

1-CONVEX TRANSFERABLE UTILITY GAMES, A REAPPRAISAL

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1-convex transferable utility games, a reappraisal

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Abstract

1-convex games have been introduced by Theo Driessen in his 1985 PhD dissertation. They form an interesting class of games for at least one reason: the core of a 1-convex n -player game is a regular simplex of dimension $n - 1$ or a single point. As a consequence, its nucleolus is the center of gravity of the core. We recall and extend the results obtained by Driessen and provide examples and applications.

Keywords: transferable utility games, core, nucleolus, Shapley value

JEL Classification: C71

1. Introduction

1-convex transferable utility games were introduced in 1985 by Theo Driessen in his PhD dissertation.¹ They form a class of balanced games whose core is a full-dimensional regular simplex or a single point. Driessen has published several papers developing further the concept of 1-convex game and providing some applications.² One remarkable consequence of the simple geometric structure of the core of a 1-convex game is that the nucleolus coincides with its center of gravity, simplifying considerably its computation.

In the present paper, we recall the major propositions proved by Driessen and add some new results. Most of these new results are related to the duals of 1-convex games that are games in which forming a coalition other than the grand coalition is not rational from an individual point of view: players may gain only by getting all together. Alongside, it explains why only the marginal contributions to the grand coalition enter into the definition of the core.

The paper is organized as follows. Basic definitions applying to transferable utility games are recalled in Section 2. The next two sections are devoted to the definition of 1-convex games and their properties. The relation between 1-convexity and convexity is the object of Section 5. Section 6 is devoted to duals of 1-convex games. The core and the nucleolus of 1-convex games are the subject of Sections 7 and 8. The last section reviews a number of applications.

2. Transferable utility games

Given a set of players $N = \{1, \dots, n\}$, $n \geq 2$, a *transferable utility game* is defined by a set function v , called *characteristic function*, that associates real numbers to subsets: $v(S)$ is the "worth" of coalition S . It could be a gain or a loss that measures what coalition S *could* or *should* obtain if it decided to form. As pointed out by Aumann (2010), there are two points of view: "could" applies to transferable utility games that are derived from games in strategic form, while "should" relies on some given criterion. By convention, $v(\emptyset) = 0$.

Notation. Finite sets are denoted by upper-case letters. Lower-case letters are used to denote their cardinals: $t = |T|$, $s = |S|$, ... For a vector x , $x(S)$ denotes the sum of its coordinates over the subset S . By convention, a sum over an empty set is zero. Inclusion and strict inclusion are denoted by $\subset$ and $\subsetneq$ respectively. Braces are sometimes omitted for coalitions: $v(\{i, j, k, \dots\})$ is written $v(i, j, k)$ instead. The characteristic function can be written as a list of numbers. Games with 3 or 4 players will be written in the following compact and self-explanatory form:

$$v = (1, 2, 3 \mid 12, 13, 23 \mid 123) \text{ for } n = 3,$$

$$v = (1, 2, 3, 4 \mid 12, 13, 14, 23, 24, 34 \mid 123, 124, 134, 234 \mid 1234) \text{ for } n = 4.$$

The question addressed by the theory of transferable utility games concerns the division of the payoff that results from the joint and coordinated actions of all players. The set of allocations associated to a game (N, v) is given by: $X(N, v) = \{x \in \mathbb{R}^{2^n-1} \mid x(N) = v(N)\}$.

We denote by $G(N)$ the set of all characteristic functions on a set of player N . It can be identified to the vector space $\mathbb{R}^{2^n-1}$. There are two possibilities:

$$\text{either } \sum_{i \in N} v(i) \leq v(N), \tag{1}$$

¹ "Contributions to the theory of games: the τ -value and k -convex games", University of Nijmegen.

² Driessen has actually developed the concept of k -convex game. Here, we limit ourselves to the most interesting case, namely the case where $k = 1$.

$$\text{or } \sum_{i \in N} v(i) \geq v(N). \quad (2)$$

Often but not necessarily, the first type of games covers *surplus-sharing* situations while the second type covers *cost-sharing* situations. In these situations, the inequalities (1) and (2) say that there is collectively no loss in forming the grand coalition. Under (1), it is possible to allocate the total surplus in such a way that no one *receives less* than his stand-alone surplus. Under (2), it is possible to allocate the total cost in such a way that no one *pays more* than his stand-alone cost. Games for which (1) holds with equality are said to be *inessential*. A game (N, v) is *additive* if

$$v(S) = \sum_{i \in S} v(i) \text{ for all } S \subset N.$$

Additive games on N are inessential. *Quasi-additive games* are games that satisfy the above equalities for all $S \subsetneq N$. *Simple games* are games (N, v) such that $v(S) \in \{0, 1\}$ and $v(N) = 1$. A game (N, v) is *symmetric* if there exists a real-valued function f such that $f(\emptyset) = 0$ and $v(S) = f(s)$ for all $S \subset N$.

Two games (N, v) and (N, w) on a common set of player are *strategically equivalent* if there exists a real number $\alpha > 0$ and a vector $\beta \in \mathbb{R}^n$ such that $w(S) = \alpha v(S) + \beta(S)$. Strategic equivalence defines an *equivalence relation*.

There is no loss of generality in considering games where individual worth's are set to zero. The *0-normalization* of a game (N, v) is the game (N, v_0) defined by:

$$v_0(S) = v(S) - \sum_{i \in S} v(i) \text{ for all } S \subset N.$$

A game (N, v) and its 0-normalization (N, v_0) are strategically equivalent.

The *dual* (N, v^*) of a game (N, v) is defined by $v^*(S) = v(S) - v(N \setminus S)$, for all $S \subset N$ where the difference $v(N) - v(N \setminus S)$ measures the contribution of coalition S to the grand coalition.

Superadditivity or subadditivity are commonly assumed. A game (N, v) is *superadditive* if $v(S) + v(T) \leq v(S \cup T)$ for all pairs of disjoint subsets $S, T \subset N$. A game (N, v) is *subadditive* if the reversed inequalities apply. These assumptions say that splitting a coalition cannot be beneficial. *Weak superadditivity* impose that there is no gain in partitioning the set of players:

$$\sum_{h=1}^m v(S_h) \leq v(N) \text{ for all partitions } (S_1, \dots, S_m) \text{ of } N.$$

Weak subadditivity obtains if the reverse inequalities hold. (1) is verified under weak superadditivity while (2) is verified under weak subadditivity. Convexity and concavity are instead stronger assumptions. A game (N, v) is *convex* if $v(S) + v(T) \leq v(S \cup T) - v(S \cap T)$ for all pairs of subsets $S, T \subset N$. A game (N, v) is *concave* if the reversed inequalities apply.

In a game (N, v) , the *marginal contribution* of player i to a coalition S is defined by $v(S) - v(S \setminus i)$. If Π_N denotes the set of permutations of the players in N , the *marginal contributions vector* associated to the permutation $\pi = (i_1, i_2, \dots, i_n) \in \Pi_N$ is the vector $\mu^\pi(N, v)$ defined by:

$$\mu_{i_k}^\pi(N, v) = v(i_1, \dots, i_k) - v(i_1, \dots, i_{k-1}) \quad (i = 1, \dots, n)$$

In particular, $\mu_{i_1}^\pi(N, v) = v(i_1)$ and $\mu_{i_n}^\pi(N, v) = v(N) - v(i_1, \dots, i_{n-1})$. There are $n!$ marginal contribution vectors, not necessarily distinct. By construction, marginal contributions vectors are allocations: $\mu^\pi(N, v) \in X(N, v)$ for all $\pi \in \Pi_N$.

Convexity or concavity can be characterized in terms of marginal contributions. A game (N, v) is convex if marginal contributions are *non-decreasing* with coalition size:

$$v(S) - v(S \setminus i) \leq v(T) - v(T \setminus i) \text{ for all } i, S \text{ and } T \text{ such that } i \in S \subset T.$$

Similarly, a game (N, v) is concave if marginal contributions are *non-increasing* with coalition size.

When dividing a surplus, *individual rationality* is the minimal requirement to be imposed: only the allocations that give to players at least what they are worth *alone* are retained. Given a game (N, v) verifying (1), the *imputation set* is defined by: $I(N, v) = \{x \in X(N, v) \mid x_i \geq v(i) \text{ for all } i \in N\}$. (1) ensures that the imputation set is nonempty. It reduces to the status quo $(v(1), \dots, v(n))$ if the game is inessential. Geometrically, the imputation set of an essential game is a *regular simplex* of dimension $n - 1$. In the of a case game verifying (2), the inequalities defining the imputation set are reversed.

To solve a cost-sharing game (N, c) , one can work with the surplus-sharing game (N, v) defined by:

$$v(S) = \sum_{i \in S} c(i) - c(S) \text{ for all } S \subset N,$$

where $v(S)$ measures the gain of coalition S if it decided to form. It is equivalent to the game $(N, -c_0)$. An allocation x of the total surplus $v(N)$ and the associated allocation y of the total cost $c(N)$ are related by the equations $x_i + y_i = c(i)$, $i = 1, \dots, n$.

3. 1-convex games

For a given game (N, v) , we denote the *marginal contribution* of player i to the grand coalition by $b_i(N, v) = v(N) - v(N \setminus i)$. They define the *upper-vector* $b(N, v)$. The *gap function* g_v associated to a game (N, v) is defined by:

$$g_v(S) = \sum_{i \in S} b_i(N, v) - v(S) \text{ for all } S \subset N.$$

It measures the gain or loss of a coalition for being assigned its members' marginal contributions to the grand coalition.

Notice that $g_v(\emptyset) = 0$: the gap function defines a game (N, g_v) . Furthermore, the gaps associated to coalitions of size $n - 1$ coincide with the gaps associated to the grand coalition:

$$g(N \setminus i) = \sum_{j \in S} b_j(N, v) - v(N \setminus i) = \sum_{j \in N} b_j(N, v) - v(N) = g(N).$$

A game (N, v) is *1-convex* if its gaps are nonnegative and attain a *minimum* at the grand coalition:

$$0 \leq g_v(N) \leq g_v(S) \text{ for all } S \subset N, S \neq \emptyset \quad (3)$$

i.e.

$$\sum_{i \in S} b_i(N, v) \geq v(S) \text{ for all } S \subset N, S \neq \emptyset, \quad (4)$$

$$v(S) \leq v(N) - \sum_{i \in N \setminus S} b_i(N, v) \text{ for all } S \subset N, S \neq \emptyset. \quad (5)$$

In particular, $g_v(i) = b_i(N, v) - v(i) \geq 0$ implies $v(i) + v(N \setminus i) \leq v(N)$. Notice that (5) is an identity for the coalitions of sizes n and $n - 1$.

Example 1 The 3-player game given by $v = (0, 0, 0 \mid 21, 25, 28 \mid 41)$ is 1-convex: its gap function $g_v = (13, 16, 20 \mid 8, 8, 8 \mid 8)$ verifies (3).

For a simple (N, v) game, $b_i(N, v) = 1$ if and only if i is a veto player. A simple game is 1-convex if and only if there is a *single* veto player. Indeed, in the absence of veto players, $g_v(S) \leq 1 = g_v(N)$ for all $S \subset N$: the game is not 1-convex. If 1 is the unique veto player, we have $b_1(N, v) = 1$ and $b_i(N, v) = 0$ for all $i \neq 1$, implying $g_v(S) = g_v(N) = 0$ for all $S \subset N$: the game is 1-convex. If there are more than one veto player, say k , we have $g_v(N) = k - 1 > 0$ and $g_v(S) = 0$ for all S not containing the k veto players: the game is not 1-convex.

A symmetric game (N, f) is 1-convex if and only if $f(n) - f(s) \geq (n - s)\Delta_n(f)$ and $f(n) \leq n\Delta_n(f)$ for all s , $0 < s < n$, where $\Delta_n(f) = f(n) - f(n - 1)$.

A game that is *strategically equivalent* to a 1-convex game is 1-convex. Indeed, if the games (N, v) and (N, w) are related by the equations $w(S) = \alpha v(S) + \beta(S)$, we have $b_i(N, w) = \alpha b_i(N, v) + \beta_i$ and $g_w(S) = \alpha g_v(S)$. In particular, the 0-normalization of a 1-convex game produces a 1-convex game: the gap functions coincide. Furthermore, adding two 1-convex game on a common player set results in a 1-convex game. Hence, the set of 1-convex games is a *convex cone* within $G(N)$.

A game (N, v) is 1-concave if the game $(N, -v)$ is 1-convex: $g_v(S) \leq g_v(N) \leq 0$ for all $S \subset N$, $S \neq \emptyset$. If the game (N, v) is 1-convex, the game (N, g_v) defined by its gap function is 1-concave. Indeed, $b_i(N, g_v) = g_v(N) - g_v(N \setminus i) = 0$ for all $i \in N$ and, as a consequence, $g_{g_v}(S) = -g_v(S)$ for all $S \subset N$.

The following proposition is an immediate consequence of the fact that the surplus-sharing game (N, v) associated to the cost-sharing game (N, c) is defined by: $v(S) = -c_0(S)$. The two gap functions add-up to zero: $g_v(S) = -g_c(S)$ for all $S \subset N$.

Proposition 1 A cost-sharing game is 1-concave if and only if the associated surplus-sharing game is 1-convex.

4. Superadditivity and 1-convexity

1-convex games are not necessarily superadditive. They are however *weakly* superadditive.

Proposition 2 If (N, v) is a 1-convex game, $\sum_{h=1}^m v(S_h) \leq v(N)$ for all partitions $(S_1, \dots, S_m)$ of N .

Proof Consider a 1-convex game (N, v) and assume that the reversed inequality holds for some nontrivial partition $(S_1, \dots, S_m)$ of N . An impossibility then results if $g(N) \geq 0$:

$$g_v(N) = \sum_{i \in N} b_i(N, v) - v(N) > \sum_{h=1}^m \left(\sum_{i \in S_h} b_i(N, v) - v(S_h) \right) = \sum_{h=1}^m g_v(S_h) \geq m g_v(N). \quad \blacklozenge$$

Proposition 2 can be transposed to 1-concave games: they are weakly subadditive games.

2-player 1-convex games are obviously superadditive. This is also true for 3-player games, and for 4-player games but under a restriction.

Proposition 3 1-convex 3-player games are superadditive.

Proof Consider a 3-player 1-convex game (N, v) . In view of Proposition 3, we only have to prove that $v(i) + v(j) \leq v(i, j)$ for all $i \neq j$. Applying (5) to the coalitions $\{1\}$ and $\{2\}$, we get:

$$v(1) + v(2) \leq 2v(N) - b_1(N, v) - b_2(N, v) - 2b_3(N, v) = v(N) - \sum_{i=1}^3 b_i(N, v) + v(1, 2).$$

Superadditivity then follows from (4) applied to N . $\blacklozenge$

Proposition 4 A 1-convex 4-player game (N, v) is superadditive if the characteristic function of its 0-normalization takes *non-negative values*.

Proof Let (N, v) be a 4-player 1-convex game. In terms of the normalized game (N, v_0) , we have $v_0(i) + v_0(j) = 0 \leq v_0(i, j)$ for all $i \neq j$. Given what we know from Proposition 3, we only have to prove superadditivity for pairs of disjoint coalitions of the form $\{1, 2\}$ - $\{3\}$. The normalized game being 1-convex, we can apply (5) to coalition $\{1, 2\}$:

$$v(1, 2) \leq v(N) - b_3(N, v) - b_4(N, v) = v(1, 2, 3) + v(1, 2, 4) - v(N)$$

or

$$v(1, 2, 3) - v(1, 2) \geq v(N) - v(1, 2, 4) = b_3(N, v)$$

where $b_3(N, v) \geq v(3) = 0$ by (4). The 0-normalized game is therefore superadditive and so is the original game (N, v) . ♦

The 4-player game defined by $v(S) = s$ except $v(1, 2) = 1$ is 1-convex but not superadditive. Its gaps are given by $g(S) = 0$ for all $S \neq \{1, 2\}$ and $g(1, 2) = 1$. The 5-player game defined by $v(i) = 0$ for all i , $v(S) = s$ for all S of sizes 2 or more, except $v(1, 2, 3) = 1$, is 1-convex but fails to be superadditive.

It is easily verified that superadditivity (resp. subadditivity) of the game (N, v) implies subadditivity (resp. superadditivity) of the game (N, g_v) .

5. Convexity and 1-convexity

Convexity of a game can be assessed in terms of the game defined by its gap function.

Lemma 1 (Driessen, 1988, p.112) A game (N, v) is convex (resp. concave) *if and only if* the game (N, g_v) is concave (resp. convex).

Indeed, marginal contributions associated to the game (N, v) are related to the marginal contributions of the game (N, g_v) by the equations

$$g_v(S) - g_v(S \setminus i) = b_i(N, v) - (v(S) - v(S \setminus i)).$$

Furthermore, the right-hand side being non-negative by convexity, the gap function of a convex game is non-decreasing in terms of set inclusion. The following proposition then follows from (3).

Proposition 5 (Driessen, 1988, p.182) A 1-convex game is convex *if and only if* $g_v(S) = g_v(N) \geq 0$ for all $S \subset N$, $S \neq \emptyset$.

Additive games are convex *and* 1-convex: marginal contributions are constant and the gap function is identically zero. Quasi-additive games involving more than two players and verifying (1) are convex but not 1-convex. Indeed, if (N, v) is a quasi-additive game verifying (1), we have:

$$v(S) - v(S \setminus i) = v(i) \text{ for all } S \subsetneq N \text{ and } v(N) - v(N \setminus i) = v(N) - \sum_{j \neq i} v(j) \geq v(i),$$

and

$$g_v(N) = (n-1) \left(v(N) - \sum_{i \in N} v(i) \right) \geq s \left(v(N) - \sum_{i \in N} v(i) \right) = g_v(S) \geq 0 \text{ for all } S \subsetneq N.$$

We have seen that simple games are 1-convex if there is a single veto player. They are actually convex as well: the gap function is identically zero.

6. Dual 1-convex games

How do properties of a game translate into its dual? Superadditivity or subadditivity, even weak, do *not* translate into similar properties of the dual of a game, as the following examples show. The game $v = (1, 2, 3 \mid 5, 7, 9 \mid 10)$ is superadditive while its dual $v^* = (1, 3, 5 \mid 7, 8, 9 \mid 10)$ is neither subadditive nor superadditive. The game $v = (2, 3, 3 \mid 3, 5, 6 \mid 6)$ is subadditive while its dual $v^* = (0, 1, 3, \mid 3, 3, 4 \mid 6)$ is neither subadditive nor superadditive. It goes differently for convexity or concavity.

Proposition 6 The dual of a convex game is a concave game (and vice versa).

Proof Consider a convex game (N, v) and its dual (N, v^*) . Given two arbitrary subsets S and T , we have successively:

$$v^*(S) + v^*(T) = 2v(N) - v(N \setminus S) - v(N \setminus T)$$

and

$$v(N \setminus S) + v(N \setminus T) \leq v((N \setminus S) \cup (N \setminus T)) + v((N \setminus S) \cap (N \setminus T))$$

where $(N \setminus S) \cup (N \setminus T) = N \setminus (S \cap T)$ and $(N \setminus S) \cap (N \setminus T) = N \setminus (S \cup T)$. Concavity of the dual game then follows:

$$v^*(S) + v^*(T) \geq 2v(N) - v(N \setminus (S \cup T)) - v(N \setminus (S \cap T)) = v^*(S \cup T) + v^*(S \cap T) \quad \blacklozenge$$

The above examples show that the dual of a game that verifies (1) does not necessarily verify (2). It goes differently for 1-convex games.

Proposition 7 1-convex games verify (1) and their duals verify (2).

Proof The fact that 1-convex games verify (1) follows from Proposition 3. Assume now that (N, v) is a 1-convex game. Then by (4), its dual (N, v^*) verifies (2):

$$\sum_{i \in N} v^*(i) = \sum_{i \in N} (v(N) - v(N \setminus i)) = \sum_{i \in N} b_i(N, v) \geq v(N) = v^*(N). \quad \blacklozenge$$

However, the dual of a 1-convex game is generally not 1-concave and vice versa, as the following examples show.

Example 2 The *unanimity game* is the simple game (N, u) defined by $u(S) = 1$ if and only if $S = N$. It is not 1-convex. Indeed, its gap function takes nonnegative values and attains a *maximum* at the grand coalition. All coalitions have worth 1 in the dual and the gaps are all equal to -1 : the game (N, u^*) is 1-concave.

Example 1 (continued) The dual of the 1-convex game $v = (0, 0, 0 \mid 21, 25, 28 \mid 41)$ is given by $v^* = (13, 16, 20 \mid 41, 41, 41 \mid 41)$. Its gaps satisfy the inequalities $g_{v^*}(N) \leq g_{v^*}(S) \leq 0$ for all $S \subset N$.

Notice that the inequalities (3) are being reversed. It is actually the case of games (N, v) such that:

$$\sum_{i \in S} v(i) \leq v(S) \text{ for all } S \subset N. \quad (6)$$

Indeed, looking at the gap function of the dual game and using (4) and (5), we have:

$$\begin{aligned} g_{v^*}(S) &= \sum_{i \in S} (v^*(N) - v^*(N \setminus i)) - v^*(S) = \sum_{i \in S} v(i) - v(N) + v(N \setminus S) \\ &\leq \sum_{i \in S} (v(i) - b_i(N, v)) = - \sum_{i \in S} g_v(i) \leq 0 \text{ for all } S \subset N. \end{aligned}$$

Hence, $g_{v^*}(N) \leq 0$ and, under (6), we have:

$$g_{v^*}(N) - g_{v^*}(S) = \sum_{i \in N \setminus S} v(i) - v(N \setminus S) \leq 0 \text{ for all } S \subset N. \quad \blacklozenge$$

The converse is not true: the dual of a game (N, v) such that $g_v(N) \leq g_v(S) \leq 0$ for all $S \subset N$ is not necessarily 1-convex. For instance, the subadditive game $v = (5, 6, 7 \mid 10, 11, 11 \mid 13)$ satisfies the reversed inequalities while its dual is not 1-convex.

1-convexity of a game can be characterized in terms of its dual.

Proposition 8 A game (N, v) is 1-convex *if and only if* its dual verifies the following inequalities:

$$\sum_{i \in S} v^*(i) \leq v^*(S) \text{ for all } S \subsetneq N \text{ and } \sum_{i \in N} v^*(i) \geq v^*(N). \quad (7)$$

Proof The gap function of the game (N, v) is given by:

$$g_v(S) = \sum_{i \in S} b_i(N, v) - v(S) = \sum_{i \in S} v^*(i) - v^*(N) + v^*(N \setminus S).$$

We then have successively:

$$g_v(N) = \sum_{i \in N} v^*(i) - v^*(N)$$

and

$$g_v(N) - g_v(S) = \sum_{i \in N \setminus S} v^*(i) - v^*(N \setminus S) \text{ for all } S \subsetneq N. \quad \blacklozenge$$

Example 3 The game $v = (4, 2, 3, 6 \mid 7, 9, 10, 8, 10, 9 \mid 12, 12, 13, 12 \mid 13)$ satisfies (7). Its dual is the super-additive and 1-convex game $v^* = (1, 0, 1, 1 \mid 4, 3, 5, 3, 4, 6 \mid 7, 10, 11, 9 \mid 13)$. The gap function of the game (N, v) satisfies the reversed inequalities $g_v(N) \leq g_v(S) \leq 0$ for all $S \subset N$.

Proposition 8 carries over to 1-concavity. A game (N, v) is 1-concave *if and only if* its dual verifies the inequalities:

$$\sum_{i \in S} v^*(i) \geq v^*(S) \text{ for all } S \subsetneq N \text{ and } \sum_{i \in N} v^*(i) \leq v^*(N). \quad (8)$$

Notice that games that satisfy (7) are weakly subadditive. Indeed, for all nontrivial partitions $(S_1, \dots, S_m)$ of N , we have:

$$\sum_{h=1}^m v(S_h) \geq \sum_{h=1}^m \sum_{i \in S_h} v(i) = \sum_{i \in N} v(i) \geq v(N).$$

Similarly, games that satisfy (8) are weakly superadditive.

Together with Proposition 9, Proposition 5 implies that a game is 1-convex *and* convex *if and only if* its dual is a quasi-additive game verifying (2). Indeed, we have:

$$g_v(S) = g_v(N) \Rightarrow \sum_{i \in S} v^*(i) = v^*(S) \text{ for all } S \subsetneq N.$$

Similarly, a game is 1-concave *and* concave *if and only if* its dual is a quasi-additive game verifying (1).

5. The core of a 1-convex game

The reasoning behind the concept of core is to extend individual rationality to coalitions. A coalition may indeed raise an objection against an allocation that gives to the coalition as a whole less than its worth. The *core* of a game (N, v) is then defined by:

$$\bar{C}(N, v) = \{x \in \mathbb{R}^n \mid x(N) = v(N) \text{ and } x(S) \geq v(S) \text{ for all } S \subset N\},$$

a definition that applies to games verifying (1). For games verifying (2), the definition of the core must be adjusted. The so-called *anticore* is obtained by reversing the inequalities:

$$\underline{C}(N, v) = \{x \in \mathbb{R}^n \mid x(N) = v(N) \text{ and } x(S) \leq v(S) \text{ for all } S \subset N\}.$$

No coalition should get more than what it's entitled to or have to pay more than its cost.

Lemma 2 (Driessen, 1985, p.172) Given a game (N, v) verifying (1), an allocation $x \in X(N, v)$ belongs to $\bar{C}(N, v)$ if and only if

$$x_i \leq b_i(N, v) \text{ for all } i \in N, \quad (9)$$

and

$$x(S) \geq v(S) \text{ for all } S \neq \emptyset \text{ such that } g_v(S) < g_v(N). \quad (10)$$

The core of a 1-convex game is therefore entirely determined by its upper-vector. The following proposition then follows.

Proposition 9 (Driessen, 1985, p.73) The core of a 1-convex games is given by:

$$\begin{aligned} \bar{C}(N, v) &= \{x \in \mathbb{R}^n \mid x(N) = v(N), x_i \leq v(N) - v(N \setminus i) \text{ for all } i \in N\} \\ &= \{x \in \mathbb{R}^n \mid x(N) = v(N), x(N \setminus i) \geq v(N \setminus i) \text{ for all } i \in N\}. \end{aligned}$$

Notice that the non-negativity of the gaps follows from the no emptiness of the core. Indeed, (9) implies $v(S) \leq x(S) \leq \sum_{i \in S} b_i(N, v)$.

In the case where (2) applies, inequalities (9) and (10) are reversed. Proposition 9 apply to 1-concave games and to the anticore $\underline{C}(N, v)$.

The core of a 1-convex game turns out to have a simple and regular geometric structure.

Proposition 10 The core of a 1-convex n -player game is either a single point or a *regular simplex* of dimension $n - 1$.

Proof Consider a 1-convex game (N, v) and its core $\bar{C}(N, v)$. Then, the set

$$\{y \in \mathbb{R}^n \mid y = b(N, v) - x, x \in \bar{C}(N, v)\} = \left\{y \in \mathbb{R}_+^n \mid y(N) = \sum_{i \in N} b_i(N, v) - v(N)\right\} \quad (11)$$

is a full-dimensional simplex if $g_v(N) > 0$. It coincides with the $\{0\}$ if instead $g_v(N) = 0$. ♦

It follows from Proposition 10 that the core of a 1-convex game (N, v) reduces to the singleton $\{b(N, v)\}$ if and only if $g_v(N) = 0$. It is the case of additive games.

The set defined by (11) is the core of the game defined by the gap function, $C(N, g_v)$, and its n vertices are of the form $g_v(N) \cdot e^j$, $j = 1, \dots, n$. Hence, the vertices $(\theta^1, \dots, \theta^n)$ of the core $\bar{C}(N, v)$ are given by $\theta^j = b(N, v) - g_v(N) \cdot e^j$:

$$\begin{aligned}\theta_i^j(N, v) &= b_i(N, v) - g_v(N) \quad \text{if } i = j, \\ &= b_i(N, v) \quad \text{if } i \neq j.\end{aligned}$$

These are the *efficient upper-vectors* that are obtained from the upper-vectors in such a way that the resulting vector is efficient: by construction, $\theta^j \in X(N, v)$ for all j . Convexity combined with 1-convexity implies that the efficient upper-vectors coincide with the *distinct* marginal contribution vectors.

Example 1 (continued) We have seen that the game $v = (0, 0, 0 \mid 21, 25, 28 \mid 41)$ is 1-convex. The core is the regular simplex represented in the imputation triangle by Figure 1. Its vertices are the efficient upper-vectors $(5, 16, 20)$, $(13, 8, 20)$ and $(13, 16, 12)$.

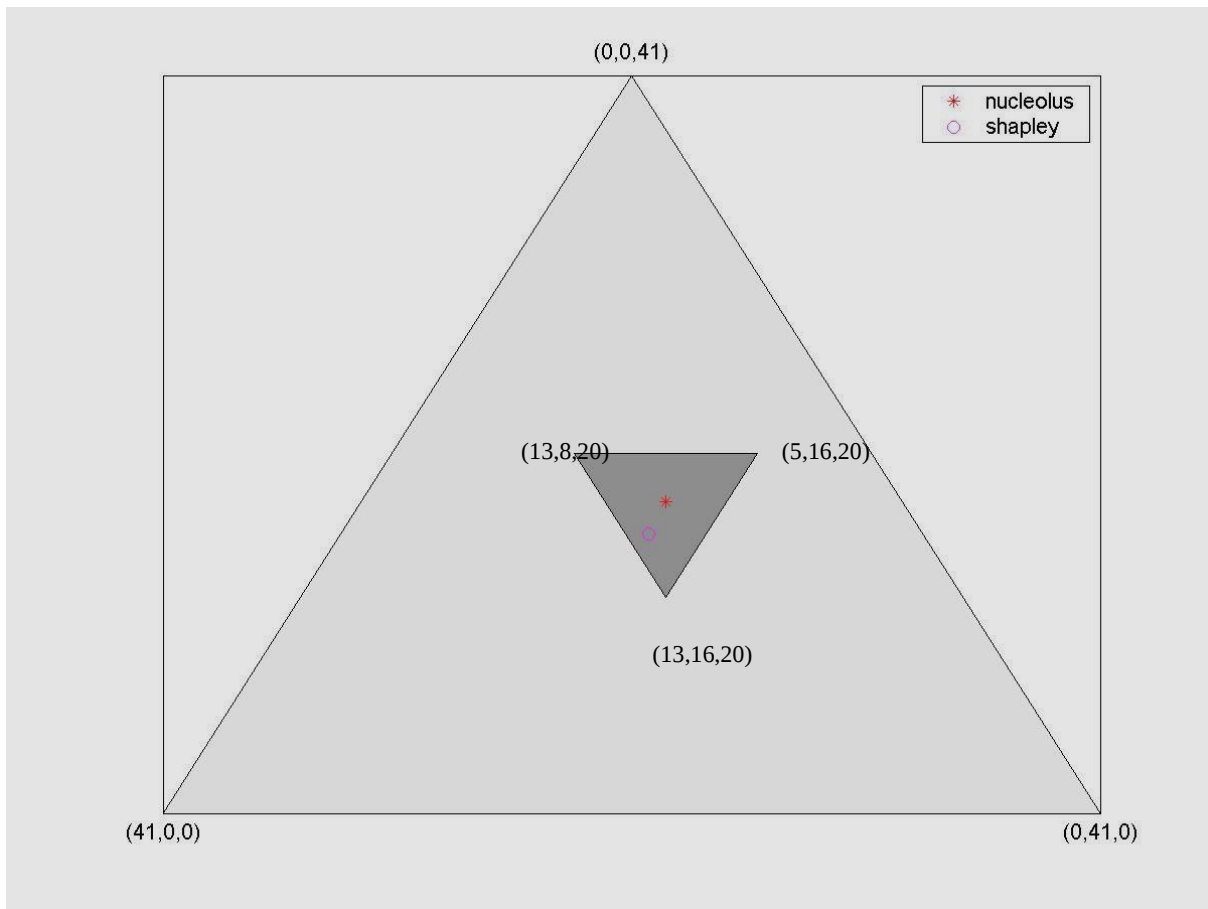


Figure 1 Core of the 1-convex game of Example 1 ³

It is well known that the core of a game is a subset (possibly empty) of the convex hull of the marginal contribution vectors, known as the *Weber set* (Weber, 1988). Here is a similar proposition involving the efficient upper-vectors.

Proposition 11 (Driessen, 1988, p.171) The core of a game is a subset of the convex hull of its efficient upper-vectors.

³ The pictures have been obtained using the platform www.tuglabweb.uvigo.es maintained by Miguel Angel Miras Calvo and Estela Sanchez Rodriguez (Universidad de Vigo).

A game is convex *if and only if* its core is the polyhedra whose vertices are the marginal contribution vectors. This is a well-known result due to Shapley (1971) and Ichiishi (1981). Here is a similar proposition for 1-convex games. It offers a characterization of 1-convexity in terms of the core.

Proposition 12 A game is 1-convex *if and only if* its core is the simplex whose vertices are the efficient upper-vectors.

It results from Proposition 11 combined with the following lemma.

Lemma 3 (Driessen, 1985, p.21) A game is 1-convex *if and only if* its efficient upper-vectors *all* belong to its core.

The core of a game can be viewed equivalently as the set of allocations that give to each coalition *not more* than its contribution to the grand coalition:

$$x(S) \geq v(S) \Leftrightarrow x(S) \leq v(N) - v(N \setminus S) = v^*(S) \text{ for all } x \in X(N, v).$$

Hence, the core of a game coincides with the anticore of its dual: $\bar{C}(N, v) = \underline{C}(N, v^*)$ and the following proposition then follows.

Proposition 13 The core of a game (N, v) whose *dual* is a 1-convex or 1-concave game is a regular simplex of dimension $n - 1$ or a single point.

Consider a game (N, v) and assume that its dual (N, v^*) is 1-convex. Then, by Proposition 9, we have:

$$\begin{aligned} \bar{C}(N, v^*) &= \{x \in \mathbb{R}^n \mid x(N) = v^*(N), x_i \leq v^*(N) - v^*(N \setminus i) \text{ for all } i \in N\} \\ &= \{x \in \mathbb{R}^n \mid x(N) = v(N), x_i \leq v(i) \text{ for all } i \in N\} = \underline{C}(N, v). \end{aligned}$$

The n vertices of $\underline{C}(N, v)$ are given by:

$$\begin{aligned} \omega_i^j(N, v) &= v(i) + h_v(N) \quad \text{if } i = j, \\ &= v(i) \quad \text{if } i \neq j, \end{aligned}$$

where $h_v(S) = \sum_{i \in S} v(i) - v(S) = \sum_{i \in S} b_i(N, v^*) - v^*(S) = g_{v^*}(S)$. Hence, $\omega_i^j(N, v) = \theta_i^j(N, v^*)$.

If the game (N, v) is inessential, $h_{v^*}(N) = g_{v^*}(N) = 0$ and the core reduces to the allocation $\{v(1), \dots, v(n)\}$. Proposition 13 applies to the case where the dual game (N, v^*) is 1-concave: $\underline{C}(N, v^*) = \bar{C}(N, v)$.

Quasi-additive games are not 1-convex but their duals are 1-convex or 1-concave. Indeed, the gaps of their duals are constant and the gap associated to the grand coalition is nonnegative or non-positive:

$$g_{v^*}(S) = \sum_{i \in S} v(i) - v(N) + v(N \setminus S) = \sum_{i \in N} v(i) - v(N).$$

Hence, the core of a quasi-additive game is a regular simplex.

Example 2 (continued) The core of the unanimity game and the core of its dual coincide with the unit simplex.

Combining Proposition 8 and Proposition 13, the core of a game satisfying (7) is a regular simplex or a single point, while not being 1-concave. Similarly, the core of a game satisfying (8) is a regular simplex or a single point, while not being 1-convex.

Example 3 (continued) The game $v = (4, 2, 3, 6 \mid 7, 9, 10, 8, 10, 9 \mid 12, 12, 13, 12 \mid 13)$ satisfies (7) and its dual is the 1-convex game $v^* = (1, 0, 1, 1 \mid 4, 3, 5, 3, 4, 6 \mid 7, 10, 11, 9 \mid 13)$. Its gaps satisfies (3) and its core is the simplex whose vertices are $(2,2,3,6)$, $(4,0,3,6)$, $(4,2,1,6)$ and $(4,2,3,4)$.

Example 4 The game $v = (2, 3, 7 \mid 5, 8, 9 \mid 15)$ satisfies (8) and its dual is the 1-concave game $v^* = (6, 7, 10 \mid 8, 12, 13 \mid 15)$. Its core is the simplex whose vertices are $(5,3,7)$, $(2,6,7)$ and $(2,3,10)$, as represented in Figure 2.

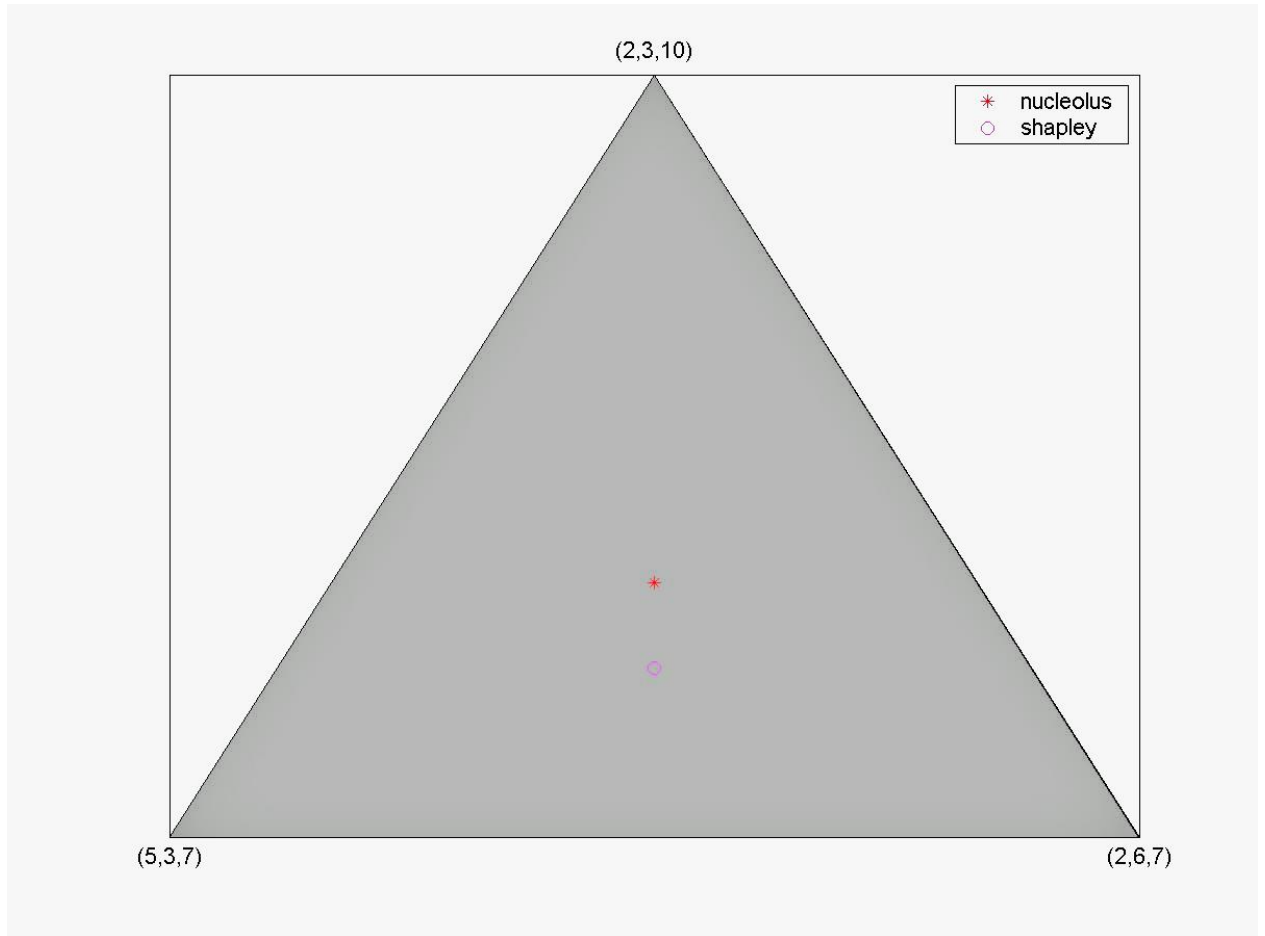


Figure 2 Core of the 1-convex game of Example 4

Proposition 8 explains why only the worth's of the coalitions of the form $N \setminus i$ enter into the definition of the core of a convex game. Indeed, looking at a game whose dual is 1-convex, (7) tells us that only the grand coalition will form and thereby only the marginal contributions to the grand coalition count. The same argument applies to games satisfying (8).

5. The nucleolus of a 1-convex game

Given an allocation x , the *excess* (or loss) of a coalition S is defined by $e(x, S) = v(S) - x(S)$, the difference between its worth and what it obtains. The ε -core introduced by Shapley and Shubik (1966) allows for allocations that induce an excess that does not exceed ε .

The *least-core* introduced by Maschler, Peleg and Shapley (1979) is the set of allocations that minimize the largest excess: $\text{Min}_{x \in X(N, v)} \text{Max}_{\substack{S \subset N \\ S \neq \emptyset, N}} e(x, S) = \varepsilon_{\min}$.

The *nucleolus* introduced by Schmeidler (1969) goes beyond the least-core by comparing the excesses lexicographically to eventually retain a single allocation. That allocation then belongs to the core if nonempty. The situation turns out to be simple when considering a 1-convex game because the least-core is a singleton and it therefore coincides with the nucleolus.

Proposition 14 The nucleolus of a 1-convex game is the *center of gravity* of its core given by:⁴

$$\text{NUC}_i(N, v) = b_i(N, v) - \frac{1}{n} \left(\sum_{i \in N} b_i - v(N) \right) \quad (i = 1, \dots, n). \quad (12)$$

Proof Consider a 1-convex game (N, v) and its core:

$$\bar{C}(N, v) = \left\{ x \in \mathbb{R}^n \mid x(N) = v(N) \text{ and } x_i \leq v(N) - v(N \setminus i) \text{ for all } i \in N \right\}.$$

The inequalities $x_i \leq v(N) - v(N \setminus i)$ are equivalent to $x(N \setminus i) \geq v(N \setminus i)$. Hence, the excesses that matter are those of the coalitions $N \setminus i$: $e(x, N \setminus i) = x_i - b_i(N, v)$. The problem

$$\text{Min}_{x \in X(N, v)} \text{Max}_{i \in N} e(x, N \setminus i) = \text{Min}_{x \in X(N, v)} \text{Max}_{i \in N} (x_i - b_i(N, v)).$$

has a unique solution and in the solution, the differences $x_i - b_i(N, v)$ are equal: $x_i - b_i(N, v) = a$ for all $i \in N$ where the real a satisfies the equation $x(N) - \sum_{i \in N} b_i(N, v) = na$. The least-core is then a singleton and it defines the nucleolus as the average of the core vertices. ♦

Example 1 (continued) The nucleolus of the game $v = (0, 0, 0 \mid 21, 25, 28 \mid 41)$ is the allocation $(31/3, 40/3, 52/3)$, as shown in Figure 1.

Example 3 (continued) The game $v = (4, 2, 3, 6 \mid 7, 9, 10, 8, 10, 9 \mid 12, 12, 13, 12 \mid 13)$ satisfies (7). Its nucleolus is therefore given by $(3.5, 1.5, 2.5, 5.5)$.

Example 4 (continued) The game $v = (2, 3, 7 \mid 5, 8, 9 \mid 15)$ satisfies (8). Its nucleolus is the allocation $(3, 4, 8)$.

Example 5 The game $v = (5, 6, 7, 8 \mid 10, 12, 12, 12, 13, 15 \mid 16, 18, 17, 20 \mid 30)$ satisfies the inequalities (8). Its dual is therefore 1-concave. As a consequence, its core is the simplex whose vertices are $(9, 6, 7, 8)$, $(5, 10, 7, 8)$, $(5, 6, 11, 8)$ and $(5, 6, 7, 12)$, and its nucleolus is the allocation $(6, 7, 8, 9)$.

Efficient upper-vectors coincide with marginal contribution vectors for games that are convex *and* 1-convex. The following proposition is therefore not a surprise although we have to take account of the fact that there are n efficient upper-vectors and $n!$ marginal contribution vectors.

Proposition 15 If a game is convex and 1-convex (or concave and 1-concave), its nucleolus and Shapley value coincide.

⁴ The center of gravity has been proposed as a core selection by González-Díaz and Sánchez-Rodríguez (2007). In the 1-concave case, it corresponds to the *egalitarian non-separable cost method*. It also coincides with the τ -value introduced by Tijs (1981) and further developed in Driessen and Tijs (1983).

Proof The core of a convex game (N, v) is the polyhedra whose vertices are the marginal contribution vectors. The Shapley value is the average marginal contribution vector, taking into account all $n!$ vectors (Shapley, 1971).

Following Proposition 2, convexity and 1-convexity combined imply that

$$v(S) = \sum_{i \in S} b_i(N, v) - g_v(N) \text{ for all } S \subset N.$$

Hence, $v(S) - v(S \setminus i) = b_i(N, v)$ for all $S \subset N, S \neq \{i\}$ and all $i \in S$. Considering the permutation $\pi = (i, i_2, \dots, i_n)$ where player i is first, we have:

$$\mu_i^\pi(N, v) = v(i) = b_i(N, v) - g_v(N) \text{ and } \mu_j^\pi(N, v) = b_j(N, v) \text{ for all } j \neq i.$$

As a consequence, all permutations in which a given player is first are identical: there are n distinct vectors, each with multiplicity $(n - 1)!$ and they coincide with the efficient upper-vectors. ♦

Example 6 The game $v = (0, 3, 7 \mid 5, 9, 12 \mid 14)$ is convex and 1-convex. Its core is the simplex whose vertices are $(0, 5, 9)$, $(2, 3, 9)$ and $(2, 5, 7)$. Its nucleolus is the allocation $(1.33, 4.33, 8.33)$ and it coincides with its Shapley value.

The Shapley value of a 1-convex game does not necessarily define a core allocation, as the following example shows.

Example 7 The game $v = (0, 0, 0 \mid 6, 4, 8 \mid 10)$ is 1-convex. Its nucleolus $(1.33, 5.33, 3.33)$ differs from the Shapley value $(2.33, 4.33, 3.33)$ as shown in Figure 3.

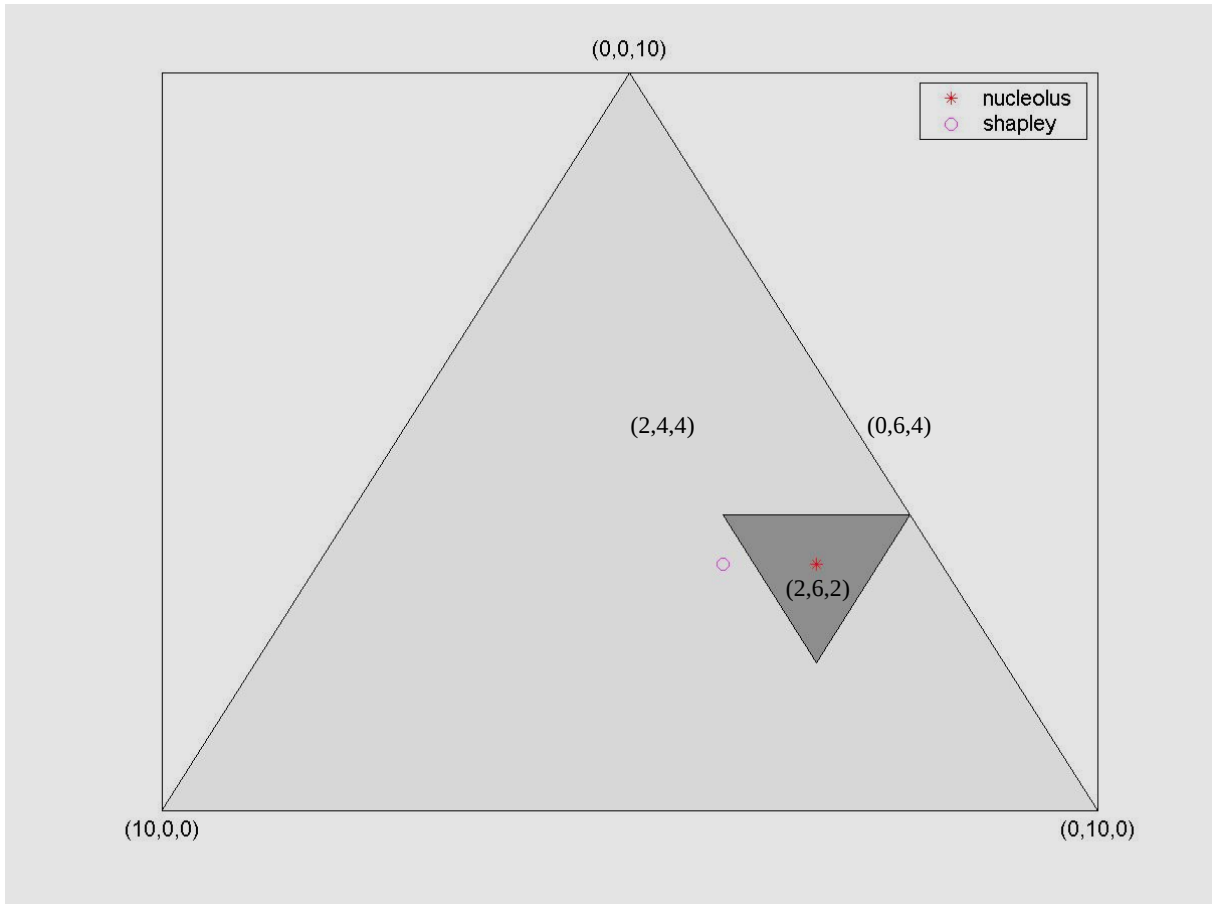


Figure 3 The core of the 1-convex game of Example 7

The Shapley value is an additive sharing rule on the class of superadditive games, while the nucleolus is not. However, the nucleolus is additive on the class of 1-convex games: considering two games (N, v_1) and (N, v_2) on a common set of players N , and applying (12), we indeed have:

$$NUC(N, v_1) + NUC(N, v_2) = NUC(N, v_1 + v_2).$$

7. Applications

Here are applications that involve 1-convex or 1-concave games. We have left aside the information games (Driessen et al., 1992 and 1995) and co-insurance games (Driessen et al., 2010) whose cores are single allocations under 1-convexity.

7.1 Bankruptcy resolution (O'Neill, 1982; Aumann and Maschler, 1985)

Consider a situation involving n creditors, each holding a claim, $d_i > 0$ for creditor i . $E > 0$ is the value of the estate to be allocated. It falls short of the total claim $d(N)$. Reasoning in terms of what a coalition could obtain, the worth of a coalition is equal to what is left for its members, when priority has been given to those outside the coalition:

$$v(S) = \text{Max}(0, E - d(N \setminus S)).$$

This defines *bankruptcy games*. It is well known that these games are convex. The upper-vector is given by:

$$b_i(N, v) = v(N) - v(N \setminus i) = \text{Min}(E, d_i).$$

Assuming that no individual debt exceeds the estate value, we have $b_i(N, v) = d_i$ for all $i \in N$ and $g_v(S) = d(S) - v(S)$, and therefore:

$$g_v(N) = d(N) - v(N) = d(N) - E \geq 0$$

and

$$g_v(N) - g_v(S) = d(N \setminus S) - E + v(S) = \text{Max}(0, d(N \setminus S) - E) \text{ for all } S \neq \emptyset.$$

Hence, a bankruptcy game is 1-convex *if and only if* no proper coalition has a debt exceeding the estate value, i.e. $d(N \setminus i) \leq E \leq d(N)$ for all $i \in S$.⁵ If it is the case, Proposition 5 applies:

$$g_v(S) = g_v(N) = d(N) - v(N) \geq 0 \text{ for all } S \subset N$$

and the core is given by:

$$\bar{C}(N, v) = \{x \in \mathbb{R}^n \mid x(N) = E, x_i \leq d_i \text{ for all } i \in N\}.$$

Its vertices are the efficient upper-vectors $d - (d(N) - E) \cdot e^i$ and the nucleolus is given by:

$$NUC_i(N, v) = d_i - \frac{1}{n}(d(N) - E) \quad (i = 1, \dots, n).$$

It coincides with the Shapley value in accordance with Proposition 15. The excess over the estate value is uniformly distributed among the claimants.

The dual of a bankruptcy game (N, v) has been considered by Aumann (2010). It is given by:

$$v^*(S) = \text{Min}(E, d(S)).$$

The worth of a coalition is equal to the sum of its members' claims, as long as it does not exceed the estate value. It is not what coalition S can get. In this formulation, coalition S has a right to $d(S)$.

⁵ See Driessen (1995).

By Proposition 7, (N, v^*) is concave as dual of a convex game. If $d(N \setminus i) \leq E$ for all $i \in S$, (N, v) is 1-convex and therefore, its dual is quasi-additive. Indeed, under that assumption, $v^*(S) = d(S)$ for all $S \subseteq N$ and $v^*(N) = E$.

Example 8 The bankruptcy situation $(E \mid d) = (1000 \mid 150, 200, 350, 400)$ gives rise to the 1-convex game $v = (50, 100, 250, 300 \mid 250, 400, 450, 450, 500, 650 \mid 600, 650, 800, 850 \mid 1000)$. The four core vertices are given by $(50, 200, 350, 400)$, $(150, 100, 350, 400)$, $(150, 200, 250, 400)$ and $(150, 200, 350, 300)$. The allocation $(125, 175, 325, 375)$ defines the nucleolus and the Shapley value. The dual is the quasi-additive game defined by $v^*(i) = 150$ and $v^*(N) = 1000$.

7.2 Contributing to a joint subscription (Sales 2002 and Driessen et al. 2012)

The problem is the joint purchase of scientific electronic journals by a consortium of universities. Knowing the journals' prices and the historical choices made by each university, the question concerns the division of the total annual bill.

Consider a situation involving a set N of universities (the players), a set M of journals and a single publisher. The prices of the journals are given by a nonnegative vector $p = (p_1, \dots, p_m)$. The historical demands are given by a $n \times m$ nonnegative matrix $E = [e_{ih}]$ where $e_{ih} \in [0, 1]$ and $e_{ih} = 1$ if and only if journal h was part of the historical demand of university i . Knowing that the universities have no choice but to buy the complete set of journals, a discount $\alpha \in [0, 1]$ applies to journals that are not included in no historical demand. The subset of journals that are not included in the historical demands of university i is denoted by $M_i = \{h \in M \mid d_{ih} = 0\}$.

The resulting cost-sharing game (N, v) is then defined by the following characteristic function:

$$v(S) = \sum_{h \in M} p_h \left(\sum_{i \in S} d_{ih} \right) + \alpha \sum_{h \in M_S} p_h$$

where M_S is the intersection of the M_i 's. In particular, we have

$$v(N) = \sum_{h \in M} n(h) p_h + \alpha \sum_{h \in M_N} p_h$$

where $n(h) = \sum_{i \in N} d_{ih}$ is the number of universities that have journal h in their historical demands. The first part is a sum of additive games, one for each journal. The second part is a 1-concave game. To see this, consider the game (N, w) defined by $w(S) = \sum_{h \in M_S} p_h$. The upper-vector is defined by:

$$b_i(N, w) = \sum_{h \in M_N} p_h - \sum_{h \in M_{N \setminus i}} p_h = - \sum_{h \in \bar{M}_i} p_h \quad (i = 1, \dots, n).$$

where $\bar{M}_i = \{h \in M \mid d_{ih} = 1 \text{ and } d_{jh} = 0 \text{ for all } j \in N \setminus i\}$ is the set of journals that *only* university i has in its historical demand. The gap function is given by:

$$g_w(S) = - \sum_{i \in S} \sum_{h \in \bar{M}_i} p_h - \sum_{h \in M_S} p_h.$$

The gaps are non-positive. In particular, $g_w(N) \leq 0$. Furthermore, we have:

$$g_w(S) - g_w(N) = \sum_{i \in N \setminus S} \sum_{h \in \bar{M}_i} p_h + \sum_{h \in M_N} p_h - \sum_{h \in M_S} p_h = \sum_{i \in N \setminus S} \sum_{h \in \bar{M}_i} p_h - \sum_{h \in M_S \setminus M_N} p_h.$$

The set of journals that appear in the first part are those that are exclusively included in the historical demands of the members of $N \setminus S$. It is included in the set of journals that appear in the second part, those in $M \setminus M_N$ that are not in the historical demands of the members of coalition S .

Hence, the above difference is non-positive and the inequalities (3) hold for all $S \neq \emptyset$. The game (N, w) is therefore 1-concave and so is the complete game (N, v) whose upper-vector is defined by:

$$b_i(N, v) = \sum_{h \in M \setminus M_i} p_h - \alpha \sum_{h \in \bar{M}_i} p_h \quad (i = 1, \dots, n).$$

The gap associated to the grand coalition is given by:

$$g_v(N) = \left(\sum_{i \in N} \sum_{h \in M \setminus M_i} p_h - \sum_{h \in M} p_h \sum_{i \in N} d_{ih} \right) - \alpha \left(\sum_{i \in N} \sum_{h \in \bar{M}_i} p_h + \sum_{h \in M_N} p_h \right) = -\alpha \left(\sum_{h \in \bar{M}_0} p_h + \sum_{h \in M_N} p_h \right)$$

where $\bar{M}_0$ is the union of the $\bar{M}_i$'s. The expression between parentheses is the cost of the journals included in the historical demands of *single* universities plus the cost of the journals for which there was no demand. The nucleolus of the complete game is then given by:

$$NUC_i(N, v) = \sum_{h \in M \setminus M_i} p_h - \alpha \sum_{h \in \bar{M}_i} p_h + \frac{\alpha}{n} \sum_{\substack{h \in M: \\ n(h) \leq 1}} p_h$$

University i pays the cost of the journals that are in its historical demand, plus a flat fee for all the journals that are absent of all historical demands ($n(h) = 0$). It also pays a flat fee for the journals that are in the historical demands of *single* universities ($n(h) = 1$). A reduction is only offered to these universities. While being different, the Shapley value has a similar structure:

$$SV_i(N, v) = \sum_{h \in M \setminus M_i} p_h - \alpha \sum_{h \in M \setminus M_i} \frac{p_h}{n(h)} + \frac{\alpha}{n} \sum_{h \in M} p_h$$

Universities pay for the journals that are in their historical demands, plus a flat fee based on all journals. And each university benefits of a reduction based on the journals that are in its historical demand.

Example 9 Consider a situation involving three universities and six journals, where the prices and historical demands are given by $p = (12, 7, 9, 10, 15, 8)$ and

$$D = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The game (N, w) is given by $w = (32, 15, 34 | 8, 17, 15 | 8)$. Its upper-vector and gap function are given by $b = -(p_2, p_3, 0) = -(7, 9, 0)$ and $g_w = -(39, 24, 34 | 24, 24, 24 | 24)$, confirming 1-concavity. For $\alpha = 1/2$, the complete game is given by $v = (45, 53.5, 44 | 79, 64.5, 80.5 | 106)$. Its upper-vector and gap function are given by $b = (25.5, 41.5, 27)$ and $g_v = -(19.5, 12, 17 | 12, 12, 12 | 12)$. The core vertices are $(37.5, 41.5, 27)$, $(25.5, 53.5, 27)$ and $(25.5, 41.5, 39)$. The nucleolus is then the allocation $(29.5, 45.5, 31)$, to be compared to the allocation $(31.16, 43.42, 31.42)$ that defines the Shapley value.

7.3 Sharing data (Dehez and Tellone, 2013)⁶

A set N of players need to have access to a set M of data that are own by one or several players. A value is attached to each data, $p_h > 0$ for data h , for instance the cost of reproducing the data. The

⁶ The background is the EU program REACH that imposes to chemical firms to submit data about their products.

subset of data owned by player i is denoted by M_i . If a coalition forms, the cost of acquiring the missing data is given by:

$$v(S) = \sum_{h \in M \setminus M_S} p_h = w_0 - \sum_{h \in M_S} p_h \text{ for all } S \subset N, S \neq \emptyset$$

where M_S is the data set detained by coalition S and w_0 is the value of the complete data set. This defines a cost-sharing game.⁷ Because $v(N) = 0$, an allocation is a set of payments $x_1, x_2, \dots, x_n$ summing to zero. Data games are subadditive. They are essential if at least one player does not detain the complete data set. Data games are 1-concave.

We denote by $\bar{M} \subset M$ the set of data detained by *single players* and by $\bar{M}_i = M_i \cap \bar{M}$ the set of data that player i is *alone* to detain. Alongside, we denote by $\bar{w}_0$ and $\bar{w}_i$ the value of the data in $\bar{M}$ and the value of the data in $\bar{M}_i$, respectively. We then have:

$$b_i(N, v) = v(N) - v(N \setminus i) = -(w_0 - \sum_{h \in M_{N \setminus i}} p_h) = -\bar{w}_i.$$

The core of a data game is non-empty: it contains the no-compensation allocation $x = 0$. If x is a core allocation, $x(N \setminus i) \leq v(N \setminus i) = \bar{w}_i$ and, because $x(N) = 0$, $x_i \geq -\bar{w}_i$. Hence, if an allocation x satisfies these inequalities for all i , we have:

$$x(S) = -x(N \setminus S) \leq \sum_{i \in N \setminus S} \bar{w}_i \leq \sum_{h \in M \setminus M_S} p_h = v(S) \text{ for all } S \subset N.$$

As a consequence, the core is given by:

$$C(N, v) = \{x \in \mathbb{R}^n \mid x(N) = 0 \text{ and } x_i \geq -\bar{w}_i \text{ for all } i \in N\}.$$

If $\bar{M} \neq \emptyset$, $\bar{w}_0 > 0$ and the core is the simplex whose vertices are the n efficient upper-vectors $(-\bar{w}_1, -\bar{w}_2, \dots, -\bar{w}_n) + \bar{w}_0 e^i$. Therefore, referring to Proposition 12, the game is 1-concave and the nucleolus is given by:

$$NUC_i(N, v) = \frac{1}{n} \sum_{h \in \bar{M}} p_h - \sum_{h \in M_i} p_h \quad (i = 1, \dots, n)$$

A player is compensated *if and only if* the value of the data he is alone to detain exceeds the *per capita* value of the data detained by single firms. If instead $\bar{M} = \emptyset$, $\bar{w}_0 = 0$ and the core consists of a single allocation, namely no-compensation allocation $x = 0$. The Shapley value instead makes no difference between data held by single or several players:

$$SV_i(N, v) = \frac{1}{n} \sum_{h \in M} p_h - \sum_{h \in M_i} \frac{p_h}{n(h)} \quad (i = 1, \dots, n)$$

where $n(h)$ denotes the number of players that hold data h . In the particular case where *every* data is held by *one and only one* player, $\bar{M} = M$ and $n(h) = 1$ for all h , the game is concave and the nucleolus coincides with the Shapley value, in accordance with Proposition 16.

Example 10 Consider the 3-player and 3-data situation defined by $M_1 = \emptyset$, $M_2 = \{1, 2\}$, $M_3 = \{2, 3\}$. The data game corresponding to $p = (6, 9, 12)$ is given by $v = (27, 12, 6 \mid 12, 6, 0 \mid 0)$. The value of the data detained by single players is $\bar{w}_0 = d_1 + d_3 = 18$ and the upper-vector is given by $\bar{w} = -(0, 6, 12)$. The core vertices are given by $(18, -6, -12)$, $(0, 12, -12)$ and $(0, -6, 6)$. The

⁷ The relations between data games and library games have been pointed out by Driessen and Khmelnitskaya (2017).

nucleolus is given then by (6, 0, -6): player 1 pays 6 to player 3, and player 2 stay put. The Shapley value is given by (9, -1.5, -7.5). Here, players 2 and 3 compensate player 1.

7.4 Sharing patents (Dehez and Poukens, 2013)

Imagine a situation where n firms operating on different markets are willing to commercialize a product that relies on patents owned by some of them. These patents are assumed to be *weak*: while being essential, they can be invented around at a known cost. The value of the patents owned by firm i is given by $p_i \geq 0$ and the profit it can generate from the commercialization of the product is given by $\pi_i \geq 0$. The associated transferable utility game (N, v) is then defined by:

$$v(S) = \sum_{i \in S} \pi_i - \sum_{i \in N \setminus S} p_i.$$

The question then concerns the redistribution of the total profit $v(N) = \pi(N)$ and a redistribution defines transfers (royalties) between firms. The game (N, v) is both convex and 1-convex. Indeed, following Proposition 2, we have:

$$b_i(N, v) = \pi_i + p_i \Rightarrow g_v(S) = p(N) \text{ for all } S \subset N.$$

The core is therefore defined by:

$$\bar{C}(N, v) = \{x \in \mathbb{R}^n \mid x(N) = \pi(N) \text{ and } x_i \leq \pi_i + p_i \text{ for all } i \in N\}.$$

Its vertices are given by $\theta^i = b(N, v) - p(N)e^i$ and, following Proposition 17, the Shapley value and the nucleolus coincide:

$$SV_i(N, v) = NUC_i(N, v) = \pi_i + p_i - \frac{1}{n} \sum_{j \in N} p_j.$$

All firms uniformly support the value of the whole patent set while the value of each patent is entirely redistributed to the firm holding it: patent holders perceives a compensation if the value of the patents they hold is superior to the *per capita* value of the whole patent set.

Example 11 Consider the 3-firm situation where $\pi = (10, 20, 30)$ and $p = (3, 7, 9)$. The associated game is given by $v = (-6, 8, 20 \mid 21, 33, 47 \mid 60)$ and its nucleolus and Shapley value are given by the allocation (6.66, 20.66, 32.66).

7.5 Provision of an indivisible public good (Dehez, 2013)

A group of n cities consider the provision of an indivisible public good, for instance a waste treatment plant. A known cost is associated to each town, $c_i \geq 0$ for city i . The question is to decide where to locate the treatment plant and what compensations to award to the city that host it.

Without loss of generality, cities are ordered in terms of their cost: $0 \leq c_1 \leq c_2 \leq \dots \leq c_n$. If a coalition forms, it will naturally rely on the player with the lowest cost. The associated cost-sharing game (N, v) is then defined by:

$$v(S) = \min_{i \in S} c_i \text{ for all } S \subset N, S \neq \emptyset.$$

This game is *subadditive* game that is *essential* if $c_{an} > 0$. Its core is non-empty: it always contains the no-compensation allocation $(c_1, 0, \dots, 0)$. It is *concave* if $c_2 = c_n$: the marginal contributions of player 1 are all equal to $c_1 - c_2$ while the marginal contributions of the other players are all equal to zero.

Notice that the function v can be written as $v(S) = c_n - w(S)$ where the game $w(S) = \text{Max}_{i \in S} (c_n - c_i)$ is an *airport game* on N with cost parameters $(0, c_n - c_{n-1}, \dots, c_n - c_1)$.⁸

The upper-vector is defined by $b_1(N, v) = c_1 - c_2$ and $b_i(N, v) = 0$ for all $i \neq 1$. In terms of the gap function, we then have:

$$g_v(N) = \sum_{i \in N} b_i(N, v) - v(N) = -c_2 \leq 0$$

and

$$\begin{aligned} g_v(N) - g_v(S) &= -c_2 - \left(\sum_{i \in S} b_i(N, v) - v(S) \right) = 0 && \text{if } 1 \in S \\ &= -c_2 + \text{Min}_{i \in S} c_i \geq 0 && \text{if } 1 \notin S \end{aligned}$$

The inequalities (3) are verified and the game (N, v) is 1-concave. The core and its vertices are given by:

$$\underline{C}(N, v) = \{x \in \mathbb{R}^n \mid x(N) = c_1, x_1 \geq c_1 - c_2 \text{ and } x_i \geq 0 \text{ for all } i \in N \setminus \{1\}\},$$

$$\theta^1 = (nc_1 - (n-1)c_2, c_1 - c_2, \dots, c_1 - c_2), \theta^2 = (0, c_2, 0, \dots, 0), \dots, \theta^n = (0, 0, \dots, c_2),$$

from which we deduce the nucleolus

$$\begin{aligned} \text{NUC}_i(N, v) &= \frac{c_2}{n} - (c_2 - c_1) = c_1 - \frac{n-1}{n} c_2 \quad \text{for } i = 1 \\ &= \frac{c_2}{n} && \text{for all } i \neq 1. \end{aligned}$$

The provider supports his cost and receives a compensation from the other players who contribute an equal amount. We observe that only the two lowest costs enter into account. This is not so for the Shapley value that may differ significantly from the nucleolus. It can be derived from the Shapley value of the associated airport game, hence the following triangular formula:

$$\text{SV}_i(N, v) = \sum_{j=i}^n \frac{c_j - c_{j+1}}{j} \quad (i = 1, \dots, n)$$

where $c_{n+1} = 0$. The Shapley value and the nucleolus coincide in the particular case where $c_2 = c_n$. The game is indeed concave in this case and Proposition 15 applies.

Example 12 The 3-player game associated to the cost vector $c = (3, 7, 25)$ is defined by $v = (3, 7, 25 \mid 3, 3, 7 \mid 3)$. Its nucleolus and Shapley value are given by:

$$\text{NUC}(N, v) = (-1.66, 2.33, 2.33),$$

$$\text{SV}(N, v) = (-4.66, -0.66, 8.34).$$

Notice that the Shapley value compensates the second player. This is not appropriate in this context: there is no reason to compensate other players than the provider. Actually, core allocations do not compensate any player other than the provider.

⁸ Airport games were introduced by Littlechild and Owen (1973).

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