A quick refresher (for my personal use) on why the smallness of the aspect ratio leads to the hydrostatic approximation in geophysical and environmental fluid flow models

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This is about the Saint-Venant — or shallow water — equations. I am outlining hereinafter how I am used to derive their inviscid version from the general Euler equations. Most of the flows I have been dealing with so far are characterised by a very small aspect ratio, which is probably why I feel inclined to see the smallness of this dimensionless ratio as the key hypothesis. Accordingly, the shallow water equations simply appear as boundary layer equations, bearing many similarities with those established in general fluid mechanics textbooks (e.g. Kundu et al. 2012).

In the present context, the aspect ratio δ is defined as follows:

$$\delta = \frac{\text{vertical length scale}}{\text{horizontal length scale}} = \frac{L_v}{L_h} \quad . \tag{1}$$

The vertical length scale cannot exceed the water column depth, whilst the horizontal length scale is at most of the order of magnitude of the (horizontal) size of the domain of interest. For the North Sea, for instance, the aspect ratio relevant to the propagation of the M2 tide is as follows:

$$\delta \approx \frac{10^2 \text{ m}}{10^2 \text{ km}} \approx 10^{-3} \quad . \tag{2}$$

For larger-scale flows, taking place in a ocean basin such the as the Atlantic or the Pacific, the aspect ratio presumably is of the same order of magnitude:

$$\delta \approx \frac{10^3 \text{ m}}{10^3 \text{ km}} \approx 10^{-3} \quad . \tag{3}$$

In a smaller-size domain of interest like the Scheldt Estuary, one obtains

$$\delta \approx \frac{10 \text{ m}}{10 \text{ km}} \approx 10^{-3} \quad . \tag{4}$$

Clearly, for most geophysical and environmental (water) flows, the aspect ratio is small.

The smallness of the aspect ratio suggests treating in a different manner the variables and differential operators associated with the vertical direction on one hand and the horizontal one on the other hand. Accordingly, the three-dimensional water velocity is to be written as follows:

$$\mathbf{v} = \mathbf{u} + w\mathbf{e}_z \quad , \tag{5}$$

where $\mathbf{u}$ denotes the horizontal velocity vector; $\mathbf{e}_z$ is a unit vector, pointing upwards, and w is the vertical component of the velocity ($w = \mathbf{v} \bullet \mathbf{e}_z$). Similarly, the del operator ∇ is to be split into its horizontal and vertical parts:

$$\nabla = \nabla_h + \mathbf{e}_z \frac{\partial}{\partial z} \quad , \tag{6}$$

where vertical coordinate z grows upward and is zero that the equilibrium altitude of the free surface.

For the sake of simplicity, the water density is assumed to be constant[1]. Therefore, using the notations (5)-(6), the continuity equation reads

$$\nabla_h \bullet \mathbf{u} + \frac{\partial w}{\partial z} = 0 \quad . \tag{7}$$

Let U and W represent the order of magnitude of the horizontal and vertical velocities. Then, the scaling of (7) yields

$$\underbrace{\nabla_h \bullet \mathbf{u}}_{\approx \frac{U}{L_h}} = -\underbrace{\frac{\partial w}{\partial z}}_{\approx \frac{W}{L_v}} \quad \Rightarrow \quad \frac{U}{L_h} \approx \frac{W}{L_v} \quad \Rightarrow \quad W \approx \underbrace{\frac{L_v}{L_h}}_{=\delta} U \approx \delta W \quad . \tag{8}$$

The vertical velocity is much smaller than the horizontal one, making it difficult to measure w in the field. This is why inverse or indirect methods are often resorted to in order to estimate the vertical velocity field.

To evaluate the order of magnitude of time derivatives, it is necessary to introduce the relevant timescale, which will be denoted T hereinafter. In many instances, it is sufficient to opt for the so-called advective timescale, i.e.

$$T \approx \frac{L_h}{U} \approx \frac{L_v}{W} \quad . \tag{9}$$

As a consequence, the material derivative operator, which is defined to be

$$D_t = \frac{\partial}{\partial t} + \mathbf{u} \bullet \nabla_h + w \frac{\partial}{\partial z} \quad , \tag{10}$$

scales as follows

$$D_t = \overbrace{\underbrace{\frac{\partial}{\partial t}}_{\approx 1/T} + \underbrace{\mathbf{u} \bullet \nabla_h}_{\approx U/L_h} + \underbrace{w \frac{\partial}{\partial z}}_{\approx W/L_v}}^{\approx 1/T \approx U/L_h \approx W/L_v} \quad . \tag{11}$$

Let p and g denote the pressure and the gravitational acceleration, respectively. The inviscid momentum equation (in a non-rotating framework) is

$$D_t \mathbf{v} = -\frac{1}{\rho} \nabla p - g \mathbf{e}_z \quad . \tag{12}$$

Splitting it into its horizontal and vertical part yields

$$D_t \mathbf{u} = -\frac{1}{\rho} \nabla_h p \tag{13}$$

and

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[1] In fact, it is sufficient to assume that the Boussinesq approximation holds valid. However, this would result in unnecessary additional complexity of the mathematical developments.

$$D_t w = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \quad . \tag{14}$$

In most cases, the gravitational acceleration is several orders of magnitude greater than the vertical acceleration $D_t w$, implying that in first estimate the fluid is in hydrostatic equilibrium. Therefore, it is appropriate to define a reference state of the fluid that is in hydrostatic equilibrium:

$$\mathbf{v}_\bullet = 0 \ , \ \ p_\bullet = -\rho_\bullet g z \quad , \tag{15}$$

where the subscript "•" refers to the abovementioned reference state. It is convenient to introduce the so-called reduced pressure,

$$q = \frac{p - p_\bullet}{\rho_\bullet} = \frac{p}{\rho_\bullet} + gz \quad , \tag{16}$$

which is a measure of the discrepancy between the actual pressure p and the reference (hydrostatic) pressure $p_\bullet$. The density was assumed to be constant. Thus, it is appropriate so set $\rho = \rho_\bullet$. Using the reduced pressure, momentum equation (13)-(14) transform to

$$D_t \mathbf{u} = -\nabla_h q \tag{17}$$

and

$$D_t w = -\frac{\partial q}{\partial z} \quad . \tag{18}$$

The range of variation of the pressure is much smaller in the horizontal direction than in the vertical one. However, it is reasonable to assume that that no such consideration applies to the reduced pressure. The latter may be safely assumed to exhibit horizontal variations that are of the same order of magnitude as the vertical ones. Accordingly, if Q denotes the range of variation of the reduced pressure, then (17) scales as follows:

$$\underbrace{D_t \mathbf{u}}_{\approx U/T} = -\underbrace{\nabla_h q}_{\approx Q/L_h} \quad , \tag{19}$$

yielding

$$Q \approx \frac{L_h U}{T} \approx \frac{L_h U}{L_h / U} \approx U^2 \quad . \tag{20}$$

Substituting this result into momentum equation (18) leads to

$$\frac{|D_t w|}{\left|\frac{\partial q}{\partial z}\right|} \approx \frac{W/T}{Q/L_v} \approx \delta^2 \quad , \tag{21a}$$

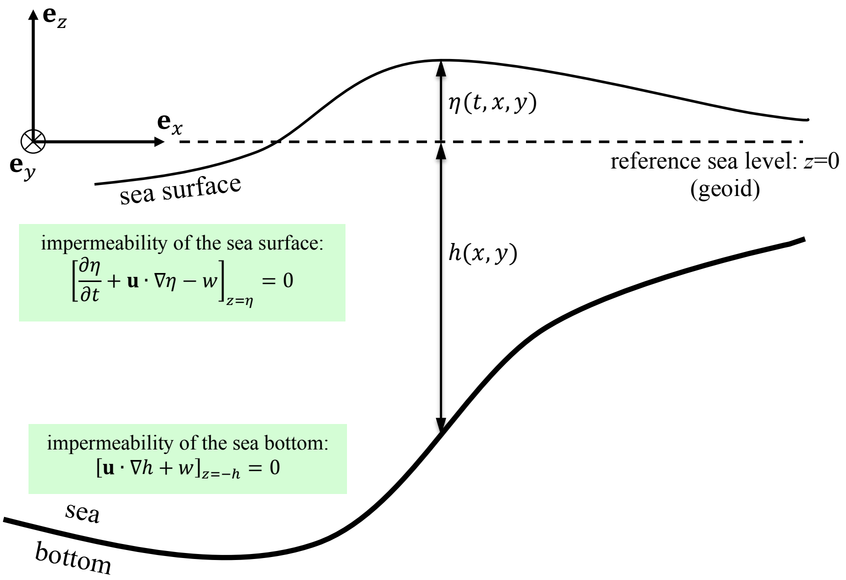
which transforms to the compact formulation

$$|D_t w| \approx \delta^2 \left|\frac{\partial q}{\partial z}\right| \tag{21b}$$

This clearly suggests that the vertical acceleration is of order δ^2 as compared to the vertical gradient of the reduced pressure. In other words, if the aspect ratio is $\delta \approx 10^{-3}$ as in the abovementioned example, then the vertical acceleration $D_t w$ is one million times smaller than the vertical gradient of the reduced pressure, $\partial q / \partial z$. Therefore, is legitimate to simplify the vertical momentum equation to the hydrostatic balance

$$0 = -\frac{\partial q}{\partial z} \tag{22}$$

It is worth stressing that the simplified vertical momentum equation (22) applies when the square of the aspect ratio (and not the aspect ratio *per se*) is small. Thus, for instance, if $\delta \approx 1/3$, a value which is not that small, the square of the aspect ratio is of order 10^{-1}, implying that the hydrostatic balance (22) still applies.



The lower boundary of the domain is located at $z=-h$, where h is the unperturbed depth of the water column. The ocean-air interface is at $z=\eta$, where η is positive if the free surface is above the reference level (see figure above). At the top of the water column, the pressure is equal to the atmospheric pressure p_a. Combining this constraint with definition (16) of the reduced pressure leads to

$$[q]_{z=\eta} = \left[\frac{p-p_\bullet}{\rho_\bullet}\right]_{z=\eta} = \left[\frac{p}{\rho_\bullet} + gz\right]_{z=\eta} = \frac{p_a}{\rho_\bullet} + g\eta \quad . \tag{23}$$

Since hydrostatic balance (22) implies that the reduced pressure is independent of the vertical coordinate, the reduced pressure and its horizontal gradient read

$$q = \frac{p_a}{\rho_\bullet} + g\eta \quad \Rightarrow \quad \nabla_h q = \frac{1}{\rho_\bullet}\nabla_h p_a + g\nabla_h \eta \quad . \tag{24}$$

Substituting (24) into (17), one obtains

$$\boxed{D_t\mathbf{u} = -\frac{1}{\rho_\bullet}\nabla_h p_a - g\nabla_h \eta} \tag{25}$$

This equation can be averaged over the height of the water column, so as to obtain the Saint-Venant (horizontal) momentum equation. When doing so, the pressure force, i.e. the right-hand side of (25) is left unchanged, for it is independent of the vertical coordinate.

From now on, the horizontal velocity will be considered depth-independent. Next, a simple inspection of continuity equation (7) indicates that the vertical velocity must be a linear function of the vertical coordinate, which may be cast into the following form

$$w \;=\; w_{-h} \;+\; \left(w_{-\eta} - w_{-h}\right)\frac{z+h}{\eta+h} \;\;, \tag{26}$$

where w_{-h} and w_{η} denote the vertical velocity at the lower and upper boundaries of the water column, respectively. These vertical velocities may be derived from the impermeability conditions of the lower and upper boundaries of the water column (see figure above):

$$w_{-h} \;=\; -\mathbf{u}\bullet\nabla_h h \;=\; -D_t h \;\;, \tag{27}$$

and (e.g. Deleersnijder 2014)

$$w_{\eta} \;=\; \frac{\partial\eta}{\partial t} + \mathbf{u}\bullet\nabla_h \eta \;=\; D_t \eta \;\;. \tag{28}$$

The relative height is defined as follows:

$$\sigma \;=\; \frac{z+h}{\eta+h} \;\;. \tag{29}$$

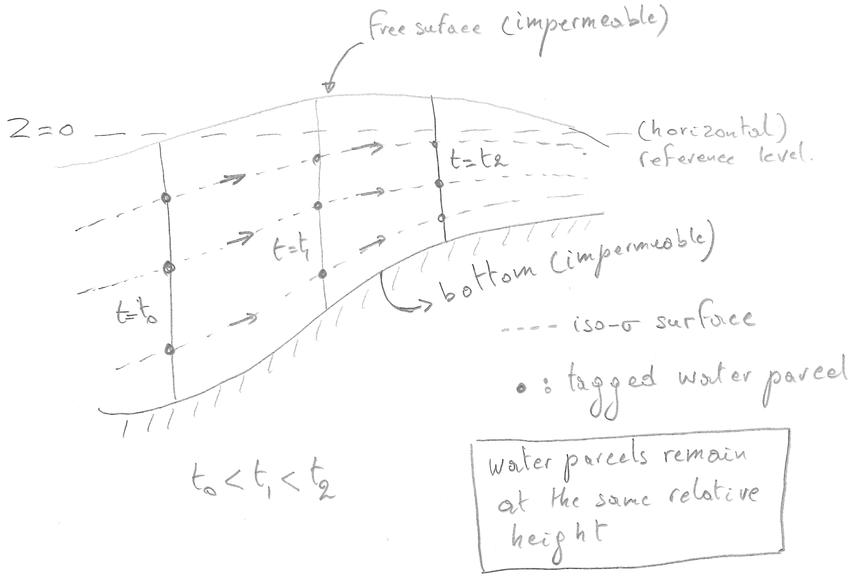
For instance, the relative height of a water parcel located at the bottom, in the middle or at the top of the water column is 0, 1/2 and 1, respectively. Then, using (26)-(29), it may be seen that the following relation holds valid:

$$\begin{aligned}(\eta+h)D_t\sigma \;&=\; \overbrace{D_t z}^{w} \;+\; \overbrace{D_t h}^{-w_{-h}} \;-\; \sigma\,\overbrace{D_t \eta}^{w_{\eta}} \;-\; \sigma\,\overbrace{D_t h}^{-w_{-h}} \\ &=\; w \;-\; \underbrace{\left[w_{-h} + \left(w_{\eta} - w_{-h}\right)\sigma\right]}_{=w,\ \text{see (26) and (29)}} \;=\; 0\end{aligned} \tag{30}$$

As a consequence, the relative height of a water parcel is conserved during the course of its motion:

$$D_t\sigma \;=\; 0 \tag{31}$$

Since the horizontal velocity was assumed to be independent of the vertical coordinate, this result implies that a water column moves as a whole. In other words, water parcels initially located on a vertical line will remain at the same relative height on a vertical line during the motion of the latter, which takes place at velocity $\mathbf{u}$. This is illustrated by the figure below.



References

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