

ASSESSMENT OF DIFFERENT 1D NUMERICAL SCHEMES FOR STEADY FLOWS OVER A GEOMETRICAL DISCONTINUITY

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ABSTRACT

This paper focuses on the numerical modeling of steady flows and proposes a comparison of three 1D finite-volume schemes. These schemes are all adapted to arbitrary topographies. Two are based on the HLL scheme and one on Roe's approach for the flux calculation. The main goal of the work presented here is to identify the best scheme, between those three, to be extended to sediment transport in rivers. This selected scheme should, in pure hydrodynamics, possess the following characteristics: (1) be well-balanced and (2) predict accurately the water level and discharge even with a coarse mesh.

To compare those schemes, two types of numerical tests are used. First, the well-balanced property is verified by maintaining water at rest over an irregular topography. Then, a steady flow in a convergent / divergent channel is simulated.

These tests highlighted the limits of HLL based schemes as they are unable to provide accurate results without a proper mesh refinement. The best identified scheme is thus the Augmented Roe's scheme with energy balance. However this scheme showed its limitations in the treatment of shocks. A study of this particular issue is also presented in this paper.

Keywords: finite-volumes; 1D models; Arbitrary cross-sections; well-balanced; steady flows

1. INTRODUCTION

One-dimensional schemes are widely used to simulate fast transient flows in rivers. These schemes have the advantages of being faster and requiring less data than 2D or 3D schemes. However, modeling this kind of flows in 1D presents some difficulties because of the strong influence of the source terms. A particular attention should thus be given to their integration. A correct integration can be tested using simple steady flows. To assess the accuracy of a numerical model, two characteristics should be checked: (1) the well-balanced property, in other words, the ability of the model to maintain water at rest, (2) an accurate prediction of both the water level and discharge. The discharge is particularly important if the model is further extended to sediment transport, because the calculation of the sediment discharge depends on the liquid discharge. Another characteristic is important if the model is used in real rivers: the solution should have as less dependence on the mesh refinement as possible, because for real rivers, coarse meshes are often used.

The currently most used schemes solve an approximate Riemann problem to compute the intercell fluxes. Among those, two families can be identified: (1) HLL-type schemes that solve the problem using approximate values of the wave celerities, first developed by Harten et al. (1983) and (2) Roe-type scheme that solve an approximate problem exactly, first developed by Roe (1981). Both approaches have their advantages and limits. Two of latest developments of the HLL scheme are the LHLL adapted to arbitrary topographies (Petaccia et al., 2013) and the HLLS (Murillo and Garcia-Navarro, 2012). The former uses Nujic (1995) variation for the mass flux and distributes the source terms on both sides of the interface considering wave celerities. The latter includes a stationary wave to represent the source terms. It also uses Roe's celerities instead of the classical HLL ones. It was initially developed for rectangular and prismatic sections and is, here, extended to arbitrary topographies. For Roe-based scheme, one of the latest developments is the Augmented Roe's solver with energy balance by Murillo and Garcia-Navarro (2014). Like the HLLS, this scheme adds a stationary wave to represent the source terms. However, it is entirely based on Roe's approach. Moreover, it considers the conservation of energy to compute the source terms.

The main goal of the work presented hereafter is to study and compare the results of the three schemes in simulating steady flows.

This paper is organized as follow. In section 2, the governing equations are presented. Section 3 presents the steady-flow tests cases, then the three schemes are presented and tested on these test-cases in section 4. Finally, the results are discussed in section 5.

2. GOVERNING EQUATIONS

The flow is represented by the one-dimensional shallow-water equations. Those equations consider mass and momentum conservation. Written for cross-sections of arbitrary shape, they read:

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial x} Q = 0 \quad [1]$$

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left(\frac{Q^2}{A} + g I_1 \right) = g [I_2 + A(S_0 - S_f)] \quad [2]$$

where Q is the discharge, A the wetted area, g the gravity, S_0 the bed slope represented by

$$S_0 = -\frac{dz}{dx} \quad [3]$$

The hydrostatic pressure is calculated from the first moment of inertia I_1 (see Figure 1) that is defined by

$$I_1 = \int_0^h [h - \eta] b(x, \eta) d\eta \quad [4]$$

with the water depth h and the width b

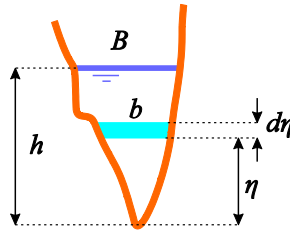


Figure 1. Definition of the hydrostatic pressure

In the source term, I_2 represents the lateral pressures due to the longitudinal width variation and is defined as

$$I_2 = \int_0^h [h - \eta] \frac{\partial b}{\partial x} d\eta \quad [5]$$

The friction is represented using Manning formula

$$S_f = n^2 \frac{Q^2 P^{4/3}}{A^{10/3}} \quad [6]$$

The system of equations [1] and [2] will be hereafter written in vector form as

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = S \quad [7]$$

with the vector of conserved variables $U = (A, Q)^T$, the vector of fluxes $F = (Q, Q^2/A + g I_1)^T$ and the source terms $S = (0, g[I_2 + A(S_0 - S_f)])^T$

3. STEADY-FLOW TEST-CASES

The first test focuses on the well-balanced property in a trapezoidal channel with varying bed level and width (Murillo and García-Navarro, 2014). The shape of the sections is defined by

$$B(x) = B_0(x) + 0.2(B_0(x) - 9.5)h \quad [8]$$

with B_0 the width of the channel at $h = 0$. It is illustrated, along with the bed elevation, in Figure 2.

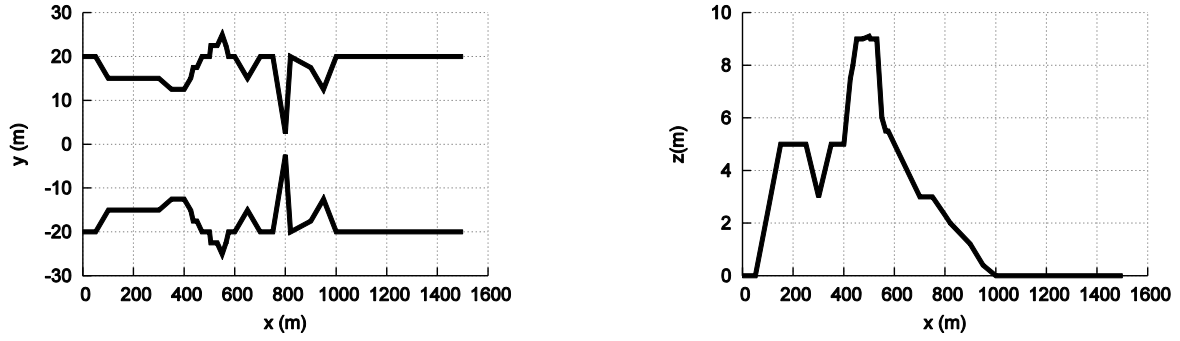


Figure 2. Top view and bed elevation

Then, test cases of a steady flow in a convergent-divergent channel are used to assess the behavior of numerical schemes. In the present case, the domain is frictionless, 25m long. The bed is horizontal and the width (see Figure 3) is described by

$$B(x) = \min(0.8 + 0.05(x - 10)^2; 1) \quad [9]$$

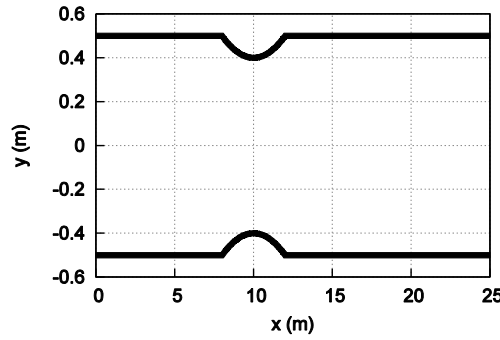


Figure 3. Top view

Different steady flows are obtained by changing the boundary conditions. (1) To obtain a transcritical flow without a shock, a discharge of 4.42 m³/s is selected. The flow being supercritical, nothing is imposed downstream. The models are tested using two different mesh sizes: $\Delta x = 0.5$ m and $\Delta x = 0.05$ m. (2) To obtain a transcritical flow followed by a jump. The discharge is still 4.42 m³/s but, now, downstream, the water level is fixed at 1.9 m. The models are tested using two different mesh sizes: $\Delta x = 0.025$ m and $\Delta x = 0.05$ m.

4. NUMERICAL MODELS

Using a first-order finite volumes scheme, system [7] is discretized as

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x} (F_{i-1/2}^* - F_{i+1/2}^*) \quad [10]$$

The next three sections describe the computation of the intercell fluxes $F_{i\pm 1/2}^*$.

4.1 Lateralized HLL

This scheme was developed by Petaccia et al. (2013) for arbitrary cross-sections. The mass flux is written considering Nujic (1995) variation. A lateralization of the momentum equation is introduced through a difference between the right and left side of an interface. This difference is defined by the source terms and the wave celerities (Fraccarollo et al., 2003).

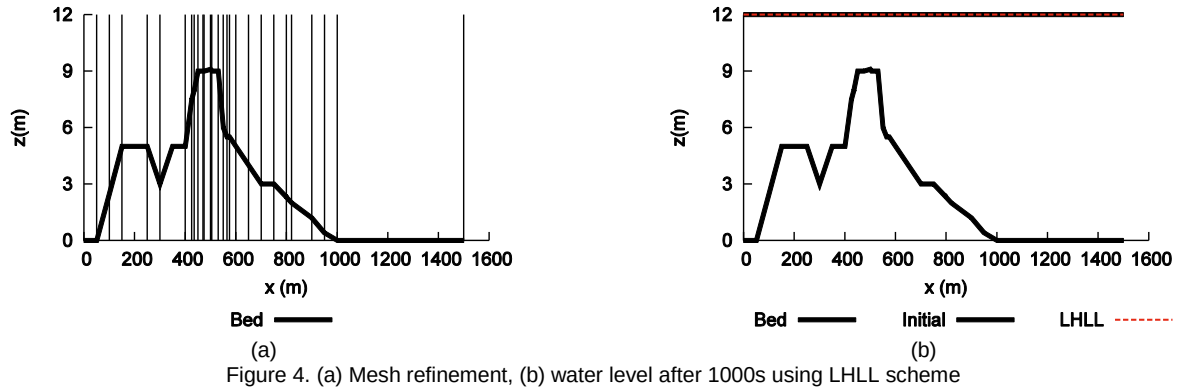
The fluxes are defined as

$$Q_{i-1/2,L}^* = \frac{\lambda^+ Q_{i-1} - \lambda^- Q_i + \lambda^+ \lambda^- (z_{w,i} - z_{w,i-1})}{\lambda^+ - \lambda^-} \quad [11]$$

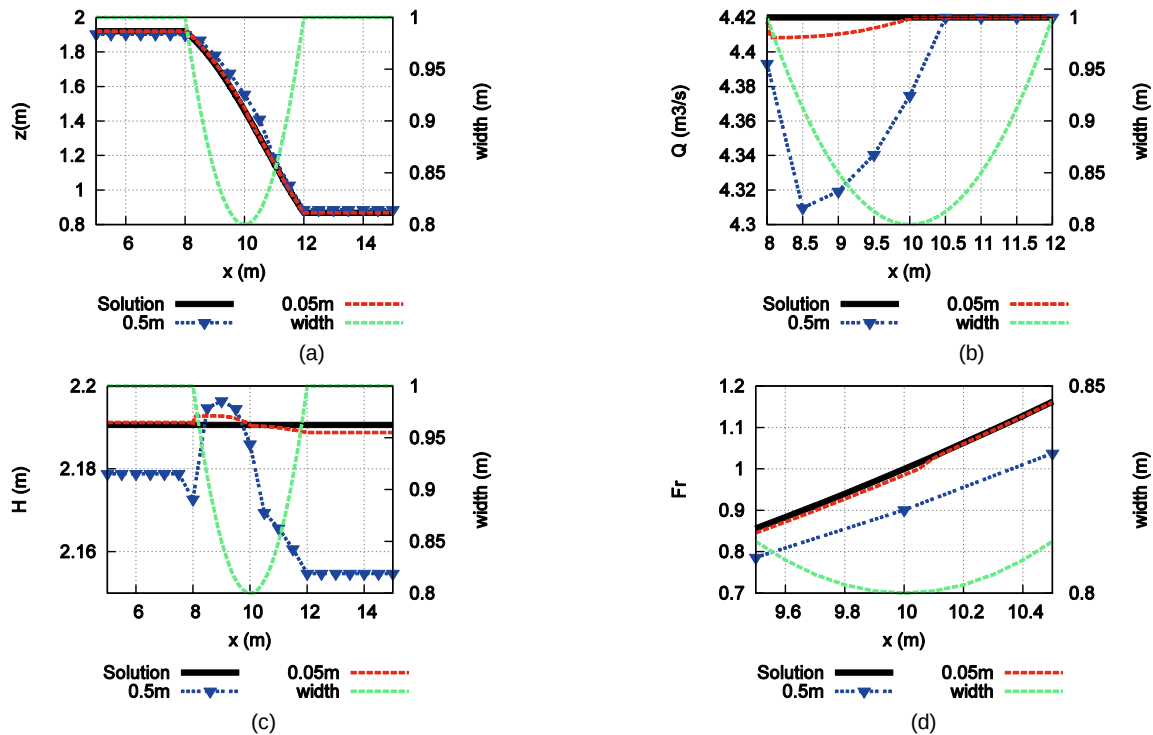
$$F_{i-1/2,L}^* = \frac{\lambda^+ F_{i-1} - \lambda^- F_i + \lambda^+ \lambda^- (U_i - U_{i-1}) + \lambda^- S}{\lambda^+ - \lambda^-} \quad [12]$$

$$F_{i-1/2,R}^* = \frac{\lambda^+ F_{i-1} - \lambda^- F_i + \lambda^+ \lambda^- (U_i - U_{i-1}) + \lambda^+ S}{\lambda^+ - \lambda^-} \quad [13]$$

As can be seen in Figure 4, this scheme is able to maintain water at rest even with a coarse mesh.



Concerning the steady cases of transcritical flow without shock, the LHLL is able to represent accurately the water level with a fine mesh, but not with the coarse mesh of 0.5 m, as can be observed in Figure 5a: the water level at critical section is not well predicted. As regards the discharge (Figure 5b), a significant error is observed in the subcritical part of the flow (i.e. $x < 10$ m) with the coarse mesh, and the error is still important even with the refined mesh. Figure 5c represents the total head that should be constant because no friction losses are considered in this case. Again, it can be observed that even with a finer mesh this is not achieved. Finally, the Froude number (Figure 5d) appears to be very poorly predicted with the coarse mesh, the critical section being located downstream of the actual critical depth ($x = 10$ m). With the fine mesh, the Froude number is underestimated in the upstream part where the flow is subcritical, and correctly predicted in the downstream, supercritical, part.



In the case of a transcritical flow with a shock, only fine-mesh results ($\Delta x = 0.025$ m and $\Delta x = 0.05$ m) are presented in order to avoid an excessive diffusion of the shock. Indeed, the shock position and amplitude are well predicted (Figure 6a). However, the discharge (Figure 6b) is far from constant, with a “peak” is observed at the location of the shock.

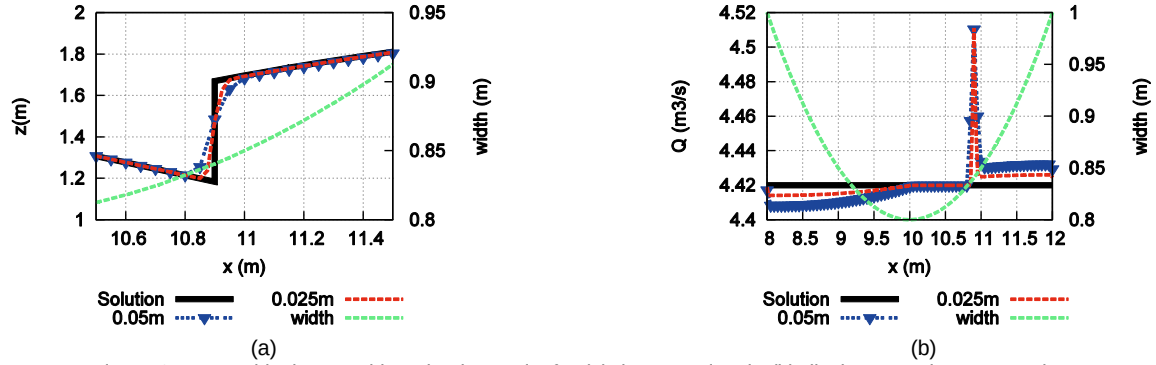


Figure 6. Transcritical case with a shock, results for (a) the water level , (b) discharge using LHLL scheme

4.2 HLLS

This scheme was developed by Murillo and García-Navarro (2012). It includes the source terms by adding a stationary wave to the system. The approach developed by the authors considers rectangular and prismatic sections. It is here extended to arbitrary sections.

The fluxes are defined as

$$F_{i-1/2,L}^* = \frac{\lambda^+ F_{i-1} - \lambda^- F_i + \lambda^+ \lambda^- (U_i - U_{i-1}) + \lambda^- (S - \lambda^+ H)}{\lambda^+ - \lambda^-} \quad [14]$$

$$F_{i-1/2,R}^* = \frac{\lambda^+ F_{i-1} - \lambda^- F_i + \lambda^+ \lambda^- (U_i - U_{i-1}) + \lambda^+ (S - \lambda^- H)}{\lambda^+ - \lambda^-} \quad [15]$$

with the source terms written as

$$S = \begin{bmatrix} 0 \\ S_g + S_\tau \end{bmatrix} = \begin{bmatrix} 0 \\ -g \bar{A} \delta z_w + \delta I_1 + S_\tau \end{bmatrix} \quad [16]$$

$$H = \frac{-(S_g + S_\tau)}{\lambda^+ \lambda^-} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad [17]$$

with S_τ defined using Manning equation. In Eq. [16], the term I_2 related to the lateral pressure due to changes in the channel width is replaced by $-g \bar{A} \delta h + \delta I_1$. The celerities used in this scheme are Roe's ones

$$\lambda^+ = \tilde{u} + \tilde{c} \quad [18]$$

$$\lambda^- = \tilde{u} - \tilde{c} \quad [19]$$

Like the LHLL scheme, the HLLS is able to maintain water at rest (see Figure 7)

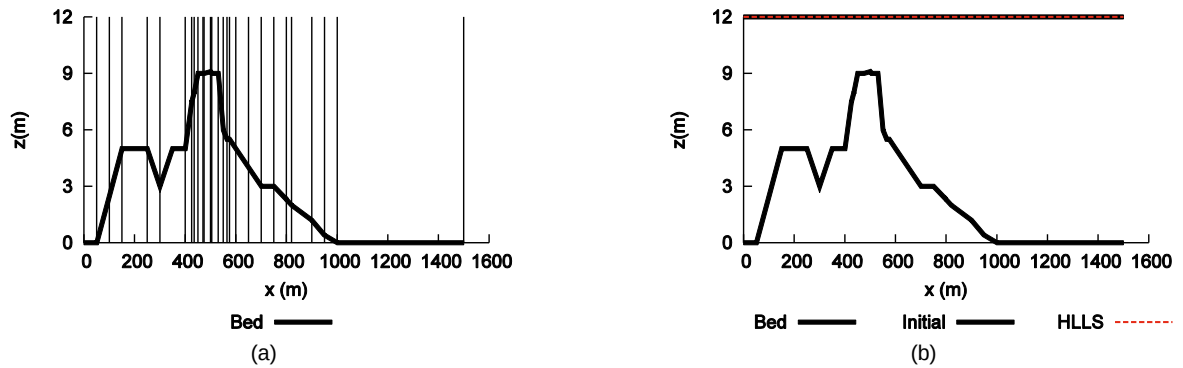


Figure 7. (a) Mesh refinement (a), (b) water level after 1000s using HLLS scheme

For the transcritical case without a shock (Figure 8a), the HLLS scheme predicts the correct water level, with a significant improvement for the simulation run on a coarse mesh. Figure 8b shows that the discharge is conserved, with good results also for the coarse mesh. However, it is incapable with the coarse mesh to predict the correct head (Figure 8c) and Froude number (Figure 8d).

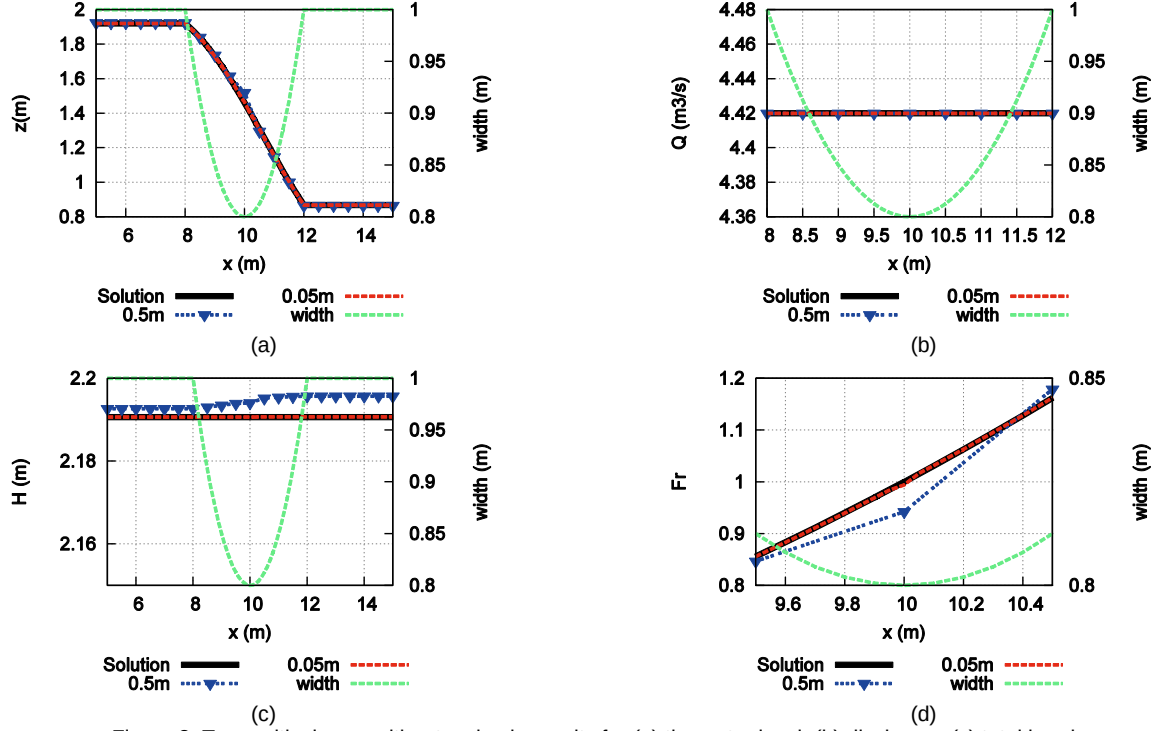


Figure 8. Transcritical case without a shock, results for (a) the water level, (b) discharge, (c) total head and (d) Froude number using HLLS scheme

The transcritical case with a shock is globally well represented (Figure 9): the water level is correct and the discharge is conserved everywhere except in one cell, located at the jump.

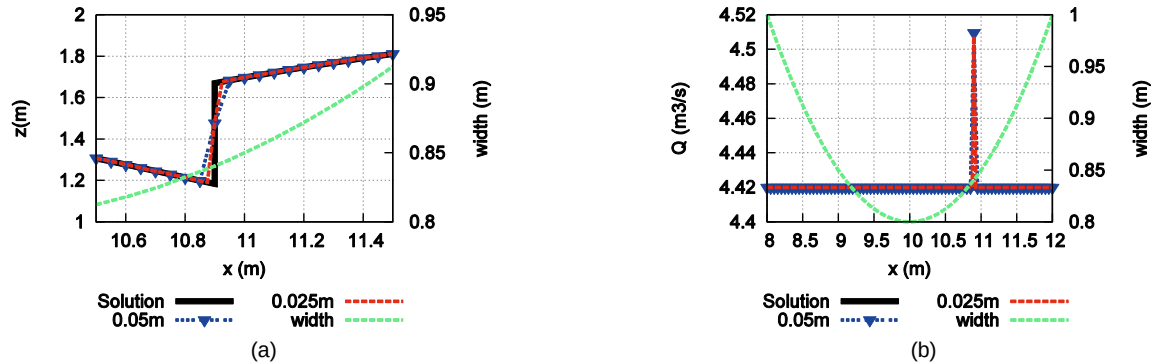


Figure 9. Transcritical case with a shock, results for (a) the water level, (b) discharge using HLLS scheme

4.3 Augmented Roe's solver with energy balance

This model, like the precedent, includes an additional wave to represent the source terms. It was developed by Murillo and García-Navarro (2014) for arbitrary cross-sections. Unlike the two previous schemes, it is based on Roe's solver (Roe, 1981).

The fluxes are defined as

$$F_{i-1/2,L}^* = F_{i-1} + \sum_{\lambda^m < 0}^m (\lambda \alpha \theta e)_{i-1/2}^m \quad [20]$$

$$F_{i-1/2,R}^* = F_i - \sum_{\lambda^m > 0}^m (\lambda \alpha \theta e)_{i-1/2}^m \quad [21]$$

with Roe's wave celerities,

$$\lambda_1 = \tilde{u} - \tilde{c} \quad [22]$$

$$\lambda_2 = \tilde{u} + \tilde{c} \quad [23]$$

and the following expressions for the wave strengths and eigenvectors

$$\alpha_1 = \frac{\lambda_2 \delta A - \delta Q}{2\tilde{c}} \quad [24]$$

$$\alpha_2 = -\frac{\lambda_1 \delta A - \delta Q}{2\tilde{c}} \quad [25]$$

$$e_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} \quad [26]$$

$$e_2 = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix} \quad [27]$$

The source terms are introduced in Eq. [20] and [21] using a stationary wave represented as

$$\theta_1 = 1 + \frac{1}{\lambda_1 \alpha_1} \frac{S}{2\tilde{c}} \quad [28]$$

$$\theta_2 = 1 - \frac{1}{\lambda_2 \alpha_2} \frac{S}{2\tilde{c}} \quad [29]$$

They are computed using a combination of two possible expressions ensuring energy conservation, as written in Eq. [30]. The weight coefficient A_e is defined in Eq. [31]

$$S = \begin{bmatrix} 0 \\ S_g + S_\tau \end{bmatrix} = \begin{bmatrix} 0 \\ (1 - A_e)S_{ga} + A_e S_{gb} + S_\tau \end{bmatrix} \quad [30]$$

$$A_e = \frac{\delta \left(\frac{Q^2}{A} \right) - \tilde{A} \delta \left(\frac{Q^2}{2A^2} \right)}{S_{gb} - S_{ga}} \quad [31]$$

$$S_{ga} = -g\tilde{A}\delta z_w + \tilde{c}^2 \delta A \quad [32]$$

$$S_{gb} = -gA_{min}\delta z_w + \tilde{c}^2 \delta A \quad [33]$$

In case of a jump, where the energy is dissipated, the weigh coefficient A_e is set to zero.

Like the previous scheme, the Augmented Roe's solver with energy balance is able to maintain water at rest (Figure 10)

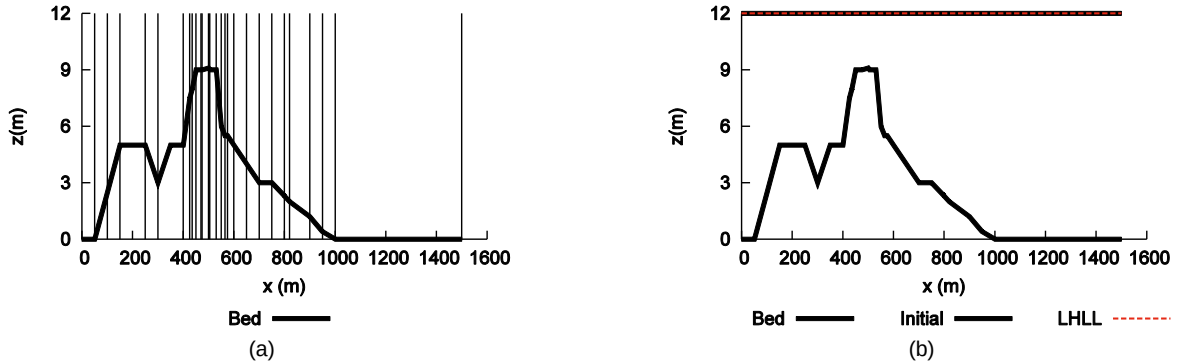


Figure 10. (a) Mesh refinement, (b) water level after 1000s using Augmented Roe's scheme with energy balanced

The results for the transcritical case without shock are really good (Figure 11). The model is able to predict the correct water level (Figure 11a) and discharge (Figure 11b). Also the head (Figure 11c) and the Froude number (Figure 11d) are much better predicted than with the previous schemes even with a coarser mesh.

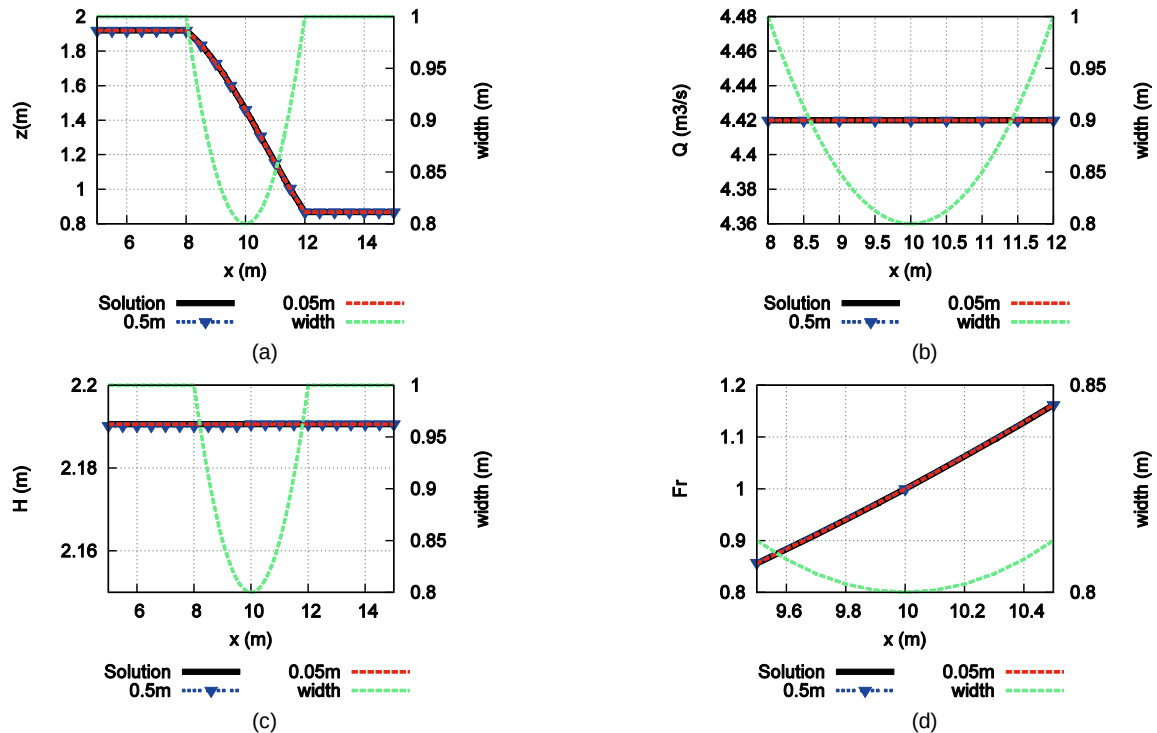


Figure 11. Transcritical case without a shock, results for (a) the water level, (b) discharge, (c) total head and (d) Froude number using Augmented Roe's scheme with energy balance

However, the results are not so good for the transcritical case with shock (Figure 12). Even if the water level is globally well represented, the position of the shock is not correct (Figure 12a). As regards the discharge, the Augmented Roe's solver with energy balance conserves the discharge everywhere but in one cell, just like the HLLS.

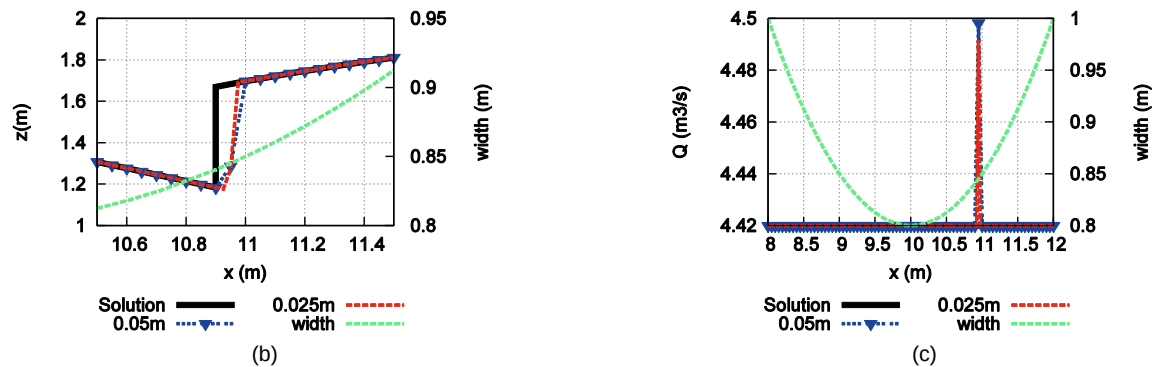


Figure 12. Transcritical case with a shock, results for (a) the water level, (b) discharge using Augmented Roe's scheme with energy balance

In order to better understand the shock modeling in the Augmented Roe's solver with energy balance and to improve it, several changes are made. First, to assess if the error on the jump position is created by the energy balance property, the latter is removed. The results are presented in Figure 13: as can be observed, removing the energy balance does not solve the problem of the discharge conservation (Figure 13b) but solves the problem of the jump position (Figure 13a). However, by removing the energy balance, the results are now more dependent on the mesh size.

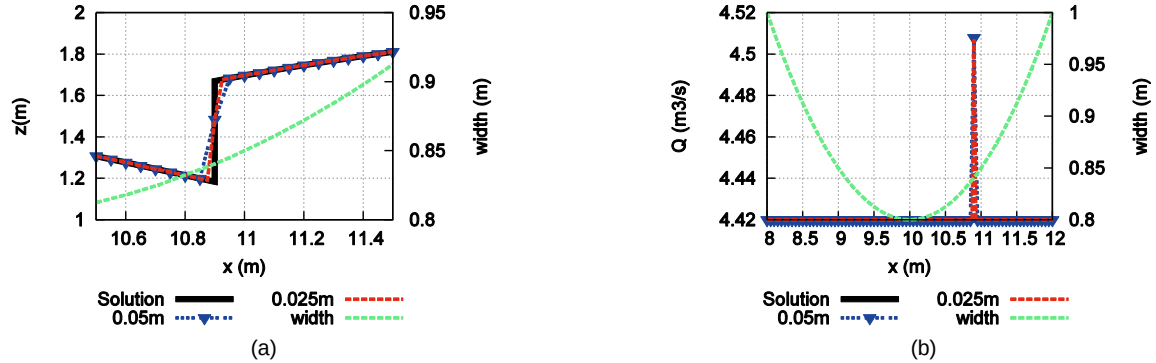


Figure 13. Transcritical case with a shock, results for (a) the water level , (b) discharge using Augmented Roe's scheme without energy balance

Another possibility would be to impose a critical interface ($\lambda^- = 0$) at the location of the jump. However, as shown in Figure 14, this does not really work, the jump is now located too far downstream. Adding a threshold on λ^- to this imposition seems to improve the solution (see Figure 15). However, the position of the jump is worse than without the energy balance. Also, the limit of this approach is that this threshold has to be calibrated. Moreover, by enforcing a critical interface, the Riemann problem is changed.

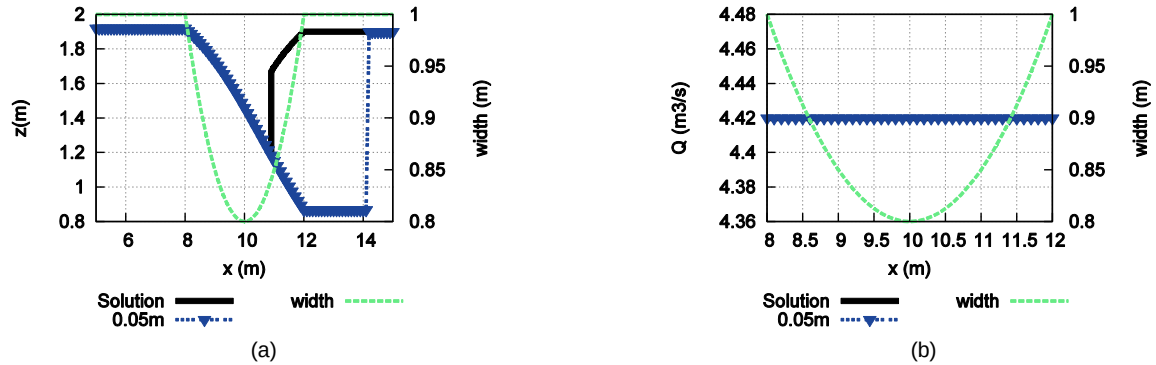


Figure 14. Transcritical case with a shock, results for (a) the water level , (b) discharge using Augmented Roe's scheme with energy balance and a critical interface

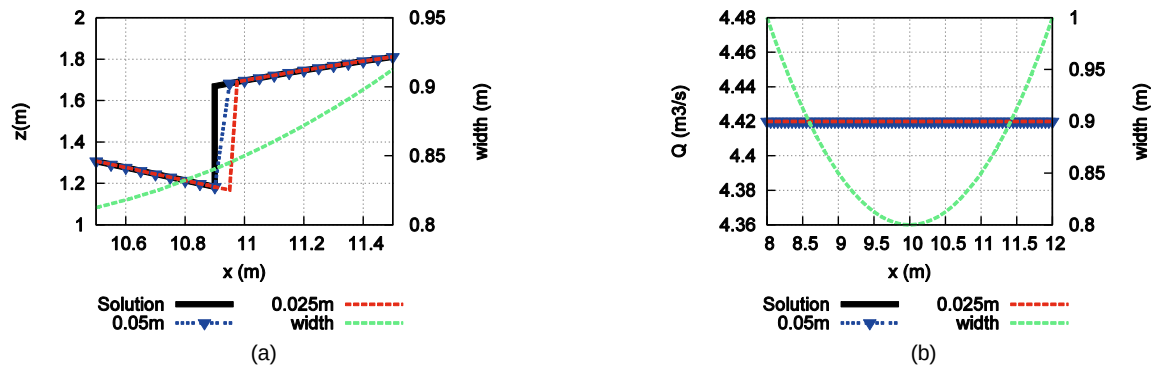


Figure 15. Transcritical case with a shock, results for (a) the water level, (b) discharge using Augmented Roe's scheme with energy balance and a critical interface using a threshold

Another solution is to impose the discharge conservation at the jump using the weight coefficient A_e , it is now written as

$$A_e = \frac{\delta \left(\frac{Q^2}{A} \right) + g \tilde{A} \delta(z_w)}{S_{gb} - S_{ga}} \quad [34]$$

Figure 16 shows the result of this approach which also considers a threshold. The results are similar than the previous ones (Figure 15). However, this solution has the advantage that it does not change the Riemann problem. This solution is thus the most promising one, but a key issue still has to be solved as the jump position appears now better predicted with a coarser mesh (0.050 m) than with a finer mesh (0.025 m).

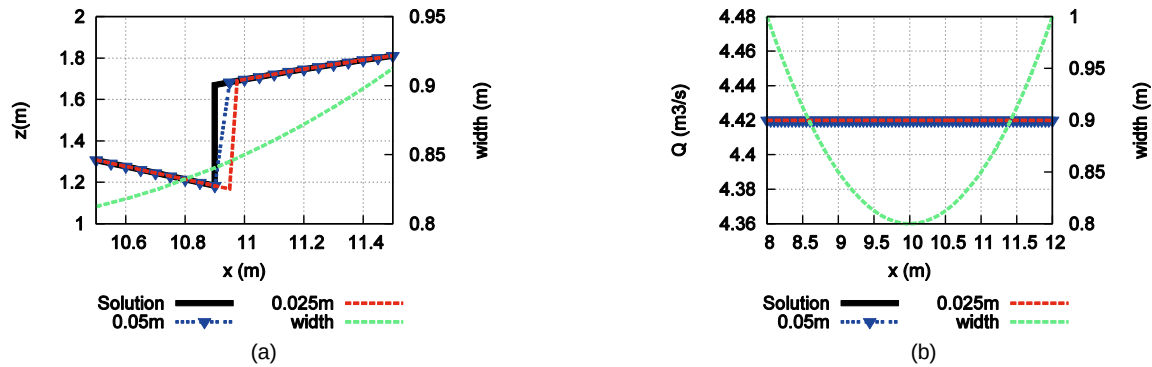


Figure 16 Transcritical case with a shock, results for (a) the water level, (b) discharge using Augmented Roe's scheme with energy balance and conservation of discharge at the jump using a threshold

5. Discussion

The results of the three test cases are summarized in Table 1. The HLLS is clearly an improvement of the LHLL scheme. The Augmented Roe's scheme with energy balance gives the best results. However, its limitation is the prediction of the jump position and, also, of the discharge at the jump. A study was made about this limitation in order to highlight the impact of different modification on the scheme. However, none gives accurate results. Nonetheless, it has been shown that the wrong position of the jump is generated by the energy balance property and, moreover, by the determination of the weight coefficient at the jump. A deeper study on this weight coefficient could possibly solve the problem.

Table 1. Summary of the capabilities of each scheme

	REST	LARGE DX	DISCHARGE	WATER LEVEL	ENERGY	JUMP POSITION	JUMP DISCHARGE	Fr=1
LHLL	OK			OK		OK		
HLLS	OK		OK	OK		OK		
AROE+E	OK	OK	OK	OK	OK			OK

6. CONCLUSIONS

Three schemes have been presented and compared in this paper: the LHLL, HLLS and Augmented Roe's solver with energy balance.

The LHLL showed good results for the water level with a fine mesh however it is unable to conserve the discharge. The main advantage of this scheme is the simplicity of the determination of the wave celerities. It can thus be used when the water level prediction is the more important and an accurate prediction of the discharge is not. In other words, it should not be used for sediment transport as the transport capacity is a function of the discharge, and would thus be poorly predicted.

The HLLS scheme was extended to arbitrary topography. It revealed to be an improvement of the LHLL scheme as it can conserve the discharge. However, its incapacity to give the exact solution makes it less precise than the Augmented Roe's scheme with energy balance.

The Augmented Roe's solver with energy balance provides accurate results for both the water level and the discharge. It is also able to give accurate results for the Froude number. This, combined with the energy balance property, makes it less dependent on the grid refinement. Nevertheless, one limitation of this model was highlighted: the prediction of the jump position and discharge for the contraction/expansion case. A study of this issue showed that the position of the jump can be corrected if the energy balance is neglected. However, removing this property makes the scheme depend on the grid refinement. Moreover, several changes were made to the Augmented Roe's solver to present some possible improvement of the model.

A scheme used to represent sediment transport in rivers should, in hydrodynamics, give accurate prediction of the water level and the discharge. As it is extremely expensive to obtain the data on the field for a river, the scheme should provide accurate results even using a coarse mesh. The only scheme which shows these characteristics is the Augmented Roe's solver with energy balance. Nonetheless, the limitation of the jump position and discharge should be deeply investigated. Moreover, tests should be conducted in order to assess the behavior of the scheme in highly complex topography such as a real river. Finally, this approach should be extended to take into account the sediment transport.

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