Categorical structures enriched in a quantaloid : categories and semicategories

Isar Stubbe

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This thesis consists of two parts: a synthesis of the theory of categories enriched in a quantaloid; and a weakening of this theory for it to include semicategories describing ordered sheaves on a quantaloid.

A synthesis of, and supplements to, results in the literature concerning the theory of categories enriched in a quantaloid $\mathscr{Q}$ (as particular case of categories enriched in a bicategory) is contained in the first chapters. This theory is built with $\mathscr{Q}$-categories, functors and distributors, and contains such notions as, for example, adjoint functors, weighted colimits, presheaves, Kan extensions, Cauchy completions and Morita equivalence, and so on. The literature does not provide an overview of these matters, so it was necessary to provide one here.

Then the necessary theory is developed to arrive at an elementary description of "ordered sheaves on a quantaloid $\mathscr{Q}$", henceforth referred to as $\mathscr{Q}$-orders. As there is no "topos of sheaves on a quantaloid", $\mathscr{Q}$-orders cannot be defined as ordered objects in such a topos. Instead a description of $\mathscr{Q}$-orders as categorical structures enriched in the quantaloid $\mathscr{Q}$ is proposed. The well-known ordered sheaves on a locale Ω (i.e. ordered objects in the topos of sheaves on Ω) should of course be a particular example of the general theory, taking $\mathscr{Q}$ to be the (one-object suspension of) Ω. Then it turns out that the theory of $\mathscr{Q}$-categories has to be weakened to include "categories without units", i.e. $\mathscr{Q}$-semicategories. But for $\mathscr{Q}$-semicategories to admit a convenient distributor calculus, a "regularity" condition has to be imposed. And for those regular $\mathscr{Q}$-semicategories to admit a reasonable theory of Cauchy completions and Morita equivalence, the even stronger condition of "total regularity" has to be imposed. The former notion has been studied before for semicategories enriched in a symmetric monoidal closed category; the latter notion is new, and is introduced via the intuitively clear idea of "stability of objects". The point is then that precisely the Cauchy complete totally regular $\mathscr{Q}$-semicategories are the $\mathscr{Q}$-orders; for a locale Ω they are indeed the ordered objects in the topos of sheaves on Ω. A (bi)equivalent description of those $\mathscr{Q}$-orders can be given in terms of categories enriched in the split-idempotent completion of the quantaloid $\mathscr{Q}$: a totally regular semicategory enriched in $\mathscr{Q}$ corresponds in a precise sense to a category enriched in the split-idempotent completion of $\mathscr{Q}$. Applying this once more to a locale Ω instead of a quantaloid $\mathscr{Q}$, these results thus deepen the work of the Louvain-la-Neuve school, and reconcile it with that of the Sydney school, on the description of (ordered) sheaves on a locale as enriched categorical structures.

The extended introduction gives a compact yet intuitive presentation of the developments contained in the thesis.

“He not busy being born is busy dying!”

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Contents

Introduction

The chapters that follow this introduction contain a study of categorical structures enriched in a quantaloid. First the existing material on this subject, shattered in the literature, is brought together and streamlined to yield a powerful formalism of categories enriched in a quantaloid. Then this formalism is used, and adapted where necessary, to analyze in detail the concept of, loosely speaking, "sheaves on a quantaloid". As it happens, a theory of quantaloid-enriched semicategories – a weakening of the notion of category – had to be developed to capture the essential elements of this subject.

Unavoidably, the text contains many technicalities. But nevertheless the underlying ideas and principles – and their results – are all quite transparent when looked at from a certain angle. It therefore seemed useful to provide here a compact yet intuitive presentation of the developments contained in the text—a <u>thesis</u>, really. References to the literature are added, underlining that this work must be seen in the context of a rather recent but nonetheless fruitful mathematical tradition.

A quantaloid as base for enriched categories

The theory of categories enriched in a symmetric monoidal closed category $\mathscr{V}$ is, by now, well known [Bénabou, 1963, 1965; Eilenberg and Kelly, 1966; Lawvere, 1973; Kelly, 1982]. For such a $\mathscr{V}$ with "enough" (co)limits the theory of $\mathscr{V}$-categories, distributors and functors can be pushed as far as needed: it includes such things as (weighted) (co)limits in a $\mathscr{V}$-category, $\mathscr{V}$-presheaves on a $\mathscr{V}$-category, Kan extensions of enriched functors, Morita theory for $\mathscr{V}$-categories, and so on. We will argue below that "$\mathscr{V}$-category theory" can be generalized to "$\mathscr{Q}$-category theory", where now $\mathscr{Q}$ denotes a quantaloid. A quantaloid is a kind of bicategory, so this is really a particular case of the theory of "$\mathscr{W}$-category theory", with $\mathscr{W}$ a bicategory, as pioneered by [Bénabou, 1967; Walters, 1981; Street, 1983] and later further developed [Betti *et al.*, 1983; Gordon and Power, 1997].

From monoidal categories to bicategories

Monoidal categories are precisely one-object bicategories [Bénabou, 1967]. To wit, for a monoidal category $\mathscr{V}$, denote $\mathsf{susp}(\mathscr{V})$ – and call it the suspension of $\mathscr{V}$ – for the following bicategory: it has one object; its arrows are the objects of $\mathscr{V}$, their composition in $\mathsf{susp}(\mathscr{V})$ is given by the tensor in $\mathscr{V}$, and in particular is the identity arrow on the single object of $\mathsf{susp}(\mathscr{V})$ precisely the identity object for the tensor in $\mathscr{V}$; and the 2-cells in $\mathsf{susp}(\mathscr{V})$ correspond precisely to the arrows of $\mathscr{V}$. That $\mathsf{susp}(\mathscr{V})$ is in general a bicategory, and not a 2-category, is due to the fact that the tensor on $\mathscr{V}$ is associative, and has left and right identity, only up to natural isomorphism. (Viewing a monoidal category $\mathscr{V}$ as a one-object bicategory has some analogues that may be more familiar: for instance, a monoid is a one-object category, or more in particular, a group is a one-object groupoid.)

Let $\mathscr{W}$ now denote a general bicategory. Given the above it is natural to ask in how far $\mathscr{V}$-category theory can be generalized to $\mathscr{W}$-category theory. Let us therefore first try to make sense of the usual conditions on a base monoidal category $\mathscr{V}$ – symmetry, closedness, (co)completeness – in the case of a bicategory $\mathscr{W}$.

In $\mathscr{V}$-category theory one usually assumes the symmetry of the tensor in $\mathscr{V}$; this is essential to show that $\mathscr{V}$ is itself a $\mathscr{V}$-category with hom-objects given by the right adjoint to tensoring. But the definitions for $\mathscr{V}$-category, distributor or functor do not require the symmetry of the tensor in $\mathscr{V}$, and moreover, many important features of $\mathscr{V}$-category theory can still be developed in the non-symmetric case (sometimes one has to work a little harder though). This is encouraging, because in passing from a monoidal category $\mathscr{V}$ to a general bicategory $\mathscr{W}$ we will have to sacrifice this symmetry: tensoring objects in $\mathscr{V}$ corresponds to composing morphisms in $\mathscr{W}$ and in general it simply does not make sense for the composition $g \circ f$ of two arrows f, g to be "symmetric" (the domain of f may be different from the codomain of g)[1]!

On the other hand, we can successfully translate the notion of closedness of a monoidal category $\mathscr{V}$ to the more general setting of a bicategory $\mathscr{W}$. The tensor on objects in $\mathscr{V}$ corresponds to the composition of morphisms in $\mathscr{W}$, so we simply ask that, for any object X of $\mathscr{W}$ and any arrow $f : A \longrightarrow B$ in $\mathscr{W}$, both functors[2]

$$- \circ f : \mathscr{W}(B, X) \longrightarrow \mathscr{W}(A, X) : x \mapsto x \circ f, \tag{1}$$

$$f \circ - : \mathscr{W}(X, A) \longrightarrow \mathscr{W}(X, B) : x \mapsto f \circ x \tag{2}$$

[1] It may be argued however that the symmetry of the tensor in $\mathscr{V}$ is adequately generalized by an involution on $\mathscr{W}$.

[2] Composition of arrows $f : A \longrightarrow B$ and $g : B \longrightarrow C$ is written in the usual manner, as $g \circ f : A \longrightarrow C$. And $\mathscr{W}(A, B)$ denotes the hom-category whose objects are arrows with domain A and codomain B.

have right adjoints. We'll denote these as

$$\{f,-\} : \mathscr{W}(A,X) \longrightarrow \mathscr{W}(B,X) : y \mapsto \{f,y\}, \tag{3}$$

$$[f,-] : \mathscr{W}(X,B) \longrightarrow \mathscr{W}(X,A) : y \mapsto [f,y]. \tag{4}$$

(Indeed, we have to ask for both adjoints!) Such a bicategory $\mathscr{W}$ is said to be closed. Some call an arrow such as $\{f,y\}$ a (right) extension and $[f,y]$ a (right) lifting (of y through f).

Finally, by saying that $\mathscr{V}$ has "enough limits and colimits" is in practice often meant that $\mathscr{V}$ has small limits and small colimits. (In particular are the colimits required to compose distributors between (small) $\mathscr{V}$-categories. And the limits are crucial if one wants to speak of a $\mathscr{V}$-object of $\mathscr{V}$-natural transformations between two given parallel functors.) In a bicategory $\mathscr{W}$ the analogue for these (co)completeness conditions is straightforward: recalling that in passing from a monoidal category $\mathscr{V}$ to a bicategory $\mathscr{W}$ the rôle of the objects of $\mathscr{V}$ is played by the arrows of $\mathscr{W}$, we must now ask for $\mathscr{W}$ to have in its hom-categories small limits and small colimits (i.e. $\mathscr{W}$ is locally complete and cocomplete).

So, to summarize, when trying to develop category theory over a base bicategory $\mathscr{W}$ rather than a base monoidal category $\mathscr{V}$, it seems reasonable to work with a base bicategory which is closed, locally complete and locally cocomplete. Note that in such a bicategory $\mathscr{W}$, due to its closedness, composition always distributes on both sides over colimits of morphisms:

$$f \circ (\mathrm{colim}_{i\in I}\, g_i) \cong \mathrm{colim}_{i\in I}(f \circ g_i), \tag{5}$$

$$(\mathrm{colim}_{j\in J}\, f_j) \circ g \cong \mathrm{colim}_{j\in J}(f_j \circ g). \tag{6}$$

That is to say, the local colimits are stable under composition. (But this does not hold in general for local limits!)

Quantales and quantaloids

We will focus on a special case of these closed, locally complete and locally cocomplete bicategories: namely, we study such bicategories whose hom-categories are moreover small and skeletal. Thus the hom-categories are simply complete lattices: a small, skeletal complete (cocomplete) category is precisely an ordered set with arbitrary infima (suprema), and the suprema (infima) then come for free. We will write the local structure as an order, and local limits and colimits of morphisms as their infimum, resp. supremum—so for arrows with same domain and codomain we have things like $f \leq f'$, $\bigvee_{i\in I} f_i$, $\bigwedge_{j\in J} g_j$, etc. In particular (5) and (6) become

$$f \circ (\textstyle\bigvee_i g_i) = \bigvee_i (f \circ g_i), \tag{7}$$

$$(\textstyle\bigvee_j f_j) \circ g = \bigvee_j (f_j \circ g). \tag{8}$$

The adjoint functor theorem says that the existence of the adjoints (3) and (4) to the composition functors (1) and (2) (not only implies but also) is implied by their distributing over suprema of morphisms as in (7) and (8). Such bicategories – whose hom-categories are complete lattices and whose composition distributes on both sides over arbitrary suprema – are called quantaloids. A one-object quantaloid is a quantale[3]. So a quantaloid $\mathcal{Q}$ is a Sup-enriched category and a quantale is monoid in Sup. (Sup denotes the symmetric monoidal closed category of complete lattices and morphisms that preserve suprema). This at once fixes the notion of a homomorphism between quantaloids: it is a Sup-functor between the quantaloids viewed as Sup-categories. (But due to the local structure of quantaloids, this notion of homomorphism can be "relaxed"; in particular does J. Bénabou's notion of "morphism of bicategories", i.e. the Australian's "lax functor", make sense too.)

To gain some intuition about quantales and quantaloids it may be useful to put them in some perspective. Let us therefore look at some examples.

Of course Sup is the example *par excellence* of a quantaloid. And it follows from standard arguments for symmetric monoidal closed categories that, for any sup-lattice L, $\mathsf{Sup}(L, L)$ is a monoid in Sup, i.e. a quantale, and that any quantale is canonically a subquantale of some $\mathsf{Sup}(L, L)$.

Thinking of Sup as some sort of "infinitary version" of Ab, a quantale (a monoid in Sup) is then much like a ring (a monoid in Ab): instead of a finitary sum $a_1 + a_2 + ... + a_n$ it comes with an infinitary one, namely the supremum $\bigvee_{i \in I} a_i$. (Of course, $\bigvee$ is an idempotent operation—so it is more than just an "infinitary sum".) In particular, the distributivity of product over sums in a ring generalizes to the distributivity of composition over suprema in a quantale. On the other hand, thinking of Sup as some "simplified version" of Cat confirms the fact that a quantaloid (a Sup-category) is a simple kind of 2-category (a Cat-category); note that indeed every diagram of 2-cells commutes in a quantaloid (and this is obviously not the case in any 2-category).

Quantales can be viewed as "non-commutative locales". Recall that a locale Ω is a complete lattice in which finitary infima distribute over arbitrary suprema: $\forall x, (y_i)_{i \in I} \in \Omega$,

$$x \wedge (\bigvee_{i \in I} y_i) = \bigvee_{i \in I} (x \wedge y_i).$$

[3]It was C. Mulvey [1986] who introduced the word 'quantale' in his work on (non-commutative) C^*-algebras to contrast with the word 'locale'—see further. In today's literature though the word 'quantale' often means different things to different authors. Let us therefore underline that throughout this text a quantale always has a unit for its multiplication ([Rosenthal, 1990] and [Kruml, 2002] do not ask this), but that on the contrary it should not necessarily be "idempotent" nor "right-sided" (as [Borceux, Rosický and Van den Bossche, 1989] ask).

For a topological space $(X,\mathscr{T})$ the topology $\mathscr{T}$ is a locale with infimum given by intersection and supremum by union. Although not every locale can be obtained in this way, it is for this reason that they are often thought of as "pointless topologies" [Johnstone, 1982, 1983]. A locale is in particular a complete lattice for which binary infimum and the top element define a compatible structure of (commutative) monoid. A quantale is, in the same terms, a complete lattice equipped with a compatible monoid structure—but not necessarily with the infimum as multiplication nor the top element as unit. Therefore we may think of a quantale as a "(pointless) non-commutative topology". (This is easier said than done; the "spatiality" of a quantale is highly non-trivial! It is one of the research themes of the Brno school [Rosický, 1995; Paseka and Rosický, 2000; Kruml, 2002].)

Let us, to be a bit more concrete, briefly sketch how quantales and quantaloids quite naturally arise in ring theory as such strange "topologies". (This is in some sense (a modern account of) the "historical example" of quantales and quantaloids. Indeed, the importance of ordered sets with a compatible multiplication as structure *an sich* was first recognized in the study of ideals in a ring [Krull, 1924; Ward and Dilworth, 1939].) Thereto, let R denote a (not necessarily commutative) ring (with unit). The ring R is canonically a right module over itself, and a right ideal I is by definition just a (right) submodule[4]. Similarly one defines left-sided ideals, and two-sided ideals, in R. These form respective sets $\mathsf{Ridl}(R)$, $\mathsf{Lidl}(R)$ and $\mathsf{Tidl}(R)$. (For a commutative R these sets obviously coincide, and we may simply denote $\mathsf{Idl}(R)$ for the ideals in R.) Any two additive subgroups $I,J\subseteq R$ can be multiplied, by the formula

$$I\cdot J=\{\text{finite sums } i_1j_1+...+i_nj_n \text{ with all } i_k\in I \text{ and all } j_k\in J\}, \qquad (9)$$

and a new additive subgroup of R is obtained. It is easily seen that this multiplication is internal on each of the sets $\mathsf{Ridl}(R)$, $\mathsf{Lidl}(R)$ and $\mathsf{Tidl}(R)$, and that the latter is then even a monoid with unit R. Moreover, for any family $(I_k)_{k\in K}$ of additive subgroups of R we may consider the subgroup generated by (the set-theoretic union of) this family: it is the sum

$$\sum_{k\in K} I_k=\{\text{finite sums of elements in } \bigcup_{k\in K} I_k\}. \qquad (10)$$

In particular, the ordered sets $(\mathsf{Ridl}(R),\subseteq)$, $(\mathsf{Lidl}(R),\subseteq)$ and $(\mathsf{Tidl}(R),\subseteq)$ are complete lattices with this sum as supremum (and top element R). The multiplication in (9) distributes on both sides over the arbitrary sums in (10); so $\mathsf{Tidl}(R)$ is a quantale, and $\mathsf{Ridl}(R)$ and $\mathsf{Lidl}(R)$ are "almost" quantales (they lack a two-sided unit for the multiplication).

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[4]So R is a "space", and I is a "subspace".

At first sight it thus seems that the separate structures of right-sided, left-sided and two-sided ideals in a ring R each provide some kind of "(pointless) non-commutative topology". But actually it is not necessary to keep these different kinds of ideals separated, as G. Van den Bossche [1995] points out (but she credits B. Lawvere for this idea): they organize themselves quite naturally in one single "quantaloid of ideals" of R. Indeed, observe that:

when $I \in \mathsf{Lidl}(R)$ and $J \in \mathsf{Ridl}(R)$, then $I \cdot J \in \mathsf{Tidl}(R)$;

when $I \in \mathsf{Ridl}(R)$ and $J \in \mathsf{Tidl}(R)$, then $I \cdot J \in \mathsf{Ridl}(R)$;

when $I \in \mathsf{Tidl}(R)$ and $J \in \mathsf{Lidl}(R)$, then $I \cdot J \in \mathsf{Lidl}(R)$.

When $I \in \mathsf{Ridl}(R)$ and $J \in \mathsf{Lidl}(R)$, then $I \cdot J$ still makes sense, but it is not any longer an ideal in R. However, it is an additive subgroup of R which is a module on the center $\mathscr{Z}(R)$ of R. The collection of these still forms a quantale with multiplication as in (9) (and $\mathscr{Z}(R)$ as unit) and suprema as in (10) (with top R). (Note that any additive subgroup of R which is a left $\mathscr{Z}(R)$-module is also a right $\mathscr{Z}(R)$-module, and conversely. And that for a commutative R, the center being the whole of R, these coincide with the ideals.) This leads to the consideration of the following quantaloid: $\mathscr{Q}_R$ has two objects, 0 and 1; the hom-objects are

$$\mathscr{Q}_R(0,0) = \text{additive subgroups of } R \text{ which are } \mathscr{Z}(R)\text{-modules},$$
$$\mathscr{Q}_R(0,1) = \mathsf{Lidl}(R), \quad \mathscr{Q}_R(1,0) = \mathsf{Ridl}(R), \quad \mathscr{Q}_R(1,1) = \mathsf{Tidl}(R)$$

(which are all complete lattices with suprema as in (10) and, in particular, top element R); composition in $\mathscr{Q}_R$ is the multiplication as in (9) which we read from right to left (so $I \circ J = I \cdot J$); and the identity arrow on 1 is R and that on 0 is $\mathscr{Z}(R)$. (Note that, for a commutative R, this quantaloid $\mathscr{Q}_R$ is equivalent as Sup-category to the quantale $\mathsf{Idl}(R)$ of ideals in R!) This quantaloid is then a "(multi-typed, non-commutative) topology" for the ring R[5].

There are many more examples of quantales and quantaloids arising in different branches of mathematics, but this is not the place for an extensive discussion. But just to give an idea of the range of subjects where quantales and quantaloids show up, let us mention that:

- Quantales arise naturally when studying rings and algebras by ordered structures of ideals or other substructures. In particular can at least certain classes of C^*-algebras be recovered from a suitable quantale of ideals [Akemann, 1970; Giles and Kummer, 1971; Borceux, Rosický and Van den Bossche, 1989; Mulvey and Pelletier, 2001, 2002].

[5] K. Rosenthal [1996] has a slightly different construction for the "quantaloid of ideals" of a ring R: it is like Van den Bossche's except for the hom-lattice $\mathscr{Q}_R(0,0)$ which he takes to be all additive subgroups of R. He too attributes this construction to Lawvere. But this construction has the drawback that, for a commutative R, it is not equivalent to $\mathsf{Idl}(R)$.

- When thinking of a quantale as the order-theoretic version of a ring, it makes sense to develop a theory of modules over a quantale. One may for example investigate conditions on such modules to make them behave like (an order-theoretic version of) a Hilbert space [Paseka, 2002].

- The category of relations in a (Grothendieck) topos is a quantaloid, and difficult calculations with toposes can be reduced to simpler ones by considering precisely those quantaloids of relations [Pitts, 1988].

- Abstract categories of relations have been studied (under different names: ‘allegories’, ‘distributive categories of relations’) and they are if not quantaloids then at least locally ordered categories [Carboni and Walters, 1987; Freyd and Scedrov, 1990].

- Certain classes of quantales have been recognized as semantics for propositional linear logic [Girard, 1987; Rosenthal, 1990].

- In computer science quantaloids can be regarded as algebras of typed experimental observations and are as such of use in the theory of process semantics [Abramsky and Vickers, 1993; Resende, 2000].

- More recently quantales and quantaloids have been recognized to play an important rôle in dynamic operational quantum logic [Amira *et al.*, 1998; Coecke and Stubbe, 1999; Coecke *et al.*, 2001]. More generally, a “process logic” may be viewed as a category enriched in a “quantaloid of processes”, the enrichment thus providing an external implication to the logic [Coecke and Smets, 2001; Stubbe and Sourbron, 2002; Stubbe, 2003].

In 2000 the AMS decided to consider the theory of quantales as a subject in its own right, and to classify it henceforth under MSC 06F07.

Enriched categories, distributors and functors

Now that we've agreed on the kind of bicategory that we'd like to use as base for enriched categories, namely quantaloids, let us see how the familiar definitions for $\mathscr{V}$-enriched category, distributor and functor – as found in [Kelly, 1982; Borceux, 1994] for example – can be generalized.

Let us first recall that, for $\mathscr{V}$ a symmetric monoidal closed category, a small $\mathscr{V}$-enriched category $\mathbb{A}$ consists of:

- a set $\mathbb{A}_0$ of objects $a, a', a'', ...$;

- for every two objects $a, a' \in \mathbb{A}_0$ a hom-object[6] $\mathbb{A}(a', a)$ in $\mathscr{V}$;

[6]As for the notation: in the base category $\mathscr{V}$ a hom-object like $\mathscr{V}(A, B)$ contains arrows from A to B; but in a $\mathscr{V}$-category $\mathbb{A}$ the hom-object $\mathbb{A}(a', a)$ is thought of as containing arrows which go from a to a'. This may seem awkward at first, but it will make perfect sense when we will replace $\mathscr{V}$ by a quantaloid $\mathscr{Q}$...

- for every three objects $a, a', a'' \in \mathbb{A}_0$ a composition morphism in $\mathscr{V}$

$$\mu_{a'',a',a}\colon \mathbb{A}(a'',a') \otimes \mathbb{A}(a',a) \longrightarrow \mathbb{A}(a'',a);$$

- for every object $a \in \mathbb{A}_0$ an unit morphism in $\mathscr{V}$

$$\eta_a\colon I \longrightarrow \mathbb{A}(a,a);$$

and these data have to satisfy some diagrammatic axioms. For two $\mathscr{V}$-categories $\mathbb{A}$ and $\mathbb{B}$, a functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ is

- an object mapping $\mathbb{A}_0 \longrightarrow \mathbb{B}_0\colon a \mapsto Fa$;

- for any two objects $a, a' \in \mathbb{A}_0$ a $\mathscr{V}$-morphism

$$F_{a',a}\colon \mathbb{A}(a',a) \longrightarrow \mathbb{B}(Fa',Fa);$$

satisfying axioms that express precisely the preservation of composition and units in $\mathbb{A}$. A distributor (which some also call module or profunctor) $\Phi\colon \mathbb{A} \,{\circ}\!\!\longrightarrow \mathbb{B}$ is usually[7] defined to be a functor $\Phi\colon \mathbb{B}^\star \otimes \mathbb{A} \longrightarrow \mathscr{V}$. This requires the notion of dual $\mathscr{V}$-category $\mathbb{B}^\star$, which itself requires the symmetry of the tensor in $\mathscr{V}$. In view of the generalization we wish to make, where we no longer dispose of a "symmetric tensor" in the base (bi)category, it is therefore appropriate to note that a distributor $\Phi\colon \mathbb{A} \,{\circ}\!\!\longrightarrow \mathbb{B}$ is equivalently given by:

- for every two objects $a \in \mathbb{A}_0$, $b \in \mathbb{B}_0$, an object $\Phi(b,a)$ in $\mathscr{V}$;

- for every three objects $a, a' \in \mathbb{A}_0$, $b \in \mathbb{B}_0$, a $\mathscr{V}$-morphism

$$\Phi^b_{a',a}\colon \Phi(b,a') \otimes \mathbb{A}(a',a) \longrightarrow \Phi(b,a);$$

- for every three objects $a \in \mathbb{A}_0$, $b, b' \in \mathbb{B}_0$, a $\mathscr{V}$-morphism

$$\Phi^{b,b'}_a\colon \mathbb{B}(b,b') \otimes \Phi(b',a) \longrightarrow \Phi(b,a);$$

where these morphisms $\Phi^b_{a',a}$ and $\Phi^{b,b'}_a$ – expressing a right action of $\mathbb{A}$, and a left action of $\mathbb{B}$, on Φ – satisfy certain compatibility conditions with composition and units in $\mathbb{A}$ and $\mathbb{B}$.

[7] It should be noted that a distributor $\Phi\colon \mathbb{A} \,{\circ}\!\!\longrightarrow \mathbb{B}$ is equivalently given by a $\mathscr{V}$-functor from $\mathbb{A}$ into $\widehat{\mathbb{B}}$, the latter being the $\mathscr{V}$-category of contravariant $\mathscr{V}$-presheaves on $\mathbb{B}$. This is actually J. Bénabou's original concept behind the notion of distributor that he gave for ordinary categories [Bénabou, 1973]. However, in the context of our generalization to a base quantaloid $\mathscr{Q}$, the problem remains that symmetry of the tensor in $\mathscr{V}$ is required to define the $\mathscr{V}$-category of contravariant $\mathscr{V}$-presheaves on $\mathbb{B}$.

$$\begin{tikzcd}
ta \arrow[r, "\mathbb{A}(a',a)"] \arrow[dr, "\mathbb{A}(a'',a)"'] & ta' \arrow[d, "\mathbb{A}(a'',a')"] \\
& ta''
\end{tikzcd}$$

Diagram 0.1

How can these familiar definitions be understood when we work over a base quantaloid $\mathcal{Q}$ instead of a symmetric monoidal closed $\mathcal{V}$? We should keep in mind that it are the arrows of $\mathcal{Q}$ that will play the rôle of the objects of $\mathcal{V}$, with the composition of arrows being like the tensor of $\mathcal{V}$-objects, and that the 2-cells in $\mathcal{Q}$ must be thought of as the morphisms between objects in $\mathcal{V}$. It may also be recalled that, due to the hom-categories of a quantaloid being ordered sets, every diagram of 2-cells in a quantaloid $\mathcal{Q}$ commutes.

A small $\mathcal{Q}$-enriched category $\mathbb{A}$ will surely have a set of objects, let us denote $a, a', a'', ... \in \mathbb{A}_0$. For reasons that will soon be clear, we must associate to each $a \in \mathbb{A}_0$ some object of $\mathcal{Q}$; we will systematically denote for an $\mathbb{A}$-object a this associated $\mathcal{Q}$-object by ta, and we'll refer to it as the type[8] of a in $\mathcal{Q}$. (Conversely, given an object A of $\mathcal{Q}$, we say that an object $a \in \mathbb{A}_0$ lies over A precisely when $ta = A$.) So we have that

$$\mathbb{A}_0 \text{ is a } \mathcal{Q}_0\text{-typed set: to each } a \in \mathbb{A}_0 \text{ is associated some } ta \in \mathcal{Q}_0. \tag{11}$$

Such a $\mathcal{Q}_0$-typed set $\mathbb{A}_0$ can now be enriched in $\mathcal{Q}$: for every two objects $a, a' \in \mathbb{A}_0$, ask for an arrow $\mathbb{A}(a', a)$ in $\mathcal{Q}$ to play the rôle of "hom from a to a'". Again, for reasons that will soon be clear, we'll ask that this arrow has domain ta and codomain ta':

$$\text{for all } a, a' \in \mathbb{A}_0, \text{ a hom-arrow } \mathbb{A}(a', a)\colon ta \longrightarrow ta' \text{ in } \mathcal{Q}. \tag{12}$$

Now suppose that we have three objects a, a', a''; we can consider diagram 0.1 in $\mathcal{Q}$ of the hom-arrows between these objects. To say that there is a "composition from $\mathbb{A}(a'', a') \circ \mathbb{A}(a', a)$ to $\mathbb{A}(a'', a)$ in the $\mathcal{Q}$-enriched category $\mathbb{A}$", we must ask for a composition 2-cell in $\mathcal{Q}$:

$$\text{for all } a, a' \in \mathbb{A}_0 : \mathbb{A}(a'', a') \circ \mathbb{A}(a', a) \leq \mathbb{A}(a'', a) \text{ in } \mathcal{Q}(ta, ta''). \tag{13}$$

Likewise, on any object $a \in \mathbb{A}_0$ there is an endo-hom-arrow $\mathbb{A}(a, a)\colon ta \longrightarrow ta$ in $\mathcal{Q}$, but there is also the identity arrow $1_{ta}\colon ta \longrightarrow ta$ in $\mathcal{Q}$. To say that there is a

[8] So when $\mathcal{Q}$ has a (small) set of objects, the "typing" of objects is simply a function $t\colon \mathbb{A}_0 \longrightarrow \mathcal{Q}_0$.

"unit on a in the $\mathscr{Q}$-enriched category $\mathbb{A}$", we must ask for a unit 2-cell in $\mathscr{Q}$:

$$\text{for all } a \in \mathbb{A}_0 : 1_{ta} \leq \mathbb{A}(a,a) \text{ in } \mathscr{Q}(ta,ta). \tag{14}$$

We do not need to impose any axioms about "composition being associative with left and right units": since "composition" and "units" in $\mathbb{A}$ are determined by 2-cells in $\mathscr{Q}$, the appropriate (diagrammatic) axioms automatically hold! So the above is the appropriate definition of a $\mathscr{Q}$-enriched category $\mathbb{A}$. Note that, quite unlike for $\mathscr{V}$-categories, the composition and unit 2-cells in a $\mathscr{Q}$-category $\mathbb{A}$ are uniquely determined. More precisely, given a $\mathscr{Q}_0$-typed set $\mathbb{A}_0$ as in (11) together with hom-arrows $\mathbb{A}(a',a)\colon ta \longrightarrow ta'$ in $\mathscr{Q}$ as in (12), it is a property, rather than an extra structure, that these objects and hom-arrows form (or not) a $\mathscr{Q}$-category $\mathbb{A}$, i.e. that they satisfy (or not) the composition and unit inequalities of (13) and (14).

For two $\mathscr{Q}$-enriched categories $\mathbb{A}$ and $\mathbb{B}$, a functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ should somehow consist of a mapping of objects together with an action on the hom-arrows. For such an action on hom-arrows to make sense, it will be required that the mapping of objects preserves the types of the objects:

$$\text{an object mapping } \mathbb{A}_0 \longrightarrow \mathbb{B}_0\colon a \mapsto Fa \text{ such that } ta = t(Fa). \tag{15}$$

To express the action of F on the hom-arrows of $\mathbb{A}$, we must ask for action 2-cells in $\mathscr{Q}$:

$$\text{for all } a,a' \in \mathbb{A}_0 : \mathbb{A}(a',a) \leq \mathbb{B}(Fa',Fa) \text{ in } \mathscr{Q}(ta,ta'). \tag{16}$$

The axioms for the "functoriality" of F are certain diagrams of 2-cells in $\mathscr{Q}$, so they are always satisfied. Therefore, given an object mapping as in (15), it is again a property, rather than extra structure, that this mapping determines a functor, i.e. that it satisfies the functor inequalities of (16).

Finally, let us consider a distributor between $\mathscr{Q}$-categories $\mathbb{A}$ and $\mathbb{B}$. In analogy with the above, such a $\Phi\colon \mathbb{A} \multimap \mathbb{B}$ consists of a collection of $\mathscr{Q}$-arrows:

$$\text{for all } a \in \mathbb{A}_0, b \in \mathbb{B}_0, \text{ an arrow } \Phi(b,a)\colon ta \longrightarrow tb \text{ in } \mathscr{Q}. \tag{17}$$

And to express the "action of $\mathbb{A}$ on the right, and $\mathbb{B}$ on the left, on Φ" we must ask for action 2-cells in $\mathscr{Q}$:

$$\text{for all } a,a' \in \mathbb{A}_0, b \in \mathbb{B}_0 : \Phi(b,a') \circ \mathbb{A}(a',a) \leq \Phi(b,a) \text{ in } \mathscr{Q}(ta,tb); \tag{18}$$

$$\text{for all } a \in \mathbb{A}_0, b,b' \in \mathbb{B}_0 : \mathbb{B}(b,b') \circ \Phi(b',a) \leq \Phi(b,a) \text{ in } \mathscr{Q}(ta,tb). \tag{19}$$

Once more, the compatibility of these action 2-cells with the composition and unit 2-cells of both $\mathbb{A}$ and $\mathbb{B}$ is automatic, for these compatibilities are expressed by diagrams of 2-cells in $\mathscr{Q}$. Thus, given a family of $\mathscr{Q}$-arrows as in (17), it is a property, rather than extra structure, that this family determines a distributor $\Phi\colon \mathbb{A} \xrightarrow{\circ} \mathbb{B}$, i.e. satisfies the action inequalities of (18) and (19).

Before we go any further, it is probably useful to discuss briefly some examples.

(a) The extended non-negative reals $\mathbb{R}^+ \cup \{\infty\}$ form a (symmetric) quantale for the opposite order and addition as binary operation. In his famous paper on "Metric spaces, generalized logic and closed categories" B. Lawvere recognized that a category enriched in (the one-object suspension of) this quantale is a "generalized metric space": the enrichment itself is a binary distance function taking values in the positive reals. In particular is the composition-inequality in such an enriched category the triangular inequality. A functor between such generalized metric spaces is a distance decreasing application. (These metric spaces are "generalized" in that the distance function is not symmetric, that the distance between two points being zero doesn't imply their being identical, and that the distance between two points may be infinite.)

(b) By **2** we will denote (the one-object suspension of) the two-element Boolean algebra—obviously a particular example of a quantale (or one-object quantaloid). Writing its elements as 0 and 1, the composition in **2** is the conjunction of these truth values, and their disjunction is the supremum. A **2**-enriched category $\mathbb{A}$ is precisely a preordered set: its elements are the objects of $\mathbb{A}$, and the enrichment classifies the preorder:

$$\text{for } x, x' \in \mathbb{A}_0 : \mathbb{A}(x, x') = 1 \text{ if and only if } x \leq x'.$$

The transitivity of the preorder corresponds to the composition-inequality in $\mathbb{A}$, and its reflexivity to the unit-inequality. In the same way, a distributor $\Phi\colon \mathbb{A} \xrightarrow{\circ} \mathbb{B}$ between **2**-categories is a so-called ideal relation between preorders: the relation $\Phi_0 \subseteq \mathbb{A}_0 \times \mathbb{B}_0$ defined by

$$\text{for } (x, y) \in \mathbb{A}_0 \times \mathbb{B}_0 : \Phi(y, x) = 1 \text{ if and only if } (x, y) \in \Phi_0$$

satisfies – due to the action-inequalities for the distributor – the condition that

$$\text{if } x \leq x' \text{ in } \mathbb{A}_0,\ y' \leq y \text{ in } \mathbb{B}_0 \text{ and } (x, y) \in \Phi_0,\ \text{then } (x', y') \in \Phi_0.$$

That is to say, the relation Φ_0 between the preorders $(\mathbb{A}_0, \leq)$ and $(\mathbb{B}_0, \leq)$ is "up-closed" in $\mathbb{A}_0$ and "down-closed" in $\mathbb{B}_0$. Finally, a functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ between **2**-categories is exactly a preorder-preserving map.

(c) Let R be a commutative ring (with unit), then the ideals in R form a quantale $\mathsf{Idl}(R)$. The elements of the ring itself are the objects of an $\mathsf{Idl}(R)$-enriched category whose homs are given by annihilators: for $r, s \in R$ put the hom from r to s to be $\mathsf{Ann}(r - s) = \{x \in R \mid (r - s)x = 0\}$. For a non-commutative R

we have seen on p. 5 how the quantale of ideals generalizes to a quantaloid $\mathscr{Q}_R$; and G. Van den Bossche [1995] points out how such a non-commutative ring R determines a $\mathscr{Q}_R$-enriched category by generalizing annihilators to commutators: $\mathsf{Comm}(r,s) = \{x \in R \mid rx = xs\}$. Note first that if r is an element of $\mathscr{Z}(R)$ (the center of R) then $\mathsf{Comm}(r,s)$ is a left ideal in R; if s is an element of $\mathscr{Z}(R)$ then $\mathsf{Comm}(r,s)$ is a right ideal; if both $r,s \in \mathscr{Z}(R)$ then obviously $\mathsf{Comm}(r,s)$ is a two-sided ideal; and if neither r nor s is in $\mathscr{Z}(R)$ then $\mathsf{Comm}(r,s)$ is just an additive subgroup of R which is a $\mathscr{Z}(R)$-module. That is to say, R determines a $\mathscr{Q}_R$-enriched structure – which we denote as Comm_R – whose objects of type 0 are the elements of R and whose objects of type 1 are the elements of the center of R, and the homs of which are given by commutators.

(d) A quantaloid $\mathscr{Q}$ itself determines $\mathscr{Q}$-enriched categories of arrows with common codomain, as follows. Fix an object $A \in \mathscr{Q}_0$, and consider all $\mathscr{Q}$-arrows with codomain A; this already defines a $\mathscr{Q}_0$-typed set

$$(\mathbb{Q}_A)_0 = \coprod_{X \in \mathscr{Q}_0} \mathscr{Q}(X,A) \text{ with types } tf = \mathsf{dom}(f).$$

For two such arrows $X \xrightarrow{f} A \xleftarrow{g} Y$ the lifting $[g,f]\colon X \longrightarrow Y$ is a $\mathscr{Q}$-arrow from $\mathsf{dom}(f)$ to $\mathsf{dom}(g)$, and actually

$$\mathbb{Q}_A(g,f) = [g,f]$$

defines a $\mathscr{Q}$-category on $(\mathbb{Q}_A)_0$. Similarly, the data

$$(\mathbb{Q}^A)_0 = \coprod_{X \in \mathscr{Q}_0} \mathscr{Q}(A,X), \;\; tf = \mathsf{cod}(f), \;\; \mathbb{Q}^A(g,f) = \{g,f\}$$

is a $\mathscr{Q}^{\mathsf{op}}$-category (but not a $\mathscr{Q}$-category!).

(e) By the "opposite" of a $\mathscr{Q}$-category $\mathbb{A}$ is of course meant the new structure "$\mathbb{A}^{\mathsf{op}}$" with the same objects as $\mathbb{A}$, but whose hom-arrows are reversed:

$$\text{for objects } a,a' : \mathbb{A}^{\mathsf{op}}(a',a) := \mathbb{A}(a,a').$$

But note that this does not define a $\mathscr{Q}$-category, but rather a $\mathscr{Q}^{\mathsf{op}}$-category!

Matrices, monads and bimodules

To see how $\mathscr{Q}$-categories and distributors organize themselves in a categorical structure, it is useful to reconsider once more their respective definitions. In particular we will make a very clear distinction between two different aspects of these definitions. A first observation is that both a $\mathscr{Q}$-category $\mathbb{A}$ and a distributor $\Phi\colon \mathbb{A} \mathrel{-\!\!\circ\!\!\rightarrow} \mathbb{B}$ consist in some way of a "matrix whose elements are arrows in $\mathscr{Q}$":

the hom-arrows of a $\mathcal{Q}$-category $\mathbb{A}$ as in (12) form a square matrix, with $|\mathbb{A}_0|$ rows and $|\mathbb{A}_0|$ columns, whose entry at position $(a', a) \in \mathbb{A}_0 \times \mathbb{A}_0$ is precisely $\mathbb{A}(a', a)$; and the elements of a distributor $\Phi\colon \mathbb{A} \relbar\joinrel\mapstochar\joinrel\rightarrow \mathbb{B}$ as in (17) form a matrix, with $|\mathbb{B}_0|$ rows and $|\mathbb{A}_0|$ columns, whose entry at position $(b, a) \in \mathbb{B}_0 \times \mathbb{A}_0$ is $\Phi(b, a)$. A second and quite separate aspect is then the translation of the composition and unit inequalities (13) and (14) (for a category) and the action inequalities (18) and (19) (for a distributor) to these "matrices". It turns out that categories are monads, and distributors bimodules, in an appropriate quantaloid of matrices whose elements are arrows in $\mathcal{Q}$ [Betti *et al.*, 1983; Carboni *et al.*, 1987].

More precisely, let us write (X, t), (Y, t), (Z, t), ... for $\mathcal{Q}_0$-typed sets: simply sets to every element of which is associated a type in $\mathcal{Q}_0$ (as in (11)). Given two such $\mathcal{Q}_0$-typed sets (X, t) and (Y, t), a matrix with elements in $\mathcal{Q}$ with $|Y|$ rows and $|X|$ columns – written as $\mathbb{M}\colon (X, t) \longrightarrow (Y, t)$ – is a collection of $\mathcal{Q}$-arrows like so:

$$\Big(\mathbb{M}(y, x)\colon tx \longrightarrow ty\Big)_{(x,y)\in X\times Y}. \tag{20}$$

Two matrices $\mathbb{M}\colon (X, t) \longrightarrow (Y, t)$ and $\mathbb{N}\colon (Y, t) \longrightarrow (Z, t)$ can be composed: the new matrix $\mathbb{N} \circ \mathbb{M}\colon (X, t) \longrightarrow (Z, t)$ has as element on its zth row and xth column

$$(\mathbb{N} \circ \mathbb{M})(z, x) = \bigvee_{y\in Y} \mathbb{N}(z, y) \circ \mathbb{M}(y, x). \tag{21}$$

Reading the supremum of arrows in $\mathcal{Q}$ as their "infinitary sum", and the composition as their "multiplication", this is the formula for matrix multiplication as in linear algebra! The identity matrix on some $\mathcal{Q}_0$-typed set (X, t) will be denoted as $\Delta_X\colon (X, t) \longrightarrow (X, t)$, because its elements are "Kronecker deltas":

$$\Delta_X(x', x) = \begin{cases} 1_{tx}\colon tx \longrightarrow tx \text{ when } x = x', \\ 0_{tx',tx}\colon tx \longrightarrow tx' \text{ otherwise.} \end{cases} \tag{22}$$

(An arrow like $0_{tx',tx}\colon tx \longrightarrow tx'$ denotes the smallest element of the sup-lattice $\mathcal{Q}(tx, tx')$.) It is easily verified that $\mathcal{Q}_0$-typed sets and matrices with elements in $\mathcal{Q}$ form a category—which we denote by $\mathsf{Matr}(\mathcal{Q})$. Actually, this category is a quantaloid for the "elementwise" ordering of parallel matrices: for

$$\mathbb{M}, \mathbb{M}'\colon (X, t) \rightrightarrows (Y, t)$$

put $\mathbb{M} \leq \mathbb{M}'$ when for all $X \in X$ and $y \in Y$, $\mathbb{M}(y, x) \leq \mathbb{M}'(y, x)$ as $\mathcal{Q}$-arrows.

There is an obvious embedding of the given quantaloid $\mathcal{Q}$ into the quantaloid $\mathsf{Matr}(\mathcal{Q})$ of matrices with elements in $\mathcal{Q}$: every arrow $f\colon A \longrightarrow B$ in $\mathcal{Q}$ can be seen as a one-element matrix between one-element $\mathcal{Q}_0$-typed sets. This embedding has the universal property that it is the direct-sum completion of $\mathcal{Q}$ in the (illegitimate) category of (possibly large) quantaloids and homomorphisms[9]. From this fact it is then easy to characterize those quantaloids that arise as quantaloids of matrices.

Let us now study the inequalities (13) and (14) (for $\mathcal{Q}$-categories), and (18) and (19) (for distributors) in the context of the quantaloid $\mathsf{Matr}(\mathcal{Q})$ of matrices with elements in $\mathcal{Q}$. Rewriting (13) by the equivalent

$$\text{for all } a, a'' \in \mathbb{A}_0 : \bigvee_{a' \in \mathbb{A}_0} \mathbb{A}(a'', a') \circ \mathbb{A}(a', a) \leq \mathbb{A}(a'', a) \tag{23}$$

it is readily seen that this inequality simply says that $\mathbb{A} \circ \mathbb{A} \leq \mathbb{A}$ in $\mathsf{Matr}(\mathcal{Q})$, where now $\mathbb{A}$ is viewed as a square matrix on $(\mathbb{A}_0, t)$. Likewise, equation (14) can be rewritten as

$$\text{for all } a, a' \in \mathbb{A}_0 : \Delta_{\mathbb{A}_0}(a', a) \leq \mathbb{A}(a', a) \tag{24}$$

or even as $\Delta_{\mathbb{A}_0} \leq \mathbb{A}$ in $\mathsf{Matr}(\mathcal{Q})$. These inequalities precisely say that a $\mathcal{Q}$-enriched category is a monad in the quantaloid $\mathsf{Matr}(\mathcal{Q})$! And, much in the same way, (18) and (19) say that a distributor $\Phi\colon \mathbb{A} \mathrel{-\!\!\circ\!\!\rightarrow} \mathbb{B}$ is really a matrix $\Phi\colon (\mathbb{A}_0, t) \longrightarrow (\mathbb{B}_0, t)$ such that

$$\text{for all } a, a' \in \mathbb{A}_0, b \in \mathbb{B}_0 : \textstyle\bigvee_{a' \in \mathbb{A}_0} \Phi(b, a') \circ \mathbb{A}(a', a) \leq \Phi(b, a); \tag{25}$$

$$\text{for all } a \in \mathbb{A}_0, b, b' \in \mathbb{B}_0 : \textstyle\bigvee_{b' \in \mathbb{B}_0} \mathbb{B}(b, b') \circ \Phi(b', a) \leq \Phi(b, a). \tag{26}$$

In other words, $\Phi \circ \mathbb{A} \leq \Phi$ and $\mathbb{B} \circ \Phi \leq \Phi$ in the quantaloid $\mathsf{Matr}(\mathcal{Q})$—thus a distributor $\Phi\colon \mathbb{A} \mathrel{-\!\!\circ\!\!\rightarrow} \mathbb{B}$ is a bimodule between the monads $\mathbb{A}$ and $\mathbb{B}$ in the quantaloid $\mathsf{Matr}(\mathcal{Q})$!

Indeed, an endo-arrow $A \circlearrowleft^{t}$ in a quantaloid $\mathcal{Q}$ is a monad[10] when it satisfies $t \circ t \leq t$ and $1_A \leq t$. And given two monads $A \circlearrowleft^{t}$ and $B \circlearrowleft^{s}$ in $\mathcal{Q}$, a bimodule $b\colon t \mathrel{-\!\!\circ\!\!\rightarrow} s$ is by definition a $\mathcal{Q}$-arrow $b\colon A \longrightarrow B$ satisfying $b \circ t \leq b$ and $s \circ b \leq b$. For monads $A \circlearrowleft^{t}$, $B \circlearrowleft^{s}$ and $C \circlearrowleft^{r}$, two bimodules $b\colon t \mathrel{-\!\!\circ\!\!\rightarrow} s$ and $c\colon s \mathrel{-\!\!\circ\!\!\rightarrow} r$ can be composed: the composition is written $c \otimes_s b\colon t \mathrel{-\!\!\circ\!\!\rightarrow} r$ but is actually computed simply as $c \circ b$, the composition in $\mathcal{Q}$. We use this "tensor notation" for the composition of bimodules to stress that the left and right identities for composition are not inherited from $\mathcal{Q}$: it is indeed easily verified that the identity bimodule on a

[9]The quantaloid $\mathsf{Matr}(\mathcal{Q})$ always has a class of objects, no matter how "small" $\mathcal{Q}$ is! This is our main motivation for not *a priori* restricting our attention to small quantaloids: the large ones arise as important universal constructions on small ones.

[10]The notions of monad and bimodule make sense in any bicategory [Street, 1972], but applied to a quantaloid they take these very simple forms because a quantaloid is locally ordered. In particular any monad in a quantaloid is idempotent. The notation with a "$\mathrel{-\!\!\circ\!\!\rightarrow}$" for bimodules is meant to distinguish them from arrows—and is already anticipating the notation for distributors.

monad $\overset{\frown}{A}{}^{t}$ is the $\mathcal{Q}$-arrow t itself (and not 1_A). So in particular we have, for a bimodule $b\colon t{\longrightarrow\!\!\!\!\!\!\!\!\circ\;\;\;}r$, that $b\otimes_t t = b$ and $r\otimes_r b = b$. For any quantaloid $\mathcal{Q}$ we can thus construct a category $\mathsf{Bim}(\mathcal{Q})$ whose objects are monads in $\mathcal{Q}$ and whose arrows are bimodules. Actually, $\mathsf{Bim}(\mathcal{Q})$ is a quantaloid: its local structure is inherited from $\mathcal{Q}$ in the sense that for two parallel bimodules $b,b'\colon t\rightrightarrows r$ we put that $b\leq b'$ in $\mathsf{Bim}(\mathcal{Q})$ precisely when $b\leq b'$ in $\mathcal{Q}$—so for a family $(b_i\colon t{\longrightarrow\!\!\!\!\!\!\!\!\circ\;\;\;}r)_{i\in I}$ their supremum as bimodules is calculated "as in $\mathcal{Q}$".

There is then an obvious embedding of $\mathcal{Q}$ into $\mathsf{Bim}(\mathcal{Q})$, because any morphism $f\colon A\longrightarrow B$ in $\mathcal{Q}$ determines a bimodule between trivial monads. This embedding has a universal property in the (illegitimate) category of quantaloids and homomorphisms[11]: it is the universal splitting of monads in $\mathcal{Q}$. The object over which a monad in a quantaloid splits, is both the object of its algebras (Eilenberg-Moore algebras) and the object of its free algebras (Kleisli algebras). (These notions are due to [Street, 1972].) This is due to the idempotency of any such monad. Thanks to this universal property it is easy to characterize those quantaloids which arise as quantaloids of monads and bimodules.

The point now is the following: (small) $\mathcal{Q}$-enriched categories and distributors are precisely monads and bimodules in the quantaloid of $\mathcal{Q}_0$-typed sets and matrices with elements in $\mathcal{Q}$:

$$\mathsf{Dist}(\mathcal{Q}) = \mathsf{Bim}(\mathsf{Matr}(\mathcal{Q})). \tag{27}$$

This at once dictates the rules for composition of distributors (the composition $\Psi\otimes_{\mathbb{B}}\Phi\colon\mathbb{A}{\longrightarrow\!\!\!\!\!\!\!\!\circ\;\;\;}\mathbb{C}$ of two distributors $\Phi\colon\mathbb{A}{\longrightarrow\!\!\!\!\!\!\!\!\circ\;\;\;}\mathbb{B}$ and $\Psi\colon\mathbb{B}{\longrightarrow\!\!\!\!\!\!\!\!\circ\;\;\;}\mathbb{C}$ is computed as a matrix composition) and identity distributors (the identity distributor on a $\mathcal{Q}$-category $\mathbb{A}$ is $\mathbb{A}\colon\mathbb{A}{\longrightarrow\!\!\!\!\!\!\!\!\circ\;\;\;}\mathbb{A}$ itself). As a result then, $\mathsf{Dist}(\mathcal{Q})$ is a (large) quantaloid in which $\mathcal{Q}$ can be embedded. This embedding has a particular universal property, and therefore those quantaloids arising as quantaloids of $\mathcal{Q}$-enriched categories and distributors can be characterized without much difficulty.

Moreover, $\mathsf{Dist}(\mathcal{Q})$ being a quantaloid, i.e. a particular kind of bicategory, we dispose of all kinds of bicategorical notions that are pertinent for categories and distributors: think of adjoints, liftings and extensions, etc. This will be particularly important for the development of "$\mathcal{Q}$-categorical algebra".

Categorical algebra over a quantaloid

We've seen how (small) $\mathcal{Q}$-enriched categories and distributors organize themselves in a new quantaloid $\mathsf{Dist}(\mathcal{Q})$. Functors between $\mathcal{Q}$-categories too give rise to a category: it is quite obvious how one computes the composition of two

[11] The quantaloid $\mathsf{Bim}(\mathcal{Q})$ is small whenever $\mathcal{Q}$ is.

functors $F\colon\mathbb{A}\longrightarrow\mathbb{B}$ and $G\colon\mathbb{B}\longrightarrow\mathbb{C}$:

$$G\circ F\colon\mathbb{A}\longrightarrow\mathbb{C} : a\mapsto G(F(a)),\tag{28}$$

and also the identity functor on a $\mathcal{Q}$-category $\mathbb{A}$ is easily defined:

$$1_{\mathbb{A}}\colon\mathbb{A}\longrightarrow\mathbb{A} : a\mapsto a.\tag{29}$$

We'll write $\mathsf{Cat}(\mathcal{Q})$ for the category of (small) $\mathcal{Q}$-enriched categories and functors.

A crucial lemma is that every functor determines an adjoint pair of distributors. Indeed, let $F\colon\mathbb{A}\longrightarrow\mathbb{B}$ be a functor between $\mathcal{Q}$-enriched categories. Then it is easily seen that both matrices

$$\Big(\Phi(b,a)=\mathbb{B}(b,Fa)\Big)_{(a,b)\in\mathbb{A}_0\times\mathbb{B}_0}\quad\text{and}\quad\Big(\Psi(a,b)=\mathbb{B}(Fa,b)\Big)_{(b,a)\in\mathbb{B}_0\times\mathbb{A}_0}$$

are in fact distributors, thus $\Phi\colon\mathbb{A}\xrightarrow{\circ}\mathbb{B}$ and $\Psi\colon\mathbb{B}\xrightarrow{\circ}\mathbb{A}$. And a calculation shows that Φ is left adjoint to Ψ in the quantaloid $\mathsf{Dist}(\mathcal{Q})$; that is to say, $\mathbb{A}\leq\Psi\otimes_{\mathbb{B}}\Phi$ and $\Psi\otimes_{\mathbb{A}}\Phi\leq\mathbb{B}$. Not to introduce too many unnecessary notations, we'll simply write

$$\begin{tikzcd}\mathbb{A} \arrow[r, shift right, "\mathbb{B}(-,F-)"', "\circ" marking] & \mathbb{B} \arrow[l, shift right, "\mathbb{B}(F-,-)"', "\circ" marking]\end{tikzcd}$$

for these adjoint distributors represented by the functor $F\colon\mathbb{A}\longrightarrow\mathbb{B}$.

Sending a functor $F\colon\mathbb{A}\longrightarrow\mathbb{B}$ to the distributor $\mathbb{B}(-,F-)\colon\mathbb{A}\xrightarrow{\circ}\mathbb{B}$, which is a left adjoint, determines a functor from $\mathsf{Cat}(\mathcal{Q})$ to $\mathsf{Dist}(\mathcal{Q})$ which is the identity on objects:

$$\mathsf{Cat}(\mathcal{Q})\longrightarrow\mathsf{Dist}(\mathcal{Q})\colon(\mathbb{A}\xrightarrow{\ F\ }\mathbb{B})\mapsto(\mathbb{A}\xrightarrow[\circ]{\mathbb{B}(-,F-)}\mathbb{B}).\tag{30}$$

Sending F to $\mathbb{B}(F-,-)$ of course determines a contravariant such functor. These functors provide the key to the development of $\mathcal{Q}$-categorical algebra. To get a feeling for this, let us look at some important examples.

(i) $\mathsf{Cat}(\mathcal{Q})$ as 2-category: Because of the functor in (30) the category $\mathsf{Cat}(\mathcal{Q})$ inherits the local structure of the quantaloid $\mathsf{Dist}(\mathcal{Q})$: for functors $F,G\colon\mathbb{A}\rightrightarrows\mathbb{B}$ we put

$$F\leq G\text{ in }\mathsf{Cat}(\mathcal{Q})(\mathbb{A},\mathbb{B})\iff\mathbb{B}(-,F-)\leq\mathbb{B}(-,G-)\text{ in }\mathsf{Dist}(\mathcal{Q})(\mathbb{A},\mathbb{B}).\tag{31}$$

Thus $\mathsf{Cat}(\mathcal{Q})$ becomes a locally preordered category (indeed the local order is not anti-symmetric in general, and certainly not complete), and we dispose of all possible 2-categorical notions for $\mathcal{Q}$-categories and functors too: adjoint functors, equivalent categories, etc. It may by the way be noted that (31) is – as one would expect – the correct translation of the idea of a $\mathscr{V}$-natural transformation to the context of $\mathscr{Q}$-categories, for it may be rewritten as

$$\text{for all } a \in \mathbb{A}_0 : 1_{ta} \leq \mathbb{B}(Fa, Ga) \text{ in } \mathscr{Q}(ta, ta). \tag{32}$$

There is now an evident way in which every $\mathscr{Q}$-enriched category has an "underlying preordered set" (or rather an indexed family of preordered sets), precisely like every $\mathscr{V}$-enriched category has an underlying (ordinary) category. Indeed, every object $a \in \mathbb{A}_0$ of a $\mathscr{Q}$-category $\mathbb{A}$ may be identified with a "constant functor"

$$\Delta a \colon *_A \longrightarrow \mathbb{A} \colon * \mapsto a. \tag{33}$$

(For an object A of $\mathscr{Q}$, $*_A$ denotes the one-object $\mathscr{Q}$-category whose single hom-arrow is the identity on A[12]; in the above we must necessarily put $A = ta$.) We may thus order the objects of $\mathbb{A}$ as we order the corresponding functors:

$$a \leq a' \text{ in } \mathbb{A}_0 \iff ta = ta' =: A \text{ and } \Delta a \leq \Delta a' \text{ in } \mathsf{Cat}(\mathscr{Q})(*_A, \mathbb{A}). \tag{34}$$

Hence $(\mathbb{A}_0, \leq)$ is a ($\mathscr{Q}_0$-indexed family of) preordered sets.

A little calculation shows that

$$a \leq a' \text{ in } \mathbb{A}_0 \iff ta = ta' =: A \text{ and } 1_A \leq \mathbb{A}(a, a') \text{ in } \mathscr{Q}(A, A). \tag{35}$$

Thus the local preorder in $\mathsf{Cat}(\mathscr{Q})$ proves to be "pointwise": (32) just says that

$$\text{for all } a \in \mathbb{A}_0 : Fa \leq Ga \text{ in } (\mathbb{B}_0, \leq). \tag{36}$$

So every $\mathscr{Q}$-category determines an underlying $\mathscr{Q}_0$-indexed family of preorders, and every functor determines a $\mathscr{Q}_0$-indexed family of order-preserving maps—we obtain a functor $\mathscr{U} \colon \mathsf{Cat}(\mathscr{Q}) \longrightarrow \mathsf{PreOrd}_{\mathscr{Q}_0}$ between locally (pre)ordered categories. This functor admits a left adjoint.

(ii) Weighted colimits in a $\mathscr{Q}$-category: Let $F \colon \mathbb{A} \longrightarrow \mathbb{B}$ be a functor, and $\Phi \colon \mathbb{C} \longrightarrow \mathbb{A}$ a distributor, between $\mathscr{Q}$-enriched categories; we'll say that

$$\mathbb{C} \xrightarrow{\ \Phi\ } \mathbb{A} \xrightarrow{\ F\ } \mathbb{B} \tag{37}$$

is a Φ-weighted diagram in $\mathbb{B}$. Since F determines a right adjoint distributor $\mathbb{B}(F-,-) \colon \mathbb{B} \longrightarrow \mathbb{A}$, we may consider an associated diagram of distributors of which, by closedness of the quantaloid $\mathsf{Dist}(\mathscr{Q})$, we can calculate the universal lifting—see the left hand side of diagram 0.2. If there exists a functor $G \colon \mathbb{C} \longrightarrow \mathbb{B}$ for which $\mathbb{B}(G-,-) = \big[\Phi, \mathbb{B}(F-,-)\big]$ (i.e. G represents the distributor), then such a G is essentially unique: would G' be another such functor, then G and G' are isomorphic elements in (the preordered set) $\mathsf{Cat}(\mathscr{Q})(\mathbb{C}, \mathbb{B})$. Therefore it makes sense to define the Φ-weighted colimit of F as the (essentially unique, when it exists) functor[13] $\mathrm{colim}(\Phi, F) \colon \mathbb{C} \longrightarrow \mathbb{B}$ representing the universal lifting

[12] In other words, $*_A$ is the image of $A \in \mathscr{Q}_0$ by the canonical inclusion $\mathscr{Q} \longrightarrow \mathsf{Dist}(\mathscr{Q})$.

[13] Some would probably be happier to see $\Phi \star F$ as notation instead of $\mathrm{colim}(\Phi, F)$.

$$\begin{tikzcd} \mathbb{A} & \mathbb{B} \arrow[l, "\mathbb{B}(F-,-)"', "\circ" marking] \arrow[dl, "\circ" marking, "{\left[\Phi,\mathbb{B}(F-,-)\right]}"] \\ \mathbb{C} \arrow[u, "\Phi", "\circ" marking] & \end{tikzcd} \qquad \begin{tikzcd} \mathbb{A} \arrow[r, "F"] & \mathbb{B} \\ \mathbb{C} \arrow[u, "\Phi", "\circ" marking] \arrow[ur, "{\mathrm{colim}(\Phi,F)}"'] & \end{tikzcd}$$

Diagram 0.2

$\left[\Phi,\mathbb{B}(F-,-)\right]:\mathbb{B}\xrightarrow{\circ}\mathbb{C}$. The right hand side of diagram 0.2 resumes the situation.

A $\mathscr{Q}$-category $\mathbb{B}$ is said to be cocomplete when for every (small) weighted diagram in $\mathbb{B}$ (cf. (37)) the colimit exists. Such a cocomplete $\mathscr{Q}$-category has at least as many objects as the base quantaloid $\mathscr{Q}$; in particular, if $\mathscr{Q}$ is a "large" quantaloid, then any cocomplete $\mathscr{Q}$-category will be "large" too. In principle we should thus carefully distinguish, in everything that follows, between "small" and "large" $\mathscr{Q}$-categories. To avoid this rather unpleasant – and not very instructive – technical complication, we will *from now on suppose that $\mathscr{Q}$ is a small quantaloid* unless we explicitly state otherwise.

Given data as in

$$\mathbb{A}\xrightarrow[\circ]{\Phi}\mathbb{B}\xrightarrow{F}\mathbb{C}\xrightarrow{F'}\mathbb{C}', \tag{38}$$

and supposing that the Φ-weighted colimit of F exists, then F' is said to preserve this colimit if the Φ-weighted colimit of $F'\circ F$ exists too, and moreover

$$\mathrm{colim}(\Phi,F'\circ F)\cong F'\circ\mathrm{colim}(\Phi,F). \tag{39}$$

A colimit is absolute if it is preserved by any functor, and a functor is cocontinuous if is preserves any colimit. Colimits whose weight is a left adjoint distributor are always absolute, and functors which are left adjoint are always cocontinuous.

(iii) Presheaves on a $\mathscr{Q}$-category: Let $\mathbb{A}$ be some (small) $\mathscr{Q}$-enriched category. The $\mathscr{Q}$-enriched category $\mathscr{P}\mathbb{A}$ of presheaves on $\mathbb{A}$ is defined by the following universal property: for every $\mathscr{Q}$-category $\mathbb{C}$ there is a natural equivalence of preorders[14]

$$\mathsf{Dist}(\mathscr{Q})(\mathbb{C},\mathbb{A})\simeq\mathsf{Cat}(\mathscr{Q})(\mathbb{C},\mathscr{P}\mathbb{A}). \tag{40}$$

That is to say, $\mathscr{P}\mathbb{A}$ ought to "classify" the distributors with codomain $\mathbb{A}$.

It is not hard to see that the objects of $\mathscr{P}\mathbb{A}$ are exactly distributors with codomain $\mathbb{A}$ and whose domain is a trivial one-object $\mathscr{Q}$-enriched category. We

[14]It turns out that $\mathsf{Cat}(\mathscr{Q})(\mathbb{C},\mathscr{P}\mathbb{A})$ is a partial order rather than a preorder. As $\mathsf{Dist}(\mathscr{Q})(\mathbb{C},\mathbb{A})$ is a complete lattice – it is the "hom" of a quantaloid – this equivalence is actually an isomorphism of complete lattices.

will always denote these with small Greek letters, like $\phi\colon *_A \longrightarrow\!\!\!\!\!\!\!\!\circ\;\;\; \mathbb{A}$, $\psi\colon *_B \longrightarrow\!\!\!\!\!\!\!\!\circ\;\;\; \mathbb{A}$, etc. In elementary terms, such a presheaf $\phi\colon *_A \longrightarrow\!\!\!\!\!\!\!\!\circ\;\;\; \mathbb{A}$ is thus a collection of $\mathcal{Q}$-arrows

$$\Big(\phi(x)\colon A \longrightarrow tx\Big)_{x\in\mathbb{A}_0} \tag{41}$$

satisfying the inequalities expressing the action by $\mathbb{A}$. (The action by $*_A$ is trivial!) The collection of such contravariant presheaves on $\mathbb{A}$ thus forms a $\mathcal{Q}_0$-typed set[15]: the type in $\mathcal{Q}_0$ of $\phi\colon *_A \longrightarrow\!\!\!\!\!\!\!\!\circ\;\;\; \mathbb{A}$ is $t\phi = A$. For two such presheaves $\phi\colon *_A \longrightarrow\!\!\!\!\!\!\!\!\circ\;\;\; \mathbb{A}$ and $\psi\colon *_B \longrightarrow\!\!\!\!\!\!\!\!\circ\;\;\; \mathbb{A}$ the hom-arrow $\mathscr{P}\mathbb{A}(\psi,\phi)\colon A \longrightarrow B$ is the single element of the distributor $[\psi,\phi]\colon *_A \longrightarrow\!\!\!\!\!\!\!\!\circ\;\;\; *_B$ (i.e. the lifting of ϕ through ψ).

Putting $\mathbb{C} = \mathbb{A}$ in (40) there must correspond to the identity distributor $\mathbb{A}\colon \mathbb{A} \longrightarrow\!\!\!\!\!\!\!\!\circ\;\;\; \mathbb{A}$ some functor from $\mathbb{A}$ to $\mathscr{P}\mathbb{A}$: it is of course the Yoneda embedding $Y_\mathbb{A}\colon \mathbb{A} \longrightarrow \mathscr{P}\mathbb{A}$. It can be figured out that $Y_\mathbb{A}$ sends an object $a \in \mathbb{A}_0$ onto the representable presheaf $\mathbb{A}(-,a)\colon *_{ta} \longrightarrow\!\!\!\!\!\!\!\!\circ\;\;\; \mathbb{A}$, the elements of which – as its notation suggests – are

$$\Big(\mathbb{A}(x,a)\colon ta \longrightarrow tx\Big)_{x\in\mathbb{A}_0}. \tag{42}$$

From here on one can prove such things (familiar from $\mathscr{V}$-category theory) as:

- the "Yoneda lemma" says that for any $\phi \in \mathscr{P}\mathbb{A}$ and $a \in \mathbb{A}$,

$$\mathscr{P}\mathbb{A}(Y_\mathbb{A}a, \Phi) = \phi(a); \tag{43}$$

- as a consequence, the Yoneda embedding $Y_\mathbb{A}\colon \mathbb{A} \longrightarrow \mathscr{P}\mathbb{A}$ is a fully faithful functor in the sense that, for any $a, a' \in \mathbb{A}_0$,

$$\mathbb{A}(a',a) = \mathscr{P}\mathbb{A}(Y_\mathbb{A}a', Y_\mathbb{A}a); \tag{44}$$

- every presheaf is a "colimit of representables": for a presheaf $\phi \in \mathscr{P}\mathbb{A}$ (of type $t\phi = A$, say) consider the weighted diagram

$$*_A \xrightarrow{\ \phi\ } \mathbb{A} \xrightarrow{\ Y_\mathbb{A}\ } \mathscr{P}\mathbb{A};$$

its colimit exists and is the functor which "picks out" exactly the object $\phi \in \mathscr{P}\mathbb{A}$, so we write with some abuse of notation that

$$\mathrm{colim}(\phi, Y_\mathbb{A}) = \phi; \tag{45}$$

- $Y_\mathbb{A}\colon \mathbb{A} \longrightarrow \mathscr{P}\mathbb{A}$ is the free cocompletion in $\mathsf{Cat}(\mathcal{Q})$: the inclusion of the category $\mathsf{Cocont}(\mathcal{Q})$ of cocomplete $\mathcal{Q}$-categories and cocontinuous functors (i.e. functors that preserve colimits) in $\mathsf{Cat}(\mathcal{Q})$ admits a left adjoint which is precisely the presheaf construction:

$$\mathscr{P} \dashv i\colon \mathsf{Cocont}(\mathcal{Q}) \rightleftarrows \mathsf{Cat}(\mathcal{Q}). \tag{46}$$

[15]Note that there too it is quite essential that $\mathcal{Q}$ is a *small* quantaloid: if not, $\mathscr{P}\mathbb{A}$ would have a proper class of objects. Similar remarks apply to the Cauchy completion $\mathbb{A}_{\mathsf{cc}}$ encountered further on.

(iv) Cauchy completion of a $\mathcal{Q}$-category: We've seen that every functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ determines a left adjoint distributor $\mathbb{B}(-,F-)\colon \mathbb{A} \longrightarrow\!\!\!\!\!\!\circ\;\;\, \mathbb{B}$. But the converse need not hold: not every left adjoint distributor $\Phi\colon \mathbb{A} \longrightarrow\!\!\!\!\!\!\circ\;\;\, \mathbb{B}$ is determined by a functor! Following [Lawvere, 1973; Street, 1983] we'll use the term 'Cauchy distributor' as synonym for 'left adjoint distributor', and when a Cauchy distributor $\Phi\colon \mathbb{A} \longrightarrow \mathbb{B}$ is determined by a functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ we'll say that Φ converges to F. (Such an F is necessarily essentially unique.) If all Cauchy distributors with codomain $\mathbb{B}$ converge, then $\mathbb{B}$ is said to be Cauchy complete.

For a small $\mathcal{Q}$-enriched category $\mathbb{A}$ a new $\mathcal{Q}$-category $\mathbb{A}_{\mathsf{cc}}$ is defined by the following universal property: for every $\mathcal{Q}$-category $\mathbb{C}$ there is a natural equivalence of preorders[16]

$$\mathsf{Map}(\mathsf{Dist}(\mathcal{Q}))(\mathbb{C},\mathbb{A}) \simeq \mathsf{Cat}(\mathcal{Q})(\mathbb{C},\mathbb{A}_{\mathsf{cc}}). \tag{47}$$

In other words, $\mathbb{A}_{\mathsf{cc}}$ is supposed to universally classify the Cauchy distributors into $\mathbb{A}$. (The left hand side of this equation denotes the left adjoints – the "maps" – amongst the distributors from $\mathbb{C}$ to $\mathbb{A}$.) It can be shown that this new $\mathcal{Q}$-category $\mathbb{A}_{\mathsf{cc}}$ is Cauchy complete; it is called the Cauchy completion of $\mathbb{A}$.

It turns out that $\mathbb{A}_{\mathsf{cc}}$ is really a full subcategory of $\mathscr{P}\mathbb{A}$: it is determined by those presheaves on $\mathbb{A}$ which are left adjoint [Betti and Carboni, 1983; Street, 1983]. So suppose that $\phi\colon *_A \longrightarrow\!\!\!\!\!\!\circ\;\;\, \mathbb{A}$ and $\psi\colon *_B \longrightarrow\!\!\!\!\!\!\circ\;\;\, \mathbb{A}$ (contravariant presheaves on $\mathbb{A}$) are actually Cauchy distributors (say $\phi \dashv \phi^*$ and $\psi \dashv \psi^*$ in $\mathsf{Dist}(\mathcal{Q})$), then $\mathbb{A}_{\mathsf{cc}}(\psi,\phi)\colon A \longrightarrow B$ is the $\mathcal{Q}$-arrow which is the single element of the lifting $[\psi,\phi]\colon *_A \longrightarrow\!\!\!\!\!\!\circ\;\;\, *_B$ in $\mathsf{Dist}(\mathcal{Q})$. (It is – because of the adjunction $\psi \dashv \psi^*$ – equivalently given by the single element of the composition $\psi^* \otimes_{\mathbb{A}} \phi\colon *_A \longrightarrow *_B$.)

Of course the identity distributor $\mathbb{A}\colon \mathbb{A} \longrightarrow\!\!\!\!\!\!\circ\;\;\, \mathbb{A}$ itself is Cauchy; thus by (47) it must correspond to some functor $i_{\mathbb{A}}\colon \mathbb{A} \longrightarrow \mathbb{A}_{\mathsf{cc}}$. This functor is precisely the factorization of the Yoneda embedding $Y_{\mathbb{A}}\colon \mathbb{A} \longrightarrow \mathscr{P}\mathbb{A}$ over the full subcategory $\mathbb{A}_{\mathsf{cc}} \subseteq \mathscr{P}\mathbb{A}$. Some results – familiar in $\mathscr{V}$-category theory – can now be obtained:

- $i_{\mathbb{A}}\colon \mathbb{A} \longrightarrow \mathbb{A}_{\mathsf{cc}}$ is the absolute-colimit completion of $\mathbb{A}$ in $\mathsf{Cat}(\mathcal{Q})$: the Cauchy construction is left adjoint to the full inclusion of the Cauchy complete categories in $\mathsf{Cat}(\mathcal{Q})$:

$$(-)_{\mathsf{cc}} \dashv i\colon \mathsf{Cat}_{\mathsf{cc}}(\mathcal{Q}) \underset{\longrightarrow}{\overset{\longleftarrow}{}} \mathsf{Cat}(\mathcal{Q}); \tag{48}$$

- for two $\mathcal{Q}$-categories $\mathbb{A}$ and $\mathbb{B}$ we have that

$$\mathbb{A} \cong \mathbb{B} \text{ in } \mathsf{Dist}(\mathcal{Q}) \iff \mathbb{A}_{\mathsf{cc}} \simeq \mathbb{B}_{\mathsf{cc}} \text{ in } \mathsf{Cat}(\mathcal{Q}) \iff \mathscr{P}\mathbb{A} \simeq \mathscr{P}\mathbb{B} \text{ in } \mathsf{Cat}(\mathcal{Q}),$$

and $\mathbb{A}$ and $\mathbb{B}$ are said to be Morita equivalent if (any of) these conditions hold;

- any $\mathcal{Q}$-category $\mathbb{A}$ is isomorphic in $\mathsf{Dist}(\mathcal{Q})$ to its Cauchy completion $\mathbb{A}_{\mathsf{cc}}$, so there is an equivalence of quantaloids

$$\mathsf{Dist}_{\mathsf{cc}}(\mathcal{Q}) \simeq \mathsf{Dist}(\mathcal{Q}) \tag{49}$$

[16] It turns out that this is actually an isomorphism of partial orders.

(where $\mathsf{Dist}_{\mathsf{cc}}(\mathcal{Q})$ denotes the obvious full subcategory);

- and as a result – in combination with (47) – there are equivalences of locally preordered categories

$$\mathsf{Map}(\mathsf{Dist}(\mathcal{Q})) \simeq \mathsf{Map}(\mathsf{Dist}_{\mathsf{cc}}(\mathcal{Q})) \simeq \mathsf{Cat}_{\mathsf{cc}}(\mathcal{Q}). \tag{50}$$

So, if one really only cares about "Cauchy invariant" (i.e. "Morita equivalent") constructions involving categories and functors (as one usually does), then one might as well forget about functors altogether and work with $\mathsf{Map}(\mathsf{Dist}(\mathcal{Q}))$ instead.

This is a lot of theory—and it may be useful to look at some examples here.

(a) Consider again Lawvere's generalized metric spaces (see p. 11); let $\mathbb{A}$ denote such a space. In [Lawvere, 1973] it is proved that a left adjoint distributor from a one-point space into $\mathbb{A}$ is precisely (an equivalence class of) a Cauchy sequence in $\mathbb{A}$—and it is therefore that left adjoint distributors are also called Cauchy distributors. Cauchy completeness of $\mathbb{A}$ as enriched category coincides with its Cauchy completeness as metric space ("all Cauchy sequences/distributors converge").

(b) For an object A of a quantaloid $\mathcal{Q}$ we denote $*_A$ for the one-object $\mathcal{Q}$-category whose single object is of type A and whose single hom-arrow is 1_A. A contravariant presheaf on $*_A$ is – applying (41) – just an arrow $f\colon X \longrightarrow A$ in $\mathcal{Q}$; and the hom-arrow between two such presheaves $X \xrightarrow{f} A \xleftarrow{g} Y$ is the lifting $[g, f]$. That is to say, with notations as introduced in the examples on p. 11, $\mathscr{P}*_A = \mathbb{Q}_A$.

(c) Let A be an object of the quantaloid $\mathcal{Q}$, and $\mathbb{A}$ a $\mathcal{Q}$-category. A functor $F\colon \mathbb{A}^{\mathsf{op}} \longrightarrow \mathbb{Q}^A$ between these $\mathcal{Q}^{\mathsf{op}}$-categories (which were defined in the examples on p. 11) is a mapping of objects, associating to any $a \in \mathbb{A}_0$ some arrow $Fa\colon A \longrightarrow ta$ in $\mathcal{Q}$, such that $\mathbb{A}^{\mathsf{op}}(a', a) \leq \{Fa', Fa\}$ for any two objects a, a' in $\mathbb{A}^{\mathsf{op}}$. The latter is equivalent to $\mathbb{A}(a, a') \circ Fa' \leq Fa$, so F is *precisely* the same thing as a distributor $\phi\colon *_A \mathrel{\circ\!\!\longrightarrow} \mathbb{A}$ between $\mathcal{Q}$-categories as in (41)—i.e. a contravariant presheaf on $\mathbb{A}$. (So a contravariant presheaf on $\mathbb{A}$ is a "functor on the opposite of $\mathbb{A}$ with values in $\mathcal{Q}$" after all!)

(d) As indicated previously, $\mathbf{2}$-enriched categories are actually preordered sets, functors between $\mathbf{2}$-categories are just order-preserving maps, and distributors are ideal relations. More precisely we can state that $\mathsf{Cat}(\mathbf{2}) \cong \mathsf{PreOrd}$ and $\mathsf{Dist}(\mathbf{2}) \cong \mathsf{Idl}$. It now follows quite easily that, for a $\mathbf{2}$-category $\mathbb{A}$, a presheaf on $\mathbb{A}$ is exactly a "down-closed subset" of the preorder $\mathbb{A}_0$; and the Yoneda embedding $Y_{\mathbb{A}}\colon \mathbb{A} \longrightarrow \mathscr{P}\mathbb{A}$ maps an object $a \in \mathbb{A}_0$ to the "principal down-closed subset" that it determines. This is precisely the sup-completion of $\mathbb{A}_0$! But this is entirely consistent with the general fact that the presheaf-category is the free cocompletion, for a $\mathbf{2}$-category $\mathbb{A}$ has all weighted colimits if and only if $\mathbb{A}_0$ has all suprema. To see this, consider a weighted diagram of **2**-categories

$$*_C \xrightarrow[\circ]{\phi} \mathbb{B} \xrightarrow{F} \mathbb{A}$$

(and without loss of generality we let the weight be a presheaf on $\mathbb{B}$, i.e. ϕ_0 is a down-closed subset of $\mathbb{B}_0$). Then actually the lifting

$$\Big[\phi, \mathbb{B}(F-,-)\Big] : \mathbb{A} \xrightarrow{\circ} *_C$$

corresponds to the up-closed subset of $\mathbb{A}_0$ which contains all the upper bounds of the image through F of the elements of ϕ_0:

$$\Big[\phi, \mathbb{B}(F-,-)\Big]_0 = \{a \in \mathbb{A}_0 \mid \text{for all } b \in \phi_0 : Fb \leq a\}.$$

The ϕ-weighted colimit of F exists if and only if this lifting is representable—which is precisely the case when there is a least upper bound for the family $\{Fb \in \mathbb{A}_0 \mid b \in \phi_0\}$, because this supremum is then precisely the representing object for the colimit! Similar calculations show that every **2**-category is Cauchy complete: any Cauchy presheaf into a 2-category $\mathbb{A}$ is necessarily a principle down-closed set in $\mathbb{A}_0$.

Especially this latter example seems to indicate that the theory of $\mathcal{Q}$-enriched categories, functors and distributors is like an "order-theory with truth-values in $\mathcal{Q}$". This is surely a helpful intuition, but things are a bit more complicated than that. The second part of the thesis is precisely devoted to a thorough study of these complications.

Ideals, orders and sets over a base quantaloid

As quantales may be viewed as "non-commutative locales", and quantales themselves are only quantaloids with one single object, one may wonder about "sheaves on a quantaloid" as generalization of sheaves on a locale. Given that sheaves on a locale Ω can be described in terms of so-called Ω-sets, and the latter are in some sense easier to manipulate, several authors have rather tried to generalize those Ω-sets to "$\mathcal{Q}$-sets" for a quantale or quantaloid $\mathcal{Q}$. This is not at all straightforward! Starting from the work of [Van den Bossche, 1995; Borceux and Cruciani, 1998; van der Plancke, 1998] we will analyze the situation using the formalism of quantaloid-enriched categorical structures. As it turns out, we need a theory of "categories without units" enriched in a base quantaloid—so we generalize and further develop the theory of "categories without units" enriched in a symmetric monoidal closed $\mathcal{V}$ of [Moens, 2000; Moens *et al.*, 2002].

But before we plunge ourselves in technical developments of all kinds, we should probably first ask ourselves...

Do we really need "sheaves on a quantaloid"?

To give some motivation (other than curiosity) for the study of "sheaves on a quantaloid" it may be good to quickly recall the basic idea behind so-called sheaf representations of rings; let us do this with a simple example (quoted from [Borceux, 1994]). Thereto let R denote a commutative ring (with unit). An ideal $I \subseteq R$ is radical when, for every $r \in R$ and $n \in \mathbb{N}$, if $r^n \in I$ then also $r \in I$. The radical ideals of R form a localic quotient $\mathsf{Rad}(R)$ of the quantale $\mathsf{Idl}(R)$:

$$\mathsf{Idl}(R) \longrightarrow \mathsf{Rad}(R) \colon I \mapsto \sqrt{I} = \{r \in R \mid \exists n \in \mathbb{N} : r^n \in I\}.$$

In the topos of sheaves on $\mathsf{Rad}(R)$ there is a ring-object $\mathscr{R}$ whose ring of global sections is isomorphic to R; therefore $\mathscr{R}$ is said to be a sheaf representation for R. But this sheaf $\mathscr{R}$ of rings has the interesting feature that it is a local ring in the topos of sheaves on $\mathsf{Rad}(R)$: for any $I \in \mathsf{Rad}(R)$ and $r \in \mathscr{R}(I)$ there exists a covering $I = \bigvee_{k \in K} I_k$ in $\mathsf{Rad}(R)$ such that for every $k \in K$ either $r \mid_{I_k}$ or $(1-r) \mid_{I_k}$ is invertible in $\mathscr{R}(I_k)$. This explains the interest of the sheaf representation: even though R is only a commutative ring (in the topos of sets), $\mathscr{R}$ is a local ring (in the topos of sheaves on $\mathsf{Rad}(R)$).

There are many other examples of interesting sheaf representations, and most of them make use of one or another locale of ideals of a given ring R to serve as "topology" on which R is then represented as a more-or-less interesting sheaf—and so there is absolutely no confusion about the notion of "sheaf" in these cases. But we have argued on p. 5 that the so-to-speak "generic topology" for a ring R is its quantaloid of ideals $\mathscr{Q}_R$ (which, for a commutative R, is equivalent to the quantale $\mathsf{Idl}(R)$). So it would certainly be interesting to discuss sheaf representations of R over this quantaloid. (We already know of a way in which R determines a $\mathscr{Q}_R$-enriched category – see p. 11 – but in what sense is this a "sheaf" or anything of that kind?)

This is the place to mention that several authors have put forward – sometimes quite different – ideas on this topic in the context of sheaf representations for rings and algebras:

- C. Mulvey and M. Nawaz [1995] work with an "idempotent right-sided quantale[17]", and for them a quantale-set is a sheaf on the sublocale of two-sided elements in the quantale. In the localic topos of these quantale-sets they recognize an extra "quantalic subobject classifier". (See also Nawaz' [1985] doctoral thesis.)

[17]Such an "idempotent right-sided quantale" is a complete lattice Q equipped with a binary product $-\&-\colon Q \times Q \longrightarrow Q$ which distributes on both sides over suprema, for which every element of the lattice is idempotent ($x\&x = x$), and for which the top element of the lattice is a unit-on-the-right ($x\&\top = x$). As the product is not required to have a (two-sided) unit, this is *not* a quantale in our sense!

- F. Borceux and B. Van den Bossche [1989] construct, for a given "idempotent right-sided quantale" Q, a category whose objects are the principal down-closed subsets of Q and whose morphisms are suitable "restriction maps". On this category they recognize a Grothendieck topology, and so they define a "sheaf on Q" as a sheaf on this site. In this Grothendieck topos the locales of subobjects can be provided with the additional structure of a quantale.

- U. Berni-Canani *et al.* [1989] too start from such an "idempotent right-sided quantale" and develop a theory of quantale-sets, but now these are the objects of a fibration of localic toposes over the given quantale. And the localic topos of [Mulvey and Nawaz, 1995] is then actually the stalk at the top element of the quantale.

- G. Van den Bossche [1995] takes any quantaloid $\mathscr{Q}$ and defines a "$\mathscr{Q}$-set" to be an idempotent matrix with elements in $\mathscr{Q}$. She shows how, when equipped with suitable morphisms, the category of $\mathscr{Q}_R$-sets (where $\mathscr{Q}_R$ is the "quantaloid of ideals" for a – not necessarily commutative – ring R, as mentioned on p. 5) contains the object Comm_R (see p. 11) whose "global sections" correspond to the elements of $\mathscr{Z}(R)$, and certain other "sections" correspond to the elements of R itself.

- F. van der Plancke [1998] further investigates these "$\mathscr{Q}$-sets" that [Van den Bossche, 1995] introduced, taking the point of view that such an object is a "set with an equality-predicate that takes its values in $\mathscr{Q}$". With this intuition a suitable notion of "subset of a $\mathscr{Q}$-set" is given, and – amongst other things – it is shown how the "$\mathscr{Q}$-sets of subsets" determine, and are determined by, projective objects in the category $\mathsf{Quant}(\mathscr{Q}, \mathsf{Sup})$ of Sup-presheaves on $\mathscr{Q}$.

- R. Gylys [2001] combines elements from Van den Bossche's [1995] "matrix-approach" with Borceux and Van den Bossche's [1986] "restriction-and-gluing" approach. For [Gylys, 2001] "$\mathscr{Q}$-sets" are sets with a suitable $\mathscr{Q}$-valued equality; such a $\mathscr{Q}$-set is "complete" if it allows "restriction and gluing" of its "singletons". Therefore, a $\mathscr{Q}$-set is to be thought of as a "presheaf", and a complete $\mathscr{Q}$-set as a "sheaf", on $\mathscr{Q}$.

- Given a commutative ring R, S. Ambler and D. Verity [1996] consider enrichments in the split-idempotent completion of the quantale $\mathsf{Idl}(R)$. They show how the ring R is "represented" by an enriched category – to be thought of as a "presheaf" on $\mathsf{Idl}(R)$ – and how the Cauchy completion of this category corresponds to its "sheafification". Their formalism is thus a generalization of Walters' [1981] description of sheaves on a locale as enriched categories. They also recover the results in [Borceux and Van den Bossche, 1991].

Especially the works of G. Van den Bossche, F. van der Plancke and M.-A. Moens have been of great influence on the present work. I only recently "discovered" [Ambler and Verity, 1996; Gylys, 2001]—and was pleased to find out that their results are particular cases of the more general theory.

Ordered sheaves on a locale

The topos $\mathsf{Sh}(\Omega)$ of sheaves on a locale Ω can be described in terms of so-called Ω-sets [Fourman and Scott, 1979; Higgs, 1984; Fourman and Scott, 1979; Borceux, 1994; Johnstone, 2002]. An Ω-set $(A, [\,\cdot = \cdot\,])$ consists of a set A equipped with an "Ω-valued equality"

$$[\,\cdot = \cdot\,]\colon A \times A \longrightarrow \Omega$$

satisfying for all $a, a', a'' \in A$,

$$[a = a'] \wedge [a' = a''] \leq [a = a''], \tag{51}$$

$$[a = a'] = [a' = a]. \tag{52}$$

Clearly (51) is a transitivity axiom, and (52) is symmetry. There is no reflexivity axiom, for it would read

$$\forall a \in A\colon \top \leq [a = a] \tag{53}$$

which would force all elements of $(A, [\,\cdot = \cdot\,])$ to be "global" ($\top$ denotes the top of the locale). Together with a "good" notion of morphism, these Ω-sets form a category equivalent to $\mathsf{Sh}(\Omega)$.

With the intuition that a "non-symmetric equality" is an "inequality", F. Borceux and R. Cruciani [1998] have worked out a theory of "Ω-orders" (or Ω-posets as those authors call them). Such an Ω-order $(A, [\,\cdot \leq \cdot\,])$ consists of a set A together with an "Ω-valued inequality"

$$[\,\cdot \leq \cdot\,]\colon A \times A \longrightarrow \Omega$$

satisfying for all $a, a', a'' \in A$,

$$[a \leq a'] \wedge [a' \leq a''] \leq [a \leq a''], \tag{54}$$

$$[a \leq a'] \leq [a \leq a], \tag{55}$$

$$[a \leq a'] \leq [a' \leq a']. \tag{56}$$

Here again (54) says that the "inequality" is transitive, and symmetry is gotten rid off (and there is still no reflexivity)—but two new axioms (55) and (56) have to be imposed "to make things work" (to quote F. Borceux). Together with a "good" notion of morphism, Ω-orders form a category equivalent to $\mathsf{Ord}(\mathsf{Sh}(\Omega))$, the category of ordered objects in the topos $\mathsf{Sh}(\Omega)$.

It may now be observed that both Ω-sets and Ω-orders are categorical structures enriched in (the one-object suspension of) the locale Ω. Take an Ω-order

$(A, [\,\cdot \leq \cdot\,])$; it quite naturally determines an object set $\mathbb{A}_0 = A$ and an enrichment in Ω:

$$\Big(\mathbb{A}(a, a') = [a \leq a']\Big)_{(a,a') \in \mathbb{A}_0 \times \mathbb{A}_0}.$$

The transitivity axiom (54) then says that there are composition-inequalities (cf. (13)) but the lack of reflexivity (53) means that there are no unit-inequalities (cf. (14)): $\mathbb{A}$ is not a category but only a *semi*category enriched in (the one-object suspension of) Ω. Thus an Ω-order (and *a fortiori* an Ω-set) is an Ω-semicategory satisfying some extra axioms ((55) and (56) for orders, (52) for sets).

The point is that – when rewritten in a suitable manner – the axioms for Ω-orders still make sense when we replace the locale Ω by a quantaloid $\mathcal{Q}$. The axioms for Ω-sets do not make sense for a base quantaloid $\mathcal{Q}$ (because of the symmetry in (52)), unless it is endowed with an involution (see p. 47). Our aim here is to give a very precise analysis of these matters by means of the formalism of categorical structures enriched in a quantaloid.

Categories without units over a base quantaloid

By a $\mathcal{Q}$-enriched semicategory we will mean precisely a "category without units" enriched in the quantaloid $\mathcal{Q}$: a $\mathcal{Q}$-semicategory $\mathbb{A}$ thus consists of a $\mathcal{Q}_0$-typed set $\mathbb{A}_0$, as in (11), which is enriched in $\mathcal{Q}$, as in (12), satisfying composition-inequalities, as in (13)—and that's it. In terms of matrices, $\mathbb{A}$ is an endo-arrow in the quantaloid $\mathsf{Matr}(\mathcal{Q})$ that satisfies $\mathbb{A} \circ \mathbb{A} \leq \mathbb{A}$; it is a "monad without unit" in $\mathsf{Matr}(\mathcal{Q})$.

As neither the definition for "functor between $\mathcal{Q}$-categories" nor that for "distributor between $\mathcal{Q}$-categories" (see p. 10) refer in any way to the units of the involved $\mathcal{Q}$-categories, they still make sense for $\mathcal{Q}$-semicategories. (Essentially this is due to the fact that both the "functoriality" of a functor and the "compatibility of action with unit and composition axioms" of a distributor are diagrammatic axioms of two-cells in the base quantaloid—and any such diagram is commutative!) Thus, for two $\mathcal{Q}$-semicategories $\mathbb{A}$ and $\mathbb{B}$, a morphism $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ is by definition an object mapping $\mathbb{A}_0 \longrightarrow \mathbb{B}_0\colon a \mapsto Fa$ that preserves types, as in (15), satisfying the action-inequalities for the hom-arrows, as in (16). And a distributor $\Phi\colon \mathbb{A} \multimap \mathbb{B}$ is just a $|\mathbb{B}_0| \times |\mathbb{A}_0|$ matrix with elements in $\mathcal{Q}$, as in (17), satisfying the action-inequalities of (18) and (19). In other words, such a distributor is just a matrix $\Phi\colon \mathbb{A}_0 \longrightarrow \mathbb{B}_0$ together with action-inequalities $\Phi \circ \mathbb{A} \leq \Phi$ and $\mathbb{B} \circ \Phi \leq \Phi$; it is a "bimodule" between "monads without units" in $\mathsf{Matr}(\mathcal{Q})$.

It is quite trivial to verify that $\mathcal{Q}$-semicategories and morphisms form a category; the composition and identities are as for functors (see (28) and (29)). We will denote this category as $\mathsf{SCat}(\mathcal{Q})$, and from the definitions it is clear there is a (non-trivial[18]) embedding

$$\mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{SCat}(\mathcal{Q}). \tag{57}$$

This embedding is full: the reason, ultimately, is that in the definition for "functor" there is no reference whatsoever to the unit-inequalities for the domain and codomain $\mathcal{Q}$-categories. Moreover, this embedding has a left adjoint. Explicitly, given a $\mathcal{Q}$-semicategory $\mathbb{A}$, one defines a $\mathcal{Q}$-category $\overline{\mathbb{A}}$ by freely adding units: $\overline{\mathbb{A}}$ has the same objects as $\mathbb{A}$, but for the hom-arrows we put

$$\overline{\mathbb{A}}(a',a) = \begin{cases} \mathbb{A}(a',a) \text{ if } a \neq a', \\ \mathbb{A}(a,a) \bigvee 1_{ta} \text{ if } a = a'. \end{cases}$$

In other terms, viewing $\mathbb{A}$ as endo-matrix on $\mathbb{A}_0$, $\overline{\mathbb{A}}$ is the supremum with the identity matrix on $\mathbb{A}_0$ (22): $\overline{\mathbb{A}} = \mathbb{A} \bigvee \Delta_{\mathbb{A}_0}$. It is now a fact that, for a semicategory $\mathbb{A}$ and a category $\mathbb{C}$, an object mapping $\mathbb{A}_0 \longrightarrow \mathbb{C}_0 \colon a \mapsto Fa$ is a morphism between $\mathbb{A}$ and $\mathbb{C}$ if and only if the *same* object mapping is a functor between $\overline{\mathbb{A}}$ and $\mathbb{C}$; so we obtain that

$$\mathsf{Cat}(\mathcal{Q})(\overline{\mathbb{A}}, \mathbb{C}) = \mathsf{SCat}(\mathcal{Q})(\mathbb{A}, \mathbb{C}). \tag{58}$$

In sum we've described the left adjoint

$$\overline{(-)} \colon \mathsf{SCat}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q}). \tag{59}$$

In particular, for a morphism $F \colon \mathbb{A} \longrightarrow \mathbb{C}$ – with $\mathbb{A}$ a semicategory and $\mathbb{C}$ a category – we will write $\overline{F} \colon \overline{\mathbb{A}} \longrightarrow \mathbb{C}$ for the corresponding functor.

Let us now look at the distributors between $\mathcal{Q}$-semicategories. For two $\mathcal{Q}$-semicategories $\mathbb{A}$ and $\mathbb{B}$ it is certainly true that the collection of distributors from $\mathbb{A}$ to $\mathbb{B}$ forms a complete lattice:

$$\mathsf{Distrib}(\mathbb{A}, \mathbb{B}) = \{\Phi \in \mathsf{Matr}(\mathcal{Q})(\mathbb{A}, \mathbb{B}) \mid \Phi \circ \mathbb{A} \leq \Phi \text{ and } \mathbb{B} \circ \Phi \leq \Phi\}$$

is indeed a sublattice of $\mathsf{Matr}(\mathcal{Q})(\mathbb{A}, \mathbb{B})$, inheriting the suprema. But it is not true that these complete lattices are the hom-objects of some quantaloid of "$\mathcal{Q}$-semicategories and distributors": although the formula for the composition of distributors (21) still makes sense, and although a $\mathcal{Q}$-semicategory $\mathbb{A}$ is a distributor $\mathbb{A} \colon \mathbb{A} \mathrel{-\!\!\circ\!\!\rightarrow} \mathbb{A}$, it is not true in general that $\mathbb{A}$ is the *identity* distributor on itself. On the other hand, it follows from elementary calculations in the quantaloid $\mathsf{Matr}(\mathcal{Q})$ that, for any two $\mathcal{Q}$-semicategories $\mathbb{A}$ and $\mathbb{B}$, a matrix from $\mathbb{A}_0$ to $\mathbb{B}_0$ is a distributor between the semicategories $\mathbb{A}$ and $\mathbb{B}$ if and only if the *same* matrix is a distributor between the free categories $\overline{\mathbb{A}}$ and $\overline{\mathbb{B}}$:

$$\mathsf{Distrib}(\mathbb{A}, \mathbb{B}) = \mathsf{Dist}(\mathcal{Q})(\overline{\mathbb{A}}, \overline{\mathbb{B}}). \tag{60}$$

[18] For example, a strictly ordered set $(P, <)$ determines a **2**-semicategory which is not a category. More examples on p. 33.

When working with $\mathscr{Q}$-semicategories, it are especially (58) and (60) that are of great importance. To illustrate this, let us consider the presheaf-construction for semicategories—we will see how it reduces to the known presheaf construction for categories.

Indeed, let $\mathbb{A}$ and $\mathbb{B}$ be $\mathscr{Q}$-semicategories. It is then a matter of pasting together (58), (60) and (40) to see that

$$\mathsf{Distrib}(\mathbb{A},\mathbb{B}) = \mathsf{Dist}(\mathscr{Q})(\overline{\mathbb{A}},\overline{\mathbb{B}}) \cong \mathsf{Cat}(\mathscr{Q})(\overline{\mathbb{A}},\mathscr{P}\overline{\mathbb{B}}) = \mathsf{SCat}(\mathscr{Q})(\mathbb{A},\mathscr{P}\overline{\mathbb{B}}). \tag{61}$$

This indicates that $\mathscr{P}\overline{\mathbb{B}}$ – the $\mathscr{Q}$-category of (contravariant) presheaves on the free $\mathscr{Q}$-category $\overline{\mathbb{B}}$ – is the correct $\mathscr{Q}$-categorical structure whose objects are the (contravariant) presheaves on the $\mathscr{Q}$-semicategory $\mathbb{B}$: it "classifies" the distributors with codomain $\mathbb{B}$ (and as domain any $\mathscr{Q}$-semicategory $\mathbb{A}$). For a $\mathscr{Q}$-category $\mathbb{C}$ we know that $\overline{\mathbb{C}}=\mathbb{C}$ so it follows that $\mathscr{P}\mathbb{C}=\mathscr{P}\overline{\mathbb{C}}$ and there can thus be no confusion about the notion of "presheaf" on a $\mathscr{Q}$-category.

Since $\mathbb{B}$ itself is a distributor $\mathbb{B}\colon\mathbb{B}\mathrel{-\!\!\circ\!\!\rightarrow}\mathbb{B}$, there is a corresponding morphism $Y_{\mathbb{B}}\colon\mathbb{B}\longrightarrow\mathscr{P}\overline{\mathbb{B}}$—which we of course will call the Yoneda morphism[19]. For an object $b\in\mathbb{B}_0$ there is quite evidently a distributor $\mathbb{B}(-,b)\colon *_{tb}\mathrel{-\!\!\circ\!\!\rightarrow}\mathbb{B}$ with as elements the $\mathscr{Q}$-arrows

$$\Big(\mathbb{B}(x,b)\colon tb\longrightarrow tx\Big)_{x\in\mathbb{B}_0} \tag{62}$$

and by (60) there is a corresponding distributor $\mathbb{B}(-,b)\colon *_{tb}\mathrel{-\!\!\circ\!\!\rightarrow}\overline{\mathbb{B}}$ (which *as matrix* is identical, but note that the codomain has changed!). The latter is the image of b by $Y_{\mathbb{B}}$: it is the presheaf represented by b. But – unfortunately – two major properties for presheaves on a $\mathscr{Q}$-category fail for presheaves on a $\mathscr{Q}$-semicategory. For one thing, there is no "Yoneda lemma" for all presheaves on a given $\mathscr{Q}$-semicategory $\mathbb{B}$. And a second important difference is that a presheaf on a $\mathscr{Q}$-semicategory is not necessarily a "colimit of representables".

Still, for certain presheaves on a $\mathscr{Q}$-semicategory the "Yoneda lemma" may hold (we call these appropriately 'Yoneda presheaves'), and certain presheaves may be "colimit of representables" (and we call these 'regular presheaves'). Moens *et al.* [2002] have recognized the importance of – in particular – regular presheaves in the context of semicategories enriched in a symmetric monoidal closed category $\mathscr{V}$. In the next few paragraphs we will generalize their insights to $\mathscr{Q}$-enriched semicategories[20].

(i) Yoneda presheaves: Let $\mathbb{A}$ be a $\mathscr{Q}$-semicategory, and $\phi\in\mathscr{P}\overline{\mathbb{A}}$ a presheaf

[19] One must be careful not to confuse the Yoneda morphism $Y_{\mathbb{A}}\colon\mathbb{A}\longrightarrow\mathscr{P}\overline{\mathbb{A}}$ with the Yoneda functor $Y_{\overline{\mathbb{A}}}\colon\overline{\mathbb{A}}\longrightarrow\mathscr{P}\overline{\mathbb{A}}$; in particular is the latter quite different from the functor $\overline{Y_{\mathbb{A}}}\colon\overline{\mathbb{A}}\longrightarrow\mathscr{P}\overline{\mathbb{A}}$!

[20] In [Van den Bossche, 1995; van der Plancke, 1998] the "regularity" of presheaves plays an important rôle too, but this "regularity" is not recognized and studied *as such*.

$$
\begin{tikzcd}
*_A \arrow[r, "\phi"] \arrow[dr, "{\left[\mathbb{A},\phi\right]}"'] & \overline{\mathbb{A}} \\
 & \overline{\mathbb{A}} \arrow[u, "\mathbb{A}"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
*_A \arrow[r, "\phi"] \arrow[dr, "\mathbb{A}\otimes_{\overline{\mathbb{A}}}\phi"'] & \overline{\mathbb{A}} \arrow[d, "\mathbb{A}"] \\
 & \overline{\mathbb{A}}
\end{tikzcd}
$$

Diagram 0.3

on $\mathbb{A}$. We'll say that ϕ is a Yoneda presheaf if for all $a \in \mathbb{A}_0$

$$\mathscr{P}\overline{\mathbb{A}}(Y_{\mathbb{A}}a, \phi) = \phi(a) \text{ in } \mathscr{Q}. \tag{63}$$

That is to say, we explicitly ask the validity of the "Yoneda lemma" (43) for $\phi \in \mathscr{P}\overline{\mathbb{A}}$. Another way of putting this, is as follows: consider $\mathbb{A}$ as distributor from $\overline{\mathbb{A}}$ to $\overline{\mathbb{A}}$, then it is required that

$$\left[\mathbb{A}, \phi\right] = \phi \text{ in } \mathsf{Dist}(\mathscr{Q}); \tag{64}$$

see the left-hand side of diagram 0.3.

The Yoneda presheaves on a given $\mathscr{Q}$-semicategory $\mathbb{A}$ define a full subcategory of the category of all presheaves; let us denote it as $\mathscr{P}^y\mathbb{A}$. It is then an easy corollary that the Yoneda embedding $Y_{\mathbb{A}}\colon \mathbb{A} \longrightarrow \mathscr{P}\overline{\mathbb{A}}$ factors over the full embedding $\mathscr{P}^y\mathbb{A} \longrightarrow \mathscr{P}\overline{\mathbb{A}}$ if and only if $\mathbb{A}$ is a category rather than a semicategory—that is to say, precisely when all representable presheaves on $\mathbb{A}$ are Yoneda presheaves.

(ii) Regular presheaves: We will say that a presheaf $\phi\colon *_A \longrightarrow \overline{\mathbb{A}}$ on a semicategory $\mathbb{A}$ is regular if it is "a colimit of representables". More precisely, given ϕ we may consider the weighted diagram

$$*_A \xrightarrow{\ \phi\ } \overline{\mathbb{A}} \xrightarrow{\ \overline{Y_{\mathbb{A}}}\ } \mathscr{P}\overline{\mathbb{A}} \tag{65}$$

in the $\mathscr{Q}$-category $\mathscr{P}\overline{\mathbb{A}}$ (note that the functor involved is determined – through the universal property of the free category on $\mathbb{A}$ – by $Y_{\mathbb{A}}\colon \mathbb{A} \longrightarrow \mathscr{P}\overline{\mathbb{A}}$). This diagram must have a colimit, since $\mathscr{P}\overline{\mathbb{A}}$ is cocomplete! This colimit is thus a functor $\mathrm{colim}(\phi, \overline{Y_{\mathbb{A}}})\colon *_A \longrightarrow \mathscr{P}\overline{\mathbb{A}}$ which "picks out" exactly one object of $\mathscr{P}\overline{\mathbb{A}}$. When this object is precisely ϕ itself, we'll say that the presheaf $\phi \in \mathscr{P}\overline{\mathbb{A}}$ is regular: thus we explicitly ask explicitly that

$$\mathrm{colim}(\phi, \overline{Y_{\mathbb{A}}}) = \phi. \tag{66}$$

Straightforward calculations show that the colimit of the weighted diagram (65) in $\mathscr{P}\overline{\mathbb{A}}$ is actually equal to the composition $\mathbb{A} \otimes_{\overline{\mathbb{A}}} \phi$ ($\mathbb{A}$ is regarded as endodistributor on $\overline{\mathbb{A}}$), so that the requirement for ϕ to be a regular presheaf on $\mathbb{A}$

becomes quite simply that

$$\mathbb{A} \otimes_{\overline{\mathbb{A}}} \phi = \phi \text{ in } \mathsf{Dist}(\mathcal{Q}); \tag{67}$$

see the right-hand side of diagram 0.3.

The regular presheaves on a $\mathcal{Q}$-semicategory $\mathbb{A}$ define a full subcategory of $\mathcal{P}\overline{\mathbb{A}}$; we'll denote it as $\mathcal{P}^\star\mathbb{A}$. We will say that the semicategory $\mathbb{A}$ is regular when the Yoneda embedding $Y_\mathbb{A}: \mathbb{A} \longrightarrow \mathcal{P}\overline{\mathbb{A}}$ factors over the full inclusion $\mathcal{P}^\star\mathbb{A} \longrightarrow \mathcal{P}\overline{\mathbb{A}}$. In other words, $\mathbb{A}$ is a regular $\mathcal{Q}$-semicategory if and only if the representable presheaves are regular [Moens *et al.*, 2002]. It is clear that every $\mathcal{Q}$-category is a regular semicategory, but the converse is not true[21]. So 'regular semicategory' is a strictly weaker notion than 'category', but still strong enough to admit a "good behavior" of presheaves[22].

(It is true that our definitions for 'Yoneda presheaf' and 'regular presheaf' on a $\mathcal{Q}$-semicategory are different from – but equivalent to! – those in [Moens, 2000; Moens *et al.*, 2002]. Indeed, in discussing presheaves on a $\mathcal{Q}$-semicategory $\mathbb{A}$ we always passed over to the free category $\overline{\mathbb{A}}$: we used the theory of $\mathcal{Q}$-categories to say something about semicategories. But there are completely equivalent descriptions for presheaves, Yoneda presheaves and regular presheaves on semicategories that do not refer in any way to free categories—i.e. a more *ad hoc* treatment of the matter. The original definitions of M.-A. Moens [2000] – given for $\mathcal{V}$-enriched categories – are then precisely these alternative ones.)

In a sense, the requirement for a presheaf ϕ on a semicategory $\mathbb{A}$ to be 'Yoneda' (64) is "adjoint" to the requirement for it to be 'regular' (67)—compare both sides of diagram 0.3. Actually, for a regular $\mathcal{Q}$-semicategory $\mathbb{A}$ the inclusion

$$i: \mathcal{P}^\star\mathbb{A} \longrightarrow \mathcal{P}\overline{\mathbb{A}}: \phi \mapsto \phi$$

is left adjoint to

$$j: \mathcal{P}\overline{\mathbb{A}} \longrightarrow \mathcal{P}^\star\mathbb{A}: \psi \mapsto \mathbb{A} \otimes_{\overline{\mathbb{A}}} \phi$$

which in turn is left adjoint to

$$k: \mathcal{P}^\star\mathbb{A} \longrightarrow \mathcal{P}\overline{\mathbb{A}}: \theta \mapsto [\mathbb{A}, \theta].$$

Both i and k are fully faithful; and the image of $k: \mathcal{P}^\star\mathbb{A} \longrightarrow \mathcal{P}\overline{\mathbb{A}}$ is precisely $\mathcal{P}^y\mathbb{A}$. (So we have here what B. Lawvere [1996] calls a situation of "unity and identity of opposites".) As a result, the $\mathcal{Q}$-category $\mathcal{P}^\star\mathbb{A}$ is cocomplete. A similar reasoning holds to prove that $\mathcal{P}^y\mathbb{A}$ is cocomplete too.

[21] Example of a regular semicategory which is not a category: see p. 33.

[22] In fact, these regular $\mathcal{Q}$-enriched semicategories are exactly what van der Plancke [1998] calls "$\mathcal{Q}$-sets". A regular presheaf is then what he calls a "part" (or "subset") of a $\mathcal{Q}$-set, and he observes that "every part is the union of singletons"—that is to say, every regular presheaf is a colimit of representables.

Regular semicategories: a distributor calculus

A $\mathcal{Q}$-enriched semicategory $\mathbb{A}$ is precisely a "monad without units" in the quantaloid $\mathsf{Matr}(\mathcal{Q})$: it is an endo-matrix $\mathbb{A}\colon \mathbb{A}_0 \longrightarrow \mathbb{A}_0$ for which $\mathbb{A} \circ \mathbb{A} \leq \mathbb{A}$. We decided to call such a semicategory 'regular' when all representable presheaves on $\mathbb{A}$ are regular. In terms of matrices this comes down to asking for the endo-matrix $\mathbb{A}\colon \mathbb{A}_0 \longrightarrow \mathbb{A}_0$ to be idempotent[23].

Indeed, let $\mathbb{A}$ be a $\mathcal{Q}$-semicategory. For an object $a \in \mathbb{A}_0$ the representable presheaf $\mathbb{A}(-,a)\colon *_{ta} \multimap \overline{\mathbb{A}}$ is the matrix whose elements are the $\mathcal{Q}$-arrows

$$\Big(\mathbb{A}(x,a)\colon ta \longrightarrow tx\Big)_{x \in \mathbb{A}_0}.$$

And – cf. (67) – this representable is regular when $\mathbb{A} \otimes_{\overline{\mathbb{A}}} \mathbb{A}(-,a) = \mathbb{A}(-,a)$, or, when computing this composition of distributors,

$$\forall x \in \mathbb{A}_0 : \bigvee_{y \in \mathbb{A}_0} \mathbb{A}(x,y) \circ \mathbb{A}(y,a) = \mathbb{A}(x,a)$$

as $\mathcal{Q}$-arrows. Thus $\mathbb{A}$ is a regular $\mathcal{Q}$-semicategory if and only if

$$\forall a, a' \in \mathbb{A}_0 : \bigvee_{x \in \mathbb{A}_0} \mathbb{A}(a',x) \circ \mathbb{A}(x,a) = \mathbb{A}(a',a),$$

i.e. the matrix $\mathbb{A}\colon \mathbb{A}_0 \longrightarrow \mathbb{A}_0$ is an idempotent in $\mathsf{Matr}(\mathcal{Q})$.

The theory of regular $\mathcal{Q}$-semicategories is thus the theory of idempotents in $\mathsf{Matr}(\mathcal{Q})$. Let us therefore first consider – quite generally – idempotents in any quantaloid (small or large, that doesn't matter here), to afterwards specify our findings to the quantaloid $\mathsf{Matr}(\mathcal{Q})$.

For two idempotents $A \circlearrowleft^{e}$ and $B \circlearrowleft^{f}$ in a quantaloid $\mathcal{Q}$ we say that a $\mathcal{Q}$-arrow $b\colon A \longrightarrow B$ is a regular bimodule if $b \circ e = b$ and $f \circ b = b$, and we'll write this as $b\colon e \multimap f$. Such bimodules can be composed: given $b\colon e \multimap f$ and $c\colon f \multimap g$ we denote $c \otimes_f b\colon e \multimap g$ for the composition which is computed simply as $c \circ b$ in $\mathcal{Q}$. An idempotent e is a regular bimodule on itself, indeed it is the identity bimodule: $e\colon e \multimap e$. We thus obtain a category $\mathsf{Bim}^{\star}(\mathcal{Q})$ whose objects are idempotents and whose arrows are regular bimodules; it is in fact a quantaloid because the supremum of regular bimodules $(b_i\colon e \multimap f)_{i \in I}$ may be calculated as supremum of (the underlying) $\mathcal{Q}$-arrows. (Note that $\mathsf{Bim}^{\star}(\mathcal{Q})$ is small whenever $\mathcal{Q}$ is.)

[23] In [Van den Bossche, 1995] this is precisely the definition of "$\mathcal{Q}$-set". F. van der Plancke [1995] was apparently aware of the fact that his "$\mathcal{Q}$-sets" – which he defined in more elementary terms, see a previous footnote – are precisely these idempotent matrices, but he didn't exploit this fact.

Recall that $\mathsf{Bim}(\mathcal{Q})$ denoted the quantaloid of monads and bimodules in $\mathcal{Q}$ (see p. 15). Quite evidently, every bimodule between monads is a regular bimodule between idempotents. That is to say, there are full embeddings

$$\mathcal{Q} \longrightarrow \mathsf{Bim}(\mathcal{Q}) \longrightarrow \mathsf{Bim}^\star(\mathcal{Q}). \tag{68}$$

We have seen that $\mathcal{Q} \longrightarrow \mathsf{Bim}(\mathcal{Q})$ is the universal splitting of monads, and it is thus not too surprising that $\mathcal{Q} \longrightarrow \mathsf{Bim}^\star(\mathcal{Q})$ is the universal splitting of idempotents [Freyd, 1964]. This universal property of $\mathsf{Bim}^\star(\mathcal{Q})$ makes it easy to characterize those quantaloids which arise as quantaloids of idempotents and regular bimodules: they are the quantaloids that are Cauchy complete *as category* [Borceux and Dejean, 1986].

We can now perform the splitting of idempotents in the quantaloid $\mathsf{Matr}(\mathcal{Q})$: we obtain a new quantaloid

$$\mathsf{Dist}^\star(\mathcal{Q}) = \mathsf{Bim}^\star(\mathsf{Matr}(\mathcal{Q})) \tag{69}$$

whose objects are precisely the regular $\mathcal{Q}$-enriched semicategories, and whose morphisms we will obviously call 'regular distributors'. Explicitly, a regular distributor $\Phi\colon \mathbb{A} \xrightarrow{\circ} \mathbb{B}$ between regular $\mathcal{Q}$-semicategories is thus a matrix, say some $\Phi\colon (\mathbb{A}_0, t) \longrightarrow (\mathbb{B}_0, t)$, such that

$$\text{for all } a \in \mathbb{A}_0, b \in \mathbb{B}_0 : \bigvee_{a' \in \mathbb{A}_0} \Phi(b, a') \circ \mathbb{A}(a', a) = \Phi(b, a); \tag{70}$$

$$\text{for all } a \in \mathbb{A}_0, b \in \mathbb{B}_0 : \bigvee_{b' \in \mathbb{B}_0} \mathbb{B}(b, b') \circ \Phi(b', a) = \Phi(b, a). \tag{71}$$

That is to say, a *regular* distributor is one for which the action-inequalities of (25) and (26) "saturate" to equalities. This quantaloid $\mathsf{Dist}^\star(\mathcal{Q})$ is the appropriate "distributor calculus" for (regular) $\mathcal{Q}$-enriched semicategories. The base quantaloid $\mathcal{Q}$ can obviously be embedded in $\mathsf{Dist}^\star(\mathcal{Q})$; more precisely, there are full embeddings

$$\mathcal{Q} \longrightarrow \mathsf{Dist}(\mathcal{Q}) \longrightarrow \mathsf{Dist}^\star(\mathcal{Q}) \tag{72}$$

in analogy with (68). For a small $\mathcal{Q}$ the embedding $\mathcal{Q} \longrightarrow \mathsf{Dist}^\star(\mathcal{Q})$ has a particularly interesting universal property: it is the Cauchy completion of $\mathcal{Q}$ *as quantaloid* [van der Plancke, 1998].

Since regular $\mathcal{Q}$-semicategories and regular distributors form a quantaloid $\mathsf{Dist}^\star(\mathcal{Q})$, it now makes sense to speak of 'adjoint regular distributors'—and we may ask ourselves whether "morphisms between regular semicategories induce adjoints pairs of regular distributors" (as is the case for functors between categories). Unfortunately, the answer is negative: let $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ be a morphism between regular $\mathcal{Q}$-semicategories, then surely it determines two distributors

$$\mathbb{B}(-, F-)\colon \mathbb{A} \xrightarrow{\circ} \mathbb{B} \text{ and } \mathbb{B}(F-, -)\colon \mathbb{B} \xrightarrow{\circ} \mathbb{A} \tag{73}$$

$$\begin{tikzcd}
\mathsf{Cat}(\mathcal{Q}) \arrow[r] \arrow[d] & \mathsf{SCat}^\star(\mathcal{Q}) \arrow[d] \\
\mathsf{Dist}(\mathcal{Q}) \arrow[r] & \mathsf{Dist}^\star(\mathcal{Q})
\end{tikzcd}$$

Diagram 0.4

but these needn't be regular[24]! However, would these distributors be regular, then they prove to be adjoint:

$$\mathbb{B}(-,F-) \dashv \mathbb{B}(F-,-) \text{ in } \mathsf{Dist}^\star(\mathcal{Q}).$$

We will thus say that $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ is regular when both distributors in (73) are regular—in elementary terms, F should satisfy

$$\text{for all } a \in \mathbb{A}_0, b \in \mathbb{B}_0 : \bigvee_{a' \in \mathbb{A}_0} \mathbb{B}(b, Fa') \circ \mathbb{A}(a', a) = \mathbb{B}(b, Fa); \tag{74}$$

$$\text{for all } a \in \mathbb{A}_0, b \in \mathbb{B}_0 : \bigvee_{a' \in \mathbb{A}_0} \mathbb{A}(a, a') \circ \mathbb{B}(Fa', a) = \mathbb{B}(Fa, b). \tag{75}$$

(The other two equations that one might expect, are trivial.) It is a fact that regular $\mathcal{Q}$-semicategories and regular morphisms form a category, and we denote it as $\mathsf{SCat}^\star(\mathcal{Q})$. This category is on the one hand a (non-full) subcategory of $\mathsf{SCat}(\mathcal{Q})$ (the category of *all* $\mathcal{Q}$-semicategories and *all* morphisms), and on the other hand it contains $\mathsf{Cat}(\mathcal{Q})$ ($\mathcal{Q}$-categories and functors) as (full) subcategory:

$$\mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{SCat}^\star(\mathcal{Q}) \longrightarrow \mathsf{SCat}(\mathcal{Q}). \tag{76}$$

Further, sending a regular $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ to the left adjoint $\mathbb{B}(-,F-)\colon \mathbb{A} \xrightarrow{\circ} \mathbb{B}$ preserves composition and identities, so that

$$\mathsf{SCat}^\star(\mathcal{Q}) \longrightarrow \mathsf{Dist}^\star(\mathcal{Q}) : (\mathbb{A} \xrightarrow{F} \mathbb{B}) \mapsto (\mathbb{A} \xrightarrow[\circ]{\mathbb{B}(-,F-)} \mathbb{B}) \tag{77}$$

is a functor (which is the identity on objects). (Sending F to $\mathbb{B}(F-,-)$ determines a contravariant such functor.) The various inclusions and embeddings of (30), (72), (76) and (77) commute as in diagram 0.4.

After all this theory, let us look at some examples of such regular semicategories.

(a) To see that 'regular semicategory' is a strictly weaker notion than that of 'category', we may consider the strict ordering on the real numbers as a **2**-semicategory: that is, we think of $\mathbb{R}$ as the **2**-enriched structure whose objects are the real numbers, and whose hom-arrows are given by

$$\mathbb{R}(y,x) = 1 \text{ if and only if } y < x.$$

[24]For a counterexample, see a bit further.

The transitivity of the order is then precisely the composition-inequality in $\mathbb{R}$, and the order being strict means that the unit-inequality cannot hold. So this is not a **2**-category, but a semicategory. However, it is a regular **2**-semicategory, due to the "density[25]" of the reals:

$$\bigvee_{y\in\mathbb{R}} \mathbb{R}(z,y)\wedge\mathbb{R}(y,x)=\mathbb{R}(z,x)$$

holds because for every $z<x$ there exists a y such that $z<y<x$!

(b) More about this example. The free category on the semicategory $(\mathbb{R},<)$ is simply $(\mathbb{R},\leq)$. A presheaf on $(\mathbb{R},<)$ is by definition a presheaf on $(\mathbb{R},\leq)$, i.e. a down-closed subset. But to illustrate the difference between the Yoneda embeddings, note that:

$$\begin{aligned} &Y_{(\mathbb{R},<)}\colon \mathbb{R}\longrightarrow\mathscr{P}\mathbb{R}\colon r\mapsto {\downarrow\!\!\!\downarrow} r=\{x\in\mathbb{R}\mid x<r\} \text{ for } \mathbb{R}\text{-as-semicategory,}\\ &Y_{(\mathbb{R},\leq)}\colon \mathbb{R}\longrightarrow\mathscr{P}\mathbb{R}\colon r\mapsto {\downarrow} r=\{x\in\mathbb{R}\mid x\leq r\} \text{ for } \mathbb{R}\text{-as-category.}\end{aligned}$$

A down-closed set $D\subseteq\mathbb{R}$ corresponds to a regular presheaf on $(\mathbb{R},<)$ whenever for every $x\in D$ there exists a $y\in D$ for which $x<y$ (so D has no maximum); and D is a Yoneda presheaf on $(\mathbb{R},<)$ whenever ${\downarrow\!\!\!\downarrow} x\subseteq D$ implies that $x\in D$ too. Clearly every ${\downarrow\!\!\!\downarrow} r$ is a regular presheaf on $(\mathbb{R},<)$, which is consistent with the previous observation that $(\mathbb{R},<)$ is a regular semicategory.

(c) More generally, a continuously ordered set $(P,\leq)$ with "way-below" relation $\ll$ (see for example [Gierz *et al.*, 1980] for a precise definition) may be viewed as a regular **2**-semicategory $\mathbb{P}$ when putting $\mathbb{P}_0=P$ as object-set and

$$\text{for } x,y\in P:\mathbb{P}(y,x)=1 \text{ if and only if } y\ll x.$$

The point is that "$\ll$" is an idempotent relation on P. And quite surprisingly (the underlying order of) the **2**-category $\mathscr{P}^{\star}\mathbb{P}$ of regular presheaves on $\mathbb{P}$ is (isomorphic to) the Scott topology on P [van der Plancke, 1998].

(d) Inspired by the previous example, we may now give an example of a morphism between regular semicategories which is not regular. Thereto, consider on the one hand the strict ordering of the reals $(\mathbb{R},<)$ and on the other the (partial) ordering on an interval, say $([-1,1],\leq)$. They are both examples of regular **2**-semicategories (the latter is even a **2**-category). The mapping

$$F\colon\mathbb{R}\longrightarrow[-1,1]\colon x\mapsto\begin{cases}1 \text{ if } x>0\\ -1 \text{ otherwise}\end{cases}$$

is surely a morphism between regular **2**-semicategories: if $x<y$ in $\mathbb{R}$ then $Fx\leq Fy$. For F to be a regular morphism, it is required that

[25] J. Koslowski [1997] and R. Rosebrugh and R. Wood [1994] would probably prefer to see the word "interpolation property" here.

- for all $x \in \mathbb{R}$ and $a \in [-1,1]$ satisfying $a \leq Fx$ there exists $x' \in \mathbb{R}$ such that $x' < x$ and $a \leq Fx'$;

- for all $x \in \mathbb{R}$ and $a \in [-1,1]$ satisfying $Fx \leq a$ there exist $x' \in \mathbb{R}$ such that $x < x'$ and $Fx' \leq a$.

The first of these sentences is true, but the second is false (take $x = 0$ and $a = -1$), so F cannot be a regular morphism!

(e) We have observed (see p. 25) that for a locale Ω, an Ω-set $(A, [\,\cdot = \cdot\,])$ may be viewed as an Ω-enriched semicategory whose objects are the elements of A, and for two objects a, a' the hom is $[a' = a]$. Actually, it is a regular semicategory[26], because for any $a, a' \in A$ we have due to symmetry of the Ω-valued equality that $[a' = a] = [a' = a] \wedge [a = a'] \leq [a' = a']$ and therefore

$$[a' = a] = [a' = a'] \wedge [a' = a] \leq \bigvee_{x \in A} \Big([a' = x] \wedge [x = a]\Big) \leq [a' = a]$$

(using the transitivity axiom, i.e. the composition inequality). Rewriting the definition of 'regular presheaf' on such an Ω-set in terms of the Ω-valued equality, it is easily seen to be precisely a mapping $\phi\colon A \longrightarrow \Omega$ satisfying for all $a \in A$ that

$$\bigvee_{x \in A} [a = x] \wedge \phi(x) = \phi(a).$$

This is precisely the correct notion of (the "characteristic function" of) an 'Ω-subset' of the given Ω-set: thinking of the Ω-set as a sheaf on Ω, these Ω-subsets correspond to subsheaves [Borceux, 1994]. In particular is the (underlying order of the) Ω-category of regular presheaves on a given Ω-set (viewed as regular semicategory) exactly the locale of subsheaves of that Ω-set (viewed as sheaf).

Cauchy completion of semicategories: total regularity

'Regular $\mathcal{Q}$-semicategory', 'regular distributor' and 'regular morphism' are strictly weaker notions than '$\mathcal{Q}$-category', 'distributor' and 'functor', but they are just strong enough to still allow for a distributor quantaloid $\mathsf{Dist}^\star(\mathcal{Q})$ and for every regular morphism $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ in $\mathsf{SCat}^\star(\mathcal{Q})$ to induce an adjoint pair of regular distributors. As the similar functor (30) was the key to the development of the theory of $\mathcal{Q}$-categories, we may now ask ourselves in how far (77) allows for a useful theory of regular $\mathcal{Q}$-semicategories. Particularly, it is quite essential to find a decent notion of 'Cauchy completion' for such semicategories; after all, any categorical structure is just a presentation of its Cauchy completion and so in principle a semicategory too should be 'Morita equivalent' to its 'Cauchy completion'.

[26] It is even more than that, as we will see further on...

Recall that for a $\mathscr{Q}$-category $\mathbb{C}$ its Cauchy completion $\mathbb{C}_{\mathsf{cc}}$ is the category classifying the Cauchy distributors with codomain $\mathbb{C}$; it is the full subcategory of the presheaf category $\mathscr{P}\mathbb{C}$ determined by the Cauchy presheaves. The Yoneda embedding $Y_{\mathbb{C}}\colon\mathbb{C}\longrightarrow\mathscr{P}\mathbb{C}$ factors fully over the full inclusion $\mathbb{C}_{\mathsf{cc}}\subseteq\mathscr{P}\mathbb{C}$: every object $c\in\mathbb{C}_0$ can be "pointed at" by means of a constant functor $\Delta c\colon *_{tc}\longrightarrow\mathbb{C}$, which in turn determines the (Cauchy) representable presheaf $\mathbb{C}(-,c)\colon *_{tc}\xrightarrow{\ \circ\ }\mathbb{C}$ (adjoint to $\mathbb{C}(c,-)\colon\mathbb{C}\xrightarrow{\ \circ\ }*_{tc}$) that thus converges to Δc itself. That is to say, $\mathbb{C}$ can be thought of as a full subcategory of $\mathbb{C}_{\mathsf{cc}}$ because "representable presheaves on $\mathbb{C}$ converge to constant functors into $\mathbb{C}$". (See p. 20 and further.)

Unfortunately, for a (regular) $\mathscr{Q}$-semicategory $\mathbb{A}$ it is no longer true that "representables converge to constants"! Indeed, even though any object $a\in\mathbb{A}_0$ does indeed determine a representable presheaf $\mathbb{A}(-,a)\colon *_{ta}\xrightarrow{\ \circ\ }\overline{\mathbb{A}}$, it is simply not true that this representable is left adjoint to $\mathbb{A}(a,-)\colon\overline{\mathbb{A}}\xrightarrow{\ \circ\ }*_{ta}$: for then necessarily $1_{ta}\leq\mathbb{A}(a,a)$ by the unit of the adjunction! Put differently, it is impossible in general to "point at" an object of $\mathbb{A}$ with a (regular) morphism like $F\colon *_A\longrightarrow\mathbb{A}$: this would imply that $1_A\leq\mathbb{A}(a,a)$, by the action of the morphism F on the hom-arrows! If we *per se* want an object $a\in\mathbb{A}_0$ of a (regular) $\mathscr{Q}$-semicategory $\mathbb{A}$ to be "pointed at" by a constant (regular) morphism, so that this morphism determines a left adjoint "representable presheaf" that in turn converges to the constant, then we will explicitly have to ask this right from the start—and we will have to be willing to consider a constant morphism whose domain is not a category.

All this motivates the following definition: an object a of a regular $\mathscr{Q}$-semicategory[27] $\mathbb{A}$ is *stable* if there exists a regular morphism from a one-object regular $\mathscr{Q}$-semicategory to $\mathbb{A}$ "pointing at" $a\in\mathbb{A}_0$.

Note that such a one-object regular $\mathscr{Q}$-semicategory is, simply, the "suspension" of an idempotent in $\mathscr{Q}$; for an idempotent $A\overset{e}{\circlearrowleft}$ we will denote $*_e$ for the regular semicategory whose single object $*$ is of type $t*=\mathsf{dom}(e)$ and whose single hom-arrow is precisely e. In the above definition we thus ask, for the object $a\in\mathbb{A}_0$, the existence of some idempotent e_a in $\mathscr{Q}$ and a regular morphism

$$F_a\colon *_{e_a}\longrightarrow\mathbb{A}\colon *\mapsto a. \tag{78}$$

As a consequence of the regularity of this morphism we then have an adjoint pair of distributors

$$\mathbb{A}(-,a)\dashv\mathbb{A}(a,-)\colon\mathbb{A}\underset{\ \circ\ }{\overset{\ \circ\ }{\leftrightarrows}} *_{e_a}\,. \tag{79}$$

This is precisely what we wanted: to "point at" an object a with a constant morphism, and to have a left adjoint "representable" that converges to this constant morphism.

[27]Since we want to speak of "adjoint distributors", it is clear that this will only make sense for a regular $\mathscr{Q}$-semicategory $\mathbb{A}$: then we can work in the quantaloid $\mathsf{Dist}^{\star}(\mathscr{Q})$.

This definition can be expressed "equationally": indeed, an object a in $\mathbb{A}$ is stable if and only if

$$\text{for all } a' \in \mathbb{A}_0 : \begin{cases} \mathbb{A}(a',a) \circ \mathbb{A}(a,a) = \mathbb{A}(a',a), \\ \mathbb{A}(a,a) \circ \mathbb{A}(a,a') = \mathbb{A}(a,a'). \end{cases} \tag{80}$$

These equations imply immediately that $\mathbb{A}(a,a)\colon ta \longrightarrow ta$ is an idempotent which itself may play the rôle of the idempotent e_a in (78) and (79) above.

We are now obviously interested in regular semicategories all objects of which are stable: we will call such a semicategory 'totally regular'. Using (80) we may thus spell out that a regular $\mathcal{Q}$-semicategory $\mathbb{A}$ is totally regular if and only if

$$\text{for all } a, a' \in \mathbb{A}_0 : \begin{cases} \mathbb{A}(a',a) \circ \mathbb{A}(a,a) = \mathbb{A}(a',a), \\ \mathbb{A}(a,a) \circ \mathbb{A}(a,a') = \mathbb{A}(a,a'). \end{cases} \tag{81}$$

(These conditions make sense for any – not necessarily regular – $\mathcal{Q}$-semicategory $\mathbb{A}$: they actually imply $\mathbb{A}$'s regularity.) Every $\mathcal{Q}$-category is a totally regular semicategory, but the converse is not true; and not every regular semicategory is totally regular (see p. 40 for examples).

It is easily verified that a distributor $\Phi\colon \mathbb{A} \mathrel{-\!\!\circ\!\!\rightarrow} \mathbb{B}$ between totally regular $\mathcal{Q}$-semicategories is regular – sensu (70) and (71) – if and only if

$$\text{for all } a \in \mathbb{A}_0, b \in \mathbb{B}_0 : \begin{cases} \Phi(b,a) \circ \mathbb{A}(a,a) = \Phi(b,a), \\ \mathbb{B}(b,b) \circ \Phi(b,a) = \Phi(b,a). \end{cases} \tag{82}$$

Denoting henceforth $\mathsf{Dist}^{\star\star}(\mathcal{Q})$ for the full sub-quantaloid of $\mathsf{Dist}^{\star}(\mathcal{Q})$ determined by the totally regular $\mathcal{Q}$-semicategories, we thus have full embeddings of quantaloids like

$$\mathsf{Dist}(\mathcal{Q}) \longrightarrow \mathsf{Dist}^{\star\star}(\mathcal{Q}) \longrightarrow \mathsf{Dist}^{\star}(\mathcal{Q}). \tag{83}$$

We can apply (82) to morphisms: for totally regular semicategories $\mathbb{A}$ and $\mathbb{B}$, a morphism $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ is regular – sensu (74) and (75) – if and only if

$$\text{for all } a \in \mathbb{A}_0, b \in \mathbb{B}_0 : \begin{cases} \mathbb{B}(b,Fa) \circ \mathbb{A}(a,a) = \mathbb{B}(b,Fa), \\ \mathbb{A}(a,a) \circ \mathbb{B}(Fa,b) = \mathbb{B}(Fa,b). \end{cases} \tag{84}$$

As we denoted $\mathsf{Dist}^{\star\star}(\mathcal{Q})$ for the full sub-quantaloid of $\mathsf{Dist}^{\star}(\mathcal{Q})$ determined by the totally regular semicategories, we will denote $\mathsf{SCat}^{\star\star}(\mathcal{Q})$ for the full sub-2-category of $\mathsf{SCat}^{\star}(\mathcal{Q})$ with those objects. It is quite obvious that we have (strict) embeddings

$$\mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{SCat}^{\star\star}(\mathcal{Q}) \longrightarrow \mathsf{SCat}^{\star}(\mathcal{Q}) \tag{85}$$

(all of which are full), and that diagram 0.5 commutes: the arrows from left to

$$\begin{tikzcd}
\mathsf{Cat}(\mathcal{Q}) \arrow[r] \arrow[d] & \mathsf{SCat}^{\star\star}(\mathcal{Q}) \arrow[r] \arrow[d] & \mathsf{SCat}^{\star}(\mathcal{Q}) \arrow[d] \\
\mathsf{Dist}(\mathcal{Q}) \arrow[r] & \mathsf{Dist}^{\star\star}(\mathcal{Q}) \arrow[r] & \mathsf{Dist}^{\star}(\mathcal{Q})
\end{tikzcd}$$

Diagram 0.5

right are full embeddings, those from top to bottom indicate that a regular morphism, and in particular a functor, induces an adjoint pair of regular distributors.

The point is now that a totally regular $\mathcal{Q}$-semicategory is indeed 'Morita equivalent' to its 'Cauchy completion' in a way that is very similar to, but at the same time quite different from, what we know for $\mathcal{Q}$-categories. More precisely we can proceed as follows.

- Say that a totally regular $\mathcal{Q}$-semicategory $\mathbb{B}$ is *Cauchy complete* if every left adjoint regular distributor with codomain $\mathbb{B}$, like $\Phi\colon \mathbb{A} \multimap \mathbb{B}$, is determined by a (necessarily essentially unique) regular morphism $F\colon \mathbb{A} \longrightarrow \mathbb{B}$. That is to say, "all Cauchy distributors into $\mathbb{B}$ converge".

- For any given totally regular $\mathcal{Q}$-semicategory $\mathbb{B}$ there exists a new $\mathcal{Q}$-categorical structure $\mathbb{B}_{\mathsf{cc}}$ with the universal property that, for all totally regular $\mathbb{A}$,

$$\mathsf{Map}(\mathsf{Dist}^{\star\star}(\mathcal{Q}))(\mathbb{A}, \mathbb{B}) \simeq \mathsf{SCat}^{\star\star}(\mathcal{Q})(\mathbb{A}, \mathbb{B}_{\mathsf{cc}}) \tag{86}$$

(natural in $\mathbb{A}$). This $\mathbb{B}_{\mathsf{cc}}$ can be constructed as follows: its objects are the left adjoint (regular) distributors into $\mathbb{B}$ whose domain is a one-object totally regular semicategory, i.e. like $\phi\colon *_e \multimap \mathbb{B}$ with $\phi \dashv \phi^*$ in $\mathsf{Dist}^{\star\star}(\mathcal{Q})$; the type of such an object is $t(\phi\colon^{\star}_{e} \multimap \mathbb{B}) = \mathsf{dom}(e)$; and for two objects $\psi\colon *_e \multimap \mathbb{B}$ and $\psi\colon *_f \multimap \mathbb{B}$ the hom-arrow $\mathbb{B}_{\mathsf{cc}}(\psi, \phi)\colon \mathsf{dom}(e) \longrightarrow \mathsf{dom}(f)$ is the single element of the regular distributor $[\psi, \phi] = \psi^* \otimes_{\mathbb{B}} \phi$ (lifting and composition in the quantaloid $\mathsf{Dist}^{\star\star}(\mathcal{Q})$). This $\mathbb{B}_{\mathsf{cc}}$ is a Cauchy complete totally regular $\mathcal{Q}$-semicategory.

- Since $\mathbb{B}\colon \mathbb{B} \multimap \mathbb{B}$ is a (regular) left adjoint distributor, there must be a corresponding (regular) morphism $j_{\mathbb{B}}\colon \mathbb{B} \longrightarrow \mathbb{B}_{\mathsf{cc}}$: this is the *Cauchy completion* of $\mathbb{B}$, it sends an object $b \in \mathbb{B}_0$ onto the left adjoint regular distributor

$$\mathbb{B}(-, b)\colon *_{\mathbb{B}(b,b)} \multimap \mathbb{B}.$$

By the naturality of (86) it is the unit of the left adjoint to the obvious full embedding:

$$(-)_{\mathsf{cc}} \dashv i\colon \mathsf{SCat}^{\star\star}_{\mathsf{cc}}(\mathcal{Q}) \rightleftarrows \mathsf{SCat}^{\star\star}(\mathcal{Q}). \tag{87}$$

- The (regular) morphism $j_{\mathbb{B}}$ in turn determines (an adjoint pair of) distributors $\mathbb{B}_{\mathsf{cc}}(-, j_{\mathbb{B}}-)\colon \mathbb{B} \multimap \mathbb{B}_{\mathsf{cc}}$ and $\mathbb{B}_{\mathsf{cc}}(j_{\mathbb{B}}-, -)\colon \mathbb{B}_{\mathsf{cc}} \multimap \mathbb{B}$, which are actually each

$$\begin{tikzcd} \mathsf{SCat}^{\star\star}(\mathscr{Q}) \arrow[r, "(-)_{\mathsf{cc}}"] & \mathsf{SCat}^{\star\star}(\mathscr{Q}) \\ \mathsf{Cat}(\mathscr{Q}) \arrow[u] \arrow[r, "(-)_{\mathsf{cc}}"'] \arrow[ur, phantom, "\neq"] & \mathsf{Cat}(\mathscr{Q}) \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{Dist}^{\star\star}(\mathscr{Q}) \arrow[r, "(-)_{\mathsf{cc}}"] & \mathsf{Dist}^{\star\star}(\mathscr{Q}) \\ \mathsf{Dist}(\mathscr{Q}) \arrow[u] \arrow[r, "(-)_{\mathsf{cc}}"'] \arrow[ur, phantom, "\neq"] & \mathsf{Dist}(\mathscr{Q}) \arrow[u] \end{tikzcd}$$

Diagram 0.6

others' inverse; so $\mathbb{B}$ is isomorphic to $\mathbb{B}_{\mathsf{cc}}$ in $\mathsf{Dist}^{\star\star}(\mathscr{Q})$. That is to say, $\mathbb{B}$ is *Morita equivalent* to its Cauchy completion. And it follows that

$$\mathsf{Dist}^{\star\star}_{\mathsf{cc}}(\mathscr{Q}) \simeq \mathsf{Dist}^{\star\star}(\mathscr{Q}) \tag{88}$$

is an equivalence of quantaloids (where the former is the obvious full subquantaloid of the latter).

- It now follows from (86) and (88) that

$$\mathsf{Map}(\mathsf{Dist}^{\star\star}(\mathscr{Q})) \simeq \mathsf{Map}(\mathsf{Dist}^{\star\star}_{\mathsf{cc}}(\mathscr{Q})) \simeq \mathsf{SCat}^{\star\star}_{\mathsf{cc}}(\mathscr{Q}); \tag{89}$$

so, if one is only interested in "Cauchy invariant" properties of (or "Morita equivalent" constructions on) totally regular $\mathscr{Q}$-semicategories, one may take the left adjoint (regular) distributors for "morphisms".

A warning is appropriate here: for a $\mathscr{Q}$-category $\mathbb{C}$ (which is trivially a totally regular $\mathscr{Q}$-semicategory), its Cauchy completion *as category* is very different from its Cauchy completion *as totally regular semicategory*. In fact, the latter isn't even a $\mathscr{Q}$-category anymore (but a totally regular semicategory)! Our notations do not distinguish between $\mathbb{C}$'s "Cauchy completion as category" and its "Cauchy completion as totally regular semicategory"; we write both as $\mathbb{C}_{\mathsf{cc}}$ and hope that the context makes clear which of both is meant. But – to make things perfectly clear – the squares in diagram 0.6 are not commutative! The theory of (Cauchy complete) totally regular $\mathscr{Q}$-semicategories is thus not at all "merely a straightforward generalization" of the theory of $\mathscr{Q}$-enriched categories. (Another such situation that makes this apparent, is that for a totally regular $\mathscr{Q}$-semicategory $\mathbb{A}$, its Cauchy completion is not a full substructure of $\mathscr{P}^\star\mathbb{A}$ (or even $\mathscr{P}\overline{\mathbb{A}}$), whereas for a category its Cauchy completion (as category) is a full subcategory of its presheaf category.)

It may be useful to mention that the construction of "$\mathbb{B}_{\mathsf{cc}}$" as outlined below (86) makes perfectly sense for any (not necessarily *totally*) regular $\mathscr{Q}$-semicategory $\mathbb{B}$ (although it may not have the desired properties). But $\mathbb{B}_{\mathsf{cc}}$ is always totally regular, and if $\mathbb{B}$ is fully faithfully embedded in $\mathbb{B}_{\mathsf{cc}}$ (a property that one certainly wants from a "Cauchy completion"!) then $\mathbb{B}$ must be totally regular too. This justifies – in retrospect – why a workable notion of "Cauchy completion" only applies to totally regular $\mathscr{Q}$-semicategories, and not to the merely regular ones.

Here are some examples.

(a) Recall that the strictly ordered reals $(\mathbb{R}, <)$ may be viewed as a regular $\mathbf{2}$-semicategory. But it is not totally regular: for $x < y$ it is clear that $\mathbb{R}(x, y) \wedge \mathbb{R}(y, y) \neq \mathbb{R}(x, y)$. So not every regular semicategory is totally regular. On the other hand it is obvious that not every totally regular semicategory is a category.

(b) Every totally regular $\mathbf{2}$-semicategory is Morita-equivalent to a true and honest $\mathbf{2}$-category, in the following way. Suppose that $\mathbb{A}$ is a totally regular $\mathbf{2}$-semicategory, and that $a \in \mathbb{A}_0$ is an object for which $1 \not\leq \mathbb{A}(a, a)$. Then it follows by total regularity that

$$\text{for all } a' \in \mathbb{A}_0 : \mathbb{A}(a', a) = 0 = \mathbb{A}(a, a'),$$

that is to say, those objects are "isolated" in $\mathbb{A}$. Consider now the full substructure of $\mathbb{A}$ determined by those objects which are not isolated; let $i\colon \widehat{\mathbb{A}} \longrightarrow \mathbb{A}$ denote the full inclusion. Then $\widehat{\mathbb{A}}$ is a $\mathbf{2}$-category, and the adjoint distributors $\mathbb{A}(-, i-)\colon \widehat{\mathbb{A}} \multimap \mathbb{A}$ and $\mathbb{A}(i-, -)\colon \mathbb{A} \multimap \widehat{\mathbb{A}}$ are actually each others' inverse in $\mathsf{Dist}^{\star\star}(\mathbf{2})$. So the full embedding $\mathsf{Dist}(\mathbf{2}) \longrightarrow \mathsf{Dist}^{\star\star}(\mathbf{2})$ is an equivalence, and it follows that $\mathsf{Cat}_{\mathsf{cc}}(\mathbf{2}) \simeq \mathsf{SCat}^{\star\star}_{\mathsf{cc}}(\mathbf{2})$ too. (Recalling that every $\mathbf{2}$-category is Cauchy complete, we may even state that $\mathsf{Cat}(\mathbf{2}) \simeq \mathsf{SCat}^{\star\star}_{\mathsf{cc}}(\mathbf{2})$.)

(c) A similar remark applies to the structures enriched in the extended non-negative reals (see p. 11). If $\mathbb{A}$ is a totally regular semicategory enriched in $\mathbb{R}^+ \cup \{\infty\}$, and $a \in \mathbb{A}_0$ is an object for which $0 \not\geq \mathbb{A}(a, a)$, then necessarily

$$\text{for all } a' \in \mathbb{A}_0 : \mathbb{A}(a', a) = \infty = \mathbb{A}(a, a').$$

That is to say, a is an "isolated" object ("infinitely far" from every other object—even from itself!). But $\mathbb{A}$ is Morita-equivalent to its full subcategory determined by the non-isolated objects.

(d) About the $\mathscr{Q}_R$-enriched category Comm_R determined by some ring R, introduced on p. 11. On the one hand this $\mathscr{Q}_R$-category may be Cauchy completed *as category*, but on the other hand we may consider it to be a totally regular $\mathscr{Q}_R$-semicategory, and Cauchy complete it *as semicategory*. The outcomes of both procedures are different! In the case of a commutative ring R, [Ambler and Verity, 1996] argue – in a setting that we will shortly prove to be equivalent to ours – that precisely the Cauchy completion of Comm_R as totally regular semicategory gives the correct "(generic) sheaf representation of R over $\mathscr{Q}_R$"; their paper contains the result that $(\mathsf{Comm}_R)_{\mathsf{cc}}$ is a "ring-object" in $\mathsf{SCat}^{\star\star}_{\mathsf{cc}}(\mathscr{Q}_R)$. The generalization to a non-commutative R has not yet been studied in this fashion (and it must be noted that Van den Bossche's [1995] result is quite different in nature from Ambler and Verity's [1996] work!).

The most important example – that which motivated us to consider semicategories in the first place – deserves a subsection of its own: totally regular semicategories enriched in a locale Ω.

Finally, the answer to the initial question

We are now – finally! – at the point where we can answer our initial question: "what precisely are those ordered objects in $\mathsf{Sh}(\Omega)$ as Borceux and Cruciani [1998] describe them?"

We observed that the ordered objects in $\mathsf{Sh}(\Omega)$ are those semicategories enriched in (the one-object suspension of) the locale Ω that satisfy two extra ("mysterious") conditions (55) and (56). But these conditions can be rewritten as

$$[a \le a] \wedge [a \le a'] = [a \le a'], \tag{90}$$

$$[a \le a'] \wedge [a' \le a'] = [a \le a']. \tag{91}$$

And then it is clear that an "Ω-order" is exactly a totally regular Ω-semicategory! Further, the "morphisms between $\mathcal{Q}$-orders" of which Borceux and Cruciani [1998] speak, are exactly left adjoint regular distributors between totally regular semicategories. So the (2-)category that they prove to be equivalent to $\mathsf{Ord}(\mathsf{Sh}(\Omega))$ is precisely what we denote by $\mathsf{Map}(\mathsf{Dist}^{\star\star}(\Omega))$.

The proof of this result relies totally on the theory of Cauchy complete totally regular Ω-semicategories. In fact,

- F. Borceux and R. Cruciani first define what we now recognize to be the category $\mathsf{Map}(\mathsf{Dist}^{\star\star}(\Omega))$;

- then they prove that "every Ω-order is isomorphic to a complete Ω-order", i.e. every totally regular semicategory is Morita equivalent to its Cauchy completion, so it follows that $\mathsf{Map}(\mathsf{Dist}^{\star\star}(\Omega)) \simeq \mathsf{Map}(\mathsf{Dist}^{\star\star}_{\mathsf{cc}}(\Omega))$;

- the authors then show that "for those complete Ω-orders the morphisms are equivalently given by certain object-mappings", which is precisely the equivalence $\mathsf{Map}(\mathsf{Dist}^{\star\star}_{\mathsf{cc}}(\Omega)) \simeq \mathsf{SCat}^{\star\star}_{\mathsf{cc}}(\Omega)$;

- and finally, by a direct argument, $\mathsf{SCat}^{\star\star}_{\mathsf{cc}}(\Omega) \simeq \mathsf{Ord}(\mathsf{Sh}(\Omega))$.

Interpreting (87) in this context we have explicitly that

$$(-)_{\mathsf{cc}} \dashv i\colon \mathsf{SCat}^{\star\star}_{\mathsf{cc}}(\Omega) \underset{\longrightarrow}{\overset{\longleftarrow}{}} \mathsf{SCat}^{\star\star}(\Omega)$$

behaves like a "sheafification": the objects of $\mathsf{SCat}^{\star\star}(\Omega)$ are to be understood as "ordered presheaves on Ω", the Cauchy complete such semicategories are "ordered sheaves", and the Cauchy completion of a semicategory is then like the sheafification of a presheaf.

Encouraged by this example, we now propose to define the "quantaloid of $\mathscr{Q}$-orders and ideals" precisely as

$$\mathsf{Idl}(\mathscr{Q}) := \mathsf{Dist}^{\star\star}_{\mathsf{cc}}(\mathscr{Q}), \tag{92}$$

and the "locally (pre)ordered category of $\mathscr{Q}$-orders and order-preserving maps" as

$$\mathsf{Ord}(\mathscr{Q}) := \mathsf{SCat}^{\star\star}_{\mathsf{cc}}(\mathscr{Q}). \tag{93}$$

That is to say, Cauchy complete totally regular $\mathscr{Q}$-semicategories are "ordered sheaves on $\mathscr{Q}$", a regular distributor is an "ideal relation" between ordered sheaves, and a regular morphism an "order-preserving map". It is then a result that $\mathsf{Ord}(\mathscr{Q}) = \mathsf{Map}(\mathsf{Idl}(\mathscr{Q}))$, i.e. that "left adjoint ideals are (the graphs of) order-preserving maps".

At this point we must make a comment on categorical structures enriched in $\mathbf{2}$. We have argued earlier that $\mathbf{2}$-*categories* are (pre)ordered sets (and functors are order-preserving maps and distributors are ideal relations). But the statements above imply that we should rather take the *Cauchy complete totally regular* $\mathbf{2}$-*semicategories* (and the regular morphisms and distributors) to be the correct understanding of preorders in this context. So which of both is it? Well, as indicated in an example on p. 40, it is "accidentally" so that every totally regular $\mathbf{2}$-semicategory is Morita-equivalent to a true and honest $\mathbf{2}$-category; and moreover every $\mathbf{2}$-category is Cauchy complete. Therefore the possible confusion is easily solved:

$$\mathsf{Ord}(\mathbf{2}) \simeq \mathsf{Cat}(\mathbf{2}) \text{ and } \mathsf{Idl}(\mathbf{2}) \simeq \mathsf{Dist}(\mathbf{2}).$$

(And for similar reasons, Cauchy complete generalized metric spaces are precisely "orders over the quantale of extended reals".)

From now on we use "$\mathscr{Q}$-order" as synonym for "Cauchy complete totally regular $\mathscr{Q}$-semicategory" (which is quite a mouthful).

Restrictions and gluing

To support the idea that $\mathscr{Q}$-orders are indeed "sheaves", let us indicate in what sense a "family of pairwise compatible elements admits a unique gluing".

Let $\mathbb{A}$ denote a totally regular $\mathscr{Q}$-semicategory. Thinking of $\mathbb{A}$ as a "sheaf" on $\mathscr{Q}$, we read a hom-arrow like $\mathbb{A}(a', a)$ as the "greatest level at which a is smaller than a'". In particular, an object $a \in \mathbb{A}_0$ is a "section of $\mathbb{A}$ over $\mathbb{A}(a, a)$". (This is entirely consistent with the case of Ω-sets: thinking of such an $(A, [\,\cdot = \cdot\,])$ as a sheaf on Ω, $[a = a]$ is precisely the element of Ω over which a is a section.) Note that (81) says in particular that any $\mathbb{A}(a, a)$ is an idempotent, and any $\mathbb{A}(a', a)$ a regular bimodule between such idempotents, in $\mathscr{Q}$. We thus have to

$$\begin{tikzcd}
A \arrow[loop left, "\mathbb{A}(a_i,a_i)"] \arrow[r, shift right, "x_i"'] \arrow[dr, "\mathbb{A}(a_j,a_i)"'] & A \arrow[l, shift right, "x_i^*"'] \arrow[loop right, "e=\mathbb{A}(a,a)"] \arrow[d, shift left, "x_j^*"] \\
& A \arrow[u, shift left, "x_j"] \arrow[loop right, "\mathbb{A}(a_j,a_j)"]
\end{tikzcd}$$

Diagram 0.7

try to generalize such things as restrictions, coverings, compatible families and gluings – which are well known for sheaves on a locale Ω – to this situation where idempotents in $\mathcal{Q}$ seem to play the rôle that the elements of a locale Ω play in the theory of sheaves on Ω.

Whenever for two idempotents $A \circlearrowleft^{e}$ and $A \circlearrowleft^{f}$ in $\mathcal{Q}$ there exists a left adjoint regular bimodule $x\colon e -\!\!\circ\!\!\!\to f$ (with right adjoint denoted $x^*\colon f -\!\!\circ\!\!\!\to e$) that expresses e as a section of f in $\mathsf{Bim}^{\star}(\mathcal{Q})$ (i.e. $x^* \otimes_f x = e$ and $x \otimes_e x^* \leq f$), then we think of $x\colon e -\!\!\circ\!\!\!\to f$ as a "restriction of idempotents in $\mathcal{Q}$". Suppose now that for an object $a \in \mathbb{A}_0$ in a totally regular $\mathcal{Q}$-semicategory $\mathbb{A}$, $x\colon e -\!\!\circ\!\!\!\to \mathbb{A}(a,a)$ is such a restriction of idempotents. The composition of left adjoint regular distributors

$$*_e \xrightarrow[\circ]{(x)} *_{\mathbb{A}(a,a)} \xrightarrow[\circ]{\mathbb{A}(-,a)} \mathbb{A}$$

(where (x) denotes the one-element distributor between one-object semicategories in the obvious way) gives a new left adjoint in $\mathsf{Dist}^{\star\star}(\mathcal{Q})$. If this distributor converges, then the object ("picked out" by the regular morphism) to which it converges (i.e. the representing object for this left adjoint) is the "restriction of a through $x\colon e -\!\!\circ\!\!\!\to \mathbb{A}(a,a)$". Clearly, if $\mathbb{A}$ is Cauchy complete then all such restrictions exist.

Suppose now that for some idempotent $A \circlearrowleft^{e}$ in $\mathcal{Q}$ there is a family of objects $(a_i)_{i\in I}$ in $\mathbb{A}$ for which

(i) for each $i \in I$ there is a restriction of idempotents $x_i\colon \mathbb{A}(a_i,a_i) -\!\!\circ\!\!\!\to e$ such that $e = \bigvee_{i\in I} x_i \circ x_i^*$, and

(ii) for each $i,j \in I$, $x_j^* \circ x_i \leq \mathbb{A}(a_j,a_i)$.

We may then think of the $\mathbb{A}(a_i,a_i)$'s as a "covering" of e (because of the first sentence above), and the a_i's as a "family of pairwise compatible elements" (because of the second sentence). A "gluing" of such a family is then – whenever it exists – an object $a \in \mathbb{A}_0$ such that $\mathbb{A}(a,a) = e$ and the restriction of this a through each of the $x_i\colon \mathbb{A}(a_i,a_i) -\!\!\circ\!\!\!\to \mathbb{A}(a,a)$ (exists and) equals precisely the corresponding a_i—see diagram 0.7.

Another way of expressing this, is as follows: let $\mathbb{E}$ denote the $\mathcal{Q}$-enriched structure with

- objects: $\mathbb{E}_0 = I$ and types $ti = A$ (for all $i \in I$),
- hom-arrows: for $i, j \in I$, $\mathbb{E}(j, i) = x_j^* \circ x_i$.

Due to condition (i) above, $\mathbb{E}$ is a totally regular semicategory which is Morita-equivalent to the one-object totally regular semicategory $*_e$: the distributor $\Theta \colon *_e \xrightarrow{\circ} \mathbb{E}$ with elements $\Theta(i, *) = x_i$ has as inverse $\Theta^{-1} \colon \mathbb{E} \xrightarrow{\circ} *_e$ with elements $\Theta^{-1}(*, i) = x_i^*$. Due to condition (ii) the object-mapping

$$F \colon \mathbb{E}_0 \longrightarrow \mathbb{A}_0 \colon i \mapsto Fi := a_i$$

is a regular morphism; it induces a left adjoint regular distributor $\mathbb{B}(-, F-)$. The composition of the left adjoints

$$*_e \xrightarrow[\circ]{\Theta} \mathbb{E} \xrightarrow[\circ]{\mathbb{B}(-,F-)} \mathbb{A}$$

yields a new left adjoint. If it is representable (i.e. if it converges) then the representing object is the gluing of the compatible family $(a_i)_i$. So for a Cauchy complete $\mathbb{A}$ all families of pairwise compatible objects admit an essentially unique gluing!

These definitions of "restriction", "covering", "gluing" etc. can of course be applied to Ω-sets (viewed as totally regular Ω-enriched semicategories): they then coincide with the usual notions for the Ω-sets (viewed as sheaves on Ω).

Cauchy completion of, and enriching in, a base quantaloid

Concerning the calculus of regular $\mathcal{Q}$-semicategories and regular distributors, i.e. the quantaloid $\mathsf{Dist}^\star(\mathcal{Q})$, it requires a simple calculation – using the definitions (27) and (69), and the universal properties of the matrix-construction, the idempotent-construction, and the monad-construction – to see that

$$\mathsf{Dist}^\star(\mathcal{Q}) \simeq \mathsf{Dist}(\mathsf{Dist}^\star(\mathcal{Q})). \tag{94}$$

That is to say, "regular semicategories are categories"... over a different base quantaloid[28]!

It is natural to ask wether such a property also holds for the totally regular $\mathcal{Q}$-semicategories—and the answer to that question is affirmative! To see what is happening it helps to first underline the important rôle that idempotents and regular bimodules in $\mathcal{Q}$ play in definitions (81) and (82):

[28] It would however be too easy to do away with the theory of regular semicategories altogether, because clearly $\mathsf{Cat}(\mathsf{Dist}^\star(\mathcal{Q})) \not\simeq \mathsf{SCat}^\star(\mathcal{Q})$.

- a $\mathcal{Q}$-semicategory is totally regular if and only if every endo-hom-arrow is an idempotent and every hom-arrow is a regular bimodule between these idempotents;

- a distributor between totally regular semicategories is regular if and only if every element of the distributor is a regular bimodule between the endo-hom-arrows.

As idempotents and regular bimodules in $\mathcal{Q}$ are themselves the objects and arrows of the quantaloid $\mathsf{Bim}^\star(\mathcal{Q})$ (see p. 32), it is now maybe not too surprising that totally regular $\mathcal{Q}$-semicategories can be described in terms of an enrichment in $\mathsf{Bim}^\star(\mathcal{Q})$.

Indeed, let $\mathbb{A}$ be a totally regular $\mathcal{Q}$-semicategory, define then a $\mathsf{Bim}^\star(\mathcal{Q})$-category $\widehat{\mathbb{A}}$ as follows: the object-set is $(\widehat{\mathbb{A}})_0 = \mathbb{A}_0$ but the types of the objects are now $\hat{t}a = \mathbb{A}(a,a)$ (which is indeed an object of $\mathsf{Bim}^\star(\mathcal{Q})$); and for the hom-arrows put $\widehat{\mathbb{A}}(a',a) = \mathbb{A}(a',a)$ in $\mathcal{Q}$ (which is indeed an arrow in $\mathsf{Bim}^\star(\mathcal{Q})$ between $\hat{t}a$ and $\hat{t}a'$). Similarly, for a regular distributor between totally regular semicategories like $\Phi\colon \mathbb{A} \xrightarrow{\circ} \mathbb{B}$ it is easily seen that $\widehat{\Phi}(b,a) = \Phi(b,a)$ defines a distributor $\widehat{\Phi}\colon \widehat{\mathbb{A}} \xrightarrow{\circ} \widehat{\mathbb{B}}$ between $\mathsf{Bim}^\star(\mathcal{Q})$-categories. This defines a fully faithful homomorphism of quantaloids

$$(\widehat{-})\colon \mathsf{Dist}^{\star\star}(\mathcal{Q}) \longrightarrow \mathsf{Dist}(\mathsf{Bim}^\star(\mathcal{Q})).$$

This homomorphism proves to be essentially surjective on objects[29], so it is an equivalence of quantaloids:

$$\mathsf{Dist}^{\star\star}(\mathcal{Q}) \simeq \mathsf{Dist}(\mathsf{Bim}^\star(\mathcal{Q})). \tag{95}$$

Therefore, "totally regular semicategories are categories"... but obviously over a quite different base quantaloid[30]!

On p. 32 we mentioned that $\mathsf{Bim}^\star(\mathcal{Q})$ is the quantaloid which is the Cauchy completion of $\mathcal{Q}$ *as category*, and that $\mathsf{Dist}^\star(\mathcal{Q})$ is the Cauchy completion of $\mathcal{Q}$ *as quantaloid.* And of course there are full embeddings

$$\mathcal{Q} \longrightarrow \mathsf{Bim}^\star(\mathcal{Q}) \longrightarrow \mathsf{Dist}^\star(\mathcal{Q}).$$

To stress that these Cauchy completions of $\mathcal{Q}$ (be it as category or as quantaloid) are only determined up to equivalence, we may rather use notations

$$\mathcal{Q} \longrightarrow \mathcal{Q}_{\mathsf{cc}} \longrightarrow \mathcal{Q}_{\mathsf{CC}}$$

[29] This is elementary, but depends on the fact that monads split in the base quantaloid $\mathsf{Bim}^\star(\mathcal{Q})$: the point is to prove that every $\mathsf{Bim}^\star(\mathcal{Q})$-category is Morita-equivalent to one whose endo-hom-arrows are identities.

[30] It follows that $\mathsf{SCat}^{\star\star}_{\mathsf{cc}}(\mathcal{Q}) \simeq \mathsf{Cat}_{\mathsf{cc}}(\mathsf{Bim}^\star(\mathcal{Q}))$ are biequivalent locally (pre)ordered categories. But again it must be noted that $\mathsf{SCat}^{\star\star}(\mathcal{Q}) \not\simeq \mathsf{Cat}(\mathsf{Bim}^\star(\mathcal{Q}))$, which underlines the importance of 'Morita equivalence' and 'Cauchy completions'.

$$\begin{tikzcd}
& \mathscr{Q} \arrow[rr] \arrow[dl, equal] \arrow[dd] & & \mathsf{Bim}^{\star}(\mathscr{Q}) \arrow[rr] \arrow[dl, equal] \arrow[dd] & & \mathsf{Dist}^{\star}(\mathscr{Q}) \arrow[dl, equal] \arrow[dd, equal] \\
\mathscr{Q} \arrow[rr] \arrow[dd] & & \mathscr{Q}_{\mathsf{cc}} \arrow[rr] \arrow[dd] & & \mathscr{Q}_{\mathsf{CC}} \arrow[dd, equal] & \\
& \mathsf{Dist}(\mathscr{Q}) \arrow[rr] \arrow[dl, equal] & & \mathsf{Dist}^{\star\star}(\mathscr{Q}) \arrow[rr] \arrow[dl, equal] & & \mathsf{Dist}(\mathsf{Dist}^{\star}(\mathscr{Q})) \arrow[dl, equal] \\
\mathsf{Dist}(\mathscr{Q}) \arrow[rr] & & \mathsf{Dist}(\mathscr{Q}_{\mathsf{cc}}) \arrow[rr] & & \mathsf{Dist}(\mathscr{Q}_{\mathsf{CC}}) &
\end{tikzcd}$$

Diagram 0.8

for any choice of Cauchy completion of $\mathscr{Q}$ as category, resp. as quantaloid. Enriching in these quantaloids then gives rise to embeddings

$$\mathsf{Dist}(\mathscr{Q}) \longrightarrow \mathsf{Dist}(\mathscr{Q}_{\mathsf{cc}}) \longrightarrow \mathsf{Dist}(\mathscr{Q}_{\mathsf{CC}})$$

and in particular can we now bring the various base quantaloids and enrichments together in one big diagram 0.8 in which all arrows are quantaloid homomorphisms, and the equalities should be read as equivalences. In this diagram, on the foreground, in the top row $\mathscr{Q}$ gets Cauchy completed and in the bottom row $\mathscr{Q}$ and its Cauchy completions are the base for enriched categories and distributors. In the background we then have a restatement of this scheme "in elementary terms".

As a result, we may now state several equivalent expressions for the quantaloid of $\mathscr{Q}$-orders and ideals, cf. (92):

$$\mathsf{Idl}(\mathscr{Q}) \simeq \mathsf{Dist}_{\mathsf{cc}}(\mathscr{Q}_{\mathsf{cc}}) \simeq \mathsf{Dist}(\mathscr{Q}_{\mathsf{cc}}).$$

By taking left adjoints one obtains as many equivalent expressions for $\mathsf{Ord}(\mathscr{Q})$.

The observation made here actually bridges the gap between the description of (ordered) sheaves on a locale Ω of, on the one hand, [Borceux and Cruciani, 1998] (see p. 25 and 41), and on the other hand [Walters, 1981]. Indeed, B. Walters identified sheaves on Ω as symmetric[31] Cauchy complete categories enriched in – as he described it – $\mathsf{Rel}(\Omega)$, leaving it understood that a (not-necessarily symmetric) $\mathsf{Rel}(\Omega)$-category should be thought of as an "ordered sheaf" on Ω. But

[31]The symmetry just means that in such an enriched category $\mathbb{A}$ one has for every pair of objects a, a' that $\mathbb{A}(a', a) = \mathbb{A}(a, a')$—which makes sense over $\mathsf{Rel}(\Omega)$ but not in general, see further.

$\mathsf{Rel}(\Omega)$ is precisely the split idempotent completion of Ω, because every element of Ω is idempotent of course, so Walters is working within $\mathsf{Cat}_{\mathsf{cc}}(\Omega_{\mathsf{cc}})$. This latter 2-category is up to equivalence precisely $\mathsf{SCat}^{\star\star}_{\mathsf{cc}}(\Omega)$, i.e. what we have shown to be Borceux and Cruciani's description of ordered sheaves on Ω.

Symmetric orders: towards sets

Having understood what "orders over a base quantaloid" are, it would be interesting to know more about "symmetric orders", i.e. "sets over a base quantaloid". Without digging deeply in this matter (which is not really part of this thesis) we can still try to outline the subject for future study.

First of all, let $\mathbb{A}$ be a $\mathcal{Q}$-order for some quantaloid $\mathcal{Q}$. For this order to be "symmetric" it would have to satisfy the condition on its hom-arrows that

$$\text{for all } a,a'\in\mathbb{A}_0 : \mathbb{A}(a',a)=\mathbb{A}(a,a')...$$

Which clearly doesn't make sense in general: $\mathbb{A}(a',a)$ and $\mathbb{A}(a,a')$ are arrows in the base quantaloid $\mathcal{Q}$ that go in the opposite direction! Therefore we will need an extra ingredient that allows us to compare arrows in $\mathcal{Q}$ that go in opposite directions: an involution on $\mathcal{Q}$.

An involution on a quantaloid $\mathcal{Q}$ is a homomorphism $\tau\colon\mathcal{Q}^{\mathsf{op}}\longrightarrow\mathcal{Q}$ which is the identity on objects and its own inverse on arrows. For simplicity we write $f^\tau\colon B\longrightarrow A$ for the effect of τ on some $\mathcal{Q}$-arrow $f\colon A\longrightarrow B$, and read f^τ as the "transpose" of f. Having a quantaloid plus involution $(\mathcal{Q},\tau)$ we may now define a 'τ-symmetric $\mathcal{Q}$-order' $\mathbb{A}$ as one for which

$$\text{for all } a,a'\in\mathbb{A}_0 : \mathbb{A}(a',a)=\mathbb{A}(a,a')^\tau. \tag{96}$$

As a particular example note that for a locale Ω the identity id_Ω is an involution (viewing Ω as a one-object quantaloid!), and that the id_Ω-symmetric Ω-orders are exactly the Ω-sets. Therefore we speak in general of a '$(\mathcal{Q},\tau)$-set' instead of a 'τ-symmetric $\mathcal{Q}$-order'.

More generally, if a quantaloid $\mathcal{Q}$ is endowed with an involution τ, then the quantaloid $\mathsf{Matr}(\mathcal{Q})$ of $\mathcal{Q}_0$-typed sets and matrices with elements in $\mathcal{Q}$ (see p. 12) inherits this involution: for a matrix $\mathbb{M}\colon(X,t)\longrightarrow(Y,t)$ one simply puts $\mathbb{M}^\tau\colon(Y,t)\longrightarrow(X,t)$ to be the matrix whose elements are $\mathbb{M}^\tau(x,y):=\mathbb{M}(y,x)^\tau$. A matrix $\mathbb{M}$ is then said to be 'τ-symmetric' whenever $\mathbb{M}^\tau=\mathbb{M}$—in which case $\mathbb{M}$ has to be an endo-matrix. It is then obvious that each $\mathcal{Q}$-enriched structure (category, semicategory, etc.) has a "τ-symmetric variant". It would thus be interesting to understand how those "τ-symmetric variants" organize themselves in substructures of $\mathsf{Dist}(\mathcal{Q})$, $\mathsf{Cat}(\mathcal{Q})$, $\mathsf{Dist}^\star(\mathcal{Q})$ and so on, and what (universal) properties those new structures have.

Of course, of particular interest are the τ-symmetric totally regular $\mathcal{Q}$-semicategories, i.e. the $(\mathcal{Q},\tau)$-sets. It is noteworthy that for such a $(\mathcal{Q},\tau)$-set $\mathbb{A}$, every endo-hom-arrow is a τ-symmetric idempotent in $\mathcal{Q}$. Recalling that a $\mathcal{Q}$-order can be seen as a $\mathcal{Q}_{\mathsf{cc}}$-enriched category (cf. p. 45), it thus seems that the τ-symmetric ones can be seen as τ-symmetric $\mathcal{Q}_{\tau\mathsf{cc}}$-enriched categories, where by $\mathcal{Q}_{\tau\mathsf{cc}}$ we mean the universal splitting of τ-symmetric idempotents in $\mathcal{Q}$ (as opposed to $\mathcal{Q}_{\mathsf{cc}}$ which is the universal splitting of *all* idempotents).

Another point that should be cleared out is: what is the appropriate Cauchy completion of a $(\mathcal{Q},\tau)$-set? This is not as obvious as it may seem at first sight, because the Cauchy completion of a τ-symmetric totally regular $\mathcal{Q}$-semicategory as explained on p. 38 and further is not necessarily τ-symmetric[32]! But it may be observed that for a regular morphism $F\colon\mathbb{A}\longrightarrow\mathbb{B}$ between two $(\mathcal{Q},\tau)$-sets, $\mathbb{B}(-,F-)^{\tau} = \mathbb{B}(F-,-)$ (due to the τ-symmetry of $\mathbb{B}$): the transpose of the left adjoint distributor induced by F is actually its right adjoint. It therefore seems reasonable to think that the Cauchy completion of a $(\mathcal{Q},\tau)$-set $\mathbb{A}$ should not classify *all* left adjoint regular distributors into $\mathbb{A}$, but only those regular distributors that are left adjoint to their transpose.

Ideally these considerations should lead to full substructures

$$\mathsf{Dist}^{\star\star}(\mathcal{Q},\tau)\longrightarrow\mathsf{Dist}^{\star\star}(\mathcal{Q}) \quad\text{and}\quad \mathsf{SCat}^{\star\star}(\mathcal{Q},\tau)\longrightarrow\mathsf{SCat}^{\star\star}(\mathcal{Q})$$

determined by the τ-symmetric objects; and there should be equivalences

$$\mathsf{SCat}^{\star\star}_{\mathsf{cc}}(\mathcal{Q},\tau)\simeq\mathsf{Map}_{\tau}(\mathsf{Dist}^{\star\star}_{\mathsf{cc}}(\mathcal{Q},\tau))\simeq\mathsf{Map}_{\tau}(\mathsf{Dist}^{\star\star}(\mathcal{Q},\tau))$$

where by "Map_{τ}" is meant: taking those left adjoints whose right adjoint is their transpose. It would then make sense to define "sets, relations and functions over a base quantaloid endowed with an involution" as

$$\mathsf{Rel}(\mathcal{Q},\tau):=\mathsf{Dist}_{\mathsf{cc}}(\mathcal{Q},\tau) \quad\text{and}\quad \mathsf{Set}(\mathcal{Q},\tau):=\mathsf{Map}_{\tau}(\mathsf{Rel}(\mathcal{Q},\tau)).$$

And it would then seem that $\mathsf{Rel}(\mathcal{Q},\tau)$ is a full subquantaloid of $\mathsf{Idl}(\mathcal{Q})$, but $\mathsf{Set}(\mathcal{Q},\tau)$ would not be a full subcategory of $\mathsf{Ord}(\mathcal{Q})$.

Acknowledgment

When, as an 18 year old, I first went to the Vrije Universiteit Brussel, it was almost by default that I chose to study mathematics and physics. But then I learned about group theory, and a little later about order theory, and those subjects convinced me that I had made the right choice back then in '94. It

[32] A similar observation holds for τ-symmetric $\mathcal{Q}$-categories [Betti and Walters, 1982; Labella and Schmitt, 2002].

is therefore not such a great surprise that category theory, the most natural generalization of both groups and orders, became my favorite subject, and it is even a little ironic that ultimately this thesis is again about orders (albeit quite exotic ones). Without Francis Buekenhout, Bob Coecke, Eva Colenbunders, Rudger Kieboom, Steven Sourbron, Frank Valckenborgh, Michel Van den Bergh and Bart Van Steirteghem, things wouldn't have been the same there in Brussels.

Then, four years and a diploma in mathematics later, I arrived at the Université Catholique de Louvain. The first couple of years were, I admit, quite difficult for me. If I somehow managed to fit in after all, then that is in great extent due to Jacques Boel, Francis and Christiane Borceux, Claudia Centazzo, Mathieu Dupont, Marino Gran, Mamuka Jibladze, the late René Lavendhomme, Ivan Le Creurer, Bachuki Mesablishvili, Marie-Anne Moens, Jean-Roger Roisin and Enrico Vitale. Furthermore there were the many students with whom I have worked in various exercise classes: at times we had lots of fun!

Among the many opportunities that the category theory group in Louvain-la-Neuve offered me, were my two unforgettable visits to Sydney. During my first visit in August-September '99, when I spent 10 weeks at Sydney University, I had many discussions with Marie-Anne about her thesis, and I think now that it is fair to say that it was there and then that some pre-embryonal ideas for this thesis took shape. Last January-February I spent another 5 weeks at Macquarie University, and some of the results in this work were finalized on that occasion. Many, many thanks and warm feelings go to Ruth Corran, Max and Imogen Kelly, Steve Lack, Daniel Steffen, and Ross and Margery Street.

I have always thought that it is kind of silly to dedicate a work of mathematics to someone, but I think that I now understand why people do that. This work is dedicated to Annelore, Carine and Hendrik.

I can only hope that I didn't make too big a mess of it all.

1

Preliminaries on quantaloids

Basic definitions are recalled and notation is fixed. For later use, the construction of (lax) (co)limits in a quantaloid is discussed.

1.1 Quantaloids and their homomorphisms

A sup-lattice is an ordered set $(X, \leq)$ for which every subset $X' \subseteq X$ has a supremum $\bigvee X'$ in X ('ordered' always means 'partially ordered'). A morphism of sup-lattices $f\colon (X, \leq) \longrightarrow (Y, \leq)$ is a map $f\colon X \longrightarrow Y$ that preserves arbitrary suprema. It is well-known that sup-lattices and sup-morphisms constitute a symmetric monoidal closed category Sup.

Definition 1.1.1 *A* quantaloid $\mathcal{Q}$ *is a* Sup*-enriched category. A* homomorphism $F\colon \mathcal{Q} \longrightarrow \mathcal{Q}'$ *of quantaloids is a* Sup*-enriched functor.*

We will usually denote the objects of a given quantaloid $\mathcal{Q}$ as A, B, ... and the class[1] of all objects as $\mathcal{Q}_0$; its arrows as f, g, ... ; and local structure like $f \leq g$, $\bigvee_i f_i$, ... The identity arrow on an object $A \in \mathcal{Q}_0$ is written $1_A\colon A \longrightarrow A$. Composition of arrows is written in the usual manner:

$$A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C = A \xrightarrow{\ g \circ f\ } C,$$

and $\mathcal{Q}(A, B)$ is the hom-lattice of arrows from A to B.

The bottom element of a sup-lattice $\mathcal{Q}(A, B)$ will typically be denoted by $0_{A,B}$. With the notation for identity arrow and bottom arrow being fixed now, we introduce a "Kronecker delta" as follows:

$$\delta_{A,B}\colon A \longrightarrow B = \begin{cases} 1_A\colon A \longrightarrow A & \text{if } A = B, \\ 0_{A,B}\colon A \longrightarrow B & \text{otherwise.} \end{cases}$$

[1] In principle we don't mind a quantaloid $\mathcal{Q}$ having a proper class of objects. See also the remarks on p. 92.

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] & C \\ B \arrow[ur, "g"'] & \end{tikzcd}$$

Diagram 1.1

In elementary terms, a quantaloid $\mathcal{Q}$ is thus a category whose hom-sets are actually sup-lattices, in which composition distributes on both sides over arbitrary suprema of morphisms:

$$f \circ (\bigvee_i g_i) = \bigvee_i (f \circ g_i),$$

$$(\bigvee_j f_j) \circ g = \bigvee_j (f_j \circ g).$$

In the same vein, a homomorphism $F: \mathcal{Q} \longrightarrow \mathcal{Q}'$ is a functor of (the underlying) categories that preserves arbitrary suprema of morphisms:

$$F(\bigvee_i f_i) = \bigvee_i F(f_i).$$

As we may consider Sup to be a "simplified version of Cat", we may regard quantaloids as "simplified bicategories". Notably, a quantaloid $\mathcal{Q}$ then has small hom-categories with stable local colimits, and therefore it is closed. Our notation for the respective adjoints is $- \circ f \dashv \{f, -\}$ and $g \circ - \dashv [g, -]$. The explicit formulae for these adjoints are, by the adjoint functor theorem,

$$[g, h] = \bigvee \{x: A \longrightarrow B \mid g \circ x \leq h\},$$

$$\{f, h\} = \bigvee \{y: B \longrightarrow C \mid y \circ f \leq h\},$$

with f, g and h as in diagram 1.1. That is to say,

$$g \circ f \leq h \iff f \leq [g, h] \iff g \leq \{f, h\}.$$

Sometimes it will be useful to explicitly mention the quantaloid $\mathcal{Q}$ in which these adjoints are calculated; we then use $[g, h]^{\mathcal{Q}}$ and $\{f, h\}^{\mathcal{Q}}$ as notation.

Quantaloids and homomorphisms form an illegitimate category QUANT; small quantaloids define a (true) subcategory Quant. A quantaloid with one object is often thought of as a monoid in Sup, and is called a *quantale*.

1.2 Technical lemmas for later use

Lemma 1.2.1 *In any quantaloid $\mathcal{Q}$, consider $A \underset{g}{\overset{f}{\rightleftarrows}} B$; the following are equivalent:*

1. $f \dashv g\colon B \rightleftarrows A$ *in* $\mathcal{Q}$ *(i.e.* $1_A \leq g \circ f$ *and* $f \circ g \leq 1_B$*);*
2. $(f \circ -) \dashv (g \circ -)\colon \mathcal{Q}(-,B) \rightleftarrows \mathcal{Q}(-,A)$ *in* Sup*;*
3. $(- \circ g) \dashv (- \circ f)\colon \mathcal{Q}(B,-) \rightleftarrows \mathcal{Q}(A,-)$ *in* Sup*;*
4. $(g \circ -) = [f,-]\colon \mathcal{Q}(-,B) \longrightarrow \mathcal{Q}(-,A)$ *in* Sup*;*
5. $(- \circ f) = \{g,-\}\colon \mathcal{Q}(B,-) \longrightarrow \mathcal{Q}(A,-)$ *in* Sup.

Lemma 1.2.2 *An arrow $A \xrightarrow{f} B$ in a quantaloid $\mathcal{Q}$ has a right adjoint if and only if for every diagram $B \xrightarrow{m} C \xleftarrow{n} D$,*

$$[n,m] \circ f = [n, m \circ f].$$

In this case the right adjoint to f is $[f, 1_B]$.

Dually, an arrow $A \xleftarrow{g} B$ in $\mathcal{Q}$ has a left adjoint if and only if for every diagram $B \xleftarrow{v} C \xrightarrow{w} D$,

$$g \circ \{w,v\} = \{w, g \circ v\}.$$

In this case the left adjoint to g is $\{g, 1_B\}$.

Lemma 1.2.3 *Let $f, f'\colon A \rightrightarrows B$ and $g, g'\colon B \rightrightarrows A$ form adjoint pairs in a quantaloid $\mathcal{Q}$: $f \dashv g$ and $f' \dashv g'$. Then $f \leq f'$ if and only if $g' \leq g$.*

Lemma 1.2.4 *In any quantaloid $\mathcal{Q}$, any arrow $f\colon A \longrightarrow B$ induces, for every $X \in \mathcal{Q}_0$, an adjunction*

$$[-,f] \dashv \{-,f\}\colon \mathcal{Q}(X,B)^{\mathsf{op}} \rightleftarrows \mathcal{Q}(A,X).$$

Here, by $\mathcal{Q}(X,B)^{\mathsf{op}}$ is meant: the sup-lattice $\mathcal{Q}(X,B)$ with opposite order.

Lemma 1.2.5 *For arrows in a quantaloid as in diagram 1.2 we have the (dual) identities*

$$[f,[g,h]] = [g \circ f, h] \quad \textit{and} \quad \{k,\{m,n\}\} = \{k \circ m, n\}.$$

And for arrows as in diagram 1.3 we have the (self-dual) identity

$$[x,\{y,z\}] = \{z,[x,y]\}.$$

$$\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[r, "g"] & C & Z & Y \arrow[l, "k"'] & X \arrow[l, "m"'] \arrow[d, "n"] \\
 & & D \arrow[u, "h"] & & & W
\end{tikzcd}$$

Diagram 1.2

$$\begin{tikzcd}
K & L \arrow[l, "y"'] \arrow[d, "z"] \\
M \arrow[u, "x"] & N
\end{tikzcd}$$

Diagram 1.3

Lemma 1.2.6 *Consider arrows in a quantaloid as in diagram 1.4, then the following identities hold:*

$$[h,g] \circ [g,f] \leq [h,f] \text{ and dually } \{l,m\} \circ \{k,l\} \leq \{k,m\}.$$

Further, for any arrow $d\colon E \longrightarrow F$,

$$1_E \leq [d,d] \text{ and dually } 1_F \leq \{d,d\}.$$

Lemma 1.2.7 *For a sup-lattice X and some subset $Y \subseteq X$, Y is a sub-sup-lattice (i.e. a sup-lattice in its own right for the inherited order) if and only if for all $Y' \subseteq Y$ the supremum $\bigvee Y'$ as calculated in X is actually an element of Y—i.e. the order-preserving and -reflecting inclusion $Y \hookrightarrow X$ reflects suprema.*

Lemma 1.2.8 *Any bijection between sup-lattices that preserves arbitrary suprema is an isomorphism of these sup-lattices in* Sup.

Lemma 1.2.9 *Let $F, G\colon \mathcal{Q} \rightrightarrows \mathsf{Sup}$ be homomorphisms between quantaloids, and $\alpha\colon F \overset{\sim}{\Longrightarrow} G$ a natural bijection between these homomorphisms viewed as functors, i.e. $F \cong G$ in CAT. Then α is an isomorphism from F to G as homomorphisms (i.e. $\alpha\colon F \overset{\sim}{\Longrightarrow} G$ in QUANT) if and only if every component of the natural bijection α preserves suprema.*

Lemma 1.2.10 *In a quantaloid $\mathcal{Q}$ let $\begin{tikzcd} A \arrow[loop above, "e"] \end{tikzcd}$ and $\begin{tikzcd} B \arrow[loop above, "f"] \end{tikzcd}$ be arrows such that $e \circ e \leq e$ and $f \circ f \leq f$, and consider an arrow $A \xrightarrow{k} B$. Denoting*

$$\overline{e} = e \bigvee 1_A \quad \text{and} \quad \overline{f} = f \bigvee 1_B$$

$$
\begin{tikzcd}
A \arrow[r, "f"] & D & & X & W \arrow[l, "k"'] \arrow[dl, "l"'] \arrow[d, "m"] \\
B \arrow[ur, "g"] & C \arrow[u, "h"'] & & Y & Z
\end{tikzcd}
$$

Diagram 1.4

we have that

$$k \circ e \leq k \iff k \circ \overline{e} \leq k \iff k \circ \overline{e} = k,$$
$$f \circ k \leq k \iff \overline{f} \circ k \leq k \iff \overline{f} \circ k = k.$$

1.3 A primer on (lax) (co)limits in a quantaloid

The pertinent definition

Definition 1.3.1 *Let $\mathscr{Q}$ denote a quantaloid, and $\mathscr{D}$ a small category. A* lax functor *$F: \mathscr{D} \longrightarrow \mathscr{Q}$ is a mapping of objects $\mathscr{D}_0 \longrightarrow \mathscr{Q}_0: D \mapsto FD$ together with an action on hom-sets $\mathscr{D}(D, D') \longrightarrow \mathscr{Q}(FD, FD'): f \mapsto Ff$ such that*

- *for all $D \in \mathscr{D}_0$, $1_{FD} \leq F1_D$;*
- *for all composable $f, g \in \mathscr{D}$, $(Fg \circ Ff) \leq F(g \circ f)$.*

Given two such lax functors $F, G: \mathscr{D} \rightrightarrows \mathscr{Q}$, a family of $\mathscr{Q}$-arrows

$$\Big(\theta_D: FD \longrightarrow GD\Big)_{D \in \mathscr{D}_0}$$

constitutes a lax-natural transformation *$\theta: F \Longrightarrow G$ if*

- *for all $f: D \longrightarrow D'$ in $\mathscr{D}$, $(\theta_{D'} \circ Ff) \geq (Gf \circ \theta_D)$;*

and the family is said to constitute a oplax-natural transformation *if*

- *for all $f: D \longrightarrow D'$ in $\mathscr{D}$, $(\theta_{D'} \circ Ff) \leq (Gf \circ \theta_D)$.*

The collection of lax transfos, resp. oplax transfos, between two parallel lax functors $F: \mathscr{D} \longrightarrow \mathscr{Q}$ and $G: \mathscr{D} \longrightarrow \mathscr{Q}$ constitutes a sup-lattice (with "componentwise supremum"); we'll denote it as $\mathsf{Lax}(F, G)$, resp. $\mathsf{OpLax}(F, G)$. Lax transfos (oplax transfos) may also be composed: this too is done "componentwise".

The notions of 'lax transfo' and 'oplax transfo' are dual in the following sense: two lax functors $F, G: \mathscr{D} \rightrightarrows \mathscr{Q}$ may be thought of as $F^{\mathsf{op}}, G^{\mathsf{op}}: \mathscr{D}^{\mathsf{op}} \rightrightarrows \mathscr{Q}^{\mathsf{op}}$, and then we have that

$$\mathsf{Lax}(F, G) = \mathsf{OpLax}(G^{\mathsf{op}}, F^{\mathsf{op}}). \tag{1.1}$$

Any (ordinary) functor $F\colon\mathscr{D}\longrightarrow\mathscr{Q}$ is trivially lax, and any (ordinary) natural transformation $\alpha\colon F\Longrightarrow G$ between functors is trivially (op)lax. Thus the set of natural transformations between two functors $F,G\colon\mathscr{D}\rightrightarrows\mathscr{Q}$ is a subset of the set of (op)lax transformations. In fact $\mathsf{Nat}(F,G)$ is a sub-sup-lattice of both the sup-lattices $\mathsf{Lax}(F,G)$ and $\mathsf{OpLax}(F,G)$; it is their intersection.

In the rest of this section, $\Delta_A\colon\mathscr{D}\longrightarrow\mathscr{Q}$ will denote the constant functor at the object $A\in\mathscr{Q}_0$ (on whatever category $\mathscr{D}$). For an arrow $k\colon A\longrightarrow B$ in $\mathscr{Q}$ the constant transformation from Δ_A to Δ_B will be denoted as Δ_k.

Definition 1.3.2 *For a lax functor on a small domain into a quantaloid, say $F\colon\mathscr{D}\longrightarrow\mathscr{Q}$, a* lax cone *over F consists of an object $L\in\mathscr{Q}_0$ and a lax transfo $\pi\in\mathsf{Lax}(\Delta_L,F)$. Such a lax cone (L,π) is the (necessarily essentially unique)* lax limit *of F if it is universal: for all $A\in\mathscr{Q}_0$ there is an isomorphism*[2] *of sup-lattices induced by composition with $\pi\colon\Delta_L\Longrightarrow F$,*

$$\mathscr{Q}(A,L)\xrightarrow{\ \sim\ }\mathsf{Lax}(\Delta_A,F)\colon k\mapsto\pi\circ\Delta_k. \tag{1.2}$$

Lax cocone *and* lax colimit[3] *are the dual notions: a lax cocone on F is an oplax transfo $\sigma\in\mathsf{OpLax}(F,\Delta_C)$, and it is the lax colimit if it is universal, i.e. for all $A\in\mathscr{Q}_0$*

$$\mathscr{Q}(C,A)\xrightarrow{\ \sim\ }\mathsf{OpLax}(F,\Delta_A)\colon k\mapsto\Delta_k\circ\sigma. \tag{1.3}$$

So a lax functor $F\colon\mathscr{D}\longrightarrow\mathscr{Q}$ admits a lax limit (lax colimit) in $\mathscr{Q}$ if and only if $F^{\mathsf{op}}\colon\mathscr{D}^{\mathsf{op}}\longrightarrow\mathscr{Q}^{\mathsf{op}}$ admits a lax colimit (lax limit) in $\mathscr{Q}^{\mathsf{op}}$—that is the duality between limits and colimits. In the rest of the section we will only consider lax limits, for it is clear how "by duality" the results are valid for lax colimits too.

Equivalent expressions

The definition of 'lax limit' is all about representability of quantaloid homomorphisms.

Lemma 1.3.3 *For a given lax functor $F\colon\mathscr{D}\longrightarrow\mathscr{Q}$ into a quantaloid $\mathscr{Q}$, the prescription*

$$\mathsf{Lax}(\Delta_-,F):\mathscr{Q}^{\mathsf{op}}\longrightarrow\mathsf{Sup}\colon\begin{pmatrix}A\\ \uparrow f\\ B\end{pmatrix}\mapsto\begin{pmatrix}\mathsf{Lax}(\Delta_A,F)\\ \downarrow -\circ\Delta_f\\ \mathsf{Lax}(\Delta_B,F)\end{pmatrix} \tag{1.4}$$

[2] Actually the definition of "lax (co)limit" in a general 2-category, or even bicategory, asks only for an *equivalence*, and not for an isomorphism. But in the case of a quantaloid, due to its local partial order, every equivalence is actually an isomorphism.

[3] In Australia this is also called a 'collage'.

defines a homomorphism of quantaloids; and $F\colon \mathscr{D}\longrightarrow\mathscr{Q}$ *has a lax limit if and only if this homomorphism is representable:*

$$\exists L\in\mathscr{Q} : \mathscr{Q}(-,L)\cong \mathsf{Lax}(\Delta_-,F) \textit{ in } \mathsf{QUANT}(\mathscr{Q}^{\mathsf{op}},\mathsf{Sup}).$$

Proof: Note that the members of a natural family of sup-isomorphisms

$$\Big(\alpha_A\colon \mathscr{Q}(A,L)\xrightarrow{\sim}\mathsf{Lax}(\Delta_A,F)\Big)_{A\in\mathscr{Q}^{\mathsf{op}}}$$

are necessarily defined by $\alpha_A(f)=\alpha_L(1_L)\circ\Delta_f$; it is $\alpha_L(1_L)\in\mathsf{Lax}(\Delta_L,F)$ that is the lax limit of F, cf. 1.3.2. And conversely, the sup-isomorphisms in (1.2) constitute a natural isomorphism $\mathscr{Q}(-,L)\overset{\sim}{\Longrightarrow}\mathsf{Lax}(\Delta_-,F)$. $\Box$

Yet another way of presenting lax limits, is as final object in an appropriate category of lax cones on a given lax functor $F\colon\mathscr{D}\longrightarrow\mathscr{Q}$. Consider the collection of all lax cones over F, and say that for two such lax cones (A,θ) and (A',θ') a morphism of lax cones $k\colon(A,\theta)\longrightarrow(A',\theta')$ is a $\mathscr{Q}$-arrow $k\colon A\longrightarrow A'$ such that for all $D\in\mathscr{D}_0$ we have $\theta'_D\circ k=\theta_D$. Then lax cones and their morphisms form a category $\mathsf{LaxCones}(F)$.

Lemma 1.3.4 *For a lax functor* $F\colon\mathscr{D}\longrightarrow\mathscr{Q}$ *into a quantaloid* $\mathscr{Q}$*, the final object of the category* $\mathsf{LaxCones}(F)$ *is the lax limit of* F*—whenever it exists.*

Proof: To say that a lax cone (L,π) on F is the final object of $\mathsf{LaxCones}(F)$ is to say that $\pi\in\mathsf{Lax}(\Delta_L,F)$ is such that for every $\theta\in\mathsf{Lax}(\Delta_A,F)$ there exists a unique $k\colon A\longrightarrow L$ such that for all $D\in\mathscr{D}$, $\pi_D\circ k=\theta_D$. But this means precisely that $\pi\circ\Delta_k=\theta$, here regarding $\Delta_k\colon\Delta_A\Longrightarrow\Delta_L$ as constant natural transformation. So the equivalence with the existence of a sup-isomorphism as in (1.2) is straightforward. $\Box$

When are lax limits really limits?

In general, for an ordinary functor $F\colon\mathscr{D}\longrightarrow\mathscr{Q}$, the limit on F is not necessarily its lax limit: whereas any cone $\alpha\in\mathsf{Nat}(\Delta_L,F)$ is trivially also a lax cone, it is not true that the universal property for such an α to be a *limit* implies the universal property for α to be a *lax* limit. In this subsection we explain a case where lax limits do coincide with ordinary ones. (And by duality we can apply the result to lax colimits, of course.)

Lemma 1.3.5 *Consider a lax functor* $F\colon\mathscr{D}\longrightarrow\mathscr{Q}$ *on a small category* $\mathscr{D}$ *into a quantaloid* $\mathscr{Q}$*. For any lax cone on* F*, say* $\pi\in\mathsf{Lax}(\Delta_A,F)$*, and any object* $X\in\mathscr{Q}_0$*, the mapping*

$$\mathscr{Q}(X,A)\longrightarrow\mathsf{Lax}(\Delta_X,F) : k\mapsto\pi\circ\Delta_k \tag{1.5}$$

is a sup-morphism. Whenever F is an ordinary functor, and π is an ordinary cone, the mapping (1.5) restricts to a sup-morphism

$$\mathcal{Q}(X,A)\longrightarrow\mathsf{Nat}(\Delta_X,F):k\mapsto\pi\circ\Delta_k. \tag{1.6}$$

Proof: Consider an lax transfo $\pi\colon\Delta_A\Longrightarrow F$; suppose that $k=\bigvee_j k_j$ in $\mathcal{Q}(X,A)$. An easy calculation with $\mathcal{Q}$-arrows shows that, for any $D\in\mathcal{D}_0$:

$$\begin{aligned}(\pi\circ\Delta_k)_D &= \pi_D\circ k\\ &= \pi_D\circ(\bigvee_j k_j)\\ &= \bigvee_j(\pi_D\circ k_j)\\ &= \bigvee_j(\pi\circ\Delta_{k_j})_D\end{aligned}$$

and since suprema in $\mathsf{Lax}(\Delta_X,F)$ are calculated "componentwise", the latter expression equals the component at D of $\bigvee_j(\pi\circ\Delta_{k_j})$. So (1.5) preserves arbitrary suprema. Clearly, whenever F is a functor and π is a natural transformation, $\pi\circ\Delta_k$ is a natural transformation too; in this case (1.5) thus restricts to (1.6). □

Proposition 1.3.6 *Suppose that, for some lax functor $F_1\colon\mathcal{D}_1\longrightarrow\mathcal{Q}$ and some (ordinary) functor $F_2\colon\mathcal{D}_2\longrightarrow\mathcal{Q}$ into a quantaloid $\mathcal{Q}$, for all $X\in\mathcal{Q}_0$ there is a sup-isomorphism*

$$\mathsf{Lax}(\Delta_X,F_1)\xrightarrow{\sim}\mathsf{Nat}(\Delta_X,F_2) \tag{1.7}$$

which is natural in X. Then some $\pi\in\mathsf{Lax}(\Delta_L,F_1)$ is the lax limit of F_1 if and only if $\pi'\in\mathsf{Nat}(\Delta_L,F_2)$, the image of π under the isomorphism above (taking $X=L$), is the limit of F_2.

Proof: We have seen in 1.3.3 that the lax functor $F_1\colon D\longrightarrow\mathcal{Q}$ has a lax limit in $\mathcal{Q}$ if and only if the *homomorphism* in (1.4) is naturally isomorphic in QUANT to some representable $\mathcal{Q}(-,L)$—i.e. whenever there is a natural family of *sup-isomorphisms*

$$\beta\colon\mathcal{Q}(-,L)\overset{\sim}{\Longrightarrow}\mathsf{Lax}(\Delta_-,F_1).$$

These bijections are then actually induced by composition with $\beta_L(1_L)$.

On the other hand, surely the following prescription

$$\mathsf{Nat}(\Delta_-,F_2):\mathcal{Q}^{\mathsf{op}}\longrightarrow\mathsf{Sup}:\begin{pmatrix}A\\ \uparrow f\\ B\end{pmatrix}\mapsto\begin{pmatrix}\mathsf{Nat}(\Delta_A,F_2)\\ \downarrow -\circ\Delta_f\\ \mathsf{Nat}(\Delta_B,F_2)\end{pmatrix} \tag{1.8}$$

defines a functor; and the functor $F_2\colon \mathscr{D} \longrightarrow \mathscr{Q}$ has a limit in $\mathscr{Q}$ if and only if the *functor* in (1.8) is naturally isomorphic in CAT to some representable $\mathscr{Q}(-, L')$—i.e. whenever there is a natural family of *bijections*

$$\beta'\colon \mathscr{Q}(-, L') \overset{\sim}{\Longrightarrow} \mathsf{Nat}(\Delta_{-}, F_2).$$

These bijections are then actually induced by composition with $\beta'_{L'}(1_{L'})$.

However, by 1.3.5 this *functor* turns out to be a *homomorphism*; and since the bijections $(\beta_X)_{X \in \mathscr{Q}}$ are induced by composition in $\mathscr{Q}$ they preserve arbitrary suprema. So by 1.2.8 and 1.2.9 we find that $F_2\colon \mathscr{D} \longrightarrow \mathscr{Q}$ has a limit in $\mathscr{Q}$ if and only if the *homomorphism* in (1.8) is naturally isomorphic *as a homomorphism* to some representable $\mathscr{Q}(-, L)$.

Now the proof is almost complete: it suffices to remark that the hypotheses assure that

$$\mathsf{Lax}(\Delta_{-}, F_1)\colon \mathscr{Q}^{\mathsf{op}} \longrightarrow \mathsf{Sup} \quad \text{and} \quad \mathsf{Nat}(\Delta_{-}, F_2)\colon \mathscr{Q}^{\mathsf{op}} \longrightarrow \mathsf{Sup}$$

are isomorphic in QUANT, so that if one is representable in QUANT by some object L of $\mathscr{Q}$, then so is the other—and looking at $1_L \in \mathscr{Q}(L, L)$ then shows that, indeed, lax limit and limit correspond under the isomorphism (1.7). $\square$

2

Categories enriched in a quantaloid

Viewing categories as "monadic matrices", the quantaloid of $\mathcal{Q}$-categories and distributors is defined as universal construction on a base quantaloid $\mathcal{Q}$. The universality of this construction implies the effect of a change of base. The definition of a functor between $\mathcal{Q}$-categories is given; each functor induces an adjoint pair of distributors. This fact is exploited to study isomorphisms in, and the underlying preorder of, a $\mathcal{Q}$-enriched category. This in turn helps to understand some elementary properties of adjoint functors between $\mathcal{Q}$-categories.

2.1 Matrices with elements in a quantaloid

Direct sums in a quantaloid

A set-indexed family $(A_i)_{i\in I}$ of objects in a quantaloid $\mathcal{Q}$ can be identified with (the image of) the functor

$$F\colon I \longrightarrow \mathcal{Q} : i \mapsto A_i \tag{2.1}$$

whose domain is the index-set regarded as discrete category. But precisely because this diagram is discrete, any lax cone on F is in fact an ordinary cone; and any lax cocone on F is just an ordinary cocone. That is to say, in this particular case we have for any object $A \in \mathcal{Q}_0$ that

$$\mathsf{Nat}(\Delta_A, F) = \mathsf{Lax}(\Delta_A, F) \text{ and } \mathsf{Nat}(F, \Delta_A) = \mathsf{OpLax}(F, \Delta_A). \tag{2.2}$$

This observation allows for the following result.

Proposition 2.1.1 *For a set-indexed family $(A_i)_{i\in I}$ of objects in a quantaloid $\mathcal{Q}$, the following are equivalent:*

1. *$(A_i)_{i\in I}$ has a limit (i.e. product) in $\mathcal{Q}$;*
2. *$(A_i)_{i\in I}$ has a colimit (i.e. coproduct) in $\mathcal{Q}$;*

3. *$(A_i)_{i\in I}$ has a lax limit in $\mathcal{Q}$;*

4. *$(A_i)_{i\in I}$ has a lax colimit in $\mathcal{Q}$;*

5. *there exist an object A and arrows $(A \xrightarrow{p_i} A_i \xrightarrow{s_i} A)_{i\in I}$ in $\mathcal{Q}$ such that*

   - $\forall i, j \in I : p_j \circ s_i = \delta_{A_i,A_j}$;
   - $\bigvee_{i\in I}(s_i \circ p_i) = 1_A$.

The cone in 5. is then the (lax) limit of the given family, and the cocone in 5. is its (lax) colimit. For each $i \in I$ the coprojection $A_i \xrightarrow{s_i} A$ is left adjoint to the projection $A \xrightarrow{p_i} A_i$.

Proof: The equivalence ($1 \iff 2 \iff 5$) is well-known. As for ($1 \iff 3$) and ($2 \iff 4$), these follow from the observation made in (2.2) and lemma 1.3.6. The adjunctions $s_i \dashv p_i$ follow since $1_{A_i} = p_i \circ s_i$ and $s_i \circ p_i \leq 1_A$ by 5. $\square$

As usual, we will refer to limits-colimits (in a quantaloid $\mathcal{Q}$) as *direct sums* and we write them as $\oplus_i A_i$.

Recall that a quantaloid homomorphism $F\colon \mathcal{Q} \longrightarrow \mathcal{Q}'$ is a functor that preserves local suprema. This implies that any diagram of arrows in $\mathcal{Q}$ that can be presented by equations possibly involving suprema, is preserved by F. This holds in particular for the "equational presentation" of direct sums—cf. statement 5. of the proposition above.

Corollary 2.1.2 *Any quantaloid homomorphism preserves direct sums.*

The direct sum completion of a quantaloid

We will now construct and study a new quantaloid built from a given quantaloid $\mathcal{Q}$, namely the quantaloid of matrices with elements in $\mathcal{Q}$. This quantaloid $\mathsf{Matr}(\mathcal{Q})$ turns out to be the $\oplus$-completion of $\mathcal{Q}$ in QUANT.

Proposition 2.1.3 *Given a quantaloid $\mathcal{Q}$, the following defines a new quantaloid*[1] $\mathsf{Matr}(\mathcal{Q})$ *of "matrices with elements in $\mathcal{Q}$":*

- *objects: are sets X, Y, Z, ... to every element of which we assign a "type in $\mathcal{Q}$": to every $x \in X$ we associate an object $tx \in \mathcal{Q}_0$*[2]*;*
- *arrows: an arrow $\mathbb{M}\colon (X, t) \longrightarrow (Y, t)$ is a matrix*

$$\Big(\mathbb{M}(y, x)\colon tx \longrightarrow ty\Big)_{x\in X, y\in Y}$$

  with $|X|$ columns and $|Y|$ rows whose elements are $\mathcal{Q}$-arrows;

[1] Note that $\mathsf{Matr}(\mathcal{Q})$ has a large class of objects, however small $\mathcal{Q}$ is!

[2] Such a "$\mathcal{Q}$-typed set" (X, t) is thus simply a way of writing a small discrete diagram $(A_x = tx)_{x\in X}$ in $\mathcal{Q}$.

- *two-cells: matrices are ordered "elementwise", so*

$$\mathbb{M} \leq \mathbb{N} \textit{ in } \mathsf{Matr}(\mathcal{Q})((X,t),(Y,t))$$

$$\overset{\underline{def.}}{\Longleftrightarrow}$$

$$\forall\, x \in X, \forall\, y \in Y : \mathbb{M}(y,x) \leq \mathbb{N}(y,x) \textit{ in } \mathcal{Q}(tx,ty);$$

- *composition: write*

$$(X,t)\xrightarrow{\mathbb{M}}(Y,t)\xrightarrow{\mathbb{N}}(Z,t) = (X,t)\xrightarrow{\mathbb{N}\circ\mathbb{M}}(Z,t)$$

for the composition of matrices, and compute the elements of $\mathbb{N}\circ\mathbb{M}$ *with the "linear algebra formula" for matrix multiplication:*

$$(\mathbb{N}\circ\mathbb{M})(z,x) = \bigvee_{y\in Y} \mathbb{N}(z,y)\circ\mathbb{M}(y,x) \tag{2.3}$$

- *identities: the identity matrix* $\Delta_X\colon(X,t)\longrightarrow(X,t)$ *has Kronecker deltas as elements:* $\Delta_X(x',x) = \delta_{tx,tx'}$.

Clearly, $\mathcal{Q}$ can be embedded in $\mathsf{Matr}(\mathcal{Q})$: every arrow $f\colon A\longrightarrow B$ in $\mathcal{Q}$ determines a one-element matrix between one-element $\mathcal{Q}$-typed sets:

$$\mathbb{F}\colon \Big(\{*\}, t* = A\Big)\longrightarrow\Big(\{*\}, t* = B\Big) \text{ with } \mathbb{F}(*,*) = f.$$

For short we will simply write this embedding as

$$i_{\mathcal{Q}}\colon \mathcal{Q}\longrightarrow\mathsf{Matr}(\mathcal{Q})\colon (A\xrightarrow{f}B)\mapsto(*_A\xrightarrow{(f)}*_B). \tag{2.4}$$

Note further that any homomorphism of quantaloids $F\colon\mathcal{Q}\longrightarrow\mathcal{Q}'$ determines a homomorphism

$$\mathsf{Matr}(F)\colon\mathsf{Matr}(\mathcal{Q})\longrightarrow\mathsf{Matr}(\mathcal{Q}')\colon \begin{pmatrix}(X,t)\\ \downarrow\mathbb{M}\\ (Y,t)\end{pmatrix}\mapsto\begin{pmatrix}(X,F\circ t)\\ \downarrow F\mathbb{M}\\ (Y,F\circ t)\end{pmatrix} \tag{2.5}$$

that is to say,

- a $\mathcal{Q}$-typed set (X,t) is sent to the $\mathcal{Q}'$-typed set $(X,F\circ t)$ which consists of the same set X for which the type of an element x is changed from $tx\in\mathcal{Q}_0$ to $F(tx)\in\mathcal{Q}'_0$;

- a $\mathcal{Q}$-matrix $\mathbb{M}\colon(X,t)\longrightarrow(Y,t)$ is sent to the $\mathcal{Q}'$-matrix $F\mathbb{M}$ with elements $F\mathbb{M}(y,x) = F(\mathbb{M}(y,x))$.

$$\begin{tikzcd}
\mathcal{Q} \arrow[r, "F"] \arrow[d, "i_{\mathcal{Q}}"'] & \mathcal{Q}' \arrow[d, "i_{\mathcal{Q}'}"] \\
\mathsf{Matr}\mathcal{Q} \arrow[r, "\mathsf{Matr}(F)"'] & \mathsf{Matr}(\mathcal{Q}')
\end{tikzcd}$$

Diagram 2.1

This homomorphism $\mathsf{Matr}(F)$ makes diagram 2.1 commute.

Proposition 2.1.4 *For any quantaloid $\mathcal{Q}$,*

1. *the embedding (2.4) is a fully faithful homomorphism of quantaloids;*
2. *this embedding is an equivalence of quantaloids if and only if $\mathcal{Q}$ has all direct sums;*
3. *applying the matrix construction to $\mathsf{Matr}(\mathcal{Q})$ itself, the embedding*

$$i_{\mathsf{Matr}(\mathcal{Q})}\colon \mathsf{Matr}(\mathcal{Q}) \longrightarrow \mathsf{Matr}(\mathsf{Matr}(\mathcal{Q}))$$

   is always an equivalence;

4. *$\mathcal{Q}$ is equivalent to $\mathsf{Matr}(\mathcal{Q}')$ for some quantaloid $\mathcal{Q}'$ if and only if $\mathcal{Q}$ has direct sums;*
5. *the embedding (2.4) describes the universal direct sum completion of $\mathcal{Q}$ in* QUANT.

Proof: (1) Is obvious: $\mathcal{Q}(A,B) \cong \mathsf{Matr}(\mathcal{Q})(*_A, *_B)$ as sup-lattices.

(2) By (1), the embedding in (2.4) is an equivalence of quantaloids if and only if every $\mathcal{Q}$-typed set (X,t) is isomorphic to some singleton $\mathcal{Q}$-typed set $*_A = \Big(\{*\}, t* = A\Big)$ in $\mathsf{Matr}(\mathcal{Q})$: there exist matrices $\mathbb{P}\colon *_A \longrightarrow (X,t)$ and $\mathbb{S}\colon (X,t) \longrightarrow *_A$ such that $\mathbb{S} \circ \mathbb{P} = \Delta_{*_A}$ and $\mathbb{P} \circ \mathbb{S} = \Delta_X$. But regarding the $\mathcal{Q}$-typed set (X,t) as discrete diagram $\{A_x = tx \mid x \in X\}$ in $\mathcal{Q}$ we're thus asking for a cone $(p_x\colon A \longrightarrow A_x)_{x\in X}$ (the elements of $\mathbb{P}$) and a cocone $(s_x\colon A_x \longrightarrow A)_{x \in X}$ (the elements of $\mathbb{S}$) such that $\bigvee_{x\in X} s_x \circ p_x = 1_A$ and $p_{x'} \circ s_x = \delta_{A_x, A_{x'}}$ for all $x, x' \in X$. Using 2.1.1 we can therefore conclude.

(3) Observe first that, $\mathsf{Matr}(\mathcal{Q})$ being a quantaloid itself, the construction $\mathsf{Matr}(\mathsf{Matr}(\mathcal{Q}))$ indeed makes sense. We'll prove that $\mathsf{Matr}(\mathcal{Q})$ has all products. Thereto, consider a set-indexed family of objects $\{(X_i, t) \mid i \in I\}$ in $\mathsf{Matr}(\mathcal{Q})$, and define:

(X, t) is the set $X = \coprod_{i \in I} X_i$ – so $x \in X$ if and only if $x \in X_j$ for some unique $j \in I$ – in which the type of x in X is that of x in X_j.

This defines an object (X, t), and the jth projection $\mathbb{P}_j\colon (X, t) \longrightarrow (X_j, t)$ is defined, quite naturally, to have elements

$$\mathbb{P}_j(x', x) = \begin{cases} 1_{tx} & \text{if } x, x' \in X_j \subseteq X \text{ and } x = x', \\ 0_{tx,tx'} & \text{otherwise.} \end{cases}$$

Thus, $\mathbb{P}_j$ can be thought of as the matrix with $|X|$ columns and $|X_j|$ rows that "identifies" X_j as part of X:

$$\mathbb{P}_j = \left(\begin{array}{c|c|c} \mathbb{O} & \Delta_{X_j} & \mathbb{O} \end{array} \right)$$

Given another family of matrices

$$\Big(\mathbb{F}_i\colon (Y, t) \longrightarrow (X_i, t)\Big)_{i \in I}$$

its unique factorization through the projections $(\mathbb{P}_i)_{i \in I}$, call it $\mathbb{F}\colon (Y, t) \longrightarrow (X, t)$, can be pictured as

$$\mathbb{F} = \begin{pmatrix} \mathbb{F}_1 \\ \mathbb{F}_2 \\ \vdots \\ \mathbb{F}_i \\ \vdots \end{pmatrix}$$

More precisely, for $y \in Y$ and $x \in X$, put

$\mathbb{F}(x, y) = \mathbb{F}_j(x, y)$ with j such that X_j is the unique part of X to which x belongs.

It is obvious that indeed $\mathbb{P}_j \circ \mathbb{F} = \mathbb{F}_j$; and unicity of this factorization $\mathbb{F}$ is straightforward.

(4) The "if" was proved in (2). For the "only if", the hypothesis in particular says that $\mathscr{Q}$ is equivalent to a category admitting all small (co)products; and therefore so does $\mathscr{Q}$ itself (using the essential surjectivity and the fully faithfulness of the equivalence).

(5) We must prove that for any homomorphism $F\colon \mathscr{Q} \longrightarrow \mathscr{Q}'$ into a $\mathscr{Q}'$ which has all direct sums there exists an essentially unique factorization through the homomorphism $i_{\mathscr{Q}}\colon \mathscr{Q} \longrightarrow \mathsf{Matr}(\mathscr{Q})$. But for a $\mathscr{Q}'$ with direct sums the embedding $i_{\mathscr{Q}'}\colon \mathscr{Q}' \longrightarrow \mathsf{Matr}(\mathscr{Q}')$ is an equivalence of quantaloids, and thus (2.1) reduces to a factorization, unique up to natural isomorphism, (with some abuse of notation) in QUANT—see diagram 2.2. Since, again by direct sum-completeness, any

$$\begin{tikzcd}
\mathscr{Q} \arrow[r, "F"] \arrow[d, "i_{\mathscr{Q}}"'] \arrow[dr, phantom, "\stackrel{\sim}{\Longrightarrow}"] & \mathscr{Q}' \\
\mathsf{Matr}\mathscr{Q} \arrow[ur, "\mathsf{Matr}(F)"'] &
\end{tikzcd}$$

Diagram 2.2

homomorphism $G\colon \mathsf{Matr}(\mathscr{Q})\longrightarrow\mathscr{Q}'$ is up to natural isomorphism determined by its action on one-element matrices – i.e. by its action on the image of $\mathscr{Q}$ along the embedding $\mathscr{Q}\longrightarrow\mathsf{Matr}(\mathscr{Q})$ – the factorization in diagram 2.2 is essentially unique. $\square$

Part 4. of 2.1.4 above may be read as a "representation theorem" for $\oplus$-complete quantaloids; and in 5. one may read a "definition by universal property" of the quantaloid $\mathsf{Matr}(\mathscr{Q})$ of matrices with elements in some given $\mathscr{Q}$.

To conclude this section, let us say a few words on the closedness of $\mathsf{Matr}(\mathscr{Q})$. It is interesting to note that, in a way, the closedness of the quantaloid $\mathsf{Matr}(\mathscr{Q})$ is inherited from that of $\mathscr{Q}$, and that it is crucial for the explicit description of the adjoints for composition in $\mathsf{Matr}(\mathscr{Q})$ to dispose also of the (albeit non-stable!) local limits in $\mathscr{Q}$.

Proposition 2.1.5 *For matrices $\mathbb{M}\colon (X,t)\longrightarrow(Y,t)$ and $\mathbb{N}\colon (Y,t)\longrightarrow(Z,t)$ with elements in a quantaloid $\mathscr{Q}$, the adjoints to composition in $\mathsf{Matr}(\mathscr{Q})$,*

$$-\circ\mathbb{M}\dashv\{\mathbb{M},-\}\ \textit{and}\ \mathbb{N}\circ-\dashv[\mathbb{N},-],$$

are computed as follows: for $\mathbb{K}\colon (X,t)\longrightarrow(Z,t)$,

$$\{\mathbb{M},\mathbb{K}\}(z,y)=\bigwedge_{x\in X}\Big\{\mathbb{M}(y,x),\mathbb{K}(z,x)\Big\}^{\mathscr{Q}},$$

$$[\mathbb{N},\mathbb{K}](y,x)=\bigwedge_{z\in Z}\Big[\mathbb{N}(z,y),\mathbb{K}(z,x)\Big]^{\mathscr{Q}}.$$

Proof: We have the following equivalences:

$$\begin{aligned}
&\mathbb{N}\circ\mathbb{M}\leq\mathbb{K}\text{ in }\mathsf{Matr}(\mathscr{Q})\\
\iff&\forall x\in X,\forall z\in Z:(\mathbb{N}\circ\mathbb{M})(z,x)\leq\mathbb{K}(z,x)\text{ in }\mathscr{Q}(tx,tz)\\
\iff&\forall x\in X,\forall z\in Z:\textstyle\bigvee_{y\in Y}\mathbb{N}(z,y)\circ\mathbb{M}(y,x)\leq\mathbb{K}(z,x)\text{ in }\mathscr{Q}(tx,tz)\\
\iff&\forall x\in X,\forall y\in Y,\forall z\in Z:\mathbb{N}(z,y)\circ\mathbb{M}(y,x)\leq\mathbb{K}(z,x)\text{ in }\mathscr{Q}(tx,tz)\\
\iff&\forall x\in X,\forall y\in Y,\forall z\in Z:\mathbb{M}(y,x)\leq\Big[\mathbb{N}(z,y),\mathbb{K}(z,x)\Big]\text{ in }\mathscr{Q}(tx,ty)\\
\iff&\forall x\in X,\forall y\in Y:\mathbb{M}(y,x)\leq\textstyle\bigwedge_{z\in Z}\Big[\mathbb{N}(z,y),\mathbb{K}(z,x)\Big]\text{ in }\mathscr{Q}(tx,ty).
\end{aligned}$$

Thus, if we define a $\mathscr{Q}$-matrix $[\mathbb{N},\mathbb{K}]\colon (X,t) \longrightarrow (Y,t)$ by putting

$$[\mathbb{N},\mathbb{K}](y,x) = \bigwedge_{z\in Z} \Big[\mathbb{N}(z,y),\mathbb{K}(z,x)\Big]$$

then we have indeed described the (necessarily unique!) matrix with elements in $\mathscr{Q}$ for which $\mathbb{N}\circ\mathbb{M} \le \mathbb{K} \iff \mathbb{M} \le [\mathbb{N},\mathbb{K}]$.

Likewise for $\{\mathbb{M},\mathbb{K}\}$. $\square$

2.2 Monads and bimodules in a quantaloid

Monads and their algebra objects in a quantaloid

A monad in a quantaloid $\mathscr{Q}$ is – by the general definition for "monad in a 2-category" – simply an endo-arrow $A \overset{t}{\circlearrowleft}$ such that $t\circ t \le t$ and $1_A \le t$. It is immediate that in a quantaloid all monads are idempotents.

Lemma 2.2.1 *An endo-arrow $A \overset{t}{\circlearrowleft}$ in a quantaloid $\mathscr{Q}$ is a monad if and only if $t\circ t = t$ and $1_A \le t$.*

Such a monad $A \overset{t}{\circlearrowleft}$ can be identified with (the image of) a lax functor

$$F\colon \mathsf{Ob} \longrightarrow \mathscr{Q}\colon * \mapsto A, 1_* \mapsto t. \tag{2.6}$$

Here Ob stands for the "generic object": the category with one object $*$ and one arrow 1_*. On the other hand, the arrow t can be considered as diagram in $\mathscr{Q}$, thus identified with (the image of) an ordinary functor

$$F'\colon \mathsf{Ar} \longrightarrow \mathscr{Q} : *_1 \mapsto A, *_2 \mapsto A, \alpha \mapsto t \tag{2.7}$$

where Ar stands for the "generic arrow": the category with two objects, $*_1$ and $*_2$, and exactly one arrow that is not an identity, $\alpha\colon *_1 \longrightarrow *_2$. Precisely because $1_A \le t$, to give a lax cone on F is actually the same as to give an ordinary cone on F'; and lax cocones on F coincide with cocones on F'. More precisely, we have for any object $X \in \mathscr{Q}_0$ identical sup-lattices

$$\mathsf{Nat}(\Delta_X, F') = \mathsf{Lax}(\Delta_X, F) \text{ and } \mathsf{Nat}(F', \Delta_X) = \mathsf{OpLax}(F, \Delta_X). \tag{2.8}$$

Much like for discrete diagrams in $\mathscr{Q}$, for monads too we can apply 1.3.6 to prove the equivalence of, and an "equational presentation" for, the notions of lax limit, ordinary limit, lax colimit and ordinary colimit.

Proposition 2.2.2 *For a monad $A \overset{t}{\circlearrowleft}$ in a quantaloid $\mathscr{Q}$, the following are equivalent:*

$$\begin{tikzcd}
B \arrow[r, "p"] \arrow[dr, "1_B"'] & A \arrow[d, "s"] \arrow[dr, "t"] & \\
 & B \arrow[r, "p"'] & A
\end{tikzcd}$$

Diagram 2.3

1. *the diagram* $\begin{tikzcd} A \arrow[loop above, "t"] \end{tikzcd}$ *has a limit (i.e.* $A \overset{t}{\underset{1_A}{\rightrightarrows}} A$ *has an equalizer) in* $\mathcal{Q}$*;*
2. *the diagram* $\begin{tikzcd} A \arrow[loop above, "t"] \end{tikzcd}$ *has a colimit (i.e.* $A \overset{t}{\underset{1_A}{\rightrightarrows}} A$ *has a coequalizer) in* $\mathcal{Q}$*;*
3. *the diagram* $\begin{tikzcd} A \arrow[loop above, "t"] \end{tikzcd}$ *has a lax limit in* $\mathcal{Q}$*;*
4. *the diagram* $\begin{tikzcd} A \arrow[loop above, "t"] \end{tikzcd}$ *has a lax colimit in* $\mathcal{Q}$*;*
5. *there exist an object* $B \in \mathcal{Q}_0$ *and arrows* $A \xrightarrow{s} B \xrightarrow{p} A$ *such that diagram 2.3 commutes.*

If the above is the case for the given monad t*, then* $B \xrightarrow{p} A$ *in 5. is actually its (lax) limit, and* $A \xrightarrow{s} B$ *its (lax) colimit. And the coprojection* $A \xrightarrow{s} B$ *is left adjoint to the projection* $B \xrightarrow{p} A$*.*

Proof: ($1 \iff 2 \iff 5$) is well-known for idempotents in a category, so in particular applies to monads in a quantaloid (by 2.2.1). ($1 \iff 3$) and ($2 \iff 4$) follow from the observation made in (2.8) and 1.3.6. $\square$

For a monad $t\colon A \longrightarrow A$ in a 2-category $\mathscr{K}$ viewed as a lax functor $\mathbf{1} \longrightarrow \mathscr{K}$, its lax limit (whenever it exists) is often called the "object of t-algebras", or the "Eilenberg-Moore object for t". The reason is that for $\mathscr{K} = \mathsf{Cat}$ the Eilenberg-Moore object for a monad $T\colon \mathscr{A} \longrightarrow \mathscr{A}$ is indeed $\mathscr{A}^T$, the category of Eilenberg-Moore algebras. For similar reasons, the lax colimit (whenever it exists) for a monad $t\colon A \longrightarrow A$ in $\mathscr{K}$ is the "object of free t-algebras", or the "Kleisli object".

We will adopt this terminology, and mimic the notations, for monads in a quantaloid $\mathcal{Q}$: for a monad $t\colon A \longrightarrow A$ in $\mathcal{Q}$ we'll denote by A^t its "object of algebras", and by A_t its "object of free algebras". By the proposition above, A^t exists if and only if A_t exists—and then they're isomorphic: thus we may say that "every t-algebra is free". This is particular to quantaloids—and surely does not hold for monads in 2-categories in general!

Corollary 2.2.3 *Any quantaloid homomorphism preserves all (free) algebra objects.*

More about monads in a quantaloid

Every monad $\overset{t}{A \circlearrowleft}$ in $\mathcal{Q}$ gives rise to two families of (ordinary) monads in Sup: for every $X \in \mathcal{Q}_0$,

$$\mathcal{Q}(X,t)\colon \mathcal{Q}(X,A) \longrightarrow \mathcal{Q}(X,A),$$
$$\mathcal{Q}(t,X)\colon \mathcal{Q}(A,X) \longrightarrow \mathcal{Q}(A,X).$$

For these (ordinary) monads we can consider the (ordinary) categories of Eilenberg-Moore and Kleisli algebras (viewing Sup as subcategory of Cat), respectively

$$\mathcal{Q}(X,A)^{\mathcal{Q}(X,t)} \text{ and } \mathcal{Q}(X,A)_{\mathcal{Q}(X,t)},$$
$$\mathcal{Q}(A,X)^{\mathcal{Q}(t,X)} \text{ and } \mathcal{Q}(A,X)_{\mathcal{Q}(t,X)}.$$

In other words, on the representables $\mathcal{Q}(-,A)$ and $\mathcal{Q}(A,-)$ there are monadic sup-natural transformations, respectively $\mathcal{Q}(-,t)$ and $\mathcal{Q}(t,-)$, that both admit an object of Eilenberg-Moore algebras and an object of Kleisli algebras in $\mathsf{QUANT}(\mathcal{Q}^{\mathsf{op}},\mathsf{Sup})$, resp. $\mathsf{QUANT}(\mathcal{Q},\mathsf{Sup})$.

Proposition 2.2.4 *For a monad $\overset{t}{A \circlearrowleft}$ in a quantaloid $\mathcal{Q}$, the following are equivalent:*

1. $\mathcal{Q}(-,A)^{\mathcal{Q}(-,t)}\colon \mathcal{Q}^{\mathsf{op}} \longrightarrow \mathsf{Sup}$ *is representable,*
2. $\mathcal{Q}(-,A)_{\mathcal{Q}(-,t)}\colon \mathcal{Q}^{\mathsf{op}} \longrightarrow \mathsf{Sup}$ *is representable,*
3. $\mathcal{Q}(A,-)^{\mathcal{Q}(t,-)}\colon \mathcal{Q} \longrightarrow \mathsf{Sup}$ *is representable,*
4. $\mathcal{Q}(A,-)_{\mathcal{Q}(t,-)}\colon \mathcal{Q} \longrightarrow \mathsf{Sup}$ *is representable.*

A representing object for any of these homomorphisms represents every other homomorphism too, and is the object of (free) algebras for t in $\mathcal{Q}$.

Proof: ($1 \iff 2$), respectively ($3 \iff 4$), hold because the categories $\mathcal{Q}(X,A)$, respectively $\mathcal{Q}(A,X)$ are thin: for every monad on a thin category all algebras are free! Further it is easy to see that

$$\mathcal{Q}(X,A)^{\mathcal{Q}(X,t)} = \mathsf{Lax}(\Delta_X, F) \text{ and } \mathcal{Q}(A,X)^{\mathcal{Q}(t,X)} = \mathsf{OpLax}(F, \Delta_X)$$

where F denotes the lax functor of (2.6) and the result follows by 1.3.3. $\square$

Still another way of studying and understanding monads (in a 2-category), is via the adjunctions that they generate and are generated by. In the case of $\mathcal{K} = \mathsf{Cat}$ for a monad $T\colon \mathcal{A} \longrightarrow \mathcal{A}$ both the Eilenberg-Moore category $\mathcal{A}^T$ and the Kleisli category $\mathcal{A}_T$ give rise to adjunctions that generate T:

- $\mathcal{A} \xrightarrow{F} \mathcal{A}^T \xrightarrow{U} \mathcal{A}$, $F \dashv U$, $U \circ F = T$;

- $\mathscr{A} \xrightarrow{G} \mathscr{A}_T \xrightarrow{V} \mathscr{A}$, $G \dashv V$, $V \circ G = T$.

Moreover, in a precise sense the Eilenberg-Moore adjunction is maximal, and the Kleisli adjunction is minimal, among all adjunctions that "generate" the given monad T. It is useful to make this precise for monads in a quantaloid too.

Definition 2.2.5 *For a monad* $A \circlearrowleft^{t}$ *in a quantaloid* $\mathscr{Q}$*, say that an adjoint pair of* $\mathscr{Q}$*-arrows*

$$f \dashv u \colon X \rightleftarrows A \text{ (i.e. } 1_A \leq u \circ f \text{ and } f \circ u \leq 1_X\text{)}$$

generates *the given monad if* $u \circ f = t$.

Suppose that both $f \dashv u \colon X \rightleftarrows A$ and $g \dashv v \colon Y \rightleftarrows A$ generate the monad $A \circlearrowleft^{t}$; then consider $\mathscr{Q}$-arrows $k \colon X \longrightarrow Y$ such that $k \circ f = g$ and $v \circ k = u$. This suggests the data for a quantaloid $\mathsf{Adj}(t)$, in which composition and identities are as in $\mathscr{Q}$, and also the local structure is inherited from $\mathscr{Q}$.

Proposition 2.2.6 *For any monad* $A \circlearrowleft^{t}$ *in a quantaloid* $\mathscr{Q}$*, let* $\mathsf{Adj}(t)$ *denote the quantaloid of adjunctions generating* t*. Then the following are equivalent:*

1. $\mathsf{Adj}(t)$ *has an initial object;*
2. $\mathsf{Adj}(t)$ *has a final object;*
3. $\mathsf{Adj}(t)$ *has a zero object;*
4. *there is an adjunction* $f_0 \dashv u_0 \colon Z \rightleftarrows A$ *generating* $A \circlearrowleft^{t}$ *for which* $f_0 \circ u_0 = 1_Z$.

The object Z *in 4. is then the object of (free) algebras of* t *in* $\mathscr{Q}$.

Proof: ($1 \iff 2 \iff 3$) hold in any quantaloid (the initial-final-zero object is the direct sum of the empty diagram—whenever it exists).

For ($1 \implies 4$) note that for any object $f \dashv u \colon X \rightleftarrows A$ of $\mathsf{Adj}(t)$ both $\mathscr{Q}$-arrows $1_X \colon X \longrightarrow X$ and $f \circ u \colon X \longrightarrow X$ are morphisms in $\mathsf{Adj}(t)$. So, if $f_0 \dashv u_0 \colon Z \rightleftarrows A$ is initial in $\mathsf{Adj}(t)$ then it must follow that $f_0 \circ u_0 = 1_Z$.

Now for ($4 \implies 3$), consider any other object $g \dashv v \colon X \rightleftarrows A$ in $\mathsf{Adj}(t)$: it is then easily seen that $g \circ u_0 \colon Z \longrightarrow X$ is the unique arrow in $\mathsf{Adj}(t)$ from $f_0 \dashv u_0$ to $g \dashv v$. And likewise, $f_0 \circ v \colon X \longrightarrow Z$ is the unique arrow in $\mathsf{Adj}(t)$ from $g \dashv v$ to $f_0 \dashv u_0$.

The adjunction in 4. satisfies precisely the conditions in 5. of 2.2.2—so Z is indeed the (free) algebra object for t. $\square$

The algebra object completion of a quantaloid

For a given quantaloid $\mathcal{Q}$ we will now construct its algebra-object completion: in elementary terms, this new quantaloid $\mathsf{Bim}(\mathcal{Q})$ has monads in $\mathcal{Q}$ as objects, and bimodules as arrows.

Proposition 2.2.7 *Given a quantaloid $\mathcal{Q}$, the following defines a new quantaloid $\mathsf{Bim}(\mathcal{Q})$ of "bimodules in $\mathcal{Q}$":*

- *objects: are the monads $A \overset{t}{\circlearrowleft}$, $B \overset{s}{\circlearrowleft}$, $C \overset{r}{\circlearrowleft}$, ... in $\mathcal{Q}$;*
- *arrows: an arrow $b\colon t \xrightarrow{\;\circ\;} s$ is a bimodule from t to s, i.e. a $\mathcal{Q}$-arrow $b\colon A \longrightarrow B$ with actions*
$$\begin{cases} b \circ t \leq b, \\ s \circ b \leq b; \end{cases} \tag{2.9}$$
- *two-cells: put $b \leq c$ in $\mathsf{Bim}(\mathcal{Q})(t,s)$ whenever $b \leq c$ in $\mathcal{Q}(A,B)$;*
- *composition: write composition as*
$$t \xrightarrow[\circ]{b} s \xrightarrow[\circ]{c} r = t \xrightarrow[\circ]{c \otimes_s b} r$$
and compute it simply as $c \otimes_s b = c \circ b$ in $\mathcal{Q}$;
- *identity: on a monad $A \overset{t}{\circlearrowleft}$, $t\colon t \xrightarrow{\;\circ\;} t$ itself is the identity bimodule.*

The composition of bimodules may seem to be "precisely that of $\mathcal{Q}$", and thus it may seem a bit complicated to introduce a different notation. But there lies a big difference in the identities for the composition!

We've remarked before (see 2.2.1) that every monad in $\mathcal{Q}$ is an idempotent; a similar observation can be made for bimodules, in that the action-inequalities (2.9) saturate to equalities.

Lemma 2.2.8 *For $A \overset{t}{\circlearrowleft}$ and $B \overset{s}{\circlearrowleft}$, two monads in a quantaloid $\mathcal{Q}$, an arrow in $\mathcal{Q}$ like $b\colon A \longrightarrow B$ is a bimodule $b\colon t \xrightarrow{\;\circ\;} s$ if and only if $b \circ t = b$ and $s \circ b = b$.*

The quantaloid $\mathcal{Q}$ can be embedded in $\mathsf{Bim}(\mathcal{Q})$ by sending an object A onto the identity 1_A and regarding an arrow $f\colon A \longrightarrow B$ as bimodule from 1_A to 1_B:

$$j_{\mathcal{Q}}\colon \mathcal{Q} \longrightarrow \mathsf{Bim}(\mathcal{Q})\colon (A \xrightarrow{f} B) \mapsto (1_A \xrightarrow[\circ]{f} 1_B). \tag{2.10}$$

For any homomorphism $F\colon \mathcal{Q} \longrightarrow \mathcal{Q}'$,

$$\mathsf{Bim}(F)\colon \mathsf{Bim}(\mathcal{Q}) \longrightarrow \mathsf{Bim}(\mathcal{Q}')\colon (t \xrightarrow[\circ]{b} s) \mapsto (Ft \xrightarrow[\circ]{Fb} Fs) \tag{2.11}$$

is a homomorphism making diagram 2.4 commute.

$$\begin{tikzcd} \mathcal{Q} \arrow[r, "F"] \arrow[d, "j_{\mathcal{Q}}"'] & \mathcal{Q}' \arrow[d, "j_{\mathcal{Q}'}"] \\ \mathsf{Bim}\mathcal{Q} \arrow[r, "\mathsf{Bim}(F)"'] & \mathsf{Bim}(\mathcal{Q}') \end{tikzcd}$$

Diagram 2.4

Proposition 2.2.9 *For any quantaloid $\mathcal{Q}$,*

1. *the embedding (2.10) is a fully faithful homomorphism of quantaloids;*

2. *this embedding is an equivalence of quantaloids if and only if $\mathcal{Q}$ admits all (free) algebra objects;*

3. *applying the bimodules construction to $\mathsf{Bim}(\mathcal{Q})$ itself, the embedding*

$$j_{\mathsf{Bim}(\mathcal{Q})}\colon \mathsf{Bim}(\mathcal{Q}) \longrightarrow \mathsf{Bim}(\mathsf{Bim}(\mathcal{Q}))$$

   is always an equivalence;

4. *$\mathcal{Q}$ is equivalent to $\mathsf{Bim}(\mathcal{Q}')$ for some quantaloid $\mathcal{Q}'$ if and only if $\mathcal{Q}$ admits all (free) algebra objects;*

5. *the embedding (2.10) describes the universal (free) algebra object completion of $\mathcal{Q}$ in* QUANT.

Proof: (1) Clearly, $\mathcal{Q}(A,B) \cong \mathsf{Bim}(\mathcal{Q})(1_A, 1_B)$ in Sup.

(2) By (1), the embedding (2.10) is an equivalence if and only if every monad $A \xrightarrow{t} A$ in $\mathcal{Q}$ is isomorphic to an identity monad – say $B \xrightarrow{1_B} B$ – in the quantaloid $\mathsf{Bim}(\mathcal{Q})$: there exist bimodules $t \overset{f}{\multimap} 1_B$ and $1_B \overset{u}{\multimap} t$ such that $u \otimes_{1_B} f = t$ and $u \otimes_t f = 1_B$. In terms of the quantaloid $\mathcal{Q}$ this means that for the monad t we must have arrows $f\colon A \longrightarrow B$ and $u\colon B \longrightarrow A$ such that

- $f \circ t \leq f$ (because $f\colon t \multimap 1_B$);

- $t \circ u \leq u$ (because $u\colon 1_B \multimap t$);

- $u \circ f = t$ (because $u \otimes_{1_B} f = t$);

- $f \circ u = 1_B$ (because $f \otimes_t u = 1_B$).

Using 2.2.8 (to see that, actually, $f \circ t = f$ and $t \circ u = u$) and 2.2.2 the result follows.

$$\begin{tikzcd}
s_0 \arrow[r, "s_2", "\circ" marking] \arrow[dr, "s_0"', "\circ" marking] & t \arrow[d, "s_1", "\circ" marking] \arrow[dr, "s", "\circ" marking] & \\
 & s_0 \arrow[r, "s_2"', "\circ" marking] & t
\end{tikzcd}$$

Diagram 2.5

$$\begin{tikzcd}
\mathscr{Q} \arrow[r, "F"] \arrow[d, "j_{\mathscr{Q}}"'] \arrow[dr, phantom, "\overset{\sim}{\Longrightarrow}"] & \mathscr{Q}' \\
\mathsf{Bim}(\mathscr{Q}) \arrow[ur, "\mathsf{Bim}(F)"'] &
\end{tikzcd}$$

Diagram 2.6

(3) As observed, $\mathsf{Bim}(\mathscr{Q})$ is itself a quantaloid, so we can indeed iterate the process of "taking bimodules". We will prove that all monads in $\mathsf{Bim}(\mathscr{Q})$ split. So let $t\circlearrowleft^{s}$ be a monad in $\mathsf{Bim}(\mathscr{Q})$, on a monad $A\circlearrowleft^{t}$ in $\mathscr{Q}$. Then the "underlying $\mathscr{Q}$-arrow" of s, which is $A\circlearrowleft^{s}$, is a monad in $\mathscr{Q}$ too:

- $t \leq s$ and $1_A \leq t$ imply $1_A \leq s$;

- $s \otimes_t s \leq s$ and $s \otimes_t s = s \circ s$ imply $s \circ s \leq s$.

To avoid confusion, let us denote $A\circlearrowleft^{s_0}$ for "s as monad in $\mathscr{Q}$". Further, the arrow $A \xrightarrow{s} A$ is (the underlying arrow of) a bimodule $t \xrightarrow{\circ} s_0$ but also of a bimodule $s_0 \xrightarrow{\circ} t$; let us denote these as $s_1\colon t \xrightarrow{\circ} s_0$ and $s_2\colon s_0 \xrightarrow{\circ} t$. So actually $t\circlearrowleft^{s}$ splits over $s_0\circlearrowleft^{s_0}$ (the identity bimodule on the monad $A\circlearrowleft^{s_0}$) as shown in diagram 2.5.

(4) The "if" was taken care of in (2). For the "only if", since $\mathscr{Q}$ is equivalent to a quantaloid in which all monads split, it follows straightforwardly that monads split in $\mathscr{Q}$ too (using the essential surjectivity and the fully faithfulness of the equivalence).

(5) We must prove that any homomorphism $F\colon \mathscr{Q} \longrightarrow \mathscr{Q}'$ into a $\mathscr{Q}'$ that admits (free) algebra objects factorizes in an essentially unique way over the embedding $j_{\mathscr{Q}}\colon \mathscr{Q} \longrightarrow \mathsf{Bim}(\mathscr{Q})$. But if $\mathscr{Q}'$ admits (free) algebra objects, the embedding $j_{\mathscr{Q}'}\colon \mathscr{Q}' \longrightarrow \mathsf{Bim}(\mathscr{Q}')$ is an equivalence, and $\mathsf{Bim}(F)$ can be seen as a factorization-up-to-natural-isomorphism in QUANT (with some abuse of notation)—see diagram 2.6. Suppose now that there is another factorization $G\colon \mathsf{Bim}(\mathscr{Q}) \longrightarrow \mathscr{Q}'$. Since an object $A\circlearrowleft^{t}$ of $\mathsf{Bim}(\mathscr{Q})$ may be regarded as object of (free) algebras for the monad $1_A \xrightarrow[\circ]{t} 1_A$ in $\mathsf{Bim}(\mathscr{Q})$, Gt must be the object of (free) algebras for

the monad $G1_A \xrightarrow{Gt} G1_A = FA \xrightarrow{Ft} FA$ in $\mathcal{Q}$... So this G agrees with $\mathsf{Bim}(F)$ on objects; it follows that G agrees with $\mathsf{Bim}(F)$ also on morphisms. So the factorization $\mathsf{Bim}(F)$ is essentially unique. $\square$

The following will be useful later on.

Lemma 2.2.10 *If a quantaloid $\mathcal{Q}$ has direct sums, then so does $\mathsf{Bim}(\mathcal{Q})$. In other words, for any quantaloid $\mathcal{Q}$ the canonical inclusion*

$$i_{\mathsf{Bim}(\mathsf{Matr}(\mathcal{Q}))} \colon \mathsf{Bim}(\mathsf{Matr}(\mathcal{Q})) \longrightarrow \mathsf{Matr}(\mathsf{Bim}(\mathsf{Matr}(\mathcal{Q})))$$

is always an equivalence.

Proof: For a collection $(A_i \circlearrowleft^{t_i})_{i \in I}$ of monads in $\mathcal{Q}$ we dispose by assumption of an object A and arrows $A \xrightarrow{p_i} A_0 \xrightarrow{s_i} A$ in $\mathcal{Q}$ satisfying

$$\bigvee_i s_i \circ p_i = 1_A \text{ and } p_i \circ s_j = \begin{cases} 1_{A_i} \text{ if } i = j, \\ 0_{A_i, A_j} \text{ otherwise.} \end{cases}$$

Considering the arrows $A \xrightarrow{p_i} A_i \xrightarrow{t_i} A_i \xrightarrow{s_i} A$ we put $t := \bigvee_i s_i \circ t_i \circ p_i$ and it is easily verified that $t \colon A \longrightarrow A$ is thus a monad in $\mathcal{Q}$[3]. Further it is easily seen that, for each $i \in I$, $p_i \circ t = t_i \circ p_i$ and $t \circ s_i = s_i \circ t_i$. Putting now $\pi_i := p_i \circ t$ and $\sigma_i := t \circ s_i$ it is then straightforward to see that these are bimodules $t \xrightarrow{\pi_i} t_i \xrightarrow{\sigma_i} t$ between monads in $\mathcal{Q}$, and moreover

$$\bigvee_i \sigma_i \circ \pi_i = t \text{ and } \pi_i \circ \sigma_j = \begin{cases} t_i \text{ if } i = j, \\ 0 \text{ otherwise.} \end{cases}$$

That is to say, t is the direct sum of the t_i in $\mathsf{Bim}(\mathcal{Q})$. $\square$

Because it is a quantaloid, $\mathsf{Bim}(\mathcal{Q})$ is closed; actually, its closedness is the "same" as that of $\mathcal{Q}$.

Proposition 2.2.11 *For bimodules $t \xrightarrow{f} s$ and $s \xrightarrow{g} r$ between monads in a quantaloid $\mathcal{Q}$, the adjoints to composition in $\mathsf{Bim}(\mathcal{Q})$,*

$$- \otimes_s f \dashv \{f, -\} \text{ and } g \otimes_s - \dashv [g, -],$$

are computed as follows: for $h \colon t \longrightarrow r$,

$$\{f, h\}^{\mathsf{Bim}(\mathcal{Q})} = \{f, h\}^{\mathcal{Q}},$$

$$[g, h]^{\mathsf{Bim}(\mathcal{Q})} = [g, h]^{\mathcal{Q}}.$$

[3] Precisely as the object A is the direct sum of the A_i, the arrow t is the direct sum of the t_i in $\mathcal{Q}$. The point is to prove that t is also the direct sum of the t_i in $\mathsf{Bim}(\mathcal{Q})$.

Proof: We can calculate the right adjoint to $g \otimes_s -$ as follows:

$$\begin{aligned}
[g,h]^{\mathsf{Bim}(\mathcal{Q})} &= \bigvee\{f\colon t \!-\!\!\circ\!\!\!\rightarrow s \text{ in } \mathsf{Bim}(\mathcal{Q}) \mid g \otimes_s f \leq h\} \\
&= \bigvee\{f\colon A \longrightarrow B \text{ in } \mathcal{Q} \mid f \circ t \leq f, s \circ f \leq f, g \circ f \leq h\} \\
&= \bigvee\{f\colon A \longrightarrow B \text{ in } \mathcal{Q} \mid f \circ t \leq f, s \circ f \leq f, f \leq [g,h]^{\mathcal{Q}}\},
\end{aligned}$$

so $[g,h]^{\mathsf{Bim}(\mathcal{Q})} \leq [g,h]^{\mathcal{Q}}$ as $\mathcal{Q}$-arrows. But actually $[g,h]^{\mathcal{Q}}\colon A \longrightarrow B$ turns out to be a bimodule from t to s; indeed, its actions can be verified as

$$\begin{aligned}
[g,h]^{\mathcal{Q}} \circ t &= \left(\bigvee\{x\colon A \longrightarrow B \text{ in } \mathcal{Q} \mid g \circ x \leq h\}\right) \circ t \\
&= \bigvee\{x \circ t \mid g \circ x \leq h\} \\
&\leq \bigvee\{f \mid g \circ f \leq h\} \\
&= [g,h]^{\mathcal{Q}},
\end{aligned}$$

and likewise for $s \circ [g,h]^{\mathcal{Q}} \leq [g,h]^{\mathcal{Q}}$. Since composition in $\mathsf{Bim}(\mathcal{Q})$ is computed "as in $\mathcal{Q}$", it is easily seen that $g \otimes_s [g,h]^{\mathcal{Q}} \leq h$; therefore, $[g,h]^{\mathcal{Q}} \leq [g,h]^{\mathsf{Bim}(\mathcal{Q})}$. So $[g,h]^{\mathsf{Bim}(\mathcal{Q})} = [g,h]^{\mathcal{Q}}$.

In the same way, $\{f,h\}^{\mathsf{Bim}(\mathcal{Q})} = \{f,h\}^{\mathcal{Q}}$. $\square$

2.3 Categories and distributors

Definition 2.3.1 *A* (small) $\mathcal{Q}$-enriched category $\mathbb{A}$ *(or simply* $\mathcal{Q}$-category *for short) is given by*

- *objects: a (small) set* $\mathbb{A}_0$ *to every element of which is associated a type in* $\mathcal{Q}_0$*:* $\forall a \in \mathbb{A}_0 : ta \in \mathcal{Q}_0$*;*
- *hom-arrows*[4]*:* $\forall a, a' \in \mathbb{A}_0$*:* $\mathbb{A}(a', a) \in \mathcal{Q}(ta, ta')$*;*

satisfying

- *composition-inequalities:* $\forall a, a', a'' \in \mathbb{A}_0 :\ \mathbb{A}(a'', a') \circ \mathbb{A}(a', a) \leq \mathbb{A}(a'', a)$*;*
- *identity-inequalities:* $\forall a \in \mathbb{A}_0$*:* $1_{ta} \leq \mathbb{A}(a, a)$*.*

Definition 2.3.2 *A* distributor $\Phi\colon \mathbb{A} \!-\!\!\circ\!\!\!\rightarrow \mathbb{B}$ *between* $\mathcal{Q}$*-categories* $\mathbb{A}$ *and* $\mathbb{B}$ *is a collection of* $\mathcal{Q}$*-arrows*

$$\Big(\Phi(b,a)\colon ta \longrightarrow tb\Big)_{a \in \mathbb{A}_0, b \in \mathbb{B}_0}$$

satisfying

[4]Note the "right-to-left" notation for the hom-arrows.

- *action-inequalities:* $\forall a, a' \in \mathbb{A}_0$ *and* $b, b' \in \mathbb{B}_0$:

$$\begin{cases} \Phi(b,a') \circ \mathbb{A}(a',a) \leq \Phi(b,a), \\ \mathbb{B}(b,b') \circ \Phi(b',a) \leq \Phi(b,a). \end{cases}$$

It is quite obvious that a $\mathcal{Q}$-category is a monad, and a distributor a bimodule, in $\mathsf{Matr}(\mathcal{Q})$.

Definition 2.3.3 *For a given quantaloid* $\mathcal{Q}$*, we define that*

$$\mathsf{Dist}(\mathcal{Q}) = \mathsf{Bim}(\mathsf{Matr}(\mathcal{Q})).$$

So, explicitly, this new quantaloid $\mathsf{Dist}(\mathcal{Q})$ consists of

- objects: small $\mathcal{Q}$-enriched categories;
- arrows: distributors;
- two-cells: elementwise ordering as for matrices;
- composition: with the "linear algebra formula" (2.3) as for matrices;
- identities: the identity distributor on $\mathbb{A}$ is $\mathbb{A}\colon \mathbb{A} \xrightarrow{\circ} \mathbb{A}$ itself.

The following is then an easy consequence of 2.2.1 and 2.2.8.

Lemma 2.3.4 *For any* $\mathcal{Q}$*-category* $\mathbb{A}$*,*

$$\mathbb{A} \otimes_{\mathbb{A}} \mathbb{A} = \mathbb{A};$$

and for any distributor $\Phi\colon \mathbb{A} \xrightarrow{\circ} \mathbb{B}$*,*

$$\Phi \otimes_{\mathbb{A}} \mathbb{A} = \Phi \quad \textit{and} \quad \mathbb{B} \otimes_{\mathbb{B}} \Phi = \Phi.$$

It follows from the previous sections that $\mathsf{Dist}(\mathcal{Q})$ is the $\oplus$-and-algebra-object completion of $\mathcal{Q}$ in QUANT.

Proposition 2.3.5 *For any quantaloid* $\mathcal{Q}$*,*

1. *the obvious embedding*

$$k_{\mathcal{Q}}\colon \mathcal{Q} \longrightarrow \mathsf{Dist}(\mathcal{Q})\colon (A \xrightarrow{f} B) \mapsto (*_A \xrightarrow[\circ]{(f)} *_B), \tag{2.12}$$

(where now $*_A$ *stands for the one-object* $\mathcal{Q}$*-category whose single object is of type* $A \in \mathcal{Q}_0$ *and whose hom is the identity* 1_A*, and* (f) *is the one-element distributor from* $*_A$ *to* $*_B$ *whose single entry is* f*) is a fully faithful homomorphism of quantaloids;*

$$\begin{tikzcd} \mathscr{Q} \arrow[r, "i_{\mathscr{Q}}"] \arrow[dr, "k_{\mathscr{Q}}"'] & \mathsf{Matr}(\mathscr{Q}) \arrow[d, "j_{\mathsf{Matr}(\mathscr{Q})}"] \\ & \mathsf{Bim}(\mathsf{Matr}(\mathscr{Q})) \end{tikzcd}$$

Diagram 2.7

2. *this embedding is an equivalence if and only if $\mathscr{Q}$ has all direct sums and admits (free) algebra objects;*

3. *applying the distributor construction to* $\mathsf{Dist}(\mathscr{Q})$ *itself, the embedding*

$$k_{\mathsf{Dist}(\mathscr{Q})}\colon \mathsf{Dist}(\mathscr{Q}) \longrightarrow \mathsf{Dist}(\mathsf{Dist}(\mathscr{Q}))$$

is always an equivalence;

4. *$\mathscr{Q}$ is equivalent to* $\mathsf{Dist}(\mathscr{Q}')$ *for some quantaloid $\mathscr{Q}'$ if and only if $\mathscr{Q}$ has all direct sums and admits all (free) algebra objects;*

5. *the embedding in (2.12) describes the universal direct sum-(free) algebra object completion of $\mathscr{Q}$ in* QUANT.

Proof: (1) This embedding is the composition of the embeddings in (2.4) and (2.10) as shown in diagram 2.7 Since both of these are fully faithful quantaloid homomorphisms, so is (2.12).

(2) Saying that $k_{\mathscr{Q}} = j_{\mathsf{Matr}(\mathscr{Q})} \circ i_{\mathscr{Q}}$ is an equivalence implies that $j_{\mathsf{Matr}(\mathscr{Q})}$ is surjective on objects—but it is always fully faithful, thus it is itself an equivalence! But then also $i_{\mathscr{Q}}$ must be an equivalence. This proves the “only if”. The “if” is obvious.

(3, 4, 5) Are obvious. □

Corollary 2.3.6 *Every quantaloid homomorphism $F\colon \mathscr{Q} \longrightarrow \mathscr{Q}'$ determines an essentially unique homomorphism*

$$\mathsf{Dist}(F)\colon \mathsf{Dist}(\mathscr{Q}) \longrightarrow \mathsf{Dist}(\mathscr{Q}')$$

that makes diagram 2.8 commute.

That is to say, the universality of the Dist-construction at once implies that every quantaloid homomorphism $F\colon \mathscr{Q} \longrightarrow \mathscr{Q}'$ induces a “change of base” homomorphism $\mathsf{Dist}(F)\colon \mathsf{Dist}(\mathscr{Q}) \longrightarrow \mathsf{Dist}(\mathscr{Q}')$. It is easy to figure out that, for a

$$
\begin{tikzcd}
\mathcal{Q} \arrow[r, "k_{\mathcal{Q}}"] \arrow[d, "F"'] & \mathsf{Dist}(\mathcal{Q}) \arrow[d, "\mathsf{Dist}(F)"] \\
\mathcal{Q}' \arrow[r, "k_{\mathcal{Q}'}"'] & \mathsf{Dist}(\mathcal{Q}')
\end{tikzcd}
$$

Diagram 2.8

$\mathcal{Q}$-category $\mathbb{A}$, the corresponding $\mathcal{Q}'$-category $\mathsf{Dist}(F)(\mathbb{A})$ is obtained by applying F to the types and the hom-arrows of $\mathbb{A}$—and likewise for distributors. See (2.5) and (2.11).

The adjoints to composition in $\mathsf{Dist}(\mathcal{Q}) = \mathsf{Bim}(\mathsf{Matr}(\mathcal{Q}))$ are calculated "as in $\mathsf{Matr}(\mathcal{Q})$"—this is a corollary of 2.2.11. So we can conclude – with 2.1.5 – as follows.

Proposition 2.3.7 *For $\mathcal{Q}$-categories and distributors $\mathbb{A} \overset{\Phi}{\longrightarrow\!\!\!\!\!\circ\;} \mathbb{B} \overset{\Psi}{\longrightarrow\!\!\!\!\!\circ\;} \mathbb{C}$, the adjoints to composition in $\mathsf{Dist}(\mathcal{Q})$,*

$$- \otimes_{\mathbb{B}} \Phi \dashv \{\Phi, -\} \text{ and } \Psi \otimes_{\mathbb{B}} - \dashv [\Psi, -],$$

are computed as follows: for $\mathbb{A} \overset{\Theta}{\longrightarrow\!\!\!\!\!\circ\;} \mathbb{C}$,

$$\Big\{\Phi, \Theta\Big\}^{\mathsf{Dist}(\mathcal{Q})}(z, y) = \Big\{\Phi, \Theta\Big\}^{\mathsf{Matr}(\mathcal{Q})}(z, y) = \bigwedge_{x \in \mathbb{A}_0} \Big\{\Phi(y, x), \Theta(z, x)\Big\}^{\mathcal{Q}},$$

$$\Big[\Psi, \Theta\Big]^{\mathsf{Dist}(\mathcal{Q})}(y, x) = \Big[\Psi, \Theta\Big]^{\mathsf{Matr}(\mathcal{Q})}(y, x) = \bigwedge_{z \in \mathbb{C}_0} \Big[\Psi(z, y), \Theta(z, x)\Big]^{\mathcal{Q}}.$$

2.4 Functors

Functors induce adjoint pairs of distributors

From the elementary description of a $\mathcal{Q}$-enriched category given above, it is clear what a "functor" should be: a mapping of objects together with an appropriate action on homs.

Definition 2.4.1 *For $\mathcal{Q}$-categories $\mathbb{A}$ and $\mathbb{B}$, a* functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ *is a mapping of objects*

$$\mathbb{A}_0 \longrightarrow \mathbb{B}_0 \colon a \mapsto Fa$$

satisfying

- *type-equalities: $\forall a \in \mathbb{A}_0 : \ t(Fa) = ta$;*

- *action-inequalities:* $\forall a, a' \in \mathbb{A}_0: \ \mathbb{A}(a', a) \leq \mathbb{B}(Fa', Fa)$.

It is clearly not necessary to impose any extra conditions on F for it to preserve "composition" and/or "identities" in $\mathbb{A}$. Remark that a functor is completely determined by its mapping of objects.

Functors compose in the expected manner: given $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ and $G\colon \mathbb{B} \longrightarrow \mathbb{C}$,

$$G \circ F\colon \mathbb{A} \longrightarrow \mathbb{C}\colon a \mapsto G(Fa)$$

is their composite. This composition law is clearly associative, and the identity on a $\mathcal{Q}$-category $\mathbb{A}$ is provided by $1_{\mathbb{A}}\colon \mathbb{A} \longrightarrow \mathbb{A}\colon a \mapsto a$.

Proposition 2.4.2 *$\mathcal{Q}$-enriched categories and functors form a category* $\mathsf{Cat}(\mathcal{Q})$.

Every functor induces an adjoint pair of distributors, and the resulting inclusion of the functor category in the distributor category – although straightforward – will prove to be a key element for the development of the theory of $\mathcal{Q}$-enriched categories.

Lemma 2.4.3 *For $\mathcal{Q}$-categories and functors as in $\mathbb{A} \xrightarrow{F} \mathbb{B} \xleftarrow{G} \mathbb{C}$ the matrix*

$$\Big(\mathbb{B}(G-, F-)(c, a) := \mathbb{B}(Gc, Fa)\Big)_{a \in \mathbb{A}_0, c \in \mathbb{C}_0}$$

is actually a distributor $\mathbb{B}(G-, F-)\colon \mathbb{A} \multimap \mathbb{B}$.

Proof: Clearly the matrix above is well-defined, and goes from $\mathbb{A}_0$ to $\mathbb{C}_0$. As for its actions, we have by functoriality of F and multiplication in $\mathbb{B}$ that, for any $a, a' \in \mathbb{A}_0$ and $c \in \mathbb{C}_0$

$$\mathbb{B}(Gc, Fa') \circ \mathbb{A}(a', a) \leq \mathbb{B}(Gc, Fa') \circ \mathbb{B}(Fa', Fa) \leq \mathbb{B}(Gc, Fa);$$

and likewise for $\mathbb{C}(c', c) \circ \mathbb{B}(Gc, Fa) \leq \mathbb{B}(Gc', Fa)$. $\square$

Proposition 2.4.4 *Any functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ determines an adjoint pair of distributors,*

$$\mathbb{B}(1_{\mathbb{B}}-, F-) \dashv \mathbb{B}(F-, 1_{\mathbb{B}}-).$$

Proof: We must verify two inequations in $\mathsf{Dist}(\mathcal{Q})$, but this is easily done using the formula for the composition of distributors: on the one hand

$$\begin{aligned}
\mathbb{B}(F-, 1_{\mathbb{B}}-) \otimes_{\mathbb{B}} \mathbb{B}(1_{\mathbb{B}}-, F-) &= \bigvee_{b \in \mathbb{B}_0} \mathbb{B}(F-, 1_{\mathbb{B}}b) \circ \mathbb{B}(1_{\mathbb{B}}b, F-) \\
&= \bigvee_{b \in \mathbb{B}_0} \mathbb{B}(F-, b) \circ \mathbb{B}(b, F-) \\
&\overset{*}{=} \mathbb{B}(F-, F-) \\
&\geq \mathbb{A}(-, -)
\end{aligned}$$

where $(*)$ follows from 2.3.4 and the inequation is the action-inequality of F; and on the other hand

$$\begin{aligned}
\mathbb{B}(1_\mathbb{B}-,F-)\otimes_\mathbb{A}\mathbb{B}(F-,1_\mathbb{B}-) &= \bigvee_{a\in\mathbb{A}_0}\mathbb{B}(1_\mathbb{B}-,Fa)\circ\mathbb{B}(Fa,1_\mathbb{B}-) \\
&\leq \bigvee_{b\in\mathbb{B}_0}\mathbb{B}(1_\mathbb{B}-,b)\circ\mathbb{B}(b,1_\mathbb{B}-) \\
&= \bigvee_{b\in\mathbb{B}_0}\mathbb{B}(-,b)\circ\mathbb{B}(b,-) \\
&\stackrel{*}{=} \mathbb{B}(-,-)
\end{aligned}$$

where the inequality now follows because $\{Fa \mid a\in\mathbb{A}_0\}\subseteq\mathbb{B}_0$, and the $(*)$ again follows from 2.3.4. $\square$

Of course we will always abbreviate notations as

$$\mathbb{B}(-,F-):=\mathbb{B}(1_\mathbb{B}-,F-) \text{ and } \mathbb{B}(F-,-):=\mathbb{B}(F-,1_\mathbb{B}-).$$

Proposition 2.4.5 *Sending a functor to the left adjoint distributor that it induces, as in*

$$\mathsf{Cat}(\mathcal{Q})\longrightarrow\mathsf{Dist}(\mathcal{Q})\colon(\mathbb{A}\xrightarrow{F}\mathbb{B})\mapsto(\mathbb{A}\xrightarrow{\mathbb{B}(-,F-)}\mathbb{B}), \tag{2.13}$$

determines a functor from $\mathsf{Cat}(\mathcal{Q})$ *into* $\mathsf{Dist}(\mathcal{Q})$*. Sending a functor to the right adjoint determines a similar contravariant functor.*

Proof: It is trivial that the identity functor $1_\mathbb{A}\colon\mathbb{A}\longrightarrow\mathbb{A}$ is mapped onto the identity distributor $\mathbb{A}\colon\mathbb{A}\longrightarrow\mathbb{A}$. As for composition, suppose that $F\colon\mathbb{A}\longrightarrow\mathbb{B}$ and $G\colon\mathbb{B}\longrightarrow\mathbb{C}$ are functors, then

$$\mathbb{C}(-,G-)\otimes_\mathbb{B}\mathbb{B}(-,F-)=\mathbb{C}(-,G\circ F-)$$

follows simply from 2.3.4: $\mathbb{C}(-,G-)$ is a distributor, and the equation above may be read as the action of its domain! $\square$

Since $\mathsf{Dist}(\mathcal{Q})$ has local structure, the category $\mathsf{Cat}(\mathcal{Q})$ inherits this structure via the functor (2.13). Thus every $\mathsf{Cat}(\mathcal{Q})(\mathbb{A},\mathbb{B})$ is really a preordered set[5], and we may in particular speak of "isomorphic functors" (i.e. equivalent objects in the preorder), or an "essentially unique functor" (one which is determined "up to isomorphism"), etc. Some other equivalent expressions for "$F\leq G$" will be useful later on; they are all easy consequences of 1.2.3 and 1.2.1 (i.e. calculations with adjoints in the quantaloid $\mathsf{Dist}(\mathcal{Q})$).

[5]But not a locally partially ordered one, let alone a quantaloid!

$$\begin{tikzcd}
\mathsf{Cat}(\mathcal{Q}) \arrow[r, "\mathsf{Cat}(F)"] \arrow[d] & \mathsf{Cat}(\mathcal{Q}') \arrow[d] \\
\mathsf{Dist}(\mathcal{Q}) \arrow[r, "\mathsf{Dist}(F)"'] & \mathsf{Dist}(\mathcal{Q}')
\end{tikzcd}$$

Diagram 2.9

Lemma 2.4.6 *For $F, G\colon \mathbb{A} \rightrightarrows \mathbb{B}$ in $\mathsf{Cat}(\mathcal{Q})$,*

$$\begin{aligned}
F \le G \quad &\Longleftrightarrow \quad \mathbb{B}(-, F-) \le \mathbb{B}(-, G-) \text{ in } \mathsf{Dist}(\mathcal{Q}) && (2.14)\\
&\Longleftrightarrow \quad \mathbb{B}(G-, -) \le \mathbb{B}(F-, -) \text{ in } \mathsf{Dist}(\mathcal{Q}) && (2.15)\\
&\Longleftrightarrow \quad \mathbb{A}(-, -) \le \mathbb{B}(F-, G-) \text{ in } \mathsf{Dist}(\mathcal{Q}). && (2.16)
\end{aligned}$$

and consequently

$$\begin{aligned}
F \cong G \quad &\Longleftrightarrow \quad F \le G \text{ and } G \le F \text{ in } \mathsf{Cat}(\mathcal{Q}) && (2.17)\\
&\Longleftrightarrow \quad \mathbb{B}(-, F-) = \mathbb{B}(-, G-) \text{ in } \mathsf{Dist}(\mathcal{Q}) && (2.18)\\
&\Longleftrightarrow \quad \mathbb{B}(F-, -) = \mathbb{B}(G-, -) \text{ in } \mathsf{Dist}(\mathcal{Q}) && (2.19)\\
&\Longleftrightarrow \quad \begin{cases} \mathbb{A}(-, -) \le \mathbb{B}(F-, G-) \\ \mathbb{A}(-, -) \le \mathbb{B}(G-, F-) \end{cases} \text{ in } \mathsf{Dist}(\mathcal{Q}) && (2.20)
\end{aligned}$$

Corollary 2.4.7 *We have that $\mathsf{Cat}(\mathcal{Q})$ is a locally preordered 2-category; and the functor of (2.13) turns out to be a 2-functor which is the identity on objets, "essentially faithful" (but not full), and locally fully faithful.*

As $\mathsf{Cat}(\mathcal{Q})$ is a 2-category, we now dispose of all possible 2-categorical notions, such as adjoint functors, equivalent categories, etc. We will discuss these on p. 83 and further.

To end this section with, we mention a corollary of 2.3.6 and 2.4.7 about the "change of base" for categories and functors.

Corollary 2.4.8 *A quantaloid homomorphism $F\colon \mathcal{Q} \longrightarrow \mathcal{Q}'$ canonically determines a 2-functor $\mathsf{Cat}(F)\colon \mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q}')$ that makes diagram 2.9 commute: the vertical arrows denote 2-functors as in 2.4.7, the bottom horizontal arrow is the "change of base" homomorphism for the Dist-construction as in 2.3.6.*

Proof: The point is that a functor between $\mathcal{Q}$-enriched categories, say $K\colon \mathbb{A} \longrightarrow \mathbb{B}$, is an object-mapping $\mathbb{A}_0 \longrightarrow \mathbb{B}_0\colon a \mapsto Ka$ that preserves types and satisfies certain inequalities, and that the *same* object mapping is a functor between the $\mathcal{Q}'$-categories $\mathsf{Dist}(F)(\mathbb{A})$ and $\mathsf{Dist}(F)(\mathbb{B})$. $\square$

Underlying preorder of a category

For a $\mathcal{Q}$-category $\mathbb{A}$ and an object $a \in \mathbb{A}_0$ we denote

$$\Delta a \colon *_{ta} \longrightarrow \mathbb{A} \tag{2.21}$$

for the functor defined on the one-object $\mathcal{Q}$-category $*_{ta}$ (whose single object is of type ta and whose hom is the identity 1_{ta}) that sends the single object $*$ to the given object $a \in \mathbb{A}_0$. The functoriality of Δa derives from the monadicity of $\mathbb{A}$ as endo-matrix on the $\mathcal{Q}$-typed set $\mathbb{A}_0$: $\Delta_{\mathbb{A}_0} \leq \mathbb{A}$ implies indeed that $1_{ta} \leq \mathbb{A}(a,a)$.

Consider now two objects a, a' of some $\mathcal{Q}$-category $\mathbb{A}$, and suppose that they are of the same type: $ta = ta' =: A \in \mathcal{Q}_0$. To these objects thus correspond functors

$$\Delta a \colon *_A \longrightarrow \mathbb{A} \quad \text{and} \quad \Delta a' \colon *_A \longrightarrow \mathbb{A},$$

and these functors are elements of the preorder $\mathsf{Cat}(\mathcal{Q})(*_A, \mathbb{A})$. Thus we can think of a preorder on the set $\mathbb{A}_0$ of objects of $\mathbb{A}$, induced by the preorder on the corresponding constant functors; we will speak of the *underlying preorder* $(\mathbb{A}_0, \leq)$ of the $\mathcal{Q}$-category $\mathbb{A}$. In particular, whenever two objects of $\mathbb{A}$ are equivalent in $\mathbb{A}$'s underlying preorder then we say that they are *isomorphic objects*.

It is useful to record – much like 2.4.6 – some equivalent expressions for the "underlying preorder" on the objects of a $\mathcal{Q}$-category.

Lemma 2.4.9 *Considering two objects of a $\mathcal{Q}$-category with the same type, say $a, a' \in \mathbb{A}_0$ such that $ta = ta' =: A \in \mathcal{Q}_0$, we have the following equivalences:*

$$
\begin{aligned}
a \leq a' \text{ in } \mathbb{A}_0 \quad &\iff \quad \Delta a \leq \Delta a' \text{ in } \mathsf{Cat}(\mathcal{Q})(*_A, \mathbb{A}) &(2.22)\\
&\iff \quad \mathbb{A}(-,a) \leq \mathbb{A}(-,a') \text{ in } \mathsf{Dist}(*_A, \mathbb{A}) &(2.23)\\
&\iff \quad \mathbb{A}(a',-) \leq \mathbb{A}(a,-) \text{ in } \mathsf{Dist}(\mathbb{A}, *_A) &(2.24)\\
&\iff \quad 1_A \leq \mathbb{A}(a,a') \text{ in } \mathcal{Q} &(2.25)
\end{aligned}
$$

And, consequently,

$$
\begin{aligned}
a \cong a' \text{ in } \mathbb{A}_0 \quad &\iff \quad a \leq a' \text{ and } a' \leq a &(2.26)\\
&\iff \quad \mathbb{A}(-,a) = \mathbb{A}(-,a') \text{ in } \mathsf{Dist}(*_A, \mathbb{A}) &(2.27)\\
&\iff \quad \mathbb{A}(a',-) = \mathbb{A}(a,-) \text{ in } \mathsf{Dist}(\mathbb{A}, *_A) &(2.28)\\
&\iff \quad \left\{ \begin{array}{l} 1_A \leq \mathbb{A}(a,a') \\ 1_A \leq \mathbb{A}(a',a) \end{array} \right. \text{ in } \mathcal{Q} &(2.29)\\
&\iff \quad \left\{ \begin{array}{l} \mathbb{A}(a,a') = \mathbb{A}(a,a) \\ \mathbb{A}(a',a) = \mathbb{A}(a',a') \end{array} \right. \text{ in } \mathcal{Q} &(2.30)\\
&\iff \quad \left\{ \begin{array}{l} \mathbb{A}(a,a') = \mathbb{A}(a',a') \\ \mathbb{A}(a',a) = \mathbb{A}(a,a) \end{array} \right. \text{ in } \mathcal{Q} &(2.31)
\end{aligned}
$$

Proposition 2.4.10 *Any functor $F:\mathbb{A}\longrightarrow\mathbb{B}$ between $\mathcal{Q}$-categories determines an order-preserving map between the underlying preorders:*

$$F_0:\mathbb{A}_0\longrightarrow\mathbb{B}_0: a\mapsto Fa$$

(i.e. F_0 is just the object-mapping of F). And for $F,G:\mathbb{A}\rightrightarrows\mathbb{B}$ in $\mathsf{Cat}(\mathcal{Q})$ we have that

$$F\leq G \textit{ in } \mathsf{Cat}(\mathcal{Q})(\mathbb{A},\mathbb{B}) \iff \forall a\in\mathbb{A}_0 : Fa\leq Ga \textit{ in } \mathbb{B}_0 \tag{2.32}$$

so consequently

$$F\cong G \textit{ in } \mathsf{Cat}(\mathcal{Q})(\mathbb{A},\mathbb{B}) \iff \forall a\in\mathbb{A}_0 : Fa\cong Ga \textit{ in } \mathbb{B}_0. \tag{2.33}$$

Proof: The first part really follows from the action-inequality of F and (2.25): for two objects $a,a'\in\mathbb{A}_0$ of type A,

$$a\leq a' \iff 1_A\leq\mathbb{A}(a,a')\Longrightarrow 1_{FA}\leq\mathbb{B}(Fa,Fa') \iff Fa'\leq Fa.$$

The "pointwise" expression for the preorder of functors basically follows from (2.14) and (2.23):

$$F\leq G \iff \forall a\in\mathbb{A}_0:\mathbb{B}(-,Fa)\leq\mathbb{B}(-,Ga) \iff Fa\leq Ga.$$

The alternative for $F\cong G$ is then obvious. □

More about functors

As $\mathsf{Cat}(\mathcal{Q})$ is a 2-category (see 2.4.7) we can speak of such things as, for example, adjoint functors.

Proposition 2.4.11 *For $\mathcal{Q}$-categories and functors $\mathbb{A}\underset{F}{\overset{G}{\leftrightarrows}}\mathbb{B}$,*

$$\begin{aligned}
F\dashv G \textit{ in } \mathsf{Cat}(\mathcal{Q}) &\iff \begin{cases} 1_{\mathbb{A}}\leq G\circ F \\ F\circ G\leq 1_{\mathbb{B}}\end{cases} \textit{in } \mathsf{Cat}(\mathcal{Q}) && (2.34)\\
&\iff \mathbb{B}(-,F-)\dashv\mathbb{A}(-,G-) \textit{ in } \mathsf{Dist}(\mathcal{Q}) && (2.35)\\
&\iff \mathbb{B}(F-,-)\dashv\mathbb{A}(G-,-) \textit{ in } \mathsf{Dist}(\mathcal{Q}) && (2.36)\\
&\iff \mathbb{B}(F-,-)=\mathbb{A}(-,G-) \textit{ in } \mathsf{Dist}(\mathcal{Q}) && (2.37)
\end{aligned}$$

Proof: Statement (2.34) is simply the definition of adjointness in the locally preordered 2-category $\mathsf{Cat}(\mathcal{Q})$; (2.35) and (2.36) are due to the 2-functor of (2.13) (and its contravariant counterpart) which preserves and reflects the (pre)order of functors, resp. distributors; finally, (2.37) is a consequence of the unicity of adjoints in $\mathsf{Dist}(\mathcal{Q})$—recall that $\mathbb{B}(-,F-)\dashv\mathbb{B}(F-,-)$ and use (2.35). □

Further there is a notion of "equivalent objects" in the 2-category $\mathsf{Cat}(\mathcal{Q})$. For ordinary categories and functors it is well known that $F\colon\mathscr{A}\longrightarrow\mathscr{B}$ is an equivalence if and only if F is fully faithful and essentially surjective on objects. This holds for $\mathcal{Q}$-enriched categories too.

Definition 2.4.12 *A functor $F\colon\mathbb{A}\longrightarrow\mathbb{B}$ between $\mathcal{Q}$-enriched categories is* fully faithful *if*

$$\forall\ a,a'\in\mathbb{A}_0:\mathbb{A}(a',a)=\mathbb{B}(Fa',Fa)\ \textit{in}\ \mathcal{Q};$$

and F is essentially surjective on objects *whenever*

$$\forall\ b\in\mathbb{B}_0,\ \exists\ a\in\mathbb{A}_0:Fa\cong b\ \textit{in}\ \mathbb{B}.$$

Lemma 2.4.13 *Consider a functor $F\colon\mathbb{A}\longrightarrow\mathbb{B}$ between $\mathcal{Q}$-categories, and the adjoint distributors that it determines: $\mathbb{B}(-,F-)\dashv\mathbb{B}(F-,-)$.*

1. *F is fully faithful if and only if the unit of this adjunction saturates to an equality.*
2. *If F is essentially surjective on objects then necessarily the co-unit of this adjunction saturates to an equality[6].*

Proposition 2.4.14 *For a functor between $\mathcal{Q}$-categories $F\colon\mathbb{A}\longrightarrow\mathbb{B}$ the following are equivalent:*

1. *F is an equivalence, i.e. there exists a $G\colon\mathbb{B}\longrightarrow\mathbb{A}$ such that $G\circ F\cong 1_{\mathbb{A}}$ and $F\circ G\cong 1_{\mathbb{B}}$;*
2. *F is fully faithful and essentially surjective on objects;*

in which case also

3. *G is fully faithful and essentially surjective on objects.*

Proof: It is clear that (3) follows from the equivalence of (1) and (2).

(1 $\Longrightarrow$ 2) By functoriality of F and G, and $G\circ F\cong 1_{\mathbb{A}}$,

$$\mathbb{A}(a',a)\leq\mathbb{B}(Fa',Fa)\leq\mathbb{A}(GFa',GFa)=\mathbb{A}(a',a).$$

Further, by $F\circ G\cong 1_{\mathbb{B}}$ it is obvious that $b\cong FGb$, and $Gb\in\mathbb{A}_0$ as required.

[6]The converse does not hold: a functor $F\colon\mathbb{A}\longrightarrow\mathbb{B}$ for which the co-unit of the induced adjunction is an equality – which is sometimes said to have a "dense image" – is not necessarily essentially surjective on objects.

($2 \Longrightarrow 1$) For every $b \in \mathbb{B}_0$ choose one particular $Gb \in \mathbb{A}_0$ for which $F(Gb) \cong b$; then $\mathbb{B}_0 \longrightarrow \mathbb{A}_0 \colon b \mapsto Gb$ is the required inverse equivalence, whose functoriality is guaranteed by

$$\mathbb{B}(b', b) = \mathbb{B}(FGb', FGb) = \mathbb{A}(Gb', Gb)$$

using the fully faithfulness of F. $\square$

The following is well known for ordinary categories; it will be useful further on.

Proposition 2.4.15 *Suppose that* $F \dashv G \colon \mathbb{B} \overset{\longleftarrow}{\longrightarrow} \mathbb{A}$ *in* $\mathsf{Cat}(\mathcal{Q})$. *Then*

1. *F is fully faithful*
   $\iff G \circ F \cong 1_{\mathbb{A}}$ *in the preorder* $\mathsf{Cat}(\mathcal{Q})(\mathbb{A}, \mathbb{A})$
   $\iff$ *for all* $a \in \mathbb{A}_0$: $G(Fa) \cong a$ *in* $\mathbb{A}$
   $\iff \mathbb{A}(-,-) = \mathbb{A}(-, G \circ F-)$ *in* $\mathsf{Dist}(\mathbb{A}, \mathbb{A})$
   $\iff \mathbb{A}(-,-) = \mathbb{A}(G \circ F-, -)$ *in* $\mathsf{Dist}(\mathbb{A}, \mathbb{A})$.

2. *G is fully faithful*
   $\iff F \circ G \cong 1_{\mathbb{B}}$ *in the preorder* $\mathsf{Cat}(\mathcal{Q})(\mathbb{B}, \mathbb{B})$
   $\iff$ *for all* $b \in \mathbb{B}_0$: $F(Gb) \cong b$ *in* $\mathbb{B}$
   $\iff \mathbb{B}(-,-) = \mathbb{B}(-, F \circ G-)$ *in* $\mathsf{Dist}(\mathcal{Q})(\mathbb{B}, \mathbb{B})$
   $\iff \mathbb{B}(-,-) = \mathbb{B}(F \circ G-, -)$ *in* $\mathsf{Dist}(\mathcal{Q})(\mathbb{B}, \mathbb{B})$.

And if moreover $G \dashv H \colon \mathbb{A} \overset{\longleftarrow}{\longrightarrow} \mathbb{B}$ *in* $\mathsf{Cat}(\mathcal{Q})$ *then*

3. *F is fully faithful if and only if H is fully faithful.*

Proof: (1) F is fully faithful if and only if $\mathbb{A}(-,-) = \mathbb{B}(F-, F-)$ and by $F \dashv G$ it follows that $\mathbb{B}(F-, F-) = \mathbb{B}(-, G \circ F-)$. With previous results the other equivalent statements follow easily.

(2) is similar.

(3) By $F \dashv G \dashv H$ we have $\mathbb{A}(G \circ F-, -) = \mathbb{B}(F-, H-) = \mathbb{A}(-, G \circ H-)$ and therefore also

$$\mathbb{A}(-,-) = \mathbb{B}(G \circ F-, -) \quad \iff \quad \mathbb{A}(-,-) = \mathbb{A}(-, G \circ H)$$

so the result follows. $\square$

Skeletal categories

To any preorder corresponds an order by passing to equivalence classes of elements of the preorder. For $\mathcal{Q}$-enriched categories this can be imitated. First an easy lemma which follows directly from 2.4.6, 2.4.9 and 2.4.10.

Lemma 2.4.16 *For a $\mathcal{Q}$-category $\mathbb{A}$, the following are equivalent:*

1. *no two different objects in $\mathbb{A}$ are isomorphic, i.e. the underlying preorder $(\mathbb{A}_0, \leq)$ is an order (i.e. it is anti-symmetric);*

2. *for any $\mathcal{Q}$-category $\mathbb{C}$, $\mathsf{Cat}(\mathcal{Q})(\mathbb{C},\mathbb{A})$ is an (anti-symmetric) order rather than a preorder;*

3. *for any $\mathcal{Q}$-category $\mathbb{C}$ the (pre)order-preserving mapping*

$$\mathsf{Cat}(\mathcal{Q})(\mathbb{C},\mathbb{A}) \longrightarrow \mathsf{Dist}(\mathcal{Q})(\mathbb{C},\mathbb{A})\colon F \mapsto \mathbb{A}(-,F-)$$

is injective (instead of "injective up to isomorphism").

Definition 2.4.17 *A $\mathcal{Q}$-category $\mathbb{A}$ is* skeletal *if the (equivalent) conditions in 2.4.16 are fulfilled.*

Proposition 2.4.18 *Let $\mathbb{A}$ be a $\mathcal{Q}$-enriched category.*

1. *The following data define a skeletal $\mathcal{Q}$-category $\mathbb{A}_{\mathsf{skel}}$:*
   - *objects: $(\mathbb{A}_{\mathsf{skel}})_0 = \Big\{[a] = \{x \in \mathbb{A}_0 \mid x \cong a\}\Big| a \in \mathbb{A}_0\Big\}$, with type function $t[a] = ta$;*
   - *hom-arrows: for any $[a],[a'] \in (\mathbb{A}_{\mathsf{skel}})_0$, $\mathbb{A}_{\mathsf{skel}}([a'],[a]) = \mathbb{A}(a',a)$.*

2. *The object mapping*
$$[-]\colon \mathbb{A} \longrightarrow \mathbb{A}_{\mathsf{skel}}\colon x \mapsto [x] \tag{2.38}$$
is a functor of $\mathcal{Q}$-categories which is an equivalence in $\mathsf{Cat}(\mathcal{Q})$.

3. *Moreover $\mathbb{A}$ is isomorphic to $\mathbb{A}_{\mathsf{skel}}$ in the quantaloid $\mathsf{Dist}(\mathcal{Q})$.*

Proof: (1) The construction of $\mathbb{A}_{\mathsf{skel}}$ is well-defined, because:

- as a set the quotient of $\mathbb{A}_0$ by $\cong$ is well-defined (i.e. $\cdot \cong \cdot$ is an equivalence relation);

- $t[a]$ is well-defined, because all elements of the equivalence class $[a]$ necessarily have the same type since they are isomorphic;

- for any $a' \cong b'$ and $a \cong b$ in $\mathbb{A}$, $\mathbb{A}(b',b) = \mathbb{A}(a',a)$, so $\mathbb{A}_{\mathsf{skel}}([a'],[a])$ is well-defined (i.e. does not depend on the representer chosen in $[a']$ and $[a]$).

Clearly, $\mathbb{A}_{\mathsf{skel}}$ inherits "composition" and "identities" from $\mathbb{A}$. And $\mathbb{A}_{\mathsf{skel}}$ is skeletal: if $[a] \cong [a']$ in $(\mathbb{A}_{\mathsf{skel}})_0$ then necessarily $a \cong a'$ in $\mathbb{A}_0$, which by definition implies that $[a] = [a']$.

(2) It is easily seen that the mapping of objects

$$\mathbb{A}_0 \longrightarrow (\mathbb{A}_{\mathsf{skel}})_0\colon x \mapsto [x]$$

$$\begin{tikzcd}
\mathsf{Cat}_{\mathsf{skel}}(\mathcal{Q}) \arrow[r] \arrow[d] & \mathsf{Dist}_{\mathsf{skel}}(\mathcal{Q}) \arrow[d] \\
\mathsf{Cat}(\mathcal{Q}) \arrow[r] & \mathsf{Dist}(\mathcal{Q})
\end{tikzcd}$$

Diagram 2.10

determines indeed a $\mathcal{Q}$-functor, which is obviously fully faithful and essentially surjective on objects. By 2.4.14 we thus know that $\mathbb{A} \simeq \mathbb{A}_{\mathsf{skel}}$ in $\mathsf{Cat}(\mathcal{Q})$.

(3) By the 2-functor in (2.13) we know that the equivalence $\mathbb{A} \simeq \mathbb{A}_{\mathsf{skel}}$ in the 2-category $\mathsf{Cat}(\mathcal{Q})$ must imply an equivalence $\mathbb{A} \simeq \mathbb{A}_{\mathsf{skel}}$ in the 2-category $\mathsf{Dist}(\mathcal{Q})$. But the latter being locally skeletal, any equivalence is in fact an isomorphism. So $\mathbb{A} \cong \mathbb{A}_{\mathsf{skel}}$ in $\mathsf{Dist}(\mathcal{Q})$. □

Skeletal $\mathcal{Q}$-categories can be taken as objects of a full sub-quantaloid of $\mathsf{Dist}(\mathcal{Q})$, respectively a full sub-2-category of $\mathsf{Cat}(\mathcal{Q})$:

$$\mathsf{Dist}_{\mathsf{skel}}(\mathcal{Q}) \longrightarrow \mathsf{Dist}(\mathcal{Q}),\ \ \mathsf{Cat}_{\mathsf{skel}}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q}).$$

By 2.4.16 $\mathsf{Cat}_{\mathsf{skel}}(\mathcal{Q})$ is then a *locally ordered* category rather than a locally pre-ordered one like $\mathsf{Cat}(\mathcal{Q})$.

Corollary 2.4.19 $\mathsf{Cat}_{\mathsf{skel}}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q})$ *is a biequivalence of 2-categories, and* $\mathsf{Dist}_{\mathsf{skel}}(\mathcal{Q}) \longrightarrow \mathsf{Dist}(\mathcal{Q})$ *is an equivalence of quantaloids. The 2-functor (2.13) restricts to*

$$\mathsf{Cat}_{\mathsf{skel}}(\mathcal{Q}) \longrightarrow \mathsf{Dist}_{\mathsf{skel}}(\mathcal{Q})\colon (\mathbb{A} \xrightarrow{F} \mathbb{B}) \mapsto (\mathbb{A} \xrightarrow{\mathbb{B}(-,F-)} \mathbb{B}) \tag{2.39}$$

which is the identity on objects, faithful (but not full), and locally fully faithful. Diagram 2.10 commutes.

2.5 A couple of remarks

The opposite of a category

It is a simple yet significant observation that in general the "opposite" of a $\mathcal{Q}$-category is not again a $\mathcal{Q}$-category, but rather a $\mathcal{Q}^{\mathsf{op}}$-category. Explicitly, for $\mathbb{A}$ a $\mathcal{Q}$-category we define a new structure $\mathbb{A}^{\mathsf{op}}$ with

- object-set: $(\mathbb{A}^{\mathsf{op}})_0 = \mathbb{A}_0$, with the same types too;

- hom-arrows: for $a, a' \in (\mathbb{A}^{\mathsf{op}})_0$, put $\mathbb{A}^{\mathsf{op}}(a', a) = \mathbb{A}(a, a')$.

Then indeed $\mathbb{A}^{\mathsf{op}}$ is a $\mathcal{Q}^{\mathsf{op}}$-category. And $(\mathbb{A}^{\mathsf{op}})^{\mathsf{op}}$ is $\mathbb{A}$ again, of course.

For a distributor $\Phi\colon \mathbb{A} \mathrel{-\!\!\circ\!\!\to} \mathbb{B}$ between $\mathcal{Q}$-categories we may define an "opposite" distributor between the "opposite" categories over the "opposite" base... But this distributor will go in the "opposite" direction! Indeed, putting

- for $a \in \mathbb{A}_0^{\mathsf{op}}$ and $b \in \mathbb{B}_0^{\mathsf{op}}$, $\Phi^{\mathsf{op}}(a,b) = \Phi(b,a)$

we obtain $\Phi^{\mathsf{op}}\colon \mathbb{B}^{\mathsf{op}} \mathrel{-\!\!\circ\!\!\to} \mathbb{A}^{\mathsf{op}}$ in $\mathsf{Dist}(\mathcal{Q}^{\mathsf{op}})$. And for $\Phi \leq \Psi\colon \mathbb{A} \rightrightarrows \mathbb{B}$ in $\mathsf{Dist}(\mathcal{Q})$ it is quite obvious that $\Phi^{\mathsf{op}} \leq \Psi^{\mathsf{op}}\colon \mathbb{B}^{\mathsf{op}} \rightrightarrows \mathbb{A}^{\mathsf{op}}$ in $\mathsf{Dist}(\mathcal{Q}^{\mathsf{op}})$. Clearly, applying the "op" twice gives back the original distributor.

Finally, let $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ be a functor between $\mathcal{Q}$-categories, then obviously the *same* object mapping $a \mapsto Fa$ determines the arrow $F^{\mathsf{op}}\colon \mathbb{A}^{\mathsf{op}} \longrightarrow \mathbb{B}^{\mathsf{op}}\colon a \mapsto Fa$ of $\mathsf{Cat}(\mathcal{Q}^{\mathsf{op}})$. But if $F \leq G\colon \mathbb{A} \longrightarrow \mathbb{B}$ in $\mathsf{Cat}(\mathcal{Q})$ then $G^{\mathsf{op}} \leq F^{\mathsf{op}}\colon \mathbb{A}^{\mathsf{op}} \longrightarrow \mathbb{B}^{\mathsf{op}}$ in $\mathsf{Cat}(\mathcal{Q}^{\mathsf{op}})$—because $\mathbb{B}(-,F-) \leq \mathbb{B}(-,G-)$ is equivalent to $\mathbb{B}(G-,-) \leq \mathbb{B}(F-,-)$ which in turn means that $\mathbb{B}^{\mathsf{op}}(-,G-) \leq \mathbb{B}^{\mathsf{op}}(-,F-)$. It is obvious that $(F^{\mathsf{op}})^{\mathsf{op}} = F$ in $\mathsf{Cat}(\mathcal{Q})$.

Let us record this as follows.

Proposition 2.5.1 *"Taking opposites" determines an isomorphism of quantaloids and an isomorphism of* PreOrd*-categories, respectively,*

$$\mathsf{Dist}(\mathcal{Q}) \cong \mathsf{Dist}(\mathcal{Q}^{\mathsf{op}})^{\mathsf{op}} \textit{ and } \mathsf{Cat}(\mathcal{Q}) \cong \mathsf{Cat}(\mathcal{Q}^{\mathsf{op}})^{\mathsf{co}}$$

(where the "co*" means: reversing order in the homs).*

Free categories on preorders

Two objects of a different type of some $\mathcal{Q}$-category $\mathbb{A}$ are *incomparable* in the underlying preorder $(\mathbb{A}_0, \leq)$:

$$a, b \in \mathbb{A}_0 \text{ with } ta \neq tb \quad \Longrightarrow \quad a \not\leq b \text{ and } b \not\leq a.$$

It is therefore natural to think of $(\mathbb{A}_0, \leq)$ as a "$\mathcal{Q}_0$-indexed family of preorders". And the underlying object-map of a functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ is a "$\mathcal{Q}_0$-indexed family of order-preserving maps" between such $\mathcal{Q}_0$-indexed family of preorders, because a functor preserves the "types" of the objects. We will denote $\mathsf{PreOrd}_{\mathcal{Q}_0}$ for the 2-category whose objects are families of preordered sets indexed by a subset of $\mathcal{Q}_0$[7], and arrows are families of order-preserving maps (with the obvious local structure); we may then record the following proposition.

[7]Indeed, would we speak of "$\mathcal{Q}_0$-indexed families of preorders" then – since $\mathcal{Q}_0$ may be a proper class – such a family would have a class of elements too. We wish to exclude this, so we only allow "families of preorders indexed by a *subset* of $\mathcal{Q}_0$".

Proposition 2.5.2 *Sending a $\mathcal{Q}$-category $\mathbb{A}$ on its underlying preorder and a functor $F:\mathbb{A}\longrightarrow\mathbb{B}$ on its underlying object-map determines a forgetful 2-functor*

$$\mathsf{Cat}(\mathcal{Q})\longrightarrow\mathsf{PreOrd}_{\mathcal{Q}_0}$$

which admits a left adjoint.

Proof: The first part of the proposition is clear by the preceding remarks and 2.4.10.

As for the construction of the free $\mathcal{Q}$-category on a $\mathcal{Q}_0$-indexed family of preorders: for a subset $\mathcal{X}\subseteq\mathcal{Q}_0$ of $\mathcal{Q}$-objects and a family of preordered sets $(A,\leq)=(A_X,\leq_X)_{X\in\mathcal{X}}$ consider the $\mathcal{Q}$-category $\mathbb{A}$ determined by the following data:

$$\mathbb{A}_0=\coprod_{X\in\mathcal{X}}A_X \text{ with types } ta=X \text{ for } a\in A_X,$$

$$\mathbb{A}(a',a)=\begin{cases}1_X \text{ when } ta=ta'=X \text{ and } a'\leq_X a \text{ in } A_X,\\ 0_{X,X'} \text{ otherwise .}\end{cases}$$

It is then obvious that $\mathbb{A}$ is a (small) $\mathcal{Q}$-category, whose underlying family of preorders is the given family: $(\mathbb{A}_0,\leq)=(A,\leq)$. And for any $\mathcal{Q}$-category $\mathbb{B}$,

$$\mathsf{Cat}(\mathcal{Q})(\mathbb{A},\mathbb{B})=\mathsf{PreOrd}_{\mathcal{Q}_0}(\mathbb{A}_0,\mathbb{B}_0),$$

so that the adjunction is obtained. $\square$

Of course we could also "forget the types" of the elements in the underlying preorder of a $\mathcal{Q}$-category, i.e. compose $\mathsf{Cat}(\mathcal{Q})\longrightarrow\mathsf{PreOrd}_{\mathcal{Q}_0}$ with the obvious $\mathsf{PreOrd}_{\mathcal{Q}_0}\longrightarrow\mathsf{PreOrd}$... But this functor does not admit a left adjoint (unless $\mathcal{Q}$ has only one object).

By the way, if in analogy with the notation "$\mathsf{PreOrd}_{\mathcal{Q}_0}$" we use $\mathsf{Ord}_{\mathcal{Q}_0}$ for the obvious full sub-2-category, we can add a little corollary here.

Corollary 2.5.3 *The "underlying preorder functor" in 2.5.2 restricts to*

$$\mathsf{Cat}_{\mathsf{skel}}(\mathcal{Q})\longrightarrow\mathsf{Ord}_{\mathcal{Q}_0},$$

which admits a left adjoint.

The commutative diagram 2.11 of 2-functors between 2-categories resumes the situation; the equalities are biequivalences.

$$\begin{array}{ccccc} \mathsf{Cat}_{\mathsf{skel}}(\mathcal{Q}) & \longrightarrow & \mathsf{Ord}_{\mathcal{Q}_0} & \longrightarrow & \mathsf{Ord} \\ \| & & \| & & \| \\ \mathsf{Cat}(\mathcal{Q}) & \longrightarrow & \mathsf{PreOrd}_{\mathcal{Q}_0} & \longrightarrow & \mathsf{PreOrd} \end{array}$$

Diagram 2.11

Ideal relations between (pre)orders

We have seen that a $\mathcal{Q}$-category has an underlying ($\mathcal{Q}_0$-indexed) preorder (which is an order if and only if the $\mathcal{Q}$-category is skeletal), and that a functor determines an order-preserving map between such (pre)orders. But also a distributor between $\mathcal{Q}$-categories determines a familiar "order-theoretic notion" between the underlying (pre)orders.

For preorders $(P, \leq)$ and $(Q, \leq)$ consider a relation $R \subseteq P \times Q$; say that this relation is *ideal* when for all $p, p' \in P$ and $q, q' \in Q$,

$$p \leq p', q' \leq q, (p, q) \in R \text{ implies that } (p', q') \in R.$$

It is easily seen that preordered sets together with ideal relations form a quantaloid (the local structure is just inclusion of relations). Let PreIdl denote this quantaloid.

If $\Phi\colon \mathbb{A} \longrightarrow \mathbb{B}$ is a distributor between $\mathcal{Q}$-enriched categories, then the clause

$$(a, b) \in \Phi_0 \iff ta = tb =: A \text{ and } 1_A \leq \Phi(b, a)$$

then defines an ideal relation between the underlying preorders of $\mathbb{A}$ and $\mathbb{B}$. It is easily understood how this amounts to a homomorphism of quantaloids

$$\mathsf{Dist}(\mathcal{Q}) \longrightarrow \mathsf{PreIdl}.$$

Diagram 2.12 then suggests how the "underlying order-theoretic structures" of (skeletal) $\mathcal{Q}$-categories, distributors and functors are (pre)orders, ideal relations and order-preserving maps relate; the notation Idl stands for the full subquantaloid of PreIdl whose objects are (anti-symmetrically) ordered sets; the equalities must be read s (bi)equivalences. (This diagram may be refined by working with "$\mathsf{PreIdl}_{\mathcal{Q}_0}$" and "$\mathsf{Idl}_{\mathcal{Q}_0}$", but this is not necessary to make our point here.)

This state of affairs may give the intuition that the theory of $\mathcal{Q}$-categories is that of "(pre)orders relative to a base quantaloid $\mathcal{Q}$"... but this is not entirely true. In a later chapter we will argue that the correct notion of "(pre)order relative to a base quantaloid $\mathcal{Q}$" is precisely that of 'Cauchy complete totally regular $\mathcal{Q}$-semicategory', which is a strictly weaker notion than that of '$\mathcal{Q}$-category' (see p. 151 and further).

$$
\begin{tikzcd}
 & \mathsf{Cat}(\mathscr{Q}) \arrow[rr] \arrow[dd] & & \mathsf{Dist}(\mathscr{Q}) \arrow[dd] \\
\mathsf{Cat}_{\mathsf{skel}}(\mathscr{Q}) \arrow[ur, equal] \arrow[rr] \arrow[dd] & & \mathsf{Dist}_{\mathsf{skel}}(\mathscr{Q}) \arrow[ur, equal] \arrow[dd] & \\
 & \mathsf{PreOrd} \arrow[rr] & & \mathsf{PreIdl} \\
\mathsf{Ord} \arrow[ur, equal] \arrow[rr] & & \mathsf{Idl} \arrow[ur, equal] &
\end{tikzcd}
$$

Diagram 2.12

More about the "change of base"

In 2.3.6 and 2.4.8 we have indicated how a homomorphism $F\colon \mathscr{Q} \longrightarrow \mathscr{Q}'$ between quantaloids canonically determines a "change of base" from $\mathscr{Q}$-categories to $\mathscr{Q}'$-categories (and distributors and functors). But not only a *homomorphism* determines a "change of base", also a *lax functor* does so[8].

For example, for any quantaloid $\mathscr{Q}$ we may consider a lax functor from $\mathscr{Q}$ into the quantaloid $\mathbf{2}$ (which has one object $*$, whose hom-lattice is the two-element boolean algebra $0 \leq 1$, and the composition law is simply the "conjunction of the truth values 0 and 1") as follows:

$$F\colon \mathscr{Q} \longrightarrow \mathbf{2}\colon (A \xrightarrow{\ f\ } B) \mapsto \begin{cases} 1 \text{ if } A = B \text{ and } 1_A \leq f, \\ 0 \text{ otherwise.} \end{cases}$$

(More precisely, F is a "normal lax functor": for $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathscr{Q}$ it is clear that $F(g \circ f) \geq Fg \circ Ff$ and $1_{FA} = F(1_A)$.)

It is now easy to see that $\mathsf{Cat}(\mathbf{2})$ is precisely PreOrd, and the induced "change of base"

$$\mathsf{Cat}(F)\colon \mathsf{Cat}(\mathscr{Q}) \longrightarrow \mathsf{Cat}(\mathbf{2}) = \mathsf{PreOrd}$$

is the "underlying preorder functor" as encountered previously. In the same way, $\mathsf{Dist}(\mathbf{2})$ is the quantaloid of preordered sets and ideal relations, and the "change of base"

$$\mathsf{Dist}(F)\colon \mathsf{Dist}(\mathscr{Q}) \longrightarrow \mathsf{Dist}(\mathbf{2}) = \mathsf{PreIdl}$$

is that of the previous section.

[8] This point of view was advocated in [Bénabou], and it is one of the reasons why the notion of 'lax functor', or 'morphism of bicategories' as it was called, was put forward.

Matters of size

In the definition of 'quantaloid' – see 1.1.1 – we explicitly allowed Sup-categories with a proper class of objects. Our main reason for doing so is that "large" quantaloids arise naturally from "small" ones anyway when we consider the Matr-construction: even for a small quantaloid $\mathcal{Q}$ the new Sup-enriched category $\mathsf{Matr}(\mathcal{Q})$ has a proper class of objects.

In particular is $\mathsf{Dist}(\mathcal{Q})$ always "large". But its objects, i.e. the $\mathcal{Q}$-enriched categories as defined in 2.3.1, are always "small": their objects form a (small) set. This complicates matters when developing categorical algebra over a "large" quantaloid: we will argue in the next chapter that a "cocomplete $\mathcal{Q}$-category" has at least as many objects as the base quantaloid... so a small $\mathcal{Q}$-category cannot be cocomplete when $\mathcal{Q}$ is large (see 3.1.2). A crucial example of such a "cocomplete category" is the "category of contravariant presheaves on a small category" (treated in detail on p. 99 and further).

We could develop a theory of $\mathcal{Q}$-enriched categories that very carefully distinguishes between "small" and "large" $\mathcal{Q}$-categories. But not to get involved in situations the technical complications of which are not justified by their interest, *we will work from now on with a small base quantaloid*, unless we explicitly state otherwise.

3

Some categorical algebra

The association of an adjoint pair of distributors to each functor between $\mathcal{Q}$-categories – as discussed in the previous chapter – is taken as starting point for the development of some $\mathcal{Q}$-categorical algebra, which is to include such things as weighted colimits in a $\mathcal{Q}$-category, presheaves on a $\mathcal{Q}$-category, Kan extensions of functors, Cauchy completion and Morita-equivalence of $\mathcal{Q}$-categories.

3.1 Weighted colimits

We consider $\mathcal{Q}$-categories $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$, some functor $F: \mathbb{A} \longrightarrow \mathbb{B}$ and a distributor $\Theta: \mathbb{C} \xrightarrow{\circ} \mathbb{A}$. Since F determines a distributor $\mathbb{B}(F-,-): \mathbb{B} \xrightarrow{\circ} \mathbb{A}$ and by closedness of the quantaloid $\mathsf{Dist}(\mathcal{Q})$ we can thus calculate the universal lifting $[\Theta, \mathbb{B}(F-,-)]: \mathbb{B} \xrightarrow{\circ} \mathbb{C}$ as indicated in diagram 3.1.

Definition 3.1.1 *In the situation of diagram 3.1, a functor $G: \mathbb{C} \longrightarrow \mathbb{B}$ is the* Θ-weighted colimit of F *if:*

$$\mathbb{B}(G-,-) = \Big[\Theta, \mathbb{B}(F-,-)\Big].$$

If the Θ-weighted colimit of F exists, then it is necessarily essentially unique. It therefore makes sense to speak of "the" colimit and to denote it by $\mathrm{colim}(\Theta, F)$.

We wish to speak of "those $\mathcal{Q}$-categories that admit all weighted colimits", i.e. *cocomplete* $\mathcal{Q}$-categories. But there is a small problem (the word is well-chosen).

Lemma 3.1.2 *A cocomplete $\mathcal{Q}$-category $\mathbb{B}$ has at least as many objects as the base quantaloid $\mathcal{Q}$. So would the base quantaloid have a proper class of objects, then so would all cocomplete enriched categories.*

Proof: We can for each $X \in \mathcal{Q}_0$ consider the empty diagram in $\mathbb{B}$ weighted by the empty distributor with domain $*_X$:

$$*_X \xrightarrow[\circ]{\emptyset} \emptyset \xrightarrow{\emptyset} \mathbb{B}.$$

$$
\begin{tikzcd}
\mathbb{A} & & \mathbb{B} \arrow[ll, "\mathbb{B}(F-,-)"', "\circ" marking] \arrow[lldd, "{\left[\Theta, \mathbb{B}(F-,-)\right]}", "\circ" marking] \\
 & \leq & \\
\mathbb{C} \arrow[uu, "\Theta", "\circ" marking] & &
\end{tikzcd}
$$

Diagram 3.1

$$
\begin{tikzcd}
\mathbb{A} \arrow[rr, "F"] & & \mathbb{B} \\
\mathbb{C} \arrow[u, "\Theta", "\circ" marking] \arrow[urr, dotted, "{\mathrm{colim}(\Theta, F)}"] & & \\
\mathbb{D} \arrow[u, "\Phi", "\circ" marking] \arrow[uu, bend left=60, "\Theta \otimes_{\mathbb{C}} \Phi", "\circ" marking] \arrow[uurr, dotted, "{\mathrm{colim}(\Theta \otimes_{\mathbb{C}} \Phi, F) \cong \mathrm{colim}(\Phi, \mathrm{colim}(\Theta, F))}"'] & &
\end{tikzcd}
$$

Diagram 3.2

$\mathrm{colim}(\emptyset, \emptyset)\colon *_X \longrightarrow \mathbb{B}$ must exist by cocompleteness of $\mathbb{B}$: it "picks out" an object of type X in $\mathbb{B}$—thus $\mathbb{B}$ must have such an object in the first place. $\square$

This is a problem in the framework of this text, because we didn't even bother defining "large" $\mathcal{Q}$-categories, let alone develop a theory of $\mathcal{Q}$-categories that is sensitive to these size-related issues. (The ultimate reason for our ignorance is that the extra complications that such a size-sensitive theory would inevitably be burdened with, are not justified by their interest.) *Therefore we are happy to work with a small base quantaloid $\mathcal{Q}$.* Indeed, when $\mathcal{Q}$ has a *set* of objects, the problem of "small" versus "large" disappears: also "small" $\mathcal{Q}$-categories can be cocomplete.

Next up is a collection of lemmas that will help us calculate colimits. We use the following notation: as before, for an object $a \in \mathbb{A}_0$ of a $\mathcal{Q}$-category $\mathbb{A}$, there is the functor $\Delta a\colon *_{ta} \longrightarrow \mathbb{A}$ that "picks out" a; and for a functor $F\colon *_C \longrightarrow \mathbb{A}$ on a one-object $\mathcal{Q}$-category we will write $\nabla F \in \mathbb{A}_0$ for its single value. With this notation it is clear that $\nabla\Delta a = a$ and $\Delta\nabla F = F$.

Lemma 3.1.3 *1. For* $\mathbb{D} \xrightarrow{\Phi} \mathbb{C} \xrightarrow{\Theta} \mathbb{A} \xrightarrow{F} \mathbb{B}$, *suppose that* $\mathrm{colim}(\Theta, F)$ *exists. Then* $\mathrm{colim}(\Phi, \mathrm{colim}(\Theta, F))$ *exists if and only if* $\mathrm{colim}(\Theta \otimes_{\mathbb{C}} \Phi, F)$ *exists, in which case they are isomorphic—see diagram 3.2.*

$$\begin{tikzcd} \mathbb{A} \arrow[r, "F"] & \mathbb{B} \\ \mathbb{A} \arrow[u, "\mathbb{A}", "\circ"'] \arrow[ur, dotted, "\mathrm{colim}(\mathbb{A},F)\cong F"'] & \end{tikzcd}$$

Diagram 3.3

2. *For* $\mathbb{C} \overset{\Theta}{\longrightarrow\!\!\!\!\!\!\circ\ \ } \mathbb{A} \xrightarrow{F} \mathbb{B}$, $\mathrm{colim}(\Theta, F)$ *exists if and only if, for all objects* $c \in \mathbb{C}_0$, $\mathrm{colim}(\Theta(-,c), F)$ *exists; in this case,*
$$\mathrm{colim}(\Theta, F)(c) \cong \nabla\, \mathrm{colim}(\Theta(-,c), F).$$

3. *For any* $\mathbb{A} \xrightarrow{F} \mathbb{B}$, $\mathrm{colim}(\mathbb{A}, F)$ *exists and is isomorphic to* F *– see diagram 3.3 – and for any* $a \in \mathbb{A}_0$, $\mathrm{colim}(\mathbb{A}(-,a), F)$ *exists and is isomorphic* $\Delta F a$;

4. *Suppose that* $F \cong G\colon \mathbb{A} \rightrightarrows \mathbb{B}$; *for every* $\Theta\colon \mathbb{C} \longrightarrow\!\!\!\!\!\!\circ\ \ \mathbb{A}$, *if* $\mathrm{colim}(\Theta, F)$ *exists, then* $\mathrm{colim}(\Theta, G)$ *exists, and* $\mathrm{colim}(\Theta, F) \cong \mathrm{colim}(\Theta, G)$.

Proof: (1) By a simple calculation, using 1.2.5:
$$\begin{aligned} \mathbb{B}\Big(\mathrm{colim}(\Phi, \mathrm{colim}(\Theta, F))-,-\Big) &= \Big[\Phi, \mathbb{B}\Big(\mathrm{colim}(\Theta, F))-,-\Big)\Big] \\ &= \Big[\Phi, \Big[\Theta, \mathbb{B}(F-,-)\Big]\Big] \\ &= \Big[\Theta \otimes_{\mathbb{C}} \Phi, \mathbb{B}(F-,-)\Big]. \end{aligned}$$

(2) Necessity is a corollary of (1), since $\Theta(-,c) = \Theta \otimes_{\mathbb{C}} \mathbb{C}(-,c)$. As for sufficiency, we will first show that the action on objects
$$\mathbb{C}_0 \longrightarrow \mathbb{B}_0\colon c \mapsto \nabla(\mathrm{colim}(\Theta(-,c), F))$$
is functorial: diagram 3.4[1] shows, by calculation with "$[-,-]$" and "$\{-,-\}$" in $\mathsf{Dist}(\mathcal{Q})$, that
$$\mathbb{C}(c', c) \le \Big\{\Big[\Theta(-,c), \mathbb{B}(F-,-)\Big], \Big[\Theta(-,c'), \mathbb{B}(F-,-)\Big]\Big\}.$$
Introducing the notation $Kc := \nabla(\mathrm{colim}(\Theta(-,c), F))$,
$$\begin{aligned} \mathbb{C}(c', c) &\le \Big\{\mathbb{B}(Kc,-), \mathbb{B}(Kc',-)\Big\} \\ &= \mathbb{B}(Kc',-) \otimes_{\mathbb{B}} \mathbb{B}(-,Kc) \\ &= \mathbb{B}(Kc', Kc). \end{aligned}$$

[1]Slight abuse of notation: we write $\mathbb{C}(c',c)\colon *_{tc} \longrightarrow\!\!\!\!\!\!\circ\ \ *_{tc'}$ for the distributor whose single element is the $\mathcal{Q}$-arrow $\mathbb{C}(c',c)\colon tc \longrightarrow tc'$.

$$\begin{tikzcd}
& \mathbb{A} \arrow[r, "F"] & \mathbb{B} \\
*_{tc} \arrow[ur, "\Theta(-,c)"] \arrow[rr, "\mathbb{C}(c',c)"'] & \arrow[u, phantom, "\geq"] & *_{tc'} \arrow[ul, "\Theta(-,c')"']
\end{tikzcd}$$

Diagram 3.4

by 1.2.1 and 2.3.4. So indeed

$$K\colon \mathbb{C} \longrightarrow \mathbb{B}\colon c \mapsto Kc = \nabla(\mathrm{colim}(\Theta(-,c), F))$$

is a functor. But by its construction it is now true that, for all $c \in \mathbb{C}_0$,

$$\mathbb{B}(Kc, -) = [\Theta(-,c), \mathbb{B}(F-,-)]$$

as distributors from $*_{tc}$ to $\mathbb{B}$; so if we let c "vary" in $\mathbb{C}$ then

$$\mathbb{B}(K-,-) = \Big[\Theta(-,-), \mathbb{B}(F-,-)\Big]$$

which proves that K is – to within isomorphism – the Θ-weighted colimit of F.

(3) By the trivial adjunction $\mathbb{A} \dashv \mathbb{A}\colon \mathbb{A} \leftrightarrows \mathbb{A}$ it follows by 1.2.1 that

$$\begin{aligned}
\Big[\mathbb{A}(-,-), \mathbb{B}(F-,-)\Big] &= \mathbb{A}(-,-) \otimes_{\mathbb{A}} \mathbb{B}(F-,-) \\
&= \mathbb{B}(F-,-),
\end{aligned}$$

thus $F \cong \mathrm{colim}(\mathbb{A}, F)$. The second part of the statement is then a corollary of (2).

(4) $F \cong G$ if and only if $\mathbb{B}(F-,-) = \mathbb{B}(G-,-)$, so

$$\begin{aligned}
\mathbb{C}(\mathrm{colim}(\Theta, F)-,-) &= \Big[\Theta, \mathbb{B}(F-,-)\Big] \\
&= \Big[\Theta, \mathbb{B}(G-,-)\Big]
\end{aligned}$$

which proves that $\mathrm{colim}(\Theta, F)$ also satisfies the universal property of the Θ-weighted colimit of G. $\square$

We point out one important consequence, which follows from part 2. of the lemma above.

Corollary 3.1.4 *A $\mathcal{Q}$-category $\mathbb{B}$ is cocomplete if and only if $\mathbb{B}$ admits colimits of all diagrams weighted by a distributor whose domain is a one-object $\mathcal{Q}$-category. (Later on we will call those distributors "contravariant presheaves".)*

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "F"] \arrow[rr, bend left, "F' \circ F"] & \mathbb{B} \arrow[r, "F'"] & \mathbb{B}' \\
\mathbb{C} \arrow[u, "\Theta", "\circ" marking] \arrow[ur, dotted, "\mathrm{colim}(\Theta, F)"] \arrow[urr, dotted, "\mathrm{colim}(\Theta, F' \circ F) \cong F' \circ \mathrm{colim}(\Theta, F)"']
\end{tikzcd}$$

Diagram 3.5

For later use we now introduce the appropriate terminology concerning "functors that preserve colimits".

Definition 3.1.5 *Given a distributor and a functor as in* $\mathbb{C} \xrightarrow{\Theta} \mathbb{A} \xrightarrow{F} \mathbb{B}$, *and such that* $\mathrm{colim}(\Theta, F)\colon \mathbb{C} \longrightarrow \mathbb{B}$ *exists, a functor* $F'\colon \mathbb{B} \longrightarrow \mathbb{B}'$ *is said to* preserve *this colimit if* $\mathrm{colim}(\Theta, F' \circ F)$ *exists and is isomorphic to* $F \circ \mathrm{colim}(\Theta, F)$ *(see diagram 3.5). The functor* F' *is* cocontinuous *if it preserves all colimits that exist in* $\mathbb{B}$. *And the colimit* $\mathrm{colim}(\Theta, F)$ *is* absolute *if it is preserved by any functor* $F'\colon \mathbb{B} \longrightarrow \mathbb{B}'$.

Proposition 3.1.6 *If* $\Theta\colon \mathbb{C} \longrightarrow \mathbb{A}$ *is a left adjoint in* $\mathsf{Dist}(\mathcal{Q})$, *then any colimit with weight* Θ *is absolute*[2].

Proof: If $\Theta \dashv \Psi$ in $\mathsf{Dist}(\mathcal{Q})$ and $\mathrm{colim}(\Theta, F)$ exists, then necessarily

$$\begin{aligned}
\mathbb{B}\Big(\mathrm{colim}(\Theta, F)-, -\Big) &= \Big[\Theta, \mathbb{B}(F-, -)\Big] \\
&= \Psi \otimes_{\mathbb{A}} \mathbb{B}(F-, -).
\end{aligned}$$

From this it is obvious that, for any $F'\colon \mathbb{B} \longrightarrow \mathbb{B}'$,

$$\begin{aligned}
\mathbb{B}\Big((F' \circ \mathrm{colim}(\Theta, F)-, -\Big) &= \mathbb{B}\Big(\mathrm{colim}(\Theta, F)-, -\Big) \otimes_{\mathbb{B}} \mathbb{B}'(F-, -) \\
&= \Psi \otimes_{\mathbb{A}} \mathbb{B}(F-, -) \otimes_{\mathbb{B}} \mathbb{B}'(F'-, -) \\
&= \Psi \otimes_{\mathbb{A}} \mathbb{B}\Big((F' \circ F)-, -\Big) \\
&= \Big[\Theta, \mathbb{B}\Big((F' \circ F)-, -\Big)\Big]
\end{aligned}$$

so $\mathrm{colim}(\Theta, F' \circ F)$ exists and is isomorphic to $F' \circ \mathrm{colim}(\Theta, F)$. □

Proposition 3.1.7 *Any functor* $F'\colon \mathbb{B} \longrightarrow \mathbb{B}'$ *with a right adjoint in* $\mathsf{Cat}(\mathcal{Q})$ *is cocontinuous.*

[2] A converse can be proved too: if any Θ-weighted colimit is absolute, then Θ is left adjoint [Street, 1983].

Proof: Suppose that $F' \dashv G$ in $\mathsf{Cat}(\mathcal{Q})$—i.e. $\mathbb{B}'(F'-,-) \dashv \mathbb{B}(G-,-)$ in $\mathsf{Dist}(\mathcal{Q})$ (see 2.4.11). For any colimit

$$\begin{tikzcd} \mathbb{A} \arrow[r, "F"] & \mathbb{B} \arrow[r, "F'"] & \mathbb{B}' \\ \\ \mathbb{C} \arrow[uu, "\Theta", "\circ"'] \arrow[uur, dotted, "\mathrm{colim}(\Theta, F)"'] \end{tikzcd}$$

we can then use 1.2.1 and 1.2.5 to calculate

$$\begin{aligned} \mathbb{B}'(F' \circ \mathrm{colim}(\Theta, F)-,-) &= \mathbb{B}(\mathrm{colim}(\Theta, F)-,-) \otimes_{\mathbb{B}} \mathbb{B}'(F'-,-) \\ &= \Big[\Theta, \mathbb{B}(F-,-)\Big] \otimes_{\mathbb{B}} \mathbb{B}'(F'-,-) \\ &= \Big\{\mathbb{B}(G-,-), \Big[\Theta, \mathbb{B}(F-,-)\Big]\Big\} \\ &= \Big[\Theta, \Big\{\mathbb{B}(G-,-), \mathbb{B}(F-,-)\Big\}\Big] \\ &= \Big[\Theta, \mathbb{B}(F-,-) \otimes_{\mathbb{B}} \mathbb{B}'(F'-,-)\Big] \\ &= \Big[\Theta, \mathbb{B}'((F' \circ F)-,-)\Big] \end{aligned}$$

which shows that $\mathrm{colim}(\Theta, F' \circ F)$ exists and equals $F' \circ \mathrm{colim}(\Theta, F)$. $\square$

Corollary 3.1.8 *Suppose that $\mathbb{A}$ and $\mathbb{B}$ are $\mathcal{Q}$-categories for which there are functors $i\colon \mathbb{A} \longrightarrow \mathbb{B}$, $j\colon \mathbb{B} \longrightarrow \mathbb{A}$ and $k\colon \mathbb{A} \longrightarrow \mathbb{B}$ such that $i \dashv j \dashv k$, and i (and thus also k) is fully faithful. If $\mathbb{B}$ is cocomplete then so is $\mathbb{A}$.*

Proof: First note that, by 2.4.15, the fully faithfulness of i is equivalent to the requirement that $j \circ i \cong 1_{\mathbb{A}}$. Now, for any weighted diagram in $\mathbb{A}$, like

$$\mathbb{C} \xrightarrow[\circ]{\Phi} \mathbb{C} \xrightarrow{F} \mathbb{A},$$

$\mathrm{colim}(\Phi, i \circ F)$ exists in $\mathbb{B}$ by cocompleteness of $\mathbb{B}$. Since j is a left adjoint, $j \circ \mathrm{colim}(\Phi, i \circ F) \cong \mathrm{colim}(\Phi, j \circ i \circ F) \cong \mathrm{colim}(\Phi, F)$ is the required colimit in $\mathbb{A}$. $\square$

Corollary 3.1.9 *If two functors $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ and $G\colon \mathbb{B} \longrightarrow \mathbb{A}$ constitute an equivalence in $\mathsf{Cat}(\mathcal{Q})$, then both F and G are cocontinuous. Moreover, $\mathbb{A}$ is cocomplete if and only if $\mathbb{B}$ is.*

Proof: By assumption $G \circ F \cong 1_{\mathbb{A}}$ and $F \circ G \cong 1_{\mathbb{B}}$. But this is actually precisely the case when $F \dashv G$ and $G \dashv F$ (because $\mathsf{Cat}(\mathcal{Q})$ is locally preordered), thus both functors are left adjoint, hence cocontinuous. In particular is $F \dashv G \dashv F$ and $G \circ F \cong 1_{\mathbb{A}}$ and by application of 3.1.8 the cocompleteness of $\mathbb{B}$ then implies that of $\mathbb{A}$. The converse follows from $G \dashv F \dashv G$ and $F \circ G \cong 1_{\mathbb{B}}$. $\square$

3.2 Presheaves

Principally the "contravariant presheaves on a $\mathcal{Q}$-category $\mathbb{A}$" should form the collection of objects of a new $\mathcal{Q}$-category, say $\mathscr{P}\mathbb{A}$, which "classifies" distributors with codomain $\mathbb{A}$: for every $\mathcal{Q}$-category $\mathbb{C}$,

$$\mathsf{Dist}(\mathcal{Q})(\mathbb{C},\mathbb{A}) \simeq \mathsf{Cat}(\mathcal{Q})(\mathbb{C},\mathscr{P}\mathbb{A}). \tag{3.1}$$

Here we think of the sup-lattice $\mathsf{Dist}(\mathcal{Q})(\mathbb{C},\mathbb{A})$ and the preorder $\mathsf{Cat}(\mathcal{Q})(\mathbb{C},\mathscr{P}\mathbb{A})$ and as categories, and ask their equivalence[3].

Since for any $\mathcal{Q}$-category $\mathbb{A}$ a functor $F\colon *_C \longrightarrow \mathbb{A}$ can be identified with the object $F*$ of $\mathbb{A}$ (necessarily of type C) that it picks out, putting $\mathbb{C} = *_C$ in (3.1) dictates that the collection of objects of $\mathscr{P}\mathbb{A}$ is:

$$(\mathscr{P}\mathbb{A})_0 = \{ *_C \xrightarrow{\Phi} \mathbb{A} \text{ in } \mathsf{Dist}(\mathcal{Q}) \mid C \in \mathcal{Q}_0 \} \tag{3.2}$$

with type function $t(\Phi\colon *_C \multimap \mathbb{A}) = C$.

Indeed, a distributor $\Phi\colon *_C \multimap \mathbb{A}$ is "contravariant in $\mathbb{A}$" in the sense that the action on the right (of $\mathbb{A}$) can be written as

$$\mathbb{A}(a',a) \circ \Phi(a,*) \leq \Phi(a',*) \iff \mathbb{A}(a',a) \leq \{\Phi(a,*),\Phi(a',*)\}. \tag{3.3}$$

The action on the left being trivial, we will simplify in what follows our notation for contravariant presheaves: we will systematically use small greek letters instead of capital ones (so ϕ, ψ, ... instead of Φ, Ψ, ...) and the elements of such a distributor $\phi\colon *_C \multimap \mathbb{A}$ will be written as $\phi(a)\colon C \longrightarrow ta$ (instead of $\phi(a,*)$). With these notations (3.3) becomes even more suggestive:

$$\mathbb{A}(a',a) \leq \{\phi(a),\phi(a')\}; \tag{3.4}$$

and maybe one would like to think of a contravariant presheaf on $\mathbb{A}$ as a "functor on $\mathbb{A}^{\mathsf{op}}$ with values in $\mathcal{Q}$" sending a to $\phi(a)$. Even though certain constructions in this sense are possible, one should not forget that $\mathbb{A}^{\mathsf{op}}$ is in general not a $\mathcal{Q}$-category, and neither is $\mathcal{Q}$—so that "$\mathsf{Cat}(\mathcal{Q})(\mathbb{A}^{\mathsf{op}},\mathcal{Q})$" makes no sense at all!

For clarity's sake, let us record the pertinent definition.

Definition 3.2.1 *A* contravariant presheaf *of type $A \in \mathcal{Q}_0$ on a $\mathcal{Q}$-category $\mathbb{A}$ is a distributor $\phi\colon *_A \multimap \mathbb{A}$.*

Since we agreed to work over a *small* base quantaloid $\mathcal{Q}$, the collection (3.2) of contravariant presheaves on a $\mathcal{Q}$-category $\mathbb{A}$ forms a $\mathcal{Q}$-typed set[4]. We now need

[3] But it will turn out that also $\mathsf{Cat}(\mathcal{Q})(\mathbb{C},\mathscr{P}\mathbb{A})$ is a partial order, so we're actually asking for an isomorphism of partial orders here, and in particular is $\mathsf{Cat}(\mathcal{Q})(\mathbb{C},\mathscr{P}\mathbb{A})$ thus a sup-lattice.

[4] Would the objects of the base category $\mathcal{Q}$ form a class, then the collection of contravariant presheaves on $\mathbb{A}$ would be large! The reason – ultimately – is that $\mathscr{P}\mathbb{A}$ is a cocomplete category—see 3.1.2 and 3.2.4.

$$\begin{tikzcd} C \arrow[r, "\phi", "\circ"{marking}] & \mathbb{A} \\ & C' \arrow[u, "\psi"', "\circ"{marking}] \end{tikzcd}$$

Diagram 3.6

to enrich this set of $\mathcal{Q}$-typed objects in $\mathcal{Q}$, i.e. for every two presheaves as in diagram 3.6 we need to define an arrow from C to C' in $\mathcal{Q}$, playing the rôle of the "hom" of the eventual $\mathcal{Q}$-category $\mathscr{P}\mathbb{A}$. It is the obvious candidate that will do the trick.

Proposition 3.2.2 *For any $\mathcal{Q}$-category $\mathbb{A}$, the following defines a new $\mathcal{Q}$-category $\mathscr{P}\mathbb{A}$ of contravariant presheaves on $\mathbb{A}$:*

- *objects:* $(\mathscr{P}\mathbb{A})_0 = \{*_C \overset{\phi}{\longrightarrow\!\!\!\!\!\!\circ\;\,} \mathbb{A}$ *in* $\mathsf{Dist}(\mathcal{Q}) \mid C \in \mathcal{Q}_0\}$ *with type function* $t(\phi\colon *_C \longrightarrow\!\!\!\!\!\!\circ\;\, \mathbb{A}) = C$*;*

- *hom-arrows: for ϕ, ψ as in diagram 3.6, put $\mathscr{P}\mathbb{A}(\psi,\phi)\colon C \longrightarrow C'$ to be the single element of the distributor $[\psi,\phi]\colon *_C \longrightarrow\!\!\!\!\!\!\circ\;\, *_{C'}$, thus*

$$\mathscr{P}\mathbb{A}(\psi,\phi) = \bigwedge_{a\in\mathbb{A}_0} \Big[\psi(a),\phi(a)\Big]^{\mathcal{Q}}.$$

This $\mathcal{Q}$-category is skeletal, and it satisfies (3.1).

Proof: With the properties of "$[-,-]$" it is easily verified that $\mathscr{P}\mathbb{A}$ is indeed a $\mathcal{Q}$-category—see 1.2.6. This $\mathcal{Q}$-category is easily seen to be skeletal: actually, the preordering of the objects of the $\mathcal{Q}$-category $\mathscr{P}\mathbb{A}$ coincides with their ordering as arrows in the quantaloid $\mathsf{Dist}(\mathcal{Q})$—so this is an order (i.e. antisymmetric)! As for the equivalence of categories (3.1), or rather the isomorphism of ordered sets, observe first that the following defines a bijection between both orders:

- given $\Phi\colon \mathbb{C} \longrightarrow\!\!\!\!\!\!\circ\;\, \mathbb{A}$, put

$$F_\Phi\colon \mathbb{C} \longrightarrow \mathscr{P}\mathbb{A}\colon c \mapsto (*_{tc} \xrightarrow{\Phi(-,c)} \mathbb{A}); \tag{3.5}$$

- given $F\colon \mathbb{C} \longrightarrow \mathscr{P}\mathbb{A}$, put

$$\Phi_F\colon \mathbb{C} \longrightarrow\!\!\!\!\!\!\circ\;\, \mathbb{A} \text{ with elements } \Phi_F(a,c) = (Fc)(a). \tag{3.6}$$

This bijection, then, preserves order in both directions:

$$\begin{aligned}
&\Phi \leq \Psi \text{ in } \mathsf{Dist}(\mathbb{C}, \mathbb{A})\\
&\iff \forall a \in \mathbb{A}_0, c \in \mathbb{C}_0 : \Phi(a,c) \leq \Psi(a,c) \text{ in } \mathscr{Q}(tc, ta)\\
&\iff \forall c \in \mathbb{C}_0 : F_\Phi(c) \leq F_\Psi(c) \text{ in } ((\mathscr{P}\mathbb{A})_0, \leq)\\
&\iff F_\Phi \leq F_\Psi \text{ in } \mathsf{Cat}(\mathscr{Q})(\mathbb{C}, \mathscr{P}\mathbb{A})
\end{aligned}$$

and likewise

$$\begin{aligned}
&F \leq G \text{ in } \mathsf{Cat}(\mathscr{Q})(\mathbb{C}, \mathscr{P}\mathbb{A})\\
&\iff \forall c \in \mathbb{C}_0 : F(c) \leq G(c) \text{ in } ((\mathscr{P}\mathbb{A})_0, \leq)\\
&\iff \forall a \in \mathbb{A}_0, c \in \mathbb{C}_0 : \Phi_F(a,c) \leq \Phi_G(a,c) \text{ in } \mathscr{Q}(tc, ta)\\
&\iff \Phi_F \leq \Phi_G \text{ in } \mathsf{Dist}(\mathbb{C}, \mathbb{A}).
\end{aligned}$$

We used 2.4.10 for the description of the ordering of functors. □

To the identity distributor on a $\mathscr{Q}$-category $\mathbb{A}$, $\mathbb{A}(-,-)\colon \mathbb{A} \xrightarrow{\circ} \mathbb{A}$, corresponds according to (3.1) and (3.5) the functor

$$Y_{\mathbb{A}}\colon \mathbb{A} \longrightarrow \mathscr{P}\mathbb{A} : a \mapsto (*_{ta} \xrightarrow[\circ]{\mathbb{A}(-,a)} \mathbb{A}) \tag{3.7}$$

which is, of course, the *Yoneda embedding* for $\mathbb{A}$. The contravariant presheaf $\mathbb{A}(-,a)\colon *_{ta} \xrightarrow{\circ} \mathbb{A}$ is said to be *represented* by $a \in \mathbb{A}_0$. We can now prove the "Yoneda lemma" for presheaves on a $\mathscr{Q}$-enriched category.

Proposition 3.2.3 *For any $\mathscr{Q}$-category $\mathbb{A}$,*

1. *for any object $a \in \mathbb{A}_0$ and any presheaf $\phi \in (\mathscr{P}\mathbb{A})_0$,*

$$\mathscr{P}\mathbb{A}(Y_{\mathbb{A}}a, \phi) = \phi(a);$$

2. *the Yoneda embedding is fully faithful:*

$$\forall\ a, a' \in \mathbb{A}_0 :\ \mathbb{A}(a', a) = \mathscr{P}\mathbb{A}(Y_{\mathbb{A}}a', Y_{\mathbb{A}}a).$$

Proof: (1) Using that $\mathbb{A}(-,a) \dashv \mathbb{A}(a,-)$ in $\mathsf{Dist}(\mathscr{Q})$, we have by definition of $\mathscr{P}\mathbb{A}(-,-)$ and 1.2.1 and 2.3.4,

$$\mathscr{P}\mathbb{A}(Y_{\mathbb{A}}a, \phi) = [\mathbb{A}(-,a), \phi] = \mathbb{A}(a,-) \otimes_{\mathbb{A}} \phi = \phi(a).$$

(2) Is an easy corollary. □

Proposition 3.2.4 *For every $\mathscr{Q}$-category $\mathbb{B}$, the category $\mathscr{P}\mathbb{B}$ of contravariant presheaves on $\mathbb{B}$ is cocomplete.*

Proof: The construction of colimits in $\mathscr{P}\mathbb{B}$ goes as follows: given

$$\mathbb{C} \xrightarrow[\circ]{\Theta} \mathbb{A} \xrightarrow{F} \mathscr{P}\mathbb{B},$$

to $F\colon \mathbb{A} \longrightarrow \mathscr{P}\mathbb{B}$ corresponds a unique distributor $\Phi\colon \mathbb{A} \longrightarrow \mathbb{B}$ (as in (3.6)) and, in turn, to $\Phi \otimes_{\mathbb{A}} \Theta\colon \mathbb{C} \longrightarrow \mathbb{B}$ corresponds a unique functor $G\colon \mathbb{C} \longrightarrow \mathscr{P}\mathbb{B}$ (as in (3.5)). This latter functor is in fact the colimit of F weighted by Θ (to within isomorphism, of course). Indeed, for $c \in \mathbb{C}_0$ and $\psi \in \mathscr{P}\mathbb{B}$,

$$\begin{aligned}
\Big[\Theta, \mathscr{P}\mathbb{B}(F-,-)\Big](c,\psi) &= \bigwedge_{a\in\mathbb{A}_0} \Big[\Theta(a,c), \mathscr{P}\mathbb{B}(Fa,\psi)\Big] \\
&= \bigwedge_{a\in\mathbb{A}_0} \Big[\Theta(a,c), \bigwedge_{b\in\mathbb{B}_0} \Big[(Fa)(b), \psi(b)\Big]\Big] \\
&= \bigwedge_{a\in\mathbb{A}_0} \bigwedge_{b\in\mathbb{B}_0} \Big[\Theta(a,c), \Big[(Fa)(b), \psi(b)\Big]\Big] \\
&= \bigwedge_{a\in\mathbb{A}_0} \bigwedge_{b\in\mathbb{B}_0} \Big[(Fa)(b) \circ \Theta(a,c), \psi(b)\Big] \\
&\stackrel{*}{=} \bigwedge_{a\in\mathbb{A}_0} \bigwedge_{b\in\mathbb{B}_0} \Big[\Phi(b,a) \circ \Theta(a,c), \psi(b)\Big] \\
&= \bigwedge_{b\in\mathbb{B}_0} \Big[\bigvee_{a\in\mathbb{A}_0} \Phi(b,a) \circ \Theta(a,c), \psi(b)\Big] \\
&= \bigwedge_{b\in\mathbb{B}_0} \Big[(\Phi \otimes_{\mathbb{A}} \Theta)(b,c), \psi(b)\Big] \\
&\stackrel{**}{=} \bigwedge_{b\in\mathbb{B}_0} \Big[(Gc)(b), \psi(b)\Big] \\
&= \mathscr{P}\mathbb{B}(Gc,\psi) \\
&= \mathscr{P}\mathbb{B}(G-,-)(c,\psi).
\end{aligned}$$

This is the wanted universal property. In $(*)$ we used the definition of $\Phi\colon \mathbb{A} \xrightarrow{\circ} \mathbb{B}$ as the unique distributor corresponding to $F\colon \mathbb{A} \longrightarrow \mathscr{P}\mathbb{B}$; and in $(**)$ it is the definition of $G\colon \mathbb{C} \longrightarrow \mathscr{P}\mathbb{B}$ as unique functor corresponding to $\Theta \otimes_{\mathbb{A}} \Phi\colon \mathbb{C} \xrightarrow{\circ} \mathbb{B}$ that does the trick. Further we essentially used 1.2.4 and 1.2.5 to make the calculations with the "$[-,-]$". $\square$

Whenever the weight considered in a colimit diagram is a contravariant presheaf, as in

$$*_C \xrightarrow[\circ]{\theta} \mathbb{A} \xrightarrow{F} \mathbb{B}$$

then, if $\mathrm{colim}(\theta, F)$ exists, it can be identified with an object of type C of the $\mathscr{Q}$-category $\mathbb{B}$. In particular, we have the following.

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "F"] \arrow[d, "G"'] \arrow[dr, phantom, "\leq"] & \mathbb{B} \\
\mathbb{C} \arrow[ur, "K = \langle F, G\rangle"'] &
\end{tikzcd}$$

Diagram 3.7

Corollary 3.2.5 *For a $\mathcal{Q}$-category $\mathbb{A}$ and a contravariant presheaf $\phi \in (\mathscr{P}\mathbb{A})_0$, the colimit of*

$$*_C \xrightarrow{\ \phi\ } \mathbb{A} \xrightarrow{\ Y_{\mathbb{A}}\ } \mathscr{P}\mathbb{A}$$

(exists and) is simply the constant functor $\Delta\phi\colon *_C \longrightarrow \mathscr{P}\mathbb{A}$ which "points at" $\phi \in (\mathscr{P}\mathbb{A})_0$.

This corollary says that "every presheaf is (canonically) the colimit of representables".

3.3 Kan extensions

Definition 3.3.1 *Given $\mathcal{Q}$-categories and functors $\mathbb{C} \xleftarrow{\ G\ } \mathbb{A} \xrightarrow{\ F\ } \mathbb{B}$, the* left Kan extension of F along G *is, whenever it exists, a functor $K\colon \mathbb{C} \longrightarrow \mathbb{B}$ such that $K \circ G \geq F$ in a universal way: whenever $K'\colon \mathbb{C} \longrightarrow \mathbb{B}$ satisfies $K' \circ G \geq F$, then $K' \geq K$.*

The left Kan extension of F along G may not exist—but if it does, then it is essentially unique; we will use the notation $\langle F, G\rangle$ as in diagram 3.7.

Lemma 3.3.2 *The left Kan extension $\langle F, G\rangle\colon \mathbb{C} \longrightarrow \mathbb{B}$ is the reflection of the object $F \in \mathsf{Cat}(\mathcal{Q})(\mathbb{A}, \mathbb{B})$ along the functor*[5]

$$- \circ G\colon \mathsf{Cat}(\mathcal{Q})(\mathbb{C}, \mathbb{B}) \longrightarrow \mathsf{Cat}(\mathcal{Q})(\mathbb{A}, \mathbb{B})\colon H \mapsto H \circ G. \tag{3.8}$$

Would, in the situation of 3.3.1, $\mathbb{B}$ be cocomplete, then clearly the left Kan extension of F along G exists, and

$$\langle F, G\rangle \cong \mathrm{colim}(\mathbb{C}(G-, -), F).$$

By lemma 3.1.3 we know that in this case the action of $\langle F, G\rangle\colon \mathbb{C} \longrightarrow \mathbb{B}$ on objects is given by the colimit formula

$$\langle F, G\rangle(c) \cong \nabla(\mathrm{colim}(\mathbb{C}(G-, c), F)).$$

[5]This functor is simply an order-preserving map between preordered classes—cf. 2.4.7.

Such a left Kan extension is therefore said to be pointwise: a *pointwise* left Kan extension is a colimit weighted by the left adjoint distributor determined by some functor. A $\mathcal{Q}$-category is said to *admit all pointwise left Kan extensions* when it admits all colimits of this special kind.

So a cocomplete $\mathcal{Q}$-category admits all pointwise left Kan extensions. But surprisingly enough[6] the converse is also true. It is the Yoneda embedding of a small $\mathcal{Q}$-category into the $\mathcal{Q}$-category of its contravariant presheaves $Y_{\mathbb{A}}\colon \mathbb{A}\longrightarrow\mathscr{P}\mathbb{A}$ that plays a crucial rôle.

Lemma 3.3.3 *Consider $\mathcal{Q}$-categories $\mathbb{A}$ and $\mathbb{B}$, a functor $F\colon\mathbb{A}\longrightarrow\mathbb{B}$ and a contravariant presheaf on $\mathbb{A}$, say $\phi\colon *_C \xrightarrow{\ \circ\ }\mathbb{A}$. Suppose that the left Kan extension of F along $Y_{\mathbb{A}}\colon\mathbb{A}\longrightarrow\mathscr{P}\mathbb{A}$ exists, and is pointwise:*

$$\langle F, Y_{\mathbb{A}}\rangle \cong \mathrm{colim}(\mathscr{P}\mathbb{A}(Y_{\mathbb{A}}-,-),F).$$

Then also the ϕ-weighted colimit of F exists, and is given – up to isomorphism – by

$$\mathrm{colim}(\phi,F)\colon *_C\longrightarrow\mathbb{B}\colon *\mapsto\langle F,Y_{\mathbb{A}}\rangle(\phi). \tag{3.9}$$

Proof: It suffices to show that $\mathbb{B}(\langle F,Y_{\mathbb{A}}\rangle(\phi),-)=[\phi,\mathbb{B}(F-,-)]$. But

$$\begin{aligned}\mathbb{B}(\langle F,Y_{\mathbb{A}}\rangle(\phi),-) &= [\mathscr{P}\mathbb{A}(Y_{\mathbb{A}}-,\phi),\mathbb{B}(F-,-)]\\ &= [\phi,\mathbb{B}(F-,-)]\end{aligned}$$

by (3.9) and 3.2.3. □

By 3.1.4 we may now state the following conclusion.

Proposition 3.3.4 *A $\mathcal{Q}$-category is cocomplete it and only if it admits all pointwise left Kan extensions.*

Corollary 3.3.5 *For a cocomplete $\mathcal{Q}$-category $\mathbb{B}$, any functor $G\colon\mathbb{A}\longrightarrow\mathbb{C}$ determines an adjunction*

$$\langle -,G\rangle\dashv(-\circ G)\colon\mathsf{Cat}(\mathcal{Q})(\mathbb{C},\mathbb{B})\rightleftarrows\mathsf{Cat}(\mathcal{Q})(\mathbb{A},\mathbb{B}). \tag{3.10}$$

E. Dubuc [1970] has called (3.10) the “meta-adjointness” associated to the concept of Kan extension. But there is not only this meta-adjointness that matters: also the fact that some functor $F\colon\mathbb{A}\longrightarrow\mathbb{B}$ may have a right adjoint can be expressed in terms of Kan extensions. We must first say a word about functors that preserve Kan extensions.

Definition 3.3.6 *Let $\mathcal{Q}$-categories and functors be given as in*

$$\mathbb{C}\xleftarrow{G}\mathbb{A}\xrightarrow{F}\mathbb{B}\xrightarrow{F'}\mathbb{B}'$$

and suppose that $\langle F,G\rangle\colon\mathbb{C}\longrightarrow\mathbb{B}$ exists. Then F' is said to preserve *$\langle F,G\rangle$ if $\langle F'\circ F,G\rangle$ exists too, and is isomorphic to $F'\circ\langle F,G\rangle$, as in diagram 3.8. The*

[6]But then again... “all concepts are Kan extensions!” [Mac Lane, 1998]

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "F"] \arrow[rr, bend left, "F' \circ F"] \arrow[d, "G"'] & \mathbb{B} \arrow[r, "F'"] & \mathbb{B}' \\
\mathbb{C} \arrow[ur, dotted, "\langle F, G\rangle"] \arrow[urr, dotted, "\langle F' \circ F, G\rangle \cong F' \circ \langle F, G\rangle"'] & &
\end{tikzcd}$$

Diagram 3.8

left Kan extension is absolute *if it is preserved by any* $F' \colon \mathbb{B} \longrightarrow \mathbb{B}'$*.*

These notions are reminiscent of the "absolute colimits" in 3.1.5, and for pointwise Kan extensions both notions of "absoluteness" coincide.

Proposition 3.3.7 *For a functor* $F \colon \mathbb{A} \longrightarrow \mathbb{B}$ *between* $\mathcal{Q}$*-categories the following are equivalent:*

1. *F has a right adjoint;*
2. *$\langle 1_{\mathbb{A}}, F\rangle$ exists and is absolute;*
3. *$\langle 1_{\mathbb{A}}, F\rangle$ exists and is preserved by F itself.*

In this case, $\langle 1_{\mathbb{A}}, F\rangle$ is the right adjoint of F.

Proof: ($1 \implies 2$) Suppose that $F \dashv F' \colon \mathbb{B} \leftrightarrows \mathbb{A}$—i.e. $\mathbb{A}(F'-,-) \dashv \mathbb{B}(F-,-)$. Then, using 1.2.1 and 1.2.5,

$$\begin{aligned}
\mathbb{A}(F'-,-) &= \mathbb{A}(F'-,-) \otimes_{\mathbb{A}} \mathbb{A}(-,-) \\
&= \Big[\mathbb{B}(F-,-), \mathbb{A}(1_{\mathbb{A}}-,-)\Big]
\end{aligned}$$

thus $F' \cong \operatorname{colim}(\mathbb{B}(F-,-), 1_{\mathbb{A}})$ and in particular $F' \colon \mathbb{B} \longrightarrow \mathbb{A}$ is the left Kan extension of $1_{\mathbb{A}}$ along F. The rest of the proof is analogous to the proof of 3.1.6.

($2 \implies 3$) is trivial.

And for ($3 \implies 1$) it suffices to prove that $F \dashv \langle 1_{\mathbb{A}}, F\rangle \colon \mathbb{B} \leftrightarrows \mathbb{A}$; that is to say

$$\begin{cases} 1_{\mathbb{A}} \leq \langle 1_{\mathbb{A}}, F\rangle \circ F; \\ F \circ \langle 1_{\mathbb{A}}, F\rangle \leq 1_{\mathbb{B}}. \end{cases}$$

The first of these inequalities is true by the very definition of left Kan extension. The second inequality is a consequence of $\langle 1_{\mathbb{A}}, F\rangle$ being preserved by F: $1_{\mathbb{B}} \circ F = F \implies \langle F, F\rangle \leq 1_{\mathbb{B}} \implies F \circ \langle 1_{\mathbb{A}}, F\rangle = \langle F, F\rangle \leq 1_{\mathbb{B}}$. $\square$

The following will be used in the upcoming sections.

Proposition 3.3.8 *Let $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ be a functor, and suppose that for all $\phi \in \mathscr{P}\mathbb{A}$ the ϕ-weighted colimit of F exists. Then*

$$\langle F, Y_{\mathbb{A}}\rangle \dashv \langle Y_{\mathbb{A}}, F\rangle \colon \mathbb{B} \rightleftarrows \mathscr{P}\mathbb{A}$$

and in particular

$$\begin{cases} \langle F, Y_{\mathbb{A}}\rangle(\phi) & \cong & \nabla\,\mathrm{colim}(\phi, F); \\ \langle Y_{\mathbb{A}}, F\rangle(b) & \cong & \mathbb{B}(F-, b). \end{cases}$$

Proof: By cocompleteness of $\mathscr{P}\mathbb{A}$ it follows that $\langle Y_{\mathbb{A}}, F\rangle$ exists, is pointwise, and can thus be computed as

$$\begin{aligned} \langle Y_{\mathbb{A}}, F\rangle(b) &\cong \nabla(\mathrm{colim}(\mathbb{B}(F-, b), Y_{\mathbb{A}})) \\ &\cong \nabla(\Delta\mathbb{B}(F-, b)) \\ &= \mathbb{B}(F-, b). \end{aligned}$$

In fact, $\mathbb{B}$ allowing colimits of F weighted by presheaves on $\mathbb{A}$, is sufficient to compute the left Kan extension of F along $Y_{\mathbb{A}}$: if $\mathrm{colim}(\mathscr{P}\mathbb{A}(Y_{\mathbb{A}}-, -), F)$ exists, then it is isomorphic to $\langle F, Y_{\mathbb{A}}\rangle$ (which is then pointwise). But $\mathrm{colim}(\mathscr{P}\mathbb{A}(Y_{\mathbb{A}}-, -), F)$ exists if and only if $\mathrm{colim}(\mathscr{P}\mathbb{A}(Y_{\mathbb{A}}-, \phi), F)$ exists for every $\phi \in \mathscr{P}\mathbb{A}$ (see 3.1.3). And by 3.2.3 $\mathscr{P}\mathbb{A}(Y_{\mathbb{A}}-, \phi) = \phi$ so by assumptions on $\mathbb{B}$ we are assured that $\langle F, Y_{\mathbb{A}}\rangle$ exists and can be computed pointwise as

$$\langle F, Y_{\mathbb{A}}\rangle(\phi) = \nabla\,\mathrm{colim}(\phi, F).$$

It is now a matter of applying 2.4.11 to obtain the result:

$$\begin{aligned} &\langle F, Y_{\mathbb{A}}\rangle \dashv \langle Y_{\mathbb{A}}, F\rangle \\ \iff\ & \forall \phi \in \mathscr{P}\mathbb{A}, \forall b \in \mathbb{B} : \mathbb{B}\Big(\langle F, Y_{\mathbb{A}}\rangle(\phi), b\Big) = \mathscr{P}\mathbb{A}\Big(\phi, \langle Y_{\mathbb{A}}, F\rangle(b)\Big) \\ \iff\ & \forall \phi \in \mathscr{P}\mathbb{A}, \forall b \in \mathbb{B} : \mathbb{B}\Big(\nabla\,\mathrm{colim}(\phi, F), b\Big) = \Big[\phi, \mathbb{B}(F-, b)\Big] \end{aligned}$$

which holds by the definition of colimit. $\square$

Remark that the result in 3.3.8 depends to a large extent on the very special rôle played by $Y_{\mathbb{A}}\colon \mathbb{A} \longrightarrow \mathscr{P}\mathbb{A}$, and in particular the "Yoneda lemma" 3.2.3! Indeed, in general – say for cocomplete $\mathbb{B}$ and $\mathbb{C}$ – the existence of diagram 3.9 in $\mathsf{Cat}(\mathcal{Q})$ is assured, but in no way can an adjunction between the Kan extensions be derived! For would one be able to prove that $\langle G, F\rangle \dashv \langle F, G\rangle$ using only the assumptions of cocompleteness of $\mathbb{B}$ and $\mathbb{C}$, then by symmetry of the situation also $\langle F, G\rangle \dashv \langle G, F\rangle$ would follow, and in particular $\mathbb{B} \simeq \mathbb{C}$ since $\mathsf{Cat}(\mathcal{Q})$ is locally preordered.

$$\begin{tikzcd} \mathbb{A} \arrow[r, "F"] \arrow[d, "G"'] & \mathbb{B} \arrow[dl, shift left, "{\langle G,F\rangle}"] \\ \mathbb{C} \arrow[ur, shift left, "{\langle F,G\rangle}"] & \end{tikzcd}$$

Diagram 3.9

3.4 Free cocompletion: presheaves revisited

Proposition 3.4.1 *For two $\mathcal{Q}$-categories $\mathbb{A}$ and $\mathbb{B}$, with $\mathbb{B}$ cocomplete, any functor $F\colon \mathbb{A}\longrightarrow\mathbb{B}$ factors through $Y_{\mathbb{A}}\colon \mathbb{A}\longrightarrow\mathscr{P}\mathbb{A}$ by an essentially unique cocontinuous*[7] *functor $G\colon \mathscr{P}\mathbb{A}\longrightarrow\mathbb{B}$, given explicitly by*

$$G\colon \mathscr{P}\mathbb{A}\longrightarrow\mathbb{B}\colon \psi\mapsto \nabla\operatorname{colim}(\psi,F).$$

Proof: It was shown in 3.3.8 that

$$\langle F, Y_{\mathbb{A}}\rangle = G\colon \mathscr{P}\mathbb{A}\longrightarrow\mathbb{B}\colon \psi\mapsto \nabla\operatorname{colim}(\psi,F)$$

is a left adjoint functor, its right adjoint being $\langle Y_{\mathbb{A}}, F\rangle\colon \mathscr{P}\mathbb{A}\longrightarrow\mathbb{B}$. Thus, in particular, G is cocontinous—see lemma 3.1.7.

Further, for $a\in\mathbb{A}$ we have that

$$\begin{aligned} G(Y_{\mathbb{A}}a) &= G(\mathbb{A}(-,a)) \\ &= \nabla\operatorname{colim}(\mathbb{A}(-,a),F) \\ &\cong Fa \end{aligned}$$

by 3.1.3 so indeed $F\cong G\circ Y_{\mathbb{A}}$.

Would now $G'\colon \mathscr{P}\mathbb{A}\longrightarrow\mathbb{B}$ be another cocontinous functor such that $F\cong G'\circ Y_{\mathbb{A}}$ then – by cocontinuity and using the fact that any presheaf $\phi\in\mathscr{P}\mathbb{A}$ is colimit of representables – G' and G agree on all objects of $\mathscr{P}\mathbb{A}$:

$$\begin{aligned} G'(\phi) &\cong G'(\nabla\operatorname{colim}(\phi,Y_{\mathbb{A}})) \\ &\cong \nabla\operatorname{colim}(\phi, G'\circ Y_{\mathbb{A}}) \\ &\cong \nabla\operatorname{colim}(\phi,F) \\ &= G(\phi) \end{aligned}$$

(using 3.1.3 (4) to pass from the second line to the third). □

[7] The factorization even proves to be left adjoint—see also corollary 3.4.2.

Corollary 3.4.2 *For $\mathcal{Q}$-categories $\mathbb{A}$ and $\mathbb{B}$, with $\mathbb{B}$ cocomplete, the cocontinuous functors $\mathscr{P}\mathbb{A}\longrightarrow\mathbb{B}$ coincide with the left adjoint ones.*

Proof: Any left adjoint is cocontinuous—see 3.1.7. Conversely, if $G\colon\mathscr{P}\mathbb{A}\longrightarrow\mathbb{B}$ is cocontinuous then it is the factorization of $G\circ Y_{\mathbb{A}}$ through $Y_{\mathbb{A}}$... and such a factorization is left adjoint. $\square$

Below $\mathsf{Cocont}(\mathcal{Q})$ denotes the sub-2-category of $\mathsf{Cat}(\mathcal{Q})$ whose objects are the cocomplete $\mathcal{Q}$-categories, and whose morphisms are the cocontinuous functors; so there is a 2-functor

$$\mathscr{U}\colon\mathsf{Cocont}(\mathcal{Q})\longrightarrow\mathsf{Cat}(\mathcal{Q})$$

which is the identity on objects, faithful (but not full), and locally fully faithful.

Theorem 3.4.3 *For $\mathcal{Q}$-categories $\mathbb{A}$ and $\mathbb{B}$, with $\mathbb{B}$ cocomplete, the Yoneda embedding $Y_{\mathbb{A}}\colon\mathbb{A}\longrightarrow\mathscr{P}\mathbb{A}$ induces a natural equivalence of preorders*

$$\mathsf{Cocont}(\mathcal{Q})(\mathscr{P}\mathbb{A},\mathbb{B})\xrightarrow{\sim}\mathsf{Cat}(\mathcal{Q})(\mathbb{A},\mathbb{B})\colon(\mathscr{P}\mathbb{A}\xrightarrow{G}\mathbb{B})\mapsto(\mathbb{A}\xrightarrow{G\circ Y_{\mathbb{A}}}\mathbb{B})$$

with inverse $F\mapsto\langle F,Y_{\mathbb{A}}\rangle$. That is to say, the presheaf construction provides a left (bi)adjoint to the inclusion of $\mathsf{Cocont}(\mathcal{Q})$ in $\mathsf{Cat}(\mathcal{Q})$:

$$\mathscr{P}\dashv\mathscr{U}\colon\mathsf{Cocont}(\mathcal{Q})\underset{\longrightarrow}{\overset{\longleftarrow}{}}\mathsf{Cat}(\mathcal{Q}).$$

Proof: This is really just another way of stating 3.4.1. $\square$

Corollary 3.4.4 *A $\mathcal{Q}$-category $\mathbb{A}$ is cocomplete if and only if $Y_{\mathbb{A}}\colon\mathbb{A}\longrightarrow\mathscr{P}\mathbb{A}$ is an equivalence in $\mathsf{Cat}(\mathcal{Q})$, in which case it is actually an equivalence in $\mathsf{Cocont}(\mathcal{Q})$.*

Proof: Follows from the above; the inverse equivalence is $\langle 1_{\mathbb{A}},Y_{\mathbb{A}}\rangle$. (Recall that – due to the local preordering in $\mathsf{Cat}(\mathcal{Q})$ – any equivalence functor and its inverse equivalence are always left adjoint, hence cocontinuous.) $\square$

Theorem 3.4.5 *For $\mathcal{Q}$-categories $\mathbb{A}$ and $\mathbb{B}$ there is an equivalence of preorders*

$$\mathsf{Dist}(\mathcal{Q})(\mathbb{A},\mathbb{B})\xrightarrow{\sim}\mathsf{Cocont}(\mathcal{Q})(\mathscr{P}\mathbb{A},\mathscr{P}\mathbb{B})\colon(\mathbb{A}\overset{\Phi}{\longrightarrow\!\!\!\!\!\!\!\circ\;\;\;}\mathbb{B})\mapsto(\mathscr{P}\mathbb{A}\xrightarrow{\mathscr{P}\Phi}\mathscr{P}\mathbb{B})$$

with inverse $F\mapsto\Phi_F$, where

- *$\mathscr{P}\Phi\colon\mathscr{P}\mathbb{B}\longrightarrow\mathscr{P}\mathbb{A}\colon\psi\mapsto\mathrm{colim}(\psi,F_{\Phi})$ for $F_{\Phi}\colon\mathbb{B}\longrightarrow\mathscr{P}\mathbb{A}\colon b\mapsto\Phi(-,b)$,*

- *$\Phi_F\colon\mathbb{A}\longrightarrow\!\!\!\!\!\!\circ\;\;\mathbb{B}$ has elements $\Phi_F(b,a)=F(Y_{\mathbb{A}}a)(b)$.*

Actually, the action

$$\mathsf{Dist}(\mathcal{Q})\longrightarrow\mathsf{Cocont}(\mathcal{Q})\colon(\mathbb{A}\overset{\Phi}{\longrightarrow\!\!\!\circ\!\!\!\rightarrow}\mathbb{B})\mapsto(\mathscr{P}\mathbb{A}\xrightarrow{\mathscr{P}\Phi}\mathscr{P}\mathbb{B})$$

is a biequivalence of 2-categories.

Proof: The equivalence of these preorders is a corollary of 3.4.3 and 3.2.2, and we already know that any cocomplete $\mathcal{Q}$-category is equivalent in $\mathsf{Cocont}(\mathcal{Q})$ to the category of its presheaves; so if we prove that the action $\Phi\mapsto\mathscr{P}\Phi$ is functorial then we're done.

Firstly[8], for an identity distributor $\mathbb{A}\colon\mathbb{A}\longrightarrow\!\!\!\circ\!\!\!\rightarrow\mathbb{A}$ in $\mathsf{Dist}(\mathcal{Q})$ and $\phi\in\mathscr{P}\mathbb{A}$,

$$\begin{array}{rcl}\mathscr{P}\mathbb{A}(\phi) & = & \nabla\,\mathrm{colim}(\phi,F_{\mathbb{A}})\\ & = & \nabla\,\mathrm{colim}(\phi,Y_{\mathbb{A}})\\ & \cong & \phi\end{array}$$

using the Yoneda lemma in the last step, and the fact that $F_{\mathbb{A}}=Y_{\mathbb{A}}$; so at least $\mathscr{P}\mathbb{A}\cong 1_{\mathscr{P}\mathbb{A}}$ in $\mathsf{Cat}(\mathcal{Q})(\mathscr{P}\mathbb{A},\mathscr{P}\mathbb{A})$. But $\mathscr{P}\mathbb{A}$ being skeletal assures that this isomorphism is an identity, which means that identities are mapped onto identities.

As for $\mathbb{A}\overset{\Phi}{\longrightarrow\!\!\!\circ\!\!\!\rightarrow}\mathbb{B}\overset{\Theta}{\longrightarrow\!\!\!\circ\!\!\!\rightarrow}\mathbb{C}$ in $\mathsf{Dist}(\mathcal{Q})$, to check that $\mathscr{P}\Theta\circ\mathscr{P}\Phi=\mathscr{P}(\Theta\otimes_{\mathbb{B}}\Phi)$ in $\mathsf{Cat}(\mathcal{Q})(\mathscr{P}\mathbb{A},\mathscr{P}\mathbb{C})$, first note that for $a\in\mathbb{A}_0$,

$$\begin{array}{rcl}(\mathscr{P}(\Theta\otimes_{\mathbb{B}}\Phi)\circ Y_{\mathbb{A}})(a) & = & \mathscr{P}(\Theta\otimes_{\mathbb{B}}\Phi)(\mathbb{A}(-,a))\\ & = & \nabla\,\mathrm{colim}(\mathbb{A}(-,a),F_{\Theta\otimes_{\mathbb{B}}\Phi})\\ & \cong & F_{\Theta\otimes_{\mathbb{B}}\Phi}(a)\\ & = & (\Theta\otimes_{\mathbb{B}}\Phi)(-,a)\\ & = & \Theta(-,-)\otimes_{\mathbb{B}}\Phi(-,a)\\ & \overset{*}{\cong} & \nabla\,\mathrm{colim}(\Phi(-,a),F_{\Theta})\\ & = & \mathscr{P}\Theta(\Phi(-,a))\\ & = & \mathscr{P}\Theta(F_{\Phi}(a))\\ & = & \mathscr{P}\Theta(\nabla\,\mathrm{colim}(\mathbb{A}(-,a),F_{\Phi}))\\ & = & \mathscr{P}\Theta(\mathscr{P}\Phi(\mathbb{A}(-,a)))\\ & = & (\mathscr{P}\Theta\circ\mathscr{P}\Phi\circ Y_{\mathbb{A}})(a).\end{array}$$

(in $(*)$ it is the construction of colimits in a presheaf category that is used, cf. 3.2.4) so these functors agree – up to isomorphism – on representables. But

[8] At this point there is an unpleasant confusion of notations: here by $\mathscr{P}\mathbb{A}$ is meant the functor $\mathscr{P}\mathbb{A}\longrightarrow\mathscr{P}\mathbb{A}$ determined by the distributor $\mathbb{A}\colon\mathbb{A}\longrightarrow\!\!\!\circ\!\!\!\rightarrow\mathbb{A}$ as in the statement of the corollary, and not the distributor $\mathscr{P}\mathbb{A}\colon\mathscr{P}\mathbb{A}\longrightarrow\!\!\!\circ\!\!\!\rightarrow\mathscr{P}\mathbb{A}$.

since the functors $\mathscr{P}\Phi$, $\mathscr{P}\Theta$ and $\mathscr{P}(\Theta\otimes_{\mathbb{B}}\Phi)$ are cocontinuous and every object of $\mathscr{P}\mathbb{A}$ is a colimit of representables, they must also agree (again, up to isomorphism) on any presheaf; so at least $\mathscr{P}\Theta\circ\mathscr{P}\Phi\cong\mathscr{P}(\Theta\otimes_{\mathbb{B}}\Phi)$ in $\mathsf{Cat}(\mathscr{Q})(\mathscr{P}\mathbb{A},\mathscr{P}\mathbb{C})$. The codomain category $\mathscr{P}\mathbb{C}$ being skeletal, this isomorphism is in fact an equality—which means that the image of a composition is the composition of the images. □

3.5 Cauchy completion and Morita equivalence

Cauchy complete categories

A distributor $\Phi\colon\mathbb{C}\mathrel{-\!\!\circ\!\!\rightarrow}\mathbb{A}$ which is a left adjoint is by some called a *Cauchy distributor*[9] (into $\mathbb{A}$); we adopt this habit. Such a Cauchy distributor is said to *converge* (to a functor $F\colon\mathbb{C}\longrightarrow\mathbb{A}$) if (there exists a functor $F\colon\mathbb{C}\longrightarrow\mathbb{A}$ such that) $\Phi=\mathbb{B}(-,F-)$. A Cauchy distributor $\Phi\colon\mathbb{C}\mathrel{-\!\!\circ\!\!\rightarrow}\mathbb{A}$ need not converge, but if it does then the functor $F\colon\mathbb{C}\longrightarrow\mathbb{A}$ that it converges to, is essentially unique.

In what follows we will systematically denote the right adjoint of a Cauchy distributor $\Phi\colon\mathbb{C}\mathrel{-\!\!\circ\!\!\rightarrow}\mathbb{A}$ by $\Phi^*\colon\mathbb{A}\mathrel{-\!\!\circ\!\!\rightarrow}\mathbb{C}$. In particular, would a contravariant presheaf on a $\mathscr{Q}$-category $\mathbb{A}$, i.e. some $\phi\in\mathscr{P}\mathbb{A}$, have a right adjoint (so would it be "Cauchy"), then its right adjoint will be denoted as $\phi^*\colon\mathbb{A}\mathrel{-\!\!\circ\!\!\rightarrow}*_{t\phi}$ with elements written simply as $\phi^*(a)\colon ta\longrightarrow t\phi$.

We will now study those $\mathscr{Q}$-categories in which "all Cauchy distributors converge".

Proposition 3.5.1 *For a $\mathscr{Q}$-category $\mathbb{A}$, the following are equivalent:*

1. *for all $\mathbb{C}$, any Cauchy distributor $\Phi\colon\mathbb{C}\mathrel{-\!\!\circ\!\!\rightarrow}\mathbb{A}$ converges;*
2. *for all $C\in\mathscr{Q}_0$, any Cauchy distributor $\Phi\colon *_C\mathrel{-\!\!\circ\!\!\rightarrow}\mathbb{A}$ converges.*

Proof: We only need to prove ($2\Longrightarrow 1$): If $\Phi\dashv\Phi^*\colon\mathbb{A}\rightleftarrows\mathbb{C}$ then also, for each $c\in\mathbb{C}_0$, $\Phi(-,c)\dashv\Phi^*(c,-)\colon\mathbb{A}\rightleftarrows\mathbb{C}$, simply because

$$\Phi\otimes_{\mathbb{C}}\mathbb{C}(-,c)=\Phi(-,c)\text{ and }\mathbb{C}(c,-)\otimes_{\mathbb{C}}\Phi^*=\Phi^*(c,-).$$

and the composition of adjoints is an adjoint. By assumption there exists thus for each $c\in\mathbb{C}_0$ a functor $Fc\colon *_{tc}\longrightarrow\mathbb{A}$ – that we will henceforth identify with the single element of type tc that it picks out – such that

$$\mathbb{A}(-,Fc)=\Phi(-,c)\text{ or equivalently, }\mathbb{A}(Fc,-)=\Phi^*(c,-).$$

[9]If one takes inspiration from ring theory then "small-projective" is a valuable alternative [Kelly, 1982]; and thinking of relations between sets one might prefer to call them simply "maps".

But the object mapping $\mathbb{C}_0 \longrightarrow \mathbb{A}_0: c \mapsto Fc$ is functorial: since $\mathbb{C} \leq \Phi^* \otimes_{\mathbb{A}} \Phi$ it follows that

$$\begin{aligned} \mathbb{C}(c',c) \leq \Phi^*(c',-) \otimes_{\mathbb{A}} \Phi(-,c) &= \mathbb{A}(Fc',-) \otimes_{\mathbb{A}} \mathbb{A}(-,Fc) \\ &= \mathbb{A}(Fc',Fc). \end{aligned}$$

Clearly $\Phi \dashv \Phi^*$ converges to the functor $F: \mathbb{C} \longrightarrow \mathbb{A}: c \mapsto Fc$. $\square$

Definition 3.5.2 *A $\mathscr{Q}$-category $\mathbb{A}$ is* Cauchy complete *if the (equivalent) conditions of 3.5.1 hold.*

Cauchy completion

By the above, a $\mathscr{Q}$-category $\mathbb{A}$ is Cauchy complete if and only if for every other $\mathscr{Q}$-category $\mathbb{C}$ there is an equivalence of preorders

$$\mathsf{Map}(\mathsf{Dist}(\mathscr{Q}))(\mathbb{C},\mathbb{A}) \simeq \mathsf{Cat}(\mathscr{Q})(\mathbb{C},\mathbb{A}).$$

Not every $\mathscr{Q}$-category is Cauchy complete, but still we are interested in “classifying” the Cauchy distributors on a given $\mathscr{Q}$-category: for any $\mathbb{A}$ we want to construct a new $\mathscr{Q}$-category $\mathbb{A}_{\mathsf{cc}}$ such that there is an equivalence of preorders for every category $\mathbb{C}$, like so:

$$\mathsf{Map}(\mathsf{Dist}(\mathscr{Q}))(\mathbb{C},\mathbb{A}) \simeq \mathsf{Cat}(\mathscr{Q})(\mathbb{C},\mathbb{A}_{\mathsf{cc}}). \tag{3.11}$$

Here $\mathsf{Map}(\mathsf{Dist}(\mathscr{Q}))(\mathbb{C},\mathbb{A})$ stands for the full subcategory of $\mathsf{Dist}(\mathscr{Q})(\mathbb{C},\mathbb{A})$ whose objects are the Cauchy distributors (“maps”). Much like the construction of “$\mathscr{P}\mathbb{A}$”, we have the following proposition. (The proof is much like that of 3.2.2, so it is omitted.)

Proposition 3.5.3 *For any $\mathscr{Q}$-category $\mathbb{A}$, a new $\mathscr{Q}$-category $\mathbb{A}_{\mathsf{cc}}$ called the “Cauchy completion of $\mathbb{A}$”, is defined to be the full subcategory of $\mathscr{P}\mathbb{A}$ determined by the Cauchy presheaves on $\mathbb{A}$:*

$$(\mathbb{A}_{\mathsf{cc}})_0 = \{\phi \in (\mathscr{P}\mathbb{A})_0 \mid \phi \textit{ is Cauchy}\}.$$

This $\mathscr{Q}$-category is skeletal and it satisfies (3.11). And the Yoneda embedding $Y_{\mathbb{A}}: \mathbb{A} \longrightarrow \mathscr{P}\mathbb{A}$ factors fully faithfully over the full inclusion $\mathbb{A}_{\mathsf{cc}} \subseteq \mathscr{P}\mathbb{A}$ as

$$i_{\mathbb{A}}: \mathbb{A} \longrightarrow \mathbb{A}_{\mathsf{cc}}: a \mapsto (*_{ta} \xrightarrow[\circ]{\mathbb{A}(-,a)} \mathbb{A}) \textit{ (see diagram 3.10).}$$

Note that the homs in $\mathbb{A}_{\mathsf{cc}}$ are identical to those in $\mathscr{P}\mathbb{A}$—but since objects $\phi, \psi \in \mathbb{A}_{\mathsf{cc}}$ have right adjoints we may – by 1.2.1 – write (with slight abuse of notation)

$$\mathbb{A}_{\mathsf{cc}}(\psi,\phi) = \psi^* \otimes_{\mathbb{A}} \phi. \tag{3.12}$$

This will be a helpful trick in many of the calculations that follow, to begin with in the proof of the following important proposition.

$$\begin{tikzcd} \mathbb{A} \arrow[r, "i_{\mathbb{A}}"] \arrow[dr, "Y_{\mathbb{A}}"'] & \mathbb{A}_{\mathsf{cc}} \arrow[d, "j_{\mathbb{A}}"] \\ & \mathscr{P}\mathbb{A} \end{tikzcd}$$

Diagram 3.10

Proposition 3.5.4 *Any $\mathcal{Q}$-category $\mathbb{A}$ is isomorphic to its Cauchy completion in the quantaloid* $\mathsf{Dist}(\mathcal{Q})$.

Proof: We know that the Cauchy completion $i_{\mathbb{A}}\colon \mathbb{A} \longrightarrow \mathbb{A}_{\mathsf{cc}}$ is a fully faithful functor, so the unit of the induced adjoint pair of distributors

$$\mathbb{A} \underset{\mathbb{A}_{\mathsf{cc}}(i_{\mathbb{A}}-,-)}{\overset{\mathbb{A}_{\mathsf{cc}}(-,i_{\mathbb{A}}-)}{\rightleftarrows}} \mathbb{A}_{\mathsf{cc}}$$

is an equality. It thus suffices to prove that also its co-unit is an equality rather than an inequality.

Thereto, consider two objects, $\phi \dashv \phi^*$ and $\psi \dashv \psi^*$, of $\mathbb{A}_{\mathsf{cc}}$ and calculate with the aid of (3.12) that

$$\begin{aligned} \mathbb{A}_{\mathsf{cc}}(\psi, i_{\mathbb{A}}-) \otimes_{\mathbb{A}} \mathbb{A}_{\mathsf{cc}}(i_{\mathbb{A}}-,\phi) &= \bigvee_{a\in\mathbb{A}_0} \mathbb{A}_{\mathsf{cc}}(\psi, \mathbb{A}(-,a)) \circ \mathbb{A}_{\mathsf{cc}}(\mathbb{A}(-,a),\phi) \\ &= \psi^* \otimes_{\mathbb{A}} \bigvee_{a\in\mathbb{A}_0} \Big(\mathbb{A}(-,a)\circ\mathbb{A}(a,-)\Big)\otimes_{\mathbb{A}}\phi \\ &= \psi^* \otimes_{\mathbb{A}} \mathbb{A} \otimes_{\mathbb{A}} \phi \\ &= \psi^* \otimes_{\mathbb{A}} \phi \\ &= \mathbb{A}_{\mathsf{cc}}(\psi,\phi). \end{aligned}$$

That is to say, $\mathbb{A}_{\mathsf{cc}}(-,i_{\mathbb{A}}-) \otimes_{\mathbb{A}} \mathbb{A}_{\mathsf{cc}}(i_{\mathbb{A}}-,-) = \mathbb{A}_{\mathsf{cc}}$ so indeed $\mathbb{A}_{\mathsf{cc}}(-,i_{\mathbb{A}}-) = \Big(\mathbb{A}_{\mathsf{cc}}(i_{\mathbb{A}}-,-)\Big)^{-1}$. $\square$

We now wish to show that, for any $\mathcal{Q}$-category $\mathbb{A}$, the category $\mathbb{A}_{\mathsf{cc}}$ is itself Cauchy complete. Thereto we must show that for any adjoint pair of distributors like

$$\Omega \dashv \Omega^*\colon \mathbb{A}_{\mathsf{cc}} \rightleftarrows *_Z \tag{3.13}$$

there exists an object of $\mathbb{A}_{\mathsf{cc}}$ of type Z, that is, an adjoint pair like

$$\omega \dashv \omega^*\colon \mathbb{A} \rightleftarrows *_Z,$$

such that $\Omega \dashv \Omega^*$ is represented by $\omega \dashv \omega^*$:

$$\mathbb{A}_{\mathsf{cc}}(-,\omega) = \Omega, \text{ or equivalently } \mathbb{A}_{\mathsf{cc}}(\omega,-) = \Omega^*. \tag{3.14}$$

Explicitly, for every $\phi \dashv \phi^*\colon \mathbb{A} \overset{\leftarrow\!\circ\!-}{\underset{-\!\circ\!\rightarrow}{}} *_C$ we want[10]

$$\begin{cases} \omega^* \otimes_{\mathbb{A}} \phi = \Omega^*(\phi), \\ \phi^* \otimes_{\mathbb{A}} \omega = \Omega(\phi). \end{cases} \tag{3.15}$$

Taking in particular $\phi = \mathbb{A}(-,a)$ for $a \in \mathbb{A}_0$ (so $\phi^* = \mathbb{A}(a,-)$) it now follows that, if we want to define for the given pair $\Omega \dashv \Omega^*$ a representing pair $\omega \dashv \omega^*$, we have no other choice than to put

$$\begin{cases} \omega\colon *_Z \xrightarrow{\circ} \mathbb{A} \text{ with elements } \omega(a) = \Omega(\mathbb{A}(-,a)), \\ \omega^*\colon \mathbb{A} \xrightarrow{\circ} *_Z \ \text{ with elements } \omega^*(a) = \Omega^*(\mathbb{A}(-,a)). \end{cases} \tag{3.16}$$

We will now verify that both ω and ω^* are indeed distributors forming an adjoint pair representing Ω and Ω^* in the sense of 3.14.

Lemma 3.5.5 *Let $\mathbb{A}$ be a $\mathcal{Q}$-category, and consider a distributor $\Omega\colon *_Z \xrightarrow{\circ} \mathbb{A}_{\mathsf{cc}}$; then for any $\phi \dashv \phi^*\colon \mathbb{A} \overset{\leftarrow\!\circ\!-}{\underset{-\!\circ\!\rightarrow}{}} *_C$ (so $\phi \in \mathbb{A}_{\mathsf{cc}}$),*

$$\Omega(\phi) = \phi^* \otimes_{\mathbb{A}} \Omega(i_{\mathbb{A}}-). \tag{3.17}$$

And dually, for a distributor $\Theta\colon \mathbb{A}_{\mathsf{cc}} \xrightarrow{\circ} *_Z$

$$\Theta(\phi) = \Theta(i_{\mathbb{A}}-) \otimes_{\mathbb{A}} \phi. \tag{3.18}$$

Proof: First note that, for $\phi \dashv \phi^*$ and $\tau \dashv \tau^*$ in $\mathbb{A}_{\mathsf{cc}}$, and with slight abuse of notation, $\mathbb{A}_{\mathsf{cc}}(\phi,\tau) = \phi^* \otimes_{\mathbb{A}} \tau = \phi^* \otimes_{\mathbb{A}} \mathbb{A}_{\mathsf{cc}}(i_{\mathbb{A}}-,\tau)$ simply because $\mathbb{A}_{\mathsf{cc}}(i_{\mathbb{A}}(a),\tau) = \tau(a)$. It can then easily be verified that

$$\begin{aligned} \Omega(\phi) &= \mathbb{A}_{\mathsf{cc}}(\phi,-) \otimes_{\mathbb{A}_{\mathsf{cc}}} \Omega(-) \\ &= \phi^* \otimes_{\mathbb{A}} \mathbb{A}_{\mathsf{cc}}(i_{\mathbb{A}}-,-) \otimes_{\mathbb{A}_{\mathsf{cc}}} \Omega(-) \\ &= \phi^* \otimes_{\mathbb{A}} \Omega(i_{\mathbb{A}}-). \end{aligned}$$

Likewise for the other identity. $\square$

Lemma 3.5.6 *For a $\mathcal{Q}$-category $\mathbb{A}$, consider $\Omega \dashv \Omega^*\colon \mathbb{A}_{\mathsf{cc}} \overset{\leftarrow\!\circ\!-}{\underset{-\!\circ\!\rightarrow}{}} *_Z$ as in (3.13) and define $\omega\colon *_Z \xrightarrow{\circ} \mathbb{A}$ and $\omega^*\colon \mathbb{A} \xrightarrow{\circ} *_Z$ as in (3.16). Then $\omega \dashv \omega^*$ (thus $\omega \in \mathbb{A}_{\mathsf{cc}}$) and (3.14) holds.*

Proof: The legitimacy of the distributors ω and ω^* was shown in 3.5.5. To see that $\omega \dashv \omega^*$, we must verify both inequalities

$$1_Z \le \bigvee_{x \in \mathbb{A}_0} \omega^*(x) \circ \omega(x), \tag{3.19}$$

[10] Slight abuse of notation: the $\mathcal{Q}$-arrow $\Omega(\phi)\colon Z \longrightarrow C$ must equal the single element of the $\mathcal{Q}$-distributor $[\phi,\omega]\colon *_Z \xrightarrow{\circ} *_C$, etc.

$$\forall a, a' \in \mathbb{A} : \omega(a') \circ \omega^*(a) \leq \mathbb{A}(a', a). \tag{3.20}$$

But (3.20) is easy, because of the co-unit of the adjunction $\Omega \dashv \Omega^*$ and the ("fully faithful") inclusion functor $i_{\mathbb{A}}\colon \mathbb{A} \longrightarrow \mathbb{A}_{\mathsf{cc}}$:

$$\begin{aligned} \omega(a') \circ \omega^*(a) &= \Omega(\mathbb{A}(-, a')) \circ \Omega^*(\mathbb{A}(-, a)) \\ &\leq \mathbb{A}_{\mathsf{cc}}(\mathbb{A}(-, a'), \mathbb{A}(-, a)) \\ &= \mathbb{A}(a', a). \end{aligned}$$

And (3.19) can be shown relying on the unit of $\Omega \dashv \Omega^*$ and 3.5.5 as follows:

$$\begin{aligned} 1_Z &\leq \bigvee_{\phi \in \mathbb{A}_{\mathsf{cc}}} \Omega^*(\phi) \circ \Omega(\phi) \\ &= \bigvee_{\phi \dashv \phi^* \in \mathbb{A}_{\mathsf{cc}}} \Omega^*(i_{\mathbb{A}}-) \otimes_{\mathbb{A}} \phi \otimes_{*Z} \phi^* \otimes_{\mathbb{A}} \Omega(i_{\mathbb{A}}-) \\ &\leq \Omega^*(i_{\mathbb{A}}-) \otimes_{\mathbb{A}} \mathbb{A} \otimes_{\mathbb{A}} \Omega(i_{\mathbb{A}}-) \\ &= \Omega^*(i_{\mathbb{A}}-) \otimes_{\mathbb{A}} \Omega(i_{\mathbb{A}}-) \\ &= \omega^* \otimes_{\mathbb{A}} \omega. \end{aligned}$$

It remains to prove that the adjunction $\omega \dashv \omega^*$ (i.e. the object $\omega \in \mathbb{A}_{\mathsf{cc}}$) represents the adjunction $\Omega \dashv \Omega^*$. But for any $\phi \dashv \phi^*\colon \mathbb{A} \overset{\longleftarrow\!\!\circ\!\!-}{\underset{-\!\!\circ\!\!\longrightarrow}{}} *_C$ (so $\phi \in \mathbb{A}_{\mathsf{cc}}$) it follows by equations (3.17) and (3.18) that

$$\begin{aligned} \mathbb{A}_{\mathsf{cc}}(\phi, \omega) &= [\phi, \omega] \\ &= \phi^* \otimes_{\mathbb{A}} \omega \\ &= \phi^* \otimes_{\mathbb{A}} \Omega(i_{\mathbb{A}}-) \\ &= \Omega(\phi). \end{aligned}$$

And likewise (or rather, equivalently) $\mathbb{A}_{\mathsf{cc}}(\omega, \phi) = \Omega^*(\phi)$. $\square$

Proposition 3.5.7 *For any $\mathbb{A}$, the Cauchy completion $\mathbb{A}_{\mathsf{cc}}$ is Cauchy complete.*

The last – but not least! – property of $\mathbb{A}_{\mathsf{cc}}$, the "Cauchy completion" of a small $\mathcal{Q}$-category $\mathbb{A}$, that we need to investigate is its universality. The proof of the following is analogous to that of 3.4.1, and therefore it is omitted.

Proposition 3.5.8 *For a small $\mathcal{Q}$-category $\mathbb{A}$ and a (possibly large) Cauchy complete $\mathbb{B}$, any functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ factors through $i_{\mathbb{A}}\colon \mathbb{A} \longrightarrow \mathbb{A}_{\mathsf{cc}}$ by an essentially unique functor $G\colon \mathbb{A}_{\mathsf{cc}} \longrightarrow \mathbb{B}$ given explicitly by*

$$G\colon \mathbb{A}_{\mathsf{cc}} \longrightarrow \mathbb{B}\colon \phi \mapsto \nabla \operatorname{colim}(\phi, F).$$

Assembling the results

Assembling the results in this section, we may conclude as follows.

Theorem 3.5.9 *For any quantaloid $\mathcal{Q}$,*

1. $\mathsf{Dist}(\mathcal{Q})$ *is equivalent to its full subcategory* $\mathsf{Dist}_{\mathsf{cc}}(\mathcal{Q})$ *determined by the Cauchy complete categories;*

2. *denoting* $\mathsf{Cat}_{\mathsf{cc}}(\mathcal{Q})$ *for the full sub-2-category of* $\mathsf{Cat}(\mathcal{Q})$ *determined by the Cauchy complete categories, Cauchy completing provides a left (2-)adjoint to the full embedding:*

$$(-)_{\mathsf{cc}} \dashv \mathscr{V} \colon \mathsf{Cat}_{\mathsf{cc}}(\mathcal{Q}) \underset{\longrightarrow}{\overset{\longleftarrow}{}} \mathsf{Cat}(\mathcal{Q});$$

3. *the following 2-categories are biequivalent:*

$$\mathsf{Cat}_{\mathsf{cc}}(\mathcal{Q}) \simeq \mathsf{Map}(\mathsf{Dist}_{\mathsf{cc}}(\mathcal{Q})) \simeq \mathsf{Map}(\mathsf{Dist}(\mathcal{Q})).$$

Proof: (1) Is a direct consequence of 3.5.4.

(2) By 3.5.8, we have for $\mathcal{Q}$-categories $\mathbb{A}$ and $\mathbb{B}$, with $\mathbb{B}$ Cauchy complete, a natural equivalence of preorders induced by composition with $i_{\mathbb{A}} \colon \mathbb{A} \longrightarrow \mathbb{A}_{\mathsf{cc}}$:

$$\mathsf{Cat}_{\mathsf{cc}}(\mathcal{Q})(\mathbb{A}_{\mathsf{cc}}, \mathbb{B}) \xrightarrow{\sim} \mathsf{Cat}(\mathcal{Q})(\mathbb{A}, \mathbb{B}) \colon (\mathbb{A}_{\mathsf{cc}} \xrightarrow{G} \mathbb{B}) \mapsto (\mathbb{A} \xrightarrow{G \circ i_{\mathbb{A}}} \mathbb{B})$$

with inverse $F \mapsto \mathrm{colim}(-, F)$.

(3) The second equivalence follows from (1) of course. As for the first biequivalence, the 2-functor

$$\mathsf{Cat}_{\mathsf{cc}}(\mathcal{Q}) \longrightarrow \mathsf{Map}(\mathsf{Dist}_{\mathsf{cc}}(\mathcal{Q})) \colon (\mathbb{A} \xrightarrow{F} \mathbb{B}) \mapsto (\mathbb{A} \xrightarrow{\mathbb{B}(-, F-)} \mathbb{B})$$

is the identity on objects, and by Cauchy completeness of $\mathbb{B}$ the action of this 2-functors induces equivalences

$$\mathsf{Cat}_{\mathsf{cc}}(\mathcal{Q})(\mathbb{A}, \mathbb{B}) \simeq \mathsf{Map}(\mathsf{Dist}(\mathcal{Q}))(\mathbb{A}, \mathbb{B})$$

of the hom-preorders. □

Corollary 3.5.10 *A $\mathcal{Q}$-category $\mathbb{A}$ is Cauchy complete if and only if the Cauchy completion $i_{\mathbb{A}} \colon \mathbb{A} \longrightarrow \mathbb{A}_{\mathsf{cc}}$ is an equivalence in* $\mathsf{Cat}(\mathcal{Q})$.

Finally we wish to discuss the "interplay" between $\mathbb{A}$, $\mathscr{P}\mathbb{A}$ and $\mathbb{A}_{\mathsf{cc}}$.

Proposition 3.5.11 *For $\mathcal{Q}$-categories $\mathbb{A}$ and $\mathbb{B}$, the following are equivalent:*

1. $\mathbb{A} \cong \mathbb{B}$ *in* $\mathsf{Dist}(\mathcal{Q})$*;*

2. $\mathbb{A}_{\mathsf{cc}} \simeq \mathbb{B}_{\mathsf{cc}}$ *in* $\mathsf{Cat}(\mathcal{Q})$*;*

3. $\mathscr{P}\mathbb{A} \simeq \mathscr{P}\mathbb{B}$ *in* $\mathsf{Cat}(\mathcal{Q})$*.*

Proof: ($1 \iff 2$) Since $\mathbb{A} \cong \mathbb{A}_{\mathsf{cc}}$ and $\mathbb{B} \cong \mathbb{B}_{\mathsf{cc}}$ in $\mathsf{Dist}(\mathcal{Q})$, we have that $\mathbb{A} \cong \mathbb{B}$ in $\mathsf{Dist}(\mathcal{Q})$ if and only if $\mathbb{A}_{\mathsf{cc}} \cong \mathbb{B}_{\mathsf{cc}}$ in $\mathsf{Dist}_{\mathsf{cc}}(\mathcal{Q})$, which by the biequivalence in 3.5.9 (3) is the same as an equivalence $\mathbb{A}_{\mathsf{cc}} \simeq \mathbb{B}_{\mathsf{cc}}$ in $\mathsf{Cat}_{\mathsf{cc}}(\mathcal{Q})$ or in $\mathsf{Cat}(\mathcal{Q})$ (since $\mathsf{Cat}_{\mathsf{cc}}(\mathcal{Q})$ is fully embedded in $\mathsf{Cat}(\mathcal{Q})$).

($1 \iff 3$) Follows from similar arguments, but now using the biequivalence in 3.4.5. $\square$

Definition 3.5.12 *Whenever for two $\mathcal{Q}$-categories $\mathbb{A}$ and $\mathbb{B}$ the (equivalent) conditions in 3.5.11 are fulfilled, then $\mathbb{A}$ is* Morita equivalent *to $\mathbb{B}$, denoted $\mathbb{A} \sim_M \mathbb{B}$.*

The morality of 3.5.9 is then that "a $\mathcal{Q}$-category is but a presentation of its Cauchy completion"; in particular, if one only cares about $\mathcal{Q}$-category theory "up to Cauchy completion" (i.e. "up to Morita equivalence"), then one can forget about functors and Cauchy completion and Morita equivalence altogether, and treat $\mathsf{Map}(\mathsf{Dist}(\mathcal{Q}))$ as "the 2-category of $\mathcal{Q}$-categories and functors".

More about Cauchy completeness

Here is an alternative for the classifying property in (3.11).

Proposition 3.5.13 *A distributor $\Phi\colon \mathbb{C} \multimap \mathbb{A}$ is Cauchy if and only if the corresponding functor $F_\Phi\colon \mathbb{C} \longrightarrow \mathscr{P}\mathbb{A}$ (see 3.2.2) factors through the full inclusion $j_{\mathbb{A}}\colon \mathbb{A}_{\mathsf{cc}} \longrightarrow \mathscr{P}\mathbb{A}$. In particular, a presheaf $\phi \in \mathscr{P}\mathbb{A}$ is an object of $\mathbb{A}_{\mathsf{cc}}$ if and only if the functor $\Delta\phi\colon *_{t\phi} \longrightarrow \mathscr{P}\mathbb{A}$ factors through $j_{\mathbb{A}}$.*

Proof: $\Phi\colon \mathbb{C} \multimap \mathbb{A}$ has a right adjoint if and only if for all $c \in \mathbb{C}_0$ the distributor $\Phi(-, c)\colon *_{tc} \multimap \mathbb{A}$ has a right adjoint, i.e. is an object of $\mathbb{A}_{\mathsf{cc}}$. $\square$

Presheaves on $\mathbb{A}$ are colimit in $\mathscr{P}\mathbb{A}$ of representables, but each representable is an object of $\mathbb{A}_{\mathsf{cc}}$—so the following is not a surprise.

Proposition 3.5.14 *Any $\phi \in \mathbb{A}_{\mathsf{cc}}$ is canonically a colimit of representables in $\mathbb{A}_{\mathsf{cc}}$—see diagram 3.11.*

Proof: Obviously, for every $\tau \in \mathbb{A}_{\mathsf{cc}}$,

$$\mathbb{A}_{\mathsf{cc}}(i_{\mathbb{A}}(a), \tau) = \mathbb{A}(a, -) \otimes_{\mathbb{A}} \tau(-) = \tau(a)$$

$$
\begin{tikzcd}
\mathbb{A} \arrow[r, "i_{\mathbb{A}}"] \arrow[d, shift left, "\phi^*", "\circ"'] & \mathbb{A}_{\mathsf{cc}} \\
*_C \arrow[u, shift left, "\phi", "\circ"'] \arrow[ur, dotted, "\mathrm{colim}(\phi, i_{\mathbb{A}}) = \Delta\phi"']
\end{tikzcd}
$$

Diagram 3.11

and therefore, with some abuse of notation,

$$
\begin{aligned}
[\phi, \mathbb{A}_{\mathsf{cc}}(i_{\mathbb{A}}-, \tau)] &= \phi^* \otimes_{\mathbb{A}} \mathbb{A}_{\mathsf{cc}}(i_{\mathbb{A}}-, \tau) \\
&= \phi^* \otimes_{\mathbb{A}} \tau \\
&= \mathbb{A}_{\mathsf{cc}}(\phi, \tau)
\end{aligned}
$$

which shows that $\Delta\phi$ has the universal property of $\mathrm{colim}(\phi, i_{\mathbb{A}})$. □

The next proposition explains Cauchy completeness as "admitting all Cauchy-weighted colimits".

Proposition 3.5.15 *A $\mathcal{Q}$-category is Cauchy complete if and only if it admits all colimits weighted by a Cauchy distributor.*

Proof: Let $\mathbb{B}$ be Cauchy complete, and consider a diagram of $\mathcal{Q}$-categories, distributors $\Phi \dashv \Phi^*$ and a functor like so:

$$
\begin{tikzcd}
\mathbb{C} \arrow[r, shift left, "\Phi"] & \mathbb{A} \arrow[l, shift left, "\Phi^*"] \arrow[r, "F"] & \mathbb{B}.
\end{tikzcd}
$$

Then, by assumption, the adjoint distributors

$$\Big(\mathbb{B}(-, F-) \otimes_{\mathbb{A}} \Phi\Big) \dashv \Big(\Phi^* \otimes_{\mathbb{A}} \mathbb{B}(F-, -)\Big)$$

must converge to a functor $K\colon \mathbb{C} \longrightarrow \mathbb{B}$. But then,

$$\mathbb{B}(K-, -) = \Phi^* \otimes_{\mathbb{A}} \mathbb{B}(F-, -) = \Big[\Phi, \mathbb{B}(F-, -)\Big]$$

so that K has the universal property of $\mathrm{colim}(\Theta, F)$.

Conversely, suppose that $\mathbb{B}$ admits all Cauchy weighted colimits. Given an adjoint pair $\Phi \dashv \Phi^*\colon \mathbb{B} \rightleftarrows \mathbb{C}$ we can now consider the colimit in diagram 3.12 whose universal property is

$$
\begin{aligned}
\mathbb{B}\Big(\mathrm{colim}(\Phi, 1_{\mathbb{B}})-, -\Big) &= \Big[\Phi, \mathbb{B}(1_{\mathbb{B}}-, -)\Big] \\
&= \Phi^* \otimes_{\mathbb{B}} \mathbb{B}(-, -) \\
&= \Phi^*.
\end{aligned}
$$

$$\begin{tikzcd}
\mathbb{B} \arrow[r, "1_{\mathbb{B}}"] \arrow[d, shift left, "\Phi^*"] & \mathbb{B} \\
\mathbb{C} \arrow[u, shift left, "\Phi"] \arrow[ur, dotted, "\mathrm{colim}(\Phi, 1_{\mathbb{B}})"'] &
\end{tikzcd}$$

Diagram 3.12

So $\Phi \dashv \Phi^*$ converges to $\mathrm{colim}(\Phi, 1_{\mathbb{B}})$. □

Finally, let us record the following tautology of the definition of 'Cauchy complete category', see 3.5.1 and 3.5.2.

Proposition 3.5.16 *A $\mathcal{Q}$-category $\mathbb{B}$ is Cauchy complete if and only if, for any $\mathcal{Q}$-category $\mathbb{A}$, the (preorder-preserving) map*

$$\mathsf{Cat}(\mathcal{Q})(\mathbb{A}, \mathbb{B}) \longrightarrow \mathsf{Map}(\mathsf{Dist}(\mathcal{Q}))(\mathbb{A}, \mathbb{B}) \colon F \mapsto \mathbb{B}(-, F-)$$

is surjective.

Recalling the skeletality of $\mathcal{Q}$-categories (see p. 86), and in particular (5) of 2.4.16, we may conclude with a word or two about skeletal Cauchy complete categories.

Corollary 3.5.17 *For a skeletal Cauchy complete $\mathcal{Q}$-category $\mathbb{B}$ there is an isomorphism of ordered sets (rather than an equivalence of preordered sets)*

$$\mathsf{Cat}(\mathcal{Q})(\mathbb{A}, \mathbb{B}) \longrightarrow \mathsf{Map}(\mathsf{Dist}(\mathcal{Q}))(\mathbb{A}, \mathbb{B}) \colon F \mapsto \mathbb{B}(-, F-).$$

That is to say, there is an isomorphism of (locally ordered) 2-categories

$$\mathsf{Cat}_{\mathsf{skel,cc}}(\mathcal{Q}) \cong \mathsf{Map}(\mathsf{Dist}_{\mathsf{skel,cc}}(\mathcal{Q})).$$

4

Regular semicategories

A '$\mathcal{Q}$-enriched semicategory' is a "$\mathcal{Q}$-category without units". Studying the presheaves on such $\mathcal{Q}$-semicategories, the notions of 'regular presheaf' and 'regular $\mathcal{Q}$-semicategory' are recognized to be important. Those regular $\mathcal{Q}$-semicategories turn out to be the idempotents in $\mathsf{Matr}(\mathcal{Q})$*; this observation gives rise to a calculus of $\mathcal{Q}$-enriched regular semicategories and regular distributors. Further, with a suitable notion of 'regular morphism' it is so that "every regular morphism between regular semicategories induces an adjoint pair of regular distributors". Thus the theory of regular $\mathcal{Q}$-semicategories is a strictly weaker alternative for the theory of $\mathcal{Q}$-categories.*

4.1 Semicategories, distributors and morphisms

By a 'semicategory' we mean a "category-without-identities", i.e. a multiplicative graph. We are interested in the theory of "semicategories over a base quantaloid", so we take the definition for $\mathcal{Q}$-category (see 2.3.1 on p. 75) and leave out the unit-inequalities.

Definition 4.1.1 *Let $\mathcal{Q}$ be a quantaloid. A* (small) $\mathcal{Q}$-enriched semicategory *(or* $\mathcal{Q}$-semicategory *for short) $\mathbb{A}$ consists of*

- *objects: a (small) set $\mathbb{A}_0$ to every element of which is associated a type in $\mathcal{Q}_0$: $\forall a \in \mathbb{A}_0 : ta \in \mathcal{Q}_0$;*

- *hom-arrows: $\forall a, a' \in \mathbb{A}_0$: $\mathbb{A}(a', a) \in \mathcal{Q}(ta, ta')$;*

satisfying

- *composition-inequalities: $\forall a, a', a'' \in \mathbb{A}_0$: $\mathbb{A}(a'', a') \circ \mathbb{A}(a', a) \leq \mathbb{A}(a'', a)$.*

Any $\mathcal{Q}$-category is thus a $\mathcal{Q}$-semicategory; the converse is obviously false.

Definition 4.1.2 *For $\mathcal{Q}$-semicategories $\mathbb{A}$ and $\mathbb{B}$, a* distributor $\Phi\colon\mathbb{A}\longrightarrow\!\!\!\!\!\!\!\!\circ\;\;\;\mathbb{B}$ *is a collection of $\mathcal{Q}$-arrows*

$$\Big(\Phi(b,a)\colon ta\longrightarrow tb\Big)_{a\in\mathbb{A}_0,b\in\mathbb{B}_0}$$

satisfying

- *action-inequalities: $\forall a,a'\in\mathbb{A}_0$ and $b,b'\in\mathbb{B}_0$:*

$$\begin{cases}\Phi(b,a')\circ\mathbb{A}(a',a)\leq\Phi(b,a),\\ \mathbb{B}(b,b')\circ\Phi(b',a)\leq\Phi(b,a).\end{cases}$$

This definition for 'distributor' is an exact copy of the definition of 'distributor between $\mathcal{Q}$-categories' in 2.3.2. Finally, as definition 2.4.1 for 'functor between $\mathcal{Q}$-categories' makes no reference whatsoever to the unit-inequalities in its domain and codomain[1], we may simply "copy-and-paste" it for semicategories.

Definition 4.1.3 *For $\mathcal{Q}$-semicategories $\mathbb{A}$ and $\mathbb{B}$, a* morphism $F\colon\mathbb{A}\longrightarrow\mathbb{B}$ *is a mapping of objects $\mathbb{A}_0\longrightarrow\mathbb{B}_0\colon a\mapsto Fa$ satisfying*

- *type-equalities: $\forall a\in\mathbb{A}_0:\ t(Fa)=ta$;*
- *action-inequalities: $\forall a,a'\in\mathbb{A}_0:\ \mathbb{A}(a',a)\leq\mathbb{B}(Fa',Fa)$.*

Having defined these three basic notions – semicategories, morphisms, distributors – we now investigate in how far they organize themselves in categorical structures.

Clearly morphisms can be composed by composing the object-mappings, and the identity object-mapping is the identity morphism for this composition, precisely as was the case for functors.

Proposition 4.1.4 *$\mathcal{Q}$-enriched semicategories and morphisms form a category $\mathsf{SCat}(\mathcal{Q})$.*

Note that for two $\mathcal{Q}$-categories a *morphism* between these categories-viewed-as-semicategories is the same thing as a *functor*, and *vice versa*. Therefore categories and functors form a full subcategory of semicategories and morphisms. But there is more.

Proposition 4.1.5 *The full embedding $\mathsf{Cat}(\mathcal{Q})\longrightarrow\mathsf{SCat}(\mathcal{Q})$ admits a left adjoint; the "free $\mathcal{Q}$-category" $\overline{\mathbb{A}}$ on a $\mathcal{Q}$-semicategory $\mathbb{A}$ is defined in the following way:*

[1]... which itself is a consequence of the base quantaloid $\mathcal{Q}$ being locally ordered!

- *objects:* $(\overline{\mathbb{A}})_0 = \mathbb{A}_0$, *with the same types;*

- *hom-arrows: for* $a, a' \in (\overline{\mathbb{A}})_0$, $\overline{\mathbb{A}}(a', a) = \begin{cases} \mathbb{A}(a', a) & \textit{if } a' \neq a, \\ \mathbb{A}(a, a) \bigvee 1_{ta} & \textit{if } a' = a. \end{cases}$

Proof: The "inclusion morphism" $i_{\mathbb{A}}\colon \mathbb{A} \longrightarrow \overline{\mathbb{A}}\colon a \mapsto a$ induces by composition the required (order-preserving) bijection of (preordered) sets. □

Explicitly, for any $\mathcal{Q}$-semicategory $\mathbb{A}$ and $\mathcal{Q}$-category $\mathbb{C}$,

$$\mathsf{Cat}(\mathcal{Q})(\overline{\mathbb{A}}, \mathbb{C}) = \mathsf{SCat}(\mathcal{Q})(\mathbb{A}, \mathbb{C}). \tag{4.1}$$

As notation we will often use, for a given morphism $F\colon \mathbb{A} \longrightarrow \mathbb{C}$ from a semicategory into a category, $\overline{F}\colon \overline{\mathbb{A}} \longrightarrow \mathbb{C}$ for the corresponding functor (even though it is the *same* object-mapping). It is further quite obvious that $\overline{\mathbb{A}} = \mathbb{A}$ if and only if $\mathbb{A}$ is a $\mathcal{Q}$-category rather than a semicategory.

As for distributors, given two $\mathcal{Q}$-semicategories $\mathbb{A}$ and $\mathbb{B}$ we will denote $\mathsf{Distrib}(\mathbb{A}, \mathbb{B})$ for the set of distributors with domain $\mathbb{A}$ and codomain $\mathbb{B}$; that is to say,

$$\mathsf{Distrib}(\mathbb{A}, \mathbb{B}) = \{\Phi \in \mathsf{Matr}(\mathcal{Q})(\mathbb{A}_0, \mathbb{B}_0) \mid \Phi \circ \mathbb{A} \leq \Phi, \mathbb{B} \circ \Phi \leq \Phi\}.$$

Proposition 4.1.6 *For any two $\mathcal{Q}$-semicategories $\mathbb{A}$ and $\mathbb{B}$,* $\mathsf{Distrib}(\mathbb{A}, \mathbb{B})$ *is a sub-sup-lattice of* $\mathsf{Matr}(\mathcal{Q})$. *Moreover, considering the free categories $\overline{\mathbb{A}}$ and $\overline{\mathbb{B}}$ as in 4.1.5 we have that*

$$\mathsf{Distrib}(\mathbb{A}, \mathbb{B}) = \mathsf{Dist}(\mathcal{Q})(\overline{\mathbb{A}}, \overline{\mathbb{B}}); \tag{4.2}$$

that is to say, a matrix $\Phi\colon \mathbb{A}_0 \longrightarrow \mathbb{B}_0$ is a distributor between the semicategories $\mathbb{A}$ and $\mathbb{B}$ if and only if the same *matrix is a distributor between the categories $\overline{\mathbb{A}}$ and $\overline{\mathbb{B}}$.*

Proof: The first part is elementary: the supremum of a (possibly empty) family of matrices from $\mathbb{A}_0$ to $\mathbb{B}_0$ that happen to be distributors from $\mathbb{A}$ to $\mathbb{B}$, is again such a distributor. The second part follows from 1.2.10 (which is elementary too), since, as endo-matrices on the object set $\mathbb{A}_0$, $\overline{\mathbb{A}} = \mathbb{A} \bigvee \Delta_{\mathbb{A}_0}$. □

Unfortunately, these sup-lattices of distributors between semicategories are not the hom-lattices of some "quantaloid of all semicategories and distributors". The reason is that, even though the "linear algebra formula" for the composition of distributors (2.3) still makes sense, and even though any $\mathcal{Q}$-semicategory $\mathbb{A}$ still determines a distributor $\mathbb{A}(-,-)\colon \mathbb{A} \longrightarrow\!\!\!\!\!\!\circ\;\; \mathbb{A}$ on itself, this endo-distributor is not the *identity* distributor for the composition: for example, take $\Phi \in \mathsf{Distrib}(\mathbb{A}, \mathbb{B})$

and $a \in \mathbb{A}_0$, $b \in \mathbb{B}_0$, then

$$\begin{aligned}(\Phi \otimes_{\mathbb{A}} \mathbb{A})(b,a) &\stackrel{\text{def.}}{=} \bigvee_{x \in \mathbb{A}_0} \Phi(b,x) \circ \mathbb{A}(x,a) \\ &\leq \Phi(b,a)\end{aligned}$$

but this inequality is in general not an equality[2]!

In the next section we discuss *in extenso* the notion of 'presheaf on a semicategory'. We will essentially use 4.1.5 and 4.1.6, because they allow us to say something about *semicategories* by considering the *free categories*.

4.2 Presheaves on, and regularity of, a semicategory

Presheaves

The category of contravariant presheaves on a $\mathcal{Q}$-category $\mathbb{C}$, which we denoted $\mathcal{P}\mathbb{C}$, was constructed with the purpose of classifying distributors with codomain $\mathbb{C}$—see p. 99:

$$\mathsf{Dist}(\mathcal{Q})(-,\mathbb{C}) \cong \mathsf{Cat}(\mathcal{Q})(-,\mathcal{P}\mathbb{C}).$$

It is natural to ask whether for a $\mathcal{Q}$-semicategory $\mathbb{B}$ a similar construction is possible: is there a new $\mathcal{Q}$-categorical structure that classifies distributors with codomain $\mathbb{B}$? But using (4.1) and (4.2) it is easily seen that

$$\begin{aligned}\mathsf{Distrib}(\mathbb{A},\mathbb{B}) &= \mathsf{Dist}(\mathcal{Q})(\overline{\mathbb{A}},\overline{\mathbb{B}}) \\ &\cong \mathsf{Cat}(\mathcal{Q})(\overline{\mathbb{A}},\mathcal{P}\overline{\mathbb{B}}) \\ &= \mathsf{SCat}(\mathcal{Q})(\mathbb{A},\mathcal{P}\overline{\mathbb{B}}),\end{aligned}$$

so that actually the presheaf category on the free category $\overline{\mathbb{B}}$ does indeed classify distributors with codomain $\mathbb{B}$: every distributor $\Phi\colon \mathbb{A} \!-\!\!\!\circ\!\!\!\rightarrow \mathbb{B}$ between semicategories corresponds to a unique morphism $F\colon \mathbb{A} \longrightarrow \mathcal{P}\overline{\mathbb{B}}$, and *vice versa*.

Definition 4.2.1 *A* (contravariant) presheaf *on a $\mathcal{Q}$-semicategory $\mathbb{B}$ is an object of the $\mathcal{Q}$-category $\mathcal{P}\overline{\mathbb{B}}$ of presheaves on the free $\mathcal{Q}$-category $\overline{\mathbb{B}}$.*

That is, a presheaf on $\mathbb{B}$ is a distributor from a one-object $\mathcal{Q}$-category into the $\mathcal{Q}$-category $\overline{\mathbb{B}}$. We will use similar notations for presheaves on semicategories as for those on categories, i.e. $\phi\colon *_A \!-\!\!\!\circ\!\!\!\rightarrow \overline{\mathbb{B}}$, $\psi\colon *_B \!-\!\!\!\circ\!\!\!\rightarrow \overline{\mathbb{B}}$, ... typically denote presheaves on $\mathbb{B}$. Further it is quite obvious that the presheaves on $\mathbb{B}$ organize themselves in the $\mathcal{Q}$-category $\mathcal{P}\overline{\mathbb{B}}$, which we will call the '$\mathcal{Q}$-category of presheaves on the semicategory $\mathbb{B}$' (and which is identical to the $\mathcal{Q}$-category of presheaves on the category $\overline{\mathbb{B}}$).

[2]In the definition of 'regular semicategory' and 'regular distributor' – see p. 143 – it is precisely this problem which is fixed.

Proposition 4.2.2 *The category of presheaves on a $\mathcal{Q}$-semicategory $\mathbb{B}$ "classifies" distributors with codomain $\mathbb{B}$ in the sense that*

$$\mathsf{Distrib}(\mathbb{A},\mathbb{B}) \cong \mathsf{SCat}(\mathcal{Q})(\mathbb{A},\mathscr{P}\overline{\mathbb{B}}) \tag{4.3}$$

for every $\mathcal{Q}$-semicategory $\mathbb{A}$.

Any $\mathcal{Q}$-semicategory is a distributor on itself, like $\mathbb{B}\colon\mathbb{B}\xrightarrow{\circ}\mathbb{B}$. By (4.3) there must be a corresponding morphism from $\mathbb{B}$ to $\mathscr{P}\overline{\mathbb{B}}$; in analogy with the theory of $\mathcal{Q}$-categories we will note this morphism as $Y_{\mathbb{B}}\colon\mathbb{B}\longrightarrow\mathscr{P}\overline{\mathbb{B}}$ and call it the 'Yoneda morphism' for $\mathbb{B}$. Its action on objects is as follows: any object $b\in\mathbb{B}_0$ determines a distributor $\mathbb{B}(-,b)\colon *_{tb}\xrightarrow{\circ}\mathbb{B}$; this distributor in turn determines a distributor $\mathbb{B}(-,b)\colon *_{tb}\xrightarrow{\circ}\overline{\mathbb{B}}$ (which as matrix is identical to $\mathbb{B}(-,b)$!); this latter distributor is the image of b by $Y_{\mathbb{B}}$. That is to say,

$$Y_{\mathbb{B}}\colon\mathbb{B}\longrightarrow\mathscr{P}\overline{\mathbb{B}}\colon b\mapsto(*_{tb}\xrightarrow[\circ]{\mathbb{B}(-,b)}\overline{\mathbb{B}}). \tag{4.4}$$

A word of caution is in order here. By the universal property of the free category $\overline{\mathbb{B}}$ the Yoneda morphism $Y_{\mathbb{B}}\colon\mathbb{B}\longrightarrow\mathscr{P}\overline{\mathbb{B}}$ corresponds to a functor

$$\overline{Y_{\mathbb{B}}}\colon\overline{\mathbb{B}}\longrightarrow\mathscr{P}\overline{\mathbb{B}}.$$

On the other hand, there is the (true and honest) Yoneda embedding

$$Y_{\overline{\mathbb{B}}}\colon\overline{\mathbb{B}}\longrightarrow\mathscr{P}\overline{\mathbb{B}}$$

(for the $\mathcal{Q}$-category $\overline{\mathbb{B}}$)... But the functors $\overline{Y_{\mathbb{B}}}$ and $Y_{\overline{\mathbb{B}}}$ are very different!

Yoneda presheaves

For presheaves on a $\mathcal{Q}$-enriched category $\mathbb{C}$ the important "Yoneda lemma" holds (see 3.2.3 on p. 101): for any $\phi\in(\mathscr{P}\mathbb{C})_0$ and $c\in\mathbb{C}_0$, $Y_{\mathbb{C}}\colon\mathbb{C}\longrightarrow\mathscr{P}\mathbb{C}$ satisfies

$$\mathscr{P}\mathbb{C}(Y_{\mathbb{C}}c,\phi)=\phi(c).$$

As a consequence, the Yoneda embedding $Y_{\mathbb{C}}\colon\mathbb{C}\longrightarrow\mathscr{P}\mathbb{C}$ is fully faithful.

For a $\mathcal{Q}$-semicategory $\mathbb{A}$, the Yoneda morphism $Y_{\mathbb{A}}\colon\mathbb{A}\longrightarrow\mathscr{P}\overline{\mathbb{A}}$ cannot be fully faithful, unless $\mathbb{A}$ is actually a category (because its codomain is a category). Even worse, the "Yoneda lemma" simply does not hold in general for presheaves on $\mathbb{A}$! Therefore, those presheaves for which the Yoneda lemma happens to hold, deserve some special attention.

Definition 4.2.3 *A* Yoneda presheaf *on a $\mathcal{Q}$-semicategory $\mathbb{A}$ is a presheaf $\phi\in\mathscr{P}\overline{\mathbb{A}}$ satisfying, for all $a\in\mathbb{A}_0$,*

$$\mathscr{P}\overline{\mathbb{A}}(Y_{\mathbb{A}}a,\phi)=\phi(a)\ \textit{in}\ \mathcal{Q}.$$

$$\begin{tikzcd} *_A \arrow[r, "\phi", "\circ" marking] \arrow[dr, "\circ" marking, "{\phi = \left[\mathbb{A}, \phi\right]}"'] & \overline{\mathbb{A}} \\ & \overline{\mathbb{A}} \arrow[u, "\circ" marking, "\mathbb{A}"'] \end{tikzcd}$$

Diagram 4.1

$$\begin{tikzcd} \mathbb{A} \arrow[r, "Y_{\mathbb{A}}"] \arrow[dr, dotted, "Y^y_{\mathbb{A}}"'] & \mathscr{P}\overline{\mathbb{A}} \\ & \mathscr{P}^y\mathbb{A} \arrow[u, "\text{full embedding}"'] \end{tikzcd}$$

Diagram 4.2

The following characterization of Yoneda presheaves will be useful later on.

Proposition 4.2.4 *Given a $\mathscr{Q}$-semicategory $\mathbb{A}$, consider it via (4.2) as an endo-distributor on $\overline{\mathbb{A}}$; a presheaf $\phi\colon *_A \multimap \overline{\mathbb{A}}$ on $\mathbb{A}$ is Yoneda if and only if*

$$\left[\mathbb{A}, \phi\right] = \phi \textit{ in } \mathsf{Dist}(\mathscr{Q}) \textit{ (see diagram 4.1).}$$

Proof: By definition of $\mathscr{P}\overline{\mathbb{A}}(-,-)$ we have for every $a \in \mathbb{A}_0$, that

$$\left[\mathbb{A}, \phi\right](a) = \mathscr{P}\overline{\mathbb{A}}(Y_{\mathbb{A}}a, \phi)$$

so it is then clear that $\left[\mathbb{A}, \phi\right] = \phi \iff \forall a \in \mathbb{A}_0 : \mathscr{P}\overline{\mathbb{A}}(Y_{\mathbb{A}}a, \phi) = \phi(a)$. $\square$

We will denote by $\mathscr{P}^y\mathbb{A}$ the "full subcategory" of $\mathscr{P}\overline{\mathbb{A}}$ determined by the Yoneda presheaves on $\mathbb{A}$.

Proposition 4.2.5 *For a $\mathscr{Q}$-semicategory $\mathbb{A}$ the following are equivalent:*

1. *$\mathbb{A}$ is a category;*
2. *all representable presheaves on $\mathbb{A}$ are Yoneda presheaves;*
3. *$Y_{\mathbb{A}}\colon \mathbb{A} \longrightarrow \mathscr{P}\overline{\mathbb{A}}$ factors over the full embedding $\mathscr{P}^y\mathbb{A} \longrightarrow \mathscr{P}\mathbb{A}$ as in diagram 4.2.*

Proof: For $a \in \mathbb{A}$, $Y_{\mathbb{A}}a$ is a Yoneda presheaf if and only if

$$\forall x \in \mathbb{A}_0 : \ (Y_{\mathbb{A}}a)(x) = \mathscr{P}\overline{\mathbb{A}}(Y_{\mathbb{A}}x, Y_{\mathbb{A}}a)$$

so that, when this is the case, in particular

$$\mathbb{A}(a,a) = (Y_{\mathbb{A}}a)(a) = \mathscr{P}\overline{\mathbb{A}}(Y_{\mathbb{A}}a, Y_{\mathbb{A}}a) \geq 1_{ta}.$$

The statements above now follow straightforwardly. $\square$

Regular presheaves

For a $\mathscr{Q}$-category $\mathbb{C}$ we know that every object of $\mathscr{P}\mathbb{C}$ is a "colimit of representables" (see 3.2.5 on p. 103): for $\phi \in \mathscr{P}\mathbb{C}$ we have verified that

$$\mathrm{colim}(\phi, Y_{\mathbb{C}}) = \Delta\phi.$$

Consider now a $\mathscr{Q}$-semicategory $\mathbb{A}$, and its Yoneda morphism $Y_{\mathbb{A}} \colon \mathbb{A} \longrightarrow \mathscr{P}\overline{\mathbb{A}}$. We may – via the induced functor $\overline{Y_{\mathbb{A}}}$ – consider, for any presheaf $\phi \in \mathscr{P}\overline{\mathbb{A}}$, the weighted diagram

$$*_A \xrightarrow{\ \phi\ } \overline{\mathbb{A}} \xrightarrow{\ \overline{Y_{\mathbb{A}}}\ } \mathscr{P}\overline{\mathbb{A}}$$

which must admit a colimit since $\mathscr{P}\overline{\mathbb{A}}$ is cocomplete. But... this colimit is in general different from (the constant functor pointing at) ϕ! Therefore, those presheaves on a $\mathscr{Q}$-semicategory that are indeed "colimit of representables" merit a name of their own.

Definition 4.2.6 *A* regular presheaf *on a $\mathscr{Q}$-semicategory $\mathbb{A}$ is a presheaf, like $\phi \colon *_A \longrightarrow \overline{\mathbb{A}}$, for which* $\mathrm{colim}(\phi, \overline{Y_{\mathbb{A}}}) = \Delta\phi$.

As we actually know how to compute colimits in $\mathscr{P}\overline{\mathbb{A}}$ – cf. 3.2.4 – we may characterize the regular presheaves on a $\mathscr{Q}$-semicategory in the following way.

Proposition 4.2.7 *Consider a given $\mathscr{Q}$-semicategory $\mathbb{A}$ as endo-distributor on $\overline{\mathbb{A}}$ via (4.2). A presheaf $\phi \colon *_A \longrightarrow \overline{\mathbb{A}}$ on $\mathbb{A}$ is regular if and only if*

$$\mathbb{A} \otimes_{\overline{\mathbb{A}}} \phi = \phi \textit{ in } \mathsf{Dist}(\mathscr{Q}) \textit{ (see diagram 4.3).}$$

Proof: By 3.2.4 it follows that $\mathrm{colim}(\phi, \overline{Y_{\mathbb{A}}})$ (exists and) is given by the constant functor $*_A \longrightarrow \mathscr{P}\overline{\mathbb{A}}$ pointing at the composition in $\mathsf{Dist}(\mathscr{Q})$ of the following distributors:

$$*_A \xrightarrow{\ \phi\ } \overline{\mathbb{A}} \xrightarrow{\ \Theta\ } \overline{\mathbb{A}},$$

$$\begin{tikzcd}
*_A \arrow[r, "\phi", "\circ" marking] \arrow[dr, "\circ" marking, "\phi = \mathbb{A} \otimes_{\overline{\mathbb{A}}} \phi"'] & \overline{\mathbb{A}} \arrow[d, "\circ" marking, "\mathbb{A}"] \\
& \overline{\mathbb{A}}
\end{tikzcd}$$

Diagram 4.3

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "Y_{\mathbb{A}}"] \arrow[dr, dotted, "Y^{\star}_{\mathbb{A}}"'] & \mathscr{P}\overline{\mathbb{A}} \\
& \mathscr{P}^{\star}\mathbb{A} \arrow[u, "\text{full embedding}"']
\end{tikzcd}$$

Diagram 4.4

where $\Theta\colon \overline{\mathbb{A}} \mathrel{-\!\!\circ\!\!\rightarrow} \overline{\mathbb{A}}$ corresponds to $\overline{Y_{\mathbb{A}}}\colon \overline{\mathbb{A}} \longrightarrow \mathscr{P}\overline{\mathbb{A}}$ by the universal property of $\mathscr{P}\overline{\mathbb{A}}$... that is to say, $\Theta = \mathbb{A}$ as element of $\mathsf{Dist}(\mathscr{Q})(\overline{\mathbb{A}}, \overline{\mathbb{A}})$! So

$$\mathrm{colim}(\phi, \overline{Y_{\mathbb{A}}}) = \Delta\Big(\mathbb{A} \otimes_{\overline{\mathbb{A}}} \phi\Big),$$

and the result follows. $\square$

By $\mathscr{P}^{\star}\mathbb{A}$ we will denote the "full subcategory" of $\mathscr{P}\overline{\mathbb{A}}$ determined by the regular presheaves.

Proposition 4.2.8 *For a $\mathscr{Q}$-semicategory $\mathbb{A}$ the following are equivalent:*

1. *all representable presheaves on $\mathbb{A}$ are regular presheaves;*
2. *$Y_{\mathbb{A}}\colon \mathbb{A} \longrightarrow \mathscr{P}\overline{\mathbb{A}}$ factors over the full embedding $\mathscr{P}^{\star}\mathbb{A} \longrightarrow \mathscr{P}\overline{\mathbb{A}}$ as in diagram 4.4.*

Proof: This is really a tautology. $\square$

Definition 4.2.9 *A $\mathscr{Q}$-semicategory $\mathbb{A}$ satisfying the (equivalent) conditions of 4.2.8 is* regular.

Corollary 4.2.10 *A $\mathscr{Q}$-enriched semicategory $\mathbb{A}$ is regular if and only if, when viewed as endo-distributor on $\overline{\mathbb{A}}$ via (4.2),*

$$\mathbb{A} \otimes_{\overline{\mathbb{A}}} \mathbb{A} = \mathbb{A}.$$

Proof: For $a \in \mathbb{A}_0$, $Y_{\mathbb{A}}a = \mathbb{A}(-,a)\colon *_{ta} \xrightarrow{\;\circ\;} \overline{\mathbb{A}}$ is a regular presheaf if and only if (via 4.2.7) $\mathbb{A} \otimes_{\overline{\mathbb{A}}} Y_{\mathbb{A}}a = Y_{\mathbb{A}}a$, or equivalently

$$\bigvee_{x \in (\overline{\mathbb{A}})_0} \mathbb{A}(-,x) \circ \mathbb{A}(x,a) = \mathbb{A}(-,a).$$

$\mathbb{A}$ is a regular $\mathcal{Q}$-semicategory if and only if this equation holds for all $a \in \mathbb{A}_0$, and moreover $(\overline{\mathbb{A}})_0 = \mathbb{A}_0$, so indeed

$$\bigvee_{x \in (\overline{\mathbb{A}})_0} \mathbb{A}(-,x) \circ \mathbb{A}(x,?) = \mathbb{A}(-,?)$$

is necessary and sufficient. $\square$

Every $\mathcal{Q}$-category is a regular semicategory, but the converse is not true. And neither is every $\mathcal{Q}$-semicategory regular. So the notion of 'regular $\mathcal{Q}$-semicategory' is strictly weaker than that of '$\mathcal{Q}$-category', but still *exactly* strong enough to allow for an interesting notion of "presheaf which is the colimit of representables".

Unity and identity of opposites

Theorem 4.2.11 *For a regular $\mathcal{Q}$-semicategory $\mathbb{A}$ the full embedding*

$$i\colon \mathcal{P}^{\star}\mathbb{A} \longrightarrow \mathcal{P}\overline{\mathbb{A}}\colon \phi \mapsto \phi$$

is left adjoint to

$$j\colon \mathcal{P}\overline{\mathbb{A}} \longrightarrow \mathcal{P}^{\star}\mathbb{A}\colon \psi \mapsto \mathbb{A} \otimes_{\overline{\mathbb{A}}} \psi$$

which in turn is left adjoint to

$$k\colon \mathcal{P}^{\star}\mathbb{A} \longrightarrow \mathcal{P}\overline{\mathbb{A}}\colon \theta \mapsto [\mathbb{A}, \theta].$$

Moreover, i and k are fully faithful, and the image of k is precisely $\mathcal{P}^{y}\mathbb{A}$. (In the above map-prescriptions, $\mathbb{A}$ is seen as endo-distributor on $\overline{\mathbb{A}}$ via (4.2).)

Proof: Let us first check that i, j and k are indeed functors between $\mathcal{Q}$-categories:

- For i this is trivial.
- That j is a well-defined object-mapping follows from 4.2.7 and 4.2.10: since

$$\mathbb{A} \otimes_{\overline{\mathbb{A}}} \left(\mathbb{A} \otimes_{\overline{\mathbb{A}}} \psi\right) = \left(\mathbb{A} \otimes_{\overline{\mathbb{A}}} \mathbb{A}\right) \otimes_{\overline{\mathbb{A}}} \psi = \mathbb{A} \otimes_{\overline{\mathbb{A}}} \psi$$

  i.e. $\mathbb{A} \otimes_{\overline{\mathbb{A}}} \psi$ is a regular presheaf; and clearly j preserves types. As for its action-inequalities on hom-arrows: for $\phi, \psi \in (\mathcal{P}\overline{\mathbb{A}})_0$,

$$\begin{aligned} & \mathcal{P}\overline{\mathbb{A}}(\psi,\phi) \leq \mathcal{P}^{\star}\mathbb{A}(j\psi, j\phi) \text{ in } \mathcal{Q} \\ \iff\ & [\psi,\phi] \leq [\mathbb{A} \otimes_{\overline{\mathbb{A}}} \psi, \mathbb{A} \otimes_{\overline{\mathbb{A}}} \phi] \text{ in } \mathsf{Dist}(\mathcal{Q}) \\ \iff\ & \mathbb{A} \otimes_{\overline{\mathbb{A}}} \psi \otimes_{*_{t\psi}} [\psi,\phi] \leq \mathbb{A} \otimes_{\overline{\mathbb{A}}} \phi \text{ in } \mathsf{Dist}(\mathcal{Q}) \end{aligned}$$

  which is obviously true.

- Also for k it is trivial that it is a well-defined object mapping. And for $\phi, \psi \in (\mathscr{P}^\star\mathbb{A})_0$,

$$\begin{aligned}
&\mathscr{P}^\star\mathbb{A}(\psi,\phi) \leq \mathscr{P}\overline{\mathbb{A}}([\mathbb{A},\psi],[\mathbb{A},\phi]) \text{ in } \mathscr{Q}\\
&\iff [\psi,\phi] \leq \Big[[\mathbb{A},\psi],[\mathbb{A}\phi]\Big] \text{ in } \mathsf{Dist}(\mathscr{Q})\\
&\iff [\mathbb{A},\psi] \otimes_{*_{t\psi}} [\psi,\phi] \leq [\mathbb{A},\phi] \text{ in } \mathsf{Dist}(\mathscr{Q})
\end{aligned}$$

  which holds by 1.2.6, so k is a functor too.

Next we check the adjunctions:

- $i \dashv j$ means that for all $\phi \in \mathscr{P}^\star\mathbb{A}$ and all $\psi \in \mathscr{P}\overline{\mathbb{A}}$

$$\mathscr{P}\overline{\mathbb{A}}(i\phi,\psi) = \mathscr{P}^\star\mathbb{A}(\phi, j\psi) \text{ in } \mathscr{Q},$$

  or equivalently $[\phi,\psi] = [\phi, \mathbb{A} \otimes_{\overline{\mathbb{A}}} \psi]$ in $\mathsf{Dist}(\mathscr{Q})$. But "$\leq$" is implied by the regularity of ϕ as follows:

$$\begin{aligned}
&\phi \otimes_{*_{t\phi}} [\phi,\psi] \leq \psi\\
&\Longrightarrow \mathbb{A} \otimes_{\overline{\mathbb{A}}} \phi \otimes_{*_{t\phi}} [\phi,\psi] \leq \mathbb{A} \otimes_{\overline{\mathbb{A}}} \psi\\
&\Longrightarrow \phi \otimes_{*_{t\phi}} [\phi,\psi] \leq \mathbb{A} \otimes_{\overline{\mathbb{A}}} \psi\\
&\Longrightarrow [\phi,\psi] \leq [\psi, \mathbb{A} \otimes_{\overline{\mathbb{A}}} \psi];
\end{aligned}$$

  and "$\geq$" is implied by $\mathbb{A} \otimes_{\overline{\mathbb{A}}} \psi \leq \overline{\mathbb{A}} \otimes_{\overline{\mathbb{A}}} \psi \leq \psi$ and the isotonicity of $[\phi, -]$.

- $j \dashv k$ means that for all $\psi \in \mathscr{P}\overline{\mathbb{A}}$ and all $\theta \in \mathscr{P}^\star\mathbb{A}$,

$$\mathscr{P}^\star\mathbb{A}(j\psi,\theta) = \mathscr{P}\overline{\mathbb{A}}(\psi, k\theta) \text{ in } \mathscr{Q},$$

  or equivalently $[\mathbb{A} \otimes_{\overline{\mathbb{A}}} \psi, \theta] = \Big[\psi, [\mathbb{A},\theta]\Big]$ in $\mathsf{Dist}(\mathscr{Q})$. But the latter identity is valid in any quantaloid – see 1.2.5 – thus also in $\mathsf{Dist}(\mathscr{Q})$.

The fully faithfulness of (the full inclusion) i is trivial; and that of k thus follows from the adjunctions—see 2.4.15.
Finally, we need to show that $\phi \in \mathscr{P}^y\mathbb{A}$ if and only if there exists a $\psi \in \mathscr{P}^\star\mathbb{A}$ such that $\phi = k\psi$, i.e. $\phi = [\mathbb{A}, \psi-]$ in $\mathsf{Dist}^\star(\mathscr{Q})$. To relate this expression to the adjunctions $i \dashv j \dashv k$, note first that

$$\phi = [\mathbb{A},\psi] \iff \phi = \mathscr{P}\overline{\mathbb{A}}(Y_{\mathbb{A}}-,\psi).$$

We now show necessity and sufficiency of this condition for ϕ to be 'Yoneda':

- for a $\phi \in \mathscr{P}^y\mathbb{A}$ we have that $\forall a \in \mathbb{A}_0 :\ \mathscr{P}\overline{\mathbb{A}}(Y_{\mathbb{A}}a,\phi) = \phi(a)$
  and moreover we know that

$$\begin{aligned}
\mathscr{P}\overline{\mathbb{A}}(Y_{\mathbb{A}}a,\phi) &= \mathscr{P}\overline{\mathbb{A}}(i(Y_{\mathbb{A}}),\phi)\\
&= \mathscr{P}^\star\mathbb{A}(Y_{\mathbb{A}}a, j\phi)\\
&= \mathscr{P}\overline{\mathbb{A}}(Y_{\mathbb{A}}a, j\phi)\\
&= k(j(\phi))(a)
\end{aligned}$$

hence $\phi = k(j(\phi))$, and $j(\phi) \in \mathscr{P}^\star\mathbb{A}$;

- conversely, suppose that $\phi = \mathscr{P}\overline{\mathbb{A}}(Y_{\mathbb{A}}-,\psi)$ for some $\psi \in \mathscr{P}^\star\mathbb{A}$, then

$$\begin{aligned}\mathscr{P}\overline{\mathbb{A}}(Y_{\mathbb{A}}a,\phi) &= \mathscr{P}\overline{\mathbb{A}}(Y_{\mathbb{A}}a,k(\psi)) \\ &= \mathscr{P}^\star\mathbb{A}(j(Y_{\mathbb{A}}a),\psi) \\ &= \mathscr{P}^\star\mathbb{A}(Y_{\mathbb{A}}a,\psi) \\ &= \phi(a)\end{aligned}$$

where we used that $j(Y_{\mathbb{A}}a) = Y_{\mathbb{A}}a$—i.e. that $\mathbb{A}$ is regular!

This concludes the proof! $\square$

By 4.2.11 and 3.1.8 we may record that...

Corollary 4.2.12 *For a regular $\mathcal{Q}$-semicategory $\mathbb{A}$, the $\mathcal{Q}$-category $\mathscr{P}^\star\mathbb{A}$ is cocomplete.*

Similarly to 4.2.11 also the following can be shown.

Theorem 4.2.13 *For a regular $\mathcal{Q}$-semicategory $\mathbb{A}$ the full inclusion functor*

$$i'\colon \mathscr{P}^y\mathbb{A} \longrightarrow \mathscr{P}\overline{\mathbb{A}}\colon \phi \mapsto \phi$$

is right adjoint to

$$j'\colon \mathscr{P}\overline{\mathbb{A}} \longrightarrow \mathscr{P}^y\mathbb{A}\colon \psi \mapsto [\mathbb{A},\psi]$$

which in turn is right adjoint to

$$k'\colon \mathscr{P}^y\mathbb{A} \longrightarrow \mathscr{P}\overline{\mathbb{A}}\colon \theta \mapsto \mathbb{A} \otimes_{\overline{\mathbb{A}}} \theta.$$

Moreover, i' and k' are fully faithful, and the image of k' is precisely $\mathscr{P}^\star\mathbb{A}$.

Corollary 4.2.14 *For a regular $\mathcal{Q}$-semicategory $\mathbb{A}$, the $\mathcal{Q}$-category $\mathscr{P}^y\mathbb{A}$ is cocomplete.*

4.3 Alternative descriptions of presheaves and regularity

Until now we have dealt with presheaves on a $\mathcal{Q}$-semicategory $\mathbb{A}$ by passing over to the free $\mathcal{Q}$-category $\overline{\mathbb{A}}$. This is both an advantage and a disadvantage: the advantage is that we can apply the theory of $\mathcal{Q}$-categories as developed in previous chapters; the disadvantage is that this theory of $\mathcal{Q}$-categories had to be developed and understood in the first place. But the use of $\mathcal{Q}$-categories can be avoided—that is, there is a more *ad hoc* way of studying presheaves on a $\mathcal{Q}$-semicategory.

Working with underlying matrices

A contravariant presheaf on a $\mathcal{Q}$-semicategory $\mathbb{A}$ is, by definition, an object of $\mathscr{P}\overline{\mathbb{A}}$, i.e. an element of $\mathsf{Dist}(\mathcal{Q})(*_A, \overline{\mathbb{A}})$ for some $A \in \mathcal{Q}_0$. But by (4.2) such a distributor (between $\mathcal{Q}$-categories) is equivalently given by an element of $\mathsf{Distrib}(*_A, \mathbb{A})$ (a distributor between $\mathcal{Q}$-semicategories). Moreover, given two presheaves $\phi, \psi \in \mathscr{P}\overline{\mathbb{A}}$, there is no difference between the lifting $[\psi, \phi]$ as calculated in $\mathsf{Dist}(\mathcal{Q})$ or in $\mathsf{Matr}(\mathcal{Q})$—this follows from 2.2.11.

These observations now result in the following alternative description for the presheaf category on a $\mathcal{Q}$-semicategory.

Proposition 4.3.1 *For a $\mathcal{Q}$-semicategory $\mathbb{A}$, let $\mathscr{P}\mathbb{A}$ denote the $\mathcal{Q}$-enriched structure with*

- *the object-set $(\mathscr{P}\mathbb{A})_0 = \{\phi \in \mathsf{Distrib}(*_A, \mathbb{A}) \mid A \in \mathcal{Q}_0\}$ and the types of the elements are $t(*_A \xrightarrow{\phi} \mathbb{A}) = A$;*
- *for any $\phi, \psi \in (\mathscr{P}\mathbb{A})_0$ the hom-arrow*

$$\mathscr{P}\mathbb{A}(\psi, \phi) = \text{ single element of } \Big[\psi, \phi\Big]^{\mathsf{Matr}(\mathcal{Q})}.$$

Then $\mathscr{P}\mathbb{A} = \mathscr{P}\overline{\mathbb{A}}$ as $\mathcal{Q}$-enriched categories.

In particular is the Yoneda morphism now given as

$$Y_{\mathbb{A}} \colon \mathbb{A} \longrightarrow \mathscr{P}\mathbb{A} \colon b \mapsto (*_{tb} \xrightarrow{\mathbb{A}(-,b)} \mathbb{A}).$$

It is not a great surprise now that also the notions of 'Yoneda presheaf' and 'regular presheaf' on a $\mathcal{Q}$-semicategory can be restated in terms of matrices between semicategories (rather than distributors between categories).

Lemma 4.3.2 *Let $\mathbb{A}$ be a $\mathcal{Q}$-semicategory and $\phi \in \mathsf{Distrib}(*_A, \mathbb{A})$. Then*

$$[\mathbb{A}, \phi] = \phi \text{ in } \mathsf{Dist}(\mathcal{Q}) \iff [\mathbb{A}, \phi] = \phi \text{ in } \mathsf{Matr}(\mathcal{Q});$$
$$\mathbb{A} \otimes_{\overline{\mathbb{A}}} \phi = \phi \text{ in } \mathsf{Dist}(\mathcal{Q}) \iff \mathbb{A} \circ \phi = \phi \text{ in } \mathsf{Matr}(\mathcal{Q}).$$

where, when working in $\mathsf{Dist}(\mathcal{Q})$, $\mathbb{A}$ is viewed as endo-distributor on $\overline{\mathbb{A}}$ and ϕ as element of $\mathsf{Dist}(\mathcal{Q})(*_A, \overline{\mathbb{A}})$, and, when working in $\mathsf{Matr}(\mathcal{Q})$, $\mathbb{A}$ is viewed as endo-matrix on $\mathbb{A}_0$ and ϕ as element of $\mathsf{Matr}(*_A, \mathbb{A}_0)$.

Proof: The equivalences follow from 2.2.11 and 2.2.7: lifting and composition in $\mathsf{Dist}(\mathcal{Q}) = \mathsf{Bim}(\mathsf{Matr}(\mathcal{Q}))$ are calculated as in $\mathsf{Matr}(\mathcal{Q})$. $\square$

Proposition 4.3.3 *For a $\mathcal{Q}$-semicategory $\mathbb{A}$, the $\mathcal{Q}$-category of Yoneda presheaves $\mathscr{P}^y\mathbb{A}$ may alternatively be described as the full subcategory of $\mathscr{P}\mathbb{A}$ whose objects are those $\phi \in (\mathscr{P}\mathbb{A})_0$ for which*

$$\left[\mathbb{A}, \phi\right] = \phi \textit{ in } \mathsf{Matr}(\mathcal{Q}).$$

And the $\mathcal{Q}$-category of regular presheaves $\mathscr{P}^\star\mathbb{A}$ may be described as the full subcategory of $\mathscr{P}\mathbb{A}$ whose of objects are those $\phi \in (\mathscr{P}\mathbb{A})_0$ for which

$$\mathbb{A} \circ \phi = \phi \textit{ in } \mathsf{Matr}(\mathcal{Q}).$$

Proof: Using 4.3.2 we "translated" the necessary and sufficient conditions in 4.2.4 and 4.2.7. □

The following corollary must be compared with 4.2.10.

Corollary 4.3.4 *A $\mathcal{Q}$-semicategory $\mathbb{A}$ is regular if and only if*

$$\mathbb{A} \circ \mathbb{A} = \mathbb{A} \textit{ in } \mathsf{Matr}(\mathcal{Q}),$$

i.e. it is an idempotent matrix.

It is now quite obvious how also the results on p. 127 and further can be translated in terms of "underlying matrices"; we will not repeat all that.

The observation in 4.3.4 – although trivial – is very important for the rest of this chapter: it identifies *the theory of $\mathcal{Q}$-semicategories* as that of idempotents in the quantaloid $\mathsf{Matr}(\mathcal{Q})$. But before we turn to a systematic study of "idempotents in a quantaloid", we will first explain how the notions of 'regular presheaf' on a $\mathcal{Q}$-semicategory, and *a fortiori* 'regular $\mathcal{Q}$-semicategory', correspond to those that were introduced (under the same name) in M.-A. Moens' thesis (and thus we give credit where it is due).

Regularity à la Moens

The point is the following: for any $\mathcal{Q}$-semicategory $\mathbb{A}$ and $\phi \in \mathsf{Distrib}(*_A, \mathbb{A})$ (a presheaf on $\mathbb{A}$ in the sense of 4.3.1) we may consider a "weighted diagram without units" in the $\mathcal{Q}$-category $\mathscr{P}\mathbb{A}$ (again as in 4.3.1) as follows:

$$*_A \xrightarrow{\ \phi\ } \mathbb{A} \xrightarrow{\ Y_\mathbb{A}\ } \mathscr{P}\mathbb{A}.$$

We wish to show that ϕ is a regular presheaf on $\mathbb{A}$ (in the sense of 4.3.3) if and only if it is the colimit of this diagram... But then we first need to define the notion of a "weighted colimit of a diagram without units"!

In general, let $\mathbb{A}$ and $\mathbb{B}$ be $\mathcal{Q}$-semicategories, $\mathbb{C}$ a $\mathcal{Q}$-category, $\Phi\colon \mathbb{B} \to \mathbb{A}$ a distributor and $F\colon \mathbb{A} \longrightarrow \mathbb{C}$ a morphism of semicategories; this data constitutes a

$$
\begin{tikzcd}
\mathbb{A} \arrow[r, "F"] & \mathbb{C} \\
\mathbb{B} \arrow[u, "\Phi", "\circ" marking]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\overline{\mathbb{A}} \arrow[r, "\overline{F}"] & \mathbb{C} \\
\overline{\mathbb{B}} \arrow[u, "\Phi", "\circ" marking]
\end{tikzcd}
$$

Diagram 4.5

"weighted diagram without units" in the $\mathcal{Q}$-category $\mathbb{C}$, as on the left-hand side of diagram 4.5. To produce a universal cocone on this "weighted diagram without units" we consider the "free weighted diagram with units" as on the right-hand side of 4.5. Here $\overline{F}$ is the functor determined by the universal property of the free $\mathcal{Q}$-category $\overline{\mathbb{A}}$; and recall that (the underlying matrix of) Φ is indeed a distributor between those free categories.

Definition 4.3.5 *Given a distributor* $\Phi\colon \mathbb{A} \!\multimap\! \mathbb{B}$ *between* $\mathcal{Q}$*-semicategories, and a morphism* $F\colon \mathbb{A} \longrightarrow \mathbb{C}$ *whose codomain is a* $\mathcal{Q}$*-category, the* Φ-weighted colimit of F *is – whenever it exists – the (necessarily essentially unique) morphism*

$$\mathrm{colim}(\Phi, F)\colon \mathbb{B} \longrightarrow \mathbb{C}$$

for which, by the universal property of $\overline{\mathbb{B}}$*,*

$$\overline{\mathrm{colim}(\Phi, F)} = \mathrm{colim}(\Phi, \overline{F}) \ \textit{in}\ \mathsf{Cat}(\mathcal{Q})(\overline{\mathbb{B}}, \mathbb{C}).$$

Quite straightforwardly this is a generalization of 3.1.1: when $\Phi\colon \mathbb{A} \!\multimap\! \mathbb{B}$ happens to be a distributor between $\mathcal{Q}$-*categories*, the above definition reduces to the known definition for 'weighted colimit'. So there is no ambiguity when discussing colimits of weighted diagrams "with units". (This follows of course from the fact that $\mathbb{A}$ and $\mathbb{B}$ are $\mathcal{Q}$-categories if and only if $\overline{\mathbb{A}} = \mathbb{A}$ and $\overline{\mathbb{B}} = \mathbb{B}$, in which case F is a functor.)

It is essential that in 4.3.5 we do consider a *category* $\mathbb{C}$ (whereas $\mathbb{A}$ and $\mathbb{B}$ may be semicategories). If $\mathbb{C}$ weren't a category but merely a semicategory then we couldn't necessarily associate to a functor $\overline{G}\colon \overline{\mathbb{B}} \longrightarrow \mathbb{C}$ a morphism $G\colon \mathbb{B} \longrightarrow \mathbb{C}$, and *vice versa.* The universal property of the free category $\overline{\mathbb{B}}$ on a semicategory $\mathbb{B}$

$$\mathsf{Cat}(\mathcal{Q})(\overline{\mathbb{B}}, \mathbb{C}) = \mathsf{SCat}(\mathcal{Q})(\mathbb{B}, \mathbb{C})$$

requires that $\mathbb{C}$ be a $\mathcal{Q}$-category!

The universal property of such a colimit on a diagram without units may be stated without reference to the free categories, i.e. in terms of matrices.

Proposition 4.3.6 *Given $\mathcal{Q}$-semicategories $\mathbb{A}$ and $\mathbb{B}$, and a $\mathcal{Q}$-categorory $\mathbb{C}$, together with a distributor $\Phi\colon\mathbb{A}\mathrel{-\!\!\circ\!\!\!\rightarrow}\mathbb{B}$ and a morphism $F\colon\mathbb{B}\longrightarrow\mathbb{C}$, a morphism $G\colon\mathbb{A}\longrightarrow\mathbb{C}$ is the Φ-weighted colimit of F if and only if*

$$\mathbb{C}(G-,-)=\Big[\Phi,\mathbb{C}(F-,-)\Big]\ \textit{in}\ \mathsf{Matr}(\mathcal{Q}).$$

(Here $\mathbb{C}(G-,-)$, Φ and $\mathbb{C}(F-,-)$ are thought of as matrices.)

Proof: It suffices to make the universal property in 4.3.5 explicit: G is the Φ-weighted colimit of F if and only if

$$\mathbb{C}(\overline{G}-,-)=\Big[\Phi,\mathbb{C}(\overline{F}-,-)\Big]\ \text{in}\ \mathsf{Dist}(\mathcal{Q})...$$

But since $\mathbb{C}(\overline{G}-,-)=\mathbb{C}(G-,-)$ and $\mathbb{C}(\overline{F}-,-)=\mathbb{C}(F-,-)$ *as matrices* (recall that F and $\overline{F}$, resp. G and $\overline{G}$, are determined by the *same* underlying object-mapping!), and moreover the liftings in $\mathsf{Dist}(\mathcal{Q})=\mathsf{Bim}(\mathsf{Matr}(\mathcal{Q}))$ are calculated as in $\mathsf{Matr}(\mathcal{Q})$ (see once more 2.2.11) this is equivalent to the requirement in the statement of this proposition. $\square$

Now, finally, we may explain how the original definition of 'regular presheaf on a semicategory' in [Moens, 2000] corresponds to ours.

Proposition 4.3.7 *A presheaf $\phi\in\mathsf{Distrib}(*_A,\mathbb{A})$ on a $\mathcal{Q}$-semicategory $\mathbb{A}$ is regular (in the sense of 4.3.3) if and only if $\Delta\phi\colon *_A\longrightarrow\mathscr{P}\mathbb{A}\colon *\mapsto\phi$ is the colimit of the weighted diagram without units*

$$*_A\xrightarrow{\ \phi\ }\mathbb{A}\xrightarrow{\ Y_{\mathbb{A}}\ }\mathscr{P}\mathbb{A}.$$

That is to say, for every $\psi\in\mathscr{P}\mathbb{A}$ we have that

$$\mathscr{P}\mathbb{A}(\phi,\psi)=\Big[\phi,\mathscr{P}\mathbb{A}(Y_{\mathbb{A}}-,\psi)\Big]^{\mathsf{Matr}(\mathcal{Q})}.$$

Proof: $\Delta\phi\colon *_A\longrightarrow\mathscr{P}\mathbb{A}$ is the ϕ-weighted colimit of $Y_{\mathbb{A}}\colon\mathbb{A}\longrightarrow\mathscr{P}\mathbb{A}$ if and only if $\overline{\Delta\phi}=\Delta\phi$ is the ϕ-weighted colimit of $\overline{Y_{\mathbb{A}}}\colon\overline{\mathbb{A}}\longrightarrow\mathscr{P}\mathbb{A}=\mathscr{P}\overline{\mathbb{A}}$—which is precisely the regularity of ϕ as previously defined. $\square$

This proposition translates precisely what can be found on pp. 28–30 in M.-A. Moens' [2000] thesis from $\mathscr{V}$-enriched semicategories to $\mathcal{Q}$-enriched semicategories (see also [Moens *et al.*, 2002]).

4.4 Idempotents in a quantaloid

Regular semicategories are strictly weaker $\mathcal{Q}$-enriched structures that are still strong enough to allow for a good behavior of the regular presheaves. We have

$$\begin{tikzcd}
A_e \arrow[r, "p"] \arrow[dr, "1_{A_e}"'] & A \arrow[d, "s"] \arrow[dr, "e"] & \\
& A_e \arrow[r, "p"'] & A
\end{tikzcd}$$

Diagram 4.6

seen in 4.3.4 that such a regular $\mathscr{Q}$-semicategory is precisely an idempotent in the quantaloid $\mathsf{Matr}(\mathscr{Q})$. Therefore we will in this section study idempotents in a quantaloid; and later, in the next section, apply the results to the quantaloid $\mathsf{Matr}(\mathscr{Q})$.

In studying the splitting of idempotents in a quantaloid $\mathscr{Q}$ we may take inspiration from the work already done for the "(free) algebra object completion" $\mathscr{Q} \longrightarrow \mathsf{Bim}(\mathscr{Q})$ on p. 71 and further. Indeed, for a monad $A \circlearrowleft^{t}$ in $\mathscr{Q}$ we wanted a (free) algebra object to exist in $\mathscr{Q}$; but any monad in $\mathscr{Q}$ is an idempotent, and the existence of a (free) algebra object for t is equivalent to the existence of a splitting of this idempotent. We will adapt the construction $\mathscr{Q} \longrightarrow \mathsf{Bim}(\mathscr{Q})$ to a (larger) universal embedding $\mathscr{Q} \longrightarrow \mathsf{Bim}^{\star}(\mathscr{Q})$ to provide the wanted "splitting of idempotents" completion.

The definition itself of a "monad" requires a 2-categorical structure; this is not necessary for "idempotent": the concept makes sense in any (ordinary) category $\mathscr{C}$. We will therefore first study the splitting of idempotents in $\mathscr{C}$, and then consider the case where $\mathscr{C} = \mathscr{Q}$ is a quantaloid.

Splitting idempotents in a category

An idempotent in a category $\mathscr{C}$ is, of course, an endo-arrow $A \circlearrowleft^{e}$ such that $e \circ e = e$. The following is well-known (and was actually used in 2.2.2 on p. 67 on the splitting of monads).

Proposition 4.4.1 *For an idempotent $A \circlearrowleft^{e}$ in a category $\mathscr{C}$, the following are equivalent:*

1. *the diagram $A \circlearrowleft^{e}$ has a limit (i.e. $A \underset{1_A}{\overset{e}{\rightrightarrows}} A$ has an equalizer) in $\mathscr{C}$;*
2. *the diagram $A \circlearrowleft^{e}$ has a colimit (i.e. $A \underset{1_A}{\overset{e}{\rightrightarrows}} A$ has a coequalizer) in $\mathscr{C}$;*
3. *$A \circlearrowleft^{e}$ splits in $\mathscr{C}$, i.e. there exist an object $A_e \in \mathscr{C}_0$ and arrows $A \xrightarrow{s} A_e$ and $A_e \xrightarrow{p} A$ such that diagram 4.6 commutes.*

Corollary 4.4.2 *Any functor preserves the splitting of idempotents.*

$$
\begin{tikzcd}
\mathscr{C} \arrow[r, "F"] \arrow[d, "u_{\mathscr{C}}"'] & \mathscr{C}' \arrow[d, "u_{\mathscr{C}'}"] \\
\mathsf{Bim}^{\star}(\mathscr{C}) \arrow[r, "\mathsf{Bim}^{\star}(F)"'] & \mathsf{Bim}^{\star}(\mathscr{C}')
\end{tikzcd}
$$

Diagram 4.7

We now adapt the bimodules construction for monads in $\mathscr{Q}$ to the case of idempotents in $\mathscr{C}$.

Proposition 4.4.3 *Given a category $\mathscr{C}$, the following defines a new category $\mathsf{Bim}^{\star}(\mathscr{C})$ of "idempotents and regular bimodules" in $\mathscr{C}$:*

- *objects: are all the idempotents $A \circlearrowleft^{e}$, $B \circlearrowleft^{f}$, $C \circlearrowleft^{g}$, ... in $\mathscr{C}$;*
- *arrows: a regular bimodule $b\colon e \,{-\!\!\circ\!\!\rightarrow}\, f$ is a $\mathscr{Q}$-arrow $b\colon A \longrightarrow B$ with saturated actions*
$$\begin{cases} b \circ e = b, \\ f \circ b = b; \end{cases} \tag{4.5}$$
- *composition: write composition as*
$$e \xrightarrow[\circ]{b} f \xrightarrow[\circ]{c} g = e \xrightarrow[\circ]{c \otimes_f b} g$$
and compute it simply as $c \otimes_f b = c \circ b$ in $\mathscr{C}$;
- *identity: on an idempotent $A \circlearrowleft^{e}$, $e\colon e \,{-\!\!\circ\!\!\rightarrow}\, e$ itself is the identity regular bimodule.*

The category $\mathscr{C}$ can be embedded in $\mathsf{Bim}^{\star}(\mathscr{C})$ by sending an object A onto the identity idempotent $A \circlearrowleft^{1_A}$ and an arrow $f\colon A \longrightarrow B$ to the trivial bimodule $f\colon 1_A \,{-\!\!\circ\!\!\rightarrow}\, 1_B$:

$$u_{\mathscr{C}} : \mathscr{C} \longrightarrow \mathsf{Bim}^{\star}(\mathscr{C})\colon (A \xrightarrow{f} A) \mapsto (1_A \xrightarrow[\circ]{f} 1_B) \tag{4.6}$$

Any functor $F\colon \mathscr{C} \longrightarrow \mathscr{C}'$ determines a new functor

$$\mathsf{Bim}^{\star}(F)\colon \mathsf{Bim}^{\star}(\mathscr{C}) \longrightarrow \mathsf{Bim}^{\star}(\mathscr{C}')\colon (e \xrightarrow[\circ]{b} f) \mapsto (Fe \xrightarrow[\circ]{Fb} Ff) \tag{4.7}$$

making diagram 4.7 commute.

Proposition 4.4.4 *For any category $\mathscr{C}$,*

$$\begin{tikzcd}
u_0 \arrow[r, "u_2", "\circ" description] \arrow[dr, "u_0"', "\circ" description] & e \arrow[d, "u_1", "\circ" description] \arrow[dr, "u", "\circ" description] & \\
 & u_0 \arrow[r, "u_2"', "\circ" description] & e
\end{tikzcd}$$

Diagram 4.8

1. *the embedding in (4.6) is a fully faithful functor;*

2. *this embedding is an equivalence of categories if and only if every idempotent in $\mathscr{C}$ splits;*

3. *applying the regular-bimodules construction to* $\mathsf{Bim}^\star(\mathscr{C})$ *itself, the embedding*

$$u_{\mathsf{Bim}^\star(\mathscr{C})} \colon \mathsf{Bim}^\star(\mathscr{C}) \longrightarrow \mathsf{Bim}^\star(\mathsf{Bim}^\star(\mathscr{C}))$$

is always an equivalence;

4. *$\mathscr{C}$ is equivalent to* $\mathsf{Bim}^\star(\mathscr{C}')$ *for some category $\mathscr{C}'$ if and only if all idempotents in $\mathscr{C}$ split;*

5. *the embedding (4.6) describes the universal splitting-of-idempotents completion of $\mathscr{C}$ in* CAT.

Proof: (1) Is obvious.

(2) By (1), the functor in (4.6) is an equivalence if and only if every idempotent $A \circlearrowleft^{e}$ in $\mathscr{C}$ is isomorphic to an identity $B \circlearrowleft^{1_B}$ as objects in $\mathsf{Bim}^\star(\mathscr{C})$! That is, we want $b\colon e \mathrel{-\!\circ\!\!\rightarrow} 1_B$ and $c\colon 1_B \mathrel{-\!\circ\!\!\rightarrow} e$ in $\mathsf{Bim}^\star(\mathscr{C})$ such that $c \otimes_{1_B} b = e$ and $b \otimes_e c = 1_B$. This "translates" to $\mathscr{C}$ as asking for $b\colon A \longrightarrow B$ and $c\colon B \longrightarrow A$ such that $c \circ b = e$ and $b \circ c = 1_B$—that is, a splitting of the idempotent e in $\mathscr{C}$.

(3) We prove that all idempotents in $\mathsf{Bim}^\star(\mathscr{C})$ split. Suppose thus that $e \circlearrowleft^{u}$ is an idempotent in $\mathsf{Bim}^\star(\mathscr{C})$ on an idempotent $A \circlearrowleft^{e}$ in $\mathscr{C}$: $u \otimes_e u = u$. Then, in terms of $\mathscr{C}$-arrows, we have $A \circlearrowleft^{e}$ and $A \circlearrowleft^{u}$ such that $e \circ e = e$, $e \circ u = u$, $u \circ e = u$ and $u \circ u = u$. This means that $A \circlearrowleft^{u}$ is (the underlying $\mathscr{C}$-arrow of) an idempotent in $\mathscr{C}$, but also (the underlying arrow of) a regular bimodule $e \mathrel{-\!\circ\!\!\rightarrow} u$ and a regular bimodule $u \mathrel{-\!\circ\!\!\rightarrow} e$; let us notationally distinguish these by writing $A \circlearrowleft^{u_0}$ for the idempotent, and $u_1\colon e \mathrel{-\!\circ\!\!\rightarrow} u_0$ and $u_2\colon u_0 \mathrel{-\!\circ\!\!\rightarrow} e$ for the bimodules. It is then easily verified that u splits over u_0 in $\mathsf{Bim}^\star(\mathscr{C})$—see diagram 4.8.

(4) One direction is obvious. Conversely, let $F\colon \mathscr{C} \xrightarrow{\sim} \mathsf{Bim}^\star(\mathscr{C}')$ be an equivalence and consider an idempotent $A \circlearrowleft^{e}$ in $\mathscr{C}$. Then surely $FA \circlearrowleft^{Fe}$ is idempotent

$$
\begin{tikzcd}
\mathscr{C} \arrow[r, "F"] \arrow[d, "u_{\mathscr{C}}"'] \arrow[dr, phantom, "\overset{\sim}{\Longrightarrow}"] & \mathsf{Bim}^{\star}(\mathscr{C}') \\
\mathsf{Bim}^{\star}(\mathscr{C}) \arrow[ur, dotted, "\mathsf{Bim}^{\star}(F)"'] &
\end{tikzcd}
$$

Diagram 4.9

in $\mathsf{Bim}^{\star}(\mathscr{C}')$, thus it splits:

$$\text{there exists a splitting } B \underset{f}{\overset{g}{\rightleftarrows}} FA \text{ of } Fe \text{ in } \mathsf{Bim}^{\star}(\mathscr{C}').$$

Using now that F is essentially surjective on objects we may consider an isomorphism $k\colon B \xrightarrow{\sim} FC$ in $\mathsf{Bim}^{\star}(\mathscr{C}')$, for some $C \in \mathscr{C}$. It is then easily verified that

$$\text{also } FC \underset{k \circ f}{\overset{g \circ k^{-1}}{\rightleftarrows}} FA \text{ is a splitting of } Fe \text{ in } \mathsf{Bim}^{\star}(\mathscr{C}').$$

Since F is full we know that

$$\begin{cases} k \circ f = Fx \text{ for some } A \xrightarrow{x} C \text{ in } \mathscr{C}, \\ g \circ k^{-1} = Fy \text{ for some } C \xrightarrow{y} A \text{ in } \mathscr{C}, \end{cases}$$

and by faithfulness of F moreover

$$\begin{cases} y \circ x = e \text{ because } F(y \circ x) = Fy \circ Fx = Fe, \\ x \circ y = 1_C \text{ because } F(x \circ y) = Fx \circ Fy = F(1_C). \end{cases}$$

Thus we obtained a splitting of the idempotent $A \circlearrowleft^{e}$ in $\mathscr{C}$.

(5) We must show that any functor $F\colon \mathscr{C} \longrightarrow \mathscr{C}'$ whose codomain admits splittings of all idempotents, factors through $u_{\mathscr{C}}\colon \mathscr{C} \longrightarrow \mathsf{Bim}^{\star}(\mathscr{C})$ "up to natural isomorphism", the factorization itself being "essentially unique". But by (4) we may replace $\mathscr{C}'$ by the equivalent $\mathsf{Bim}^{\star}(\mathscr{C}')$, and with some abuse of notation it then follows that $\mathsf{Bim}^{\star}(F)\colon \mathsf{Bim}^{\star}(\mathscr{C}) \longrightarrow \mathsf{Bim}^{\star}(\mathscr{C}')$ is at least one such factorization—see diagram 4.9. Suppose now that some functor $G\colon \mathsf{Bim}^{\star}(\mathscr{C}) \longrightarrow \mathsf{Bim}^{\star}(\mathscr{C}')$ is another such factorization: $G \circ u_{\mathscr{C}} \cong F$. This means precisely that G and F (and $\mathsf{Bim}^{\star}(F)$) agree up to natural isomorphism on those objects of $\mathsf{Bim}^{\star}(\mathscr{C})$ which are identities in $\mathscr{C}$, and the regular bimodules between such objects:

$$\forall A \in \mathscr{C}_0 : G(1_A) \cong 1_{FA} = \mathsf{Bim}^{\star}(F)(1_A),$$

$$\forall A, B \in \mathscr{C}_0 : \mathsf{Bim}^{\star}(\mathscr{C}')(G(1_A), G(1_B)) \cong \mathsf{Bim}^{\star}(\mathscr{C}')(1_{FA}, 1_{FB}).$$

Now consider any object e in $\mathsf{Bim}^\star(\mathscr{C})$, i.e. an idempotent $A \overset{e}{\circlearrowleft}$ in $\mathscr{C}$. Since e is the object over which the $\mathsf{Bim}^\star(\mathscr{C})$-idempotent $1_A \overset{e}{\circlearrowleft}$ splits in $\mathsf{Bim}^\star(\mathscr{C})$, Ge must be the object over which $G(1_A) \overset{Ge}{\circlearrowleft}$ splits. But the latter corresponds, via the natural isomorphism $G(1_A) \cong 1_{FA}$, to $1_{FA} \overset{Fe}{\circlearrowleft}$, which itself splits in $\mathsf{Bim}^\star(\mathscr{C}')$ over the $\mathscr{C}'$-idempotent Fe. Since splittings are unique up to isomorphism, it follows that $Ge \cong Fe$ as objects in $\mathsf{Bim}^\star(\mathscr{C}')$. Hence G and $\mathsf{Bim}^\star(F)$ are naturally isomorphic. $\Box$

The following lemma (due to P. Freyd) will be useful later on.

Lemma 4.4.5 *For any category $\mathscr{C}$, if $\mathscr{C}$ admits (co)products then so does the category $\mathsf{Bim}^\star(\mathscr{C})$.*

Proof: For a family of objects $(e_i)_{i\in I}$ in $\mathsf{Bim}^\star(\mathscr{C})$, i.e. a family of $\mathscr{C}$-idempotents $(A_i \overset{e_i}{\circlearrowleft})_{i\in I}$, we may calculate in $\mathscr{C}$, supposing that $\mathscr{C}$ admits products, $A = \prod_i A_i$ with projections $A \xrightarrow{p_i} A_i$. Considering now the family of $\mathscr{C}$-arrows $(\pi_i := e_i \circ p_i \colon A \longrightarrow A_i)_{i\in I}$ there must be a unique factorization $e\colon A \longrightarrow A$ through the $(p_i\colon A \longrightarrow A_i)_{i\in I}$. It is then a routine calculation to see that e is an idempotent in $\mathscr{C}$, and that $(\pi_i\colon e \xrightarrow{\circ} e_i)_{i\in I}$ is a family of regular bimodules forming a universal cone – thus the product of the family $(e_i)_{i\in I}$ – in $\mathsf{Bim}^\star(\mathscr{Q})$.

The proof for coproducts is similar. $\Box$

Splitting idempotents in a quantaloid

If we take $\mathscr{C} = \mathscr{Q}$ to be a quantaloid in the above, then we can say a bit more about the regular-bimodules construction in 4.4.3.

Lemma 4.4.6 *For any quantaloid $\mathscr{Q}$,*

1. *the category $\mathsf{Bim}^\star(\mathscr{Q})$ is a quantaloid too, with*
    - *two-cells: for idempotents $A \overset{e}{\circlearrowleft}$ and $B \overset{f}{\circlearrowleft}$, put $b \le c$ in $\mathsf{Bim}^\star(\mathscr{Q})(e,f)$ whenever $b \le c$ in $\mathscr{Q}(A,B)$;*
2. *the embedding*
$$u_{\mathscr{Q}}\colon \mathscr{Q} \longrightarrow \mathsf{Bim}^\star(\mathscr{Q})\colon (A \xrightarrow{f} B) \mapsto (1_A \xrightarrow[\circ]{(f)} 1_B) \tag{4.8}$$
   is a quantaloid homomorphism;
3. *for a given quantaloid homomorphism $F\colon \mathscr{Q} \longrightarrow \mathscr{Q}'$ the induced functor*
$$\mathsf{Bim}^\star(F)\colon \mathsf{Bim}^\star(\mathscr{Q}) \longrightarrow \mathsf{Bim}^\star(\mathscr{Q}')\colon (e \xrightarrow[\circ]{b} f) \mapsto (Fe \xrightarrow[\circ]{Fb} Ff)$$
   is a quantaloid homomorphism too.

Proof: (1) For any two idempotents $\overset{e}{\circlearrowleft}A$ and $\overset{f}{\circlearrowleft}B$ in $\mathcal{Q}$, and any family of regular bimodules $(b_i\colon e \multimap f)_{i\in I}$, the supremum $\bigvee_i b_i$ as calculated in $\mathcal{Q}$ is still a regular bimodule (because composition in $\mathcal{Q}$ distributes on both sides over the supremum). And since $\bigvee_i b_i$ is the least upper bound of the $(b_i)_i$ in $\mathcal{Q}(A,B)$, it is surely their least upper bound in $\mathsf{Bim}^\star(\mathcal{Q})(e,f) \subseteq \mathcal{Q}(A,B)$.

(2) Follows from (1): suprema in $\mathcal{Q}$ and $\mathsf{Bim}^\star(\mathcal{Q})$ "are the same".

(3) When F preserves suprema of $\mathcal{Q}$-arrows, then

$$F(e \xrightarrow{\bigvee_i b_i} f) = Fe \xrightarrow{F(\bigvee_i b_i)} Ff = Fe \xrightarrow{\bigvee_i Fb_i} Ff.$$

$\square$

Corollary 4.4.7 *For any quantaloid $\mathcal{Q}$,*

1. *the embedding of (4.8) is a fully faithful quantaloid homomorphism;*
2. *this embedding is an equivalence if and only if all idempotents split in $\mathcal{Q}$;*
3. *the embedding*

$$u_{\mathsf{Bim}^\star(\mathcal{Q})}\colon \mathsf{Bim}^\star(\mathcal{Q}) \longrightarrow \mathsf{Bim}^\star(\mathsf{Bim}^\star(\mathcal{Q}))$$

   is always an equivalence of quantaloids;

4. *$\mathcal{Q}$ is equivalent to $\mathsf{Bim}^\star(\mathcal{Q}')$ for some quantaloid $\mathcal{Q}'$ if and only if all idempotents split in $\mathcal{Q}$;*
5. *actually, (4.8) describes the universal split-idempotent completion of $\mathcal{Q}$ in* QUANT.

It is clear by 2.2.1 and 2.2.8 that a bimodule between monads in $\mathcal{Q}$ is a very special case of a regular bimodule between idempotents. More precisely we may state the following.

Corollary 4.4.8 *For any quantaloid $\mathcal{Q}$,*

1. $\mathsf{Bim}(\mathcal{Q})$ *is fully embedded in* $\mathsf{Bim}^\star(\mathcal{Q})$*, and the* Bim*-construction and the* $\mathsf{Bim}^\star$*-construction on $\mathcal{Q}$ commute with the full embedding as displayed in diagram 4.10;*
2. *if $u_{\mathcal{Q}}\colon \mathcal{Q} \longrightarrow \mathsf{Bim}^\star(\mathcal{Q})$ is an equivalence, then so is $j_{\mathcal{Q}}\colon \mathcal{Q} \longrightarrow \mathsf{Bim}(\mathcal{Q})$;*
3. *$j_{\mathsf{Bim}^\star(\mathcal{Q})}\colon \mathsf{Bim}^\star(\mathcal{Q}) \longrightarrow \mathsf{Bim}(\mathsf{Bim}^\star(\mathcal{Q}))$ is always an equivalence;*
4. *$i_{\mathsf{Bim}^\star(\mathsf{Matr}(\mathcal{Q}))}\colon \mathsf{Bim}^\star(\mathsf{Matr}(\mathcal{Q})) \longrightarrow \mathsf{Matr}(\mathsf{Bim}^\star(\mathsf{Matr}(\mathcal{Q})))$ is always an equivalence.*

$$\begin{tikzcd}[column sep=small] & \mathcal{Q} \arrow[dl, "j_{\mathcal{Q}}"'] \arrow[dr, "u_{\mathcal{Q}}"] & \\ \mathsf{Bim}(\mathcal{Q}) \arrow[rr, dotted, "\text{full embedding}"'] & & \mathsf{Bim}^{\star}(\mathcal{Q}) \end{tikzcd}$$

Diagram 4.10

Proof: (1) Is quite evident from the constructions involved.

(2) If in diagram 4.10 $u_{\mathcal{Q}}$ is an equivalence, then the obvious full embedding $\mathsf{Bim}(\mathcal{Q}) \longrightarrow \mathsf{Bim}^{\star}(\mathcal{Q})$ is essentially surjective on objects, hence an equivalence too. But then necessarily $j_{\mathcal{Q}}$ is an equivalence as well. (This really says that "if all idempotents split in $\mathcal{Q}$, then surely the monads do to"!)

(3) Apply (2) on $\mathsf{Bim}^{\star}(\mathcal{Q})$.

(4) $i_{\mathsf{Bim}^{\star}(\mathsf{Matr}(\mathcal{Q}))}$ is an equivalence if and only if $\mathsf{Bim}^{\star}(\mathsf{Matr}(\mathcal{Q}))$ has all direct sums, but this follows from 4.4.5: $\mathsf{Matr}(\mathcal{Q})$ has all direct sums. □

To end this paragraph with, let us calculate the adjoints to the composition in the quantaloid $\mathsf{Bim}^{\star}(\mathcal{Q})$.

Proposition 4.4.9 *For regular bimodules $e \overset{b}{\multimap} f$ and $f \overset{c}{\multimap} g$ between idempotents in a quantaloid $\mathcal{Q}$, the adjoints to composition in $\mathsf{Bim}^{\star}(\mathcal{Q})$*

$$- \otimes_f b \dashv \{b, -\} \textit{ and } c \otimes_f - \dashv [c, -]$$

are computed as follows: for $d\colon e \multimap g$,

$$\{b, d\}^{\mathsf{Bim}^{\star}(\mathcal{Q})} = g \circ \{b, d\}^{\mathcal{Q}} \circ f,$$

$$[c, d]^{\mathsf{Bim}^{\star}(\mathcal{Q})} = f \circ [c, d]^{\mathcal{Q}} \circ e.$$

Proof: First observe that $f \circ [c, d]^{\mathcal{Q}} \circ e$ is indeed a regular bimodule from e to f, and that

$$\begin{aligned} c \otimes_f \left(f \circ [c, d]^{\mathcal{Q}} \circ e\right) &= c \circ f \circ [c, d]^{\mathcal{Q}} \circ e \\ &= c \circ [c, d]^{\mathcal{Q}} \circ e \\ &\leq d \circ e \\ &= d. \end{aligned}$$

So certainly $f \circ [c, d]^{\mathcal{Q}} \circ e \leq [c, d]^{\mathsf{Bim}^{\star}(\mathcal{Q})}$. For the converse, note that for any $k\colon e \multimap f$

$$c \otimes_f k \leq d \iff c \circ k \leq d$$

$$
\begin{aligned}
&\iff && k \le [c,d]^{\mathscr{Q}} \\
&\implies && f \circ k \circ e \le f \circ [c,d]^{\mathscr{Q}} \circ e \\
&\iff && k \le f \circ [c,d]^{\mathscr{Q}} \circ e.
\end{aligned}
$$

So in particular this holds for $k = [c,d]^{\mathsf{Bim}^\star(\mathscr{Q})}$, i.e. $[c,d]^{\mathsf{Bim}^\star(\mathscr{Q})} \le f \circ [c,d]^{\mathscr{Q}} \circ e$.

The proof for "$\{-,-\}$" is similar. □

More about the splitting of idempotents

For an idempotent $A \circlearrowleft^{e}$ in a category $\mathscr{C}$ it is quite obvious that for every object X in $\mathscr{C}$ there are idempotents

$$\mathscr{C}(X,e)\colon \mathscr{C}(X,A) \longrightarrow \mathscr{C}(X,A)$$

$$\mathscr{C}(e,X)\colon \mathscr{C}(A,X) \longrightarrow \mathscr{C}(A,X)$$

in Set. These idempotents always split over the set of their fixpoints (or any isomorphic set); let us write (a choice of) these splittings as

$$\mathscr{C}(X,A) \underset{\sigma_X}{\overset{\pi_X}{\rightleftarrows}} F_X \quad \text{and} \quad \mathscr{C}(A,X) \underset{\sigma'_X}{\overset{\pi'_X}{\rightleftarrows}} F'_X.$$

In other words, the functors

$$F\colon \mathscr{C}^{\mathsf{op}} \longrightarrow \mathsf{Set}\colon (X \xleftarrow{\ f\ } Y) \mapsto (F_X \xrightarrow{\pi_Y \circ \mathscr{C}(f,A) \circ \sigma_X} F_Y)$$

$$F'\colon \mathscr{C} \longrightarrow \mathsf{Set}\colon (X \xrightarrow{\ f\ } Y) \mapsto (F'_X \xrightarrow{\pi'_Y \circ \mathscr{C}(A,f) \circ \sigma'_X} F'_Y)$$

are the "presheaves of fixpoints" over which the idempotent natural transformations

$$\mathscr{C}(-,e)\colon \mathscr{C}(-,A) \Longrightarrow \mathscr{C}(-,A), \text{ resp. } \mathscr{C}(e,-)\colon \mathscr{C}(A,-) \Longrightarrow \mathscr{C}(A,-)$$

split in $\mathsf{CAT}(\mathscr{C}^{\mathsf{op}},\mathsf{Set})$, resp. $\mathsf{CAT}(\mathscr{C},\mathsf{Set})$. (So F and F' are determined up to natural isomorphism). We maintain these notations in the following statement.

Proposition 4.4.10 *For an idempotent $A \circlearrowleft^{e}$ in a category $\mathscr{C}$, the following are equivalent:*

1. $F\colon \mathscr{C}^{\mathsf{op}} \longrightarrow \mathsf{Set}$ *is representable;*
2. $F'\colon \mathscr{C} \longrightarrow \mathsf{Set}$ *is representable.*

A representing object for one of these functors represents the other functor too, and it is actually an object over which $A \circlearrowleft^{e}$ splits in $\mathscr{C}$.

Proof: The representability of F is equivalent to the existence of a colimit for the diagram $\overset{\curvearrowright}{A}{}^{e}$ in $\mathscr{C}$. And F' is representable if and only if the diagram $\overset{\curvearrowright}{A}{}^{e}$ has a limit in $\mathscr{C}$. Now compare with 4.4.1. $\square$

For a quantaloid $\mathscr{Q}$ instead of a category $\mathscr{C}$ the statements above about Set-valued functors on $\mathscr{C}$ "lift" to Sup-valued homomorphisms on $\mathscr{Q}$. This then must be compared with 2.2.4 where it was argued that the splitting of a monad $\overset{\curvearrowright}{A}{}^{t}$ in a quantaloid $\mathscr{Q}$ is about the representability of certain Sup-valued homomorphisms on $\mathscr{Q}$... That is to say, 2.2.4 is a special case of 4.4.10 (for $\mathscr{C} = \mathscr{Q}$), simply because a monad in $\mathscr{Q}$ is a special kind of idempotent.

All this adds to the feeling that "being idempotent is as close as one gets to being a monad if one lives in a *category* $\mathscr{C}$".

Cauchy completion of a quantaloid

The "splitting of idempotents" in a category $\mathscr{C}$ is related to its Cauchy completion in CAT. For quantaloids there is a subtlety that deserves some attention: we must distinguish between Cauchy completeness in CAT and Cauchy completeness in QUANT! Let us first recall the definition of Cauchy complete $\mathscr{V}$-enriched category – for $\mathscr{V}$ a suitably complete and cocomplete symmetric monoidal closed category – and then look at $\mathscr{V} = \mathsf{Set}$ and $\mathscr{V} = \mathsf{Sup}$.

Let $\mathscr{A}$ be a small $\mathscr{V}$-category; $\mathscr{A}$ can be embedded in a (not necessarily small) $\mathscr{V}$-category $\mathscr{A}_{\mathsf{cc}}$, satisfying the following equivalent properties:

(i) $\mathscr{A}_{\mathsf{cc}}$ "classifies" left adjoint $\mathscr{V}$-distributors on $\mathscr{A}$;

(ii) $\mathscr{A}_{\mathsf{cc}}$ is the completion of $\mathscr{A}$ for small absolute $\mathscr{V}$-colimits.

This new $\mathscr{V}$-category $\mathscr{A}_{\mathsf{cc}}$ (which is determined up to equivalence) is called the $\mathscr{V}$-Cauchy completion of $\mathscr{A}$, or the Cauchy completion of $\mathscr{A}$ in $\mathscr{V}$-CAT[3].

For an ordinary category $\mathscr{C}$, i.e. a Set-enriched category, we have seen that $\mathscr{C} \longrightarrow \mathsf{Bim}^{\star}(\mathscr{C})$ is its universal split-idempotent completion; for small $\mathscr{C}$ it is well-known that this is (equivalent to) its Set-Cauchy completion.

Proposition 4.4.11 *For a small category $\mathscr{C}$, $\mathscr{C} \longrightarrow \mathsf{Bim}^{\star}(\mathscr{C})$ is (within equivalence) the Cauchy completion of $\mathscr{C}$ in CAT.*

The point now is that for a small quantaloid $\mathscr{Q}$ the split-idempotent completion $u_{\mathscr{Q}}\colon \mathscr{Q} \longrightarrow \mathsf{Bim}^{\star}(\mathscr{Q})$ is not its Sup-Cauchy completion—although it is its Set-Cauchy completion. Whereas surely the splitting of idempotents in $\mathscr{Q}$ is a necessary condition (any such splitting is an absolute Sup-coequalizer) it is not a

[3] We adopt the viewpoint that $\mathscr{A}$ should be *small* to ensure that it makes sense for a $\mathscr{V}$-distributor $\Phi\colon \mathscr{I} \mathrel{-\!\!\!\circ\!\!\!\rightarrow} \mathscr{A}$ to have a right adjoint.

sufficient condition for *all* small absolute Sup-colimits to exist in $\mathcal{Q}$. For example, all small coproducts in a quantaloid are Sup-absolute (see 2.1.2), so a Sup-Cauchy complete quantaloid must necessarily admit all small coproducts.

As for an elementary description of the Cauchy completion of a (small) quantaloid in QUANT, it is useful to re-read 4.4.8 (4): it implies that for any quantaloid $\mathcal{Q}$, the new quantaloid $\mathsf{Bim}^\star(\mathsf{Matr}(\mathcal{Q}))$ has all direct sums and that all idempotents split! So the embedding $\mathcal{Q} \longrightarrow \mathsf{Bim}^\star(\mathsf{Matr}(\mathcal{Q}))$ is a universal construction on $\mathcal{Q}$ adding all absolute coequalizers and all absolute coproducts. Is it then a surprise that... ?

Proposition 4.4.12 *For a small quantaloid $\mathcal{Q}$, $\mathcal{Q} \longrightarrow \mathsf{Bim}^\star(\mathsf{Matr}(\mathcal{Q}))$ is (to within equivalence) the Cauchy completion of $\mathcal{Q}$ in* QUANT.

F. Van der Plancke and B. Walters independently gave a proof of this statement (and F. Borceux has an unpublished proof). Our presentation of this result is different from Van der Plancke's and Walters', but the proof can be completed by copying the long but elementary calculations on pp. 67–68 of Van der Plancke's thesis.

4.5 Regular semicategories

Calculus of regular distributors

The first sections of this chapter motivated the notion of 'regular $\mathcal{Q}$-semicategory' as a "weakening of '$\mathcal{Q}$-category' that still allows for a good behavior of the representable presheaves". In particular, in 4.3.4 we have seen that those regular $\mathcal{Q}$-semicategories are precisely the idempotent $\mathcal{Q}$-matrices. So now that we have studied quite generally idempotents in a quantaloid, it is about time to apply the results to the quantaloid $\mathsf{Matr}(\mathcal{Q})$!

We had a definition for 'distributor' between $\mathcal{Q}$-semicategories... but those distributors were not the arrows of some "quantaloid of $\mathcal{Q}$-semicategories and distributors" (see p. 121 and further). Applying the "idempotents-and-regular-bimodules" construction of 4.4.3 to the quantaloid $\mathsf{Matr}(\mathcal{Q})$, we can understand that not only do we have to restrict our attention to *regular* $\mathcal{Q}$-semicategories but also to *regular* distributors!

Definition 4.5.1 *A distributor $\Phi\colon \mathbb{A} \multimap \mathbb{B}$ between $\mathcal{Q}$-semicategories*[4] *is* regular *when, for any $a \in \mathbb{A}_0$ and $b \in \mathbb{B}_0$,*

$$\begin{cases} \bigvee_{x\in\mathbb{A}_0} \Phi(b,x) \circ \mathbb{A}(x,a) = \Phi(b,a), \\ \bigvee_{y\in\mathbb{B}_0} \mathbb{B}(b,y) \circ \Phi(y,a) = \Phi(b,a). \end{cases} \tag{4.9}$$

[4] The definition makes sense for a distributor between not-necessarily regular $\mathcal{Q}$-semicategories, so we state it in this generality.

In a sense, we *impose* here the result in 2.3.4 for distributors between categories; indeed, any distributor between $\mathcal{Q}$-categories is regular.

We may now reconcile the different notions of 'regularity'.

Corollary 4.5.2 *Let $\mathbb{A}$ be a $\mathcal{Q}$-semicategory. Then*

1. *a distributor $\phi\colon *_A \xrightarrow{\circ} \overline{\mathbb{A}}$ is a regular presheaf (in the sense of 4.2.6) if and only if $\phi\colon *_A \xrightarrow{\circ} \mathbb{A}$ is a regular distributor (in the sense of 4.5.1);*

2. *$\mathbb{A}$ is a regular $\mathcal{Q}$-semicategory (in the sense of 4.2.9) if and only if the distributor $\mathbb{A}\colon \mathbb{A} \xrightarrow{\circ} \mathbb{A}$ is regular (in the sense of 4.5.1).*

It is clear that regular $\mathcal{Q}$-semicategories and regular distributors are idempotents and regular bimodules in $\mathsf{Matr}(\mathcal{Q})$.

Definition 4.5.3 *Given a quantaloid $\mathcal{Q}$ we define that*

$$\mathsf{Dist}^\star(\mathcal{Q}) = \mathsf{Bim}^\star(\mathsf{Matr}(\mathcal{Q})).$$

So, explicitly, in this new quantaloid $\mathsf{Dist}^\star(\mathcal{Q})$,

- objects: are (small) regular $\mathcal{Q}$-semicategories;
- arrows: are regular distributors;
- two-cells: elementwise ordering as for matrices;
- composition: with the "linear algebra formula" as for matrices;
- identities: the identity regular distributor on $\mathbb{A}$ is $\mathbb{A}\colon \mathbb{A} \xrightarrow{\circ} \mathbb{A}$ itself.

The following is then a consequence of the work done in the previous section, combined with the material in the previous chapters.

Proposition 4.5.4 *For any quantaloid $\mathcal{Q}$,*

1. *the obvious embedding*

$$v_{\mathcal{Q}}\colon \mathcal{Q} \longrightarrow \mathsf{Dist}^\star(\mathcal{Q})\colon (A \xrightarrow{f} B) \mapsto (*_A \xrightarrow[\circ]{(f)} *_B) \tag{4.10}$$

is a fully faithful homomorphism of quantaloids;

2. *this embedding is an equivalence if and only if $\mathcal{Q}$ has all direct products and all idempotents split;*

3. *applying the $\mathsf{Dist}^\star$-construction to $\mathsf{Dist}^\star(\mathcal{Q})$ itself, the embedding*

$$v_{\mathsf{Dist}^\star(\mathcal{Q})}\colon \mathsf{Dist}^\star(\mathcal{Q}) \longrightarrow \mathsf{Dist}^\star(\mathsf{Dist}^\star(\mathcal{Q}))$$

is always an equivalence;

$$\begin{tikzcd}
\mathcal{Q} \arrow[r, "i_{\mathcal{Q}}"] \arrow[dr, "v_{\mathcal{Q}}"'] & \mathsf{Matr}(\mathcal{Q}) \arrow[d, "u_{\mathsf{Matr}(\mathcal{Q})}"] \\
& \mathsf{Bim}^\star(\mathsf{Matr}(\mathcal{Q}))
\end{tikzcd}$$

Diagram 4.11

4. *$\mathcal{Q}$ is equivalent to $\mathsf{Dist}^\star(\mathcal{Q}')$ for some quantaloid $\mathcal{Q}'$ if and only if $\mathcal{Q}$ has all direct sums and all idempotents split;*

5. *the embedding in (4.10) describes the universal "direct-products and split-idempotent" completion of $\mathcal{Q}$ in* QUANT*;*

6. *for a small quantaloid $\mathcal{Q}$, the embedding in (4.10) describes the Cauchy completion of $\mathcal{Q}$ in* QUANT.

Proof: (1) $v_{\mathcal{Q}}$ is the composition of two fully faithful homomorphisms, namely $i_{\mathcal{Q}}$ of (2.4) followed by $u_{\mathsf{Matr}(\mathcal{Q})}$ is in (4.8)—see diagram 4.11.

(2) If $v_{\mathcal{Q}}$ is an equivalence, then both $i_{\mathcal{Q}}$ and $u_{\mathsf{Matr}(\mathcal{Q})}$ must be equivalences too (since they are fully faithful). But then, by 2.1.4 and 4.4.8, $\mathcal{Q}$ must admit direct sums and the splitting of idempotents. The converse follows directly from the mentioned propositions.

(3, 4, 5) Follow directly from 2.1.4 and 4.4.8.

(6) Is a tautology of 4.4.12. $\square$

The universality of the $\mathsf{Dist}^\star$-construction implies that every quantaloid homomorphism $F\colon \mathcal{Q} \longrightarrow \mathcal{Q}'$ induces a "change of base".

Corollary 4.5.5 *Every quantaloid homomorphism $F\colon \mathcal{Q} \longrightarrow \mathcal{Q}'$ determines an essentially unique homomorphism*

$$\mathsf{Dist}^\star(F)\colon \mathsf{Dist}^\star(\mathcal{Q}) \longrightarrow \mathsf{Dist}^\star(\mathcal{Q}')$$

that makes diagram 4.12 commute.

For an explicit prescription of this homomorphism $\mathsf{Dist}^\star(F)$ we must combine (2.5) on p. 63 with (4.7) on p. 135: given $\Phi\colon \mathbb{A} \multimap \mathbb{B}$ in $\mathsf{Dist}^\star(\mathcal{Q})$, let us write here – for brevity of notations – $\widehat{\Phi}\colon \widehat{\mathbb{A}} \multimap \widehat{\mathbb{B}}$ for its image by $\mathsf{Dist}^\star(F)$, then explicitly

- the object-set $\widehat{\mathbb{A}}_0$ equals $\mathbb{A}_0$, but the types of the objects are changed into $\widehat{ta} = F(ta)$;
- for $a, a' \in \widehat{\mathbb{A}}_0$, the hom-arrow is $\widehat{\mathbb{A}}(a', a) = F\Big(\mathbb{A}(a', a)\Big)$;

$$\begin{tikzcd}
\mathscr{Q} \arrow[r, "v_{\mathscr{Q}}"] \arrow[d, "F"'] & \mathsf{Dist}^\star(\mathscr{Q}) \arrow[d, "\mathsf{Dist}^\star(F)"] \\
\mathscr{Q}' \arrow[r, "v_{\mathscr{Q}'}"'] & \mathsf{Dist}^\star(\mathscr{Q}')
\end{tikzcd}$$

Diagram 4.12

$$\begin{tikzcd}
& \mathscr{Q} \arrow[dl, "k_{\mathscr{Q}}"'] \arrow[dr, "v_{\mathscr{Q}}"] & \\
\mathsf{Dist}(\mathscr{Q}) \arrow[rr, dotted, "\text{full embedding}"'] & & \mathsf{Dist}^\star(\mathscr{Q})
\end{tikzcd}$$

Diagram 4.13

- and for $a \in \widehat{\mathbb{A}}_0$ and $b \in \widehat{\mathbb{B}}_0$, $\widehat{\Phi}(b,a) = F\Big(\Phi(b,a)\Big)$.

As mentioned before, every $\mathscr{Q}$-category is a regular semicategory, and every distributor between $\mathscr{Q}$-categories is regular. More in particular can we apply 4.4.8 to $\mathsf{Matr}(\mathscr{Q})$ to obtain the following.

Corollary 4.5.6 *For any quantaloid $\mathscr{Q}$,*

1. *$\mathsf{Dist}(\mathscr{Q})$ is fully embedded in $\mathsf{Dist}^\star(\mathscr{Q})$, and the Dist-construction and the $\mathsf{Dist}^\star$-construction on $\mathscr{Q}$ commute with the full embedding as in diagram 4.13;*

2. *$k_{\mathsf{Dist}^\star(\mathscr{Q})}\colon \mathsf{Dist}^\star(\mathscr{Q}) \longrightarrow \mathsf{Dist}(\mathsf{Dist}^\star(\mathscr{Q}))$ is always an equivalence.*

(These may seem innocent remarks, but they will be very useful later on.)

Let us explicitly write the adjoints to composition in $\mathsf{Dist}^\star(\mathscr{Q})$—this is an application of 2.1.5 and 4.2.1.

Proposition 4.5.7 *For regular $\mathscr{Q}$-semicategories and regular distributors as in $\mathbb{A} \xrightarrow{\Phi}\!\!\!\!\!\!\circ\;\, \mathbb{B} \xrightarrow{\Psi}\!\!\!\!\!\!\circ\;\, \mathbb{C}$, the adjoints to composition in $\mathsf{Dist}(\mathscr{Q})$,*

$$- \otimes_{\mathbb{B}} \Phi \dashv \{\Phi, -\} \textit{ and } \Psi \otimes_{\mathbb{B}} - \dashv [\Psi, -],$$

are computed as follows: for $\mathbb{A} \xrightarrow{\Theta}\!\!\!\!\!\!\circ\;\, \mathbb{C}$,

$$\begin{aligned}
\Big\{\Phi, \Theta\Big\}^{\mathsf{Dist}^\star(\mathscr{Q})}(c,b) &= \mathbb{C} \circ \Big\{\Phi, \Theta\Big\}^{\mathsf{Matr}(\mathscr{Q})} \circ \mathbb{B}, \\
\Big[\Psi, \Theta\Big]^{\mathsf{Dist}^\star(\mathscr{Q})}(b,a) &= \mathbb{B} \circ \Big[\Psi, \Theta\Big]^{\mathsf{Matr}(\mathscr{Q})} \circ \mathbb{A}.
\end{aligned}$$

Regular morphisms

A crucial point in the development of the theory of $\mathcal{Q}$-categories is the fact that any functor induces an adjoint pair of distributors. We now study this phenomenon for morphisms between semicategories (defined in 4.1.3 on p. 120): we would like to associate to a given morphism $F\colon\mathbb{A}\longrightarrow\mathbb{B}$ between semicategories an adjoint pair of distributors "$\mathbb{B}(-,F-)\colon\mathbb{A}\!-\!\!\!\circ\!\!\!\rightarrow\mathbb{B}$" and "$\mathbb{B}(F-,-)\colon\mathbb{B}\!-\!\!\!\circ\!\!\!\rightarrow\mathbb{A}$". But for such a discussion to even make sense, we must be sure that those distributors $\mathbb{B}(-,F-)$ and $\mathbb{B}(F-,-)$ live in a suitable bicategory, for only then is the notion of 'adjunction' defined! This "suitable bicategory" of course being the quantaloid $\mathsf{Dist}^\star(\mathcal{Q})$, it is obvious that we will have to ask for the semicategories $\mathbb{A}$ and $\mathbb{B}$ to be regular... but it turns out that also the morphism F will have to satisfy an extra couple of conditions!

Lemma 4.5.8 *Let $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ be $\mathcal{Q}$-semicategories, and consider morphisms as in $\mathbb{A}\xrightarrow{F}\mathbb{B}\xleftarrow{G}\mathbb{C}$. The matrix*

$$\Big(\mathbb{B}(G-,F-)(c,a):=\mathbb{B}(Gc,Fa)\Big)_{a\in\mathbb{A}_0,c\in\mathbb{C}_0}$$

is a distributor $\mathbb{B}(G-,F-)\colon\mathbb{A}\!-\!\!\!\circ\!\!\!\rightarrow\mathbb{C}$.

Proof: This matrix is clearly well-defined. As for the actions of $\mathbb{A}$ and $\mathbb{B}$ on this matrix: for any $a,a'\in\mathbb{A}_0$ and $c\in\mathbb{C}_0$,

$$\mathbb{B}(Gc,Fa')\circ\mathbb{A}(a',a)\leq\mathbb{B}(Gc,Fa')\circ\mathbb{B}(Fa',Fa)\leq\mathbb{B}(Gc,Fa);$$

and the other action is similar. $\square$

Definition 4.5.9 *A morphism $F\colon\mathbb{A}\longrightarrow\mathbb{B}$ between $\mathcal{Q}$-semicategories*[5] *is* regular *if both distributors*

$$\mathbb{B}(1_{\mathbb{B}}-,F-)\colon\mathbb{A}\!-\!\!\!\circ\!\!\!\longrightarrow\mathbb{B}\quad\textit{and}\quad\mathbb{B}(F-,1_{\mathbb{B}}-)\colon\mathbb{B}\!-\!\!\!\circ\!\!\!\longrightarrow\mathbb{A}$$

are regular.

In the situation of 4.5.9 the notation for those distributors will be $\mathbb{B}(-,F-)$ and $\mathbb{B}(F-,-)$. Let us look at the (sub)structure that the regular morphisms form among all morphisms.

Lemma 4.5.10 *1. The identity morphism $1_{\mathbb{A}}\colon\mathbb{A}\longrightarrow\mathbb{A}$ on a $\mathcal{Q}$-semicategory is regular if and only if the semicategory itself is regular.*

[5] As the notion of 'regular distributor' also makes sense for non-regular $\mathcal{Q}$-semicategories, we can define 'regular morphisms' even between non-regular semicategories.

2. *Let* $F\colon \mathbb{A}\longrightarrow\mathbb{B}$ *and* $G\colon \mathbb{B}\longrightarrow\mathbb{C}$ *be regular morphisms between regular* $\mathcal{Q}$-*semicategories. The composition* $G\circ F\colon \mathbb{A}\longrightarrow\mathbb{C}$ *as computed in* $\mathsf{SCat}(\mathcal{Q})$ *is again a regular morphism.*

Proof: (1) Is quite straightforward.

(2) By assumption, $\mathbb{B}(-,F-)\colon \mathbb{A}\xrightarrow{\circ}\mathbb{B}$ and $\mathbb{C}(-,G-)\colon \mathbb{B}\xrightarrow{\circ}\mathbb{C}$ are regular distributors. But precisely by regularity of the latter,

$$\mathbb{C}(-,G-)\otimes_{\mathbb{B}}\mathbb{B}(-,F-)=\mathbb{C}(-,G\circ F-)$$

(with composition of distributors in $\mathsf{Dist}^\star(\mathcal{Q})$, and composition of morphisms in $\mathsf{SCat}(\mathcal{Q})$). But it implies that $\mathbb{C}(-,G\circ F-)$ is a regular distributor, as composition of two regular distributors! Similar for $\mathbb{C}(G\circ F-,-)$. $\square$

Proposition 4.5.11 *For any quantaloid* $\mathcal{Q}$ *the regular* $\mathcal{Q}$-*semicategories and regular morphisms form a category, that we will denote as* $\mathsf{SCat}^\star(\mathcal{Q})$. *It is a (non-full) subcategory of* $\mathsf{SCat}(\mathcal{Q})$, *and it contains* $\mathsf{Cat}(\mathcal{Q})$ *as a (full) subcategory:*

$$\mathsf{Cat}(\mathcal{Q})\longrightarrow\mathsf{SCat}^\star(\mathcal{Q})\longrightarrow\mathsf{SCat}(\mathcal{Q}).$$

The point now is that $\mathsf{SCat}^\star(\mathcal{Q})$ is to $\mathsf{Dist}^\star(\mathcal{Q})$, what $\mathsf{Cat}(\mathcal{Q})$ is to $\mathsf{Dist}(\mathcal{Q})$: every regular morphism between regular $\mathcal{Q}$-semicategories determines an adjoint pair of regular distributors. We can suitably adapt the statements and proofs of 2.4.4 and 2.4.5.

Proposition 4.5.12 *Any regular morphism between regular* $\mathcal{Q}$-*semicategories* $F\colon \mathbb{A}\longrightarrow\mathbb{B}$ *determines an adjoint pair of regular distributors,*

$$\mathbb{B}(-,F)\dashv\mathbb{B}(F-,-)\colon \mathbb{B}\underset{\circ}{\overset{\circ}{\leftrightarrows}}\mathbb{A}.$$

Proof: We have to show two inequalities:

$$\mathbb{B}(F-,-)\otimes_{\mathbb{B}}\mathbb{B}(-,F-)=\mathbb{B}(F-,F-)\geq\mathbb{A},$$

$$\mathbb{B}(-,F-)\otimes_{\mathbb{A}}\mathbb{B}(F-,-)\leq\mathbb{B}(-,-)\otimes_{\mathbb{B}}\mathbb{B}(-,-)=\mathbb{B}.$$

In the first line the "=" follows by regularity of the distributors and the "$\geq$" by the action-inequality of F. In the second line the "$\leq$" follows since $\{Fa\mid a\in\mathbb{A}_0\}\subseteq\mathbb{B}_0$, and the "=" is due to regularity of $\mathbb{B}$. $\square$

Proposition 4.5.13 *Sending a regular morphism to the left adjoint regular distributor that it determines, as in*

$$\mathsf{SCat}^\star(\mathcal{Q})\longrightarrow\mathsf{Dist}^\star(\mathcal{Q})\colon (\mathbb{A}\xrightarrow{\ F\ }\mathbb{B})\mapsto(\mathbb{A}\xrightarrow[\circ]{\mathbb{B}(-,F-)}\mathbb{B})\tag{4.11}$$

determines a functor from $\mathsf{SCat}^\star(\mathcal{Q})$ *into* $\mathsf{Dist}^\star(\mathcal{Q})$. *Sending* F *to* $\mathbb{B}(F-,-)$ *determines a contravariant such functor.*

$$\begin{array}{ccc}\mathsf{Cat}(\mathcal{Q}) & \longrightarrow & \mathsf{SCat}^{\star}(\mathcal{Q}) \\ \downarrow & & \downarrow \\ \mathsf{Dist}(\mathcal{Q}) & \longrightarrow & \mathsf{Dist}^{\star}(\mathcal{Q})\end{array}$$

Diagram 4.14

Proof: We must verify that $1_{\mathbb{A}}\colon\mathbb{A}\longrightarrow\mathbb{A}$ is sent to $\mathbb{A}\colon\mathbb{A}\relbar\joinrel\mapstochar\joinrel\rightarrow\mathbb{A}$, and for $F\colon\mathbb{A}\longrightarrow\mathbb{B}$ and $G\colon\mathbb{B}\longrightarrow\mathbb{C}$, that $\mathbb{C}(-,G\circ F-)=\mathbb{C}(-,G-)\otimes_{\mathbb{B}}\mathbb{B}(-,F-)$, but this was already contained in 4.5.10, so we need not repeat it here. $\square$

Precisely as for categories, $\mathsf{SCat}^{\star}(\mathcal{Q})$ now inherits local structure from $\mathsf{Dist}^{\star}(\mathcal{Q})$. In particular, for two regular morphisms between regular $\mathcal{Q}$-semicategories – say $F,G\colon\mathbb{A}\rightrightarrows\mathbb{B}$ – we put

$$F\le G \quad\Longleftrightarrow\quad \mathbb{B}(-,F-)\le\mathbb{B}(-,G-) \text{ in } \mathsf{Dist}^{\star}(\mathcal{Q}).$$

This makes $\mathsf{SCat}^{\star}(\mathcal{Q})(\mathbb{A},\mathbb{B})$ a preordered set; and in particular may we speak of "isomorphic" regular morphisms (i.e. F and G such that $F\le G$ and $G\le F$), or an "essentially unique" regular morphism, etc. (Note that no confusion can arise about the preordering of functors between $\mathcal{Q}$-categories: their ordering "as functors" coincides with their ordering "as regular morphisms".)

Corollary 4.5.14 $\mathsf{SCat}^{\star}(\mathcal{Q})$ *is a locally preordered 2-category. And the functor in (4.11) is actually a 2-functor which is the identity on objects, "essentially faithful" (but not full), and locally fully faithful.*

Let us put in a diagram how the theory of regular $\mathcal{Q}$-semicategories, regular distributors and regular morphisms contains the theory of $\mathcal{Q}$-categories, distributors and functors (the cherry on the pie).

Corollary 4.5.15 *For a quantaloid $\mathcal{Q}$ the diagram 4.14 of 2-categories and 2-functors commutes: the horizontal arrows are the full embeddings of 4.5.6 and 4.5.11, the vertical ones say that "every functor (regular morphism) induces an adjoint pair of distributors (regular distributors)" as attested by 2.4.5 and 4.5.14.*

Finally a corollary of 4.5.5 and 4.5.14 about the "change of base" induced by a homomorphism $F\colon\mathcal{Q}\longrightarrow\mathcal{Q}'$.

Corollary 4.5.16 *A quantaloid homomorphism $F\colon\mathcal{Q}\longrightarrow\mathcal{Q}'$ determines a 2-functor $\mathsf{SCat}^{\star}(F)\colon\mathsf{SCat}^{\star}(\mathcal{Q})\longrightarrow\mathsf{SCat}^{\star}(\mathcal{Q}')$ that makes diagram 4.15 commute; the vertical arrows are 2-functors as in (4.11).*

$$\begin{tikzcd}
\mathsf{SCat}^{\star}(\mathscr{Q}) \arrow[r, "\mathsf{SCat}^{\star}(F)"] \arrow[d] & \mathsf{SCat}^{\star}(\mathscr{Q}') \arrow[d] \\
\mathsf{Dist}^{\star}(\mathscr{Q}) \arrow[r, "\mathsf{Dist}^{\star}(F)"'] & \mathsf{Dist}^{\star}(\mathscr{Q}')
\end{tikzcd}$$

Diagram 4.15

Proof: On objects $\mathsf{SCat}^{\star}(F)$ does precisely the same thing as $\mathsf{Dist}^{\star}(F)$—see 4.5.5. The point here is that for a regular morphism $K\colon \mathbb{A}\longrightarrow\mathbb{B}$ in $\mathsf{SCat}^{\star}(\mathscr{Q})$, the *same* (underlying) object-mapping $\mathbb{A}_0\longrightarrow\mathbb{B}_0\colon a\mapsto Ka$ determines a regular morphism from $\mathsf{SCat}^{\star}(F)(\mathbb{A})$ to $\mathsf{SCat}^{\star}(F)(\mathbb{B})$. To avoid heavy notations, let us denote $\widehat{K}\colon\widehat{\mathbb{A}}\longrightarrow\widehat{\mathbb{B}}$ for the image by $\mathsf{SCat}^{\star}(F)$ of $K\colon\mathbb{A}\longrightarrow\mathbb{B}$ in $\mathsf{SCat}^{\star}(\mathscr{Q})$. That is to say,

- $\widehat{\mathbb{A}}_0=\mathbb{A}_0$, but $\widehat{ta}=F(ta)$ (where $\widehat{ta}$ is the type of an object a in $\widehat{\mathbb{A}}$);
- $\widehat{\mathbb{A}}(a',a)=F\Big(\mathbb{A}(a',a)\Big)$;
- and $\widehat{K}\colon\widehat{\mathbb{A}}\longrightarrow\widehat{\mathbb{B}}\colon a\mapsto Ka$.

It is obvious that $\widehat{K}$ is a morphism between regular semicategories; it is its regularity that requires a calculation:

$$\begin{aligned}
\bigvee_{x\in\widehat{\mathbb{A}}_0}\widehat{\mathbb{B}}(b,\widehat{K}x)\circ\widehat{\mathbb{A}}(x,a) &= \bigvee_{x\in\mathbb{A}_0}F\Big(\mathbb{B}(b,Kx)\Big)\circ F\Big(\mathbb{A}(x,a)\Big)\\
&= \bigvee_{x\in\mathbb{A}_0}F\Big(\mathbb{B}(b,Kx)\circ\mathbb{A}(x,a)\Big)\\
&= F\Big(\bigvee_{x\in\mathbb{A}_0}\mathbb{B}(b,Kx)\circ\mathbb{A}(x,a)\Big)\\
&= F\Big(\mathbb{B}(b,Ka)\Big)\\
&= \widehat{\mathbb{B}}(b,\widehat{K}a)
\end{aligned}$$

holds for all $a\in\widehat{\mathbb{A}}_0$ and $b\in\widehat{\mathbb{B}}_0$, so $\widehat{\mathbb{B}}(-,\widehat{K}-)\otimes_{\widehat{\mathbb{A}}}\widehat{\mathbb{A}}=\widehat{\mathbb{B}}(-,\widehat{K}-)$; and similar for the other action. $\square$

5

Totally regular semicategories

The notion of 'totally regular $\mathscr{Q}$-semicategory' is introduced via a study of the "stability of objects" in a (regular) $\mathscr{Q}$-semicategory. The point is that for totally regular $\mathscr{Q}$-semicategories there is a suitable theory of Cauchy completions: every totally regular $\mathscr{Q}$-semicategory is 'Morita equivalent' to its 'Cauchy completion' (which itself is 'Cauchy complete'). The distributor calculus for totally regular $\mathscr{Q}$-semicategories is shown to be equivalent to the distributor calculus for $\mathscr{Q}_{\mathsf{cc}}$-categories, $\mathscr{Q}_{\mathsf{cc}}$ being the split-idempotent completion of $\mathscr{Q}$. By taking left adjoint distributors this gives equivalent 2-categories of, on the one hand, Cauchy complete totally regular $\mathscr{Q}$-semicategories and regular morphisms, and on the other Cauchy complete $\mathscr{Q}_{\mathsf{cc}}$-categories and functors. Applying these constructions to a base locale Ω, these objects turn out to be the ordered objects in $\mathsf{Sh}(\Omega)$. This suggests the definition of "the 2-category of $\mathscr{Q}$-orders and order-preserving maps" as precisely either one of these equivalent constructions. Such a "$\mathscr{Q}$-order" is, in other words, an "ordered sheaf on $\mathscr{Q}$".

5.1 Total regularity

Stability of objects

In a $\mathscr{Q}$-category $\mathbb{C}$ any object $c \in \mathbb{C}_0$ can be "pointed at": if we consider the one-object $\mathscr{Q}$-category $*_{tc}$ then $\Delta c\colon *_{tc} \longrightarrow \mathbb{C}\colon * \mapsto c$ is a functor. There may be several functors "pointing at" the object c, but in applications this Δc is most often the preferred one: the left adjoint distributor that it determines is precisely the (Cauchy) presheaf on $\mathbb{C}$ represented by c.

In a $\mathscr{Q}$-semicategory it is not true in general that objects can be "pointed at". Therefore, those objects for which this is possible deserve a name.

Definition 5.1.1 *Let $\mathbb{A}$ be a regular $\mathcal{Q}$-semicategory. An object $a \in \mathbb{A}_0$ is* stable *whenever there exists a regular morphism from a one-object regular $\mathcal{Q}$-semicategory into $\mathbb{A}$ "pointing at" a.*

The one-object regular $\mathcal{Q}$-semicategories correspond precisely to the idempotents in $\mathcal{Q}$: any idempotent $A \overset{e}{\circlearrowleft}$ determines a regular semicategory $*_e$ with one object $*$ which is of type $\mathsf{dom}(e)$, and whose single hom-arrow is precisely e; conversely, the single hom-arrow of a one-object regular $\mathcal{Q}$-semicategory is – due to the regularity! – an idempotent in $\mathcal{Q}$.

Lemma 5.1.2 *For a regular $\mathcal{Q}$-semicategory $\mathbb{A}$ and an object $a \in \mathbb{A}_0$ the following are equivalent:*

1. *a is stable;*
2. *there exists an idempotent $ta \overset{e_a}{\circlearrowleft}$ in $\mathcal{Q}$ such that $e_a \leq \mathbb{A}(a,a)$ and for all $a' \in \mathbb{A}_0$*
$$\begin{cases} \mathbb{A}(a',a) \circ e_a = \mathbb{A}(a',a), \\ e_a \circ \mathbb{A}(a,a') = \mathbb{A}(a,a'); \end{cases} \tag{5.1}$$
3. *for all $a' \in \mathbb{A}_0$,*
$$\begin{cases} \mathbb{A}(a',a) \circ \mathbb{A}(a,a) = \mathbb{A}(a',a), \\ \mathbb{A}(a,a) \circ \mathbb{A}(a,a') = \mathbb{A}(a,a'). \end{cases} \tag{5.2}$$

Proof: ($1 \iff 2$) There is a morphism $F\colon *_{e_a} \longrightarrow \mathbb{A}\colon * \mapsto a$ if and only if $ta \overset{e_a}{\circlearrowleft}$ is an idempotent such that $e_a \leq \mathbb{A}(a,a)$; and this morphism is regular if and only if the equations (5.1) hold.

($2 \Longrightarrow 3$) From the composition-inequality in $\mathbb{A}$ and the hypotheses in (2) it is evident that
$$\mathbb{A}(a',a) \circ \mathbb{A}(a,a) \leq \mathbb{A}(a',a) = \mathbb{A}(a',a) \circ e_a \leq \mathbb{A}(a',a) \circ \mathbb{A}(a,a),$$
and similarly for the other equation.

($3 \Longrightarrow 2$) is (almost) trivial: it suffices to remark that $\mathbb{A}(a,a)\colon ta \longrightarrow ta$ is an idempotent in $\mathcal{Q}$ (put $a = a'$ in (5.2)). $\square$

In plain English, 5.1.2 says that an object a is stable if and only if there exists at least one idempotent $ta \overset{e_a}{\circlearrowleft}$ that behaves as an "identity at a" – as (5.1) says – and in fact one can canonically choose the $\mathcal{Q}$-arrow $\mathbb{A}(a,a)$ to play the rôle of this "identity at a".

For a stable object $a \in \mathbb{A}_0$ and any idempotent $ta \overset{e_a}{\circlearrowleft}$ as in part 2. of 5.1.2 there is an adjoint pair of regular distributors
$$\mathbb{A}(-,a) \dashv \mathbb{A}(a,-)\colon \mathbb{A} \mathrel{\substack{\xleftarrow{\;\circ\;} \\ \xrightarrow{\;\circ\;}}} *_{e_a}$$

which is actually induced by the regular morphism

$$*_{e_a} \longrightarrow \mathbb{A}\colon * \mapsto a$$

(see 4.5.12). In particular may we take $e_a = \mathbb{A}(a,a)$; we then speak of the *canonical* regular morphism pointing at a which we will systematically denote as

$$\Gamma a\colon *_{\mathbb{A}(a,a)} \longrightarrow \mathbb{A}\colon * \mapsto a,$$

and it induces the *canonical* adjoint pair of distributors for a:

$$*_{\mathbb{A}(a,a)} \underset{\mathbb{A}(a,-)}{\overset{\mathbb{A}(-,a)}{\rightleftarrows}} \mathbb{A}.$$

(In particular, for an object c in a *$\mathscr{Q}$-category* $\mathbb{C}$ there is a big difference between Δc and Γc!)

Definition 5.1.3 *A regular $\mathscr{Q}$-semicategory $\mathbb{A}$ is* totally regular *when all objects of $\mathbb{A}$ are stable.*

The following is then a consequence of 5.1.2.

Proposition 5.1.4 *A $\mathscr{Q}$-enriched semicategory*[1] $\mathbb{A}$ *is* totally regular *if and only if for all $a, a' \in \mathbb{A}_0$,*

$$\begin{cases} \mathbb{A}(a,a) \circ \mathbb{A}(a,a') = \mathbb{A}(a,a'), \\ \mathbb{A}(a',a) \circ \mathbb{A}(a,a) = \mathbb{A}(a',a). \end{cases}$$

This proposition thus says that in a totally regular $\mathscr{Q}$-semicategory $\mathbb{A}$, every endo-hom-arrow $\mathbb{A}(a,a)$ is an idempotent, and every $\mathbb{A}(a',a)$ is a regular bimodule from $\mathbb{A}(a,a)$ to $\mathbb{A}(a',a')$. A totally regular $\mathscr{Q}$-enriched semicategory is (by definition, really) always regular, but the converse is not true. On the other hand, any $\mathscr{Q}$-category is totally regular, but again the converse is not true.

Full subcategories of distributors and morphisms

As we have indicated, the notion of 'totally regular $\mathscr{Q}$-semicategory' lies precisely between that of '$\mathscr{Q}$-category' (which is stronger) and 'regular $\mathscr{Q}$-semicategory' (which is weaker). It is thus easily seen that, if we denote

- $\mathsf{Dist}^{\star\star}(\mathscr{Q})$ for the full subquantaloid of $\mathsf{Dist}^{\star}(\mathscr{Q})$ whose objects are totally regular semicategories,

- and likewise $\mathsf{SCat}^{\star\star}(\mathscr{Q})$ for the full sub-2-category of $\mathsf{SCat}^{\star}(\mathscr{Q})$ whose objects are totally regular semicategories,

$$\begin{tikzcd}
\mathsf{Cat}(\mathcal{Q}) \arrow[r] \arrow[d] & \mathsf{SCat}^{\star\star}(\mathcal{Q}) \arrow[r] \arrow[d] & \mathsf{SCat}^{\star}(\mathcal{Q}) \arrow[r] \arrow[d] & \mathsf{SCat}(\mathcal{Q}) \\
\mathsf{Dist}(\mathcal{Q}) \arrow[r] & \mathsf{Dist}^{\star\star}(\mathcal{Q}) \arrow[r] & \mathsf{Dist}^{\star}(\mathcal{Q}) &
\end{tikzcd}$$

Diagram 5.1

then we obtain diagram 5.1 in which all horizontal arrows are embeddings of which only $\mathsf{SCat}^{\star}(\mathcal{Q}) \longrightarrow \mathsf{SCat}(\mathcal{Q})$ is not full, and the vertical ones say that every regular morphism – in particular, every functor – induces an adjoint pair of (regular) distributors. None of the embeddings is trivial, i.e. not every semicategory is regular, not every regular semicategory is totally regular, and not every totally regular semicategory is a category.

Besides the more "conceptual" definition 5.1.3 for 'totally regular $\mathcal{Q}$-semicategory' we have the "equational" characterization in 5.1.4. Similar expressions are possible for the arrows in $\mathsf{Dist}^{\star\star}(\mathcal{Q})$ and $\mathsf{SCat}^{\star\star}(\mathcal{Q})$.

Lemma 5.1.5 *Let $\mathbb{A}$ and $\mathbb{B}$ be two totally regular $\mathcal{Q}$-semicategories. A distributor $\Phi\colon \mathbb{A} \multimap \mathbb{B}$ is regular if and only if for all $a \in \mathbb{A}_0$ and $b \in \mathbb{B}_0$,*

$$\begin{cases} \Phi(b,a) \circ \mathbb{A}(a,a) = \Phi(b,a), \\ \mathbb{B}(b,b) \circ \Phi(b,a) = \Phi(b,a). \end{cases} \tag{5.3}$$

Proof: Suppose that $\Phi\colon \mathbb{A} \multimap \mathbb{B}$ is regular, i.e. $\Phi \otimes_{\mathbb{A}} \mathbb{A} = \Phi$ and $\mathbb{B} \otimes_{\mathbb{B}} \Phi = \Phi$. Then for $a \in \mathbb{A}_0$ and $b \in \mathbb{B}_0$

$$\begin{aligned} \Phi(b,a) \circ \mathbb{A}(a,a) &= \Big(\bigvee_{x \in \mathbb{A}_0} \Phi(b,x) \circ \mathbb{A}(x,a)\Big) \circ \mathbb{A}(a,a) \\ &= \bigvee_{x \in \mathbb{A}_0} \Phi(b,x) \circ \mathbb{A}(x,a) \circ \mathbb{A}(a,a) \\ &\stackrel{*}{=} \bigvee_{x \in \mathbb{A}_0} \Phi(b,x) \circ \mathbb{A}(x,a) \\ &= \Phi(b,a), \end{aligned}$$

where $(*)$ uses the total regularity of $\mathbb{A}$. The second equation in (5.3) is similar (and uses the total regularity of $\mathbb{B}$).

Conversely, let $\Phi\colon \mathbb{A} \multimap \mathbb{B}$ be a distributor satisfying the equations in (5.3). Then for $a \in \mathbb{A}_0$ and $b \in \mathbb{B}_0$,

$$\Phi(b,-) \otimes_{\mathbb{A}} \mathbb{A}(-,a) = \bigvee_{x \in \mathbb{A}_0} \Phi(b,x) \circ \mathbb{A}(x,a)$$

[1]The conditions in this proposition make sense for any $\mathcal{Q}$-enriched semicategory, i.e. even for a non-regular one. But it is easily seen that these conditions imply regularity!

$$\begin{aligned} &\geq \Phi(b,a) \circ \mathbb{A}(a,a) \\ &= \Phi(b,a) \end{aligned}$$

and $\Phi(b,-) \otimes_{\mathbb{A}} \mathbb{A}(-,a) \leq \Phi(b,a)$ holds because Φ is a distributor; so indeed $\Phi \otimes_{\mathbb{A}} \mathbb{A} = \Phi$. Likewise for $\mathbb{B} \otimes_{\mathbb{B}} \Phi = \Phi$. So Φ is regular. $\square$

This lemma says that every element $\Phi(b,a)$ of a regular distributor $\Phi\colon \mathbb{A} \longrightarrow \mathbb{B}$ between totally regular $\mathcal{Q}$-semicategories is a regular bimodule between the endo-hom-arrows $\mathbb{A}(a,a)$ and $\mathbb{B}(b,b)$ (which are idempotents).

Lemma 5.1.6 *Let $\mathbb{A}$ and $\mathbb{B}$ be two totally regular $\mathcal{Q}$-semicategories. A morphism $F\colon \mathbb{A} \xrightarrow{\circ} \mathbb{B}$ is regular if and only if for all $a \in \mathbb{A}_0$ and $b \in \mathbb{B}_0$,*

$$\begin{cases} \mathbb{B}(b,Fa) \circ \mathbb{A}(a,a) = \mathbb{B}(b,Fa), \\ \mathbb{A}(a,a) \circ \mathbb{B}(Fa,b) = \mathbb{B}(Fa,b). \end{cases}$$

Proof: A morphism $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ is regular if and only if both induced distributors

$$\mathbb{B}(-,F-)\colon \mathbb{A} \xrightarrow{\circ} \mathbb{B} \text{ and } \mathbb{B}(F-,-)\colon \mathbb{B} \xrightarrow{\circ} \mathbb{A}$$

are regular. With 5.1.5 this can be expressed by means of four sets of equations: for all $a \in \mathbb{A}_0$ and $b \in \mathbb{B}_0$

$$\begin{cases} \mathbb{B}(b,Fa) \circ \mathbb{A}(a,a) = \mathbb{B}(b,Fa), \\ \mathbb{B}(b,b) \circ \mathbb{B}(b,Fa) = \mathbb{B}(b,Fa), \\ \mathbb{A}(a,a) \circ \mathbb{B}(Fa,b) = \mathbb{B}(Fa,b), \\ \mathbb{B}(Fa,b) \circ \mathbb{B}(b,b) = \mathbb{B}(Fa,b). \end{cases}$$

The second and fourth of these are always satisfied by total regularity of $\mathbb{B}$, so only the first and third need to be asked. $\square$

Corollary 5.1.7 *Consider a "fully faithful" morphism $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ between $\mathcal{Q}$-semicategories, i.e.*

$$\forall a,a' \in \mathbb{A}_0 : \mathbb{A}(a',a) = \mathbb{B}(Fa',Fa).$$

If $\mathbb{B}$ is totally regular then so is $\mathbb{A}$, and the morphism F is then regular.

Proof: For $a,a' \in \mathbb{A}_0$ use fully faithfulness of F and total regularity of $\mathbb{B}$ to see that

$$\mathbb{A}(a',a) \circ \mathbb{A}(a,a) = \mathbb{B}(Fa',Fa) \circ \mathbb{B}(Fa,Fa) = \mathbb{B}(Fa',Fa) = \mathbb{A}(a',a),$$

and likewise for $\mathbb{A}(a',a') \circ \mathbb{A}(a',a) = \mathbb{A}(a',a)$, so $\mathbb{A}$ is totally regular. And with a similar calculation, for $a \in \mathbb{A}_0$ and $b \in \mathbb{B}_0$,

$$\mathbb{B}(b,Fa) \circ \mathbb{A}(a,a) = \mathbb{B}(b,Fa) \circ \mathbb{B}(Fa,Fa) = \mathbb{B}(b,Fa),$$

and likewise for $\mathbb{A}(a,a) \circ \mathbb{B}(Fa,b) = \mathbb{B}(Fa,b)$, so F is regular. $\square$

A little application of this corollary – that underlines once more the difference between "regular" and "totally regular" $\mathcal{Q}$-semicategories – goes as follows.

Corollary 5.1.8 *Every full subgraph of a totally regular $\mathcal{Q}$-semicategory is again a totally regular $\mathcal{Q}$-semicategory.*

Proof: Given a totally regular semicategory $\mathbb{A}$, a subset $\mathbb{A}'_0 \subseteq \mathbb{A}_0$ determines a "full sub-graph" of $\mathbb{A}$ in the sense that we can define a $\mathcal{Q}$-enriched structure $\mathbb{A}'$ by putting $\mathbb{A}'(a',a) = \mathbb{A}(a',a)$ for all $a,a' \in \mathbb{A}'_0$. The inclusion $\mathbb{A}'_0 \longrightarrow \mathbb{A}_0 \colon a \mapsto a$ satisfies the conditions in 5.1.7, so $\mathbb{A}'$ is a totally regular semicategory. $\square$

For a $\mathcal{Q}$-category it is evident that any full sub-graph is a $\mathcal{Q}$-category too. But this does not hold in general for regular semicategories: a full sub-graph of a regular semicategory is not necessarily regular!

Underlying preorders

Diagram 5.1 of 2-categories and 2-functors implies that, for any two totally regular $\mathcal{Q}$-semicategories $\mathbb{A}$ and $\mathbb{B}$, $\mathsf{SCat}^{\star\star}(\mathcal{Q})(\mathbb{A},\mathbb{B})$ is a preorder: regular morphisms $F, G \colon \mathbb{A} \rightrightarrows \mathbb{B}$ are ordered by the clause

$$F \leq G \iff \mathbb{B}(-,F-) \leq \mathbb{B}(-,G-).$$

Moreover, since in a totally regular $\mathcal{Q}$-semicategory $\mathbb{B}$ every object can be "pointed at" by a regular morphism, the objects of $\mathbb{B}$ may be ordered by ordering the corresponding regular morphisms. There may be many regular morphisms "pointing at" a given object, but it is the *canonical* one that proves to be the preferred one (compare with p. 82 and further). That is to say, for objects b, b' of a totally regular $\mathcal{Q}$-semicategory $\mathbb{B}$, we put

$$b \leq b' \iff \Gamma b \leq \Gamma b'$$

(recall that $\Gamma b \colon *_{\mathbb{B}(b,b)} \longrightarrow \mathbb{B} \colon * \mapsto b$ and likewise for $\Gamma b'$). So, in elementary terms,

$$b \leq b' \iff tb = tb',\ \mathbb{B}(b,b) = \mathbb{B}(b',b') \text{ and } \mathbb{B}(-,b) \leq \mathbb{B}(-,b').$$

Compare the following with 2.4.16 on p. 85.

Lemma 5.1.9 *For a totally regular $\mathcal{Q}$-semicategory $\mathbb{B}$, the following are equivalent:*

1. *the preorder on the objects of $\mathbb{B}$ is an order (i.e. it is anti-symmetric);*
2. *for any totally regular $\mathcal{Q}$-semicategory $\mathbb{A}$, $\mathsf{SCat}^{\star\star}(\mathcal{Q})(\mathbb{A},\mathbb{B})$ is an (anti-symmetric) order;*

3. for any totally regular $\mathscr{Q}$-semicategory $\mathbb{A}$ the (pre)order-preserving mapping

$$\mathsf{SCat}^{\star\star}(\mathscr{Q})(\mathbb{A},\mathbb{B})\longrightarrow\mathsf{Dist}^{\star\star}(\mathscr{Q})(\mathbb{A},\mathbb{B})\colon F\mapsto\mathbb{B}(-,F-)$$

is injective.

Proof: ($1 \implies 2$) Suppose that $F,G\colon\mathbb{A}\rightrightarrows\mathbb{B}$ are regular morphisms for which $\mathbb{B}(-,F-)=\mathbb{B}(-,G-)$, then for the right adjoints we have that $\mathbb{B}(G-,-)=\mathbb{B}(F-,-)$ too, and we obtain for every $a\in\mathbb{A}_0$ that

$$\mathbb{B}(Ga,Ga)=\mathbb{B}(Ga,Fa)=\mathbb{B}(Fa,Fa)=\mathbb{B}(Fa,Ga).$$

This means in particular that $Fa\leq Ga$ and $Ga\leq Fa$ in the underlying preorder of $\mathbb{B}$, so if this is antisymmetric then $Fa=Ga$ (for all $a\in\mathbb{A}_0$), i.e. $F=G$ in $\mathsf{SCat}^{\star\star}(\mathscr{Q})(\mathbb{A},\mathbb{B})$.

($2 \implies 1$) Is obvious by the definition of "$b\leq b'$" by means of the regular morphisms Γb and $\Gamma b'$.

($2 \iff 3$) Is clear by the definition of "$F\leq G$" by means of the associated distributors $\mathbb{B}(-,F)$ and $\mathbb{B}(-,G-)$. $\square$

Definition 5.1.10 *A totally regular $\mathscr{Q}$-semicategory $\mathbb{B}$ satisfying the (equivalent) conditions in 5.1.9 is* skeletal.

A warning must be inserted here: for a $\mathscr{Q}$-category $\mathbb{C}$ the preorder on the objects of $\mathbb{C}$ *as category* (see 2.4.9) is different from the preorder on $\mathbb{C}_0$ *as totally regular semicategory* (as above). The reason is simple: an object $c\in\mathbb{C}_0$ can be "pointed at" with the functor $\Delta c\colon *_{tc}\longrightarrow\mathbb{C}$ in $\mathsf{Cat}(\mathscr{Q})$, or it can be "pointed at" with the regular morphism $\Gamma c\colon *_{\mathbb{C}(c,c)}\longrightarrow\mathbb{C}$ in $\mathsf{SCat}^{\star\star}(\mathscr{Q})$. When using the local order in $\mathsf{Cat}(\mathscr{Q})$ to order $\mathbb{C}_0$ via the Δ's one obtains a quite different result from what happens when one uses the local order in $\mathsf{SCat}^{\star\star}(\mathscr{Q})$ to order $\mathbb{C}_0$ via the Γ's: the former is finer than the latter.

(By the way, also $\mathsf{SCat}^{\star}(\mathscr{Q})$ is a locally (pre)ordered category; but since objects of a regular semicategory can in general not be "pointed at", there is no natural way to use the local order in $\mathsf{SCat}^{\star}(\mathscr{Q})$ to define an order on the object-set of a regular semicategory.)

5.2 Cauchy completion

Cauchy complete totally regular semicategories

In this section we will systematically adapt the theory on 'Cauchy completions for $\mathscr{Q}$-categories' (see p. 110) to totally regular $\mathscr{Q}$-semicategories.

$$\begin{tikzcd}
*_{\mathbb{A}(a,a)} \arrow[r, shift left, "\mathbb{A}(-,a)", "\circ" marking] \arrow[rr, bend left=40, "\phi_a", "\circ" marking] & \mathbb{A} \arrow[l, shift left, "\mathbb{A}(a,-)", "\circ" marking] \arrow[r, shift left, "\Phi", "\circ" marking] & \mathbb{B}. \arrow[l, shift left, "\Phi^*", "\circ" marking] \arrow[ll, bend left=40, "\phi_a^*", "\circ" marking]
\end{tikzcd}$$

Diagram 5.2

Much like for categories, a left adjoint regular distributor between totally regular $\mathcal{Q}$-semicategories will be called a *Cauchy distributor*. Any regular morphism between totally regular semicategories, like $F\colon \mathbb{A} \longrightarrow \mathbb{B}$, determines a Cauchy distributor $\mathbb{B}(-, F-)\colon \mathbb{A} \mathrel{-\!\!\!\circ\!\!\!\rightarrow} \mathbb{B}$, but not every Cauchy distributor arises like this. Therefore it makes sense to say that a Cauchy distributor $\Phi\colon \mathbb{A} \mathrel{-\!\!\!\circ\!\!\!\rightarrow} \mathbb{B}$ between totally regular $\mathcal{Q}$-semicategories *converges* (to a regular morphism $F\colon \mathbb{A} \longrightarrow \mathbb{B}$) if (there exists a regular morphism $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ such that) $\Phi = \mathbb{B}(-, F-)$. If a Cauchy distributor $\Phi\colon \mathbb{A} \mathrel{-\!\!\!\circ\!\!\!\rightarrow} \mathbb{B}$ converges to both $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ and $G\colon \mathbb{A} \longrightarrow \mathbb{B}$, then F and G are equivalent objects in the preorder $\mathsf{SCat}^{\star\star}(\mathcal{Q})(\mathbb{A}, \mathbb{B})$.

We now study those $\mathcal{Q}$-semicategories in which all Cauchy distributors converge.

Lemma 5.2.1 *For a totally regular $\mathcal{Q}$-semicategory $\mathbb{B}$, the following are equivalent:*

1. *for every totally regular $\mathcal{Q}$-semicategory $\mathbb{A}$, every regular Cauchy distributor $\Phi\colon \mathbb{A} \mathrel{-\!\!\!\circ\!\!\!\rightarrow} \mathbb{B}$ converges;*
2. *for every idempotent $A \overset{e}{\circlearrowleft}$ in $\mathcal{Q}$, every regular Cauchy distributor $\Phi\colon *_e \mathrel{-\!\!\!\circ\!\!\!\rightarrow} \mathbb{B}$ converges.*

Proof: Of course we only need to prove $(2 \Longrightarrow 1)$. So let $\mathbb{A}$ be a totally regular $\mathcal{Q}$-semicategory, and $\Phi\colon \mathbb{A} \mathrel{-\!\!\!\circ\!\!\!\rightarrow} \mathbb{B}$ a regular distributor with right adjoint Φ^*. For any $a \in \mathbb{A}_0$ we may consider the adjunction

$$\begin{tikzcd}
*_{\mathbb{A}(a,a)} \arrow[r, shift left, "\mathbb{A}(-,a)", "\circ" marking] & \mathbb{A} \arrow[l, shift left, "\mathbb{A}(a,-)", "\circ" marking]
\end{tikzcd}$$

and so by composition of adjoints as suggested in diagram 5.2 it follows that $\phi_a = \Phi(-, a)$ and $\phi_a^* = \Phi^*(a, -)$ define a new adjunction $\phi_a \dashv \phi_a^*$ in $\mathsf{Dist}^{\star\star}(\mathcal{Q})$. By assumption every such ϕ_a converges: there exist regular morphisms $F_a\colon *_{\mathbb{A}(a,a)} \longrightarrow \mathbb{B}$ that "represent" the ϕ_a in the sense that $\mathbb{B}(-, F_a*) = \phi_a(-)$ (and $\mathbb{B}(F_a*, -) = \phi_a^*$). Consider now the object mapping $\mathbb{A}_0 \longrightarrow \mathbb{B}_0\colon a \mapsto F_a*$, then quite obviously

$$\begin{aligned}
\mathbb{A}(a', a) &\leq \Phi^*(a', -) \otimes_{\mathbb{A}} \Phi(-, a) \\
&= \phi_{a'}^* \otimes_{\mathbb{A}} \phi_a \\
&= \mathbb{B}(F_{a'}*, -) \otimes_{\mathbb{A}} \mathbb{B}(-, F_a*) \\
&= \mathbb{B}(F_{a'}*, F_a*).
\end{aligned}$$

So $F\colon \mathbb{A}\longrightarrow\mathbb{B}\colon a\mapsto Fa=F_a*$ is a morphism of $\mathcal{Q}$-semicategories. But moreover, for any $a,a'\in\mathbb{A}_0$,

$$\begin{aligned}\mathbb{B}(b,Fa)\circ\mathbb{A}(a,a) &= \Phi(b,a)\circ\mathbb{A}(a,a)\\ &= \Phi(b,a)\\ &= \mathbb{B}(b,Fa)\end{aligned}$$

and likewise for $\mathbb{A}(a,a)\circ\mathbb{B}(Fa,b)=\mathbb{B}(Fa,b)$ (by regularity of Φ and Φ^*!), so that F is actually a regular morphism. Clearly, the given adjunction $\Phi\dashv\Phi^*$ converges to F: $\mathbb{B}(b,Fa)=\Phi(b,a)$. $\square$

Definition 5.2.2 *A totally regular $\mathcal{Q}$-semicategory $\mathbb{B}$ satisfying the equivalent conditions of 5.2.1 is* Cauchy complete*.*

For a $\mathcal{Q}$-category $\mathbb{C}$ – which is trivially a totally regular $\mathcal{Q}$-semicategory – there is a big difference between, on the one hand, Cauchy completeness *as category* (cf. 3.5.2) and on the other hand Cauchy completeness *as totally regular semicategory*: the latter implies the former, but the converse is not true!

Cauchy completion of a totally regular semicategory

We will now construct the 'Cauchy completion' of a totally regular semicategory, and show that any totally regular semicategory is 'Morita equivalent' to its 'Cauchy completion'.

Proposition 5.2.3 *Given a totally regular $\mathcal{Q}$-semicategory $\mathbb{B}$, define a new $\mathcal{Q}$-enriched structure $\mathbb{B}_{\mathsf{cc}}$ as follows:*

- *the object-set is*

$$(\mathbb{B}_{\mathsf{cc}})_0=\{*_e\overset{\phi}{\longrightarrow\!\!\!\!\!\!\!\circ\;\;}\mathbb{B}\mid e\text{ is idempotent in }\mathcal{Q},\phi\text{ is Cauchy}\}$$

  with types given by $t(*_e\overset{\phi}{\longrightarrow\!\!\!\!\!\!\!\circ\;\;}\mathbb{B})=\mathsf{dom}(e)$;

- *for $\phi,\psi\in(\mathbb{B}_{\mathsf{cc}})_0$ as in diagram 5.3, the hom-arrow $\mathbb{B}_{\mathsf{cc}}(\psi,\phi)$ is the single element of the (regular) distributor $[\psi,\phi]$ (which equals $\psi^*\otimes_{\mathbb{B}}\phi$ by 1.2.1 on p. 53), this lifting (and composition) being calculated in the quantaloid $\mathsf{Dist}^{\star\star}(\mathcal{Q})$.*

Then $\mathbb{B}_{\mathsf{cc}}$ is a skeletal totally regular $\mathcal{Q}$-semicategory, and there is a "fully faithful" (hence also regular) morphism

$$k_{\mathbb{B}}\colon\mathbb{B}\longrightarrow\mathbb{B}_{\mathsf{cc}}\colon b\mapsto(*_{\mathbb{B}(b,b)}\overset{\mathbb{B}(-,b)}{\longrightarrow\!\!\!\!\!\!\!\circ\;\;}\mathbb{B}).$$

$$\begin{tikzcd} *_e \arrow[r, shift left, "\phi"] & \mathbb{B} \arrow[l, shift left, "\phi^*"] \arrow[d, shift left, "\psi^*"] \\ & *_f \arrow[u, shift left, "\psi"] \end{tikzcd}$$

Diagram 5.3

Proof: That $\mathbb{B}_{\mathsf{cc}}$ is a $\mathcal{Q}$-semicategory[2] follows from the definition of its hom-arrows and 1.2.6. Further, for $\phi, \psi \in \mathbb{B}_{\mathsf{cc}}$ as in diagram 5.3, it follows from

$$\begin{aligned} [\psi, \phi] \otimes_{*_e} [\phi, \phi] &= \psi^* \otimes_{\mathbb{B}} \phi \otimes_{*_e} \phi^* \otimes_{\mathbb{B}} \phi \\ &= \psi^* \otimes_{\mathbb{B}} \phi \\ &= [\psi, \phi] \end{aligned}$$

that $\mathbb{B}_{\mathsf{cc}}(\psi, \phi) \circ \mathbb{B}_{\mathsf{cc}}(\phi\phi) = \mathbb{B}_{\mathsf{cc}}(\psi, \phi)$, and likewise for $\mathbb{B}_{\mathsf{cc}}(\psi, \psi) \circ \mathbb{B}_{\mathsf{cc}}(\psi\phi) = \mathbb{B}_{\mathsf{cc}}(\psi, \phi)$.

Suppose now that $\phi, \psi \in \mathbb{B}_{\mathsf{cc}}$ are such that $t\phi = t\psi$ and

$$\mathbb{B}_{\mathsf{cc}}(\phi, \phi) = \mathbb{B}_{\mathsf{cc}}(\psi, \psi) \leq \begin{cases} \mathbb{B}_{\mathsf{cc}}(\psi, \phi) \\ \mathbb{B}_{\mathsf{cc}}(\phi, \psi). \end{cases}$$

But this means that $[\phi, \phi] \leq [\phi, \psi]$ which is equivalent to $\phi \leq \phi \otimes_{*_e} [\phi, \psi]$, which itself is equivalent to $\phi \leq \psi$. Likewise for $\psi \leq \phi$, which proves the skeletality of $\mathbb{B}_{\mathsf{cc}}$.

Finally, it is clear that

$$\mathbb{B}_0 \longrightarrow (\mathbb{B}_{\mathsf{cc}})_0 \colon b \mapsto (*_{\mathbb{B}(b,b)} \xrightarrow{\mathbb{B}(-,b)} \mathbb{B})$$

is a well-defined object-mapping—see 5.1.2. It is “fully faithful” because the $\mathcal{Q}$-arrow $\mathbb{B}_{\mathsf{cc}}(k_{\mathbb{B}}b', k_{\mathbb{B}}b)$ is the single element of the distributor $\mathbb{B}(b', -) \otimes_{\mathbb{B}} \mathbb{B}(-, b)$... which equals $\mathbb{B}(b', b)$! $\square$

It is a fact that the construction “$\mathbb{B}_{\mathsf{cc}}$” as in the proposition above makes sense even for a merely *regular* $\mathcal{Q}$-semicategory $\mathbb{B}$... But $\mathbb{B}_{\mathsf{cc}}$ is always a *totally regular* $\mathcal{Q}$-semicategory and therefore, would there be a fully faithful (regular) morphism $\mathbb{B} \longrightarrow \mathbb{B}_{\mathsf{cc}}$, then necessarily $\mathbb{B}$ would be *totally regular* as well (5.1.7). This justifies why the theory of Cauchy completions only “works as it is supposed to” for *totally regular $\mathcal{Q}$-semicategories.*

[2] It is not a $\mathcal{Q}$-category, because – for a $\phi \in \mathbb{B}_{\mathsf{cc}}$ as in diagram 5.3 – even though $e \leq \mathbb{B}_{\mathsf{cc}}(\phi, \phi)$ this does not imply that $1_{t\phi} \leq \mathbb{B}_{\mathsf{cc}}(\phi, \phi)$!

For a totally regular $\mathcal{Q}$-semicategory $\mathbb{B}$, the new $\mathcal{Q}$-semicategory $\mathbb{B}_{\mathsf{cc}}$ as in 5.2.3 is called its *Cauchy completion*. The following says that "every totally regular $\mathcal{Q}$-semicategory is 'Morita equivalent' to its Cauchy completion".

Proposition 5.2.4 *Any totally regular $\mathcal{Q}$-semicategory $\mathbb{B}$ is isomorphic to $\mathbb{B}_{\mathsf{cc}}$ in $\mathsf{Dist}^{\star\star}(\mathcal{Q})$.*

Proof: Since $k_{\mathbb{B}}\colon\mathbb{B}\longrightarrow\mathbb{B}_{\mathsf{cc}}$ is a fully faithful regular morphism of totally regular semicategories, we already know that

$$\mathbb{B} \underset{\mathbb{B}_{\mathsf{cc}}(k_{\mathbb{B}}-,-)}{\overset{\mathbb{B}_{\mathsf{cc}}(-,k_{\mathbb{B}}-)}{\rightleftarrows}} \mathbb{B}_{\mathsf{cc}}$$

is an adjoint pair the unit-inequality of which is an equality. So we only need to show that the co-unit saturates to an equality too. But this requires just a simple calculation with adjoints, as follows: for any two objects ψ,θ of $\mathbb{B}_{\mathsf{cc}}$ verify that

$$\begin{aligned}
\mathbb{B}_{\mathsf{cc}}(\psi,k_{\mathbb{B}}-)\otimes_{\mathbb{B}}\mathbb{B}_{\mathsf{cc}}(k_{\mathbb{B}}-,\theta) &= \bigvee_{b\in\mathbb{B}_0}\mathbb{B}_{\mathsf{cc}}(\psi,k_{\mathbb{B}}b)\otimes_{\mathbb{B}}\mathbb{B}_{\mathsf{cc}}(k_{\mathbb{B}}b,\theta)\\
&= \bigvee_{b\in\mathbb{B}_0}\psi^*\otimes_{\mathbb{B}}k_{\mathbb{B}}b\otimes_{*_{\mathbb{B}}(b,b)}(k_{\mathbb{B}}b)^*\otimes_{\mathbb{B}}\theta\\
&= \psi^*\otimes_{\mathbb{B}}\Big(\bigvee_{b\in\mathbb{B}_0}k_{\mathbb{B}}b\otimes_{*_{\mathbb{B}}(b,b)}(k_{\mathbb{B}}b)^*\Big)\otimes_{\mathbb{B}}\theta\\
&= \psi^*\otimes_{\mathbb{B}}\Big(\bigvee_{b\in\mathbb{B}_0}\mathbb{B}(-,b)\circ\mathbb{B}(b,-)\Big)\otimes_{\mathbb{B}}\theta\\
&= \psi^*\otimes_{\mathbb{B}}\mathbb{B}\otimes_{\mathbb{B}}\theta\\
&= \psi^*\otimes_{\mathbb{B}}\theta\\
&= \mathbb{B}_{\mathsf{cc}}(\psi,\theta).
\end{aligned}$$

So we obtain indeed that $\mathbb{B}_{\mathsf{cc}}(-,k_{\mathbb{B}}-)\otimes_{\mathbb{B}}\mathbb{B}_{\mathsf{cc}}(k_{\mathbb{B}}-,-)=\mathbb{B}_{\mathsf{cc}}$. $\square$

The Cauchy completion $\mathbb{B}_{\mathsf{cc}}$ of a totally regular semicategory $\mathbb{B}$ is – fortunately! – Cauchy complete. To see this we first need some technicalities.

Lemma 5.2.5 *For any totally regular $\mathcal{Q}$-semicategory $\mathbb{B}$ we have – with slight abuse of notation – that:*

1. *for any $\phi\in(\mathbb{B}_{\mathsf{cc}})_0$, $\mathbb{B}_{\mathsf{cc}}(k_{\mathbb{B}}-,\phi)=\phi$ and $\mathbb{B}_{\mathsf{cc}}(\phi,k_{\mathbb{B}}-)=\phi^*$;*

2. *for any $\phi,\psi\in(\mathbb{B}_{\mathsf{cc}})_0$,*

$$\mathbb{B}_{\mathsf{cc}}(\psi,\phi)=\psi^*\otimes_{\mathbb{B}}\mathbb{B}_{\mathsf{cc}}(k_{\mathbb{B}}-,\phi)=\mathbb{B}_{\mathsf{cc}}(\psi,k_{\mathbb{B}}-)\otimes_{\mathbb{B}}\phi.$$

$$\begin{tikzcd}[column sep=large] *_e \arrow[r, shift left, "\Phi"] \arrow[rr, bend left=40, "\phi"] & \mathbb{B}_{\mathsf{cc}} \arrow[l, shift left, "\Phi^*"] \arrow[r, shift left, "\mathbb{B}_{\mathsf{cc}}(k_{\mathbb{B}}-,-)"] & \mathbb{B}. \arrow[l, shift left, "\mathbb{B}_{\mathsf{cc}}(-,k_{\mathbb{B}}-)"] \arrow[ll, bend left=40, "\phi^*"] \end{tikzcd}$$

Diagram 5.4

Proof: (1) By its regularity we have for an object ϕ of $\mathbb{B}_{\mathsf{cc}}$ that $\mathbb{B}\otimes_{\mathbb{B}}\phi=\phi$, so in particular $\mathbb{B}(b,-)\otimes_{\mathbb{B}}\phi=\phi(b)$, or in other terms $\mathbb{B}_{\mathsf{cc}}(k_{\mathbb{B}}b,\phi)=\phi(b)$. The other result is similar.

(2) $\mathbb{B}_{\mathsf{cc}}(\psi,\phi)$ is (the single element of) $\psi^*\otimes_{\mathbb{B}}\psi$; one then applies (1). $\square$

Proposition 5.2.6 *For any totally regular $\mathscr{Q}$-semicategory $\mathbb{B}$, its Cauchy completion $\mathbb{B}_{\mathsf{cc}}$ is a (skeletal) Cauchy complete totally regular $\mathscr{Q}$-semicategory.*

Proof: By 5.2.1 it suffices to prove that every adjoint pair

$$\begin{tikzcd} *_e \arrow[r, shift left, "\Phi"] & \mathbb{B}_{\mathsf{cc}} \arrow[l, shift left, "\Phi^*"] \end{tikzcd}$$

converges. To construct the object of $\mathbb{B}_{\mathsf{cc}}$ that "represents" Φ, consider the (fully faithful) regular morphism $k_{\mathbb{B}}\colon\mathbb{B}\longrightarrow\mathbb{B}_{\mathsf{cc}}$ and the isomorphism in $\mathsf{Dist}^{\star\star}(\mathscr{Q})$ that it determines:

$$\mathbb{B}_{\mathsf{cc}}(-,k_{\mathbb{B}}-)=\Big(\mathbb{B}_{\mathsf{cc}}(k_{\mathbb{B}}-,-)\Big)^{-1}.$$

Then by composition of adjoint pairs we define a new adjoint pair $\phi\dashv\phi^*$ as in diagram 5.4 That is to say, $\phi(b)=\Phi(k_{\mathbb{B}}b)$ and $\phi^*(b)=\Phi^*(k_{\mathbb{B}}b)$ for $b\in\mathbb{B}_0$, and it follows that ϕ is an object of $\mathbb{B}_{\mathsf{cc}}$ (of type $\mathsf{dom}(e)$).

It remains to check that ϕ indeed "represents" Φ: but for any ψ in $\mathbb{B}_{\mathsf{cc}}$

$$\begin{aligned}\mathbb{B}_{\mathsf{cc}}(\psi,\phi) &= \psi^*\otimes_{\mathbb{B}}\phi\\ &= \psi^*\otimes_{\mathbb{B}}\Phi(k_{\mathbb{B}}-)\\ &= \psi^*\otimes_{\mathbb{B}}\mathbb{B}_{\mathsf{cc}}(k_{\mathbb{B}}-,-)\otimes_{\mathbb{B}_{\mathsf{cc}}}\Phi\\ &\overset{*}{=} \mathbb{B}_{\mathsf{cc}}(\psi,-)\otimes_{\mathbb{B}_{\mathsf{cc}}}\Phi\\ &= \Phi(\psi);\end{aligned}$$

in $(*)$ we used 5.2.5. So this ends the proof. $\square$

Finally, the universal property of the Cauchy completion $k_{\mathbb{B}}\colon\mathbb{B}\longrightarrow\mathbb{B}_{\mathsf{cc}}$ is as follows.

Proposition 5.2.7 *For two totally regular $\mathscr{Q}$-semicategories $\mathbb{A}$ and $\mathbb{B}$, with the latter Cauchy complete, any regular morphism $F\colon\mathbb{A}\longrightarrow\mathbb{B}$ factors through the Cauchy completion $k_{\mathbb{A}}\colon\mathbb{A}\longrightarrow\mathbb{A}_{\mathsf{cc}}$ in an "essentially unique" way.*

Proof: Suppose that, under the hypotheses in the statement, $F\colon\mathbb{A}\longrightarrow\mathbb{B}$ is a regular morphism. By assumption there is for any object ϕ of $\mathbb{A}_{\mathsf{cc}}$ an ("essentially unique") object of $\mathbb{B}_0$ to which $\mathbb{B}(-,F-)\otimes_{\mathbb{A}}\phi$ converges:

$$\forall\phi\in(\mathbb{A}_{\mathsf{cc}})_0, \exists G\phi\in\mathbb{B}_0 : \mathbb{B}(-,F-)\otimes_{\mathbb{A}}\phi=\mathbb{B}(-,G\phi).$$

This clearly defines an object-mapping $(\mathbb{A}_{\mathsf{cc}})_0\longrightarrow\mathbb{B}_0\colon\phi\mapsto G\phi$. We now claim that this object-mapping underlies a regular morphism which is the ("essentially unique") factorization of F through $k_{\mathbb{A}}$.

To see that G is a morphism of semicategories, first observe that the definition of $G\phi$ implies that $\mathbb{B}(G\phi,-)=\phi^*\otimes_{\mathbb{A}}\mathbb{B}(F-,-)$ (by uniqueness of adjoints). For two objects of $\mathbb{A}_{\mathsf{cc}}$, say ϕ and ψ, it then follows that

$$\begin{aligned}
\mathbb{A}_{\mathsf{cc}}(\psi,\phi) &= \psi^*\otimes_{\mathbb{A}}\phi \\
&= \psi^*\otimes_{\mathbb{A}}\mathbb{A}\otimes_{\mathbb{A}}\phi \\
&\leq \psi^*\otimes_{\mathbb{A}}\mathbb{B}(F-,F-)\otimes_{\mathbb{A}}\phi \\
&= \psi^*\otimes_{\mathbb{A}}\mathbb{B}(F-,-)\otimes_{\mathbb{B}}\mathbb{B}(-,F-)\otimes_{\mathbb{A}}\phi \\
&= \mathbb{B}(G\psi,-)\otimes_{\mathbb{B}}\mathbb{B}(-,G\phi) \\
&= \mathbb{B}(G\psi,G\phi)
\end{aligned}$$

so G is a morphism. To see that it is regular, it suffices (by total regularity of $\mathbb{A}_{\mathsf{cc}}$ and $\mathbb{B}$) to verify that for any $\phi\colon *_e\longrightarrow\!\!\!\!\!\circ\;\;\mathbb{A}$ in $\mathbb{A}_{\mathsf{cc}}$ and b in $\mathbb{B}$,

$$\begin{aligned}
\mathbb{B}(b,G\phi)\circ\mathbb{A}_{\mathsf{cc}}(\phi,\phi) &= \Big(\mathbb{B}(b,F-)\otimes_{\mathbb{A}}\phi\Big)\circ\Big(\phi^*\otimes_{\mathbb{B}}\phi\Big) \\
&= \mathbb{B}(b,F-)\otimes_{\mathbb{A}}\Big(\phi\otimes_{*_e}\phi^*\otimes_{\mathbb{B}}\phi\Big) \\
&= \mathbb{B}(b,F-)\otimes_{\mathbb{A}}\phi \\
&= \mathbb{B}(b,G\phi);
\end{aligned}$$

and likewise for $\mathbb{A}_{\mathsf{cc}}(\phi,\phi)\circ\mathbb{B}(G\phi,b)=\mathbb{B}(G\phi,b)$.

Next observe that for any $a\in\mathbb{A}_0$, the definition of G says that

$$\begin{aligned}
\mathbb{B}(-,G(k_{\mathbb{A}}a)) &= \mathbb{B}(-,F-)\otimes_{\mathbb{A}}k_{\mathbb{A}}a \\
&= \mathbb{B}(-,F-)\otimes_{\mathbb{A}}\mathbb{A}(-,a) \\
&= \mathbb{B}(-,Fa)
\end{aligned}$$

(by regularity of F), which proves that $\mathbb{B}(-,G\circ k_{\mathbb{A}}-)=\mathbb{B}(-,F-)$, i.e. G is indeed a factorization of F through $k_{\mathbb{A}}$.

Finally, suppose that $H\colon \mathbb{A}_{\mathsf{cc}} \longrightarrow \mathbb{B}$ is another regular morphism which satisfies $\mathbb{B}(-, H \circ k_{\mathbb{A}}-) = \mathbb{B}(-, F-)$; we must prove that $\mathbb{B}(-, H-) = \mathbb{B}(-, G-)$. But

$$\begin{aligned}
&\mathbb{B}(-, H \circ k_{\mathbb{A}}-) = \mathbb{B}(-, F-)\\
&\iff \mathbb{B}(-, G \circ k_{\mathbb{A}}-) = \mathbb{B}(-, H \circ k_{\mathbb{A}}-)\\
&\iff \mathbb{B}(-, G-) \otimes_{\mathbb{A}_{\mathsf{cc}}} \mathbb{A}_{\mathsf{cc}}(-, k_{\mathbb{A}}-) = \mathbb{B}(-, H-) \otimes_{\mathbb{A}_{\mathsf{cc}}} \mathbb{A}_{\mathsf{cc}}(-, k_{\mathbb{A}}-)\\
&\iff \mathbb{B}(-, G-) = \mathbb{B}(-, H-)
\end{aligned}$$

where the last equivalence is a consequence of $\mathbb{A}_{\mathsf{cc}}(-, k_{\mathbb{A}}-)\colon \mathbb{A} \multimap \mathbb{A}_{\mathsf{cc}}$ being an isomorphism in $\mathsf{Dist}^{\star\star}(\mathscr{Q})$—see 5.2.4. $\square$

Putting two and two together

Theorem 5.2.8 *For any quantaloid $\mathscr{Q}$,*

1. *there is an equivalence of quantaloids*

$$\mathsf{Dist}^{\star\star}_{\mathsf{cc}}(\mathscr{Q}) \simeq \mathsf{Dist}^{\star\star}(\mathscr{Q})$$

where $\mathsf{Dist}^{\star\star}_{\mathsf{cc}}(\mathscr{Q})$ is the full subquantaloid of $\mathsf{Dist}^{\star\star}(\mathscr{Q})$ defined by the Cauchy complete objects;

2. *there is an adjunction between 2-categories*

$$(-)_{\mathsf{cc}} \dashv i\colon \mathsf{SCat}^{\star\star}(\mathscr{Q}) \underset{\longrightarrow}{\overset{\longleftarrow}{}} \mathsf{SCat}^{\star\star}_{\mathsf{cc}}(\mathscr{Q})$$

where $\mathsf{SCat}^{\star\star}_{\mathsf{cc}}(\mathscr{Q})$ is the full subcategory of $\mathsf{SCat}^{\star\star}(\mathscr{Q})$ defined by the Cauchy complete objects;

3. *the following 2-categories are (bi)equivalent:*

$$\mathsf{SCat}^{\star\star}_{\mathsf{cc}}(\mathscr{Q}) \simeq \mathsf{Map}(\mathsf{Dist}^{\star\star}_{\mathsf{cc}}(\mathscr{Q})) \simeq \mathsf{Map}(\mathsf{Dist}^{\star\star}(\mathscr{Q})).$$

Proof: (1) is a consequence of the fact that every totally regular $\mathscr{Q}$-semicategory $\mathbb{B}$ is Morita-equivalent to its Cauchy completion $\mathbb{B}_{\mathsf{cc}}$, and that $\mathbb{B}_{\mathsf{cc}}$ is Cauchy complete.

(2) is precisely the universal property of the Cauchy completion: any Cauchy completion $k_{\mathbb{A}}\colon \mathbb{A} \longrightarrow \mathbb{A}_{\mathsf{cc}}$ of a totally regular semicategory $\mathbb{A}$ determines, for any Cauchy complete totally regular semicategory $\mathbb{B}$, a (natural) equivalence of pre-orders

$$\mathsf{SCat}^{\star\star}_{\mathsf{cc}}(\mathscr{Q})(\mathbb{A}_{\mathsf{cc}}, \mathbb{B}) \xrightarrow{\ \sim\ } \mathsf{SCat}^{\star\star}(\mathscr{Q})(\mathbb{A}, \mathbb{B})\colon F \mapsto F \circ k_{\mathbb{A}}.$$

(3) The second equivalence in the statement follows from (1). As for the first: it is the 2-functor

$$\mathsf{SCat}^{\star\star}_{\mathsf{cc}}(\mathscr{Q}) \longrightarrow \mathsf{Map}(\mathsf{Dist}^{\star\star}_{\mathsf{cc}}(\mathscr{Q}))\colon (\mathbb{A} \xrightarrow{F} \mathbb{B}) \mapsto (\mathbb{A} \xrightarrow{\mathbb{B}(-, F-)} \mathbb{B})$$

which is "essentially faithful", full (because of the Cauchy completeness) and the identity on objects. $\Box$

As an illustration, we can record that "totally regular semicategories are Morita equivalent if and only if their Cauchy completions are equivalent".

Proposition 5.2.9 *For totally regular $\mathcal{Q}$-semicategories $\mathbb{A}$ and $\mathbb{B}$, the following are equivalent:*

1. $\mathbb{A} \cong \mathbb{B}$ *in* $\mathsf{Dist}^{\star\star}(\mathcal{Q})$;

2. $\mathbb{A}_{\mathsf{cc}} \simeq \mathbb{B}_{\mathsf{cc}}$ *in* $\mathsf{SCat}^{\star\star}(\mathcal{Q})$.

Proof: Since $\mathbb{A} \cong \mathbb{A}_{\mathsf{cc}}$ and $\mathbb{B} \cong \mathbb{B}_{\mathsf{cc}}$ in $\mathsf{Dist}^{\star\star}(\mathcal{Q})$ we have that $\mathbb{A} \cong \mathbb{B}$ in $\mathsf{Dist}^{\star\star}(\mathcal{Q})$ if and only if $\mathbb{A}_{\mathsf{cc}} \cong \mathbb{B}_{\mathsf{cc}}$ in $\mathsf{Dist}^{\star\star}_{\mathsf{cc}}(\mathcal{Q})$, which by the biequivalence of $\mathsf{Dist}^{\star\star}_{\mathsf{cc}}(\mathcal{Q})$ with $\mathsf{SCat}^{\star\star}_{\mathsf{cc}}(\mathcal{Q})$ is the same as an equivalence $\mathbb{A}_{\mathsf{cc}} \simeq \mathbb{B}_{\mathsf{cc}}$ in $\mathsf{SCat}^{\star\star}_{\mathsf{cc}}(\mathcal{Q})$ or in $\mathsf{SCat}^{\star\star}(\mathcal{Q})$ ($\mathsf{SCat}^{\star\star}_{\mathsf{cc}}(\mathcal{Q})$ is fully embedded in $\mathsf{SCat}^{\star\star}(\mathcal{Q})$). $\Box$

5.3 Enrichment in, and completion of, the base

Semicategories are categories?!?

We have seen in 4.5.6 that for any quantaloid $\mathcal{Q}$ the quantaloids $\mathsf{Dist}^{\star}(\mathcal{Q})$ and $\mathsf{Dist}(\mathsf{Dist}^{\star}(\mathcal{Q}))$ are equivalent. We may thus say that "the calculus of regular $\mathcal{Q}$-semicategories and regular distributors is (equivalent to) the calculus of $\mathsf{Dist}^{\star}(\mathcal{Q})$-categories and distributors". But this means that regular $\mathcal{Q}$-semicategories are actually categories... over a different base quantaloid[3]!

It is now natural to ask whether a similar property holds for the more refined notion of 'totally regular $\mathcal{Q}$-semicategories'. This is indeed the case! But we will have to work a little harder to show this.

To indicate how things work, it is useful to reconsider the characterizations in 5.1.4 and 5.1.5 of 'totally regular $\mathcal{Q}$-semicategory' and 'regular distributor' between such semicategories:

- a $\mathcal{Q}$-semicategory $\mathbb{A}$ is totally regular if and only if every endo-hom-arrow $\mathbb{A}(a,a)\colon ta \longrightarrow ta$ is an idempotent in $\mathcal{Q}$, and every $\mathbb{A}(a',a)\colon ta \longrightarrow ta'$ is a regular bimodule from $\mathbb{A}(a,a)$ to $\mathbb{A}(a',a')$;

- a distributor $\Phi\colon \mathbb{A} \longrightarrow \mathbb{B}$ from one totally regular $\mathcal{Q}$-semicategory to another is regular if and only if every element $\Phi(b,a)\colon ta \longrightarrow tb$ of this distributor is a regular bimodule from $\mathbb{A}(a,a)$ to $\mathbb{B}(b,b)$.

[3]However, it would be too simple to do away with the theory of regular $\mathcal{Q}$-semicategories: it has its own very specific notion of 'regular morphism' that has no straightforward translation in terms of functors between $\mathcal{Q}_{\mathsf{CC}}$-categories.

So there is an intimate connection between the calculus of totally regular $\mathcal{Q}$-semicategories and regular distributors, and idempotents and regular bimodules in $\mathcal{Q}$. Moreover, we know that idempotents and regular bimodules organize themselves in a new quantaloid $\mathsf{Bim}^\star(\mathcal{Q})$

To make this explicit, we first need an unavoidable series of lemmas.

Lemma 5.3.1 *Suppose that $\mathbb{A}$ is a $\mathcal{Q}$-category such that for all $a \in \mathbb{A}_0$ the endo-hom-arrow $\mathbb{A}(a,a)\colon ta \longrightarrow ta$ (which is always a monad in $\mathcal{Q}$) has an algebra object in $\mathcal{Q}$. Then $\mathbb{A}$ is isomorphic in $\mathsf{Dist}(\mathcal{Q})$ to a $\mathcal{Q}$-category whose endo-hom-arrows are identities.*

Proof: Choose, for every $a \in \mathbb{A}_0$, an algebra object and corresponding adjunction for the monad $\mathbb{A}(a,a) : ta \longrightarrow ta$ in $\mathcal{Q}$, and write it as:

$$ta \underset{f_a}{\overset{u_a}{\leftrightarrows}} A_a;$$

thus $u_a \circ f_a = \mathbb{A}(a,a)$ and $f_a \circ u_a = 1_{A_a}$ (and $f_a \dashv u_a$). Now we claim that:

(i) there is a $\mathcal{Q}$-category $\widetilde{\mathbb{A}}$ consisting of

- objects: $(\widetilde{\mathbb{A}})_0 = \mathbb{A}_0$ with types $\tilde{t}a = A_a$,
- hom-arrows: $\widetilde{\mathbb{A}}(a',a) = f_{a'} \circ \mathbb{A}(a',a) \circ u_a$;

in particular, its endo-hom-arrows are identities;

(ii) there is a distributor $\Phi : \mathbb{A} \xrightarrow{\circ} \widetilde{\mathbb{A}}$ consisting of

- elements: $\Phi(a',a) = f_{a'} \circ \mathbb{A}(a',a)$;

(iii) there is another distributor $\Psi : \widetilde{\mathbb{A}} \xrightarrow{\circ} \mathbb{A}$ consisting of

- elements: $\Psi(a',a) = \mathbb{A}(a',a) \circ u_a$;

(iv) and $\Psi = \Phi^{-1}$ in $\mathsf{Dist}(\mathcal{Q})$.

We must verify that these definitions do what they are supposed to.

(i) Obviously $(\widetilde{\mathbb{A}}_0, \tilde{t})$ is a $\mathcal{Q}$-typed set and $\widetilde{\mathbb{A}}$ an endomatrix on $(\widetilde{\mathbb{A}}_0, \tilde{t})$. We have a composition-inequality for every $a, a', a'' \in \widetilde{\mathbb{A}}_0$ like so:

$$\begin{aligned}
\widetilde{\mathbb{A}}(a'',a') \circ \widetilde{\mathbb{A}}(a',a) &= \Big(f_{a''} \circ \mathbb{A}(a'',a') \circ u_{a'}\Big) \circ \Big(f_{a'} \circ \mathbb{A}(a',a) \circ u_a\Big) \\
&= f_{a''} \circ \mathbb{A}(a'',a') \circ \mathbb{A}(a',a') \circ \mathbb{A}(a',a) \circ u_a \\
&\le f_{a''} \circ \mathbb{A}(a'',a) \circ u_a \\
&= \widetilde{\mathbb{A}}(a'',a).
\end{aligned}$$

As for the endo-hom-arrows of $\widetilde{\mathbb{A}}$, they equal identities: for every $a \in \widetilde{\mathbb{A}}_0$,

$$\begin{aligned}\widetilde{\mathbb{A}}(a,a) &= f_a \circ \mathbb{A}(a,a) \circ u_a \\ &= f_a \circ u_a \circ f_a \circ u_a \\ &= 1_{\tilde{t}a} \circ 1_{\tilde{t}a} \\ &= 1_{\tilde{t}a}.\end{aligned}$$

(ii) The elements of Φ form a matrix from $(\mathbb{A}_0, t)$ to $(\widetilde{\mathbb{A}}_0, \tilde{t})$. This matrix is a distributor: for $a, a', a'' \in \mathbb{A}_0 = \widetilde{\mathbb{A}}_0$,

$$\begin{aligned}\widetilde{\mathbb{A}}(a'',a') \circ \Phi(a',a) &= f_{a''} \circ \mathbb{A}(a'',a') \circ u_{a'} \circ f_{a'} \circ \mathbb{A}(a',a) \\ &= f_{a''} \circ \mathbb{A}(a'',a') \circ \mathbb{A}(a',a') \circ \mathbb{A}(a',a) \\ &\leq f_{a''} \circ \mathbb{A}(a'',a) \\ &= \Phi(a'',a)\end{aligned}$$

and also

$$\begin{aligned}\Phi(a'',a') \circ \mathbb{A}(a',a) &= f_{a''} \circ \mathbb{A}(a'',a') \circ \mathbb{A}(a',a) \\ &\leq f_{a''} \circ \mathbb{A}(a'',a) \\ &= \Phi(a'',a).\end{aligned}$$

(iii) The verifications for Ψ are similar.

(iv) We end by checking that $\Psi = \Phi^{-1}$ in $\mathsf{Dist}(\mathcal{Q})$. For $a, a' \in \mathbb{A}_0$,

$$\begin{aligned}\Big(\Psi \otimes_{\widetilde{\mathbb{A}}} \Phi\Big)(a',a) &= \bigvee_{x \in \widetilde{\mathbb{A}}_0} \Psi(a',x) \circ \Phi(x,a) \\ &= \bigvee_{x \in \widetilde{\mathbb{A}}_0} \mathbb{A}(a',x) \circ u_x \circ f_x \circ \mathbb{A}(x,a) \\ &= \bigvee_{x \in \widetilde{\mathbb{A}}_0} \mathbb{A}(a',x) \circ \mathbb{A}(x,x) \circ \mathbb{A}(x,a) \\ &= \mathbb{A}(a',a)\end{aligned}$$

so indeed $\Psi \otimes_{\widetilde{\mathbb{A}}} \Phi$ is the identity distributor on $\mathbb{A}$. And similarly, for $a, a' \in \widetilde{\mathbb{A}}_0$,

$$\begin{aligned}\Big(\Phi \otimes_{\mathbb{A}} \Psi\Big)(a',a) &= \bigvee_{y \in \mathbb{A}_0} \Phi(a',y) \circ \Psi(y,a) \\ &= \bigvee_{y \in \mathbb{A}_0} f_{a'} \circ \mathbb{A}(a',y) \circ \mathbb{A}(y,a) \circ u_a \\ &= f_{a'} \circ \Big(\bigvee_{y \in \mathbb{A}_0} \mathbb{A}(a',y) \circ \mathbb{A}(y,a)\Big) \circ u_a \\ &= f_{a'} \circ \mathbb{A}(a',a) \circ u_a \\ &= \widetilde{\mathbb{A}}(a',a)\end{aligned}$$

so $\Psi\otimes_{\mathbb{A}}\Phi$ is the identity distributor on $\widetilde{\mathbb{A}}$. $\square$

For convenience, let us say that a $\mathcal{Q}$-category whose endo-hom-arrows are identities, is *normal*. And let us denote $\mathsf{Dist}_{\mathsf{N}}(\mathcal{Q})$ for the full subquantaloid of $\mathsf{Dist}(\mathcal{Q})$ whose objects are precisely those normal $\mathcal{Q}$-categories. It then follows directly from 5.3.1 (and 4.4.7) that...

Corollary 5.3.2 *If $\mathcal{Q}$ is a quantaloid admitting (free) algebra objects for its monads, then the full embedding*

$$\mathsf{Dist}_{\mathsf{N}}(\mathcal{Q})\longrightarrow\mathsf{Dist}(\mathcal{Q})$$

is an equivalence of quantaloids. In particular[4]*, for any quantaloid $\mathcal{Q}$ the full embedding*

$$\mathsf{Dist}_{\mathsf{N}}(\mathsf{Bim}^{\star}(\mathcal{Q}))\longrightarrow\mathsf{Dist}(\mathsf{Bim}^{\star}(\mathcal{Q}))$$

is an equivalence.

Lemma 5.3.3 *There is a fully faithful quantaloid homomorphism*

$$\widehat{(-)}:\mathsf{Dist}^{\star\star}(\mathcal{Q})\longrightarrow\mathsf{Dist}_{\mathsf{N}}(\mathsf{Bim}^{\star}(\mathcal{Q})):(\mathbb{A}\overset{\Phi}{\longrightarrow}\mathbb{B})\mapsto(\widehat{\mathbb{A}}\overset{\widehat{\Phi}}{\longrightarrow}\widehat{\mathbb{B}})$$

where

- $\widehat{\mathbb{A}}$ *is determined by:*
  - *objects:* $(\widehat{\mathbb{A}})_0=\mathbb{A}_0$ *and types* $\widehat{t}a=\mathbb{A}(a,a)$*;*
  - *hom-arrows:* $\widehat{\mathbb{A}}(a',a)=\mathbb{A}(a',a)$ *as $\mathcal{Q}$-arrows;*
- $\widehat{\Phi}:\widehat{\mathbb{A}}\longrightarrow\widehat{\mathbb{B}}$ *is determined by:*
  - *elements:* $\widehat{\Phi}(b,a)=\Phi(b,a)$ *as $\mathcal{Q}$-arrows.*

Proof: (i) Since the endo-hom-arrows of $\mathbb{A}$ are idempotents in $\mathcal{Q}$, it is true that $(\widehat{\mathbb{A}}_0,\widehat{t})$ is a $\mathsf{Bim}^{\star}(\mathcal{Q})$-typed set; and since the hom-arrows of $\mathbb{A}$ are bimodules in $\mathcal{Q}$ between de endo-hom-arrows, $\widehat{\mathbb{A}}$ is a $\mathsf{Bim}^{\star}(\mathcal{Q})$-matrix from $(\widehat{\mathbb{A}}_0,\widehat{t})$ to itself. Its endo-hom-arrows are then identities, because the identity bimodule on an idempotent in $\mathcal{Q}$ is that idempotent itself. And it is actually a $\mathsf{Bim}^{\star}(\mathcal{Q})$-category because moreover, for all $a,a',a''\in\widehat{\mathbb{A}}_0$,

$$\begin{aligned}\widehat{\mathbb{A}}(a'',a')\otimes_{\widehat{t}a'}\widehat{\mathbb{A}}(a',a) &= \widehat{\mathbb{A}}(a'',a')\circ\widehat{\mathbb{A}}(a',a)\\ &= \mathbb{A}(a'',a')\circ\mathbb{A}(a',a)\\ &\leq \mathbb{A}(a'',a)\\ &= \widehat{\mathbb{A}}(a'',a).\end{aligned}$$

[4]A similar equivalence holds for the base quantaloid $\mathsf{Bim}(\mathcal{Q})$.

(ii) $\widehat{\Phi}$ is certainly a $\mathsf{Bim}^\star(\mathcal{Q})$-matrix from $(\widehat{\mathbb{A}},\widehat{t})$ to $(\widehat{\mathbb{B}},\widehat{t})$. As for its action-inequalities, for every $a,a'\in\widehat{\mathbb{A}}_0$ and $b\in\widehat{\mathbb{B}}_0$,

$$\begin{array}{rcl}\widehat{\Phi}(b,a')\otimes_{\widehat{ta'}}\widehat{\mathbb{A}}(a',a) & = & \widehat{\Phi}(b,a')\circ\widehat{\mathbb{A}}(a',a)\\ & = & \Phi(b,a')\circ\mathbb{A}(a',a)\\ & \leq & \Phi(a'',a)\\ & = & \widehat{\Phi}(a'',a).\end{array}$$

The action on the left by $\widehat{\mathbb{B}}$ is similar.

(iii) The ordering of distributors is preserved and reflected: simply because, for two regular distributors $\Phi,\Psi:\mathbb{A}\multimap\mathbb{B}$ between totally regular $\mathcal{Q}$-semicategories,

$$\begin{array}{rcl}\Phi\leq\Psi\text{ in }\mathsf{Dist}^{\star\star}(\mathcal{Q}) & \iff & \forall a\in\mathbb{A}_0,b\in\mathbb{B}_0:\Phi(b,a)\leq\Psi(b,a)\text{ in }\mathcal{Q}\\ & \iff & \forall a\in\mathbb{A}_0,b\in\mathbb{B}_0:\widehat{\Phi}(b,a)\leq\widehat{\Psi}(b,a)\text{ in }\mathsf{Bim}^\star(\mathcal{Q})\\ & \iff & \widehat{\Phi}\leq\widehat{\Psi}\text{ in }\mathsf{Dist}(\mathsf{Bim}^\star(\mathcal{Q})).\end{array}$$

(iv) Given any $\Theta:\widehat{\mathbb{A}}\multimap\widehat{\mathbb{B}}$ in $\mathsf{Dist}(\mathsf{Bim}^\star(\mathcal{Q}))$ we claim that $\Theta=\widehat{\Phi}$ for some $\Phi:\mathbb{A}\multimap\mathbb{B}$ in $\mathsf{Dist}^{\star\star}(\mathcal{Q})$. But, for any $a\in\widehat{\mathbb{A}}_0$ and $b\in\widehat{\mathbb{B}}_0$, the element $\Theta(b,a):\widehat{ta}\longrightarrow\widehat{tb}$ is a bimodule with an underlying $\mathcal{Q}$-arrow from ta to tb (because $\widehat{ta}$ is $\mathbb{A}(a,a):ta\longrightarrow ta$ and likewise for $\widehat{tb}$). So it makes sense to put Φ to be the $\mathcal{Q}$-matrix whose elements are the underlying $\mathcal{Q}$-arrows of the elements of the $\mathsf{Bim}^\star(\mathcal{Q})$-matrix Θ:

$$\forall a\in\mathbb{A}_0,b\in\mathbb{B}_0:\Phi(b,a)=\Theta(b,a)\text{ as }\mathcal{Q}\text{-arrows.}$$

Then Φ is a distributor from $\mathbb{A}$ to $\mathbb{B}$: for $a,a'\in\mathbb{A}_0$ and $b\in\mathbb{B}_0$ there is an action-inequality

$$\begin{array}{rcl}\Phi(b,a')\circ\mathbb{A}(a',a) & = & \Theta(b,a')\otimes_{\widehat{ta'}}\widehat{\mathbb{A}}(a',a)\\ & \leq & \Theta(b,a)\\ & = & \Phi(b,a)\end{array}$$

which for endo-hom-arrows saturates to an equality

$$\begin{array}{rcl}\Phi(b,a)\circ\mathbb{A}(a,a) & = & \Theta(b,a)\otimes_{\widehat{ta}}\widehat{ta}\\ & = & \Theta(b,a)\\ & = & \Phi(b,a)\end{array}$$

because $\Theta(b,a)$ is a bimodule with domain $\widehat{ta}=\mathbb{A}(a,a)$. Likewise for actions on the left by $\mathbb{B}$. It follows that Φ is actually a regular distributor from $\mathbb{A}$ to $\mathbb{B}$. And it is obvious from the definition that $\widehat{\Phi}=\Theta$.

(v) The identity regular distributor on a totally regular $\mathscr{Q}$-semicategory $\mathbb{A}$ is $\mathbb{A}:\mathbb{A}\mathrel{\circ\!\!\!\rightarrow}\mathbb{A}$ itself. It is quite obvious that $\widehat{\mathbb{A}}:\widehat{\mathbb{A}}\mathrel{\circ\!\!\!\rightarrow}\widehat{\mathbb{A}}$ is the identity distributor in $\mathsf{Dist}(\mathsf{Bim}^\star(\mathscr{Q}))$ too.

(vi) Consider now two regular distributors between totally regular $\mathscr{Q}$-semicategories, like so:

$$\mathbb{A}\xrightarrow[\circ]{\Phi}\mathbb{B}\xrightarrow[\circ]{\Psi}\mathbb{C}.$$

Then, for any $a\in\widehat{\mathbb{A}}_0$ and $b\in\widehat{\mathbb{B}}_0$,

$$\begin{aligned}\Big(\widehat{\Psi\otimes_{\mathbb{B}}\Phi}\Big)(c,a) &= \Big(\Psi\otimes_{\mathbb{B}}\Phi\Big)(c,a)\\ &= \bigvee_{b\in\mathbb{B}_0}\Psi(c,b)\circ\Phi(b,a)\\ &= \bigvee_{b\in\widehat{\mathbb{B}}_0}\widehat{\Psi}(c,b)\otimes_{\widehat{tb}}\widehat{\Phi}(b,a)\\ &= \Big(\widehat{\Psi}\otimes_{\widehat{\mathbb{B}}}\widehat{\Phi}\Big)(c,a).\end{aligned}$$

So (i), (ii), (v) and (vi) say that we've defined a functor

$$\mathsf{Dist}^{\star\star}(\mathscr{Q})\longrightarrow\mathsf{Dist}_{\mathsf{N}}(\mathsf{Bim}^\star(\mathscr{Q})):(\mathbb{A}\xrightarrow[\circ]{\Phi}\mathbb{B})\mapsto(\widehat{\mathbb{A}}\xrightarrow[\circ]{\widehat{\Phi}}\widehat{\mathbb{B}})$$

which by (iii) and (iv) is a fully faithful quantaloid homomorphism. $\square$

Lemma 5.3.4 *The homomorphism in 5.3.3 is actually essentially surjective on objects.*

Proof: Let $\mathbb{C}$ be a normal $\mathsf{Bim}^\star(\mathscr{Q})$-category. We claim that the following data define a totally regular $\mathscr{Q}$-semicategory $\overline{\mathbb{C}}$ for which $\mathbb{C}=\widehat{\overline{\mathbb{C}}}$ (as $\mathsf{Bim}^\star(\mathscr{Q})$-categories):

- objects: $\overline{\mathbb{C}}_0=\mathbb{C}_0$ with types $\bar{t}c=\mathsf{dom}(tc)$[5];
- hom-arrows: $\overline{\mathbb{C}}(c',c)=\mathbb{C}(c',c)$ as $\mathscr{Q}$-arrows.

Verifications:

(i) Since for every $c\in\mathbb{C}_0$ its type tc is an object of $\mathsf{Bim}^\star(\mathscr{Q})$, $\bar{t}c=\mathsf{dom}(tc)$ is an object of $\mathscr{Q}$ so that $(\overline{\mathbb{C}}_0,\bar{t})$ is indeed a $\mathscr{Q}$-typed set. The hom-arrows of $\mathbb{C}$ are bimodules: $\mathbb{C}(c',c):tc\mathrel{\circ\!\!\!\rightarrow}tc'$; their underlying $\mathscr{Q}$-arrows thus go from $\mathsf{dom}(tc)$ to $\mathsf{dom}(tc')$. Therefore $\overline{\mathbb{C}}$ is a $\mathscr{Q}$-endo-matrix on $(\overline{\mathbb{C}},\bar{t})$. For this matrix we can now calculate that, for any $c,c',c''\in\overline{\mathbb{C}}_0$,

$$\begin{aligned}\overline{\mathbb{C}}(c'',c')\circ\overline{\mathbb{C}}(c',c) &= \mathbb{C}(c'',c')\circ\mathbb{C}(c',c)\\ &= \mathbb{C}(c'',c')\otimes_{tc'}\mathbb{C}(c',c)\\ &\le \mathbb{C}(c'',c)\\ &= \overline{\mathbb{C}}(c'',c)\end{aligned}$$

[5]Here we mean: the domain of the idempotent tc in $\mathscr{Q}$.

but also

$$\begin{aligned}\overline{\mathbb{C}}(c',c)\circ\overline{\mathbb{C}}(c,c) &= \mathbb{C}(c',c)\circ\mathbb{C}(c,c)\\ &= \mathbb{C}(c',c)\otimes_{tc}1_{tc}\\ &= \mathbb{C}(c',c)\otimes_{tc}tc\\ &= \mathbb{C}(c',c)\\ &= \overline{\mathbb{C}}(c',c)\end{aligned}$$

and likewise $\overline{\mathbb{C}}(c',c')\circ\overline{\mathbb{C}}(c',c) = \overline{\mathbb{C}}(c',c)$ so that $\overline{\mathbb{C}}$ is indeed a totally regular $\mathcal{Q}$-semicategory.

(ii) If we now look at $\widehat{\overline{\mathbb{C}}}$ then obviously $(\widehat{\overline{\mathbb{C}}})_0 = \mathbb{C}_0$ as sets, but moreover

$$\widehat{\overline{t}}c = \overline{\mathbb{C}}(c,c) = \mathbb{C}(c,c) = 1_{tc} = tc,$$

so as $\mathsf{Bim}^\star(\mathcal{Q})$-typed sets too $(\widehat{\overline{\mathbb{C}}},\widehat{\overline{t}})$ and $(\mathbb{C},t)$ coincide. The matrix $\widehat{\overline{\mathbb{C}}}$ has as elements, for $c,c'\in\widehat{\overline{\mathbb{C}}}_0=\mathbb{C}_0$,

$$\widehat{\overline{\mathbb{C}}}(c',c) = \overline{\mathbb{C}}(c',c) = \mathbb{C}(c',c),$$

so it coincides with the matrix $\mathbb{C}$. Therefore $\widehat{\overline{\mathbb{C}}}$ and $\mathbb{C}$ are identical $\mathsf{Bim}^\star(\mathcal{Q})$-categories. $\square$

Corollary 5.3.5 *For any quantaloid $\mathcal{Q}$, $\mathsf{Dist}^{\star\star}(\mathcal{Q})$ is equivalent as quantaloid to $\mathsf{Dist}_{\mathsf{N}}(\mathsf{Bim}^\star(\mathcal{Q}))$, hence also equivalent to $\mathsf{Dist}(\mathsf{Bim}^\star(\mathcal{Q}))$.*

Equivalent distributor calculi

We have discussed on p. 142 the Cauchy completion of a (small) quantaloid $\mathcal{Q}$, or rather, the two Cauchy completions of $\mathcal{Q}$:

- let us denote $\mathcal{Q}_{\mathsf{cc}}$ for the Cauchy completion of $\mathcal{Q}$ in CAT (the Cauchy completion of $\mathcal{Q}$ *as category*), which is given (to within equivalence) by $\mathsf{Bim}^\star(\mathcal{Q})$;

- similarly, let us denote $\mathcal{Q}_{\mathsf{CC}}$ for the Cauchy completion of $\mathcal{Q}$ in QUANT (the Cauchy completion of $\mathcal{Q}$ *as quantaloid*), which is given (to within equivalence) by $\mathsf{Dist}^\star(\mathcal{Q})$.

We may then resume the discussion on the previous pages as follows.

Theorem 5.3.6 *For a (small) quantaloid $\mathcal{Q}$ we may say that*

$$\begin{tikzcd}
& \mathscr{Q} \arrow[rr] \arrow[dl, equal] \arrow[dd] & & \mathsf{Bim}^{\star}(\mathscr{Q}) \arrow[rr] \arrow[dl, equal] \arrow[dd] & & \mathsf{Dist}^{\star}(\mathscr{Q}) \arrow[dl, equal] \arrow[dd, equal] \\
\mathscr{Q} \arrow[rr] \arrow[dd] & & \mathscr{Q}_{\mathsf{cc}} \arrow[rr] \arrow[dd] & & \mathscr{Q}_{\mathsf{CC}} \arrow[dd, equal] & \\
& \mathsf{Dist}(\mathscr{Q}) \arrow[rr] \arrow[dl, equal] & & \mathsf{Dist}^{\star\star}(\mathscr{Q}) \arrow[rr] \arrow[dl, equal] & & \mathsf{Dist}(\mathsf{Dist}^{\star}(\mathscr{Q})) \arrow[dl, equal] \\
\mathsf{Dist}(\mathscr{Q}) \arrow[rr] & & \mathsf{Dist}(\mathscr{Q}_{\mathsf{cc}}) \arrow[rr] & & \mathsf{Dist}(\mathscr{Q}_{\mathsf{CC}}) &
\end{tikzcd}$$

Diagram 5.5

1. *"the calculus of regular $\mathscr{Q}$-semicategories and distributors is (equivalent to) the calculus of $\mathscr{Q}_{\mathsf{CC}}$-categories and distributors":*

$$\mathsf{Dist}^{\star}(\mathscr{Q}) \simeq \mathsf{Dist}(\mathscr{Q}_{\mathsf{CC}});$$

2. *"the calculus of totally regular $\mathscr{Q}$-semicategories and distributors is (equivalent to) the calculus of $\mathscr{Q}_{\mathsf{cc}}$-categories and distributors":*

$$\mathsf{Dist}^{\star\star}(\mathscr{Q}) \simeq \mathsf{Dist}(\mathscr{Q}_{\mathsf{cc}}).$$

In all its conciseness this is a very attractive theorem! And we end up with diagram 5.5 in which all arrows are homomorphisms of quantaloids, and equalities must be read as equivalences of quantaloids. On the foreground, on the top row $\mathscr{Q}$ gets Cauchy completed and from the top row to the bottom $\mathscr{Q}$ and its Cauchy completions are the base for enriched categories and distributors; this is the more "conceptual" scheme of things. In the background we have an equivalent but more explicit (more "elementary") presentation of the same scheme of completions and enrichments.

How about functors and regular morphisms then?

The crux of the theory of Cauchy completions and Morita equivalence, be it for $\mathscr{Q}$-categories or totally regular $\mathscr{Q}$-semicategories, is that functors (resp. regular morphisms) can be recovered as left adjoint (regular) distributors—"up to Morita equivalence".

By 5.3.6 we know that $\mathsf{Dist}(\mathscr{Q}_{\mathsf{cc}}) \simeq \mathsf{Dist}^{\star\star}(\mathscr{Q})$ and so it follows that

$$\mathsf{Cat}_{\mathsf{cc}}(\mathscr{Q}_{\mathsf{cc}}) \simeq \mathsf{SCat}^{\star\star}_{\mathsf{cc}}(\mathscr{Q}).$$

$$\begin{tikzcd}
\mathsf{Cat}_{\mathsf{cc}}(\mathscr{Q}_{\mathsf{cc}}) \arrow[r, equal] & \mathsf{SCat}^{\star\star}_{\mathsf{cc}}(\mathscr{Q}) \\
\mathsf{Cat}(\mathscr{Q}_{\mathsf{cc}}) \arrow[u, shift left, "(-)_{\mathsf{cc}}"] \arrow[r, dotted, shift left] & \mathsf{SCat}^{\star\star}(\mathscr{Q}) \arrow[u, shift left, "(-)_{\mathsf{cc}}"] \arrow[l, dotted, shift left]
\arrow[from=1-1, to=2-1, shift left, "i"] \arrow[from=1-2, to=2-2, shift left, "i"]
\end{tikzcd}$$

Diagram 5.6

Recall that for both categories and totally regular semicategories "Cauchy completion is right adjoint to the full embedding" (see 3.5.9 and 5.2.8). So we can compose the functors in diagram 5.6 (where the equality should be read as an equivalence) to obtain a "comparison" of the (2-)categories $\mathsf{Cat}(\mathscr{Q}_{\mathsf{cc}})$ and $\mathsf{SCat}^{\star\star}(\mathscr{Q})$ by means of the dotted functors. But – and this is the point we wish to make – although $\mathsf{Cat}_{\mathsf{cc}}(\mathscr{Q}_{\mathsf{cc}}) \simeq \mathsf{SCat}^{\star\star}_{\mathsf{cc}}(\mathscr{Q})$, it does not follow that $\mathsf{Cat}(\mathscr{Q}_{\mathsf{cc}}) \simeq \mathsf{SCat}^{\star\star}(\mathscr{Q})$!

For regular $\mathscr{Q}$-semicategories matters are worse. From 5.3.6 we have that $\mathsf{Dist}^{\star}(\mathscr{Q}) \simeq \mathsf{Dist}(\mathscr{Q}_{\mathsf{CC}}) \simeq \mathscr{Q}_{\mathsf{CC}}$, and it thus follows that

$$\mathsf{Cat}_{\mathsf{cc}}(\mathscr{Q}_{\mathsf{CC}}) \simeq \mathsf{Map}(\mathsf{Dist}^{\star}(\mathscr{Q}))...$$

But since there is no suitable theory of "Cauchy completions" for regular $\mathscr{Q}$-semicategories that guarantees an equivalence like "$\mathsf{Map}(\mathsf{Dist}^{\star}(\mathscr{Q})) \simeq \mathsf{SCat}^{\star}_{\mathsf{cc}}(\mathscr{Q})$" or an adjunction like "$(-)_{\mathsf{cc}} \dashv i\colon \mathsf{SCat}^{\star}_{\mathsf{cc}}(\mathscr{Q}) \rightleftarrows \mathsf{SCat}^{\star}(\mathscr{Q})$", there is simply no natural comparison between the (locally ordered) categories $\mathsf{SCat}^{\star}(\mathscr{Q})$ and $\mathsf{Cat}(\mathscr{Q}_{\mathsf{CC}})$ (or even $\mathsf{Cat}_{\mathsf{cc}}(\mathscr{Q}_{\mathsf{CC}})$). In other words: the concept of a 'regular morphism' belongs specifically to the theory of regular semicategories.

5.4 Orders and ideals over a base quantaloid

Ordered sheaves on a locale

It is well-known that the topos $\mathsf{Sh}(\Omega)$ of sheaves on a locale Ω can be described in terms of Ω-sets. In [Borceux and Cruciani, 1998] the authors give an analogous description of the ordered objects in $\mathsf{Sh}(\Omega)$. As the material presented in this thesis was developed precisely to better understand the results obtained by F. Borceux and R. Cruciani, we'd better explain how far we got (and let the reader decide if we have achieved our goal).

First we copy some definitions and key results that can be found in the cited reference. Throughout Ω denotes a locale.

Definition 5.4.1 *A* skew Ω-set *is a pair* $(A, [\,\cdot \leq \cdot\,])$ *where* A *is a set and* $[\,\cdot \leq \cdot\,]\colon A \times A \longrightarrow \Omega$ *is a mapping satisfying*

$$\begin{aligned}&[a \leq a'] \leq [a \leq a],\\ &[a \leq a'] \leq [a' \leq a'],\\ &[a \leq a'] \wedge [a' \leq a''] \leq [a \leq a''],\end{aligned}$$

for all elements a, a', a'' *of* A.

Definition 5.4.2 *Let* A, B *be skew* Ω*-sets. A* morphism of skew Ω-sets, *denoted* $f\colon A \longrightarrow B$, *is a pair of mappings*

$$\begin{aligned}&[f\cdot \leq \cdot\,]\colon A \times B \longrightarrow \Omega\colon (a, b) \mapsto [fa \leq b],\\ &[\cdot \ \leq f\cdot\,]\colon B \times A \longrightarrow \Omega\colon (b, a) \mapsto [b \leq fa],\end{aligned}$$

satisfying

$$\begin{aligned}&\textstyle\bigvee_{a \in A} [a' \leq a] \wedge [fa \leq b] = [fa' \leq b],\\ &\textstyle\bigvee_{a \in A} [b \leq fa] \wedge [a \leq a'] = [b \leq fa'],\\ &\textstyle\bigvee_{b \in B} [fa \leq b] \wedge [b \leq b'] = [fa \leq b'],\\ &\textstyle\bigvee_{b \in B} [b' \leq b] \wedge [b \leq fa] = [b' \leq fa],\\ &[b \leq fa] \wedge [fa \leq b'] \leq [b \leq b'],\\ &[a \leq a'] \leq \textstyle\bigvee_{b \in B} [fa \leq b] \wedge [b \leq fa'],\end{aligned}$$

for all a, a' *in* A *and* b, b' *in* B.

Proposition 5.4.3 *Skew* Ω*-sets and their morphisms form a category, with composition of morphisms* $f\colon A \longrightarrow B$, $g\colon B \longrightarrow C$ *calculated with the formulae*

$$[(g \circ f)a \leq c] = \bigvee_{b \in B} [fa \leq b] \wedge [gb \leq c],$$

$$[c \leq (g \circ f)a] = \bigvee_{b \in B} [c \leq gb] \wedge [b \leq fa],$$

and the identity morphism $1_A\colon A \longrightarrow A$ *given by*

$$[1_A a \leq a'] = [a \leq a'] = [a \leq 1_A a'].$$

The category of which this proposition speaks, is denoted as Ω_{sk}-Set in [Borceux and Cruciani, 1998]. The following auxiliary notion will allow to express the "completeness" of a skew Ω-set.

Definition 5.4.4 *A* singleton s *of a skew* Ω*-set* A *is a pair of mappings*

$$[s \leq \cdot\,]\colon A \longrightarrow \Omega, \quad [\,\cdot \leq s]\colon A \longrightarrow \Omega$$

satisfying

$$\begin{array}{l}\bigvee_{a\in A}\,[s\le a]\wedge[a\le a']=[s\le a'],\\ \bigvee_{a\in A}\,[a'\le a]\wedge[a\le s]=[a'\le s],\\ [a'\le s]\wedge[s\le a]\le[a'\le a],\\ \bigvee_{a\in A}\,[s\le a]=\bigvee_{a\in A}([s\le a]\wedge[a\le s])=\bigvee_{a\in A}\,[a\le s]\end{array}$$

for all a, a' in A. The quantity in the fourth condition is called the support *of the singleton.*

Lemma 5.4.5 *For a skew Ω-set A, any element $a \in A$ determines a singleton by means of the mappings*

$$[a\le\,\cdot\,]\colon A\longrightarrow\Omega,\quad [\,\cdot\le a]\colon A\longrightarrow\Omega;$$

its support is $[a \le a]$.

Definition 5.4.6 *A skew Ω-set A is* complete *when every singleton has the form $([a\le\,\cdot\,],[\,\cdot\le a])$ for a unique element $a \in A$.*

Of course the complete skew Ω-sets determine a full subcategory of Ω_{sk}-Set; Borceux and Cruciani [1998] denote it as $\Omega^{\mathsf{c}}_{\mathsf{sk}}$-$\mathsf{Set}$.

Proposition 5.4.7 *Let A and B be two skew Ω-sets, with B complete. There exists a bijection between*

1. *the morphisms of skew Ω-sets $f\colon A\longrightarrow B$;*
2. *the actual mappings $\varphi\colon A\longrightarrow B$ satisfying*

$$[a\le a]=[\varphi a\le\varphi a],\quad [a\le a']\le[\varphi a\le\varphi a']$$

for all a, a' in A.

The category $\Omega^{\mathsf{c}}_{\mathsf{sk}}$-$\mathsf{Set}$ is equivalent to the category whose objects are complete skew Ω-sets and whose morphisms are those mappings as in condition 2. of the statement, composition and identities being the usual ones.

Proposition 5.4.8 *Every skew Ω-set is isomorphic to a complete skew Ω-set. Thus the category of skew Ω-sets is equivalent to its full subcategory of complete skew Ω-sets: Ω_{sk}-$\mathsf{Set}\simeq\Omega^{\mathsf{c}}_{\mathsf{sk}}$-$\mathsf{Set}$.*

Theorem 5.4.9 *The category of complete skew Ω-sets is equivalent to the category of ordered sets in the topos of sheaves on Ω: $\Omega^{\mathsf{c}}_{\mathsf{sk}}$-$\mathsf{Set}\simeq\mathsf{Ord}(\mathsf{Sh}(\Omega))$.*

The point is now that these definitions and results are examples of the theory of totally regular $\mathcal{Q}$-semicategories!

Proposition 5.4.10 *A skew Ω-set $(A, [\,\cdot \leq \cdot\,])$ may be viewed as an Ω-enriched*[6] *structure $\mathbb{A}$ by putting*

- *objects: $\mathbb{A}_0 = A$ (types are trivial);*
- *hom-arrows: for a, a' in $\mathbb{A}_0 = A$, put $\mathbb{A}(a', a) = [a' \leq a]$.*

This defines a totally regular Ω-semicategory.

A morphism $f\colon A \longrightarrow B$ of skew Ω-sets is then precisely a left adjoint regular distributor $\Phi\colon \mathbb{A} \multimap \mathbb{B}$ whose elements are $\Phi(b, a) = [b \leq fa]$.

Composition of morphisms and identity morphisms of skew Ω-sets correspond to composition of distributors and identity distributors.

Proof: Obviously $\mathbb{A}$ is an Ω-semicategory. The first two conditions in 5.4.1 may be rewritten as

$$[a \leq a] \wedge [a \leq a'] = [a \leq a'] \quad \text{and} \quad [a \leq a'] \wedge [a' \leq a'] = [a \leq a']$$

so the semicategory is totally regular. As for a morphism $f\colon A \longrightarrow B$, the conditions in 5.4.2 say precisely that $\Phi(-, ?) = [- \leq f?]$ is a regular distributor with right adjoint $\Phi^*(?, -) = [f? \leq -]$. And the formulae for composition and identities are identical to those for distributors! $\square$

Corollary 5.4.11 *The category of skew Ω-sets and morphisms is (isomorphic to) the (2-)category of totally regular Ω-semicategories and left adjoint regular distributors: $\Omega_{\mathsf{sk}}\text{-}\mathsf{Set} \simeq \mathsf{Map}(\mathsf{Dist}^{\star\star}(\Omega))$.*

Proposition 5.4.12 *Let $\mathbb{A}$ denote the totally regular Ω-semicategory corresponding to some skew Ω-set A. The singletons with support $u \in \Omega$ of A are in bijective correspondence with the Cauchy distributors from $*_u$ into $\mathbb{A}$, in the following way: for a singleton s with support u the corresponding Cauchy distributor $\sigma\colon *_u \multimap \mathbb{A}$ has elements $\sigma(a) = [a \leq s]$.*

Proof: The only surprise here is that every Cauchy distributor $\sigma\colon *_u \multimap \mathbb{A}$ satisfies $\sigma^* \otimes_{\mathbb{A}} \sigma = (u)$ (and not merely $\sigma^* \otimes_{\mathbb{A}} \sigma \geq (u)$)—i.e. that the unit-inequality of the adjunction is *always* an equality. But this is due to the fact that for all $a \in \mathbb{A}_0$, $\sigma(a) \leq u$ and also $\sigma^*(a) \leq u$ (which in turn is due to the action-inequalities for the distributors σ and σ^*). $\square$

[6] Or rather, a structure enriched in the one-object suspension of Ω...

Corollary 5.4.13 *The completeness of a skew Ω-set A is (equivalent to) the Cauchy completeness of the associated totally regular Ω-semicategory $\mathbb{A}$.*

Lemma 5.4.14 *Consider skew Ω-sets A and B. The following are equivalent for a mapping $\varphi\colon A\longrightarrow B\colon a \mapsto \varphi a$ (of sets):*

1. $[a \leq a] = [\varphi a \leq \varphi a]$ *and* $[a \leq a'] \leq [\varphi a \leq \varphi a']$ *for all* a, a' *in* A*;*
2. $[a \leq a'] \leq [\varphi a \leq \varphi a']$, $[a \leq a] \wedge [\varphi a \leq a'] = [\varphi a \leq a']$, *and* $[a \leq \varphi a'] \wedge [a' \leq a'] = [a \leq \varphi a']$ *for all* a, a' *in* A.

That is to say, such a map is a regular morphism $\varphi\colon \mathbb{A}\longrightarrow\mathbb{B}$ between the totally regular Ω-semicategories associated to A and B.

Proof: Note that one condition is common in both characterizations.

($1 \Longrightarrow 2$) Knowing that $[a \leq a] = [\varphi a \leq \varphi a]$ and using the conditions in 5.4.1, $[a \leq a] \wedge [\varphi a \leq a'] = [\varphi a \leq \varphi a] \wedge [\varphi a \leq a'] = [\varphi a \leq a']$; the other condition is similar.

($2 \Longrightarrow 1$) Applying $[a \leq a] \wedge [\varphi a \leq a'] = [\varphi a \leq a']$ to $a' = \varphi a$ gives $[a \leq a] \wedge [\varphi a \leq \varphi a] = [\varphi a \leq \varphi a]$ thus already $[\varphi a \leq \varphi a] \leq [a \leq a]$. The converse inequality follows from $[a \leq a'] \leq [\varphi a \leq \varphi a']$ by putting $a' = a$. So the wanted equality follows. $\square$

Corollary 5.4.15 *5.4.7 and 5.4.8 contain the statement that*

$$\mathsf{Map}(\mathsf{Dist}^{\star\star}(\Omega)) \simeq \mathsf{Map}(\mathsf{Dist}^{\star\star}_{\mathsf{cc}}(\Omega)) \simeq \mathsf{SCat}^{\star\star}_{\mathsf{cc}}(\Omega).$$

In turn, by 5.4.9 these are equivalent to the category of ordered sets in the topos $\mathsf{Sh}(\Omega)$ of sheaves on Ω.

The morality of all this is thus that the "ordered sheaves" on a locale Ω are precisely the Cauchy complete totally regular Ω-enriched semicategories, and that the "order-preserving functions" between such sheaves are no more and no less than the regular morphisms between the semicategories.

From Louvain-la-Neuve to Sydney and back

In [Walters, 1981] it is shown that Ω-sets (for a locale Ω) are exactly symmetric, skeletal Cauchy complete categories enriched in the quantaloid $\mathsf{Rel}(\Omega)$. More precisely, this quantaloid $\mathsf{Rel}(\Omega)$ has as objects the elements of Ω, for two objects $u, v \in \Omega$ the hom-lattice is $\mathsf{Rel}(\Omega)(u, v) = \{w \in \Omega \mid w \leq u \wedge v\}$, composition is finite infimum and u is the identity on u. A $\mathsf{Rel}(\Omega)$-category $\mathbb{C}$ is "symmetric" when for every two objects c, c' in $\mathbb{C}$, $\mathbb{C}(c', c) = \mathbb{C}(c, c')$. Then B. Walters' result is...

Theorem 5.4.16 *The topos* $\mathsf{Sh}(\Omega)$ *of sheaves on a locale* Ω *is equivalent to the full subcategory of "symmetric objects" in* $\mathsf{Cat}_{\mathsf{skel,cc}}(\mathsf{Rel}(\Omega))$.

$\mathsf{Rel}(\Omega)$ is the same thing as the split-idempotent completion of Ω as one-object quantaloid (since every element of Ω is idempotent). Therefore it follows that

$$\mathsf{Cat}_{\mathsf{skel,cc}}(\mathsf{Rel}(\Omega)) \simeq \mathsf{Cat}_{\mathsf{cc}}(\mathsf{Rel}(\Omega)) \simeq \mathsf{Cat}_{\mathsf{cc}}(\Omega_{\mathsf{cc}})$$

(where we used 2.4.19 on p. 87 to drop the "skel"). By 5.3.6 we may then reconcile Borceux and Cruciani's [1998] approach to (ordered) sheaves on Ω with Walters' [1981] way of doing things.

Corollary 5.4.17 *It is understood in [Walters, 1981] that* $\mathsf{Cat}_{\mathsf{cc}}(\Omega_{\mathsf{cc}})$ *is viewed as the category of "ordered sheaves on* Ω*"; (true and honest) sheaves are then the symmetrically ordered ones. This is consistent with the explicit result in [Borceux and Cruciani, 1998] that* $\mathsf{SCat}^{\star\star}_{\mathsf{cc}}(\mathcal{Q})$ *is the category of ordered sheaves on* Ω*, since we know that* $\mathsf{Cat}_{\mathsf{cc}}(\Omega_{\mathsf{cc}}) \simeq \mathsf{SCat}^{\star\star}_{\mathsf{cc}}(\Omega)$.

Ordered sheaves on a quantaloid

Inspired by the previous two subsections, we conclude this work with a definition.

Definition 5.4.18 *For a quantaloid* $\mathcal{Q}$*, we denote*

$$\mathsf{Idl}(\mathcal{Q})$$

for the quantaloid $\mathsf{Dist}^{\star\star}_{\mathsf{cc}}(\mathcal{Q})$ *of Cauchy complete totally regular* $\mathcal{Q}$*-semicategories and regular distributors; its objects are* $\mathcal{Q}$-orders *and its arrows are* ideals. *Similarly,*

$$\mathsf{Ord}(\mathcal{Q})$$

is the notation for the locally (pre)ordered category $\mathsf{SCat}^{\star\star}_{\mathsf{cc}}(\mathcal{Q})$ *of Cauchy complete totally regular* $\mathcal{Q}$*-semicategories and regular morphisms; the latter are to be thought of as* order-preserving mappings *between* $\mathcal{Q}$*-orders.*

Thus, for any quantaloid $\mathcal{Q}$,

- $\mathsf{Idl}(\mathcal{Q})$ is equivalent to $\mathsf{Dist}^{\star\star}(\mathcal{Q})$, to $\mathsf{Dist}_{\mathsf{cc}}(\mathcal{Q}_{\mathsf{cc}})$, and to $\mathsf{Dist}(\mathcal{Q}_{\mathsf{cc}})$;
- $\mathsf{Ord}(\mathcal{Q})$ is equivalent to $\mathsf{Map}(\mathsf{Idl}(\mathcal{Q}))$, i.e. every left adjoint ideal between $\mathcal{Q}$-orders is actually an order-preserving mapping.

And applied to a locale Ω, $\mathsf{Ord}(\Omega)$ is (equivalent to) the the category of ordered sheaves on Ω. (And the symmetric orders are sheaves on Ω).

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(Page numbers in italics refer to the Introduction, where all notions are discussed in a rather non-technical way, but where it is stressed how they are part of a "bigger picture". The upright page numbers refer to the chapters 1–5 of this thesis, where all notions are rigorously defined and extensively studied.)

Categorical structures enriched in a quantaloid : categories and semicategories

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