Weak Representability of Actions of Non-Associative Algebras

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Joint work with Xabier García Martínez (*Universidade de Vigo*),
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- Semi-abelian categories were introduced in order to provide a categorical setting which would capture algebraic properties of groups, rings and algebras.
- Let $\mathcal{C}$ be a semi-abelian category (i.e. $\mathcal{C}$ is pointed, Barr-exact and Bourn-protomodular with finite coproducts).
- Let X, B be objects of $\mathcal{C}$. A *split extension* of B by X is a diagram

$$0 \to X \xrightarrow{k} A \underset{\alpha}{\overset{\beta}{\leftrightarrows}} B \to 0$$

in $\mathcal{C}$ such that $\alpha \circ \beta = \mathrm{id}_B$ and (X, k) is a kernel of α.

- For any object X in $\mathcal{C}$, we consider the functor

$$\mathrm{SplExt}(-,X)\colon \mathcal{C}^{op} \to \mathbf{Set}$$

where, for every object B in $\mathcal{C}$, $\mathrm{SplExt}(B,X)$ is the set of isomorphism classes of split extensions of B by X.

- Recall that we have an an equivalence between split extensions and actions and a canonical functor isomorphism

$$\mathrm{SplExt}(-,X) \cong \mathrm{Act}(-,X).$$

Action Representable Categories

Definition (F. Borceux, G. Janelidze and G. M. Kelly, 2005)

A semi-abelian category $\mathcal{C}$ is *action representable* if for every object X in $\mathcal{C}$, the functor $\mathrm{SplExt}(-, X)$ is representable. This means that there exists an object $[X]$ of $\mathcal{C}$, called the *actor* of X, and a natural isomorphism

$$\mathrm{SplExt}(-, X) \cong \mathrm{Hom}_{\mathcal{C}}(-, [X]).$$

- The prototype examples are the category **Grp** of groups and the category $\mathbf{LieAlg}_{\mathbb{F}}$ of Lie algebras.
- In **Grp** it is well known that every split extension of B by X is represented by a homomorphism $B \to \mathrm{Aut}(X)$, where the actor $\mathrm{Aut}(X)$ of X is the group of automorphisms of the group X.
- In $\mathbf{LieAlg}_{\mathbb{F}}$, a split extension of B by X is represented by a homomorphism $B \to \mathrm{Der}(X)$, where $\mathrm{Der}(X)$ is the Lie algebra of derivations of X.

- The notion of action representable category has proven to be quite restrictive.

Theorem (X. García Martínez, M. Tsishyn, T. Van der Linden and C. Vienne, 2021)

Let $\mathbb{F}$ be an infinite field with $\mathrm{char}(\mathbb{F}) \neq 2$ and let $\mathcal{V}$ be a non-abelian variety of non-associative algebras over $\mathbb{F}$. If $\mathcal{V}$ is action representable, then $\mathcal{V}$ must be the category $\mathbf{LieAlg}_{\mathbb{F}}$ of Lie algebras over $\mathbb{F}$.

- More recently G. Janelidze introduced the notion of *weakly action representable category*, which includes a wider class of semi-abelian categories.

Weakly Action Representable categories

Definition (G. Janelidze, 2022)

A semi-abelian category $\mathcal{C}$ is *weakly action representable* if for every object X in $\mathcal{C}$, the functor $\mathsf{SplExt}(-,X)$ admits a weak representation. This means that there exist an object T of $\mathcal{C}$ and a monomorphism of functors

$$\tau\colon \ \mathsf{SplExt}(-,X) \rightarrowtail \mathsf{Hom}_{\mathcal{C}}(-,T).$$

An object T as above is called *weak actor* of X, the pair (T,τ) is called *weak representation* of $\mathsf{SplExt}(-,X)$ and $(\varphi\colon B \to T) \in \mathsf{Im}(\tau_B)$ is called *acting morphism*.

Example

Every action representable category $\mathcal{C}$ is weakly action representable. In this case $T = [X]$ is the actor of X and τ is a natural isomorphism.

Associative algebras and Leibniz algebras

- **G. Janelidze, 2022.** The category $\mathbf{AAlg}_{\mathbb{F}}$ of associative algebras over $\mathbb{F}$ is weakly action representable. Given an associative algebra X, a weak actor of X is the associative algebra of *bimultipliers* of X.

$$\mathrm{Bim}(X) = \{(f{*}-, -{*}f) \in \mathrm{End}(X) \times \mathrm{End}(X)^{op} \mid \cdots$$

$$\cdots \mid f{*}(xy) = (f{*}x)y, (xy){*}f = x(y{*}f), x(f{*}y) = (x{*}f)y, \ \forall x, y \in X\}.$$

- **A. S. Cigoli, M. M., G. Metere, 2023.** The category $\mathbf{LeibAlg}_{\mathbb{F}}$ of Leibniz algebras over $\mathbb{F}$ is weakly action representable. Given a Leibniz algebra X, a weak actor of X is the Leibniz algebra of *biderivations* of X.

$$\mathrm{Bider}(X) = \{(d, D) \in \mathrm{End}(X)^2 \mid d(xy) = d(x)y + xd(y), \cdots$$

$$\cdots \ D(xy) = D(x)y - D(y)x, \ xd(y) = xD(y), \ \forall x, y \in X\}$$

A counterexample: Jordan algebras

Definition

A *Jordan algebra* over a field $\mathbb{F}$ is a non-associative commutative algebra $(J, \cdot)$ over $\mathbb{F}$ which satisfies the *Jordan identity*

$$(xy)(xx) = x(y(xx)), \ \ \forall x, y \in J.$$

Theorem (G. Janelidze, 2022)

Every weakly action representable category is action accessible (in the sense of D. Bourn and G. Janelidze).

Remark (A. S. Cigoli and S. Mantovani, 2012)

The category of Jordan algebras is not action accessible.

Conclusion: The category of Jordan algebras over $\mathbb{F}$ is not weakly action representable.

- Let $\mathcal{V}$ be a variety of non-associative algebras over a field $\mathbb{F}$, with $\mathrm{char}(\mathbb{F}) \neq 2$;
- We suppose that $\mathcal{V}$ is *operadic*, i.e. all the identities of $\mathcal{V}$ are *multilinear*. This happens, for instance, when $\mathrm{char}(\mathbb{F}) = 0$.
- We also suppose that $\mathcal{V}$ is *action accessible* (or equivalently *algebraically coherent*);
- This is equivalent to saying that the λ/μ *rules* holds, i.e. there exists $\lambda_1, \dots \lambda_8, \mu_1, \dots \mu_8 \in \mathbb{F}$ such that

$$x(yz) = \lambda_1(xy)z + \lambda_2(yx)z + \lambda_3 z(xy) + \lambda_4 z(yx) +$$

$$+\lambda_5(xz)y + \lambda_6(zx)y + \lambda_7 y(xz) + \lambda_8 y(zx),$$

$$(yz)x = \mu_1(xy)z + \mu_2(yx)z + \mu_3 z(xy) + \mu_4 z(yx) +$$

$$+\mu_5(xz)y + \mu_6(zx)y + \mu_7 y(xz) + \mu_8 y(zx)$$

are identities in $\mathcal{V}$.

The commutative case

- Let $\mathcal{V}$ be an operadic and action accessible variety of commutative algebras over $\mathbb{F}$. The λ/μ rules reduce to

$$x(yz) = \alpha(xy)z + \beta(xz)y,$$

for some $\alpha, \beta \in \mathbb{F}$.

Proposition

$\mathcal{V}$ is associative or satisfies the Jacobi identity ($x(yz) + y(zx) + z(xy) = 0$).

Corollary

If $\mathcal{V}$ satisfies no identities of degree $k \geq 4$, then $\mathcal{V} = \mathbf{CAAlg}_{\mathbb{F}}$, $\mathcal{V} = \mathbf{JJAlg}_{\mathbb{F}}$ or $\mathcal{V} = \mathbf{Nil}_2(\mathbf{CAAlg}_{\mathbb{F}})$.

Commutative associative algebras

Remark

Let X be a commutative associative algebra. Then

$$\mathrm{SplExt}(-, X) \cong \mathrm{Hom}_{\mathbf{AAlg}_{\mathbb{F}}}(-, \mathrm{M}(X)),$$

where $\mathrm{M}(X)$ is the associative algebra of *multipliers* of X.

Theorem (X. García Martínez, M. M., T. Van der Linden, C. Vienne, 2023)

Let X be a commutative associative algebra. T.F.A.E.:

(i) $\mathrm{M}(X)$ *is a commutative associative algebra;*

(ii) *the functor* $\mathrm{SplExt}(-, X)$ *admits a weak representation;*

(iii) $\mathrm{M}(\mathrm{X})$ *is the actor of* X*, hence* $\mathrm{SplExt}(-, X)$ *is representable.*

Example

If $\mathrm{Ann}(X)$ is trivial or $X^2 = X$, then $\mathrm{M}(X)$ is a commutative associative algebra.

Example

If $X = \mathbb{F}^2$ is the abelian two-dimensional commutative associative algebra, then $\mathrm{M}(X) = \mathrm{End}(X)$ is not commutative.

Corollary (X. García Martínez, M. M., T. Van der Linden and C. Vienne, 2023)

The variety $\mathbf{CAAlg}_{\mathbb{F}}$ *of commutative associative algebras over* $\mathbb{F}$ *is not weakly action representable.*

Jacobi-Jordan algebras

Definition

A commutative algebra X is called a *Jacobi-Jordan* algebra if it satisfies the *Jacobi identity*

$$x(yz) + y(zx) + z(xy) = 0, \ \ \forall x, y, z \in X.$$

Definition

An *anti-derivation* is a linear map $d\colon X \to X$ such that

$$d(xy) = -d(x)y - d(y)x, \ \ \forall x, y \in X.$$

Example

The left multiplications $l_x = x \cdot -$, $x \in X$, are particular anti-derivations, called *inner anti-derivations*.

Split Extensions

- Let

$$0 \to X \xrightarrow{i} A \underset{\pi}{\overset{s}{\leftrightarrows}} B \to 0$$

be a split extension of Jacobi-Jordan algebras.

- We have that

$$A \cong (B \oplus X, \cdot)$$

and the Jacobi-Jordan algebra structure is given by

$$(b, x) \cdot (b', x') = (bb', xx' + b * x' + b' * x).$$

- The action of B on X is defined by

$$b * x = s(b) \cdot_A i(x), \;\; \forall b \in B, \; \forall x \in X.$$

Conversely, if we define the product $(\cdot)$ on $B \oplus X$ using the formula above, we have the following characterization.

Proposition

$(B \oplus X, \cdot)$ is a Jacobi-Jordan algebra if and only if

1. $b * (xx') = -(b * x)x' - (b * x) * x'$;
2. $(bb') * x = -b * (b' * x) - b' * (b' * x)$;

for every $b, b' \in B$ and $x, x' \in X$.

In an equivalent way, a split extension of B by X (or an action of B on X) is a linear map

$$B \to \mathrm{ADer}(X), \;\; b \mapsto b * -, \;\; \forall b \in B,$$

such that

$$(bb') * - = -\{b * -, b' * -\}, \;\; \forall b, b' \in B,$$

where $\{f, f'\} = f \circ f' + f' \circ f$ denotes the *anti-commutator* between two linear endomorphisms of X.

Remark

The space $\mathrm{ADer}(X)$ endowed with the anti-commutator $\{-,-\}$ is not in general a Jacobi-Jordan algebra.

Example

Let $X = \mathbb{F}$ be the abelian one-dimensional Jacobi-Jordan algebra. Then $\mathrm{ADer}(X) = \mathrm{End}(X) \cong \mathbb{F}$ does not satisfy the Jacoby identity.

Definition

A *partial algebra* is a vector space X endowed with a *bilinear partial operation*

$$\cdot \colon \Omega \to X,$$

where $\Omega \subseteq X \times X$. An homomorphism of partial algebras is a linear map $f \colon X \to Y$ such that $f(xy) = f(x)f(y)$, for every $x, y \in X$ such that xy and $f(x)f(y)$ are defined.

Theorem (X. García Martínez, M. M., T. Van der Linden and C. Vienne, 2023)

Let X be a Jacobi-Jordan algebra. Then

(i) $\mathrm{SplExt}(-, X) \cong \mathrm{Hom}_{\mathbf{PAlg}_{\mathbb{F}}}(-, \mathrm{ADer}(X))$;

(ii) *if* $\mathrm{ADer}(X)$ *is a Jacobi-Jordan algebra, then it is the actor of* X*;*

(iii) *if* $\mathrm{SplExt}(-, X)$ *admits a weak representation, then* $\mathrm{ADer}(X)$ *satisfies the Jacobi identity.*

Corollary (X. García Martínez, M. M., T. Van der Linden and C. Vienne, 2023)

The variety $\mathbf{JJAlg}_{\mathbb{F}}$ *of Jacobi-Jordan algebras over* $\mathbb{F}$ *is not weakly action representable.*

Two-step nilpotent commutative associative algebras

Theorem (X. García Martínez, M. M., T. Van der Linden and C. Vienne, 2023)

- *The variety* $\mathbf{Nil}_2(\mathbf{CAAlg}_{\mathbb{F}})$ *is weakly action representable;*
- *A weak actor of an object* X *of* $\mathbf{Nil}_2(\mathbf{CAAlg}_{\mathbb{F}})$ *is the abelian algebra*

$$[X]_2 = \{f \in \mathrm{End}(X) \mid f(xy) = f(x)y = 0, \ \forall x \in X\}.$$

- *An arrow* $B \to [X]_2$*, defined by* $b \mapsto b * -$*, is an acting morphism if and only if*

$$b * (b' * x) = 0, \ \ \forall b, b' \in B, \ \forall x \in X. \tag{1}$$

The non-commutative case

- Let $\mathcal{V}$ be an operadic and action accessible variety of non-associative algebras over $\mathbb{F}$ and let

$$\Phi_{k,i}(x_1,\dots,x_k)=0, \quad i=1,\dots,n,$$

where k is the degree of Φ_k, be the multilinear identities which define $\mathcal{V}$.

- We look for a vector space $[X]$ such that

$$\mathrm{Inn}(X)=\{(x\cdot -, -\cdot x)\mid x\in X\}\leq [X]\leq \mathrm{End}(X)^2$$

and we want to endow it with a bilinear partial operation

$$\langle -,-\rangle\colon \Omega\subseteq X\times X\to X,$$

such that we can associate in a natural way a morphism of partial algebras $B\to[X]$, with every object B in $\mathcal{V}$ and every split extension of B by X.

Split Extensions

- Let
$$0 \to X \xrightarrow{i} A \underset{\pi}{\overset{s}{\leftrightarrows}} B \to 0$$
be a split extension in $\mathcal{V}$.
- We have that
$$A \cong (B \oplus X, \cdot)$$
with
$$(b, x) \cdot (b', x') = (bb', xx' + b * x' + x * b')$$
- The action of B on X is defined by the pair of bilinear maps
$$l \colon B \times X \to X, \quad r \colon X \times B \to X,$$
where $b * x' = l(b, x') = s(b) \cdot_A i(x')$, $x * b' = r(x, b') = i(x) \cdot_A s(b')$.

Conversely, if we define the product $(\cdot)$ on $B \oplus X$ as before, we have:

Proposition

$(B \oplus X, \cdot)$ is an object of $\mathcal{V}$ if and only if

$$\Phi_{k,i}(\alpha_1, \dots, \alpha_k) = 0, \;\; \forall i = 1, \dots, n,$$

where $\alpha_1, \dots, \alpha_k$ are either elements of X or elements of the form $b * -$, $- * b$, with $b \in B$. The resulting algebra is the semi-direct product of B and X and it is denoted by $B \ltimes X$.

The partial algebra $[X]$

We define $[X]$ as the subspace of all pairs $(f * -, - * f) \in \mathrm{End}(X)^2$ satisfying

$$\Phi_{k,i}(\alpha_1, \dots, \alpha_k) = 0, \ \ \forall i = 1, \dots, n,$$

where $\exists\, j = 1, \dots, n$ such that $\alpha_j \in \{f * -, - * f\}$ and $\alpha_t \in X$, $\forall t \neq j$. We endow $[X]$ with the bilinear map $\langle -, - \rangle \colon [X] \times [X] \to \mathrm{End}(X)^2$

$$\langle (f * -, - * f), (g * -, - * g) \rangle = (h * -, - * h),$$

where

$$h * x = \mu_1 (x * f) * g + \cdots + \mu_8 f * (g * x),$$
$$x * h = \lambda_1 (x * f) * g + \cdots + \lambda_8 g * (f * x).$$

Example

If $\mathcal{V} = \mathbf{AAlg}_{\mathbb{F}}$ (resp. $\mathcal{V} = \mathbf{LeibAlg}_{\mathbb{F}}$)), then $[X] = \mathrm{Bim}(X)$ (resp. $[X] \cong \mathrm{Bider}(X)$).

Remark

A split extension of B by X in $\mathcal{V}$ is the same thing as a linear map

$$B \to [X], \;\; b \mapsto (b * -, - * b), \;\; \forall b \in B,$$

such that $((bb') * -, - * (bb')) = \langle (b * -, - * b), (b' * -, - * b') \rangle$ and

$$\Phi_{k,i}(\alpha_1, \dots, \alpha_k) = 0, \;\; \forall i = 1, \dots, n,$$

where $\alpha_1, \dots, \alpha_k$ are either elements of X or elements of the form $b * -$, $- * b$, with $b \in B$, and there exist $j, j' \in \{1, \dots n\}$ such that $\alpha_j \in \{b * -, - * b\}$, $\alpha_{j'} \in \{b' * -, - * b'\}$, with $b \neq b'$.

- The bilinear map
$$\langle -,-\rangle \colon [X]\times[X]\to \mathrm{End}(X)^2$$
defines a bilinear partial operation
$$\langle -,-\rangle \colon \Omega \to [X],$$
where $\Omega=\Omega'\times\Omega'$ and $\Omega'\subseteq[X]$ is the union of the images of all morphisms which describe split extensions with kernel X.
- Let $B\to[X]$ be a morphism in $\mathbf{PAlg}_{\mathbb{F}}$ which describes a split extension of B by X in $\mathcal{V}$. Then
$$(\mathrm{Im}(B),\langle -,-\rangle)$$
is an object of $\mathcal{V}$.
- An example is given by the morphism $\mathrm{Inn}\colon X\to[X]$, defined by $x\mapsto(x\cdot -, -\cdot x)$, for every $x\in X$.

Theorem (X. García Martínez, M. M., T. Van der Linden and C. Vienne, 2023)

Let $\mathcal{V}$ be an operadic and action accessible variety of non-associative algebras over $\mathbb{F}$, with $\mathrm{char}(\mathbb{F}) \neq 2$.

(i) *Let X be an object of $\mathcal{V}$. There exists a monomorphism of functors*

$$\tau\colon \mathsf{SplExt}(-,X) \rightarrowtail \mathsf{Hom}_{\mathbf{PAlg}_{\mathbb{F}}}(-,[X]).$$

(ii) *Let B, X be objects of $\mathcal{V}$. An morphism of partial algebras $B \to [X]$, defined by $b \mapsto (b * -, - * b)$, for every $b \in B$, belongs to $\mathrm{Im}(\tau_B)$ if and only if $\Phi_{k,i}(\alpha_1, \ldots, \alpha_k) = 0$, as before.*

(iii) *If $([X], \langle -,- \rangle)$ is an object of $\mathcal{V}$, then $([X], \tau)$ becomes a weak representation of $\mathsf{SplExt}(-,X)$.*

(iv) *If $\mathcal{V}$ is a variety of commutative or anti-commutative algebras, then*

$$[X] \cong \{ f \in \mathrm{End}(X) \mid \Phi_{k,i}(f, x_2, \ldots, x_k) = 0, \ \forall x_2, \ldots, x_k \in X \}.$$

- The problem of characterizing varieties of non-associative algebras which are *weakly action representable* is still open. However the construction of $[X]$ shows a possible and explicit way to represent actions in any variety of non-associative algebras over $\mathbb{F}$;
- The converse of the implication

$$\textit{weakly action representable} \Rightarrow \textit{action accessible}$$

is not valid also in the context of varieties of non-associative algebras.

Degree two id.	**Act. Accessible**	**W. Act. Rep.**	**Act. Rep.**
$xy = yx$	$\mathbf{CAAlg}_{\mathbb{F}}$, $\mathbf{JJAlg}_{\mathbb{F}}$	$\mathbf{Nil}_2(\mathbf{CAAlg}_{\mathbb{F}})$	$\mathbf{AbAlg}_{\mathbb{F}}$
$xy = -yx$	$\mathbf{Nil}_k(\mathbf{LieAlg}_{\mathbb{F}})$, $k \geq 3$	$\mathbf{Nil}_2(\mathbf{LieAlg}_{\mathbb{F}})$	$\mathbf{LieAlg}_{\mathbb{F}}$
NO	$\mathbf{Nil}_k(\mathbf{AAlg}_{\mathbb{F}})$, $k \geq 3$	$\mathbf{AAlg}_{\mathbb{F}}$, $\mathbf{LeibAlg}_{\mathbb{F}}$	**NO**

Thank you for your attention!

A. S. Cigoli, M. Mancini and G. Metere, "On the representability of actions of Leibniz algebras and Poisson algebras", *submitted. Preprint available at arXiv:2302.05175* (2023).

X. García Martínez, M. Tsishyn, T. Van der Linden, and C. Vienne, "Algebras with representable representations", *Proceedings of the Edinburgh Mathematical Society* **64** (2021), No. 2, pp. 555-573.

G. Janelidze, "Central extensions of associative algebras and weakly action representable categories", *Theory and Applications of Categories* **38** (2022), No. 36, pp 1395-1408.