3125

REPRINT

Pascal Mossay, Pierre M. Picard and Takatoshi Tabuchi

Urban structures with forward and backward linkages

Regional Sciences and Urban Economics, 83, 103522, 2020

CORE
CENTER FOR OPERATIONS RESEARCH AND ECONOMETRICS

CORE
Voie du Roman Pays 34, L1.03.01
B-1348 Louvain-la-Neuve
Tel (32 10) 47 43 04
Email:lidam-library@uclouvain.be
https://uclouvain.be/en/research-institutes/
lidam/core/reprints.html

Regional Science and Urban Economics 83 (2020) 103522

Contents lists available at ScienceDirect

Regional Science and Urban Economics

journal homepage: www.elsevier.com/locate/regec

Urban structures with forward and backward linkages[☆]

Pascal Mossay[a,b], Pierre M. Picard[c,b], Takatoshi Tabuchi[d,*]

[a] School of Economics and Trade, Kyungpook National University, Korea
[b] CORE, Université Catholique de Louvain, Belgium
[c] CREA, University of Luxembourg, Luxembourg
[d] Faculty of Economics, University of Tokyo, Japan

ARTICLE INFO

JEL classification:
R12
R14
R31

Keywords:
Urban structure
Demand and vertical linkages
Monopolistic competition
Land use

ABSTRACT

We study urban structures driven by demand and vertical linkages in the presence of increasing returns to scale. Individuals consume urban varieties and firms use these varieties to produce a national good. We prove the existence of a spatial equilibrium and obtain an invariance result according to which more intense demand or vertical linkages have the same effect on the urban structure as lower commuting costs. Various urban configurations can emerge exhibiting a monocentric, an integrated, a duocentric, or a partially integrated city structure. We discuss the role of commuting and transport costs, demand and vertical linkages, and urbanization in shaping these patterns. We show that multiple equilibria may arise involving the monocentric city and up to a couple of duocentric and partially integrated structures.

1. Introduction

In a seminal paper, Fujita and Ogawa (1982) provided theoretical foundations for the understanding of the distribution of households and firms across residential, business, and mixed areas. Urban spatial structures were shown to depend on dispersion and agglomeration forces stemming from commuting costs and spillover externalities, and to include up to multiple centers. Since that early work, the urban economics literature has studied the properties of urban structures resulting from agglomeration forces driven by spatial externalities modelled by a black-box spillover function, see e.g. Lucas and Rossi-Hansberg (2002). However, as pointed out in the New Economic Geography literature, e.g. Krugman (1993), agglomeration forces also have their origin in economic interactions such as demand and vertical linkages particularly in locations where consumers and firms get a better access to final and intermediate goods. The main objective of our paper is to focus on the economic interactions and to show how they shape urban structures.

Empirical research in urban economics provides ample evidence of agglomeration forces in terms of economies of (urban) density. Doubling the population density increases the average productivity in cities by 1%–3% (Combes and Gobillon, 2015).[1] However, as shown by Ahlfeldt et al. (2015), agglomeration effects turn out to be extremely localized in terms of both production and consumption. Agglomeration forces entice firms to co-agglomerate so as to benefit from the proximity to other firms: according to Ellison and Glaeser (1999), such vertical linkages are the strongest factor explaining co-agglomeration in the U.S. manufacturing sector. Kolko (2010) finds that service firms tend to co-agglomerate in U.S. locations with the same zip codes, while manufacturing firms rather co-locate within the same counties. Cities have also been shown to be attractive to consumers because of providing them with an access to a rich variety of service and consumer goods, and thus enhancing the quality of urban life. Glaeser et al. (2001) explain that high urban densities allow service firms to benefit from economies of scale and that agglomeration economies can therefore be attributed to economic interactions between firms and consumers. A distinctive feature of economic linkages is their impact on prices and the diversity

☆ We wish to thank two anonymous referees, Marcus Berliant, Kristian Behrens, Yashar Blouri, Masahisa Fujita, Koichi Fukumura, Jacques Thisse, Ping Wang and participants at the Osaka Conference and the Vancouver NARSC for valuable comments and suggestions.
* Corresponding author.
E-mail addresses: pascal.mossay@knu.ac.kr (P. Mossay), pierre.picard@uni.lu (P.M. Picard), ttabuchi@e.u-tokyo.ac.jp (T. Tabuchi).
[1] The estimates of the effect economies of density vary according to the estimation methods and the dependent variable (wage or total factor productivity) (Combes and Gobillon, 2015).

https://doi.org/10.1016/j.regsciurbeco.2020.103522
Received 19 June 2019; Received in revised form 21 January 2020; Accepted 21 January 2020
Available online 1 February 2020
0166-0462/

of goods and services in urban areas. As denser locations improve the access to local goods and services, consumers are expected to face lower prices in such areas. This allows them to purchase larger quantities of goods and also entices the entry of additional local producers. Handbury and Weinstein (2015) provide empirical evidence of this effect by showing that prices are indeed lower in city centers (after controlling heterogeneity bias). Couture (2016) provides further evidence of lower prices and higher diversity in denser areas using data on restaurants. Finally, both vertical and demand linkages are noticeable in sectoral accounting data in big cities. As an example, in Tokyo in 2011, 31% of total production consisted of local intermediate goods while 39% of that production represented final goods. Hence, the distinction between the local demands by firms and consumers is crucial and deserves a particular attention.

In this paper, we study how urban structures such as those studied by Fujita and Ogawa can be explained by economic forces emphasized in the New Economic Geography literature. Towards this aim, we build a framework close in spirit to Fujita and Ogawa's urban model[2] with transport costs as an agglomeration force and commuting costs as a dispersion force. For the sake of comparison with Fujita and Ogawa, we present a model with linear commuting costs, inelastic demand for land, and a national good. However, in contrast to Fujita and Ogawa, agglomeration forces are driven by differentiated urban varieties produced with increasing returns to scale under monopolistic competition. In addition to the national good, individuals consume these urban varieties. Urban firms also use these varieties as intermediate components to produce the homogeneous national good in a perfectly competitive market as is usual in the literature. Using logarithmic consumer preferences and firm production functions, the representation of agglomeration forces in our model turns out to have the same structural form as the spillover function used by Fujita and Ogawa, which makes the two models almost isomorphic, except for firm profits coming here from increasing returns to scale.

We study various city configurations addressed in the literature, namely the monocentric, integrated, partially integrated, and duocentric urban structures. Our main results are the following. The monocentric city emerges for very low commuting costs while spatial integration of business and residential districts occurs when goods are shipped at low transport costs within the city. As more intense demand or vertical linkages have the same effect as lower commuting costs, they also make the monocentric structure more likely. Furthermore, for intermediate commuting/transport costs and moderate intensities of demand/vertical linkages, various urban structures may emerge and coexist in equilibrium. We show the existence of multiple equilibria with different urban structures. For instance, for a same set of economic parameters, we show the emergence of four urban structures involving a monocentric city, a duocentric city and two different partially integrated cities. Furthermore, we highlight the impact of urbanization by analyzing the effect of population growth on urban structures. When total population rises, the city evolves from a monocentric city to an integrated city. This transition possibly passes through the emergence of duocentric or partially integrated structures depending on the magnitude of transport costs. When transport costs are high, the economy undergoes discontinuous transitions reflecting larger benefits gained by firms when production locations separate into several business districts.

The first contribution of our paper consists in providing microfoundations to spatial spillover functions used so far in the literature. It extends the existing literature on the endogenous formation of urban areas by uncovering the role of backward and forward linkages on the formation of cities. Whereas Fujita and Ogawa (1982), Berliant et al. (2002), Lucas and Rossi-Hansberg (2002) and Allen et al. (2015) discuss the implications of agglomeration benefits arising from external face-to-face communications and technological spillovers respectively, our paper provides microfoundations for agglomeration forces driven by internal increasing returns to scale of an urban production sector selling its goods under monopolistic competition. Accordingly, firms operating in the city have market power and earn a positive profit before proceeding to rent and wage payments. The presence of these internal increasing returns makes our analysis more appealing in an urban context. The equilibrium conditions of our model are very close to the ones in Fujita and Ogawa (1982). Nevertheless, our model is not exactly isomorphic to theirs due to the presence of profits of local producers.[3] Such a strong similarity suggests that vertical linkages have similar effects on the urban structure as face-to-face communications or technological spillovers. However, the distinction between these agglomeration factors has important policy implications. Although face-to-face communication may be substituted or eased by better communication technologies and technological spillovers, they are hardly affected by urban policies. By contrast, vertical linkages are reinforced by urban policies improving the transport and delivery of goods and services within the city. Also, demand linkages are strengthened by urban policies promoting better quality and accessibility of urban consumption centers.[4] Similarly, these linkages are sensitive to national tax policies on consumption and intermediate goods.

Our second contribution to the literature is to emphasize the role of demand and vertical linkages. As in Krugman and Venables (1995), we use symmetric demand functions for final and intermediate goods. As a result, the intensities of demand and vertical linkages are reflected by respectively a demand parameter and a productivity parameter for the urban sector. We show that these two demand and productivity parameters turn out to have the same effect on the urban structure meaning that demand and vertical linkages have the same spatial effects in our model. Moreover, we show that strengthening these linkages has the same effect on urban structures as a fall in commuting costs. The second contribution of the paper is thus about the equivalence between the effects of demand and vertical linkages, and that of commuting costs. In particular, we show that a rise in the parameters affecting those linkages has the same effect on the city structure as a fall in the commuting cost parameter. Therefore, such an equivalence result highlights the link between economic and transport policies.

Our third contribution lies in revisiting issues regarding the existence and the multiplicity of equilibria in urban economics. So far, the existing literature has proposed two approaches. The constructive approach by Fujita and Ogawa (1982) consists in checking whether specific urban configurations meet the conditions for spatial equilibrium. This approach permits to discuss an analytical description of spatial equilibrium properties (such as land gradient, wage profiles …) and the conditions under which particular urban configurations emerge. Although it permits to confirm the existence of one or more spatial equilibrium configurations for specific parameter values, this approach however does not ensure the existence of an equilibrium for all parameters. In contrast, the functional approach by Lucas and Rossi-Hansberg (2002) and Carlier and Ekeland (2007) address the spatial equilibrium as a fixed point problem, to which functional analysis can be applied to demonstrate the existence of a solution. Yet, this approach leaves the issue of multiple equilibria unanswered and loses generality because

[2] With more than a thousand citations, Fujita and Ogawa (1982) framework is acknowledged in the literature as the reference model for the understanding of endogenous urban structures, see e.g. Fujita et al. (1999), Fujita and Thisse (2013), or Duranton and Puga (2014).

[3] The absence of a strict isomorphism contrasts with recent quantitative spatial models that emphasize an isomorphism between models involving different production technologies and agglomeration forces (see Allen and Arkolakis, 2014). Monte et al. (2018) show that their NEG model with commuting is isomorphic to an Armington or an Eaton and Kortum (2002) model incorporating external economies of scale and commuting.

[4] See local programs like "High Streets and Town Centers 2015" by the UK Ministry of Housing, Communities & Local Government.

equilibrium properties are usually obtained by means of numerical computations and calibration of a particularly city. Our paper builds on both approaches by guaranteeing the existence of spatial equilibrium and discussing the properties of traditional urban configurations. Our existence Proof is similar to Carlier and Ekeland (2007) and applies to Fujita and Ogawa (1982). Regarding the analysis and the computation of multiple equilibria, our approach contrasts with methods relying on fixed point algorithms. We propose an innovative way of determining urban structures, which consists in determining the borders of potential business/residential districts first, and then in checking whether each structure candidate actually constitutes a spatial equilibrium or not. Finally, our paper contrasts with quantitative theory exercises (e.g. Ahlfeldt et al., 2015; Allen et al., 2015, …) because the equilibrium existence and characterization are here obtained under the assumption of homogenous space, travel and production technologies and preferences. Quantitative exercises usually introduce various types of heterogeneities to improve the match between models and microeconomic data. Such heterogeneities also relieve the modeler of the complication of discontinuities in the land use, which are explicitly dealt with in this paper (as well as in Lucas and Rossi-Hansberg, 2002, and Carlier and Ekeland, 2007, and followers).

Our final contribution lies in the discussion of the use of space by firms. Since Alonso (1964), the urban economics literature often studies city structures under the simplifying assumption of a monocentric structure that involves a spaceless central business district, which uses no land and hosts all firms. In this paper, we are able to discuss the limiting case where firms' land use tends to zero. We find that a monocentric city becomes a spatial equilibrium only for low commuting costs, high transport costs for goods and services and strong economic linkages. This suggests us to use the above monocentric assumption only in more restrictive contexts.

The paper is structured as follows. Section 2 presents the model and discusses the equilibrium in the product, labor, and land markets. Section 3 establishes the existence of a spatial equilibrium and the invariance result between demand/vertical linkages and commuting costs. Section 4 studies various spatial equilibria exhibiting monocentric, integrated, duocentric, or partially integrated structures. Section 5 presents numerical computations and results regarding these urban structures. Section 6 concludes. All proofs are relegated to Appendices A to I.

2. The model

We consider a linear city hosting a continuum of individuals and firms along the interval $\mathcal{B} \equiv [-b, b] \subset \mathbb{R}$ with unit width. The mass of individuals is exogenous and denoted by N. The location supports of individuals and firms are denoted respectively by $\mathcal{X}$ and $\mathcal{Y}$, with $\mathcal{X} \cup \mathcal{Y} = \mathcal{B}$. Individuals occupy a unit of residential land, commute to their workplace, and consume differentiated varieties supplied in the urban markets as well as a homogeneous good supplied in the national (i.e. intercity) market. In contrast to the national good, trade of urban varieties incurs iceberg transport costs. These varieties are also used as input components to produce the national good. For these two production activities, each firm hires a worker and uses s units of land in the production process. As a consequence, the equilibrium number of firms is also equal to N. Land is owned by absentee landlords as is usual in the literature in urban economics following Alonso (1964). The opportunity cost of land is set to zero for the sake of simplicity. As in the literature (see, e.g., Fujita and Ogawa (1982); Lucas and Rossi-Hansberg (2002); Berliant et al. (2002)), individuals consume a national good that is available everywhere in the city without any transport cost. Here they also consume urban varieties purchased at firm production sites. This illustrates the idea of a "consumer city" where firms sell specialized services and consumer goods to urban customers by exploiting local economies of scale, see Glaeser et al. (2001). Typically, these local firms produce books, high-tech material, fashion cloth, or are luxury shops, restaurants, theater places, discos, museums, art galleries, or trendy furniture shops, that need sufficiently high local customer bases to be profitable and able to diversify their offer in urban areas. Since individuals' workplaces are generally closer to these producers, we assume that consumers incur the transport cost from their workplace. They consume these goods either at the production site (e.g. restaurants), or bring them back home during when commuting (e.g. clothing), or order a separate shipment the cost of which is neglected in this paper (e.g. artwork). Accordingly, an individual residing at $x \in \mathcal{X}$ and working at $z \in \mathcal{Y}$ is endowed with the following log-linear utility function (Pflüger, 2004)

$$U(x,z) = \alpha \ \ln \left[\int_{\mathcal{Y}} c(z,y)^{\frac{\sigma-1}{\sigma}} \mu(y) \mathrm{d}y \right]^{\frac{\sigma}{\sigma-1}} + c_0(x,z)$$

where $c(z, y)$ her consumption of a variety produced by a firm located at $y \in \mathcal{Y}$, $\mu(y)$ the density of firms at $y \in \mathcal{Y}$, $\sigma > 1$ the elasticity of substitution between urban varieties, $\alpha \geq 0$ the expenditure intensity on these varieties, and $c_0(x, z)$ denotes her consumption of the national good at her residence place. Her budget constraint writes as

$$p_0 c_0(x,z) + \int_{\mathcal{Y}} p(y) c(z,y) e^{\tau|z-y|} \mu(y) \mathrm{d}y + t\,|x-z| + R(x) \leq w(z) + \bar{c}_0$$

where p_0 is the price of the national good, $p(y)$ the mill price of an urban variety produced at y, τ the iceberg cost, $c(z, y)e^{\tau|z-y|}$ the quantity of a variety to be purchased at y to ensure consumption $c(z, y)$, t the commuting cost per unit of distance between residence and work places, $R(x)$ the land rent at the worker's residence, $w(z)$ her wage at workplace z, and finally $\bar{c}_0$ some initial endowment of the national good. A purpose of our paper is to study the effect of urban demand linkages that are active if $\alpha > 0$. Note that our paper encompasses the case of individuals consuming a national good only, traditionally studied in the existing literature (when setting $\alpha = 0$).

On the production side, each firm hires a worker supplying a unit of labor, and produces both the national good and a single component variety. To model vertical linkages, we assume that the national good is made of the component varieties available in the urban markets. For instance, think of a city exporting wood and metal furniture: woodcraft makers export wood furniture with steel ornaments made by local steelmakers while steelmakers export steel furniture with wood ornaments made by local woodcraft subcontractors. The production of the national good and the local component is integrated within the same production plant reflecting a too high cost for the firm to manage production over several plants. Additional outsourcing considerations are beyond the scope of the present paper. Finally, the national good is assumed to be homogeneous across firms (and cities) and firms are price takers in this market.[5] In contrast, each firm is a price setter in the market of its urban variety.

More specifically, production takes place as follows. On the one hand, the national good is made of local inputs with the output of a firm located at $y \in \mathcal{Y}$ given by the production function

$$q(y) = \beta \ \ln \left[\int_{\mathcal{Y}} i(y,z)^{\frac{\sigma-1}{\sigma}} \mu(z) \mathrm{d}z \right]^{\frac{\sigma}{\sigma-1}} \tag{1}$$

where $i(y, z)$ is firm y's local component use of a variety produced by a firm located at $z \in \mathcal{Y}$ and $\beta > 0$ the intensity of vertical linkages.[6] Because of iceberg transport costs, firm y has to purchase $i(y, z)e^{\tau|z-y|}$

[5] Product differentiation and imperfect competition in the national market would not add much to our analysis.

[6] The use of the same formulation for the utility and production functions is common in the literature with final and intermediate goods (see, e.g., Krugman and Venables (1995) for CES preferences and Picard and Tabuchi (2013) for quadratic preferences). In our text, final and intermediate goods have the same elasticity of substitution. This allows the effects of final and intermediate purchases to be simply captured by the demand shifter parameters α and β.

units of a variety produced at z and sold at mill price $p(z)$. Therefore, the operating profit of the national good production in location y, i.e., the total revenue minus the cost of inputs, is given by

$$\pi^n(y) \equiv p_0 q(y) - \int_{\mathcal{Y}} p(z) i(y,z) e^{\tau|y-z|} \mu(z) \mathrm{d}z \tag{2}$$

On the other hand, the firm produces its specific component using the national good as input. Every component requires γ units of the national good. Therefore, the operating profit in location y of the urban variety component production is

$$\pi^u(y) \equiv [p(y) - \gamma] \int_{\mathcal{Y}} [c(z,y) + i(z,y)] \, e^{\tau|z-y|} \mu(z) \mathrm{d}z \tag{3}$$

Labor and land are de facto the firm's fixed costs, so that the total profit of a firm in location y can be written as

$$\pi(y) = \pi^n(y) + \pi^u(y) - w(y) - sR(y) \tag{4}$$

There is free entry and the operating profits $\pi^n(y) + \pi^u(y)$ are fully absorbed by the wage $w(y)$ and the land rent $sR(y)$ in equilibrium. The national good is chosen as the numeraire so that its price can be normalized to $p_0 = 1$.

We now turn to the product market equilibrium. For this purpose, we consider the locations of firms and individuals as fixed. The demand of an individual working at z for urban varieties sold at location y is given by

$$c(z,y) = \alpha \frac{\left[p(y) e^{\tau|z-y|}\right]^{-\sigma}}{P(z)^{1-\sigma}} \tag{5}$$

where the urban price index is given by $P(z) \equiv \left[\int_{\mathcal{Y}} p(y)^{1-\sigma} e^{\tau(1-\sigma)|z-y|} \mu(y) \mathrm{d}y\right]^{\frac{1}{1-\sigma}}$. Her spending on urban varieties is equal to αp_0 and her demand for the national good is given by $c_0(x,z) = w(z) - t|x-z| - R(x) + (\bar{c}_0 - \alpha)$. We assume that the initial endowment $\bar{c}_0$ is sufficiently large so that consumers buy a positive amount of the national good c_0. See Appendix A for more details.

A firm located at y simultaneously chooses its input mix $i(y,\cdot)$ and its own component price $p(y)$ that maximize its profit (4) subject to relation (1). Since the firm has a zero mass, it is too small to influence the other firms' component prices and consumers' demand. Because the profit in the national market has the same functional form as the individual utility, the optimal input mix is equal to $i(y,z) = (\beta/\alpha) c(y,z)$ where $c(y,z)$ is given by expression (5). The firm's spending on urban varieties is equal to β. The operating profit in the national market can therefore be written as $\beta \left[\ln(\beta/P(y)) - 1\right]$ so that a firm supplies the national market as long as $\beta > \mathrm{e}P(y)$ (where $\mathrm{e} = \exp(1)$ is the Euler/Napier number). Also, because the total demand for an urban variety produced in y, $\int_{\mathcal{Y}} [c(y,z) + i(y,z)] \, e^{\tau|z-y|} \mathrm{d}z$, has a constant price elasticity σ, the firm sets its component price to $p(y) = \gamma\sigma/(\sigma - 1) > \gamma$, which leads to a positive operating profit in the urban component market that turns out to be the same across firms.

Given the profit-maximizing prices, the price index becomes

$$P(z) = \gamma \frac{\sigma}{\sigma - 1} T(z)$$

where

$$T(z) \equiv \left[\int_{\mathcal{Y}} e^{\tau(1-\sigma)|z-y|} \mu(y) \mathrm{d}y\right]^{\frac{1}{1-\sigma}} \tag{6}$$

denotes the access cost in location z, i.e., the aggregate delivery cost from all locations in $\mathcal{Y}$ to location z. This access cost $T(z)$ increases with the transport cost τ.

Consumers' and firms' demands for urban varieties are given respectively by

$$\frac{c(z,y)}{\alpha} = \frac{i(z,y)}{\beta} = \frac{\sigma - 1}{\gamma\sigma} \frac{e^{-\sigma\tau|z-y|}}{T^{1-\sigma}(z)} \tag{7}$$

Then, the indirect utility of an individual can then be written as

$$U(x,z) = v(z) + c_0(x,z) \tag{8}$$

where

$$v(z) \equiv \alpha \ln \left[\int_{\mathcal{Y}} c(z,y)^{\frac{\sigma-1}{\sigma}} \mu(y) \mathrm{d}y\right]^{\frac{\sigma}{\sigma-1}} = \alpha \left[\ln \frac{\alpha\rho}{\gamma} - \ln T(z)\right] \tag{9}$$

denotes the urban utility, i.e., the utility from the consumption of urban varieties with $\rho \equiv 1 - 1/\sigma$. Clearly, the urban utility $v(z)$ depends negatively on the access cost $T(z)$.

Using the profit-maximizing price $p(y) = \gamma\sigma/(\sigma - 1)$, the operating profit (3) in the urban variety market can be written as

$$\pi^u(y) = (\alpha + \beta) G(y) \tag{10}$$

where the profit G is defined by

$$G(y) \equiv \frac{1}{\sigma} \int_{\mathcal{Y}} \frac{\mu(z)}{T^{1-\sigma}(z)} e^{(1-\sigma)\tau|z-y|} \mathrm{d}z \tag{11}$$

The operating profit $\pi^u(y)$ aggregates the operating profit of a firm in location y from the sales of its variety to both consumers and firms. It increases with the expenditure intensities on urban varieties of consumers (α) and firms (β).

Similarly, as the firm output in the national market is given by $q^*(y) = \beta[\ln(\beta\rho/\gamma) - \ln T(y)]$, the operating profit (2) in that market can be written as

$$\pi^n(y) = \beta \left[\ln \frac{\beta\rho}{\gamma \mathrm{e}} - \ln T(y)\right] \tag{12}$$

The firm has an incentive to produce the national good if its profit is positive, i.e. $\ln \frac{\beta\rho}{\gamma \mathrm{e}} \geq \ln T(y)$. This holds if γ is sufficiently small, which we assume from now on.[7]

The sum of the urban utility (9) and the operating profits (2) and (3) defines the following total surplus TS generated in location y

$$\mathrm{TS}(y) \equiv v(y) + \pi^n(y) + \pi^u(y)$$

Using expressions (10) and (12), the total surplus can be rewritten as

$$\mathrm{TS}(y) = \alpha \ln \frac{\alpha\rho}{\gamma} + \beta \ln \frac{\beta\rho}{\gamma \mathrm{e}} + (\alpha + \beta) S(y) \tag{13}$$

where the surplus $S(y)$ is given by

$$S(y) \equiv G(y) - \ln T(y) \tag{14}$$

The surplus $S(y)$ increases with the profit $G(y)$ in the urban variety market and accounts for the cost $T(y)$ of accessing location y.

We now turn to the land market equilibrium still assuming the distributions of consumers and firms as given. We assume that land is used as consumption or firms' input and that its opportunity cost is nil. In a competitive land market, landowners allocate their plot of land to the highest bidders among residents and firms. As individuals are homogeneous and perfectly mobile, in equilibrium, they obtain the same utility level across residential locations, say U^*. By using the urban utility (9), the bid rent of an individual residing at location x and working at $z = z(x)$ is given by

$$\Psi_r(x) = v(z) + [w(z) - t|x-z| + \bar{c}_0 - \alpha] - U^* \tag{15}$$

where the constant utility level U^* will be endogenously determined in equilibrium. Firms freely enter and therefore make no profit. By (4), a firm at location y can bid for land up to a level which makes its profit $\pi(y)$ equal to zero so that its bid rent is given by

$$\Psi_b(y) = \frac{1}{s} \left[\pi^n(y) + \pi^u(y) - w(y)\right] \tag{16}$$

As total land consumption $(1+s)N$ is equal to the total available land $2b$, the city edge is given by $b = (1+s)N/2$.

[7] That is, if $\gamma \leq \beta(1 - 1/\sigma)/(\mathrm{e} \max_{y \in \mathcal{Y}} T(y))$. Given that $\max_{y \in \mathcal{Y}} T(y) > e^{2b\tau}$, a sufficient condition is $\gamma < \gamma^{\max} \equiv \beta(1 - 1/\sigma) \mathrm{e}^{-2b\tau - 1}$.

3. Spatial equilibrium

In this section, we define and study the spatial equilibrium of our economy where both consumers and firms exploit spatial arbitrage opportunities. For this purpose, we rewrite the worker's and the firm's bid rents as follows. The worker's bid rent in location x corresponds to the maximum bid a worker can make on a parcel of land in x while receiving a wage from a firm located in $y \in \mathcal{Y}$ willing to hire her under free entry. Using expressions (15) and (4), we get

$$\Psi_r(x) = \max_{y\in\mathcal{Y}, w(y)\in\mathbb{R}^+} (v(y) + w(y) - t(x-y) + \bar{c}_0 - \alpha - U^*)$$
$$\text{s.t. } \pi(y) = \pi^n(y) + \pi^u(y) - w(y) - sR(y) \geq 0 \tag{17}$$

Similarly, the firm's bid rent corresponds to the maximum bid that it can make on a parcel of land in y while the wage it pays to a worker located in $x \in \mathcal{X}$ induces her to supply her labor to the firm given her utility U^*. Using expressions (16) and (8)

$$\Psi_b(y) = \max_{x\in\mathcal{X}, w(y)\in\mathbb{R}^+} \frac{1}{s}(\pi^n(y) + \pi^u(y) - w(y))$$
$$\text{s.t. } U(x,y) = v(y) + w(y) - t(x-y) - R(x) + \bar{c}_0 - \alpha \geq U^* \tag{18}$$

We now draw a connection with Fujita and Ogawa (1982). In their seminal paper, urban structures endogenously emerge from agglomeration effects generated by face-to-face communication between firms' managers and/or employees. This face-to-face communication generates a production surplus that depends negatively on the logarithm of the average distance between firms. One could then interpret our parameter expression $(\sigma - 1)\tau$ in $\ln T(y)$ as Fujita and Ogawa's parameter for face-to-face communication. This means that from a reduced form point of view, vertical linkages have the same role as face-to-face communication in Fujita and Ogawa. However, here our vertical linkages stem from the production of urban varieties under imperfect competition, which generates a profit $\pi^u(y)$ at location y that is absent in the face-to-face communication model. To sum up, the reduced form of our model encompasses the structure of Fujita and Ogawa (1982) in the absence of demand linkages and profit from the production of urban varieties (that is, when $\alpha = 0$ and $\pi^u(y) = G(y) = 0$). This also means that the absence of demand linkages and urban variety production in Fujita and Ogawa (1982) implies that their agglomeration force is weaker than that in our model.

In this model, the presence of positive urban profits π^u results from imperfect competition in the differentiated product market where each differentiated variety can be produced anywhere in the city. The presence of this term complicates the theoretical analysis. This means that, in our earlier example, woodcrafters and steelmakers can produce their piece of furniture or ornaments wherever they locate in the city. This contrasts with Allen et al. (2015) modelling strategy where the urban location defines the produced variety. In that framework, firms in a same location produce the same differentiated variety, which results in local perfect competition and eliminates urban profits, thus simplifying the analysis. Their view is justified by the presence of location-specific factors such as in the historical examples of tanners and brewers producing and competing in locations close to urban streams. Our paper adopts the former framework where locations do not entail specific factors except for the access to other economic agents.

Using the total surplus TS introduced in (13) and substituting the wage expression obtained from the constraints, the bid rents (17) and (18) can be rewritten as

$$\Psi_r(x) = \max_{y\in\mathcal{Y}} \left[\mathrm{TS}(y) - t\,|x-y| + \bar{c}_0 - \alpha - U^* - sR(y)\right] \tag{19}$$

$$\Psi_b(y) = \frac{1}{s}\max_{x\in\mathcal{X}} \left[\mathrm{TS}(y) - t\,|x-y| + \bar{c}_0 - \alpha - U^* - R(x)\right] \tag{20}$$

We can now define a competitive spatial equilibrium.

Definition 1. A competitive spatial equilibrium is defined by a resident density function $\lambda : \mathcal{B} \to [0,1]$, *a firm density function* $\mu : \mathcal{B} \to [0, 1/s]$, *a land rent* $R : \mathcal{B} \to \mathbb{R}^+$, *residential and business bid rent functions* Ψ_r *and* $\Psi_b : \mathcal{B} \to \mathbb{R}^+$, *and a utility level* $U^* \in \mathbb{R}$ *that satisfy (i) mass conservation* $\int_{\mathcal{B}} \lambda = N$, *(ii) land market clearing* $\mu(z) = (1 - \lambda(z))/s$ *for all* $z \in \mathcal{B}$, *(iii) land allocation to the highest bidder for all* $z \in \mathcal{B}$

$$\lambda \in \begin{cases} \{1\} & \text{if} \quad \Psi_r > \Psi_b \\ [0,1] & \text{if} \quad \Psi_r = \Psi_b \\ \{0\} & \text{if} \quad \Psi_r < \Psi_b \end{cases},$$

$R = \max\{\Psi_r, \Psi_b, 0\}$, *and (iv) zero land rent* R *at city borders* $z = \pm b$.

3.1. Economic gains and invariance

The urban economic activity generates gains for workers and landlords, firms making zero profit due to free entry. It is instructive to examine those gains and disentangle the effects of agglomeration forces stemming from demand and vertical linkages and the dispersion force due to commuting costs.

Consider a consumer residing at x and working for a firm located in y. In equilibrium, land rents correspond to bid rents: $\Psi_r(x) = R(x)$ and $\Psi_b(y) = R(y)$. Therefore, from expressions (19) or (20), the total gain of the consumer and landlords is

$$U^* + R(x) + sR(y) = \mathrm{TS}(y) - t\,|x-y| + \bar{c}_0 - \alpha \tag{21}$$

The total gain from urban economic interactions correspond to the sum of the urban utility, the operating profit from variety sales to consumers and firms, and the operating profit from the sale of the national good, diminished by commuting costs.

Observe that the spatial structure (λ, μ) has an impact only on the terms $(\alpha+\beta)S(y) - t\,|x-y|$. This means that it is possible to maintain the same equilibrium spatial structure (λ^*, μ^*) by multiplying both parameters $(\alpha+\beta)$ and t by the same factor $k > 0$. To maintain the above equality (21) for all x and y, equilibrium rents need to be multiplied by this factor k and the equilibrium utility U^* increased to the level $\left(k\alpha \, \ln \frac{\alpha k \rho}{\gamma e} + k\beta \, \ln \frac{k\beta\rho}{\gamma e}\right)$. This heuristic argument is confirmed below in a formal way.

Suppose that a spatial equilibrium exists and denote the mass of commuters moving to the right and crossing location x by $n(x) = \int_{-b}^{x} \left[\lambda(z) - \mu(z)\right] \mathrm{d}x$. In Appendix B, we show that the spatial equilibrium (λ, μ) satisfies the identity

$$\lambda(z) - \mu(z) \in \begin{cases} \{1\} & \text{if} \quad \Psi_b(z) - \Psi_r(z) < 0 \\ [-\frac{1}{s}, 1] & \text{if} \quad \Psi_b(z) - \Psi_r(z) = 0 \\ \{-\frac{1}{s}\} & \text{if} \quad \Psi_b(z) - \Psi_r(z) > 0 \end{cases} \tag{22}$$

where

$$\Psi_b(z) - \Psi_r(z) = \int_{\hat{z}}^{z} \left[\frac{1}{s}\,(\alpha+\beta)\,\frac{\mathrm{d}S}{\mathrm{d}x} - \frac{1+s}{s} t\, sign\,(n(x))\right] \mathrm{d}x, \tag{23}$$

and $\hat{z} \in \mathcal{B}$ is a solution to $\lambda(\hat{z}) - \mu(\hat{z}) = 0$ and designates a location where commuting stops or changes direction. There always exists at least one such location. Bid rents equate in those locations $\hat{z}$ since workers and firms settle on either the same land plot or two infinitely close residential and industrial land plots. The first term in the above integral (23) yields the economic surplus $S^*(x)$ -up to a constant-generated at location x. The second term reflects the travel cost of commuters crossing location x. Of course, the relationship $\lambda = 1 - s\mu$ can be used to eliminate a density profile either λ or μ.

The above relation (23) tells that workers and firms settle in locations where they place a higher bid. When they make the same bid, any composition of firms and residents is possible. In a competitive allocation, the allocation of land depends on the sign of the bid rent differential but not on the amplitude of this differential. Hence multiplying both parameters $(\alpha+\beta)$ and t by any scalar $k > 0$ has no effect on the sign of the bid differential and therefore maintains the same spatial equilibrium structure. This yields the following main result:

Proposition 1. *Cities with the same parameter ratio $(\alpha + \beta)/t$ admit identical spatial structures.*

The proposition tells that stronger demand and vertical linkages have the same impact as lower commuting costs. The economic surplus generated by any pair of worker and firm is balanced by the commuting cost of households. A proportional increase in the intensity of demand and vertical linkages and in the commuting cost does not change this spatial balance though each effect becomes equally more intense. Of course, the simple relationship in Proposition 1 is specific to the combination of log-linear preferences, log production functions and linear commuting costs. Nevertheless, it elegantly states the balance between economic linkages and commuting costs.

Relationships (22) and (23) define a fixed point problem for the firm spatial distribution μ. Indeed, the firms' density μ determines the profiles of the commuters' mass n and surplus S, which in turn determine the bid rent differential $\Psi_b - \Psi_r$ and then the firms' density μ again. This fixed point is formulated a similar way as in Lucas and Rossi-Hansberg (2002) who refer to n as the 'stock of unhoused workers'. Our Proof consists of determining, for every infinitely small urban region with non-zero measure, the stock of unhoused workers as a function of density profiles, and then, the bid rent differential as a function of this stock. The fixed point is formulated after integration across the urban space (see Appendix B). To the best of our knowledge, our fixed point formulation of a model extends to Fujita and Ogawa (1982) framework. Like in Lucas and Rossi-Hansberg (2002), we prove the existence of a spatial equilibrium. However, we are also able to derive conditions for each urban structure and show that the multiplicity of equilibria prevails in Section 4.

3.2. Existence

Relationships (22) and (23) define a fixed point problem for the firm spatial distribution μ. In Appendix C, we show the existence of a fixed point using Schauder's theorem.

Proposition 2. *A spatial equilibrium exists.*

As shown in (22), the land allocation across firms and workers is a non-continuous process (actually, a correspondence). Land is homogenous while residents and firms are endowed with homogenous preferences and technology. As a result, residents and firms react in large numbers to small differences in bid rents.[8] Such a process makes it difficult to establish the existence of spatial equilibrium. Therefore, as in Carlier and Ekeland (2007), we first show the existence of a fixed point for smoother land allocation processes by replacing (22) by an equation where the land allocation decision across firms and workers is a continuous function of their bid rent differential $\Psi_b - \Psi_r$. In economic terms, this may be interpreted by the presence of exogenous uncertainty in landlords' choices. We then prove the convergence of the series of fixed points to that defined by (22) and (23) for less and less smooth allocation processes. Using the same interpretation, the fixed point takes place at the limit of zero uncertainty in the land allocation process.

Note that the above existence proof can be applied to Fujita and Ogawa (1982) model for which no general existence proof has been provided. As mentioned in Section 3 already, their equilibrium conditions are structurally equivalent to ours when setting α and $G(.)$ to zero and interpreting $(\sigma - 1)\tau$ as their parameter for face-to-face communication. In Fujita and Ogawa (1982), a set of economic parameters remains where no spatial equilibrium is shown. Our proposition confirms the existence of a spatial configuration for that set of economic parameters.

3.3. Numerical computation

As in Lucas and Rossi-Hansberg (2002), one may numerically iterate the fixed point expression until it converges to a spatial equilibrium for a given set of economic parameters. However, like those authors, we do not provide a proof for a fixed point contraction, which would guarantee the convergence to a numerical solution. Instead, numerical convergence obtains by chance and numerical analysis could be unsuccessful. In the case of the present model, we have applied a reformulation of the fixed point expression (see Appendix C) on a discretized geographical support. For low enough τ and t, the fixed point algorithm converges to monocentric or duocentric cities in a small number of iterations, as shown in Fig. 1. Fig. 2 provides a grid of τ and t values for which the fixed point algorithm numerically converges (before 15 iterations). In those simulations, we have observed that convergence occurs only to these two city structures. Fig. 2 reemphasizes the contribution of our theoretical approach in the following section, which consists in checking the spatial equilibrium conditions of each particular urban structure.

4. Urban structures

Our existence result in Section 3 does not imply the uniqueness of equilibrium. In this Section, we present and discuss various urban structures and determine their condition of existence.

4.1. Monocentric city

Here the business district $\mathcal{Y} = [-b_1, b_1]$ is surrounded by the residential districts $\mathcal{X} = [-b, -b_1) \cup (b_1, b]$, see Fig. 3 (a). The densities of firms and households are $\mu(y) = 1/s$ and 1 respectively. The business district edge is given by $b_1 = sN/2$. Workers commute from the residential district to the business district.

The monocentric equilibrium conditions are as follows: (*i*) firms outbid households in the business district ($\Psi_b(y) \geq \Psi_r(y)$ for all $y \in \mathcal{Y}$), (*ii*) households outbid firms in the residential district ($\Psi_r(x) \geq \Psi_b(x)$ for all $x \in \mathcal{X}$); (*iii*) the residential and the business bid rents equalize at the business district border b_1 ($\Psi_b(b_1) = \Psi_r(b_1)$); (*iv*) the zero opportunity cost of land at the city border b requiring $\Psi_r(b) = 0$.

The urban utility $v(z)$, the surpluses $S(z)$ and the total surplus $\text{TS}(z)$ defined in (9), (14) and (13) are now denoted as

$$v_M(z) = \alpha \left[\ln \frac{\alpha \rho}{\gamma} - \ln T_M(z) \right] \tag{24}$$

$$\text{TS}_M(z) = v_M(z) + \pi_M^n(z) + \pi_M^u(z) = \alpha \ \ln \frac{\alpha \rho}{\gamma} + \beta \ \ln \frac{\beta \rho}{\gamma \mathrm{e}} + (\alpha + \beta) S_M(z) \tag{25}$$

$$S_M(z) = G_M(z) - \ln T_M(z) \tag{26}$$

with the access cost T_M and the profit G_M given by

$$T_M^{1-\sigma}(z) = \frac{1}{s} \int_{-b_1}^{b_1} e^{(1-\sigma)\tau(y-z)} \mathrm{d}y, G_M(z) = \frac{1}{\sigma s} \int_{-b_1}^{b_1} \frac{e^{(1-\sigma)\tau(y-z)}}{T_M^{1-\sigma}(y)} \mathrm{d}y \tag{27}$$

Given the symmetry of the city, our analysis focuses on the employment basin in the right side of the city $[0, b]$. Our model with linear commuting costs allows us to exploit workers' arbitrage across workplaces. Because workers are perfectly mobile, firms producing at y must offer a wage $w(y)$ that is as attractive as that available in other work locations. Put differently, in equilibrium, a worker at location x obtains the same utility whether she works at $y > 0$ or $y = 0$. That is,

[8] A recent literature assumes idiosyncratic local amenities and heterogeneous preferences for land and amenities, and/or commuting (e.g. Ahlfeldt et al., 2015). This leads to a continuous land allocation process and a fixed point defined on continuous functions.

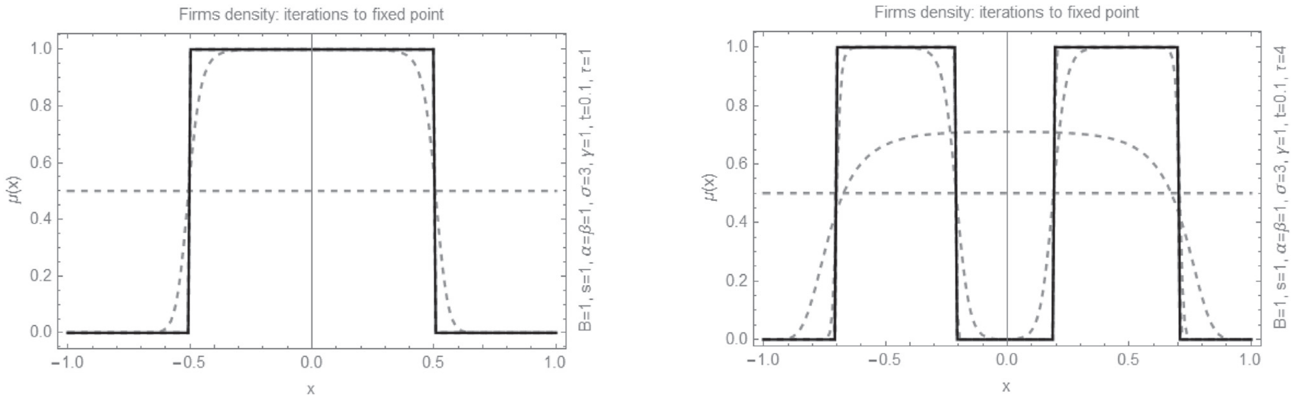


Fig. 1. Numerical examples where the fixed point converges. Note: solid curves are solutions $\mu(\mathrm{x})$ after convergence; dashed curve are iterated solutions $\mu(\mathrm{x})$. Initial condition is $\mu(\mathrm{x}) = 1/2$.

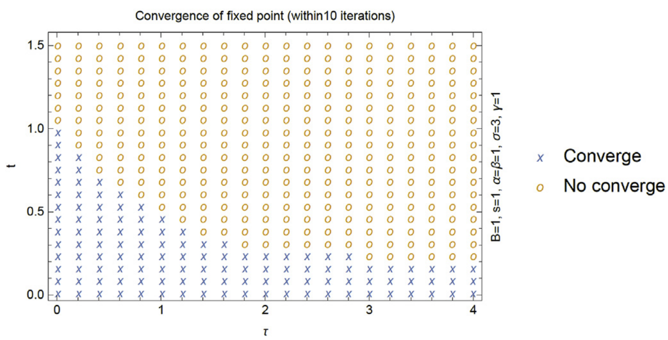


Fig. 2. Numerical convergence of fixed point. Note: x and o respectively denote the convergence and absence of convergence to a city structure after 15 iterations of the fixed point formula. Initial condition is $\mu(\mathrm{x}) = 1/2$.

$$v_M(y) - \alpha + w(y) + \bar{c}_0 - t(x-y) - R(x)$$
$$= v_M(0) - \alpha + w(0) + \bar{c}_0 - tx - R(x)$$

Solving it for the equilibrium wage $w(y)$, we get

$$w(y) = w(0) - ty + \left[v_M(0) - v_M(y)\right]$$

Hence, the wage decreases with the employer's distance to the center and the urban utility v_M. The former effect reflects wage arbitrage across locations according to which the worker's gain from a shorter commuting trip should be compensated by a lower wage. The latter effect also occurs because of arbitrage: a higher utility from the consumption of business district varieties v_M should also be compensated by a lower wage. It can be shown that the access cost to urban varieties T_M increases with the distance from the city center so that the urban utility decreases with it. Both effects then point in the same direction: the wage falls with the distance from the city center. We now present the most interesting spatial relationships and relegate computation details to Appendix D.

Using equilibrium conditions (iii) and (iv), the bid rents (15) and (16) and the wage can be rewritten as

$$\Psi_r(x) = t(b-x) \tag{28}$$

$$\Psi_b(y) = \frac{1}{s}\left\{\left[\mathrm{TS}_M(y) - \mathrm{TS}_M(b_1)\right] - t(b_1 - y) + st(b-b_1)\right\} \tag{29}$$

$$w(y) = \mathrm{TS}_M(b_1) - v_M(y) + t(b-y) - (1+s)t(b-b_1)$$

where the urban utility $v_M(y)$ and the total surpluses $\mathrm{TS}_M(y)$ are given by expressions (24) and (25). The residential bid rent Ψ_r follows the Alonso (1964) monocentric model where residents pay lower rents in locations away from the city center in proportion to their commuting cost. For very weak demand and vertical linkages ($\alpha, \beta \to 0$ so that $v_M, \mathrm{TS}_M \to 0$), our model extends Alonso's framework to the endogenous creation of a business center where firms pay lower wages and offer higher land bid rents as they locate away from the business central location. In that case, the amplitude of firms' land rent and wage gradients are mainly determined by commuting costs: firms in less central locations can indeed offer lower compensation for workers' commutes. For larger demand and vertical linkages, the total surplus TS_M generated by the production and consumption of urban varieties becomes part of wages and firm bid rents but not of residential bid rents. In a spatial equilibrium, landlords can only appropriate the rents associated to spatial differences. They therefore appropriate the commuting cost difference $t(b-x)$ between the worker's residence x and the city border b, and also the difference in total surplus $\mathrm{TS}_M(y) - \mathrm{TS}_M(b_1)$ generated by a firm and its workers at y and b_1. Because of this landlords' appropriation, each firm cashes back to its workers only the part of total surplus, $\mathrm{TS}_M(b_1)$.

Since the utility U^* is constant for any x and y, we can plug $x = b$ and $y = b_1$ into (21), which gives

$$U^* + R(b) + sR(b_1) = \mathrm{TS}_M(b_1) - t\left|b - b_1\right| + \bar{c}_0 - \alpha$$

Because $R(b) = 0$ and $R(b_1) = \Psi_r(b_1) = t(b-b_1)$, we compute the equilibrium utility as

$$U^* = \mathrm{TS}_M(b_1) - (1+s)t(b-b_1) + \bar{c}_0 - \alpha \tag{30}$$

It mainly consists of the total surplus TS_M generated by the worker and its firm at the business district edge b_1 and the commuting cost $t(b-b_1)$ that makes this worker commute from the city border. As mentioned in the previous paragraph, this outcome results from landlords' appropriation of firms' and workers' total surplus for all $y < b_1$ and resident's rents for all $x \in [b_1, b]$.

Fig. 4 illustrates the shape of bid rents in a monocentric city. The firms' bid rent curve, represented in red, is above (*resp*. below) the residents' bid rent curve (in blue) around the city center (*resp*. the city edge). Bid rents are equal to the zero opportunity cost of land at the city border and intersect at the location $y = \pm 0.5$, which separates the business district from the residential districts. Workers' bid rents are linear in the distance to the center as longer commutes are compensated by lower rents. As firms benefit from agglomeration economies, their bid rent is endogenous and depends on the location of other firms. As shown in Fig. 4, firms' bid rents are not necessarily the highest at the city center $x = 0$ as shown by Dong and Ross (2015). In that case, central firms compensate workers' commuting cost with higher wages, which dampens their benefit from better centrality.

The monocentric city configuration is a spatial equilibrium if the bid rent functions (28) and (29) also satisfy conditions (*i*) and (*ii*). Typically

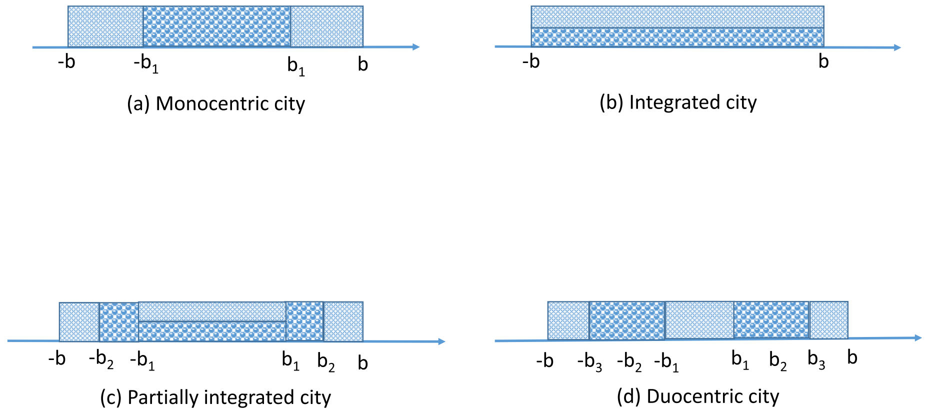


Fig. 3. Spatial urban structures. (: firms and : households).

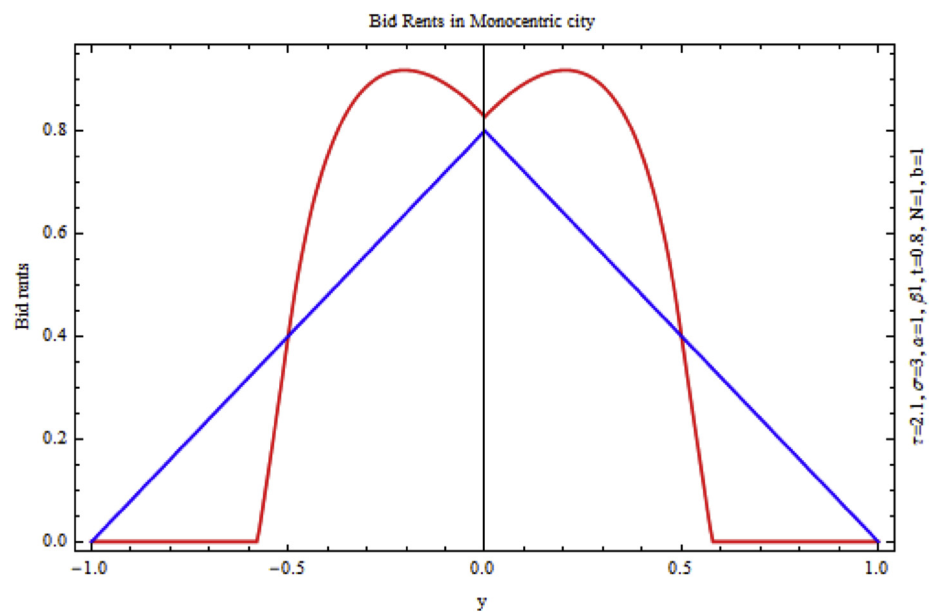


Fig. 4. Monocentric city: residential (blue) and business (red) bid rents. (For interpretation of the references to colour in this figure legend, the reader is referred to the Web version of this article.)

firms' bid rents are non-linear in the distance to the city center. A priori, it is far from obvious to guarantee a single intersection between bid rents. Nevertheless, according to the Proposition below, it turns out that only the bid rent properties at the city center and border matter for equilibrium.

Proposition 3. *The monocentric configuration is a spatial equilibrium if and only if*

$$t \le t_M \equiv \min\{\bar{t}_M, \hat{t}_M\}$$

where

$$\bar{t}_M \equiv \frac{\alpha+\beta}{1+s}\frac{S_M(0)-S_M(sN/2)}{sN/2}$$

and $$\hat{t}_M \equiv \frac{\alpha+\beta}{1+s}\frac{S_M(sN/2)-S_M((1+s)N/2)}{N/2}.$$

When $s \to 0$, $t_M = \bar{t}_M = (\alpha+\beta)\tau\,(2\sigma-1)/(2\sigma) < \hat{t}_M$.

Proof. *See* Appendix D.

While the condition $t \le \hat{t}_M$ guarantees that firms do not overbid residents at the city border ($\Psi_b(b) \le \Psi_r(b)$), the condition $t \le \bar{t}_M$ guarantees that they are not overbid by residents at the city center ($\Psi_b(0) \ge \Psi_r(0)$). Proposition 3 says that lower commuting costs favor the monocentric configuration as they allow residents to commute longer distances to the business district. Larger final demand or vertical linkages also make the monocentric configuration more likely as they increase the operating profit in the market of urban varieties due to increasing returns to scale, thus allowing firms in the business district to pay higher wages and making people more willing to commute.

The last statement in Proposition 3 implies that, even if the manufacturing sector makes no use of land ($s \to 0$), the monocentric city structure is a spatial equilibrium if and only if the commuting cost is lower than $(\alpha+\beta)\,\tau$. This result strongly qualifies the monocentric model of the urban economics literature that assumes a spaceless central business district, where firms are immobile and use no land (Alonso, 1964, among others). If firms are immobile, use almost no land ($s \to 0$) and have finite revenues, they offer almost infinite land bid rents and are able to overbid any resident close to city center, thus allowing them to occupy the city center. However, if firms are mobile, they also have an incentive to avoid high land rents in the city center and move in the residential area, typically at $x = b_1 + \varepsilon$ with small $\varepsilon > 0$, where agglomeration economies are the most intense. Such firms then attempt to attract workers by offering them higher wages, which in turn increases the wages offered by all other firms competing for the same labor. This diminishes all firms' operational profits and bid rents. This land rent arbitrage is exploited when firms and residents have equal bid rents at $x = b_1$. This arbitrage condition holds for $s \to 0$, in which case monocentric cities are not spatial equilibria for commuting costs larger than $(\alpha + \beta)\,\tau$. To become a spatial equilibrium, commuting costs should be lower, transport costs higher and economic linkages stronger. Such arguments are absent in the urban economics literature based on spaceless business district where firms are assumed to be immobile.

Finally, note that, when transport costs vanish ($\tau \to 0$), the surplus $S_M(z)$ becomes independent of location and the above thresholds become $\bar{t}_M = \hat{t}_M = 0$. Thus, the monocentric city is never a spatial equilibrium irrespective of the commuting cost $t > 0$. In this case, the demand for final and intermediate goods does not depend on location: there is no more agglomeration force and spatial concentration brings no benefit.

4.2. Integrated city

Here firms and households locate in $\mathcal{X} = \mathcal{Y} = [-b, b]$, see Fig. 3 (b). Given the unit demand for land of households and the s unit demand of firms, the density of firms is $\mu(y) = 1/(1 + s)$ and the city edge is still at $b = (1 + s)N/2$. The integrated configuration involves no commuting.

The integrated equilibrium conditions are as follows: (*i*) the business and residential bid rents equalize ($\Psi_r(z) = \Psi_b(z), \forall z \in \mathcal{X} = \mathcal{Y}$); (*ii*) the no-arbitrage relocation implying $\left|\Psi_r'(z)\right| \le t$; (*iii*) the zero opportunity cost of land at the city border b requiring $\Psi_r(b) = 0$.

The urban utility $v(z)$ and the total and variable surpluses $S(z)$ and $\mathrm{TS}(z)$ defined in (9), (14) and (13) are now denoted

$$v_I(z) = \alpha \left[\ln \frac{\alpha\rho}{\gamma} - \ln T_I(z)\right] \tag{31}$$

$$\mathrm{TS}_I(z) = v_I(z) + \pi_I^n(z) + \pi_I^u(z) = \alpha\ \ln \frac{\alpha\rho}{\gamma} + \beta\ \ln \frac{\beta\rho}{\gamma \mathrm{e}} + (\alpha+\beta)\, S_I(z) \tag{32}$$

$$S_I(z) = G_I(z) - \ln T_I(z) \tag{33}$$

with the access cost T_I and the profit G_I given by

$$T_I^{1-\sigma}(z) = \frac{1}{1+s}\int_{-b}^{b} e^{(1-\sigma)\tau(z-y)}\mathrm{d}y,$$

$$G_I(z) = \frac{1}{\sigma(1+s)}\int_{-b}^{b} \frac{e^{(1-\sigma)\tau(y-z)}}{T_I^{1-\sigma}(y)}\mathrm{d}y \tag{34}$$

These expressions are similar to those obtained in the monocentric city, except for the factor $s/(1 + s) < 1$. As a result, the access cost is larger and profits lower in the integrated city. By the same token, the surplus S_I is also lower there. This is because the urban mix of firms and residents spreads production and consumers over a larger urban area.

The equilibrium conditions (*i*) and (*iii*) yield the land bid rent and the wage

$$\Psi_r(z) = \Psi_b(z) = \frac{1}{1+s}\left[\mathrm{TS}_I(z) - \mathrm{TS}_I(b)\right]$$

$$w(y) = \mathrm{TS}_I(b) - v_I(y) + \frac{1}{1+s}\left[\mathrm{TS}_I(y) - \mathrm{TS}_I(b)\right]$$

where the surplus TS_I is given by (32).

Hence, land rents capture the surplus generated by firms and workers and are passed onto landlords. The land rent is proportional to the difference between the surplus $S_I(z)$ and that available at the city border. Workers receive wages that include the value of total surplus available TS_I at the city border b and any additional total surplus with respect to that border (term in square brackets). The wage is also diminished by the value of the urban surplus $v_I(y)$ as a lower urban surplus is compensated by a higher wage. The equilibrium utility is given by

$$U^* = \mathrm{TS}_I(b) + \bar{c}_0 - \alpha$$

which corresponds to the total surplus TS_I that firms and residents generate at the city border b. As compared to the monocentric city, the utility partly rises because workers save on commuting costs but also partly falls given that the total surplus S_I is lower due to wider dispersion of production.

Fig. 5 illustrates the shape of bid rents in an integrated city. The firms' and the residents' bid rent curves coincide at any location in the city. Bid rents vanish at the city borders due to the zero opportunity cost of land. As workers do not commute, rents do not reflect linear commuting costs but rather non-linear agglomeration economies coming from demand and input-output linkages.

In spatial equilibrium, land bid rents must satisfy condition (ii). This is guaranteed if each worker has no incentive to accept a job away from her residential location, which holds if bid rents are not too steep. This result is formalized in the following Proposition.

Proposition 4. *The integrated city is a spatial equilibrium if and only if*

$$t \ge t_I = -\frac{\alpha+\beta}{1+s} S_I'(b) > 0.$$

When $s \to 0$, $t_I = (\alpha + \beta) S_I'(b)$.

Proof. *See* Appendix E.

In Proposition 4, the equilibrium condition $t \ge t_I$ is ensured as long as commuting costs are high enough and/or demand and input-output linkages weak enough (low $\alpha + \beta$). Higher commuting costs foster the integrated configuration because very large commuting costs entice workers to take residence as close as possible to their workplace. Also, weaker final demand or vertical linkages make the integrated configuration more likely. They indeed decrease the operating profits from the production of urban varieties, which reduces workers' wages, and in turn the incentives of workers to commute. Finally, it turns out that t_I is an increasing function of τ (see Appendix E). So, *lower trade costs enlarge the domain of economic parameters for which firms integrate with residents.* For lower trade costs, input costs depend less on the distance between firms and vertical linkages lead to a weaker agglomeration force for firms. Therefore, as firms do not benefit much from clustering with other firms, the business bid can be more easily matched by workers.

When the transport cost becomes negligible ($\tau \to 0$), the surplus $S_I(z)$ is location independent so that $S_I'(z) = 0$ and $t_I = 0$. The integrated city is then a spatial equilibrium for any commuting cost $t > 0$. In this case, the demands for the final and intermediate goods are independent of location and the agglomeration forces vanish.

Finally, we prove the following corollary:

Corollary 1. *For* $s \to 0$, *the monocentric and the integrated urban configurations never coexist. That is,* $t_M < t_I$.

Proof. *See* Appendix F.

When the manufacturing sector makes no use of land, a high commuting cost is compatible only with the integrated structure. This results from the tension between backward and forward linkages and commuting costs. When backward and forward linkages are too weak, firms are unable to post land bids that are sufficiently high to foreclose workers to reside close to them. The above result excludes the possibility of multiple equilibria, where the monocentric and the integrated structures coexist for the same set of parameter values. To the best of our knowledge, this property has not been pointed out in the literature.

4.3. Partially integrated city

Here workers and firms locate with each other in the central district $(-b_1, b_1)$, which is surrounded by business districts $(-b_2, -b_1) \cup (b_1, b_2)$, themselves bordered by residential districts $(-b, -b_2) \cup (b_2, b)$, see Fig. 3 (c). The density of firms is $\mu(y) = s/(1+s)$ in the interval $(-b_1, b_1)$ and $\mu(y) = 1/s$ in the intervals $(-b_2, -b_1) \cup (b_1, b_2)$, while

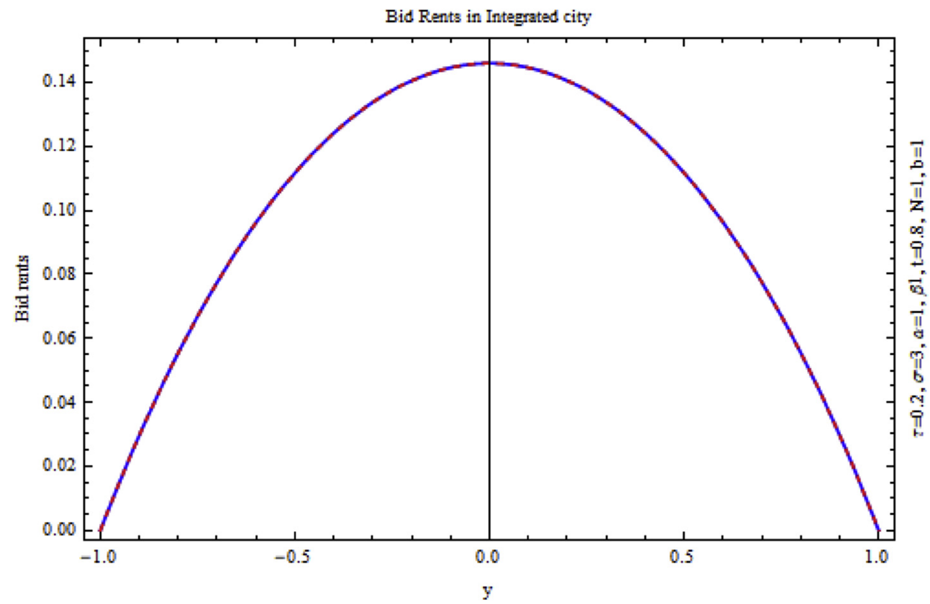


Fig. 5. Integrated city: residential (blue) and business (red) bid rents. (For interpretation of the references to colour in this figure legend, the reader is referred to the Web version of this article.)

that of workers is $\lambda(x) = 1/(1+s)$ in the interval $(-b_1, b_1)$ and $\lambda(x) = 1$ in the intervals $(-b,-b_2)\cup(b_2,b)$, with $0 < b_1 < b_2 < b$. Because the total mass of firms is N, we get

$$2(b_2-b_1)\frac{1}{s}+2b_1\frac{1}{1+s}=N$$

that is,

$$b_2=\frac{b_1}{1+s}+\frac{sN}{2} \tag{35}$$

The partially integrated equilibrium conditions are as follows: (*i*) the bid rents of households and firms equalize in the integrated district ($\Psi_r(z) = \Psi_b(z)$, $\forall z \in (-b_1, b_1)$); (*ii*) no-arbitrage relocation in the integrated district implying $\left|\Psi'(z)\right| \le t$, $\forall z \in (-b_1, b_1)$; (*iii*) firms outbid households in the business districts ($\Psi_b(y) \ge \Psi_r(y) \forall y \in [-b_2,-b_1]\cup[b_1,b_2]$); (*iv*) households outbid firms in residential districts ($\Psi_r(x) \ge \Psi_b(x)$, $\forall x \in [-b,-b_2]\cup[b_2,b]$); and (*v*) the zero opportunity cost of land at the city border b requiring $\Psi_r(b) = 0$.

The urban utility $v(z)$ and the surpluses $S(z)$ and $\mathrm{TS}(z)$ defined in (9), (14) and (13) are now denoted

$$v_P(z)=\alpha\left[\ln\frac{\alpha\rho}{\gamma}-\ln T_P(z)\right] \tag{36}$$

$$\mathrm{TS}_P(z)=v_P(z)+\pi_P^n(z)+\pi_P^u(z)=\alpha\ \ln\frac{\alpha\rho}{\gamma}+\beta\ \ln\frac{\beta\rho}{\gamma e}+(\alpha+\beta)S_P(z) \tag{37}$$

$$S_P(z)=G_P(z)-\ln T_P(z) \tag{38}$$

with the access cost T_P and the profit G_P given by

$$T_P^{1-\sigma}(z)=\frac{1}{s}\left(\int_{-b_2}^{-b_1}+\int_{b_1}^{b_2}\right)e^{(1-\sigma)\tau|z-y|}\mathrm{d}y+\frac{1}{1+s}\int_{-b_1}^{b_1}e^{(1-\sigma)\tau|z-y|}\mathrm{d}y \tag{39}$$

$$G_P(z)=\frac{1}{\sigma s}\left(\int_{-b_2}^{-b_1}+\int_{b_1}^{b_2}\right)\frac{e^{(1-\sigma)\tau|z-y|}}{T_P^{1-\sigma}(y)}\mathrm{d}y+\frac{1}{\sigma(1+s)}\int_{-b_1}^{b_1}\frac{e^{(1-\sigma)\tau|z-y|}}{T_P^{1-\sigma}(y)}\mathrm{d}y \tag{40}$$

The partially integrated configuration combines the features of monocentric and integrated cities. On the one hand, land rents and wages in the central district $(-b_1, b_1)$ are structurally similar to those in the integrated city, except that urban varieties are purchased in the business area $(-b_2, b_2)$. In Appendix G, we derive the following land bid rent and wage functions for $z \in [0, b_1)$:

$$\Psi_r(z)=\Psi_b(z)=\frac{1}{1+s}\left[\mathrm{TS}_P(z)-\mathrm{TS}_P(b_2)\right]+t(b-b_2)$$

$$w(z)=\mathrm{TS}_P(b_2)-v_P(z)+\frac{1}{1+s}\left[\mathrm{TS}_P(z)-\mathrm{TS}_P(b_2)\right]-st(b-b_2)$$

Those expressions are similar to the ones obtained in the integrated city. The surplus generated by firms and workers are here also passed onto landlords. In particular, landlords appropriate the total surplus above that obtained by the last firm at business edge b_2. In addition they appropriate the opportunity cost of commuting from the city border b to the business edge b_2. On the other hand, the city includes two commuting basins, each of which with the same structure as in the monocentric city. For instance, on its right hand side, the city includes the commuting basin (b_1, b) where workers commute from the residential district (b_2, b) to the business area (b_1, b_2). In this basin, the labor market arbitrage imposes land bid rents and wages similar to the ones obtained in monocentric city, except that urban varieties are purchased in the whole production area $(-b_2, b_2)$. The total surplus TS_P and urban utility v_P therefore accrue from the business district $(-b_2, b_2)$. The land bid rents and wages are given by

$$\Psi_b(y)=\frac{1}{s}\left[\mathrm{TS}_P(y)-\mathrm{TS}_P(b_2)-t(b-y)+(1+s)t(b-b_2)\right],\quad y\in[b_1,b_2] \tag{41}$$

$$\Psi_r(x)=t(b-x),\qquad x\in(b_2,b]$$

and

$$w(y)=\mathrm{TS}_P(b_2)-v_P(y)+t(b-y)-(1+s)t(b-b_2),\quad y\in[b_1,b_2], \tag{42}$$

where the urban utility v_P and the total surplus TS_P are given by (36) and (37). Apart from district labels and values of total surplus TS_P and urban utility v_P, those expressions are similar to those found in the monocentric city. Finally, the equilibrium utility is given by

$$U^* = \mathrm{TS}_P(b_2) - (1+s)\,t\,(b-b_2) + \bar{c}_0 - \alpha$$

This reflects the gain associated with the total surplus at the border of the business center, b_2, diminished by the commuting cost from the city edge, b.

Partially integrated urban structures are difficult to study because the access cost and the surplus functions are more intricate and equilibrium conditions have to be checked globally over many districts. Nevertheless, the following Proposition establishes the equilibrium conditions for such urban configurations.

Proposition 5. *The partially integrated city is a spatial equilibrium if and only if $\hat{t}_P \le t \le t_P^*$ where the thresholds $\hat{t}_P$ and t_P^* are given by*

$$\hat{t}_P \equiv \frac{\alpha+\beta}{1+s} \max_{z\in(0,b_1)} (S_P'(z));\; t_P^* \equiv \min(\bar{t}_P, \tilde{t}_P)$$

with

$$\tilde{t}_P \equiv \frac{\alpha+\beta}{1+s}\frac{S_P(b_2)-S_P(b)}{b-b_2}, \qquad \bar{t}_P \equiv \min_{y\in[b_1,b_2]} \frac{\alpha+\beta}{1+s}\frac{S_P(y)-S_P(b_2)}{b_2-y}$$

The district borders (b_1, b_2) satisfy

$$(1+s)t(b_2-b_1) = (\alpha+\beta)\left[S_P(b_1) - S_P(b_2)\right] \tag{43}$$

where b_2 is given by relation (35) and $0 < b_1 < b_2 < N$.

Proposition 5 shows the partially integrated configuration can only exist for intermediate values of the commuting cost. It also makes explicit the condition that district borders need to satisfy. By inspection of equation (43), the commuting cost from b_2 to b_1 is to be proportional to the surplus difference between locations b_1 and b_2.

Fig. 6 illustrates the shape of bid rents in a partially integrated city. Panel a) presents the case of balanced use of space, where firms and workers use the same amount of land ($s = 1$). Around the city center, firms and residents populate an integrated district where the firms' and the residents' bid rents coincide. This integrated area is surrounded by two business districts where firms outbid residents. Residential districts emerge at city edges where residents outbid firms. The condition $t \ge \hat{t}_P$ ensures sufficiently high commuting cost so that workers have no incentive to commute within the integrated district. The economic intuition is similar to that prevailing in the integrated city. The condition $t \le \tilde{t}_P$ guarantees that residents outbid firms in peripheral districts. In particular it expresses the sufficient condition that firms producing at $y = b_2$ have no incentives to relocate production at the city edge $y = b$. Firms indeed balance the effects of such a relocation on their revenues and wage bills. On the one hand, they lose revenues at the city edge because of the lower access to their customers. On the other hand, they offer lower wages at the city edge because workers need not be compensated for long commuting. The latter effect gets smaller and is dominated by the former when commuting costs become lower. To sum up, partially integrated cities therefore exist only for intermediate commuting costs and, by Proposition 1, only for intermediate values of demand and input-output linkages.

We provide two Corollaries exhibiting additional properties of partially integrated equilibria. The first property relates to the contiguity and intersection of the sets of parameters supporting the integrated and the partially integrated cities. In particular, we show that the partially integrated city is a spatial equilibrium for $t = t_I - \delta$ and $\delta > 0$ small enough.

Corollary 2. *The parameter sets that support the integrated and the partially integrated cities are contiguous but do not overlap.*

Proof. *See Appendix H.*

The second property is about the existence of partial integrated cities even when firms do not use space. The literature generally extends Alonso (1964) model to multiple business centers located on points (e.g. Fujita et al., 1999). The following corollary suggests that, although insightful, this strategy does not cover the exhaustive set of urban configurations. Indeed, consider infinitesimally small business districts ($s \to 0$). By Corollary 1, the commuting cost interval (t_M, t_I) is not empty. For such commuting costs, monocentric and integrated city structures are not spatial equilibria. In the following corollary, we show that spatial equilibria include partially integrated cities when businesses use no land.

Corollary 3. *Suppose firms use an infinitely small amount of land ($s \to 0$). Then, there exists a partially integrated equilibrium with an integrated district surrounded by two infinitely small business districts and residential districts in the periphery when commuting costs satisfy $\max\{t_M, \hat{t}_P^0\} \le t \le \min\{t_I, \bar{t}_P^0, \tilde{t}_P^0\}$.*

Proof. *See Appendix I.*

Panel b) in Fig. 6 presents a second example of bid rents in a partially integrated city where the use of space is unbalanced. In such a case, firms do not use much space unlike residents and business districts have a very small spatial extent ($s = 0.01$). Additional numerical exercises show that the city configuration remains identical as $s \to 0$. In that case, the two business districts become so small that the central integrated district becomes contiguous to the peripheral residential districts. Note that parameters fulfill the conditions of the above Corollary. To sum up, the partially integrated city exists even in the absence of space requirement by firms. So, although very insightful, previous discussions on city subcenters concentrated on spatial points do not encompass the full set of urban configurations.

4.4. Duocentric city

Here the city consists of two business districts $\mathcal{Y} = [-b_3, -b_1) \cup (b_1, b_3]$ and three residential districts $\mathcal{X} = [-b, -b_3) \cup [-b_1, b_1] \cup (b_3, b]$, see Fig. 3 (d). Households living in $[-b, -b_3)$ commute to $[-b_3, -b_2)$, those living in $[-b_1, 0)$ to $[-b_2, -b_1)$, those living in $[0, b_1]$ to $[b_1, b_2)$, and those living in $(b_3, b]$ to $[b_2, b_3)$. Given the total masses of firms and consumers, we have

$$b_1 = \frac{1}{s}(b_2 - b_1), \qquad b - b_3 = \frac{1}{s}(b_3 - b_2) \tag{44}$$

The density of firms is $\mu(y) = 1/s$ on $\mathcal{Y}$ and district edges satisfy $b_2 = (1+s)b_1$ and $b_3 = b_1 + sN/2$. As before, we focus on $[0, b]$.

The duocentric equilibrium conditions are as follows: (*i*) firms outbid households in the business districts ($\Psi_b(y) \ge \Psi_r(y)$ for all $y \in \mathcal{Y}$); (*ii*) households outbid firms in the residential districts ($\Psi_r(x) \ge \Psi_b(x)$ for all $x \in \mathcal{X}$); (*iii*) the business and the residential bid rents equalize at district borders ($\Psi_r(b_1) = \Psi_b(b_1)$ and $\Psi_r(b_3) = \Psi_b(b_3)$); and (*iv*) the zero opportunity cost of land at the city border requiring $\Psi_r(b) = 0$.

The urban utility $v(z)$ and the surpluses $S(z)$ and $\mathrm{TS}(z)$ defined in (9), (14) and (13) are now denoted

$$v_D(z) = \alpha\left[\ln\frac{\alpha\rho}{\gamma} - \ln T_D(z)\right]$$

$$\mathrm{TS}_D(z) = v_D(z) + \pi_D^n(z) + \pi_D^u(z) = \alpha\,\ln\frac{\alpha\rho}{\gamma} + \beta\,\ln\frac{\beta\rho}{\gamma e} + (\alpha+\beta)S_D(z)$$

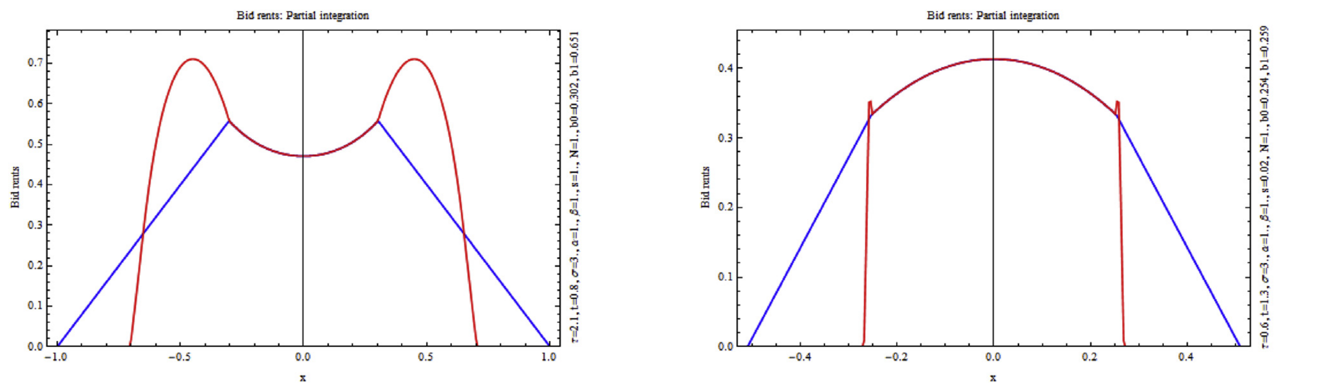
$$S_D(z) = G_D(z) - \ln T_D(z)$$

with the access cost T_D and the profit G_D given by

$$T_D^{1-\sigma}(z) = \frac{1}{s}\left[\int_{-b_3}^{-b_1} e^{(1-\sigma)\tau|z-y|}\mathrm{d}y + \int_{b_1}^{b_3} e^{(1-\sigma)\tau|z-y|}\mathrm{d}y\right]$$

$$G_D(y) = \frac{1}{\sigma s}\left[\int_{-b_3}^{-b_1} \frac{e^{(1-\sigma)\tau|z-y|}}{T_D^{1-\sigma}(z)}\mathrm{d}z + \int_{b_1}^{b_3} \frac{e^{(1-\sigma)\tau|z-y|}}{T_D^{1-\sigma}(z)}\mathrm{d}z\right]$$

The duocentric configuration exhibits monocentric features as it includes four commuting basins $(-b, -b_2)$, $(-b_2, 0)$, $(0, b_2)$ and (b_2, b), where workers commute from residential to business areas. Residents



Panel a: balanced use of space, s=1

Panel b: unbalanced use of space, s=0.01

Fig. 6. Partially integrated city: residential (blue) and business (red) bid rents. (For interpretation of the references to colour in this figure legend, the reader is referred to the Web version of this article.)

and firms display land bid rents similar to those in the monocentric configuration, except that the surplus and the urban utility accrue from the two business districts $(-b_3, -b_1)$ and (b_1, b_3). Focusing on the right-side of the city $(x, y > 0)$, Appendix J shows that equilibrium conditions (iii) and (iv) give the following bid rents and wage

$$\begin{aligned}\Psi_r(x) &= t(b-b_2) - t(x-b_2), & x \in (0,b_1) \cup (b_3,b) \\ \Psi_b(y) &= \frac{1}{s}((\alpha+\beta)(S_D(y) - S_D(b_3)) + t\hat{b} + t(y-b_2)), & y \in (b_1,b_3)\end{aligned} \tag{45}$$

$$w(y) = \mathrm{TS}_D(b_3) - v_D(y) - t(y-b_2), \qquad y \in (b_1,b_3) \tag{46}$$

where $\hat{b} = (1+s)\left[(b_3-b_2) - 2(b_2-b_1)\right] + s(b-b_2)$. Residential bid rents linearly increase with distance from city center in the central residential district while they fall in the outside residential district. Firm bid rents rise with distance from the intersection of the inner and outer employment basins, b_2. Firm bid rents also include the economic surplus above the one generated by a firm at outside border of business center, b_3. As in the monocentric city, wages include the total surplus at this border and fall with the urban surplus and the commuting cost. A main difference with other urban structures is that, in the absence of demand linkages $(\alpha = 0)$, wages have non-monotonic patterns with peak at b_2. By a continuity argument, this property holds for low enough demand linkages. Finally, the equilibrium utility writes as

$$U^* = \mathrm{TS}_D(b_3) - (1+s)\,t\,(b-b_3) + \bar{c}_0 - \alpha$$

Similarly to previous urban structures, the equilibrium utility includes the total surplus generated at the border of the business subcenter, b_3. It decreases with the commuting distance from the city border. Equilibrium conditions are difficult to study because they apply to several districts. Nevertheless, the following Proposition presents equilibrium conditions.

Proposition 6. *The duocentric city is a spatial equilibrium if and only if*

$$t \le \max\{\hat{t}_D, \bar{t}_D\}$$

$$\Psi_b(y) \ge \Psi_r(y) \text{ for all } y \in \mathcal{Y}$$

where

$$\hat{t}_D \equiv \frac{\alpha+\beta}{1+s}\frac{S_D(b_1+sN/2) - S_D(0)}{(1+2s)b_1 - sN/2} \quad \text{and}$$

$$\bar{t}_D \equiv \frac{\alpha+\beta}{1+s}\frac{S_D(b_1+sN/2) - S_D(b)}{N/2 - b_1}$$

with $b_1 \in (0, N/4]$ *satisfying*

$$t\,(N-4b_1)\,(1+s)\,s/2 = (\alpha+\beta)\left[S_D(b)_1 - S_D(b_1+sN/2)\right]. \tag{47}$$

Fig. 7 illustrates the shape of bid rents in a duocentric city. Residential districts are located at the city center and peripheries where the residents' bid rent (in blue) is above the firms' bid rent curve (in red). In between these residential districts, two business districts gather firms outbidding residents. Panel a) and b) of Fig. 7 show two different duocentric city structures for the same set of economic parameters, for which equation (47) admits two solutions. It can be checked on Fig. 7 that $\Psi_b(y) \ge \Psi_r(y)$ for all y in the business districts. So, this means that multiple duocentric equilibria can arise.

In Proposition 6, the endogenous border b_1 is the solution to (47), which has no closed-form solution. Proposition 6 also imposes the bid rent condition $\Psi_b(y) \ge \Psi_r(y)$, for any location y in the business districts $\mathcal{Y}$. As the purpose will be to determine the duocentric configurations by use of numerical computations, we shall replace the condition $\Psi_b(y) \ge \Psi_r(y)$, for all $y \in \mathcal{Y}$, by the necessary condition $\Psi_b(b_2) \ge \Psi_r(b_2)$, which is equivalent to $t \le \tilde{t}_D$ where

$$\tilde{t}_D \equiv \frac{\alpha+\beta}{1+s}\frac{S_D(b_2) - S_D(b_1)}{b_2 - b_1}$$

Note that we did not find any example where this condition was not also sufficient. So, the threshold value of the commuting cost is now given by

$$t \le t_D = \max\{\hat{t}_D, \bar{t}_D, \tilde{t}_D\}$$

5. Numerical analysis

The theoretical analysis of urban structures in the previous section allows us to determine and study the nature of urban structures in terms of economic parameters. Given the complex nature of duocentric and partially integrated configurations, we determine them by numerical computations. This is done in two steps. First, given some fixed parameter values, district borders are determined by solving (47) and (44) for the duocentric configuration (*resp*. (43) for the partially integrated configuration). Note that up to two roots are found in either case. Second, additional restrictions reflecting rent conditions at district borders are imposed. The role of these restrictions is to retain only the candidates that actually lead to an equilibrium configuration.

Equilibrium urban structures. Fig. 8 gathers the central results of our numerical analysis. It presents the equilibrium structures derived in Propositions 3, 4, 5, and 6 in the plane of transport and commuting costs (τ, t) for fixed values of other parameters, i.e.

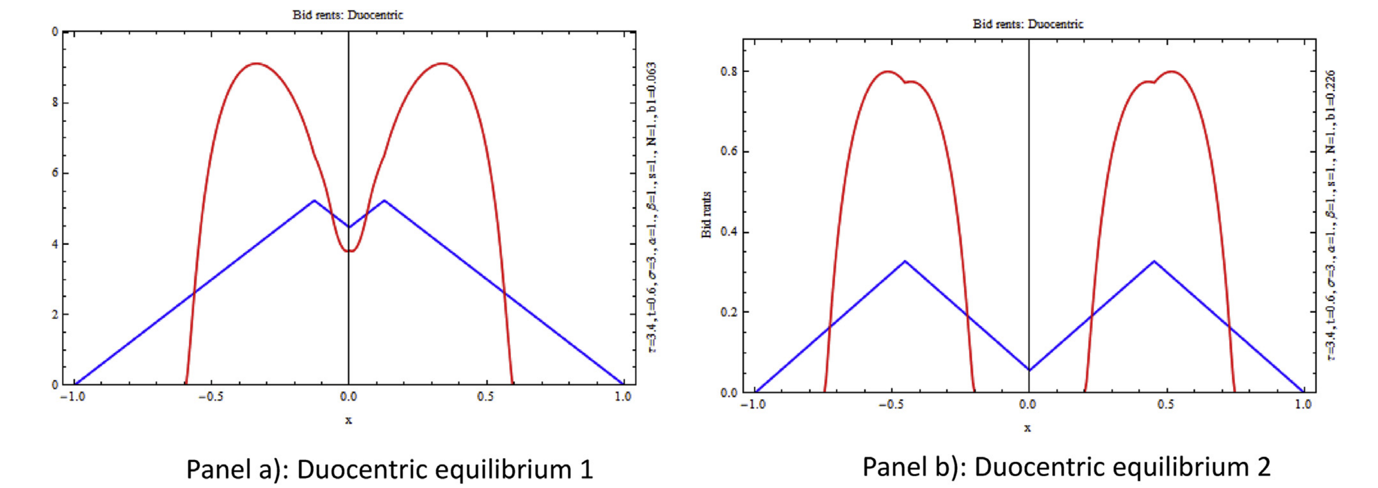


Fig. 7. Duocentric city: residential (blue) and business (red) bid rents. (For interpretation of the references to colour in this figure legend, the reader is referred to the Web version of this article.)

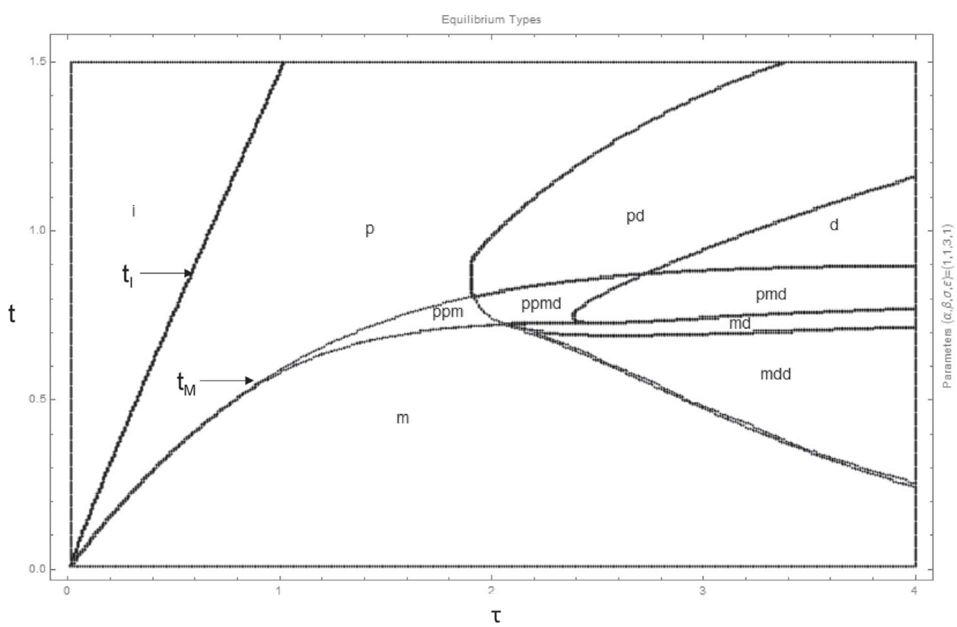


Fig. 8. Spatial equilibria, multiplicity, welfare ranking in (τ,t) plane. *Note:* Monocentric (m), partially integrated (p), integrated (i), and duocentric (d) configurations. Parameters: $\alpha = \beta = \gamma = s = 1$ and $\sigma = 3$.

$\alpha = \beta = \gamma = s = 1$ and $\sigma = 3$. The figure displays the monocentric "m", the integrated "i", the partially integrated "p", and the duocentric "d" configurations. The concatenation of several letters indicates the presence of multiple equilibria with different urban structures for the same set of parameters. For instance, "mdd" indicates the co-existence of a monocentric equilibrium and two different duocentric equilibria. For some parameter configurations, up to four equilibria can co-exist (see "dmpp" at the center of Fig. 8). In Section 2, existence of a spatial equilibrium was proved. Indeed, numerical computations always deliver at least an equilibrium for any point in (τ, t) plane. This contrasts with Fujita and Ogawa (1982) where some parameter configurations were shown to support no equilibrium.

Fig. 8 displays the curve corresponding to threshold t_M below which parameters support the monocentric city. The monocentric city arises only if commuting costs are relatively low. By definition of t_M, residents and businesses make the same land bid at the city center for any value of t lying on the curve t_M. For slightly higher commuting costs -and as long as transport costs are low enough-, residents are willing to match the firms' bid rent at the city center. Some residents and firms can then gather at the city center, while many other workers are still willing to commute. Hence the curve t_M refers to a transition between the monocentric and the partially integrated configurations.

Fig. 8 also displays the curve corresponding to the threshold t_I above which parameters support the integrated city. In the integrated city, residents work and reside in the same location. By definition of threshold t_I, when commuting costs equate t_I, some individuals are indifferent between working at their residence location and commuting to a firm located further away. For commuting costs slightly below t_I, residents located at the city border have an incentive to commute and choose a workplace between the city center and the city border, so that a partially integrated city structure emerges. Hence the curve t_I refers to a transition between the partially integrated and the integrated configurations.

The partially integrated configuration exists only for intermediate commuting costs. Consider such a partially integrated structure. When the commuting cost rise becomes too high, the business centers and the residential peripheries shrink, and ultimately vanish when the commuting cost reaches the threshold t_I. Hence the partially integrated config-

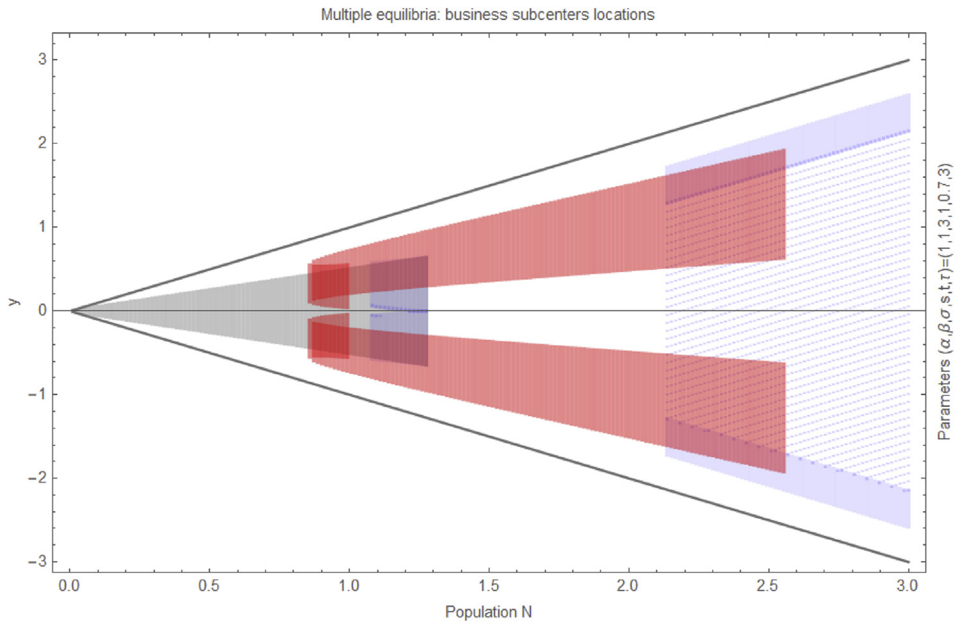


Fig. 9. City structure as population size increases. The horizontal axis is population and the vertical axis location. Diagonal straight lines display city borders. Residential districts are represented in white, monocentric business districts in grey, duocentric business districts in red, and the business and the integrated districts of the partially integrated city in plain and dash blue respectively. Overlaps indicate multiple equilibria. . (For interpretation of the references to colour in this figure legend, the reader is referred to the Web version of this article.)

uration converges smoothly to the integrated city. For low transport costs (i.e., when only one partially integrated equilibrium exists), the two business centers get closer to each other to finally join each other as the commuting cost approaches t_M from above: the partial integrated structure then converges smoothly to the monocentric configuration. For higher transport costs (i.e., when the partially integrated and the monocentric cities co-exist), one partially integrated configuration converges continuously to the monocentric structure as the commuting cost approaches t_M from below.

Duocentric cities are equilibria only for high transport costs and intermediate commuting costs (see the right-hand side of Fig. 8). Indeed, when shipping goods are costly, firms benefit from clustering their production activities in two separate business centers so as to decrease the average distance to suppliers and consumers. Consider commuting costs rising from zero (i.e., a rising path in the right-hand side of Fig. 8). When commuting costs are very low, workers accept long journeys to their workplace and benefit from a concentrated business district, i.e. the monocentric structure “m”. However, when commuting costs reach intermediate values, an urban structure with two activity centers becomes beneficial to them because long commutes can be avoided. When this happens, two duocentric configurations emerge (see transition from “m” to “mdd” in Fig. 8). As commuting costs keep on increasing, one duocentric configuration persists while the other disappears (transition from “dmd” to “dm”). Finally, for even higher commuting costs, the remaining duocentric city structure ceases to exist as households want to locate closer to their workplace, which gives rise to a transition to a partially integrated configuration.

Multiple equilibria. Fig. 8 also illustrates the multiplicity of equilibria. In many regions of the parameter space, a unique equilibrium emerges (see “m”, “i”, “p” and “d”). There are also large sets of parameters for which several structures among the monocentric, the duocentric, or the partially integrated configurations co-exist. Note that all structures can co-exist for a narrow set of parameter values. The fully integrated structure cannot co-exist with any other configuration. As mentioned in the above paragraph, multiple equilibria can involve different urban structures of the same type: either two partially integrated structures like in “mpp” and “dmpp” or two duocentric structures like in “dmd” and “mdd”. This can only happen for intermediate commuting costs. For higher commuting costs, one of the two equilibria always disappears. It is interesting to describe how two equilibria of the same type emerge. When commuting costs increase, it can be shown that the transition from “m” to “mpp” (*resp*. from “m” to “mdd”) implies that the monocentric city keeps the same structure while a partially integrated city (*resp*. a duocentric city) initially emerges and immediately separates into two distinct partially integrated structures (*resp*. a duocentric) with business centers evolving at different locations.

Population growth. Finally, we look at the impact of urbanization on spatial structures by considering gradual increases in the city size N for given commuting and transport costs. According to our numerical analysis, three cases arise depending on transport costs τ.

(i) For low transport costs τ, transitions are smooth. The configuration is initially monocentric for small N (i.e. $t < t_M$), then it becomes partially integrated for intermediate N (i.e. $t_M < t < t_I$).
(ii) For intermediate transport costs τ, the configuration is initially monocentric. Then it bifurcates into either a partially integrated or duocentric city. These structures are quite different from the monocentric city so that the transition is discontinuous, the resulting path being undetermined. In the case of a duocentric city, it will partially integrate at a later stage, and will finally approach full integration.
(iii) For large transport costs τ, the configuration is initially monocentric. When it bifurcates, it becomes a duocentric city. Then, it becomes partially integrated, and finally fully integrated. Fig. 9 plots this case with $\tau = 3$ and $t = 0.7$ for an increasing population N. It shows that the configuration is initially monocentric (black). When it breaks, it becomes duocentric (red), and then partially integrated (blue).

These results show that transitions are continuous (*resp*. discontinuous) for small (*resp*. large) transport costs. This reflects the fact that when transport costs are large, firms can gain more when firms locate in several business districts.

6. Conclusion

This paper studies endogenous urban structures driven by demand and vertical linkages. The agglomeration forces stem from the input of local varieties of goods and services in the production of firms and the consumption of the same varieties by consumers. Both firms and residents care about their access to local producers. Because of increasing returns to scale, firms' production is higher in the presence of additional urban varieties. Our model proposes microeconomic foundations for such agglomeration forces. It contrasts with a number of existing models in the urban economic literature where the agglomeration mechanism stems from externalities that are exogenous to firms activities (e.g. technological spillovers).

Our paper focuses on a demand and production structure that maintains a very close similarity to Fujita and Ogawa (1982) model with exogenous externalities. This allows an easier comparison with the existing literature and suggests that the implications derived previously from exogenous agglomeration processes turn out to be quite general. In particular, it permits to transpose the comparative statics properties and policy recommendations of economies with exogenous externalities to those with input-output linkages. For instance, policies and technological shocks that promote either technological spillovers or vertical linkages entice firms and residents to separate in different districts. Policies facilitating commuting have similar effects irrespective of the nature of agglomeration forces.

The paper shows the existence of a spatial equilibrium using a fixed point argument. As such, it contrasts with Fujita and Ogawa (1982). Our argument also allows to formulate an equivalence result between the effects of economic linkages and commuting costs. In particular, a rise in the parameters governing demand and vertical linkages is equivalent to a fall in the commuting cost parameter. Hence, all the effects on urban structures that are known for commuting costs inversely apply for demand and vertical linkages. Finally, we show that the fixed point formulation does not lead to a contraction mapping meaning that standard iterative procedures may fail to find spatial equilibria. Instead, to find equilibrium urban configurations, we propose a two-step procedure where the type of urban configuration is first postulated and then the spatial equilibrium conditions are checked. This approach also provides a theoretical discussion about wages and land rents in each urban configuration.

Various urban configurations emerge in equilibrium, exhibiting monocentric, integrated, duocentric, or partially integrated city structures. The paper discusses how commuting and transport costs and intensities of demand/vertical linkages affect these structures. It highlights the emergence of multiple equilibria: two, three or even four urban configurations can coexist for the same set of parameters. Some multiple equilibria may include the same structure: the analysis shows different spatial equilibria with two centers that are separated by residential areas of different sizes. Though this emphasizes the complexity of the endogenous formation of urban areas, it also provides interesting insights about possible urban restructuring. For instance, if one presumes that new technologies (e.g. telecommunication and fast good delivery) decrease the cost of trading goods and services within urban areas, cities may then be expected to endogenously restructure from duocentric to monocentric or partially integrated urban configurations.

Appendix A. Positive consumption of national good

The sufficient condition for each spatial configuration is as follows:

$\overline{c}_0 > \alpha - TS_M(b_1) + v_M(0) + (1+s)t(b-b_1)$ for a monocentric city,
$\overline{c}_0 > \alpha - TS_I(b) + v_I(0)$ for an integrated city,
$\overline{c}_0 > \alpha - TS_P(b_2) + v_P(0) + (1+s)t(b-b_2)$ for a partially integrated city,
$\overline{c}_0 > \alpha - TS_D(b_3) + v_D((1+s)b_3 - sb) + (1+s)^2 t(b-b_3)$ for duocentric cities,

where b's are determined by the equilibrium conditions.

7. Appendix B: Equilibrium conditions

Let $\boldsymbol{\mathcal{B}} = [-b, b]$, $b \in \mathbb{R}_0^+$, be the support of the city. The densities of residents and firms are given by $\lambda : \boldsymbol{\mathcal{B}} \to [0,1]$ and $\mu : \boldsymbol{\mathcal{B}} \to [0, 1/s]$. The land market clearing imposes that $\lambda + s\mu = 1$ while labor market clearing imposes that the mass of workers is equal to mass of firms $\int_{\boldsymbol{\mathcal{B}}} \lambda = \int_{\boldsymbol{\mathcal{B}}} \mu = N$. With the city border $b \equiv N(1+s)/2$, every worker and every firm find a job match. So, we just need to impose $\int_{\boldsymbol{\mathcal{B}}} \mu = N$.

While worker's bid rent is the maximum bid a worker makes on a parcel of land as of (19), the firm's bid rent is the maximum bid that it can make on land as of (20). The purpose of this appendix is to show the conditions for spatial equilibrium given by identities (22) and (23). Remind that $S(y) = G(y) - \ln T(y)$, where $T(y) = \left[\int_{\boldsymbol{\mathcal{B}}} e^{(1-\sigma)\tau|z-y|} \mu(z) \mathrm{d}z\right]^{\frac{1}{1-\sigma}}$ and $G(y) = \frac{1}{\sigma} \int_{\boldsymbol{\mathcal{B}}} \frac{e^{(1-\sigma)\tau|z-y|}}{T^{1-\sigma}(z)} \mu(z) \mathrm{d}z$. Note that G, T and therefore S and TS: $\boldsymbol{\mathcal{B}} \to \mathbb{R}^+$ are integrals and therefore continuous functions of y. Functions G and T are also strictly positive because their integrands are positive as μ has positive measure.

No cross commute. At every location, the flow of commuters is unidirectional. The Proof can be found in Ogawa and Fujita (1980, Property 1). The authors show that workers commuting in opposite directions have no incentives to cross each other in equilibrium.

City edges. Spatial equilibrium involves workers residing at city edges. Suppose this is not the case: $\lambda(b) = 0$ and $\mu(b) = 1/s$. Then, there would be workers at $x < b$ facing a positive land rent and commuting to a firm located at b. These workers would benefit from relocating to $b + \delta$, with $\delta > 0$ infinitely small, because of a zero land rent and an infinitely small commuting cost $t\delta$, which violates equilibrium. The same argument applies at $x = -b$.

More generally, spatial equilibrium involves more residents than firms at city edges. Otherwise, $\lambda(b) < \mu(b)$, and the above argument still applies as there would be workers at $x < b$ facing a positive land rent and commuting to a firm in b, who would benefit from relocating to $b + \delta$, with $\delta > 0$ infinitely small.

Lemma 1. *City edges host more residents than firms*: $\lambda > 0$, $\lambda \geq \mu$ *for* $z = \pm b$.

Land rent continuity. The bid rents Ψ_r and Ψ_b are the maximum values of programs (19) and (20), which involve not only the continuous functions TS and absolute value, but also the land rent function R which is not a priori continuous. Therefore, the Maximum Theorem cannot be applied in order to show the continuity of bid rents Ψ_r and Ψ_b. Instead, the continuity of bid rents is shown by the fact that they are linear functions of the land rent R.

Lemma 2. *The bid rents Ψ_r and Ψ_b in* (19) and (20) *are continuous functions $\mathcal{B} \to \mathbb{R}$. So $\Psi_r - \Psi_b$ and $R = \max\{\Psi_r, \Psi_b, 0\}$ are also continuous functions $\mathcal{B} \to \mathbb{R}$.*

Proof. *Indeed, each bid rent can be written as the function $\Psi(x) = \max_z f_1(x) + f_2(z) + f_3(x-z)$, where $f_1 : \mathcal{B} \to \mathbb{R}$ and $f_3 : [-2b, 2b] \to \mathbb{R}$ are continuous functions and $f_2 : \mathcal{B} \to \mathbb{R}$ is not necessarily a continuous function. For $\Psi = \Psi_r$, take f_1=0, $f_2 = \alpha \ln \frac{\alpha\rho}{\gamma e} + \beta \ln \frac{\beta\rho}{\gamma e} + (\alpha+\beta)S + \overline{c}_0 - U^* - sR$ and $f_3(x-z) = -t|x-z|$. For $\Psi = \Psi_b$, take $f_1 = (\alpha \ln \frac{\alpha\rho}{\gamma e} + \beta \ln \frac{\beta\rho}{\gamma e} + (\alpha+\beta)S + \overline{c}_0 - U^*)/s$, $f_2 = -R/s$* and *$f_3(x-z) = -t|x-z|/s$. Remember that S is a continuous function. In contrast, the land rent R is not a priori a continuous function. Let y and y' be the maximizers of Ψ respectively at x and x'. By the definition of a maximum, we have $\Psi(x) = f_1(x) + f_2(y) + f_3(x-y) \geq f_1(x) + f_2(y') + f_3(x-y')$ and $\Psi(x') = f_1(x') + f_2(y') + f_3(x'-y') \geq f_1(x') + f_2(y) + f_3(x'-y)$. Using the conditions, we have*

$$f_3(x-y') - f_3(x-y) \leq f_2(y) - f_2(y') \leq f_3(x'-y') - f_3(x'-y) \tag{48}$$

Then, *we compute the difference*

$$\Psi(x) - \Psi(x') = f_1(x) - f_1(x') + f_2(y) - f_2(y') + f_3(x-y) - f_3(x'-y')$$

and therefore

$$f_2(y) - f_2(y') = \Psi(x) - \Psi(x') - \big(f_1(x) - f_1(x')\big) - \big(f_3(x-y) - f_3(x'-y')\big)$$

Plugging this in (48), *we get*

$$f_1(x) - f_1(x') + f_3(x-y') - f_3(x'-y') \leq \Psi(x) - \Psi(x') = f_1(x) - f_1(x') - f_3(x'-y) + f_3(x-y)$$

Then, *because f_1 and f_3 are continuous functions,* $\lim_{x'\to x} |f_1(x) - f_1(x')| = 0$ *and* $\lim_{y'\to y} |f_3(x-y') - f_3(x'-y')| = \lim_{x'\to x} |f_3(x'-y) - f_3(x-y)| = 0$, *we have that* $\lim_{x'\to x} |\Psi(x) - \Psi(x')| = 0$. ■

Commuting flows. Let $n : \mathcal{B} \to \mathbb{R}$ be the number of commuters flowing in the right hand direction (i.e. toward positive x). Since there is no resident and no firm at the left of $x = -b$ and at the right of $x = b$, we have $n(\pm b) = 0$. This flow increases with the mass of residents who start their commuting trip in the right direction and decreases with the mass of firms that hire commuters located on their left: $\dot{n} = \lambda - \mu$. Given the definition of spatial equilibrium, we have

$$\dot{n} \in \begin{cases} \{1\} & \text{if} \quad \Psi_r > \Psi_b \\ [-\frac{1}{s}, 1] & \text{if} \quad \Psi_r = \Psi_b \\ \{-\frac{1}{s}\} & \text{if} \quad \Psi_r < \Psi_b \end{cases} \tag{49}$$

Given the absence of cross commuting, a location is a 'source' of commuters if $\dot{n} > 0$ and a 'sink' of commuters if $\dot{n} < 0$. The flow of commuters points to the right direction (residence address x < work address y) when $n > 0$ and to the left direction ($x > y$) if $n < 0$. By the first fundamental theorem of calculus, the flow of commuters $n(z) = \int_{-b}^{z} \dot{n} dz$ is a continuous function. At city edges, because $\lambda \geq \mu$ at $z = \pm b$, we have $\dot{n}(\pm b) \geq 0$. Also, when $\dot{n}(\pm b) > 0$, the function $\dot{n}$ is equal to 0 at least twice in order to satisfy $n(b) = \int_{-b}^{b} \dot{n} dz = 0$.

It is useful to introduce some terminology about the sets of sources and sinks of commuters moving to right or left direction. Let the set of sources $\mathcal{S}^\kappa = \{z \in \mathcal{B} : \dot{n} > 0 \text{ and } \kappa n > 0\}$ and the set of sinks (destinations) $\mathcal{D}^\kappa = \{z \in \mathcal{B} : \dot{n} < 0 \text{ and } \kappa n > 0\}$ where $\kappa = 1$ if commuters flow to the right direction and $\kappa = -1$ otherwise. For example, $\mathcal{S}^1$ includes source locations of commuters flowing to the right direction and $\mathcal{S}^{-1}$ those flowing to the left direction. Obviously, commuters departing from $\mathcal{S}^\kappa$ and arriving in $\mathcal{D}^\kappa$ flow to the right direction (*resp.* the left direction) if $\kappa = 1$ (*resp.* -1). Locations that are neither origin nor destination of commuters define the set $\mathcal{P} = \{z \in \mathcal{B} : \dot{n} = 0\}$. Locations that involve a change in commuting direction are denoted by the set $\mathcal{Q} = \{z \in \mathcal{B} : n = 0 \text{and} \dot{n} \neq 0\}$. Because n is a continuous function and either $\dot{n} > 0$ or $\dot{n} < 0$ for $z \in \mathcal{Q}$, any location $z \in \mathcal{Q}$ has zero measure, i.e. it is an isolated point. Finally, note that those definitions determine six disjoint sets: $\mathcal{D}^{+1}$, $\mathcal{D}^{-1}$, $\mathcal{S}^1$, $\mathcal{S}^{-1}$, $\mathcal{P}$ and $\mathcal{Q}$.

We now study the land rent properties on these sets.

Wage arbitrage in $\mathcal{S}^\kappa \cup \mathcal{D}^\kappa$. Consider two disjoint, open and convex intervals, $\mathcal{S}' \subseteq \mathcal{S}^\kappa$ and $\mathcal{D}' \subseteq \mathcal{D}^\kappa$, such that (i) workers residing in $\mathcal{S}'$ commute to firms located in $\mathcal{D}'$ and labor flows in the same direction within each interval ($\mathcal{S}' \cap \mathcal{D}' = \emptyset$). Since $n(x)$ and $n(y)$ have the same sign for $x \in \mathcal{S}'$ and $y \in \mathcal{D}'$, one readily checks that $|x-y| = (y-x)\,\text{sign}(n(x)) = y\ \text{sign}(n(y)) - x\ \text{sign}(n(x))$ where $\text{sign}(n) = 1, 0, -1$ depending on whether n is positive, zero, or negative. Since n has the same sign in each interval, each function $\text{sign}(n)$ is a constant. The first order conditions for the maximum problems (19) and (20) are given by

$$\dot{\text{TS}} - t\ \text{sign}(n) - s\dot{R} = 0, \quad y \in \mathcal{D}' \tag{50}$$

$$t\ \text{sign}(n) - \dot{R} = 0, \quad x \in \mathcal{S}' \tag{51}$$

where the upper dot $\cdot$ denotes the derivative w.r.t. to distance to center; here $\dot{\text{TS}}(y) = (d/dy)\text{TS}(y)$.

While the first identity includes only residents irrespective of their workplaces, the second one involves only firms irrespective of workers' residences. These identities apply for any infinitesimal interval where workers start to commute in the same direction and firms hire from the same side of the city. Since these conditions hold for all $\mathcal{S}' \subseteq \mathcal{S}^\kappa$ and $\mathcal{D}' \subseteq \mathcal{D}^\kappa$, they hold a.e. in $\mathcal{S}^\kappa$ and $\mathcal{D}^\kappa$, where a.e. (almost everywhere) means for all points except on a zero-measure set.

Lemma 3. *The land bid rent differential is given by*

$$\dot{\Psi}_b - \dot{\Psi}_r = \frac{1}{s}\dot{\text{TS}} - \frac{1+s}{s} t\ \text{sign}\ (n), \qquad \text{a.e. } z \in \mathcal{S}^\kappa \cup \mathcal{D}^\kappa$$

Proof. *In this proof we compare the profiles of firms' and residents' bid rents. By definition of a spatial equilibrium, the bid rent of a firm or worker actually using a land plot is equal to the equilibrium rent R while the bid rent of the other agent is given by its highest feasible transfer for the same land plot. Consider* *again the two disjoint, open and convex intervals $\mathcal{S}' \subseteq \mathcal{S}^{\kappa}$ and $\mathcal{D}' \subseteq \mathcal{D}^{\kappa}$. Then, conditions (50) and (51) imply that the equilibrium land rent R is equal to workers' and firms' bid rents. Applying the envelope theorem to (19) and (20) yields*

$$\dot{\Psi}_r = t\,sign\,(n)\,, \quad x \in \mathcal{S}'$$

$$\dot{\Psi}_b = \frac{1}{s}\left[\dot{\mathrm{TS}} - t\,sign\,(n)\right], \quad y \in \mathcal{D}'$$

We construct the 'highest losing' bids Ψ_r on $\mathcal{D}'$ and Ψ_b on $\mathcal{S}'$. We first look at the set $\mathcal{S}'$. Because workers depart from their residence $x \in \mathcal{S}'$ to work, they must reside on $\mathcal{S}'$ and win the land market competition. Because workers reside in $\mathcal{S}'$, her bid Ψ_r must be equal to the equilibrium land price R. Consider then a firm that would produce at $x \in \mathcal{S}'$ and hire a resident at $x' \in \mathcal{S}'$. Because of (51), any resident in $\mathcal{S}'$ has the same incentive to work in the firm. So, we can say that the firm hires a worker residing at $x' = x$. The firm's highest (losing) bid Ψ_b is given by (20) evaluated at $y = x$. Using this, the bid rent differential writes as

$$\Psi_b - \Psi_r = \frac{1}{s}\left[\mathrm{TS} - \alpha + \bar{c}_0 - U^*\right] - \frac{1+s}{s}R \tag{52}$$

Differentiating this expression and using (51), we have the bid rent differential

$$\dot{\Psi}_b - \dot{\Psi}_r = \frac{1}{s}\dot{\mathrm{TS}} - \frac{1+s}{s}t\,sign\,(n)\,, \quad z \in \mathcal{S}' \tag{53}$$

which holds for any $\mathcal{S}'$ in $\mathcal{S}^{\kappa}$, and thus for any measurable set in $\mathcal{S}^{\kappa}$.

We then look at the set $\mathcal{D}'$. Because $\mathcal{D}'$ hosts firms, it must be that $\Psi_b = R$. Consider a resident who would choose to reside at $y \in \mathcal{D}'$ and be hired by a firm in $\mathcal{D}'$. Because of (50), the worker is indifferent to any firm producing at $y' \in \mathcal{D}'$, so that we can focus on the case where the hiring firm produces at $y' = y$. The worker's highest (losing) bid is given by (19) evaluated at $x = y$. The bid rent difference writes as

$$\Psi_b - \Psi_r = -\mathrm{TS} + \alpha - \bar{c}_0 + U^* + (1+s)\,R \tag{54}$$

Differentiating this expression and using (51) to substitute for $\dot{R}$, we get the bid rent differential

$$\dot{\Psi}_b - \dot{\Psi}_r = \frac{1}{s}\dot{\mathrm{TS}} - \frac{1+s}{s}t\,sign\,(n)\,, \quad y \in \mathcal{D}'$$

which holds for any $\mathcal{D}'$ in $\mathcal{D}^{\kappa}$, and thus for any measurable set in $\mathcal{D}^{\kappa}$. ■

Wage arbitrage in $\mathcal{P}$. Consider a location $z \in \mathcal{P}$, which is neither a source nor a sink of commuters. Since there is no empty hinterland, this location must host both workers and firms, so that land market clearing imposes $R = \Psi_b = \Psi_r$. So,

$$R = \frac{1}{1+s}\left[\mathrm{TS} - \alpha + \bar{c}_0 - U^*\right], \quad z \in \mathcal{P}$$

Furthermore, let $\mathring{\mathcal{P}}$ be the interior set of $\mathcal{P}$ so that $z \in \mathring{\mathcal{P}}$ is not an isolated point in $\mathcal{P}$. Then, we have

$$\dot{R} = \dot{\Psi}_r = \dot{\Psi}_b = \frac{1}{1+s}\dot{\mathrm{TS}}, \quad z \in \mathring{\mathcal{P}}$$

and

$$\dot{\Psi}_b - \dot{\Psi}_r = 0, \quad z \in \mathring{\mathcal{P}} \tag{55}$$

Bid rent differential at city borders. The zero opportunity cost of land imposes $R = 0$ at the city border $z = -b$. By Lemma 1, there must be workers with residences at $z = -b$ so that the highest bid is that of workers: $\Psi_r = R = 0$ at $z = -b$. The edge location $z = -b$ also belongs to the set $\mathcal{S}^1$ since $\lambda(-b) > 0$ and workers there cannot commute to the left direction. We can therefore apply (52) and get the land rent differential

$$\Psi_b - \Psi_r = \Psi_b = \frac{1}{s}\left[\mathrm{TS} - \alpha + \bar{c}_0 - U^*\right], \quad z = -b \tag{56}$$

Equilibrium condition. We summarize the above results as follows. Let us define $\phi : \mathcal{B} \to \mathbb{R}$,

$$\phi\,(z) = \frac{1}{s}\dot{\mathrm{TS}}\,(z) - \frac{1+s}{s}t\,\mathrm{sign}(n\,(z)) \tag{57}$$

Note that $\dot{n}$ and ϕ may not be continuous in z. However ϕ is bounded because $\dot{S}$ is bounded.

A spatial equilibrium is defined by the rent differential function $\Psi_b - \Psi_r$ and the commuting function n such that $\Psi_b - \Psi_r$ is continuous everywhere on $\mathcal{B}$, $\Psi_b - \Psi_r = \Psi_b$ and $\dot{n} \geq 0 = n$ at $z \in \{-b, b\}$, and

$$\begin{array}{lcl} z \in \mathcal{S}^{\kappa} \cup \mathcal{D}^{\kappa} & \Rightarrow & \dot{\Psi}_b - \dot{\Psi}_r = \phi\,(z) \text{ a.e.} \\ z \in \mathcal{P} & \Rightarrow & \dot{\Psi}_b - \dot{\Psi}_r = 0 \text{ a.e.} \end{array} \tag{58}$$

Remind that $\mathcal{Q}$ has zero measure and therefore is not affected by the above implications.

Lemma 4. *A spatial equilibrium is such that*

$$\begin{array}{lcl} z \in \mathcal{S}^{\kappa} \cup \mathcal{D}^{\kappa} & \Longleftrightarrow & \phi\,(z) \neq 0 \text{ a.e.} \\ z \in \mathcal{P} & \Longleftrightarrow & \phi\,(z) = 0 \text{ a.e.} \end{array}$$

Note that any equilibrium commuting function n has two possible patterns: it is either such that $\dot{n} = n = \phi = 0$, $\forall z \in \mathcal{B}$ (fully integrated city) or such that the city space is partitioned with $\dot{n} > 0$ for some source locations z and $\dot{n} < 0$ for some sink locations z. In the latter case, we have

$\phi > 0$ for some z and $\phi < 0$ for some other z. This means that ϕ always accepts a zero. Let Z be the set of zeroes or changes in the sign of ϕ, $Z = \{z \in \mathcal{B} : \lim_{\epsilon\to 0}\phi(z+\epsilon)\cdot\phi(z-\epsilon) \le 0\}$.

Then, the spatial equilibrium is given by the function $n : \mathcal{B} \to \mathbb{R}$, which is the solution to

$$\dot{n}(z) \in \begin{cases} \{1\} & \text{if} \quad \Phi(z) < 0 \\ [-\frac{1}{s}, 1] & \text{if} \quad \Phi(z) = 0 \\ \{-\frac{1}{s}\} & \text{if} \quad \Phi(z) > 0 \end{cases}$$

with $\dot{n} \ge 0 = n$ at $z = \pm b$ and $\Phi(z) \equiv \int_{\hat{z}\in Z}^{z} \phi(x)\,\mathrm{d}x$. Using $n(x) = \int_{-b}^{x} (\lambda - \mu)\,\mathrm{d}z$ yields condition (22) in the text.

Invariance. To check invariance, note that $\dot{\mathrm{TS}} = (\alpha+\beta)\dot{S}$ where $\dot{S}(y) = (\mathrm{d}/\mathrm{d}y)S(y)$, which is independent of α and β.

Indeed, one can replace t and $(\alpha + \beta)$ by $\tilde{t} = mt$ and $(\widetilde{\alpha} + \widetilde{\beta}) = m(\alpha+\beta)$, $m \in \mathbb{R}_0$. This yields $\widetilde{\phi}(n) = m\phi(n)$, which gives the same set of roots $\widehat{Z} = \{z \in \mathcal{B} : \widetilde{\phi}(n) = 0\} = \{z \in \mathcal{B} : \phi(n) = 0\}$. As a result, the sign of $\widetilde{\Phi}(n)$ is the same as that of $\Phi(n)$, and thus, the equilibrium must be the same. We therefore have invariance with respect to changes in parameters as long as $(\alpha + \beta)/t$ remains constant.

Appendix C. Existence of a spatial equilibrium

Formulation of a fixed point. We denote the set of continuous functions on $\mathcal{B} = [-b, b]$ by $C^0(\mathcal{B})$ and the set of functions with continuous first derivative by $C^1(\mathcal{B})$. To formulate the fixed point problem, we work with the commuting flow function n which is continuous: that is, $n \in \mathcal{N}$ where $\mathcal{N} \equiv \{n \in C^0(\mathcal{B}) : n(-b) = n(b) = 0, \dot{n} \in [-1/s, 1]\, a.e.\}$ where a.e. means almost everywhere, that is everywhere except on a zero measure set. We also use a concise notation for operators where Mn expresses the application of operator $M : \mathcal{N} \to C^0(\mathcal{B})$ on a function $n \in \mathcal{N}$ and $Mn(z)$ expresses the value returned by Mn at $z \in \mathcal{B}$. In this Appendix, we use the indicator function $1_{\{A\}}$ that returns 1 if the predicate A is true and 0 otherwise. Integrals $\int_{-b}^{z} f(x)\cdot 1_{\{z>x\}}dx$ then write as $\int_{\mathcal{B}} f(x)\cdot 1_{\{z>x\}}dx$ where f is a function.

We redefine the total surplus as the operator $\mathrm{TS} : \mathcal{N} \to C^1(\mathcal{B})$ and its slope $\dot{\mathrm{TS}} : \mathcal{N} \to C^0(\mathcal{B})$ so that $\mathrm{TS}n(z) = \mathrm{TS}n(-b) + \int_{\mathcal{B}} \dot{\mathrm{TS}}n(x)\cdot 1_{\{z>x\}}dx$. We denote the rent differential by $\Delta\Psi : \mathcal{N} \to C^0(\mathcal{B})$, $(\Delta\Psi)(z) = \int_{\mathcal{B}} (\dot{\Psi}_b - \dot{\Psi}_r)\cdot 1_{\{z>x\}}\mathrm{d}x$. Using the value for $\dot{\Psi}_b - \dot{\Psi}_r$ in (57) and (58), integrating from $z = -b$, and using rent differential (56) at this location, we get

$$\Delta\Psi n(z) = \frac{1}{s}\left(-\alpha + \overline{c}_0 - U^*\right) + \frac{1}{s}\mathrm{TS}n(z) - \frac{1+s}{s}t\int_{-b}^{z} \mathrm{sign}(n(x))\mathrm{d}x$$

We rewrite the bid rent differential in a more compact form as

$$\Delta\Psi = \frac{1}{s}(\Lambda + \Lambda_0 - U)$$

where the operator $\Lambda : \mathcal{N} \to C^0(\mathcal{B})$ is defined by $\Lambda = \mathrm{TS} - (1+s)tL$, the operator $L : \mathcal{N} \to C^0(\mathcal{B})$ by $Ln(z) = \int_{\mathcal{B}} \mathrm{sign}(n(x))\cdot 1_{\{z>x\}}\mathrm{d}x$, and the scalar Λ_0 by $s\left(\overline{c}_0 - \alpha\right)$.

A spatial equilibrium is then defined by the function $n : [-b, b] \to \mathbb{R}$ and the scalar U which are the solutions to

$$\dot{n} \in \begin{cases} \{1\} & \text{if} \quad \Lambda n + \Lambda_0 - U < 0 \\ [-\frac{1}{s}, 1] & \text{if} \quad \Lambda n + \Lambda_0 - U = 0 \\ \{-\frac{1}{s}\} & \text{if} \quad \Lambda n + \Lambda_0 - U > 0 \end{cases} \tag{59}$$

with $\dot{n} \ge 0 = n$ at $z = \pm b$.

We now formulate the fixed point problem in terms of the continuous function n rather than the discontinuous function $\dot{n}$. Towards this aim, we integrate the previous condition (59). To take care of the three subconditions, we define the correspondence $F : \mathbb{R} \to [-1/s, 1]$ such that $F(z) = -1/s$ if $z < 0$, $F(z) = [-1/s, 1]$ if $z = 0$, and $F(z) = 1$ if $z > 0$. We define the functional $U^* : \mathcal{N} \to \mathbb{R}$ that for any function n, gives the scalar U that solves

$$0 = \int_{\mathcal{B}} F[U - \Lambda_0 - \Lambda n(z)]\mathrm{d}z$$

and the operator $O : \mathcal{N} \to \mathcal{N}$,

$$(On)(z) = \int_{\mathcal{B}} F[U^*n - \Lambda_0 - \Lambda n(x)]\cdot 1_{\{z>x\}}\mathrm{d}x \tag{60}$$

The definition of U^* guarantees that $n(b) = 0$ while $n(-b) = 0$ is trivially satisfied. By construction, the operator O returns a function n such that $\dot{n} \in [-1/s, 1]$. So, O maps $\mathcal{N}$ into itself.

Lemma 5. *The existence of a spatial equilibrium is equivalent to the existence of a fixed point $n \in On$.*

Sequences of fixed points. The Schauder theorem applies on a set of continuous mappings of continuous functions. So, we use continuous approximations of F and sign functions and then prove convergence. We define the continuous, differentiable and strictly increasing functions $F_k : \mathbb{R} \to [-1/s, 1]$, $k \in \mathbb{N}$, such that F_k converges to F as $k \to \infty$ and $0 < F_k' \le k$, and the continuous, differentiable and strictly increasing function $H_l : \mathbb{R} \to [-1, 1]$, $l \in \mathbb{N}$, such that H_l converges to *sign* as $l \to \infty$ and $0 < H_l' \le l$. In particular, we consider logistic functions with images on $[-1/s, 1]$ and $[-1, 1]$: $F_k(z) = -1 + (1+s) / \left[1 + \exp(-4kz\,/\,(s+1)\right]$ and $H_l(z) = -1 + 2/\left[1 + \exp(-2lz)\right]$. The type of convergence will be defined formally below.

We use the subset of continuous functions $\mathcal{N}$ and the sup norm $\|n\|_\infty = \sup_{z\in\mathcal{B}} |n(z)|$. We define the operators Λ_l and $L_l : \mathcal{N} \to \mathcal{C}^0(\mathcal{B})$ by

$$\Lambda_l n(z) = (\alpha + \beta) Sn(z) - (1+s)\, tL_l n(z)$$

$$L_l n(z) = \int_{\mathcal{B}} H_l(n(x)) \cdot 1_{\{z>x\}} \mathrm{d}x$$

Furthermore we define the functional $U^*_{kl} : \mathcal{N} \to \mathbb{R}$ as being the solution U to $\int_{\mathcal{B}} F_k[U - \Lambda_0 - \Lambda_l n(x)]\mathrm{d}x = 0$ and the operator $O_{kl} : \mathcal{N} \to \mathcal{N}$ by

$$O_{kl} n(z) = \int_{\mathcal{B}} F_k[U^*_{kl} n - \Lambda_0 - \Lambda_l n(x)] \cdot 1_{\{z>x\}} \mathrm{d}x$$

Note that U^*_{kl} is a unique solution because F_k is strictly increasing.

Our strategy is to prove the existence of fixed points n^*_{kl} to $n^*_{kl} = O_{kl} n^*_{kl}$ $\forall k, l \in \mathbb{N}$ and prove that the limit $n^* = \lim_{k\to\infty} \lim_{l\to\infty} n^*_{kl}$ satisfies $n^* = On^*$. We first prove that the set $\mathcal{N}$ is a closed convex set in the Banach space $C^0(\mathcal{B})$ and the mapping O_{kl} is well defined and continuous.

Lemma 6. *The set $\mathcal{N}$ is a closed convex set in the Banach space $C^0(\mathcal{B})$.*

Proof. *The space $C^0(\mathcal{B})$ is a vector space of continuous scalar functions on the real compact interval $\mathcal{B} = [-b, b]$ for the operation of addition and scalar multiplication. It is equipped with the sup norm $\|n\| = \sup_{z\in\mathcal{B}} |n|$. Every Cauchy sequence $\{n_k\}_{k=1}^{\infty}$ with $n_k \in \mathcal{N}$ converges to some function $n \in \mathcal{N}$. It is thus a Banach space. The set of continuous functions $\mathcal{N} = \{n \in C^0(\mathcal{B}) : n(-b) = n(b) = 0, \dot{n} \in [-1/s, 1]\}$ is closed and convex. Indeed, any convex combination $\alpha n + (1-\alpha)m$, $n, m \in \mathcal{N}$, belongs to $\mathcal{N}$. It is also closed because every sequence $\{n_k\}_{k=1}^{\infty}$ with $n_k \in \mathcal{N}$ converges to some $n \in \mathcal{N}$.* ■

We now check that O_{kl} is a well-defined continuous mapping. We remind that an operator is well defined if it actually maps its domain into the defined set of images. An operator $M : X \to Y$, with $X, Y \in C^0(\mathcal{B})$, is a continuous mapping if for all $\delta > 0$, there exists $k > 0$ such that for any $n, m \in X$, $\|n - m\|_\infty < 1/k \Rightarrow \|Mn - Mm\|_\infty < \delta$. It must be noted that the composition of continuous mappings is also a continuous mapping. Also, the convolution $M : X \to Y$, $Mn(z) = \int_{\mathcal{B}} f(z-x) n(x)\mathrm{d}x$ is a continuous mapping for any bounded valued function $f : \mathbb{R} \to \mathbb{R}$, $|f| < \infty$. As a result, we just need to show that TS is a well defined continuous mapping.

Lemma 7. *TS is a well defined continuous mapping.*

Proof. *We first prove the following useful statement: An operator $M : \mathcal{N} \to C^0(\mathbb{R})$, $Mn(z) = \int_{\mathcal{B}} f(x,n) g(x-z) \dot{n}(x)\mathrm{d}x$ is a continuous mapping when we consider $f : \mathcal{B} \times \mathcal{N} \to \mathbb{R}$ as a continuous function with bounded image (i.e. $|f| < f^{\max}$) and $g : \mathbb{R} \to \mathbb{R}^+$ with bounded image (i.e. $|g| < g^{\max}$). We indeed successively get*

$$\begin{aligned}
\|Mn - Mm\|_\infty &= \sup_{z\in\mathcal{B}} \left| \int_{\mathcal{B}} f(x,n) g(x-z) \dot{n}(x)\mathrm{d}x - \int_{\mathcal{B}} f(x,m) g(x-z) \dot{m}(x)\mathrm{d}x \right| \\
&= \sup_{z\in\mathcal{B}} \left| \begin{array}{l} \int_{\mathcal{B}} f(x,n) g(x-z) \dot{n}(x)\mathrm{d}x - \int_{\mathcal{B}} f(x,m) g(x-z) \dot{m}(x)\mathrm{d}x \\ - \int_{\mathcal{B}} f(x,n) g(x-z) \dot{m}(x)\mathrm{d}x + \int_{\mathcal{B}} f(x,n) g(x-z) \dot{m}(x)\mathrm{d}x \end{array} \right| \\
&\le \sup_{z\in\mathcal{B}} \left| \int_{\mathcal{B}} f(x,n) g(x-z) \left(\dot{n}(x) - \dot{m}(x)\right) \mathrm{d}x \right| + \left| \int_{\mathcal{B}} \left[f(x,m) - f(x,n)\right] g(x-z) \dot{m}(x)\mathrm{d}x \right| \\
&\le f^{\max} g^{\max} \left| \int_{\mathcal{B}} \left(\dot{n}(x) - \dot{m}(x)\right) \mathrm{d}x \right| + g^{\max} \left| \int_{\mathcal{B}} \left[f(x,m) - f(x,n)\right] \dot{m}(x)\mathrm{d}x \right|
\end{aligned}$$

Note that $\int_{\mathcal{B}} \left(\dot{n}(x) - \dot{m}(x)\right) \mathrm{d}x = \int_{-b}^{b} \mathrm{d}n(x) - \int_{-b}^{b} \mathrm{d}m(x) = n(b) - n(-b) - \left[m(b) - m(-b)\right] = 0$ *since $n(\pm b) = m(\pm b) = 0$ for any $n, m \in \mathcal{N}$. Also, for any $m \in \mathcal{N}$, one has that $\dot{m} \in [-1/s, 1]$ and therefore $|\dot{m}| \le \max\{1, 1/s\}$. Using this,*

$$\begin{aligned}
\sup_{z\in\mathcal{B}} \left| \int_{\mathcal{B}} \left[f(x,m) - f(x,n)\right] \dot{m}(x)\mathrm{d}x \right| &\le \max\{1, \frac{1}{s}\} \cdot \left| \int_{\mathcal{B}} \left[f(x,m) - f(x,n)\right] \mathrm{d}x \right| \\
&\le \max\{1, \frac{1}{s}\} \cdot 2b \sup_{z\in\mathcal{B}} \left|f(z,m) - f(z,n)\right|
\end{aligned}$$

Finally, *by continuity of f, for any $\delta > 0$, there exists $\epsilon > 0$ such that $\|m - n\| < \epsilon$ implies $\left|f(z,m) - f(z,n)\right| < \delta$ for all z, or equivalently, $\sup_{z\in\mathcal{B}} \left|f(z,m) - f(z,n)\right| < \delta$. Thus, for any δ' we can use the same ϵ such that $\|m - n\| < \epsilon \Rightarrow \|Mn - Mm\|_\infty < \delta'$ where we set $\delta = \delta' / \left[2bg^{\max} \max\{1, 1/s\}\right]$. This proves the continuity of M.*

Second, the operator $T^{1-\sigma} : \mathcal{N} \to C^0(\mathbb{R})$ with $T^{1-\sigma} n(z) = \int_{\mathcal{B}} e^{(1-\sigma)\tau|z-x|} \mu(x)\mathrm{d}x$ is a well-defined continuous operator. Indeed, using $\mu = (1 - \dot{n})/(1+s)$, we have $T^{1-\sigma} n(z) = \int_{\mathcal{B}} e^{(1-\sigma)\tau|z-x|}\mathrm{d}x/(1+s) - \int_{\mathcal{B}} e^{(1-\sigma)\tau|z-x|} \dot{n}\mathrm{d}x/(1+s)$. The first term is independent of n. Regarding the second term, just set $f(x,n)=1$ and $g(z-x) = e^{(1-\sigma)\tau|z-x|}$, which have bounded images for $x, z \in \mathcal{B}$. The operator is well defined as it returns continuous functions having images in a strictly positive range: $T^{1-\sigma} n(\mathcal{B}) \subset (2be^{(1-\sigma)2b\tau}/(1+s), 2b/s)$. For the highest bound, we simply plug $\dot{n} = -1/s$ in the first line of the above expression. For the lowest bound, we compute $T^{1-\sigma} n(y) = \int_{\mathcal{B}} e^{(1-\sigma)\tau|z-y|} \frac{1-\dot{n}(z)}{1+s} \mathrm{d}z \ge e^{(1-\sigma)2b\tau} \int_{\mathcal{B}} \frac{1-\dot{n}(z)}{1+s} \mathrm{d}z = \frac{2be^{(1-\sigma)2b\tau}}{1+s} > 0$ where we use $|z - y| < 2b$ in the inequality and $\int_{\mathcal{B}} \dot{n}\mathrm{d}z = 0$ in the last equality. Therefore, $1/\left[\sigma T^{1-\sigma} n(x)\right]$ is also a continuous function having images in a strictly positive range while $\ln\left[T^{1-\sigma} n(x)\right]$ is a continuous functions having real bounded images.

Third, the operator $G : \mathcal{N} \to C^0(\mathbb{R})$ with $Gn(z) = \frac{1}{\sigma} \int_{\mathcal{B}} \frac{e^{(1-\sigma)\tau|z-x|}}{(T^{1-\sigma} n)(x)} \mu(x)\mathrm{d}x$ is a well-defined continuous operator. Indeed, one just has to set $f(x,n) = 1/\left[\sigma T^{1-\sigma} n(x)\right]$ and $g(z-x) = e^{-\sigma\tau|z-x|}$ in the above statement. It is clear that the operator returns a function with bounded positive images.

Finally, since $S = G - \ln T = G + \frac{1}{\sigma-1} \ln T^{1-\sigma}$, S is a linear combination of well defined continuous operators. Thus, we can conclude that $S : \mathcal{N} \to C^0(\mathcal{B})$ is a well-defined continuous mapping. Being a linear combination of S and a continuous function, TS is also a well-defined continuous mapping $\mathcal{N} \to C^0(\mathcal{B})$. ■

Lemma 8. *L_l is a well-defined continuous mapping.*

Proof. *For any $\delta > 0$, we must find ϵ such that $\|m-n\|_\infty < \epsilon \Rightarrow \|L_l n - L_l m\|_\infty < \delta$. We successively get*

$$\|L_l n - L_l m\| = \sup_{z\in\mathcal{B}} \left| \int_{\mathcal{B}} H_l(n(x)) \cdot 1(z>x)\,\mathrm{d}x - \int_{\mathcal{B}} H_l(m(x)) \cdot 1_{\{z>x\}}\mathrm{d}x \right|$$

$$= \sup_{z\in\mathcal{B}} \left| \int_{\mathcal{B}} \left[H_l(n(x)) - H_l(m(x))\right] \cdot 1_{\{z>x\}}\mathrm{d}x \right|$$

$$= \sup_{z\in\mathcal{B}} \left| \int_{\mathcal{B}} [n(x) - m(x)]\, H_l'(p(x)) \cdot 1_{\{z>x\}}\mathrm{d}x \right|$$

where $p(x) \in [\min\{n(x), m(x)\}, \max\{n(x), m(x)\}]$ by the mean value theorem and the continuity and the differentiability of H_l. Using the fact that $H_l{}' \le l$ and that $n(-b) = m(-b) = 0$, we have

$$(n - L_l m)_\infty \le l\, sup_{z\in\mathcal{B}} \Big(\int_{-b}^{z} (n(x) - m(x))\mathrm{d}x \Big)$$

$$\le l\, sup_{z\in\mathcal{B}} \int_{-b}^{z} (n(x) - m(x))\mathrm{d}x$$

$$= 2bl\, sup_{z\in\mathcal{B}} (n(z) - m(z)) = 2bl(n-m)$$

So, *for any δ, choose $\epsilon < \delta/l$. Therefore, $L_l : \mathcal{N} \to C^0(\mathcal{B})$ is a continuous mapping. Since H_l has image on $[-1,1]$, the image of L_l lies in the interval $[-2b, 2b]$. So, it is well defined.* ∎

Since $\Lambda_l : \mathcal{N} \to C^0(\mathcal{B})$ is composition of well-defined continuous mappings with bounded images, it is also a well defined continuous mapping. We finally need to show the corresponding result for O_{kl} which depends on U_{kl}^*.

Lemma 9. *U_{kl}^* is a well-defined continuous functional.*

Proof. *For readability we here temporarily write U_{kl}^* as U. Then, $U : \mathcal{N} \to \mathbb{R}$ is the unique solution to $\int_{\mathcal{B}} F_k[U - \Lambda_0 - \Lambda_l n(x)]\mathrm{d}x = 0$ because F_k is strictly increasing. Also, it is clear that the solution U must lie between $\min_{x\in\mathcal{B}} \left[\Lambda_0 + \Lambda_l n(x)\right]$ and $\max_{x\in\mathcal{B}} \left[\Lambda_0 + \Lambda_l n(x)\right]$. Since $\Lambda_l n$ has bounded image, Un has also bounded images and is well defined.*

Let us consider two commuting functions n and $m : \mathcal{N} \to \mathbb{R}$. Let also p^{min} and $p^{\max} : \mathcal{N} \times \mathcal{N} \to \mathbb{R}$ be the minimum and the maximum functions of $Un - \Lambda_l n$ and $Um - \Lambda_l m$. By the mean value theorem and the differentiability of F_k, we have

$$F_k[Un - \Lambda_0 - \Lambda_l n(x)] - F_k[Um - \Lambda_0 - \Lambda_l m(x)] = F_k'(p(x)) \cdot [Un - \Lambda_l n(x) - Um + \Lambda_l m(x)]$$

where $p : \mathcal{B} \to \mathbb{R}$ such that $p \in [p^{min}(n,m), p^{max}(n,m)]$. This yields

$$F_k'(p(x)) \cdot (Un - U_{kl} m) = F_k[Un - \Lambda_0 - \Lambda_l n(x)] - F_k[Um - \Lambda_0 - \Lambda_l m(x)] + F_k'(p(x)) \cdot [\Lambda_l n(x) - \Lambda_l m(x)]$$

Integrating, applying $\int_{\mathcal{B}} F_k[U - \Lambda_0 - \Lambda_l n(x)]\mathrm{d}x = 0$ to n and m, and subtracting expressions, we get

$$(Un - Um) \cdot \int_{\mathcal{B}} F_k'(p(x))\mathrm{d}x = \int_{\mathcal{B}} F_k'(p(x)) \cdot [\Lambda_l n(x) - \Lambda_l m(x)]\mathrm{d}x$$

Since $F_k' > 0$, we have

$$(Un - Um) = \frac{\int_{\mathcal{B}} F_k'(p(x)) \cdot [\Lambda_l n(x) - \Lambda_l m(x)]\mathrm{d}x}{\int_{\mathcal{B}} F_k'(p(x))\mathrm{d}x}$$

$$\le \frac{\int_{\mathcal{B}} F_k'(p(x)) \cdot (\Lambda_l n(x) - \Lambda_l m(x))\mathrm{d}x}{\int_{\mathcal{B}} F_k'(p(x))\mathrm{d}x}$$

$$\le \frac{\int_{\mathcal{B}} F_k'(p(x)) \cdot sup_z(\Lambda_l n(z) - \Lambda_l m(z))\mathrm{d}x}{\int_{\mathcal{B}} F_k'(p(x))\mathrm{d}x}$$

$$= sup_z(\Lambda_l n(z) - \Lambda_l m(z))$$

$$= (\Lambda_l n - \Lambda_l m)_\infty$$

In other words Un is a continuous mapping of $\Lambda_l n$. Since $\Lambda_l n$ is a continuous mapping of n, Un is also a continuous mapping of n. ∎

Lemma 10. *For any $k, l \in \mathbb{N}$, $O_{kl} : \mathcal{N} \to \mathcal{N}$ is a continuous mapping.*

Proof. *The operator O_{kl} is defined by*

$$O_{kl} n(z) = \int_{\mathcal{B}} F_k[U_{kl}^* n - \Lambda_l n(x)] \cdot 1_{\{z>x\}}\mathrm{d}x \tag{61}$$

where $U_{kl}^* n$ was discussed in the previous lemma. By construction, the image functions $m \in O_{kl} n$ are continuous and satisfy $m(-b) = m(b) = 0$ and $\dot{m} \in [-1/s, 1]$. The operator O_{kl} is well defined as it is a mapping $\mathcal{N} \to \mathcal{N}$.

For readability we temporarily dispense U_{kl}^* and O_{kl} with the subscripts kl and superscript *. Using the mean value theorem and the differentiability of F_k and p defined in the previous lemma, we get

$$\|On - Om\|_\infty = \sup_{z\in\mathcal{B}} \left| \int_{\mathcal{B}} \{F_k[Un - \Lambda_l n(x)] - F_k[Un - \Lambda_l n(x)]\} \cdot 1_{\{z>x\}} \mathrm{d}x \right|$$

$$= \sup_{z\in\mathcal{B}} \left| \int_{\mathcal{B}} F_k'(p(x)) \cdot [Un - \Lambda_l n(x) - U(m) + \Lambda_l m(x)] \cdot 1_{\{z>x\}} \mathrm{d}x \right|$$

$$\leq \sup_{z\in\mathcal{B}} \left| \int_{\mathcal{B}} F_k'(p(x)) \cdot [\Lambda_l n(x) - \Lambda_l m(x)] \cdot 1_{\{z>x\}} \mathrm{d}x \right| + |Un - Um| \cdot \sup_{z\in\mathcal{B}} \left| \int_{\mathcal{B}} F_k'(p(x)) \cdot 1_{\{z>x\}} \mathrm{d}x \right|$$

We successively have

$$sup_{z\in\mathcal{B}} \int_{\mathcal{B}} (F_k'(p(x))[\Lambda_l n(x) - \Lambda_l m(x)]) \cdot 1(x<z)\mathrm{d}x \leq sup_{z\in\mathcal{B}} \int_{\mathcal{B}} (F_k'(p(x))) \cdot (\Lambda_l n(x) - \Lambda_l m(x)) \cdot 1_{\{z>x\}} \mathrm{d}x$$

$$\leq \int_{\mathcal{B}} (F_k'(p(x))) \cdot (\Lambda_l n(x) - \Lambda_l m(x)) \mathrm{d}x$$

$$\leq k \int_{\mathcal{B}} (\Lambda_l n(x) - \Lambda_l m(x)) \mathrm{d}x$$

$$\leq 2bk \sup_{z\in\mathcal{B}} (\Lambda_l n(z) - \Lambda_l m(z))$$

$$= 2bk(\Lambda_l n - \Lambda_l m)_\infty$$

and

$$(Un - Um) \cdot sup_{z\in\mathcal{B}} (\int_{\mathcal{B}} F_k'(p(x)) \cdot 1(x<z)\mathrm{d}x) \leq (\Lambda_l n - \Lambda_l m)_\infty \cdot sup_{z\in\mathcal{B}} \int_{\mathcal{B}} (F_k'(p(x))) 1(x<z) \mathrm{d}x$$

$$\leq (\Lambda_l n - \Lambda_l m)_\infty \cdot \int_{\mathcal{B}} (F_k'(p(x))) \mathrm{d}x$$

$$\leq 2bk(\Lambda_l n - \Lambda_l m)$$

Finally,

$$\|On - Om\|_\infty \leq 4bk\|\Lambda_l n - \Lambda_l m\|_\infty$$

For any δ, we can then choose $\epsilon = \delta/(4bk)$ so that $\|\Lambda_l n - \Lambda_l m\|_\infty < \epsilon$ implies $\|On - Om\|_\infty < \delta$. Hence, putting back the subscripts kl, O_{kl} is a continuous mapping of Λ_l. Since Λ_l is a continuous mapping of n, we have that O_{kl} is also a continuous mapping of n. ■

Lemma 11. *The image $O_{kl}(\mathcal{N})$ is a relatively compact subset of $\mathcal{N}$.*

Proof. *By Arzela-Ascoli theorem, this is true if $n \in O_{kl}(\mathcal{N})$ are uniformly bounded and equicontinuous on $\mathcal{B}$. To say that $n \in O_{kl}(\mathcal{N})$ are uniformly bounded means that there exists $M>0$ such that $\|n\|_\infty = \sup_{z\in\mathcal{B}} |n(z)| < M$ for all $n \in O_{kl}(\mathcal{N})$. This is true since F_k has image on $[-1/s, 1]$. Therefore, O_{kl} generates functions $n : \mathcal{B} \to \mathbb{R}$ such that $n(-b) = n(b)=0$ and $\dot{n} \in [-1/s, 1]$. These functions have image in $[-2n, 2b]$. To say that n is equicontinuous on $\mathcal{B}$ means that for every $\delta>0$ there exists an ϵ such that for every $x, y \in \mathcal{B}$ and every $n \in \mathcal{N}$, we have $|x-y| < \epsilon \Rightarrow |n(x) - n(y)| < \delta$. This follows from the continuity of $n \in C^0(\mathcal{B})$.* ■

To sum up, the set $\mathcal{N}$ is a closed convex set in the Banach space $C^0(\mathcal{B})$. The operator $O_{kl} : \mathcal{N} \to \mathcal{N}$ is a continuous mapping such that $O_{kl}(\mathcal{N})$ is a relatively compact subset of $\mathcal{N}$. We can then apply Schauder fixed point theorem.

Lemma 12. *For each $k, l \in \mathbb{N}$, there exists a fixed point $n^*_{kl} \in \mathcal{N}$ such that $n^*_{kl} = O_{kl} n^*_{kl}$.*

Convergence. We now turn to the question about whether the sequence of fixed points $\{n^*_{kl}\}_{k,l\in\mathbb{N}}$ converges to a function that is also a fixed point of (60). Because $\mathcal{N}$ is relatively compact and any n^*_{kl} belongs to $\mathcal{N}$, the sequence $\{n^*_{kl}\}_{k,l\in\mathbb{N}}$ converges uniformly on $\mathcal{B}$ to a function $n^*_\infty \in \mathcal{N}$.

Note that F is a correspondence because $F(0) = [-1/s, 1]$. In turn, the operator $O : \mathcal{N}\mathcal{N} \to$ is a correspondence. We therefore want to check whether $n^*_\infty \in On^*_\infty$. Formally, we ask whether for any $\delta > 0$, there exists a function $n^* \in On^*_\infty$ and two positive integers $\bar{k}$ and $\bar{l}$ such that we have $\left\|O_{kl} n^*_\infty - n^*\right\|_\infty < \delta$ for $k > \bar{k}$ and $l > \bar{l}$.

We here assume that the sequence $\{H_l\}_{l\in\mathbb{N}}$ converges to the sign function under the norm $\|\cdot\|_1$. Formally, for any $\delta > 0$, there exists $\bar{l} \in \mathbb{N}$ such that $\int |H_l(\xi) - \mathrm{sign}(\xi)| \mathrm{d}\xi < \delta$. Similarly, we assume that $\{F_k\}_{k\in\mathbb{N}}$ converges to F under the norm $\|\cdot\|_1$. That is, for any $\delta > 0$, there exists $\bar{k} \in \mathbb{N}$ such that $\int |F_k(\xi) - F(\xi)| \mathrm{d}\xi < \delta$. The last integral returns a scalar because the correspondence F returns a set only on the zero measure set at $\xi = 0$.

Lemma 13. *Λ_l converges to Λ uniformly under the norm $\|\cdot\|_\infty$.*

Proof. *The mapping Λ_l uniformly converges to the mapping Λ if for any $\delta>0$, we can find $\bar{l}$ such that $\|\Lambda_l - \Lambda\|_\infty \leq \delta$ for all $l > \bar{l}$. Note that $\|\Lambda_l - \Lambda\|_\infty = \|(\alpha+\beta)S - (1+s)tL_l - (\alpha+\beta)S + (1+s)tL\|_\infty = (1+s)t\|L_l - L\|_\infty$. So, we just need to check the uniform convergence of the mapping $L_l : \mathcal{N} \to \mathbb{R}$, $L_l n(z) = \int_{\mathcal{B}} H_l(n(x)) \cdot 1_{\{z>x\}} \mathrm{d}x$ to the mapping $Ln(z) = \int_{\mathcal{B}} \mathrm{sign}(n(x)) \cdot 1_{\{z>x\}} \mathrm{d}x$. By the definition of the convergence for H_l, for any $\delta>0$, we can find $\bar{l}$ such that $\int |H_l(\xi) - \mathrm{sign}(\xi)| \mathrm{d}\xi \leq \delta$ for all $l > \bar{l}$. Furthermore we have*

$$\begin{aligned}\|L_l - L\|_\infty &= \sup_{z\in\mathcal{B}}\left|\int_{\mathcal{B}}\left[H_l(n(x)) - \mathrm{sign}(n(x))\right]\cdot 1_{\{z>x\}}\mathrm{d}x\right| \\ &\le \int_{\mathcal{B}}\left|H_l(n(x)) - \mathrm{sign}(n(x))\right|\mathrm{d}x \\ &= \int\left|H_l(\xi) - \mathrm{sign}(\xi)\right|\cdot\mathcal{M}(x : \xi \le n(x) < \xi + \mathrm{d}\xi)\mathrm{d}\xi \\ &\le 2b\int\left|H_l(\xi) - \mathrm{sign}(\xi)\right|\mathrm{d}\xi \\ &\le 2b\delta\end{aligned}$$

where $\mathcal{M}(\mathcal{E})$ denotes the measure of the set $\mathcal{E} \subseteq \mathcal{B}$ so that $\mathcal{M}(\mathcal{B}) = 2b$. So, for any δ', we can use the above integer $\bar{l}$ for $\delta = \delta'/2b$ and get $\|L_l - L\|_\infty < \delta'$ for all $l > \bar{l}$. Therefore, L_l converges uniformly to L. The same conclusion applies to Λ_l and Λ. That is, the sequence $\{\Lambda_l n\}_{l\in\mathbb{N}}$ converges to Λn. ■

Let the function $\widehat{F} : \mathbb{R} \to [-1/s, 1]$ be such that $\widehat{F}(\xi) = 1$ if $\xi > 0$, $\widehat{F}(\xi) = 0$ if $\xi = 0$, and $\widehat{F}(\xi) = -1/s$ if $\xi < 0$. It is clear that $\widehat{F} \in F$ and that $\{F_k\}_{k\in\mathbb{N}}$ converges to $\widehat{F}$ under the norm $\|\cdot\|_1$. For any $\delta > 0$, there indeed exists $\bar{k} \in \mathbb{N}$ such that $\int_{-\infty}^{\infty}\left|F_k(\xi) - \widehat{F}(\xi)\right| d\xi < \delta$. This is true because $\{F_k\}_{k\in\mathbb{N}}$ converges to F and $\int_{-\infty}^{\infty}\left|F_k(\xi) - \widehat{F}(\xi)\right| d\xi = \int_{-\infty}^{\infty}\left|F_k(\xi) - F(\xi)\right| d\xi$ since F and $\widehat{F}$ differ only on a zero measure set.

Lemma 14. *For any operator $Z : \mathcal{N} \to C^0(\mathcal{B})$ and function $n \in \mathcal{N}$, $\int_{\mathcal{B}} F_k\left[Zn(x)\right]\cdot 1_{\{z>x\}}\mathrm{d}x$ converges uniformly to $\int_{\mathcal{B}}\widehat{F}\left[Zn(x)\right]\cdot 1_{\{z>x\}}\mathrm{d}x$ under the norm $\|\cdot\|_\infty$ and $\int_{\mathcal{B}}\widehat{F}\left[Zn(x)\right]\cdot 1_{\{z>x\}}\mathrm{d}x \in \int_{\mathcal{B}} F\left[Zn(x)\right]\cdot 1_{\{z>x\}}\mathrm{d}x$.*

Proof. *We first prove that for any $\delta > 0$ there exists a positive integer $\bar{k} \in \mathbb{N}$ such that*

$$I_k = \left\|\int_{\mathcal{B}}\widehat{F}(Zn(x))\cdot 1_{\{z>x\}}\mathrm{d}x - \int_{\mathcal{B}} F_k[Zn(x)]\cdot 1_{\{z>x\}}\mathrm{d}x\right\|_\infty < \delta$$

for $k > \bar{k}$. We indeed successively have

$$\begin{aligned}I_k &= \left(\int_{\mathcal{B}}(\widehat{F}(Zn(x)) - F_k[Zn(x)])\cdot 1_{\{z>x\}}\mathrm{d}x\right)_\infty \\ &= sup_{z\in\mathcal{B}}\left(\int_{\mathcal{B}}(\widehat{F}(Zn(x)) - F_k[Zn(x)])\cdot 1_{\{z>x\}}\mathrm{d}x\right) \\ &\le \int_{\mathcal{B}}(\widehat{F}(Zn(x)) - F_k[Zn(x)])\mathrm{d}x \\ &= \int(\widehat{F}(\xi) - F_k(\xi))\cdot\mathcal{M}_{\{x:\xi\le Zn(x)<\xi+\mathrm{d}\xi\}}\mathrm{d}\xi\end{aligned}$$

where $\mathcal{M}_{\mathcal{E}}$ denotes the measure of a set $\mathcal{E} \subseteq \mathcal{B}$ so that $\mathcal{M}_{\mathcal{B}} = 2b$. We can use the fact that the measure is always smaller than $2b$ and get

$$I_k \le 2b\int\left|\widehat{F}(\xi) - F_k(\xi)\right|\mathrm{d}\xi$$

Finally, using the definition of convergence of F_k to $\widehat{F}$, we can find a positive integer $\bar{k}$ such that $\int\left|\widehat{F}(\xi) - F_k(\xi)\right|\mathrm{d}\xi < \delta/2b$, so that $I_k \le \delta$ for any $k > \bar{k}$.

We finally prove that $\int_{\mathcal{B}}\widehat{F}\left[Zn(x)\right]\cdot 1_{\{z>x\}}\mathrm{d}x \in \int_{\mathcal{B}} F\left[Zn(x)\right]\cdot 1_{\{z>x\}}\mathrm{d}x$. For any $z \in \mathcal{B}$, we have

$$\int_{\mathcal{B}}\widehat{F}\left[Zn(x)\right]\cdot 1_{\{z>x\}}\mathrm{d}x = \int\widehat{F}(\xi)\cdot\mathcal{M}_{\{x:x<z,\xi\le Zn(x)<\xi+\mathrm{d}\xi,Zn(x)\neq 0\}}\mathrm{d}\xi + 0\cdot\mathcal{M}_{\{x:x<z,Zn(x)=0\}}$$

$$\int_{\mathcal{B}} F\left[Zn(x)\right]\cdot 1_{\{z>x\}}\mathrm{d}x = \int F(\xi)\cdot\mathcal{M}_{\{x:x<z,\xi\le Zn(x)<\xi+\mathrm{d}\xi,Zn(x)\neq 0\}}\mathrm{d}\xi + \int F(0)\cdot\mathcal{M}_{\{x:x<z,:Zn(x)=0\}}\mathrm{d}\xi$$

where $\mathcal{M}_{\mathcal{E}}$ denotes the measure of a set $\mathcal{E} \subseteq \mathcal{B}$. For any $z \in \mathcal{B}$,

$$\int_{\mathcal{B}} F\left[Zn(x)\right]\cdot 1_{\{z>x\}}\mathrm{d}x = \int_{\mathcal{B}}\widehat{F}\left[Zn(x)\right]\cdot 1_{\{z>x\}}\mathrm{d}x + \int F(0)\cdot\mathcal{M}_{\{x:x<z,:Zn(x)=0\}}\mathrm{d}\xi$$

The last term in the RHS is equal to either $\{0\}$ if $\mathcal{M}_{\{x:x<z,:Zn(x)=0\}} = 0$ or the interval $[-m/s, m]$ where $m = \mathcal{M}_{\{x:x<z,:Zn(x)=0\}}$ if $\mathcal{M}_{\{x:x<z,:Zn(x)=0\}} > 0$. It is then clear that $\int_{\mathcal{B}} F\left[Zn(x)\right]\cdot 1_{\{z>x\}}\mathrm{d}x \ni \int_{\mathcal{B}}\widehat{F}\left[Zn(x)\right]\cdot 1_{\{z>x\}}\mathrm{d}x$. ■

Given these lemmas, for any $n \in \mathcal{N}$ and $U \in \mathbb{R}$, we get

$$\begin{aligned}\lim_{k\to\infty}\lim_{l\to\infty} O_{kl}n(z) &= \lim_{k\to\infty}\lim_{l\to\infty}\int_{\mathcal{B}} F_k(U - \Lambda_0 - \Lambda_l n(x))\cdot 1_{\{z>x\}}\mathrm{d}x \\ &= \lim_{k\to\infty}\int_{\mathcal{B}} F_k\left[U - \Lambda_0 - \left(\lim_{l\to\infty}\Lambda_l n\right)(x)\right]\cdot 1_{\{z>x\}}\mathrm{d}x \\ &= \lim_{k\to\infty}\int_{\mathcal{B}} F_k(U - \Lambda_0 - \Lambda n(x))\cdot 1_{\{z>x\}}\mathrm{d}x \\ &= \int_{\mathcal{B}}\widehat{F}(U - \Lambda_0 - \Lambda n(x))\cdot 1_{\{z>x\}}\mathrm{d}x\end{aligned}$$

where the second equality stems from the continuity of F_k, the third one from Lemma 13, and the last one from Lemma 14. Also, from Lemma 14, we have $\int_{\mathcal{B}}\widehat{F}\left[U-\Lambda_0-\Lambda n(x)\right]\cdot 1_{\{z>x\}}\mathrm{d}x \in \int_{\mathcal{B}}F\left[U-\Lambda_0-\Lambda n(x)\right]\cdot 1_{\{z>x\}}\mathrm{d}x$. So, $\lim_{k\to\infty}\lim_{l\to\infty}O_{kl}n(z)\in On(z)$. Applying this to the fixed point solution $n_\infty^* = \lim_{k\to\infty}\lim_{l\to\infty}n_{kl}^*$ and $U_\infty^* = \lim_{k\to\infty}\lim_{l\to\infty}U_{kl}^* n_{kl}^*$, we get $n_\infty^* = \lim_{k\to\infty}\lim_{l\to\infty}O_{kl}n_\infty^* \in On_\infty^*$. Hence, n_∞^* is a fixed point of O, which proves Proposition 2.

Appendix D. Monocentric city

In this Appendix, we derive the expressions of the residential and business bid rents and the wage in terms of the total surplus $TS_M(z)$ and the urban utility $v_M(z)$. Then, we establish the conditions for which the monocentric city is a spatial equilibrium as stated in Proposition 3.

We focus our analysis on the half-segment $[0,b]$. The residential bid rent (15) can be written as

$$\Psi_r(x) = A_M(y) - tx, \qquad x\in[b_1,b]$$

where

$$A_M(y) \equiv v_M(y) + w(y) + ty + \bar{c}_0 - \alpha - U^*, \quad y\in[b_1,b]$$

As the expression $A_M(y)$ should be independent from the workplace y and the city border condition (iv) imposes $\Psi_r(b) = 0$, we have $A_M(y) = tb$. This leads to the land rent in the residential district as well as the wage in the business district

$$\Psi_r(x) = t(b-x), x\in(b_1,b) \tag{62}$$

$$w(y) = t(b-y) - v_M(y) - (\bar{c}_0 - \alpha - U^*), \quad y\in(0,b_1) \tag{63}$$

By plugging the above wage expression (63) into the business bid rent (16), we get

$$\Psi_b(y) = \frac{1}{s}\left[\mathrm{TS}_M(y) - t(b-y) + \bar{c}_0 - \alpha - U^*\right], \quad y\in(0,b_1) \tag{64}$$

By imposing condition (iii) $\Psi_b(b_1) = \Psi_r(b_1)$, we get the equilibrium utility

$$U^* = \mathrm{TS}_M(b_1) - (1+s)t(b-b_1) + \bar{c}_0 - \alpha$$

By plugging this expression of U^* into the business bid rent (64) and the wage (63), we get

$$\Psi_b(y) = \frac{1}{s}\left\{(\alpha+\beta)\left[S_M(y) - S_M(b_1)\right] + (1+s)t(b-b_1) - t(b-y)\right\} \tag{65}$$

$$w(y) = \mathrm{TS}(b_1) - (1+s)t(b-b_1) + t(b-y) - v(y), \qquad y\in(0,b_1)$$

So as to derive the conditions stated in Proposition 3, we still need to impose conditions (i) and (ii). We prove below in this Appendix that these two conditions can actually be reduced to conditions (i') $\Psi_b(0)\geq\Psi_r(0)$ and (ii') $\Psi_r(b)\geq\Psi_b(b)$ respectively. We now rewrite these two last conditions.

Using expressions (65) and (62), condition (i') and (ii') can be written respectively as

$$\frac{1}{s}\left\{(\alpha+\beta)\left[S_M(0) - S_M(b_1)\right] + st(b-b_1) - tb_1\right\} \geq tb$$

$$(\alpha+\beta)\left[S_M(b) - S_M(b_1)\right] + (1+s)t(b-b_1) \leq 0$$

that is

$$(1+s)b_1 t \leq (\alpha+\beta)\left[S_M(0) - S_M(b_1)\right]$$

$$(1+s)(b-b_1)t \leq (\alpha+\beta)\left[S_M(b_1) - S_M(b)\right]$$

and thus

$$t \leq \bar{t}_M \equiv \frac{2(\alpha+\beta)}{(1+s)sN}\left[S_M(0) - S_M(b_1)\right]$$

$$t \leq \hat{t}_M \equiv \frac{2(\alpha+\beta)}{(1+s)N}\left[S_M(b_1) - S_M(b)\right]$$

where we made use of $b_1 = sN/2$ and $b = (1+s)N/2$.

These two conditions provide the transport cost thresholds stated in Proposition 3.

When $s\to 0$, we get

$$\lim_{s\to 0}\left[\ln T_M(sN/2) - \ln T_M((1+s)N/2)\right] = -\tau N/2$$

$$\lim_{s\to 0}\left[G_M(sN/2) - G_M((1+s)N/2)\right] = \frac{1-e^{(1-\sigma)\tau N/2}}{\sigma}$$

Therefore

$$\lim_{s\to 0}\hat{t}_M \equiv \lim_{s\to 0}\frac{2(\alpha+\beta)}{(1+s)N}\left[S_M(sN/2)-S_M((1+s)N/2)\right]$$
$$=\lim_{s\to 0}\frac{2(\alpha+\beta)}{(1+s)N}\left[G_M(sN/2)-G_M((1+s)N/2)-\ln T_M(s/2)+\ln T_M((1+s)N/2)\right]$$
$$=\frac{\alpha+\beta}{N}\left(2\frac{1-e^{(1-\sigma)\tau N/2}}{\sigma}+\tau N\right)>(\alpha+\beta)\tau>\frac{(\alpha+\beta)\tau\,(2\sigma-1)}{2\sigma}=\lim_{s\to 0}\bar{t}_M$$

so that $\lim_{s\to 0}\hat{t}_M > \lim_{s\to 0}\bar{t}_M$.

We now prove that conditions (*i*) and (*ii*) can be reduced to conditions (*i*') and (*ii*'). For this, we make use of the following Lemma.

Lemma 15. *(i) S_M is convex in $\left[0,\bar{b}\right)$ then concave in $\left[\bar{b},b_1\right)$ for some $\bar{b}\in[0,b_1)$ with $S_M'(0)$=0. (ii) S_M is convex in $(b_1,b]$. (iii) $S_M\in C^1$ in the interval $[0,b]$ with $S_M'(b_1)$<0 and $S_M''(b_1-0)<0 < S_M''(b_1+0)$.*

Proof. (*i*) *The access cost T_M given by (27) can be written as*

$$T_M^{1-\sigma}(z)=\frac{2}{(\sigma-1)\tau s}\{1-e^{(1-\sigma)\tau b_1}\cosh[(\sigma-1)\tau z]\} \tag{66}$$

Because $-\cosh(z)$ is strictly concave and decreasing for $z\geq 0$, so is $T_M^{1-\sigma}$. As $(\ln T_M^{1-\sigma})''=[(T_M^{1-\sigma})''T_M^{1-\sigma}-(T_M^{1-\sigma})'^2]/T_M^{2(1-\sigma)}<0$, $\ln T_M^{1-\sigma}$ is strictly concave and decreasing for $z\geq 0$, and so is $-\ln T_M(z)$. Differentiating G_M as given by (27), we successively get

$$G_M'(z)=\frac{1}{\sigma s}\left[(1-\sigma)\tau\int_{-b_1}^{z}\frac{e^{(1-\sigma)\tau(z-y)}}{T_M^{1-\sigma}(y)}dy-(1-\sigma)\tau\int_{z}^{b_1}\frac{e^{(1-\sigma)\tau(y-z)}}{T_M^{1-\sigma}(y)}dy\right] \tag{67}$$

$$G_M''(z)=\frac{(1-\sigma)\tau}{\sigma s}\left[(1-\sigma)\tau\int_{-b_1}^{z}\frac{e^{(1-\sigma)\tau(z-y)}}{T_M^{1-\sigma}(y)}dy+(1-\sigma)\tau\int_{z}^{b_1}\frac{e^{(1-\sigma)\tau(y-z)}}{T_M^{1-\sigma}(y)}dy+\frac{2}{T_M^{1-\sigma}(z)}\right]$$
$$=(\sigma-1)^2\tau^2G_M(z)-\frac{(\sigma-1)\tau}{\sigma s}\frac{2}{T_M^{1-\sigma}(z)}=\frac{(\sigma-1)\tau}{s\sigma}\left[s\sigma(\sigma-1)\tau G_M(z)-\frac{2}{T_M^{1-\sigma}(z)}\right]$$

Given the above expression of G_M'' and the fact $2/T_M^{1-\sigma}$ is strictly increasing for $z\geq 0$, the convexity of G_M can be determined by looking at the relative positions of $s\sigma(\sigma-1)\tau G_M$ and $2/T_M^{1-\sigma}$. The curve $s\sigma(\sigma-1)\tau G_M$ is convex (resp. concave) when above (resp. below) the curve $2/T_M^{1-\sigma}$. Two possibilities actually arise: $s\sigma(\sigma-1)\tau G_M$ is either concave and below the curve $2/T_M^{1-\sigma}$ or there exists some $\bar{b}\in(0,b_1)$ such that G_M is convex on $[0,\bar{b}]$ and concave on $[\bar{b},b_1]$ with $\bar{b}$ necessarily corresponding to the single intersection point of $s\sigma(\sigma-1)\tau G_M$ with the curve $2/T_M^{1-\sigma}$. See the illustration in Figure A1. It can also be shown that $S_M'(0)=G_M'(0)$=$(-\ln T_M)'(0)$=0.

(ii) *We now prove that S_M is convex in $(b_1,b]$. For $x\in(b_1,b]$, we have*

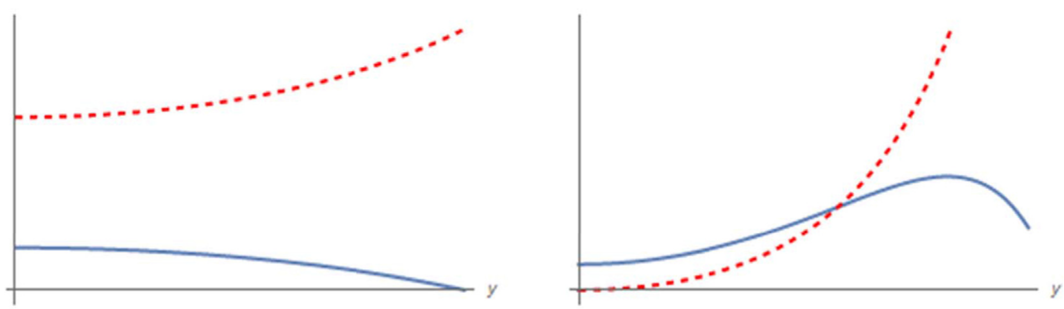


Fig. A1 Concavity and convexity of G_M. $\sigma(\sigma-1)s\tau G_M$ (*resp.* $2/T_M^{1-\sigma}$) is represented by the plain (*resp.* dashed) curve. Left panel: $\sigma=3$, $\tau=1$, $s=1$; right panel $\sigma=5$, $\tau=2.5$, $s=1$.

$$G_M(x)=\frac{1}{\sigma s}\int_{-b_1}^{b_1}\frac{e^{(1-\sigma)\tau(x-y)}}{T_M^{1-\sigma}(y)}dy$$
$$G_M'(x)=\frac{(1-\sigma)\tau}{\sigma s}\int_{-b_1}^{b_1}\frac{e^{(1-\sigma)\tau(x-y)}}{T_M^{1-\sigma}(y)}dy<0$$
$$G_M''(x)=\frac{(1-\sigma)^2\tau^2}{\sigma s}\int_{-b_1}^{b_1}\frac{e^{(1-\sigma)\tau(x-y)}}{T_M^{1-\sigma}(y)}dy>0 \tag{68}$$

and

$$\ln T_M(x) = \tau x + \frac{1}{1-\sigma} \ln\left(\frac{1}{s}\int_{-b_1}^{b_1} e^{-(1-\sigma)\tau y} \mathrm{d}y\right)$$

which is linear in x. *Hence,* S_M *is convex in* $(b_1, b]$.

(iii) *We showed that* $-\ln T_M(z)$ *is concave, decreasing and* C^1 *on* $[0, b]$. *For* $z \to b_1 - 0$, *we have*

$$G_M'(b_1-0) = \frac{(1-\sigma)\tau}{\sigma s}\int_{-b_1}^{b_1} \frac{e^{(1-\sigma)\tau(b_1-y)}}{T_M^{1-\sigma}(y)}\mathrm{d}y < 0$$

$$G_M''(b_1-0) = \frac{(\sigma-1)\tau}{\sigma s}\left[(\sigma-1)\tau\int_{-b_1}^{b_1} \frac{e^{(1-\sigma)\tau(b_1-y)}}{T_M^{1-\sigma}(y)}\mathrm{d}y - \frac{2}{T_M^{1-\sigma}(z)}\right] < 0$$

as G_M *is concave in* $z = b_1$ *by part (i). For* $z \to b_1 + 0$, *we have*

$$G_M'(b_1+0) = \frac{(1-\sigma)\tau}{\sigma s}\int_{-b_1}^{b_1} \frac{e^{(1-\sigma)\tau(b_1-y)}}{T_M^{1-\sigma}(y)}\mathrm{d}y < 0$$

$$G_M''(b_1+0) = \frac{\sigma\tau^2}{s}\int_{-b_1}^{b_1} \frac{e^{(1-\sigma)\tau(b_1-y)}}{T_M^{1-\sigma}(y)}\mathrm{d}y > 0$$

by using expression (68). Therefore, $S_M'(b_1-0) = S_M'(b_1+0)$ *but* $S_M''(b_1-0) < S_M''(b_1+0)$. *So,* S_M *is* C^1 *on* $[0, b]$. *It also appears that* $S_M'(b_1) < 0$. ■

We now show that the equilibrium conditions $\Psi_b(y) \geq \Psi_r(y)$ for all $y \in [0, b_1]$ and $\Psi_r(x) \geq \Psi_b(x)$ for all $x \in [b_1, b]$ can be reduced to $\Psi_b(0) \geq \Psi_r(0)$ and $\Psi_b(b) \leq \Psi_r(b) = 0$ respectively.

Proof. *In the interval of* $[0, b]$, *the bid rents of residents and firms are given by*

$$\Psi_r(y) = t(b-y)$$

$$\Psi_b(y) = \frac{1}{s}\left[\mathrm{TS}_M(y) - t(b-y) + \bar{c}_0 - U^* - \alpha\right]$$

Given Lemma 15(i), Ψ_b *is convex in* $\left[0, \bar{b}\right)$ *then concave in* $\left[\bar{b}, b_1\right)$ *for some* $\bar{b} \in [0, b_1)$ *with* $\Psi_b'(0) = t$. *Therefore* $\Psi_b(y) - \Psi_r(y)$ *is quasiconcave on* $[0, b_1]$. As $\Psi_b(b_1) = \Psi_r(b_1)$, *the equilibrium condition(i)* $\Psi_b(y) \geq \Psi_r(y)$ *for all* $y \in [0, b_1]$ *reduces to* $\Psi_b(0) \geq \Psi_r(0)$. *According to Lemma 15(ii),* $\Psi_b(z)$ *is convex for* $z \in [b_1, b]$. *As* $\Psi_r(z)$ *is linear and* $\Psi_b(b_1) = \Psi_r(b_1)$, *the equilibrium condition (ii)* $\Psi_b(x) \leq \Psi_r(x)$ *for all* $x \in [b_1, b]$ *reduces to* $\Psi_b(b) \leq \Psi_r(b) = 0$. ■

Appendix E. Integrated city

In this Appendix, we derive the expressions of the residential and business bid rents and the wage in terms of the total surplus $\mathrm{TS}_I(z)$ and the urban utility $v_I(z)$. Then, we establish the conditions for which the monocentric city is a spatial equilibrium as stated in Proposition 4.

As in the monocentric analysis, we focus our analysis on the half-segment $[0, b]$. We need to impose conditions (*i*)-(*iii*) on the bid rent functions. By using the bid rent expressions (15) and (16), the equilibrium condition (*i*) imposing rent equalization, $\Psi_r = \Psi_b$, leads to

$$w(y) = -\frac{s}{1+s}\left[v_I(y) + \bar{c}_0 - \alpha - U^*\right] + \frac{1}{1+s}\left[\pi_I^n(y) + \pi_I^u(y)\right] \quad (69)$$

By plugging this wage expression into (15), we get

$$\Psi(y) = \frac{1}{1+s}\left[\mathrm{TS}_I(y) + \bar{c}_0 - \alpha - U^*\right] \quad (70)$$

From the city border condition (*iii*), $\Psi_r(b) = 0$, we get

$$U^* = \mathrm{TS}_I(b) + \bar{c}_0 - \alpha$$

By inserting this equilibrium utility U^* into above the wage expression (69) and the bid rent (70), we get

$$w(y) = -v_I(y) + \mathrm{TS}_I(b) + \frac{\alpha+\beta}{1+s}\left[S_I(y) - S_I(b)\right]$$

$$\Psi(y) = \frac{\alpha+\beta}{1+s}\left[S_I(y) - S_I(b)\right]$$

According to the no-arbitrage condition (*ii*), an urban configuration is an equilibrium if workers have no incentive to commute to any other location, i.e., if $\left|R'(z)\right| = \left|(\alpha+\beta)S_I'(z)/(1+s)\right| \leq t$, for $\forall z \in [0, b]$. It is shown below that $\left|S_I'(z)\right| \leq \left|S_I'(b)\right|$, $\forall z \in [0, b]$, so that the no-arbitrage condition reduces to $\left|R'(b)\right| = \left|(\alpha+\beta)S_I'(b)/(1+s)\right| \leq t$.

We show that $\left|S_I'(b)\right| \geq \left|S_I'(z)\right|$, $\forall z \in [0, b]$.

Given the similarity of expression of T_I (34) with that corresponding to the monocentric case (27), the argument used in Appendix D showing that $\ln T_M^{1-\sigma}$ is strictly concave and decreasing in z, can be applied here. Thus, $-\ln(T_I)$ is strictly concave and decreasing for $y \geq 0$.

By differentiating $G_I(z)$ as given by expression (34), we get

$$G_I'(z) = \frac{1}{\sigma(1+s)}\left[(1-\sigma)\tau\int_{-b}^{z}\frac{e^{(1-\sigma)\tau(z-y)}}{T_I^{1-\sigma}(y)}\mathrm{dy} - (1-\sigma)\tau\int_{z}^{b}\frac{e^{(1-\sigma)\tau(y-z)}}{T_I^{1-\sigma}(y)}\mathrm{dy}\right]$$

$$= \frac{(\sigma-1)\tau}{\sigma(1+s)}\left[\int_{-b}^{z}\frac{e^{(1-\sigma)\tau(z-y)}}{T_I^{1-\sigma}(y)}\mathrm{dy} + \int_{z}^{b}\frac{e^{(1-\sigma)\tau(y-z)}}{T_I^{1-\sigma}(y)}\mathrm{dy} - 2\int_{-b}^{z}\frac{e^{(1-\sigma)\tau(z-y)}}{T_I^{1-\sigma}(y)}\mathrm{dy}\right]$$

$$= \frac{(\sigma-1)\tau}{\sigma(1+s)}\left[\sigma(1+s)G_I(z) - 2\int_{-b}^{z}\frac{e^{(1-\sigma)\tau(z-y)}}{T_I^{1-\sigma}(y)}\mathrm{dy}\right]$$

$$G_I''(z) = \frac{(\sigma-1)\tau}{\sigma(1+s)}\left[(1+s)\tau\sigma(\sigma-1)G_I(z) - \frac{2}{T_I^{1-\sigma}(y)}\right]$$

It suffices to show that $G_I'(b) + G_I'(z) \leq 0$ in order to have

$$\left|G_I'(b)\right| \geq \left|G_I'(z)\right|, \ \forall z \in [0,b] \tag{71}$$

as $G_I'(b) < 0$. To show that $G_I'(b) + G_I'(z) \leq 0$, we develop $G_I'(b) + G_I'(z)$ into

$$\frac{(\sigma-1)\tau}{\sigma(1+s)}\left[\sigma(1+s)\left[G_I(b)+G_I(z)\right] - 2\int_{-b}^{z}\frac{e^{(1-\sigma)\tau(b-y)}}{T_I^{1-\sigma}(y)}\mathrm{dy} - 2\int_{-b}^{z}\frac{e^{(1-\sigma)\tau(z-y)}}{T_I^{1-\sigma}(y)}\mathrm{dy}\right]$$

$$= \frac{(\sigma-1)\tau}{\sigma(1+s)}\left[\int_{-b}^{b}\frac{e^{(1-\sigma)\tau(b-y)}}{T_I^{1-\sigma}(y)}\mathrm{dy} + \int_{-b}^{z}\frac{e^{(1-\sigma)\tau(z-y)}}{T_I^{1-\sigma}(y)}\mathrm{dy} + \int_{z}^{b}\frac{e^{(1-\sigma)\tau(y-z)}}{T_I^{1-\sigma}(y)}\mathrm{dy} \ - 2\int_{-b}^{b}\frac{e^{(1-\sigma)\tau(b-y)}}{T_I^{1-\sigma}(y)}\mathrm{dy} - 2\int_{-b}^{z}\frac{e^{(1-\sigma)\tau(z-y)}}{T_I^{1-\sigma}(y)}\mathrm{dy}\right]$$

$$= \frac{(\sigma-1)\tau}{\sigma(1+s)}\left[\int_{z}^{b}\frac{e^{(1-\sigma)\tau(y-z)}}{T_I^{1-\sigma}(y)}\mathrm{dy} - \int_{-b}^{b}\frac{e^{(1-\sigma)\tau(b-y)}}{T_I^{1-\sigma}(y)}\mathrm{dy} + \int_{-b}^{z}\frac{e^{(1-\sigma)\tau(z-y)}}{T_I^{1-\sigma}(y)}\mathrm{dy}\right]$$

$$= \frac{(\sigma-1)\tau}{\sigma(1+s)}\left[\int_{z}^{b}\frac{e^{(1-\sigma)\tau(y-z)}}{T_I^{1-\sigma}(y)}\mathrm{dy} - \int_{-b}^{z}\frac{e^{(1-\sigma)\tau(b-y)}}{T_I^{1-\sigma}(y)}\mathrm{dy} - \int_{z}^{b}\frac{e^{(1-\sigma)\tau(b-y)}}{T_I^{1-\sigma}(y)}\mathrm{dy} - \int_{-b}^{z}\frac{e^{(1-\sigma)\tau(z-y)}}{T_I^{1-\sigma}(y)}\mathrm{dy}\right]$$

$$= \frac{(\sigma-1)\tau}{\sigma(1+s)}\left[\int_{z}^{b}\frac{e^{(1-\sigma)\tau(y-z)} - e^{(1-\sigma)\tau(b-y)}}{T_I^{1-\sigma}(y)}\mathrm{dy} - \int_{-b}^{z}\frac{e^{(1-\sigma)\tau(b-y)}}{T_I^{1-\sigma}(y)}\mathrm{dy} - \int_{-b}^{z}\frac{e^{(1-\sigma)\tau(z-y)}}{T_I^{1-\sigma}(y)}\mathrm{dy}\right]$$

As $T_I^{1-\sigma}$ is decreasing for $z \geq 0$, we have $\int_z^b (e^{(1-\sigma)\tau(y-z)} - e^{(1-\sigma)\tau(b-y)})/T_I^{1-\sigma}(y)\mathrm{dy} \leq 0$, $\forall z \geq 0$, because the increasing function $e^{(1-\sigma)\tau(y-z)}$ is the reflected image of $e^{(1-\sigma)\tau(b-y)}$ over the line $y = (z+b)/2$. This proves $G_I'(b) + G_I'(z) \leq 0$. Finally, using the fact that $-\ln T_I$ is strictly concave and decreasing and relation (71) leads to

$$\left|S_I'(b)\right| \geq \left|S_I'(z)\right|, \ \forall z \in [0,b]$$

Also, note that $S_I'(b) < 0$ as both $G_I'(b)$ and $-(\ln T_I)'(b)$ are negative. Therefore, $S_I(z)$ is concave.

We next prove that t_I is an increasing function of τ.

We need to show that $S_I'(b)$ is a decreasing function of τ. First, it can be shown that $d\left[-(\ln T)'(b)\right]/d\tau = -1$. Second, by using expression (34), we have

$$G_I'(b) = -\frac{(\sigma-1)\tau}{\sigma(1+s)}\int_{-b}^{b}\frac{e^{(1-\sigma)\tau(b-y)}}{T_I^{1-\sigma}(y)}\mathrm{dy}$$

and

$$\frac{d}{d\tau}G_I'(b) = \frac{(\sigma-1)}{(1+s)\sigma}\left[-\int_{-b}^{b}\frac{e^{(1-\sigma)\tau(b-z)}}{T_I^{1-\sigma}(z)}\mathrm{dz} + (\sigma-1)\tau\int_{-b}^{b}\frac{(b-z)e^{(1-\sigma)\tau(b-z)}}{T_I^{1-\sigma}(z)}\mathrm{dz} - \tau\int_{-b}^{b}e^{(1-\sigma)\tau(b-z)}\frac{d}{d\tau}T_I^{\sigma-1}(z)\mathrm{dz}\right]$$

As $\mathrm{d}[T_I^{1-\sigma}(z)]/\mathrm{d}\tau < 0$, it suffices to show that the sum of the first two terms is negative. We have

$$- \frac{(\sigma-1)}{(1+s)\,\sigma} \int_{-b}^{b} \frac{e^{(1-\sigma)\tau(b-z)}}{T_I^{1-\sigma}(z)} dz + \frac{(\sigma-1)^2\tau}{(1+s)\,\sigma} \int_{-b}^{b} \frac{(b-z)e^{(1-\sigma)\tau(b-z)}}{T_I^{1-\sigma}(z)} dz$$

$$= \frac{(\sigma-1)}{(1+s)\,\sigma} \int_{-b}^{b} [(b-z)(\sigma-1)\tau - 1] e^{(1-\sigma)\tau(b-z)} T_I^{\sigma-1}(z) dz$$

$$= \frac{1}{(1+s)\,\sigma\tau} \int_{0}^{2(\sigma-1)\tau b} (r-1)e^{-r} T_I^{\sigma-1} \left(b - \frac{r}{(\sigma-1)\,\tau}\right) dr$$

$$= \frac{1}{(1+s)\,\sigma\tau} \left(\int_0^1 + \int_1^{2(\sigma-1)\tau b} \right) (r-1)e^{-r} T_I^{\sigma-1} \left(b - \frac{r}{(\sigma-1)\,\tau}\right) dr$$

$$\le \frac{T_I^{\sigma-1}\left(b - \frac{1}{(\sigma-1)\tau}\right)}{(1+s)\,\sigma\tau} \left(\int_0^1 + \int_1^{2(\sigma-1)\tau b} \right) (r-1)e^{-r} dr < 0$$

$$\le \frac{T_I^{\sigma-1}\left(b - \frac{1}{\sigma\tau}\right)}{(1+s)\sigma\tau} \left(\int_0^1 + \int_1^{2\sigma\tau b} \right) (r-1)e^{-r} dr < 0$$

as $T_I^{\sigma-1}\left(b - \frac{r}{(\sigma-1)\tau}\right)$ is a decreasing function of r and $\int_0^{r_1} (r-1)e^{-r} dr < 0$ for all $r_1 > 0$.

Appendix F. Cut-offs t_M and t_I

We prove that $\lim_{s\to 0} t_M < \lim_{s\to 0} t_I$. As $s \to 0$, $t_M = \bar{t}_M = (\alpha+\beta)\,\tau$. We need to check that

$$(\ln T_I)'(b) - G_I'(b) > \tau$$

when $s \to 0$. Given the expressions (34) of $T_I^{1-\sigma}$ and G_I', we have

$$(\ln T_I)'(b) = \frac{1}{1-\sigma} \frac{d}{dz} \ln \left[1 - e^{(1-\sigma)\tau b} \cosh[(\sigma-1)\tau z] \right] \Big|_{z=b}$$

$$= \tau \frac{e^{(1-\sigma)\tau b} \sinh[(\sigma-1)\tau z]}{1 - e^{(1-\sigma)\tau b} \cosh[(\sigma-1)\tau z]} \Bigg|_{z=b}$$

$$= \tau$$

and

$$G_I'(b) = -\frac{(\sigma-1)\,\tau}{\sigma\,(1+s)} \int_{-b}^{b} \frac{e^{(1-\sigma)\tau(b-y)}}{T_I^{1-\sigma}(y)} dy < 0$$

Hence, we get $(\ln T_I)'(b) - G_I'(b) > \tau$.

Appendix G. Partially integrated city

In this Appendix, we derive the expressions of the bid rents and the wage in terms of the total surplus $\mathrm{TS}_P(z)$ and the urban utility $v_P(z)$. Then we establish the conditions for which the partially integrated city is a spatial equilibrium as stated in Proposition 5.

As in the study of previous urban configurations, we focus our analysis on the half-segment $[0, b]$. In the integrated district $[0, b_1]$, the rent equalization condition (i) imposes $R(z) = \Psi_r(z) = \Psi_b(z)$. From the bid rent expressions (15) and (16), the land rent and the wage in the integrated district can be written as

$$R(z) = \frac{1}{1+s} \left[\mathrm{TS}_P(y) + \bar{c}_0 - \alpha - U^* \right] \tag{72}$$

$$w(z) = \frac{s}{1+s} \left[-v_P(y) - \bar{c}_0 + \alpha + U^* \right] + \frac{1}{1+s} \left[\pi_P^n(y) + \pi_P^u(y) \right], \quad z \in [0, b_1) \tag{73}$$

In the residential district $[b_2, b]$, the land rent is given by (15)

$$R(x) = \Psi_r(x) = A_P(y) - tx, \qquad x \in [b_2, b]$$

where

$$A_P(y) = \alpha \left[\ln \frac{\alpha\rho}{\gamma} - \ln T_P(y) \right] + w(y) + ty + \bar{c}_0 - \alpha - U^*, \quad y \in [b_1, b_2]$$

As the expression $A_P(y)$ should be independent from the workplace y and the land rent is nil at $x = b$ according to condition (v), we have $A_P(y) = tb$. This leads to the land rent in the residential district as well as the wage in the business district

$$R(x) = t(b-x), \qquad x \in (b_2, b] \tag{74}$$

$$w(y) = t\,(b-y) - v_P(y) - \bar{c}_0 + \alpha + U^*, \quad y \in [b_1, b_2] \tag{75}$$

Inserting this wage expression (75) into the firm bid rent $\Psi_b(y)$ (16) yields the land rent in the business district

$$R(y) = \frac{1}{s}\left[\mathrm{TS}_P(y) - t(b-y) + \bar{c}_0 - \alpha - U^*\right], \quad y \in [b_1, b_2] \tag{76}$$

As the equilibrium land rent is continuous, conditions at district borders b_1 and b_2 impose $R(b_1 - 0) = R(b_1 + 0)$ and $R(b_2 - 0) = R(b_2 + 0)$. Solving these two equations for U^* and b_1 leads to

$$U^* = \mathrm{TS}_P(b_2) - (1+s)\,t\,(b - b_2) + \bar{c}_0 - \alpha \tag{77}$$

and

$$(\alpha+\beta)\left[S_P(b_1) - S_P(b_2)\right] = (1+s)\,t(b_2 - b_1)$$

The last relation shows equation (43) which determines b_1 as b_2 is given by relation (35).

By inserting the equilibrium utility expression (77) into relations (72) and (73) (*resp*. relations (76) and (75)) leads to the land rent and the wage in the integrated district (*resp*. the business district).

All these land rent and wage expressions are summarized in relations (41) and (42).

We still have to impose conditions (*ii*), (*iii*), and (*iv*). First, condition (*ii*) ensuring no commuting within any two locations in the integrated district writes as $\left|(\alpha+\beta)S_p'(z)\right| \le t(1+s), \quad z \in [0, b_1]$ that is,

$$t > \hat{t}_P \equiv \frac{\alpha+\beta}{1+s} \max_{z\in[0,b_1]} \left|S_P'(z)\right| \tag{78}$$

Second, condition (*iii*) says that firms outbid workers in the business district: $R(y) \ge \Psi_r(y) \forall y \in [b_1, b_2]$. Subtracting (74) from (41), we get

$$(\alpha+\beta)\left[S_P(y) - S_P(b_2)\right] \ge t(1+s)(b_2 - y), \quad y \in [b_1, b_2]$$

which gives

$$t \le \bar{t}_P \equiv \min_{y\in[b_1,b_2]} \frac{\alpha+\beta}{1+s} \frac{S_P(y) - S_P(b_2)}{b_2 - y} \tag{79}$$

Finally, condition (*iv*) ensuring that workers outbid firms in the residential district ($R(x) \ge \Psi_b(x), \forall x \in [b_2, b]$) gives

$$(\alpha+\beta)\left[S_P(b_2) - S_P(x)\right] - t(1+s)(x - b_2) \ge 0, \quad x \in [b_2, b] \tag{80}$$

which is similar to relation (79). Only signs and supports differ. Note that in the interval (b_2, b), $S_P(x)$ is convex because $G_P''(x) = \sigma^2\tau^2 G_P > 0$ and $(\ln T_P^{1-\sigma}(x))'' = 0$. Since the LHS of inequality (80) is zero at $x = b_2$, this inequality is satisfied for all $x \in [b_2, b]$ if and only if

$$(\alpha+\beta)\left[S_P(b_2) - S_P(b)\right] \ge t(1+s)(b - b_2) \ge 0$$

Thus, inequality (80) is equivalent to

$$t \le \tilde{t}_P \equiv \frac{\alpha+\beta}{1+s} \frac{S_P(b_2) - S_P(b)}{b - b_2} \tag{81}$$

To sum up, the spatial equilibrium is given by (b_1, b_2) that solve equations (35) and (43) and satisfy (78), (79), and (81).

Appendix H. Proof of Corollary 2

We show that the partially integrated city is a spatial equilibrium for $t = t_I - \delta$ and small enough $\delta > 0$. By continuity, $S_P(z)$ tends to $S_I(z)$ for any z when b_1 tends to b. This is also true for $S_p'(z)$ and $S_P''(z)$. It is shown in Appendix E that $S_I(z)$ is decreasing and concave in the neighborhood of $z = b$ and $\left|S_I'(b)\right| > \left|S_I'(z)\right|$, $z \in [0, b]$. Hence, we have $S_P'(z) < 0$ and $S_P''(z) < 0$ for z close enough to b while $\left|S_P'(b)\right| > \left|S_P'(z)\right|$, $\forall z \in [0, b]$. By continuity, $\left|S_P'(b_1)\right| > \left|S_P'(z)\right|$, $\forall z \in [0, b_1]$ when b_1 tends to b. In this case, $S_P(z)$ is decreasing and concave in z within the intervals $[b_1, b]$ for small $\delta > 0$. So, by concavity of $S_P(z)$, we compute

$$\bar{t}_P \equiv \min_{y\in[b_1,b_2]} \frac{\alpha+\beta}{1+s} \frac{S_P(y) - S_P(b_2)}{b_2 - y} = \frac{\alpha+\beta}{1+s} \frac{S_P(b_1) - S_P(b_2)}{b_2 - b_1}$$

$$\tilde{t}_P \equiv \frac{\alpha+\beta}{1+s} \frac{S_P(b_2) - S_P(b)}{b - b_2}$$

$$\hat{t}_P \equiv \frac{\alpha+\beta}{1+s} \max_{z\in(0,b_1)} (TS_P'(z)) = \frac{\alpha+\beta}{1+s}(S_P'(b_1))$$

while

$$t = \frac{\alpha+\beta}{1+s} \frac{S_P(b_1) - S_P(b_2)}{b_2 - b_1}$$

We now check that $\hat{t}_P \le t \le \min\{\bar{t}_P, \tilde{t}_P\}$. Indeed, $t = \bar{t}_P$ and $\hat{t}_P \le t \le \tilde{t}_P$ by concavity of $S_P(z)$.

Appendix I. Proof of Corollary 3

Towards this aim, we rewrite equation (43) as

$$t = f_s(b_1) \equiv \frac{\alpha+\beta}{1+s} \frac{S_P(b_1) - S_P(b_2)}{b_2 - b_1}$$

where b_2 is given by (35). As $s \to 0$, one can verify

$$\lim_{s\to 0} f_s(0) = t_M \text{ and } \lim_{s\to 0} f_s(b) = t_I$$

Because $t_I > t_M > 0$ and $f_s(b_1)$ is continuous in $[0, b/2]$, we can apply the intermediate value theorem. There exists at least a value of b_1^0 that solves $t = f_s(b_1)$ for all $t \in (t_M, t_I)$. One then computes the thresholds

$$\tilde{t}_P^0 = (\alpha+\beta)\frac{S_P(b_1^0) - S_P(b)}{b - b_1^0}$$

$$\bar{t}_P^0 = -(\alpha+\beta)\, S_P'(b_1^0)$$

$$\hat{t}_P^0 = (\alpha+\beta) \max_{z\in\left[0, b_1^0\right)} \left|S_P'(z)\right|$$

Then, if $\hat{t}_P^0 \le t \le \min\left\{\bar{t}_P^0, \tilde{t}_P^0\right\}$, there exists a partially integrated equilibrium with two infinitely small business districts at $y = \pm b_1^0$.

Appendix J. Duocentric city

In this Appendix, we derive the expressions of the bid rents and the wage in terms of the total surplus $\mathrm{TS}_D(z)$ and the urban utility $v_D(z)$. We also establish the conditions for which the duocentric city is a spatial equilibrium as stated in Proposition 6.

As in other cases, workers arbitrage between the different workplaces up to their commuting cost and land rent payments. As a result, a firm located at z must offer individuals a wage $w(z)$ that is as attractive as in other workplaces. Since workers commute from the left and right hand sides of the business district located on (b_1, b_3), it is convenient to denote the business center of the business district by $b_2 \in (b_1, b_3)$. This is the unique location where a firm attracts workers from each side of its location. Since the labor market divides about $y = b_2$, the left hand side's labor demand and supply balance so that $b_2 - b_1 = sb_1$ while its right hand side clears so that $b_3 - b_2 = s(b - b_3)$. Denoting the utility of a worker working at b_2 by $U_2 = v_D(b_2) - \alpha + w(b_2) + \bar{c}_0$, we can write the following wage arbitrage condition for any $y \ge 0$:

$$v_D(y) + w(y) + t\,|y - b_2| + \bar{c}_0 - \alpha = U_2$$

so that the equilibrium wage $w(y)$ satisfies

$$w(y) = U_2 - v_D(y) - t\,|y - b_2| - \bar{c}_0 + \alpha \tag{82}$$

This is also the wage that a firm should pay to attract a worker when it relocates in a residential district.

By plugging the wage expression into (15) and setting $y = b_2$, the household's bid rent can be rewritten as

$$\Psi_r(x) = U_2 - U^* - t\,|x - b_2| \tag{83}$$

Similarly, the firm's bid rent (16) writes as

$$\Psi_b(y) = \frac{1}{s}\left[\mathrm{TS}_D(y) - U_2 + \bar{c}_0 - \alpha + t\,|y - b_2|\right]$$

Landlords reap the surplus of the production and consumption of each pair of individuals and firms up to the level of the individual's utility.

As before, we focus on $[0, b]$ when imposing the equilibrium conditions (*i*)-(*iv*) on the bid rents from households and firms. Using the wage (82) from the wage arbitrage condition and applying it for $x > b_3 > y > b_2$, one can express the relations $\Psi_r(b) = 0$ and $\Psi_r(b_3) = \Psi_b(b_3)$ as functions of U_2, U^*, b_2 and b_3. Using the same wage (82) and applying it for $x < b_1 < y < b_2$, one can also express the relation $\Psi_r(b_1) = \Psi_b(b_1)$ as function of the same variables. Therefore, conditions (*iii*) and (*iv*) together with the identities $b_2 = (1+s)b_1$ and $b_3 = b_1 + sN/2$ provide five equations for the five variables U_2, U^*, b_1, b_2 and b_3. The solution for U_2 and U^* is given by

$$U_2 = \mathrm{TS}_D(b_1) + \bar{c}_0 - \alpha + t\,(b_2 - b_1) - st\,(b - 2b_2 + b_1)$$

$$U^* = \mathrm{TS}_D(b_1) + \bar{c}_0 - \alpha - (1+s)\,t\,(b - 2b_2 + b_1) \tag{84}$$

while b_1 solves (47). Note that $U_2 - U^* = t\,(b - b_2)$. Using this, we get (45) and (46).

The equilibrium inequality condition (*ii*) $\Psi_r(x) \ge \Psi_b(x)$ for all $x \in \mathcal{X}$, writes as

$$\Psi_r(x) - \Psi_b(x) = U_2 - U^* - \frac{1}{s}\left[\mathrm{TS}_D(x) - U_2 + \bar{c}_0 - \alpha\right] - \frac{1+s}{s} t\,|x - b_2| \ge 0$$

where $S_D(x)$ can be shown to be convex on the intervals $(0, b_1)$ and (b_3, b): $S_D'' = G_D'' - \frac{1}{1-\sigma}\left(\ln T_D^{1-\sigma}\right)''$ where $G_D'' = (1-\sigma)^2\tau^2 G_D > 0$ and $(\ln T_D^{1-\sigma})'' = \left[(T_D^{1-\sigma})'' T_D^{1-\sigma} - (T_D^{1-\sigma})'^2\right] / T_D^{2(1-\sigma)}$; the last expression is equal to 0 on (b_3, b) and to $\left[2(\sigma-1)\tau/(sT_D^{1-\sigma})\right]^2 \int_{-b_3}^{-b_1} e^{(1-\sigma)\tau(y-z)}dz \int_{-b_3}^{-b_1} e^{(1-\sigma)\tau(z-y)}dz > 0$ on $(0, b_1)$. Then, $\Psi_r(x) - \Psi_b(x)$ is concave on those two intervals. Since $\Psi_r(b_1) - \Psi_b(b_1) = 0$, we have $\Psi_r(x) - \Psi_b(x) \ge 0$ for all $x \in (0, b_1)$ if and only if $\Psi_r(0)\Psi_b(0) \ge 0$, which can be rewritten as $t \le \hat{t}_D \equiv (\alpha+\beta)[S_D(b_3) - S_D(0)]/[(2b_2 - b_3)(1+s)]$. Similarly, since $\Psi_r(b_3) - \Psi_b(b_3) = 0$, we have $\Psi_r(x) - \Psi_b(x) \ge 0$ for all $x \in (b_3, b)$ if and only if $\Psi_r(b) - \Psi_b(b) \ge 0$, which can be rewritten as $t \le \bar{t}_D \equiv (\alpha+\beta)\,[S_D(b_3) - S_D(b)]/\left[(b - b_3)(1+s)\right]$. This means that condition (*ii*) reduces to $t \le \max\{\hat{t}_D, \bar{t}_D\}$. The equilibrium inequality condition (*i*) remains as it is: $\Psi_b(y) \ge \Psi_r(y)$ for all $y \in \mathcal{Y}$.

It can easily be shown that b_1 should not exceed $N/4$. Otherwise, the residential bid rent would be negative at $x = 0$.

Finally, we can express the above utility U^* in (84) as a function of b_3 as follows. Using (82) and the zero profit condition $\pi(y) = \pi_D^n(y) + \pi_D^u(y) - w(y) - sR(y) = 0$ we can write

$$U_2 - t\,|y - b_2| - \bar{c}_0 + sR(y) = \mathrm{TS}_D(y) - \alpha$$

Applying this at $y = b_1$ or $y = b_3$ gives

$$-t(b_2 - b_1) + t(b_3 - b_2) + s\left[R(b_1) - R(b_3)\right] - (\alpha + \beta)\left[S(b_1) - S(b_3)\right] = 0$$

In the spatial equilibrium, the residential bid rents (83) at the residential district borders $x = b_1$ and $x = b_3$ become the equilibrium land rents so that $R(b_1) - R(b_3) = \Psi_r(b_1) - \Psi_r(b_3) = t(b_3 - b_2) - t(b_2 - b_1)$. Inserting in the previous identity gives the main relationship between borders of residential and business districts:

$$(1 + s)t\left[(b_3 - b_2) - (b_2 - b_1)\right] = (\alpha + \beta)\left[S(b_1) - S(b_3)\right]$$

The LHS corresponds to the difference in the commuting cost to workplace b_2 from district borders b_3 and b_1, while the RHS is proportional to the surplus difference between locations b_3 and b_1. This means that these commuting costs and surplus differences should balance. Then, by using (44), we get equation (47) for b_1 as stated in Proposition 6.

The utility (84) becomes

$$U^* = \mathrm{TS}_D(b_3) - (1 + s)t(b - b_3) + \overline{c}_0 - \alpha$$

Similarly,

$$U_2 = \mathrm{TS}_D(b_3) + \overline{c}_0 - \alpha - t(b_2 + bs - (1 + s)b_3)$$

$$\Psi_b(y) = \frac{1}{s}((\alpha + \beta)(S_D(y) - S_D(b_3)) + (1 + s)t((b_3 - b_2) - 2(b_2 - b_1)) + st(b - b_2) + t(y - b_2)), \quad y \in (b_1, b_3)$$

$$w(y) = \mathrm{TS}_D(b_3) + (1 + s)t(b_3 - b_2) - st(b - b_2) - v_D(y) - t(y - b_2), \quad y \in (b_1, b_3)$$

References

Ahlfeldt, G.M., Redding, S.J., Sturm, D.M., Wolf, N., 2015. The economics of density: evidence from the berlin wall. Econometrica 83 (6), 2127–2189.

Allen, T., Arkolakis, C., 2014. Trade and the topography of the spatial economy. Q. J. Econ. 129 (3), 1085–1140.

Allen, T., Arkolakis, C., Li, X., 2015. Optimal City Structure. Mimeo.

Alonso, W., 1964. Location and Land Use. Harvard University Press, Cambridge, MA.

Berliant, M., Peng, S.-K., Wang, P., 2002. Production externalities and urban configuration. J. Econ. Theor. 104, 275–303.

Carlier, G., Ekeland, I., 2007. Equilibrium structure of a bidimensional asymmetric city. Nonlinear Anal. R. World Appl. 8, 725–748.

Combes, P.P., Gobillon, L., 2015. The empirics of agglomeration economies. Handbook of Regional and Urban Economics, vol. 5, pp. 247–348.

Dong, X., Ross, S.L., 2015. Accuracy and efficiency in simulating equilibrium land-use patterns for self-organizing cities. J. Econ. Geogr. 15 (4), 707–722.

Duranton, G., Puga, D., 2014. The growth of cities. In: Handbook of Economic Growth2, pp. 781–853.

Eaton, J., Kortum, S., 2002. Technology, geography, and trade. Econometrica 70 (5), 1741–1779.

Ellison, G., Glaeser, E.L., 1999. The geographic concentration of industry: does natural advantage explain agglomeration? Am. Econ. Rev. 89 (2), 311–316.

Fujita, M., Ogawa, H., 1982. Multiple equilibria and structural transition of non-monocentric urban configurations. Reg. Sci. Urban Econ. 12, 161–196.

Fujita, M., Krugman, P., Mori, T., 1999. On the evolution of hierarchical urban systems. Eur. Econ. Rev. 43, 209–251.

Fujita, M., Thisse, J.-F., 2013. Economics of Agglomeration: Cities, Industrial Location, and Globalization. Economics of Agglomeration: Cities, Industrial Location, And Globalization. Cambridge University Press, Cambridge.

Glaeser, E.L., Kolko, J., Saiz, A., 2001. Consumer city. J. Econ. Geogr. 1 (1), 27–50.

Handbury, J., Weinstein, D.E., 2015. Goods prices and availability in cities. Rev. Econ. Stud. 82 (1), 258–296.

Kolko, J., 2010. Urbanization, agglomeration, and coagglomeration of service industries. In: Glaeser, E.L. (Ed.), Agglomeration Economics. University of Chicago Press, Chicago, pp. 151–180.

Krugman, P.R., 1993. Geography and Trade. MIT press, Cambridge, MA.

Krugman, P., Venables, A.J., 1995. Globalization and the inequality of nations. Q. J. Econ. 110 (4), 857–880.

Lucas, R., Rossi-Hansberg, E., 2002. On the internal structure of cities. Econometrica 70, 1445–1476.

Monte, F., Redding, S.J., Rossi-Hansberg, E., 2018. Commuting, migration, and local employment elasticities. Am. Econ. Rev. 108 (12), 3855–3890.

Ogawa, H., Fujita, M., 1980. Equilibrium land use patterns in a nonmonocentric city. J. Reg. Sci. 20, 455–475.

Pflüger, M.P., 2004. A simple, analytically solvable Chamberlinian agglomeration model. Reg. Sci. Urban Econ. 34, 565–573.

Picard, P.M., Tabuchi, T., 2013. On microfoundations of the city. J. Econ. Theor. 148 (6), 2561–2582.