

## Definition of timescales and related variables for the World Ocean and its leaky funnel idealisation

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### Local and global variables for the World Ocean

Assume that it is possible to evaluate the age  $\tau$  of every water particle in the World Ocean; the age is assumed to be positive definite, i.e.  $0 \leq \tau < \infty$ . At time  $t$  and location  $\mathbf{x}$ , take an arbitrarily-small sample of water, whose volume is  $\delta V$ . In this water sample, the volume of the water particles whose age lies in the interval  $[\tau, \tau + \delta\tau]$  tends to  $c(t, \mathbf{x}, \tau) \delta V \delta\tau$  as  $\delta V \rightarrow 0$  and  $\delta\tau \rightarrow 0$  (e.g. Delhez et al. 1999, Deleersnijder et al. 2001), where  $c(t, \mathbf{x}, \tau)$  is termed the *water age distribution*. The definition of the latter implies that the following constraint must be met at any time and position

$$(1) \quad \int_0^{\infty} c(t, \mathbf{x}, \tau) d\tau = 1 .$$

Then, it is readily understood that the *mean water age*, i.e. the mean age of all the water particles the water sample is made up of, reads

$$(2) \quad a(t, \mathbf{x}) = \int_0^{\infty} \tau c(t, \mathbf{x}, \tau) d\tau .$$

Global properties may also be defined. The *global water age distribution* of the water is defined to be

$$(3) \quad \mu(t, \tau) = \frac{1}{\Omega} \int_{\Omega} c(t, \mathbf{x}, \tau) d\mathbf{x} ,$$

where  $\Omega$  represents the volume of the World Ocean. In other words, the total volume of the water particles whose age lies in the interval  $[\tau, \tau + \delta\tau]$  tends to  $\mu(t, \tau) \Omega \delta\tau$  as  $\delta\tau \rightarrow 0$ . Combining (1) and (2) yields the following constraint

$$(4) \quad \int_0^{\infty} \mu(t, \tau) d\tau = 1 .$$

Then, the *global mean water age*  $\bar{a}(t)$  — which may also be called the *mean age of the World Ocean* — may be computed from either the mean water age or the global water age distribution:

$$(5) \quad \bar{a}(t) = \frac{1}{\Omega} \int_{\Omega} a(t, \mathbf{x}) d\mathbf{x} = \int_0^{\infty} \tau \mu(t, \tau) d\tau .$$

Another distribution function is also worth introducing: the *global mean water age distribution*  $\phi(t, \tau)$ , which is displayed in Figure 5 of Mouchet and Deleersnijder, where, unfortunately, this variable was given another name — which is believed to be less appropriate than that suggested herein. The following definition applies:  $\phi(t, \tau)$  is such that the volume of the World Ocean water whose mean water age lies in the interval interval  $[\tau, \tau + \delta\tau]$  tends to  $\phi(t, \tau) \Omega \delta\tau$  as  $\delta\tau \rightarrow 0$ , yielding the expression

$$(6) \quad \phi(t, \tau) = \frac{1}{\Omega} \lim_{\delta\tau \rightarrow 0} \int_{\Omega} \frac{\text{sign}[\tau + \delta\tau - a(t, \mathbf{x})] + \text{sign}[a(t, \mathbf{x}) - \tau]}{2\delta\tau} d\mathbf{x} .$$

Clearly, the key variable is the water age distribution. The latter obeys the following partial differential equation (Delhez et al. 1999, Deleersnijder et al. 2001):

$$(7) \quad \frac{\partial c}{\partial t} + \nabla \cdot (\mathbf{u}c) = \nabla \cdot (\mathbf{K} \cdot \nabla c) - \frac{\partial c}{\partial \tau} ,$$

where  $\mathbf{u}$  and  $\mathbf{K}$  denote the velocity, which is divergence-free, and the diffusivity tensor, which is symmetric and positive-definite. For the age to be used as an indicator of ventilation rates, the age of every water particle must be prescribed to be zero at the ocean surface  $\Gamma^s$ . This leads to the following boundary condition:

$$(8) \quad [c(t, \mathbf{x}, \tau)]_{\mathbf{x} \in \Gamma^s} = \delta(\tau - 0) ,$$

where  $\delta$  denotes the Dirac function. The time for a water particle to travel from the ocean surface to any point in the ocean interior is non zero. Therefore, in the ocean interior there is no water particle characterised by a zero age:

$$(9) \quad [c(t, \mathbf{x}, \tau)]_{\tau=0} = 0 .$$

The ocean bottom bottom and coastlines are impermeable. If this part of the domain boundary is denoted  $\Gamma^i$ , the associated boundary conditions are

$$(10) \quad [\mathbf{u}c \cdot \mathbf{n}]_{\mathbf{x} \in \Gamma^i} = 0$$

and

$$(11) \quad [(\mathbf{K} \cdot \nabla c) \cdot \mathbf{n}]_{\mathbf{x} \in \Gamma^i} = 0 ,$$

where  $\mathbf{n}$  is the outward unit normal vector to  $\Gamma^i$ .

Given an appropriate water age distribution at  $t=0$ ,  $c(t, \mathbf{x}, \tau)$  may be evaluated at any time and position in the World Ocean by solving the partial differential problem consisting of equation (7) and boundary conditions (8)-(11). Then, from the water age distribution, all the timescales and distributions defined above may be obtained.

### The leaky funnel model

The leaky funnel introduced by Mouchet and Deleersnijder (2008) is a semi infinite pipe whose section  $S(x)$  decreases exponentially, i.e.

$$(12) \quad S(x) = S_0 e^{-x/L}$$

where  $x$  is the distance to the entrance of the pipe ( $0 \leq x < \infty$ ), whereas the positive constants  $S_0$  and  $L$  denote the section at the pipe entrance and the length scale characterising the exponential decrease of the pipe section, respectively. The water flows through the pipe at constant velocity  $U$ . The diffusion in the  $x$ -direction is parameterized by means of a Fourier-Fick expression involving the constant diffusivity  $K$ .

All variables are assumed to be constant over every section of the pipe. In line with this hypothesis, the water age distribution in the leaky funnel,  $c(t, x, \tau)$  obeys the partial differential equation

$$(13) \quad \frac{\partial(Sc)}{\partial t} + \frac{\partial(SUc)}{\partial x} = -\frac{\partial(SU)}{\partial x}c + \frac{\partial}{\partial x} \left( SK \frac{\partial c}{\partial x} \right) + \frac{\partial(Sc)}{\partial \tau},$$

where the first term in the right-hand side member of this relation is associated with water flux leaving the funnel through its porous walls. Substituting (12) into (13) yields

$$(14) \quad \frac{\partial c}{\partial t} + U' \frac{\partial c}{\partial x} = K \frac{\partial^2 c}{\partial x^2} + \frac{\partial c}{\partial \tau},$$

with  $U' = U + K/L$ . This equation is to be solved under the following boundary conditions:

$$(15) \quad [c(t, x, \tau)]_{x=0} = \delta(\tau - 0)$$

and

$$(16) \quad [c(t, x, \tau)]_{\tau=0} = 0.$$

The World Ocean counterparts of (15) and (16) obviously are (8) and (9), respectively.

In the leaky funnel, the mean water age, the global water age distribution, the global mean water age and the global mean water age distribution are to be calculated as follows:

$$(17) \quad a(t,x) = \int_0^{\infty} \tau c(t,x,\tau) d\tau$$

$$(18) \quad \mu(t,\tau) = \frac{1}{L} \int_0^{\infty} c(t,x,\tau) e^{-x/L} dx ,$$

$$(19) \quad \bar{a}(t) = \frac{1}{L} \int_0^{\infty} a(t,x) e^{-x/L} dx = \int_0^{\infty} \tau \mu(t,\tau) d\tau ,$$

$$(20) \quad \phi(t,\tau) = \frac{1}{L} \lim_{\delta\tau \rightarrow 0} \int_{\Omega} \frac{\text{sign}[\tau + \delta\tau - a(t,x)] + \text{sign}[a(t,x) - \tau]}{2\delta\tau} e^{-x/L} dx .$$

### The steady-state solution in the leaky funnel

At a steady state, the leaky-funnel admits the following analytical solutions (see Appendix):

$$(21) \quad c(x,\tau) = \frac{x \exp\left[-\frac{(x-U'\tau)^2}{4K\tau}\right]}{\sqrt{4\pi K\tau^3}} \quad \text{(water age distribution)}$$

$$(22) \quad a(x) = \int_0^{\infty} \tau c(x,\tau) d\tau = \frac{x}{U'} = \frac{x}{U} \frac{Pe}{1+Pe} \quad \text{(mean water age)}$$

$$(23) \quad \mu(\tau) = \frac{1}{L} \int_0^{\infty} c(x,\tau) e^{-x/L} dx = \sqrt{\frac{K}{\pi L^2 \tau}} \exp\left(-\frac{U'^2 \tau}{4K}\right) + \frac{1}{\theta} \left[1 + \text{erf}\left(\frac{1}{\theta} \sqrt{\frac{L^2 \tau}{K}}\right)\right] \exp\left(-\frac{U\tau}{L}\right) \quad \text{(global water age distribution)}$$

$$(24) \quad \bar{a} = \frac{1}{L} \int_0^{\infty} a(x) e^{-x/L} dx = \int_0^{\infty} \tau \mu(\tau) d\tau = \frac{L}{U'} = \frac{L}{U} \frac{Pe}{1+Pe} \quad \text{(global mean water age)}$$

$$(25) \quad \phi(\tau) = \frac{1}{L} \lim_{\delta\tau \rightarrow 0} \int_{\Omega} \frac{\text{sign}[\tau + \delta\tau - a(x)] + \text{sign}[a(x) - \tau]}{2\delta\tau} e^{-x/L} dx = \frac{U' e^{-U'\tau/L}}{L} \quad \text{(global mean water age distribution)}$$

with  $Pe = UL/K$ ,  $\frac{1}{\theta} = \frac{U'}{2L} \left(1 - \frac{2}{Pe'}\right)$  and  $Pe' = U'L/K$ .

### Appendix: steady-state leaky funnel solutions

At a steady state, (14)-(16) reduce to

$$(A.1) \quad \frac{\partial c}{\partial \tau} + U' \frac{\partial c}{\partial x} = K \frac{\partial^2 c}{\partial x^2},$$

$$(A.2) \quad [c(x, \tau)]_{x=0} = \delta(\tau - 0),$$

$$(A.3) \quad [c(x, \tau)]_{\tau=0} = 0.$$

Upon denoting  $\tilde{c}(x, s)$  the Laplace transform of the water age distribution  $c(x, \tau)$  and using some of its basic properties (e.g. Abramowitz and Stegun 1972), it is readily seen that (A.1) and (A.2) transform to

$$(A.4) \quad s\tilde{c} - \underbrace{c(x, 0)}_{=0} + U' \frac{\partial \tilde{c}}{\partial x} = K \frac{\partial^2 \tilde{c}}{\partial x^2},$$

see (A.3)

$$(A.5) \quad \tilde{c}(0, s) = 1,$$

leading to

$$(A.6) \quad \tilde{c}(x, s) = \exp\left(-\frac{\sqrt{(U')^2 + 4Ks} - U'}{2K} x\right).$$

The original of this expression is equivalent to expression (21) (e.g. Abramowitz and Stegun 1972). To obtain the mean water age (22) and the global water age distribution (23), integrals involving the water age distribution have to be performed, a task that may be achieved with the help of Gradshteyn and Ryzhik (2000) or a symbolic calculation software such as *Mathematica*. QED.

## References

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