

Review

Variations on the Brouwer Fixed Point Theorem: A Survey

Jean Mawhin

Institut de Recherche en Mathématique et Physique, Université Catholique de Louvain, chemin du Cyclotron, 2, 1348 Louvain-la-Neuve, Belgium; jean.mawhin@uclouvain.be

Received: 25 February 2020; Accepted: 21 March 2020; Published: 2 April 2020



Abstract: This paper surveys some recent simple proofs of various fixed point and existence theorems for continuous mappings in $\mathbb{R}^n$. The main tools are basic facts of the exterior calculus and the use of retractions. The special case of holomorphic functions is considered, based only on the Cauchy integral theorem.

Keywords: Brouwer fixed point theorem; Hamadard theorem; Poincaré–miranda theorem

MSC: 55M20; 54C15; 30C15

1. Introduction

The Bolzano theorem for continuous functions $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$, which states that f has a zero in $[a, b]$ if $f(a)f(b) \leq 0$, was first proved in 1817 by Bolzano [1] and, independently and differently in 1821 by Cauchy [2]. Its various proofs are not very long, and depend only upon the order and completeness properties of $\mathbb{R}$. A consequence of the Bolzano theorem applied to $I - T$ is that $T : [-R, R] \rightarrow \mathbb{R}$, continuous, has a fixed point in $[-R, R]$ if $T(-R) \in [-R, R]$ and $T(R) \in [-R, R]$. This is the case if $T : [-R, R] \rightarrow [-R, R]$.

As $[-R, R]$ is the closed ball of center 0 and radius R in $\mathbb{R}$, a natural question is to know if, B_R denoting the closed ball $B_R \subset \mathbb{R}^n$ of center 0 and radius $R > 0$, any continuous mapping $T : B_R \rightarrow \mathbb{R}^n$ such that $T(\partial B_R) \subset B_R$ has a fixed point, and, in particular, if any continuous mapping $T : B_R \rightarrow B_R$ has a fixed point. The answer is yes, and the first result, usually called the Rothe fixed point theorem (FPT), is more correctly referred as the Birkhoff–Kellogg FPT, and the second one as the Brouwer FPT.

Many different proofs of those results have been given since the first published one of the Brouwer FPT by Hadamard in 1910 [3]. Brouwer's original proof [4], published in 1912, was topological and based on some fixed point theorems on spheres proved with the help of the topological degree introduced in the same paper. The Birkhoff–Kellogg FPT was first proved by Birkhoff and Kellogg in 1922 [5]. Its standard name Rothe FPT refers to its extension to Banach spaces by Rothe [6] in 1937.

The existing proofs use ideas from various areas of mathematics such as algebraic topology, combinatorics, differential topology, analysis, algebraic geometry, and even mathematical economics. A survey and a bibliography can be found in [7]. Even for $n = 2$, they cease to be elementary and/or can be technically complicated. The aim of this paper is to survey recent results on some elementary approaches to the Birkhoff–Kellogg and Brouwer FPT, and on how to deduce from them in a simple and systematic way other fixed point and existence theorems for mappings in $\mathbb{R}^n$. Recall that these results, combined with basic facts of functional analysis, are fundamental in obtaining useful extensions to some classes of mappings in infinite-dimensional normed spaces.

After recalling the simple concept of curvilinear integral in $\mathbb{R}^2$, we first propose in Section 2 an elementary proof of the Birkhoff–Kellogg FPT for $n = 2$, based upon such integrals. As the extension to arbitrary n , using differential $(n - 1)$ forms in $\mathbb{R}^n$, leads to very cumbersome computations, we adopt

in Section 3 a variant given in [8], using differential n -forms, which in dimension n happens to be significantly simpler than the direct extension of the approach of Section 2.

The generalizations of the Birkhoff–Kellogg and Brouwer FPT to a closed ball in $\mathbb{R}^n$ and their homeomorphic images are stated in Section 4. After the concepts of retract and retraction are introduced, the Leray–Schauder–Schaefter FPT on a closed ball is deduced from the Brouwer FPT, whose statement is also extended to retracts of a closed ball in $\mathbb{R}^n$. Finally, the equivalence of the Birkhoff–Kellogg and Brouwer FPT on a closed ball is established.

The Brouwer FPT and retractions are then used in Section 5 to prove, in a very simple and unified way inspired by the approach of [9], several conditions for the existence of zeros continuous mappings in $\mathbb{R}^n$, namely the Poincaré–Bohl theorem on a closed ball, the Hadamard theorem on a compact convex set, the Poincaré–Miranda theorem on a closed n -interval, and the Hartman–Stampacchia theorem on variational inequalities.

Finally, in Section 6, following the method introduced in [10], simple versions of the Cauchy integral theorem provide criterions for the existence of zeros of a holomorphic function in same spirit of the approach in Section 2. They allow very simple proofs of the Hadamard and Poincaré–Miranda theorems and of the Birkhoff–Kellogg and Brouwer FPT for holomorphic functions.

2. A Proof the Birkhoff–Kellogg Theorem on a Closed Disc Based on Curvilinear Integrals

Let $D \subset \mathbb{R}^2$ be open and nonempty and let $\langle \cdot, \cdot \rangle$ denote the usual inner product in $\mathbb{R}^2$. Given $f = (f_1, f_2) : D \rightarrow \mathbb{R}^2$, $x \mapsto f(x)$ and $\varphi = (\varphi_1, \varphi_2) : [a, b] \rightarrow D$, $t \mapsto \varphi(t)$ of class C^1 , we consider the corresponding **curvilinear integral** defined by $\int_a^b \langle f(\varphi(t)), \varphi'(t) \rangle dt$ where $'$ denotes the derivative with respect to t .

The following result is fundamental for our proof of the Birkhoff–Kellogg FPT on a closed disc.

Lemma 1. *If $f = (f_1, f_2) : D \rightarrow \mathbb{R}^2$ is of class C^1 and such that $\partial_1 f_2 = \partial_2 f_1$ and if $\Phi : [a, b] \times [0, 1] \rightarrow D$ is of class C^2 and such that $\Phi(b, \lambda) = \Phi(a, \lambda)$ for all $\lambda \in [0, 1]$, then $\lambda \rightarrow \int_a^b \langle f(\Phi(t, \lambda)), \partial_t \Phi(t, \lambda) \rangle dt$ is constant on $[0, 1]$.*

Proof. It suffices to prove that $\partial_\lambda \int_a^b \langle f(\Phi(t, \lambda)), \partial_t \Phi(t, \lambda) \rangle dt = 0$ for all $\lambda \in [0, 1]$. We have, with differentiation under integral sign easily justified and the use of assumptions, the Schwarz theorem and the fundamental theorem of calculus, and omitting the arguments (t, λ) for the sake of brevity

$$\begin{aligned}
 & \partial_\lambda \int_a^b \langle f(\Phi), \partial_t \Phi \rangle dt = \int_a^b \partial_\lambda [\langle f(\Phi), \partial_t \Phi \rangle] dt \\
 &= \int_a^b \{ \langle \partial_\lambda [f(\Phi)], \partial_t \Phi \rangle + \langle f(\Phi), \partial_\lambda \partial_t \Phi \rangle \} dt \\
 &= \int_a^b \left[\left\langle \sum_{j=1}^2 \partial_j f(\Phi) \partial_\lambda \Phi_j, \partial_t \Phi \right\rangle + \langle f(\Phi), \partial_t \partial_\lambda \Phi \rangle \right] dt \\
 &= \int_a^b \left[\sum_{k=1}^2 \sum_{j=1}^2 \partial_j f_k(\Phi) \partial_\lambda \Phi_j \partial_t \Phi_k + \langle f(\Phi), \partial_t \partial_\lambda \Phi \rangle \right] dt \\
 &= \int_a^b \left[\sum_{k=1}^2 \sum_{j=1}^2 \partial_k f_j(\Phi) \partial_t \Phi_k \partial_\lambda \Phi_j + \langle f(\Phi), \partial_t \partial_\lambda \Phi \rangle \right] dt \\
 &= \int_a^b \left[\sum_{j=1}^2 \partial_t f_j(\Phi) \partial_\lambda \Phi_j + \langle f(\Phi), \partial_t \partial_\lambda \Phi \rangle \right] dt \\
 &= \int_a^b \{ \langle \partial_t [f(\Phi)], \partial_\lambda \Phi \rangle + \langle f(\Phi), \partial_t \partial_\lambda \Phi \rangle \} dt \\
 &= \int_a^b \partial_t [\langle f(\Phi), \partial_\lambda \Phi \rangle] dt = f(\Phi(b, \lambda)) - f(\Phi(a, \lambda)) = 0.
 \end{aligned}$$

□

Let $B_R := \{x \in \mathbb{R}^2 : |x| \leq R\}$, with $|x|$ the Euclidian norm. We prove the **Birkhoff–Kellogg FPT on a closed disc**.

Theorem 1. Any continuous mapping $T : B_R \rightarrow \mathbb{R}^2$ such that $T(\partial B_R) \subset B_R$ has a fixed point in B_R .

Proof. Assume that T has no fixed point in B_R . Then, $|y - T(y)| > 0$ for all $y \in \partial B_R$, and, as $T(\partial B_R) \subset B_R$, $|y - \lambda T(y)| \geq |y| - \lambda |T(y)| \geq 1 - \lambda > 0$, for all $(y, \lambda) \in \partial B_R \times [0, 1]$. Similarly, $\lambda y - T(\lambda y) \neq 0$ for all $(y, \lambda) \in \partial B_R \times [0, 1]$. As T is continuous, there exists $\delta > 0$ such that $|y - \lambda T(y)| \geq \delta$ and $|\lambda y - T(\lambda y)| \geq \delta$ for all $(y, \lambda) \in \partial B_R \times [0, 1]$. From the Weierstrass approximation theorem, there is a polynomial $P : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $|T(y) - P(y)| \leq \frac{\delta}{2}$ for all $y \in B_R$. Consequently, letting $F(y, \lambda) := y - \lambda P(y)$ and $G(y, \lambda) := \lambda y - P(\lambda y)$, we have, for all $(y, \lambda) \in \partial B_R \times [0, 1]$, $|F(y, \lambda)| \geq \frac{\delta}{2}$ and $|G(y, \lambda)| \geq \frac{\delta}{2}$. Hence, there exists an open neighborhood Δ of ∂B_R such that $F(y, \lambda) \neq 0$ and $G(y, \lambda) \neq 0$ for all $(y, \lambda) \in \Delta \times [0, 1]$. If

$$f_1 : \mathbb{R}^2 \setminus \{0\} \rightarrow \mathbb{R}, x \mapsto -|x|^{-2}x_2, \quad f_2 : \mathbb{R}^2 \setminus \{0\} \rightarrow \mathbb{R}, x \mapsto |x|^{-2}x_1,$$

then $\partial_2 f_1(x) = |x|^{-4}(x_2^2 - x_1^2) = \partial_1 f_2(x)$. If $\gamma_R : [0, 2\pi] \rightarrow \mathbb{R}^2$, $t \mapsto R(\cos t, \sin t)$ is a parametric representation of ∂B_R , so that $\gamma_R(0) = \gamma_R(2\pi)$, it follows from Lemma 1 that the integrals

$$\int_0^{2\pi} \langle f[F(\gamma_R(t), \lambda)], \partial_t F(\gamma_R(t), \lambda) \rangle dt \text{ and } \int_0^{2\pi} \langle f[G(\gamma_R(t), \lambda)], \partial_t G(\gamma_R(t), \lambda) \rangle dt$$

are constant for $\lambda \in [0, 1]$. Hence, noticing that $F(\cdot, 1) = G(\cdot, 1) = I - P$,

$$\begin{aligned} & \int_0^{2\pi} \langle f[F(\gamma_R(t), 0)], \partial_t F(\gamma_R(t), 0) \rangle dt = \int_0^{2\pi} \langle f[F(\gamma_R(t), 1)], \partial_t F(\gamma_R(t), 1) \rangle dt \\ & = \int_0^{2\pi} \langle f[G(\gamma_R(t), 1)], \partial_t G(\gamma_R(t), 1) \rangle dt = \int_0^{2\pi} \langle f[G(\gamma_R(t), 0)], \partial_t G(\gamma_R(t), 0) \rangle dt. \end{aligned}$$

However, as $G(\cdot, 0) = -P(0)$ is constant and $F(\cdot, 0) = I$,

$$\begin{aligned} 0 &= \int_0^{2\pi} \langle f[G(\gamma_R(t), 0)], \partial_t G(\gamma_R(t), 0) \rangle dt \\ &= \int_0^{2\pi} \langle f[F(\gamma_R(t), 0)], \partial_t F(\gamma_R(t), 0) \rangle dt \\ &= \int_0^{2\pi} \langle f(\gamma_R(t)), \gamma_R'(t) \rangle dt = \int_0^{2\pi} (\sin^2 t + \cos^2 t) dt = 2\pi, \end{aligned}$$

a contradiction. □

A direct consequence is the **Brouwer FPT on a closed disc**.

Corollary 1. Any continuous mapping $T : B_R \rightarrow B_R$ has a fixed point in B_R .

3. A Proof of the Birkhoff–Kellogg Theorem on a Closed n -Ball Based on Differential n -Forms

The argument used in Section 2 for mappings in $\mathbb{R}^2$ can be extended to mappings in $\mathbb{R}^n$, using the basic properties of differential k -forms in $\mathbb{R}^n$. For $n = 2$, the differential 1-forms and differential $(n - 1)$ -forms coincide, and it is the last ones that are requested for extending the proof of Theorem 1 to arbitrary n . We leave to the motivated reader the work to write down this extension of the first approach and to realize that this generalization to dimension n of Lemma 1 is very cumbersome and lengthy. Fortunately a similar approach based on differential n -forms instead of $(n - 1)$ -forms has been

introduced in [8], which, for $n = 2$, has the same length and technicality as the one used in Section 2, but keeps its simplicity for arbitrary n . We describe it in this section.

For $D \subset \mathbb{R}^n$ open, bounded and nonempty, we need the concept of differential $(n - 1)$ -forms and n -forms and suppose that the reader is familiar with the notions, notations and properties of differential k -forms ($1 \leq k \leq n$) on D , wedge products, pull backs, exterior differentials and the Stokes–Cartan theorem for differential forms with compact support [11]. All the functions involved in differential forms are supposed to be of class C^2 . We associate to the functions $f_j : \overline{D} \rightarrow \mathbb{R}$ ($j = 1, \dots, n$) the **differential 1-form** $\omega_f := \sum_{j=1}^n f_j dx_j$ in D , and the **differential $(n - 1)$ -form**

$$v_f = \sum_{j=1}^n (-1)^{j-1} f_j dx_1 \wedge \dots \wedge \widehat{dx_j} \wedge \dots \wedge dx_n,$$

where $\widehat{dx_j}$ means that the corresponding term is missing. We associate also to $g : \overline{D} \rightarrow \mathbb{R}^n$ the **differential n -form** $\mu_g = g dx_1 \wedge \dots \wedge dx_n$. For example, given the function $w : D \rightarrow \mathbb{R}$ with partial derivatives $\partial_j w$, its **differential** $dw := \sum_{j=1}^n (\partial_j w) dx_j$ is the differential 1-form $\omega_{\nabla w}$.

Let $\Delta \subset \mathbb{R}^n$ be open, bounded and nonempty, $F : \Delta \times [0, 1] \rightarrow D$, $(y, \lambda) \mapsto F(y, \lambda)$. For each fixed $\lambda \in [0, 1]$,

$$\begin{aligned} F^*(\cdot, \lambda) \omega_f &= \sum_{j=1}^n [f_j \circ F(\cdot, \lambda)] dF_j(\cdot, \lambda) \\ &= \sum_{k=1}^n \left[\sum_{j=1}^n [f_j \circ F(\cdot, \lambda)] \partial_k F_j(\cdot, \lambda) \right] dy_k \quad (j = 1, \dots, n) \end{aligned}$$

is well defined. To shorten the notations, we write F_j for $F_j(\cdot, \lambda)$. We define the **derivative with respect to λ** of $F^* \omega_f$ by

$$\partial_\lambda (F^* \omega_f) := \sum_{k=1}^n \partial_\lambda \left[\sum_{j=1}^n (f_j \circ F) \partial_k F_j \right] dy_k.$$

so that

$$\begin{aligned} \partial_\lambda (F^* \omega_f) &= \sum_{k=1}^n \sum_{j=1}^n [\partial_\lambda (f_j \circ F) \partial_k F_j + (f_j \circ F) \partial_\lambda \partial_k F_j] dy_k \\ &= \sum_{j=1}^n [\partial_\lambda (f_j \circ F) dF_j + (f_j \circ F) \partial_\lambda (dF_j)]. \end{aligned}$$

Furthermore,

$$\partial_\lambda (dF_j) = \sum_{k=1}^n (\partial_\lambda \partial_k F_j) dy_k = \sum_{k=1}^n (\partial_k \partial_\lambda F_j) dy_k = d(\partial_\lambda F_j) \quad (j = 1, \dots, n).$$

On the other hand,

$$dF_1 \wedge \dots \wedge dF_n = J_F dy_1 \wedge \dots \wedge dy_n,$$

where $J_{F(\cdot, \lambda)}(y, \lambda)$ denotes the Jacobian of $F(\cdot, \lambda)$ at $(y, \lambda) \in \overline{\Delta} \times [0, 1]$, and

$$\partial_\lambda [dF_1 \wedge \dots \wedge dF_n] = \sum_{j=1}^n dF_1 \wedge \dots \wedge \partial_\lambda dF_j \wedge \dots \wedge dF_n.$$

The following two results replace Lemma 1 in Section 2. The first one shows that the differential n -form $\partial_\lambda (F^* \mu_g)$ is exact in Δ , i.e., is the exterior differential of a $(n - 1)$ -differential form in Δ .

Lemma 2. For each $\lambda \in [0, 1]$, we have

$$\partial_\lambda(F^*\mu_g) = d \left[(g \circ F) \left(\sum_{j=1}^n (-1)^{j-1} \partial_\lambda F_j dF_1 \wedge \dots \wedge \widehat{dF_j} \wedge \dots \wedge dF_n \right) \right].$$

Proof. We have

$$\begin{aligned} \partial_\lambda(F^*\mu_g) &= \partial_\lambda(g \circ F) dF_1 \wedge \dots \wedge dF_n + (g \circ F) \partial_\lambda(dF_1 \wedge \dots \wedge dF_n) \\ &= \left(\sum_{j=1}^n (\partial_j g \circ F) \partial_\lambda F_j \right) dF_1 \wedge \dots \wedge dF_n \\ &\quad + (g \circ F) \left(\sum_{j=1}^n dF_1 \wedge \dots \wedge \partial_\lambda dF_j \wedge \dots \wedge dF_n \right) \\ &= \sum_{j=1}^n (-1)^{j-1} (\partial_j g \circ F) dF_j \wedge \partial_\lambda F_j dF_1 \wedge \dots \wedge \widehat{dF_j} \wedge \dots \wedge dF_n \\ &\quad + (g \circ F) \left(\sum_{j=1}^n (-1)^{j-1} d(\partial_\lambda F_j) \wedge dF_1 \wedge \dots \wedge \widehat{dF_j} \wedge \dots \wedge dF_n \right) \\ &= \sum_{j=1}^n (-1)^{j-1} \left(\sum_{k=1}^n (\partial_k g \circ F) dF_k \right) \wedge \partial_\lambda F_j dF_1 \wedge \dots \wedge \widehat{dF_j} \wedge \dots \wedge dF_n \\ &\quad + (g \circ F) \left(\sum_{j=1}^n (-1)^{j-1} d(\partial_\lambda F_j dF_1 \wedge \dots \wedge \widehat{dF_j} \wedge \dots \wedge dF_n) \right) \\ &= d(g \circ F) \wedge \left(\sum_{j=1}^n (-1)^{j-1} \partial_\lambda F_j dF_1 \wedge \dots \wedge \widehat{dF_j} \wedge \dots \wedge dF_n \right) \\ &\quad + (g \circ F) d \left(\sum_{j=1}^n (-1)^{j-1} \partial_\lambda F_j dF_1 \wedge \dots \wedge \widehat{dF_j} \wedge \dots \wedge dF_n \right) \\ &= d \left[(g \circ F) \left(\sum_{j=1}^n (-1)^{j-1} \partial_\lambda F_j dF_1 \wedge \dots \wedge \widehat{dF_j} \wedge \dots \wedge dF_n \right) \right] := dv_{g,F}. \end{aligned}$$

□

Corollary 2. If $w \in C^2(\mathbb{R}^n, \mathbb{R})$, Δ is open, bounded and $F \in C^2(\overline{\Delta} \times [0, 1], \mathbb{R}^n)$ verify $F(\partial\Delta \times [0, 1]) \cap \text{supp } w = \emptyset$, then $\int_\Delta F^*\mu_w$ is independent of λ on $[0, 1]$.

Proof. Using Lemma 2, the assumption and Stokes–Cartan theorem, we get

$$\partial_\lambda \int_\Delta F^*\mu_w = \int_\Delta \partial_\lambda(F^*\mu_w) = \int_\Delta dv_{w,F} = \int_{\partial\Delta} v_{w,F} = 0.$$

□

Let $B_R := \{x \in \mathbb{R}^n : |x| \leq R\}$ with $|x|$ the Euclidian norm. We now show that Proposition 2 allows a simple proof of the **Birkhoff–Kellogg FPT on a closed n -ball**, quite similar to that of Theorem 1.

Theorem 2. Any continuous mapping $T : B_R \rightarrow \mathbb{R}^n$ such that $T(\partial B_R) \subset B_R$ has a fixed point in B_R .

Proof. Assume that T has no fixed point in B_R . Then, $x - T(x) \neq 0$ for $x \in \partial B_R$, and for $(x, \lambda) \in \partial B_R \times [0, 1]$, we have $|x - \lambda T(x)| \geq R - \lambda |T(x)| \geq (1 - \lambda)R > 0$. Thus, $|x - \lambda T(x)| > 0$ for all $(x, \lambda) \in \partial B_R \times [0, 1]$. On the other hand, for $(x, \lambda) \in \partial B_R \times [0, 1]$, we have $\lambda x \in B_R$, $\lambda x - T(\lambda x) \neq 0$,

and hence $|\lambda x - T(\lambda x)| > 0$ for all $(x, \lambda) \in \partial B(R) \times [0, 1]$. By continuity, there exists $\delta > 0$ such that $|x - \lambda T(x)| > \delta$ for all $(x, \lambda) \in \partial B_R \times [0, 1]$. Let $P : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a polynomial such that $\max_{B_R} |P - T| \leq \delta/2$, and define $F \in C^\infty(\mathbb{R}^n \times [0, 1], \mathbb{R}^n)$ and $G \in C^\infty(\mathbb{R}^n \times [0, 1], \mathbb{R}^n)$ by $F(x, \lambda) = \lambda x - P(\lambda x)$ and $G(x, \lambda) = x - \lambda P(x)$, so that $|F(x, \lambda)| \geq \delta/2$ and $|G(x, \lambda)| \geq \delta/2$ for all $(x, \lambda) \in \partial B_R \times [0, 1]$. Let $w \in C^2(\mathbb{R}^n, \mathbb{R})$ with $\text{supp } w \subset B(\delta/2)$, the open ball of center 0 and radius $\delta/2$, and $\int_{B_R} w(y) dy = 1$. Then, by Proposition 2 with $\bar{\Delta} = B_R$, we get

$$0 = \int_{B_R} F^*(\cdot, 0) \mu_w = \int_{B_R} F^*(\cdot, 1) \mu_w = \int_{B_R} (I - P)^* \mu_w,$$

and

$$\begin{aligned} \int_{B_R} (I - P)^* \mu_w &= \int_{B_R} G^*(\cdot, 1) \mu_w = \int_{B_R} G^*(\cdot, 0) \mu_w = \int_{B_R} \mu_w \\ &= \int_{B_R} w(y) dy = 1, \end{aligned}$$

a contradiction. $\square$

The **Brouwer FPT on a closed n -ball** is a special case.

Corollary 3. Any continuous mapping $T : B_R \rightarrow B_R$ has a fixed point in B_R .

4. Fixed Points, Homeomorphisms and Retractions in $\mathbb{R}^n$

Now, if $K \subset \mathbb{R}^n$, if there exists a homeomorphism $h : B^n \rightarrow K$, and if $T : K \rightarrow K$ is continuous, $h^{-1} \circ T \circ h : B^n \rightarrow B^n$ is continuous, has a fixed point x^* by Theorem 3, and $h(x^*) \in K$ is a fixed point of T . Consequently, we have a **Brouwer FPT for homeomorphic images of a closed n -ball**.

Theorem 3. If $K \subset \mathbb{R}^n$ is homeomorphic to B_R , any continuous mapping $T : K \rightarrow K$ has a fixed point in K .

For example, K can be any **closed n -interval** $[a_1, b_1] \times \dots \times [a_n, b_n]$, or an **n -simplex** $\mathbb{R}_+^n := \{x = \sum_{j=1}^n x_j e^j \in \mathbb{R}^n : x_j \geq 0, \sum_{j=1}^n x_j \leq 1\}$.

Remark 1. In Theorem 3, the boundedness assumption on K cannot be omitted: a translation $x \mapsto x + a$ in $\mathbb{R}^n$ with $a \neq 0$ has no fixed point. The closedness assumption on K cannot be omitted as well: $T : (0, 1) \rightarrow (0, 1)$, $x \mapsto x^2$ has no fixed point in $(0, 1)$. Theorem 3 does not hold for any closed bounded set: a nontrivial rotation of the closed annulus $A = \{x \in \mathbb{R}^2 : r_1 \leq |x| \leq r_2\}$ has no fixed point in A .

We now introduce concepts and results due to Borsuk [12] which provide another class of sets on which the Brouwer FPT holds and simple proofs of various equivalent formulations of this theorem. We say that $U \subset V \subset \mathbb{R}^n$ is a **retract** of V if there exists a continuous mapping $r : V \rightarrow U$ such that $r = I$ on U (**retraction of V in U**). For example, B_R is a retract of $\mathbb{R}^n$, with a retraction r given by

$$r(x) = \begin{cases} x & \text{if } |x| \leq R \\ R \frac{x}{|x|} & \text{if } |x| > R. \end{cases} \quad (1)$$

Similarly, for any $0 < R_1 \leq R_2$, B_{R_1} is a retract of B_{R_2} .

Remark 2. The Brouwer FPT on B_R implies the Birkhoff–Kellogg FPT on B_R . Indeed, if $T : B_R \rightarrow \mathbb{R}^n$ is continuous, $T(\partial B_R) \subset B_R$, and r is given by (1), then $r \circ T : B_R \rightarrow B_R$ is continuous and, by the Brouwer FPT 3, has a fixed point $x^* \in B_R$. If $|T(x^*)| > R$, $|x^*| = |r(T(x^*))| = R$ and $|T(x^*)| \leq R$, a contradiction. Thus, $|T(x^*)| \leq R$ and $x^* = T(x^*)$. Thus, the two statements are equivalent.

Remark 3. The Brouwer FPT has for immediate topological consequence the well-known **no-retraction theorem**, stating that ∂B_R is not a retract of B_R in $\mathbb{R}^n$. We do not repeat here the simple proof of this result and the proof of Brouwer FPT from the no-retraction theorem.

An easy consequence of Theorem 3 is the **Leray–Schauder–Schaefer fixed point theorem**, a special case of a more general result obtained in 1934 by Leray and Schauder [13]. The proof given here is due to Schaefer [14].

Theorem 4. Any continuous mapping $T : B_R \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $x \neq \lambda T(x)$ for all $(x, \lambda) \in \partial B_R \times (0, 1)$ has a fixed point in B_R .

Proof. Let $r : \mathbb{R}^n \rightarrow B_R$ be the retraction of $\mathbb{R}^n$ onto B_R defined in Equation (1). Theorem 3 implies the existence of $x^* \in B_R$ such that $x^* = r(T(x^*))$. If $|T(x^*)| > R$, then $x^* = \frac{R}{|T(x^*)|} T(x^*)$, so that $|x^*| = R$ and $x^* = \lambda^* T(x^*)$ with $\lambda^* = \frac{R}{|T(x^*)|} < 1$, a contradiction with the assumption. Hence, $|T(x^*)| \leq R$ and $x^* = T(x^*)$. $\square$

Remark 4. If $T : \partial B_R \rightarrow B_R$, it is clear that the assumption of Theorem 4 is satisfied. Thus the Leray–Schauder–Schaefer FPT implies the Birkhoff–Kellogg FPT, and hence the two statements are equivalent.

The Brouwer FPT holds for retracts of a closed ball.

Theorem 5. If $U \subset \mathbb{R}^n$ is a retract of B_R , any continuous mapping $T : U \rightarrow U$ has a fixed point.

Proof. Let $U = r(B_R)$ for some retraction $r : B_R \rightarrow U$. Then, $T \circ r : B_R \rightarrow U \subset B_R$ has a fixed point $x^* \in U$. Hence, $x^* = r(x^*)$, and $x^* = T(x^*)$. $\square$

If $C \subset \mathbb{R}^n$ is non- empty, closed and convex, the **orthogonal projection** $p_C(x)$ on C of $x \in \mathbb{R}^n$, defined by $|p_C(x) - x| = \min_{y \in C} |y - x|$, is a retraction of $\mathbb{R}^n$ onto C [15]. Consequently, C is a retract of any $B_R \supset C$, giving a **Brouwer FPT on compact convex sets**.

Corollary 4. If $C \subset \mathbb{R}^n$ is compact and convex, any continuous mapping $T : C \rightarrow C$ has a fixed point in C .

5. Zeros of Continuous Mappings in $\mathbb{R}^n$

The first theorem on the existence of a zero for a mapping from B_R into $\mathbb{R}^n$ was first stated and proved for C^1 mappings by Bohl [16] in 1904, and extended to continuous mappings by Hadamard in 1910 [3], under the name **Poincaré–Bohl theorem**. It is a reformulation of the Leray–Schauder–Schaefer FPT Theorem 4.

Theorem 6. Any continuous mapping $f : B_R \rightarrow \mathbb{R}^n$ such that $f(x) \neq \mu x$ for all $x \in \partial B_R$ and for all $\mu < 0$ has a zero in B_R .

Proof. Define the continuous mapping $T : B_R \rightarrow \mathbb{R}^n$ by $T(x) = x - f(x)$. For $(x, \lambda) \in \partial B_R \times (0, 1)$, we have, by assumption,

$$x - \lambda T(x) = (1 - \lambda)x + \lambda f(x) = \lambda \left[f(x) - \frac{\lambda - 1}{\lambda} x \right] \neq 0.$$

By Theorem 4, T has a fixed point x^* in B_R , which is a zero of f . $\square$

In 1910, two years before the publication of [4], Hadamard, informed by a letter from Brouwer of the statement of his fixed point theorem, published a simple proof based on the Kronecker index (a forerunner of the Brouwer topological degree) in an appendix to an introductory analysis book

of Tannery [3]. Hadamard's proof consisted in showing that Brouwer's assumption implies that the condition $\langle x, x - T(x) \rangle \geq 0$ holds for all $x \in \partial B_R$, where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^n$. This condition implies the existence of a zero of $I - T$, because the assumption of the Poincaré–Bohl theorem 6 is satisfied. Hadamard's reasoning using the Kronecker index does not depend upon the special structure $I - T$ of the mapping in the inner product. Hence, it is natural (although not usual) to call **Hadamard theorem** the statement of existence of a zero for a continuous mapping $f : B_R \rightarrow \mathbb{R}^n$, when $x - T(x)$ is replaced by $f(x)$ in the inequality above, a statement which became in the year 1960 a key ingredient in the theory of monotone operators in reflexive Banach spaces. Using convex analysis, we give an extension to compact convex sets.

Let $C \subset \mathbb{R}^n$ be compact and convex and $p_C : \mathbb{R}^n \rightarrow C$ be the orthogonal projection of x on C [15]. Recall that $p_C(x)$ is characterized by the condition

$$\langle x - p_C(x), y - p_C(x) \rangle \leq 0 \quad \text{for all } y \in C. \quad (2)$$

For $x \in \partial C$, the set

$$N_x := \{v \in \mathbb{R}^n : \langle v, y - x \rangle \leq 0 \text{ for all } y \in C\}$$

is nonempty and called the **normal cone** to C at x , and its elements v are called the **outer normals** to C at x . The relation in Equation (2) shows that, for each $x \notin C$, $x - p_C(x) \in N_{p_C(x)} \setminus \{0\}$. It can also be shown that each $x \in \partial C$ is the orthogonal projection of some $z \notin C$, so that $N_x = \{z \in \mathbb{R}^n \setminus C : p_C(z) = x\}$. The **Hadamard theorem on a convex compact set** follows in a similar way as Theorem 6 from the Brouwer FPT 3.

Theorem 7. *If $C \subset \mathbb{R}^n$ is a compact and convex, any continuous $f : C \rightarrow \mathbb{R}^n$ such that $\langle v, f(x) \rangle \geq 0$ for all $x \in \partial C$ and all $v \in N_x$ has a zero in C .*

Proof. Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be defined by $T = p_C - f \circ p_C$. Then, for all $x \in \mathbb{R}^n$,

$$|T(x)| \leq |p_C(x)| + |f(p_C(x))| \leq \max_{x \in C} |x| + \max_{y \in C} |f(y)| := R,$$

and T maps B_R into itself. By Theorem 3, there exists $x^* \in B_R$ such that $x^* = p_C(x^*) - f(p_C(x^*))$. If $x^* \notin C$, the assumption implies that

$$0 < |x^* - p_C(x^*)|^2 = -\langle x^* - p_C(x^*), f(p_C(x^*)) \rangle \leq 0,$$

a contradiction. Thus, $x^* \in C$, $x^* = p_C(x^*)$ and $f(x^*) = 0$. $\square$

Corollary 5. *Any continuous mapping $f : B_R \rightarrow \mathbb{R}^n$ such that $\langle x, f(x) \rangle \geq 0$ for all $x \in \partial B_R$ has a zero in B_R .*

Proof. For each $x \in \partial B_R$, $N_x = \{\lambda x : \lambda > 0\}$, and we apply Theorem 7. $\square$

Remark 5. *As shown when mentioning Hadamard's contribution, Theorem 5 implies the Brouwer FPT, and even the Birkhoff–Kellogg FPT, on B_R . Consequently, those statements are equivalent.*

Some twenty years before the publication of Brouwer's paper [4], Poincaré [17] stated in 1883 a theorem about the existence of a zero of a continuous mapping $f : P = [-R_1, R_1] \times \cdots \times [-R_n, R_n] \rightarrow \mathbb{R}^n$ when, for each $i = 1, \dots, n$, f_i takes opposite signs on the opposite faces of P

$$P_i^- := \{x \in P : x_i = -R_i\}, \quad P_i^+ := \{x \in P : x_i = R_i\} \quad (i = 1, \dots, n).$$

Poincaré's proof just told that the result was a consequence of the Kronecker index, which is correct but sketchy. The statement, forgotten for a while, was rediscovered by Cinquini [18] in 1940 with an inconclusive proof, and shown to be equivalent to the Brouwer FPT on P one year later by Miranda [19]. Many other proofs have been given since, and we again refer to [7,20] for a more complete history, variations and references, and to [21–23] for useful generalizations to more complicated sets than closed n -intervals. Here, we obtain the **Poincaré–Miranda theorem on a closed n -interval** as a special case of Theorem 7.

Corollary 6. Any continuous mapping $f : P \rightarrow \mathbb{R}^n$ such that $f_i(x) \leq 0$ for all $x \in P_i^-$ and $f_i(x) \geq 0$ for all $x \in P_i^+$ ($i = 1, \dots, n$) has a zero in P .

Proof. If x is in the (relative) interior of the face P_i^- , then $N_x = \{-\lambda e_i : \lambda > 0\}$, where $(e_1, e_2, \dots, e_n)$ is the orthonormal basis in $\mathbb{R}^n$, and the assumption of Theorem 7 becomes $-f_i(x) \geq 0$, i.e., $f_i(x) \leq 0$. Similarly, if x is in the (relative) interior to the face P_i^+ , then $N_x = \{\lambda e_i : \lambda > 0\}$, and the assumption of Theorem 7 becomes $f_i(x) \geq 0$. Of course, $-\lambda e_i$ and λe_i ($\lambda > 0$) also belong to the respective normal cones for $x \in P_i^-$ and P_i^+ respectively, and if, say, $x \in P_i^- \cap P_j^+$ then $v = -\lambda e_i + \mu e_j \in N_x$ for all $\lambda, \mu > 0$, and $\langle v, f(x) \rangle = -\lambda f_i(x) + \mu f_j(x) \geq 0$. In general, when x belongs to the intersection of several faces of P , N_x will be made of the linear combination of the e_i corresponding to the indices of the faces, with a negative coefficient for a face having symbol $-$ and positive coefficient for a face having symbol $+$, so that, using the assumption, $\langle v, f(x) \rangle \geq 0$ for all $x \in \partial P$ and all $v \in N_x$. The result follows from Theorem 7. $\square$

Remark 6. Corollary 6 implies the Brouwer FPT on P . Indeed, if $T : P \rightarrow P$ is continuous, and if we set $f = I - T$, then, as $-R_i \leq T_i(x) \leq R_i$ for all $x \in \partial P$, we have, for $x \in P$ such that $x_i = -R_i$, $f_i(x) = x_i - T_i(x) = -R_i - T_i(x) \leq 0$, and, for $x \in P$ such that $x_i = R_i$, $f_i(x) = x_i - T_i(x) = R_i - T_i(x) \geq 0$. Thus f has at least one zero in P , which is a fixed point of T . Consequently, the two statements are equivalent.

Remark 7. Both the Hadamard theorem on B_R and the Poincaré–Miranda theorem can be seen as distinct n -dimensional generalizations of the Bolzano theorem to closed ball and n -intervals respectively.

Remark 8. Using the Brouwer degree, it is easy to obtain the conclusion of the Hadamard Theorem 7 for a compact convex neighborhood of 0 under the weaker condition that for each $x \in \partial C$, there exists $v \in N_x$ such that $\langle v, f(x) \rangle \geq 0$. No proof based only upon the Brouwer FPT seems to be known.

If $C \subset \mathbb{R}^n$ is a compact convex set and $g : C \rightarrow \mathbb{R}$ is of class C^1 , then g reaches its minimum on C at some $x^* \in C$ for which

$$g(x^* + \lambda(v - x^*)) - g(x^*) \geq 0 \text{ for all } v \in C \text{ and for all } \lambda \in [0, 1],$$

so that, dividing both members by λ and letting $\lambda \rightarrow 0+$, we obtain $\langle \nabla g(x^*), v - x^* \rangle \geq 0$ for all $v \in C$, where ∇g denotes the gradient of g . For example, if $u \in \mathbb{R}^n$ is fixed and $g : C \rightarrow \mathbb{R}$ is defined by $g(x) = (1/2)|x - u|^2$, the minimization problem corresponds to the definition of $p_C(u)$, and, as $\nabla g(x) = x - u$, the inequality above is just Equation (2). In 1966, Hartman and Stampacchia [24] proved that the existence of such a x^* still holds when ∇g is replaced by an arbitrary continuous function $f : C \rightarrow \mathbb{R}^n$. When C is a simplex, the same result was proved independently the same year by Karamardian [25]. We give here a proof, due to Brezis (see [26]) and based upon Brouwer's FPT, of the **Hartman–Stampacchia theorem on variational inequalities**.

Theorem 8. If $C \subset \mathbb{R}^n$ is compact, convex and $f : C \rightarrow \mathbb{R}^n$ continuous, there exists $x^* \in C$ such that $\langle f(x^*), v - x^* \rangle \geq 0$ for all $v \in C$.

Proof. The Brouwer FPT on C (Corollary 4) applied to the continuous mapping $p_C \circ (I - f) : C \rightarrow C$ implies the existence of $x^* \in C$ such that

$$x^* = p_C(x^* - f(x^*)). \quad (3)$$

Taking $x = x^* - f(x^*)$ in Equation (2) and using Equation (3), one gets

$$\langle x^* - f(x^*) - x^*, v - x^* \rangle \leq 0 \text{ for all } v \in C,$$

which is the requested inequality. $\square$

Remark 9. The conclusion of Theorem 8 is called a **variational inequality**. In the terminology of the theory of convex sets [15], the conclusion of Theorem 8 means that there exists $x^* \in C$ such that either $f(x^*) = 0$ or $f(x^*) \neq 0$ and $H := \{y \in \mathbb{R}^n : \langle f(x^*), y - x^* \rangle = 0\}$ is a **supporting hyperplane** for C passing through x^* i.e., C is entirely contained in one of the two closed half-spaces determined by H .

Remark 10. The Brouwer FPT on C (Corollary 4) also follows from the Hartman–Stampacchia theorem. Indeed, if $T : C \rightarrow C$ is continuous and x^* is given by Theorem 8 applied to $f = I - T$, then, taking $v = T(x^*) \in C$ in the variational inequality, we obtain $0 \leq \langle x^* - T(x^*), T(x^*) - x^* \rangle = -|x^* - T(x^*)|^2 \leq 0$, so that $x^* = T(x^*)$. Hence, the two statements are equivalent.

Remark 11. If x^* and $x^\#$ are two distinct solutions of the variational inequality, then

$$\langle f(x^*), x^\# - x^* \rangle \geq 0, \quad \langle f(x^\#), x^* - x^\# \rangle \geq 0,$$

and hence $\langle f(x^*) - f(x^\#), x^* - x^\# \rangle \leq 0$. Consequently, the variational inequality has a unique solution if f satisfies the condition $\langle f(x) - f(y), x - y \rangle > 0$ for all $x \neq y \in C$, i.e., if f is **strictly monotone** on C .

6. A Direct Approach for Holomorphic Functions in $\mathbb{C}$

The assumption of the Bolzano theorem for a continuous function $f : [-R, R] \rightarrow \mathbb{R}$ can be, without loss of generality, be written $f(-R) \leq 0 \leq f(R)$ or, equivalently, $xf(x) \geq 0$ for $|x| = R$. In 1982, Shih [27] proposed a version of the Bolzano theorem for a complex function f holomorphic on a suitable bounded open neighborhood $\Omega \subset \mathbb{C}$ of 0 and continuous on $\bar{\Omega}$. He showed that f has a unique zero in Ω when $\Re[\bar{z}f(z)] > 0$ on $\partial\Omega$, using the Rouché theorem applied to $f(z)$ and $g(z) = \alpha z$ for a suitable real α . As $\Re[\bar{z}f(z)] = \Re z \cdot \Re f(z) + \Im z \cdot \Im f(z)$, Shih's condition is just Hadamard's one in Theorem 5 with strict inequality sign. Following the approach introduced in [10], we show in this section that, when the (non strict) Hadamard condition holds on the boundary of a ball, the existence of a zero of a holomorphic function results in a very simple way from an immediate consequence of the Cauchy integral theorem. The same is true for a Poincaré–Miranda theorem on a rectangle, giving another extension of the Bolzano theorem to complex functions. The Brouwer's FPT for holomorphic functions on a closed ball or a closed rectangle follow immediately.

We suppose the reader familiar with the concepts of **holomorphic function** f , **piecewise C^k cycle** γ , and **integral** $\int_\gamma f(z) dz$ **of f along γ** [28]. We denote by $B(R)$ the open disc of center 0 and radius $R > 0$ in $\mathbb{C}$, and by $\bar{B}_R$ the corresponding closed disc. Let $\gamma_R : [0, 2\pi] \rightarrow \partial B_R$, $t \mapsto Re^{it}$ be the standard C^∞ -cycle whose image is ∂B_R . The **Cauchy integral theorem on a circle** is proved here in a simple way, reminiscent of Cauchy's proof in 1825 [29], reworked by Falk in 1883 [30], and similar in spirit to the proof of Lemma 2.

Proposition 1. If $f : \bar{B}_R \rightarrow \mathbb{C}$ is continuous on $\bar{B}_R$ and holomorphic on $B(R)$, then $\int_{\gamma_R} f(z) dz = 0$.

Proof. Define $\Gamma : [0, 1] \times [0, 2\pi] \rightarrow B_R$ by $\Gamma(\lambda, t) = \lambda\gamma_R(t)$, so that $\Gamma(1, \cdot) = \gamma_R$ and $\Gamma(0, \cdot)$ is the constant zero mapping. To show that $\lambda \mapsto \int_{\Gamma(\lambda, \cdot)} f(z) dz$ is constant in $(0, 1)$, we have (with differentiation under the integral sign easily justified and $'$ denoting the derivative with respect to z)

$$\begin{aligned} \partial_\lambda \int_{\Gamma(\lambda, \cdot)} f(z) dz &= \partial_\lambda \int_0^{2\pi} f(\Gamma(\lambda, t)) \partial_t \Gamma(\lambda, t) dt \\ &= \int_0^{2\pi} [f'(\Gamma(\lambda, t)) \partial_\lambda \Gamma(s, t) \partial_t \Gamma(\lambda, t) + f(\Gamma(\lambda, t)) \partial_\lambda \partial_t \Gamma(\lambda, t)] dt \\ &= \int_0^{2\pi} [\partial_t \{f(\Gamma(\lambda, t))\} \partial_\lambda \Gamma(\lambda, t) + f(\Gamma(\lambda, t)) \partial_t \partial_\lambda \Gamma(\lambda, t)] dt \\ &= \int_0^{2\pi} \partial_t [f(\Gamma(\lambda, t)) \partial_\lambda \Gamma(\lambda, t)] dt \\ &= f(\Gamma(\lambda, 2\pi)) \partial_\lambda \Gamma(\lambda, 2\pi) - f(\Gamma(\lambda, 0)) \partial_\lambda \Gamma(\lambda, 0) = 0. \end{aligned}$$

By continuity, $\lambda \mapsto \int_{\Gamma(\lambda, \cdot)} f(z) dz$ is constant in $[0, 1]$, and hence

$$\int_{\gamma_R} f(z) dz = \int_{\Gamma(1, \cdot)} f(z) dz = \int_{\Gamma(0, \cdot)} f(z) dz = 0.$$

□

Let $a > 0$, $b > 0$, $P = \{z \in \mathbb{C} : -a \leq \Re z \leq a, -b \leq \Im z \leq b\}$ be the corresponding closed rectangle in $\mathbb{C}$, and let us introduce the continuous mapping $\rho : [0, 4] \rightarrow \partial P$ of class C^∞ on $(0, 1) \cup (1, 2) \cup (2, 3) \cup (3, 4)$ defined by

$$\rho(t) = \begin{cases} -a + 2ta - ib & \text{if } t \in [0, 1] \\ a + i[-b + 2(t-1)b] & \text{if } t \in [1, 2] \\ a - 2(t-2)a + ib & \text{if } t \in [2, 3] \\ -a + i[b - 2(t-3)b] & \text{if } t \in [3, 4], \end{cases} \quad (4)$$

whose image $\rho([0, 4]) = \partial P$. We state and prove the **Cauchy's integral theorem on the boundary of a rectangle**.

Proposition 2. If $f : P \rightarrow \mathbb{C}$ is continuous on P and holomorphic on $\text{int } P$, then $\int_\rho f(z) dz = 0$.

Proof. It is entirely similar to that of Proposition 1. If we define $R : [0, 1] \times [0, 4] \rightarrow P$ by $R(\lambda, t) = \lambda\rho(t)$, the integral $\int_{R(\lambda, \cdot)} f(z) dz$ has to be decomposed into four integrals over $[0, 1]$, $[1, 2]$, $[2, 3]$, and $[3, 4]$, respectively, of $f(R(\lambda, t)) \partial_t R(\lambda, t)$, and each integral has to be differentiated with respect to λ separately. The details are left to the reader. □

Propositions 1 and 2 immediately imply the following simple **theorem for the existence of a zero of f** .

Proposition 3. Any function $f : B_R \rightarrow \mathbb{C}$ (respectively, $f : P \rightarrow \mathbb{C}$) holomorphic on $B(R)$ (respectively, $\text{int } P$), continuous on B_R (respectively, P), different from zero on ∂B_R (respectively, ∂P) and such that

$$\int_{\gamma_R} \frac{dz}{f(z)} \neq 0 \quad \left(\text{resp. } \int_\rho \frac{dz}{f(z)} \neq 0 \right)$$

has a zero in $B(R)$ (respectively, P).

Proof. It is entirely similar in both cases and we prove it for B_R . If f has no zero in $B(R)$, then $z \mapsto \frac{1}{f(z)}$ is holomorphic on $B(R)$ and continuous on B_R . By Proposition 1, $\int_{\gamma_R} \frac{dz}{f(z)} = 0$, a contradiction to the assumption. $\square$

Proposition 3 provides a very simple proof of the **Hadamard theorem for a holomorphic function on B_R** .

Theorem 9. Any function $f : B_R \rightarrow \mathbb{C}$ holomorphic on $B(R)$, continuous on B_R and such that $\Re[\bar{z}f(z)] \geq 0$ for all $z \in \partial B_R$, has a zero in B_R .

Proof. For each integer $k \geq 1$, define $f_k : B_R \rightarrow \mathbb{C}$ by $f_k(z) = k^{-1}z + f(z)$. Each f_k has the regularity properties of f and is such that, for any $z \in \partial B_R$, $\Re[\bar{z}f_k(z)] = k^{-1}R^2 + \Re[\bar{z}f(z)] > 0$, so that $f_k(z) \neq 0$ for all $z \in \partial B_R$, and

$$\begin{aligned} \Im \left[\int_{\gamma_R} \frac{dz}{f_k(z)} \right] &= \Im \left[\int_{\gamma_R} \frac{z\bar{z}}{\bar{z}f_k(z)} \frac{dz}{z} \right] \\ &= \Im \left[\int_{\gamma_R} \frac{|z|^2 \{ \Re[\bar{z}f_k(z)] - i\Im[\bar{z}f_k(z)] \}}{|\bar{z}f_k(z)|^2} \frac{dz}{z} \right] \\ &= \Im \left[\int_0^{2\pi} \frac{i\Re[Re^{-it}f_k(Re^{it})] + \Im[Re^{-it}f(Re^{it})]}{|f_k(Re^{it})|^2} dt \right] \\ &= \int_0^{2\pi} \frac{\Re[Re^{-it}f_k(Re^{it})]}{|f_k(Re^{it})|^2} dt > 0. \end{aligned}$$

By Proposition 3, for each $k \geq 1$, f_k has a zero z_k in $B(R)$, and, by the Bolzano–Weierstrass theorem, a subsequence $(z_{k_n})_{n \geq 1}$ of $(z_k)_{k \geq 1}$ converges to some $z^* \in B_R$ such that $0 = \lim_{n \rightarrow \infty} [k_n^{-1}z_{k_n} + f(z_{k_n})] = f(z^*)$. $\square$

The **Birkhoff–Kellog FPT for a holomorphic function on a disc** is a direct consequence of Theorem 9.

Corollary 7. Any function $T : B_R \rightarrow \mathbb{C}$ continuous on B_R , holomorphic on $B(R)$ and such that $T(\partial B_R) \subset B_R$ has a fixed point in B_R .

Proof. For each $z \in \partial B_R$, one has $\Re\{\bar{z}[z - T(z)]\} \geq R^2 - |z||T(z)| \geq 0$. $\square$

Example 1. For any integer $m \geq 1$, the mapping T defined by $T(z) = \frac{z}{2}(z^m + 1)$ is such that for $|z| = 1$, $|T(z)| \leq \frac{|z|}{2}(|z|^m + 1) \leq 1$. There is no uniqueness as T has the fixed points 0 and 1 in B_1 .

Let $P_1^- = \{-a + iy : y \in [-b, b]\}$, $P_1^+ = \{a + iy : y \in [-b, b]\}$, $P_2^- = \{x - ib : x \in [-a, a]\}$ and $P_2^+ = \{x + ib : x \in [-a, a]\}$ be the opposite vertical and horizontal sides of P , respectively. Proposition 3 provides a **Poincaré–Miranda theorem for a holomorphic function on a rectangle**.

Theorem 10. Any function $f : P \rightarrow \mathbb{C}$ continuous on P , holomorphic on $\text{int } P$ and such that $\Re f(z) \leq 0$ for all $z \in P_1^-$, $\Re f(z) \geq 0$ for all $z \in P_1^+$, $\Im f(z) \leq 0$ for all $z \in P_2^-$ and $\Im f(z) \geq 0$ for all $z \in P_2^+$ has a zero in P .

Proof. For each integer $k \geq 1$, the function f_k defined on P by $f_k(z) = k^{-1}z + f(z)$ is such that $\Re f_k(z) < 0$ for $z \in P_1^-$, $\Re f_k(z) > 0$ for $z \in P_1^+$, $\Im f_k(z) < 0$ for $z \in P_2^-$, and $\Im f_k(z) > 0$ for

$z \in P_2^+$. Hence, $f_k(z) \neq 0$ for each $z \in \partial P$. Let $\rho : [0, 4] \rightarrow \Omega$ be the cycle defined by Equation (4). By the assumptions,

$$\begin{aligned} \Im \left[\int_{\rho} \frac{dz}{f_k(z)} \right] &= \Im \left\{ \int_{\rho} |f_k(z)|^{-2} [\Re f_k(z) - i \Im f_k(z)] dz \right\} \\ &= \int_{\rho} |f_k(z)|^{-2} [-\Im f_k(z) dx + \Re f_k(z) dy] \\ &= - \int_0^1 |f_k(\rho(t))|^{-2} \Im f_k[\rho(t)] 2a dt + \int_1^2 |f_k(\rho(t))|^{-2} \Re f_k[\rho(t)] 2b dt \\ &\quad - \int_2^3 |f_k(\rho(t))|^{-2} \Im f_k[\rho(t)] 2a dt + \int_3^4 |f_k(\rho(t))|^{-2} \Re f_k[\rho(t)] 2b dt \\ &= - \int_{-a}^a |f_k(s - ib)|^{-2} \Im f_k(s - ib) ds + \int_{-b}^b |f_k(a + it)|^{-2} \Re f_k(a + it) ds \\ &\quad + \int_{-a}^a |f_k(s + ib)|^{-2} \Im f_k(s + ib) dt - \int_{-b}^b |f_k(-a + is)|^{-2} \Re f_k(-a + is) ds \\ &= \int_{-a}^a \left[-|f_k(s - ib)|^{-2} \Im f_k(s - ib) + |f_k(s + ib)|^2 \Im f_k(s + ib) \right] ds \\ &\quad + \int_{-b}^b \left[|f_k(a + is)|^{-2} \Re f_k(a + is) - |f_k(-a + is)|^{-2} \Re f_k(-a + is) \right] ds > 0, \end{aligned}$$

For $k \geq 1$, Proposition 3 implies the existence of $z_k \in \text{int } P$ such that $k^{-1}z_k + f(z_k) = 0$. Using the Bolzano–Weierstrass theorem, a subsequence $(z_{k_n})_{n \geq 1}$ converges to some $z^* \in P$ such that $0 = \lim_{n \rightarrow \infty} [k_n^{-1}z_{k_n} + f(z_{k_n})] = f(z^*)$. $\square$

Example 2. Let the holomorphic function $f : \mathbb{C} \rightarrow \mathbb{C}$ be defined by $f(z) = z^3 + 4z + 1 + i$. Taking $P = \{z \in \mathbb{C} : \Re z \in [-1, 1] \text{ and } \Im z \in [-1, 1]\}$, one has

$$\begin{aligned} z \in P_1^- &\Rightarrow \Re f(z) = -4 + 3y^2 < 0, \quad z \in P_1^+ \Rightarrow \Re f(z) = 6 - 3y^2 > 0 \\ z \in P_2^- &\Rightarrow \Im f(z) = -3x^2 - 2 < 0, \quad z \in P_2^+ \Rightarrow \Im f(z) = 3x^2 + 4 > 0, \end{aligned}$$

and f has a zero in $[-1, 1] \times [-1, 1]$.

A direct consequence of Theorem 10 is the **Birkhoff–Kellogg FPT for a holomorphic function on a rectangle**.

Corollary 8. Any function $T : P \rightarrow \mathbb{C}$ continuous on P , holomorphic on $\text{int } P$, and such that $T(\partial P) \subset P$ has a fixed point in P .

Proof. Define $f : P \rightarrow \mathbb{C}$ by $f(z) = z - T(z)$ for all $z \in P$. The assumption $T(\partial P) \subset P$ is equivalent to $-a \leq \Re T(z) \leq a$ and $-b \leq \Im T(z) \leq b$ for all $z \in \partial P$, and, hence, if $z \in P_1^-$, $\Re f(z) = -a - \Re T(z) \leq 0$, if $z \in P_1^+$, $\Re f(z) = a - \Re T(z) \geq 0$, if $z \in P_2^-$, $\Im f(z) = -b - \Im T(z) \leq 0$, and if $z \in P_2^+$, $\Im f(z) = b - \Im T(z) \geq 0$. Thus, by Theorem 10, f has a zero in P and T a fixed point in P . $\square$

Funding: This research received no external funding.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Bolzano, B. *Rein analytisches Beweis des Lehrsatzes dass zwischen je zwey Werthen, die ein entgegengesetztes Resultat gewähren, wenigstens eine reelle Wurzel der Gleichung liege*; Abh. K. Gesells. Wiss.: Prag, Czech, 1817.
2. Cauchy, A. *Cours d'analyse de l'École Royale Polytechnique*; I^{re} Partie Analyse Algébrique: Debure, Paris, 1821.

3. Hadamard, J. *Sur quelques applications de l'indice de Kronecker, Note additionnelle*, 2nd ed.; 'Introduction à la théorie des Fonctions d'une Variable' de J. Tannery; Hermann: Paris, France, 1910; Volume 2, pp. 437–477.
4. Brouwer, L.E.J. Ueber Abbildungen von Mannigfaltigkeiten. *Math. Ann.* **1912**, *71*, 97–115. [\[CrossRef\]](#)
5. Birkhoff, G.D.; Kellogg, O.D. Invariant points in function space. *Trans. Amer. Math. Soc.* **1992**, *23*, 96–115. [\[CrossRef\]](#)
6. Rothe, E. Zur Theorie der topologischen Ordnung und der Vektorfelder in Banachschen Räumen. *Compos. Math.* **1937**, *5*, 177–197.
7. Mawhin, J. Le théorème du point fixe de Brouwer: Un siècle de métamorphoses. *Sci. Tech. Perspect.* **2007**, *10*, 175–220.
8. Mawhin, J. Simple proofs of various fixed point and existence theorems based on exterior calculus. *Math. Nachr.* **2005**, *278*, 1607–1614. [\[CrossRef\]](#)
9. Mawhin, J. Simple proofs of the Hadamard and Poincaré–Miranda theorems using the Brouwer fixed point theorem. *Am. Math. Mon.* **2019**, *126*, 260–263. [\[CrossRef\]](#)
10. Mawhin, J. Bolzano's theorems for holomorphic mappings. *Chin. Ann. Math.* **2017**, *38*, 563–578. [\[CrossRef\]](#)
11. Flanders, H. *Differential Forms with Applications to the Physical Sciences*; Academic Press: New York, NY, USA, 1989, Reprinted Dover: New York, NY, USA, 1989.
12. Borsuk, K. Sur les rétractes. *Fundam. Math.* **1931**, *17*, 152–170. [\[CrossRef\]](#)
13. Leray, J.; Schauder, J. Topologie et équations fonctionnelles. *Ann. Scient. École Norm. Sup.* **1934**, *51*, 45–78. [\[CrossRef\]](#)
14. Schaefer, H. Ueber die Methode der a priori Schranken. *Math. Ann.* **1955**, *129*, 415–416. [\[CrossRef\]](#)
15. Hiriart-Urruty, J.B.; Lemaréchal, C. *Convex Analysis and Minimization Algorithms I*; Springer: Berlin, Germany, 1993.
16. Bohl, P. Ueber die Bewegung eines mechanischen Systems in der Nähe einer Gleichgewichtslage. *J. Reine Angew. Math.* **1904**, *127*, 179–276.
17. Poincaré, H. Sur certaines solutions particulières du problème des trois corps. *C R. Acad. Sci. Paris* **1883**, *97*, 251–252.
18. Cinquini, S. Problemi di valori al contorno per equazioni differenziali di ordine n . *Ann. Della Sc. Norm. Super.-Pisa Sci.* **1940**, *9*, 61–77.
19. Miranda, C. Un' osservazione su un teorema di Brouwer. *Cons. Naz. Delle Ric.* **1940**, *3*, 5–7.
20. Mawhin, J. Variations on Poincaré–Miranda's theorem. *Adv. Nonlinear Stud.* **2013**, *13*, 209–217. [\[CrossRef\]](#)
21. Fonda, A.; Gidoni, P. Generalizing the Poincaré–Miranda theorem: The avoiding cones condition. *Ann. Mat. Pura Appl.* **2016**, *195*, 1347–1371. [\[CrossRef\]](#)
22. Pireddu, M.; Zanolin, F. Fixed points for dissipative-repulsive systems and topological dynamics of mappings defined on N -dimensional cells. *Adv. Nonlinear Stud.* **2005**, *5*, 411–440. [\[CrossRef\]](#)
23. Pireddu, M.; Zanolin, F. Cutting surfaces and applications to periodic points and chaotic-like dynamics. *Topol. Methods Nonlinear Anal.* **2007**, *30*, 279–319.
24. Hartman, P.; Stampacchia, G. On some non-linear elliptic differential-functional equations. *Acta Math.* **1966**, *115*, 271–310. [\[CrossRef\]](#)
25. Karamardian, S. *Duality in Mathematical Programming*; ORC 66-2; Operations Research Center, University California: Berkeley, CA, USA, 1966.
26. Kinderlehrer, D.; Stampacchia, G. *An Introduction to Variational Inequalities and Their Applications*; Academic Press: New York, NY, USA, 1980.
27. Shih, M.H. An analog of Bolzano's theorem for functions of a complex variable. *Am. Math. Mon.* **1982**, *89*, 210–211. [\[CrossRef\]](#)
28. Ahlfors, L. *Complex Analysis*, 2nd ed.; Academic Press: New York, NY, USA, 1966.
29. Cauchy, A. Mémoire sur les intégrales définies. *Mém. Acad. Sci. Paris* **1827**, *15*, 41–89.
30. Falk, M. Extrait d'une lettre adressée à M. Hermite. *Bull. Sci. Math. Astron.* **1883**, *7*, 137–139.

