G^{-1}, G) \in \mathsf{j}(A)$, where G is bounded with unbounded inverse, so that

$$\mathcal{H}(G^{-1}) \subset \mathcal{H} \subset \mathcal{H}(G).$$

In that case, one can apply the *KLMN theorem*, [1] namely,

> Given a symmetric operator $A = A^{\times}$, assume there is a metric operator $G \in \mathcal{M}_b(\mathcal{H})$ with an unbounded inverse, for which there exists a $\lambda \in \mathbb{R}$ such that $A - \lambda I$ has a boundedly invertible representative $(A - \lambda I)_{GG^{-1}} : \mathcal{H}(G^{-1}) \to \mathcal{H}(G)$. Then $A_{GG^{-1}}$ has a unique restriction to a self-adjoint operator A in the Hilbert space $\mathcal{H}$, with dense domain $D(\mathsf{A}) = \{\xi \in \mathcal{H} : A\xi \in \mathcal{H}\}$. In addition, $\lambda \in \rho(\mathsf{A})$.

If there is no bounded G as before, i.e. $(G^{-1}, G) \in \mathsf{j}(A)$, one can still use the KLMN theorem, but in the Hilbert scale $V_{\mathcal{G}}$ built on the powers of $G^{-1/2}$ or $(R_G)^{-1/2}$:

> Let $V_{\mathcal{G}} = \{\mathcal{H}_n, n \in \mathbb{Z}\}$ be the Hilbert scale built on the powers of the operator $G^{\pm 1/2}$ or $(R_G)^{-1/2}$, depending on the (un)boundedness of $G^{\pm 1} \in \mathcal{M}(\mathcal{H})$ and let $A = A^{\times}$ be a symmetric operator in $V_{\mathcal{G}}$.

(i) Assume there is a $\lambda \in \mathbb{R}$ such that $A - \lambda I$ has a boundedly invertible representative $(A - \lambda I)_{nm} : \mathcal{H}_m \to \mathcal{H}_n$, with $\mathcal{H}_m \subset \mathcal{H}_n$. Then A_{nm} has a unique restriction to a self-adjoint operator A in the Hilbert space $\mathcal{H}$, with dense domain $D(\mathsf{A}) = \{\xi \in \mathcal{H} : A\xi \in \mathcal{H}\}$. In addition, $\lambda \in \rho(\mathsf{A})$.

(ii) If the natural embedding $\mathcal{H}_m \to \mathcal{H}_n$ is compact, the operator A has a purely point spectrum of finite multiplicity, thus $\sigma(\mathsf{A}) = \sigma_p(\mathsf{A})$, $m_{\mathsf{A}}(\lambda_j) < \infty$ for every $\lambda_j \in \sigma_p(\mathsf{A})$ and $\sigma_c(\mathsf{A}) = \emptyset$.

Note, however, that there is so far no known (quasi-)similarity relation between $A_{GG^{-1}}$ or A and another operator! On the contrary, under the previous assumption $A : \mathcal{H}(G^{-1}) \to \mathcal{H}(G)$, $\mathsf{B} = G^{1/2} A_{GG^{-1}} G^{1/2}$ is bounded on $\mathcal{H}$, but $A_{GG^{-1}} \not\dashv \mathsf{B}$. Indeed, (io_1) imposes $T = G^{-1/2}$, hence unbounded, but then conditions (io_0) and (io_2) cannot be satisfied.

8 Semi-similarity of PIP-space operators

So far we have considered only the case of one metric operator G in relation to A. Assume now we take two different metric operators $G_1, G_2 \in \mathcal{M}(\mathcal{H})$. What can be said concerning A if it maps $\mathcal{H}(G_1)$ into $\mathcal{H}(G_2)$?

One possibility is to introduce, following [3], a notion slightly more general than quasi-similarity, called *semi-similarity*.

Definition 8.1 Let $\mathcal{H}, \mathcal{K}_1$ and $\mathcal{K}_2$ be three Hilbert spaces, A a closed, densely defined operator from $\mathcal{K}_1$ to $\mathcal{K}_2$, B a closed, densely defined operator on $\mathcal{H}$. Then A is said to be *semi-similar* to B, which we denote by $A \dashv\vdash B$, if there exist two bounded operators $T : \mathcal{K}_1 \to \mathcal{H}$ and $S : \mathcal{K}_2 \to \mathcal{H}$ such that (see Fig. 4):

(i) $T : D(A) \to D(B)$;

(ii) $BT\xi = SA\xi, \ \forall\, \xi \in D(A)$.

[1] KLMN stands for Kato, Lax, Lions, Milgram, Nelson.

$$\begin{tikzcd} & \mathcal{K}_1 \arrow[dl, "T"'] \arrow[dd, "A"] \\ \mathcal{H} & \\ & \mathcal{K}_2 \arrow[ul, "S"] \end{tikzcd}$$

Figure 4: The semi-similarity scheme.

The pair (T, S) is called an *intertwining couple.*

Of course, if $\mathcal{K}_1 = \mathcal{K}_2$ and $S = T$, we recover the notion of quasi-similarity and $A \dashv B$ (with a bounded intertwining operator).

Assume there exist two bounded metric operators G_1, G_2 such that $A : \mathcal{H}(G_1) \to \mathcal{H}(G_2)$ continuously. Then $\mathsf{B}_0 := G_2^{1/2} A_{G_2G_1} G_1^{-1/2}$ has a bounded extension B to $\mathcal{H}$ (its closure) and $A_{G_2G_1} \dashv\!\vdash \mathsf{B}r$, with respect to the intertwining couple $T = G_1^{1/2}, S = G_2^{1/2}$.

Take now $A = A^\times$ symmetric. Then $A : \mathcal{H}(G_1) \to \mathcal{H}(G_2)$ implies $A : \mathcal{H}(G_2^{-1}) \to \mathcal{H}(G_1^{-1})$ Assume that $G_1 \preceq G_2$, that is, $\mathcal{H}(G_1) \subset \mathcal{H}(G_2)$. Then we have

$$\mathcal{H}(G_2^{-1}) \;\subset\; \mathcal{H}(G_1^{-1}) \;\subset\; \mathcal{H} \;\subset\; \mathcal{H}(G_1) \;\subset\; \mathcal{H}(G_2).$$

It follows that the KLMN theorem applies. Assume indeed there exists $\lambda \in \mathbb{R}$ such that $A - \lambda I$ has an invertible representative $(A - \lambda I)_{G_2G_2^{-1}} : \mathcal{H}(G_2^{-1}) \to \mathcal{H}(G_2)$. Then $A_{G_2G_2^{-1}}$ has a unique restriction to a *self-adjoint* operator A in $\mathcal{H}$, hence $A_{G_2G_2^{-1}} \dashv\!\vdash \mathsf{B}$ and $\mathsf{A} \dashv\!\vdash \mathsf{B}$. A question remains open, namely, A is self-adjoint, but is the spectrum of B real?

In conclusion, there are three cases: if $A : \mathcal{H}(G_1) \to \mathcal{H}(G_2)$, then

(i) G_1 is unbounded and G_2 is bounded: then

$$\mathcal{H}(G_1) \;\subset\; \mathcal{H} \;\subset\; \mathcal{H}(G_2),$$

and A maps the small space into the large one, thus the KLMN theorem applies.

(ii) G_1 and G_2 are both unbounded, with $\mathcal{H}(G_1) \subset \mathcal{H}(G_2)$; then the KLMN theorem applies.

(iii) G_1 is bounded and G_2 is unbounded: then

$$\mathcal{H}(G_2) \;\subset\; \mathcal{H} \;\subset\; \mathcal{H}(G_1) \quad \text{and} \quad \mathcal{H}(G_1^{-1}) \;\subset\; \mathcal{H} \;\subset\; \mathcal{H}(G_2^{-1}),$$

so that, in both cases, A maps the large space into the small one; hence the KLMN theorem does *not* apply.

9 Pseudo-Hermitian Hamiltonians

Non-self-adjoint Hamiltonians appear in *Pseudo-Hermitian quantum mechanics.* In general, they are $\mathcal{PT}$-symmetric operators, that is, invariant under the joint action of space reflection ($\mathcal{P}$) and complex conjugation ($\mathcal{T}$). Typical examples are $H = p^2 + ix^3$ and $H = p^2 - x^4$, which are both $\mathcal{PT}$-symmetric, but non-self-adjoint, and have both a purely point spectrum, real and positive.

Now, the usual assumption is that H is Dieudonné-pseudo-Hermitian, that is, there exists an (unbounded) metric operator G satisfying the relation $H^*G = GH$.

Assume instead that H *pseudo-Hermitian*, that is, $D(H) \subset D(G)$ and

$$\langle H\xi | G\eta\rangle = \langle G\xi | H\eta\rangle, \ \ \forall\, \xi, \eta \in \mathcal{D}(H).$$

Then, if G is bounded, one gets $H \dashv H^*$ and $G^{1/2} H G^{-1/2}$ is self-adjoint. If G is unbounded and H is strictly quasi-Hermitian, then $H \dashv H^*$. If, in addition, G^{-1} is bounded, then $G^{-1} H^* G\eta = H\eta$, $\forall\, \eta \in \mathcal{D}(H)$, which is a restrictive form of similarity.

Finally, assume that H is a quasi-Hermitian operator which possesses a (large) set of vectors, $\phi \in \mathcal{D}^\omega_G(H)$, *analytic* in the norm $\|\cdot\|_G$ and contained in $D(G)$ [20], that is,

$$\sum_{n=0}^{\infty} \frac{\|H^n \phi\|_G}{n!}\, t^n < \infty, \ \text{for some } t \in \mathbb{R}.$$

Thus $\mathcal{D}^\omega_G(H) \subset D(H) \subset D(G) \subset D(G^{1/2}) \subset \mathcal{H}$.

Under this assumption, we can proceed to the construction of the physical system, following [3, Sec.6]. Define $\mathcal{H}_G$ as the completion of $(\mathcal{D}^\omega_G(H)$ in the norm $\|\cdot\|_G)$. This is a closed subspace of $\mathcal{H}(G)$ and one has

$$\langle \phi | H\psi\rangle_G = \langle H\phi|\psi\rangle_G, \ \forall\, \phi, \psi \in \mathcal{D}^\omega_G(H).$$

Thus H is a densely defined symmetric operator in $\mathcal{H}_G$, with a dense set of analytic vectors. Therefore, H essentially self-adjoint, according to Nelson's theorem [20].

Then the closure $\overline{H}$ of H is self-adjoint in $\mathcal{H}_G$. The pair $(\mathcal{H}_G, \overline{H})$ may be interpreted as the physical quantum system.

Next, $W_{\mathcal{D}} = G^{1/2} \upharpoonright \mathcal{D}^\omega_G(H)$ is isometric from $\mathcal{D}^\omega_G(H)$ into $\mathcal{H}$, hence it extends to an isometry $W = \overline{W_{\mathcal{D}}} : \mathcal{H}_G \to \mathcal{H}$. The range of W is a closed subspace of $\mathcal{H}$, denoted by $\mathcal{H}_{\rm phys}$, and the operator W is unitary from $\mathcal{H}_G$ to $\mathcal{H}_{\rm phys}$. Therefore, the operator $h = W\, \overline{H}\, W^{-1}$ is self-adjoint in $\mathcal{H}_{\rm phys}$. This operator h is interpreted as the genuine Hamiltonian of the system, acting in the physical Hilbert space $\mathcal{H}_{\rm phys}$.

The situation becomes simpler if $\mathcal{D}^\omega_G(H)$ is dense in $\mathcal{H}$. Then, indeed, $W(\mathcal{D}^\omega_G(H))$ is also dense, $\mathcal{H}_G = \mathcal{H}(G)$, $\mathcal{H}_{\rm phys} = \mathcal{H}$ and $W = G^{1/2}$ is unitary from $\mathcal{H}(G)$ onto $\mathcal{H}$.

Now, every eigenvector of an operator is automatically analytic, hence this construction generalizes that of [19]. This applies, for instance, to the example given there, namely, the $\mathcal{PT}$-symmetric operator $H = \frac{1}{2}(p - i\alpha)^2 + \frac{1}{2}\omega^2 x^2$ in $\mathcal{H} = L^2(\mathbb{R})$, for any $\alpha \in \mathbb{R}$, which has an orthonormal basis of eigenvectors.

A beautiful example of the situation just analyzed has been given recently by Samsonov [22], namely, the second derivative on the positive half-line, with special boundary conditions at the origin (this example stems from Schwartz [25]).

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