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Essays on Growth and Sustainability: Discounting, Habits and Externalities

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Introduction

Most models in macroeconomic and growth theory rely on a set of standard assumptions concerning intertemporal preferences. Two fundamental ones are the assumption of a constant utility discount rate and a utility function separable across time and generations. Even though, these are common assumptions in growth models, they are also particularly restrictive. In the present work, our objective is to study three theoretical models where intertemporal preferences play a larger role than in standard frameworks. We study two distinct ways to extend intertemporal preferences. The first one is through the endogenization of the utility discount rate while the other is through the inclusion of habits in the utility function. Concerning endogenous discounting, two strands of the literature can be identified. The first one suggests that the discount rate should depend on some variable under the control of the agent such as individual capital or consumption. We can for example imagine that richer individuals (owning an important capital stock) can afford to be more patient than poorer ones or that high consumption levels induce economic agents to further increase consumption suggesting a kind of addiction mechanism. A different approach suggests that social variables that are not under the control of the agent might influence the way in which economic agents value future outcomes. An intuitive way to introduce social variables is through external effects such as average consumption or average capital holdings. In this case, the surrounding environment plays an important role on the way economic agents perceive future events. Moreover, the introduction of external effects opens the door to the possibility of welfare improvements through public policy. Concerning the introduction of habits in the utility function, these can be introduced under different forms. Habits can be formed with respect to past individual behavior or with respect to the surrounding environment. If we focus on habits in consumption, agents will evaluate consumption against a frame of reference, which can be their own past consumption, the average consumption of the economy or the consumption level of past generations. In the present work, we will in turn explore some of these possibilities by introducing extended intertemporal preferences in different growth models.

In Chapter 1, we focus on an extension of the neoclassical growth model where the discount rate is endogenous and decreasing in capital accumulation while the production function is compatible with perfect competition, decreasing interest rates and endogenous growth. After showing that such a model should always restrict the utility function to the positive domain, we derive appropriate sufficiency conditions

ensuring the optimality of our dynamic path. We then choose our functional forms concerning the discount and the production function in accordance before proceeding with the dynamic analysis. The model is able to generate multiple equilibria with the existence of two steady states and one asymptotic balanced growth path. The first steady state is saddle-path stable and thus equivalent to a poverty trap while the second is unstable. In the long run, economies will converge toward the poverty trap or the asymptotic balanced growth path. This is consistent with the empirical evidence concerning the twin-peaks of economic growth. Concerning economies converging toward the asymptotic balanced growth path, we also obtain a positive correlation between increasing savings and growth rates as documented empirically. Finally, in this simple framework, it is possible that sufficiently large productivity shocks allow a given economy to escape the poverty trap thus suggesting the possibility of growth miracles and disasters. The opposite effects that decreasing interest and discount rates have on capital accumulation explain most of the results in this work.

In Chapter 2, we study an overlapping generation model with habits in consumption and the presence of environmental quality in the utility function. The modeling of habits implies that the representative adult compares present consumption with the one of his childhood (assumed to be equivalent to the one of his parents). Our main objective is to study the sustainability of such an economy where intergenerational externalities are present. Following the literature, we consider that sustainable growth paths should ensure non-decreasing utility or at least bounded below utility. We first notice that the competitive steady-state capital stock can be higher or lower compared to the standard Diamond framework. Concerning transitional dynamics, the presence of habits and environmental quality can possibly generate oscillatory behavior implying the breakdown of accepted sustainability criteria. The study of the optimal outcome shows that the latter can still be characterized by short run fluctuations even in the no-discounting case suggesting that we can solve for the environmental externalities but not the sustainability problem if we stick to a standard welfare function without any sustainability constraint. Concerning the optimal policy, given the direct link between capital and environmental quality, it might be necessary to tax investment. This goes against the policy adopted when environmental quality is not present and where investment should always be subsidized.

In Chapter 3, we focus once again on an extension of the neoclassical growth model. As before, the discount rate is decreasing in capital accumulation but we also introduce external effects driven by average consumption. In this case, the latter can be seen as a proxy for an economy's living standards and affect the discount rate as well as the production function. In the first case, following empirical evidence showing that individuals tend to compare their income or consumption levels with the one of their surrounding environment; we assume that the discount rate is actually increasing in average consumption. In the second case, we suppose that individual productivities can be positively affected by standards of living (for example

through the access to better health and nutrition standards) implying that consumption might enhance capital accumulation contrary to the standard framework. The model is able to generate multiple equilibria under fairly general conditions concerning the discount and the production function. On the dynamic side, local indeterminacy implying the existence of stationary sunspot equilibria is only possible when discounting externalities are present. In order to obtain a unique steady state, which is also indeterminate, it is necessary to combine sufficiently large production and discounting externalities. Given the presence of consumption in the production function, under some restrictions concerning the discount, the utility and the production function, unbounded growth is also a possible outcome of the model. In that case, we obtain a unique balanced growth path, which is always indeterminate. Finally, concerning the optimal outcome, while saddle-path stability prevails on the dynamic side, the magnitude of both kind of externalities is crucial in order to determine if consumption should be taxed or subsidized.

Chapter 1

Endogenous discounting and economic development

1.1 Introduction

The debates about discounting have a long history in economic growth theory even though standard models usually assume a positive and constant utility discount rate. This assumption is mostly used for analytical convenience rather than based on strong economic intuition. Authors such as Ramsey (1928) and Harrod (1948) were already against this practice and in favor of a zero rate of discount based on ethical arguments. More recently, several economists have started to focus on the possibility of an endogenous discount rate. Different approaches can be identified in this literature. A first one relies on so called social variables which are not under the control of the representative agent. This is the case in works such as Shi (1999) where the discount rate depends on average consumption, in Schmitt-Grohe and Uribe (2003) where average utility impacts the discount rate or in Meng (2006) where the discount rate is a function of both average consumption and income. A second approach mixes these social variables with private ones under the control of the agent. Drugeon (1998) combines individual and average consumption, Drugeon (1996a) and Palivos and al (1997) combine individual consumption with average capital holdings while Yanase (2011) combines individual consumption with total pollution. Finally, a last approach which will be the one followed in this paper relies uniquely on individual variables under the control of the agent. We can cite as examples the work of Ayong Le Kama and Schubert (2007) where the discount rate depends on environmental variables, the ones of Uzawa (1968), Obstfeld (1990), Drugeon (1996b) and Das (2003) where the focus is on individual consumption or the work of Schumacher (2009, 2011) and Strulik (2012) where the discount rate is a function of the individual capital stock.

The objective of the present paper is to introduce endogenous discounting in a standard model of economic growth in order to shed light on some important facts concerning the growth process. A very similar attempt has been made by Strulik (2012) who develops a model with a discount rate decreasing in capital and a produc-

tion function delivering both a decreasing interest rate and endogenous growth. The main result of the paper is the ability of the model to generate a positive correlation between increasing saving and growth rates as observed empirically. However, in his numerical simulations, Strulik restricts the utility function to take negative values. As explained in Schumacher (2011), in endogenous discounting models, the sign of the utility function matters and can reverse the results. In particular, as we will see, with a discount rate decreasing in capital, a negative utility function implies that intertemporal welfare is actually decreasing in capital which goes against economic intuition. In the present paper, we postulate a utility function that can only take positive values. This leads us to the breakdown of the standard Mangasarian second order conditions which require a negative utility function. We thus derive appropriate second order conditions and choose our functional forms in accordance. This allows us to build a model in line with three fundamental observations concerning the growth process:

- 1) The joint observation that there exist a positive correlation between increasing saving and growth rates and at the same time a decrease in the interest rate is in opposition with the standard Ramsey model.
- 2) Most empirical observations have confirmed that the discount rate is not constant but actually decreasing in wealth accumulation.
- 3) The world distribution of income has been changing from an approximately normal distribution toward a bimodal one with both growth miracles and disasters.

In this paper, we show that a discount rate decreasing in capital coupled with a standard endogenous growth model might reconcile these stylized facts with theory. The choice of capital as the discount rate variable is mostly driven by empirical evidence. Capital can directly be related to wealth accumulation which in turn is positively correlated with lower mortality rates and thus lower discount rates (Fielding and al. 2009). Moreover, authors such as Lawrence (1991) and Samwick (1998) have found that the discount rate is decreasing in capital and wealth in panel data analysis.

Most of the endogenous discounting literature of this type focuses on exogenous growth through the use of a neoclassical production function (notable exceptions are the works of Drugeon 1996b and Strulik 2012). This is a clear impediment to reproduce a positive correlation between increasing saving and growth rates since convergence toward steady-state equilibrium is in opposition with the possibility of increasing growth rates. Our objective is thus to combine this endogenous discounting framework with a production function ensuring perfect competition and delivering endogenous growth. We prefer to maintain the perfect competition assumption in order to highlight the main effect due to endogenous discounting. In order to do so, as in Strulik (2012), we choose to refer to the work of Jones and Manuelli (1990) who propose a production function combining an AK and a neoclassical part. This production function will allow us to benefit from both the convergence effect of the neoclassical production function as well as from the long run properties of the AK model. The marginal productivity of capital being decreasing in this case, we are able to study the opposite effects of capital accumulation on the discount and the

interest rate.

A brief account of the results is as follows. We first argue that the utility function should only take positive values and derive appropriate sufficiency conditions and specific functional forms in accordance with this assumption. We then derive a necessary and sufficient condition for the existence of an asymptotic balanced growth path equilibrium involving restrictions on the elasticity of marginal utility. The possible existence of two steady-state equilibria of stagnation is then studied. The one with the lowest capital level being locally stable and thus equivalent to a poverty trap while the second being unstable defines the frontier between convergence toward the poverty trap or the asymptotic balanced growth path. The positive correlation between increasing growth and saving rates is confirmed for an economy converging toward the asymptotic balanced growth path. Concerning growth miracles and disasters, these are made possible through exogenous shocks on the productivity parameter. We use this mechanism more as an example than an actual way for a country to escape a poverty trap. The main driving force behind the results seems to be due to the fact that the discount rate effect always dominates the interest rate one as capital accumulation proceeds.

The structure of the paper is the following: section 2 introduces the model and derives appropriate sufficient conditions in the case of a utility function taking only positive values. The different equilibria of this economy are studied in section 3 while section 4 focuses on transitional dynamics. Section 5 confronts the model to fundamental stylized facts concerning the growth process and section 6 finally concludes.

1.2 The model

The model is based on a representative infinitely-lived agent who maximizes consumption subject to capital accumulation. The latter is made possible through investment and reduced by consumption and constant depreciation of capital. The discount rate is endogenous and depends on the historical path of capital per capita. Furthermore, population is constant and normalized to one. The intertemporal discounted utility function is given by:

$$U(c(t), \{k(t)\}_0^t) = \int_0^\infty u(c(t))e^{-\theta t} dt \quad (1.1)$$

The discount rate $\theta(t) > 0$ depends on past and current levels of capital per capita in the following way:

$$\theta(t) = \int_0^t \rho(k(s)) ds \quad (1.2)$$

Assumption 1:

The utility function is twice continuously differentiable and has the following properties: $u'(c) > 0$, $u''(c) < 0$ and $\lim_{c \rightarrow 0} u'(c) = \infty$. The last property implies that a positive amount of consumption is needed at the optimum. Mathematically, it

guarantees an interior solution. In the forthcoming analysis, we will use a constant intertemporal elasticity of substitution (CIES) utility function which will take the following functional form:

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma} \quad (1.3)$$

with $\sigma \geq 0$ representing the inverse of the intertemporal elasticity of substitution. The discount function is twice continuously differentiable and has the following properties: $\rho'(k) < 0$ and $\rho''(k) > 0$. The discount rate is thus decreasing in capital accumulation implying that richer individuals are more patient than poorer ones. For the moment, we don't specify a particular function concerning the discount rate.

Before proceeding, we should focus on the importance of the sign of the utility function in endogenous discounting models. In the present case, preferences being recursive, a change in the capital stock will not only have an impact on present utility but also on the way the representative agent perceives future utility levels. In order to compute the marginal utilities, we will rely on the Volterra derivatives with respect to consumption and capital. The Volterra derivative gives the rate at which intertemporal utility changes with respect to a small increase in consumption or capital near a given time t . In the present case, we obtain:

$$U_{tc} = u'(c(t))e^{-\int_0^t \rho(k(s))ds} \quad (1.4)$$

$$U_{tk} = -\rho'(k(s))e^{-\int_0^t \rho(k(s))ds} \int_t^\infty u(c(s))e^{-\int_t^s \rho(k(\tau))d\tau} ds \quad (1.5)$$

It can be observed that the Volterra derivative with respect to consumption is always positive while the one with respect to capital is negative if $u(c) < 0$ and positive if $u(c) > 0$. This suggests that in the case of negative utility a higher capital level decreases intertemporal welfare. Subsequently, a higher discount rate can be seen as something good in the sense that it increases total welfare. The lack of economic intuition given by these results suggests that in the present framework, we should use a utility function that can only take positive values as stated in Schumacher (2011). Note that this is not a more general statement and that the choice of the sign of the utility function should always be based on the respective Volterra derivatives of each model.

In a similar framework, Strulik (2012) does not seem to take into account this observation and proceeds with a utility function taking negative values by assuming a CIES utility function with an elasticity of marginal utility equal to 2 for his numerical simulations.

If we choose to proceed with a utility function that only takes positive values, a different kind of problem arises. In the case of negative utility, the Mangasarian sufficiency conditions are equivalent to $u(c) < 0$ so that they are always satisfied. However, in the case of positive utility, this is not anymore the case and specific sufficiency conditions should be derived. We thus proceed with our CIES utility

function where $\sigma < 1$ and derive the appropriate sufficiency conditions. Our optimization problem can be written in the following way:

$$\max_{c(t), k(t)} \int_0^\infty u(c(t)) e^{-\int_0^t \rho(k(s)) ds} dt \quad (1.6)$$

subject to

$$\begin{cases} \dot{k}(t) = f(k(t)) - c(t) - \delta k(t) & \forall t \\ k(t) \geq 0, c(t) \geq 0 & \forall t \\ \text{with } k(0) \text{ given} \end{cases}$$

As $\frac{d\theta}{dt} = \rho(k(t))$, we can write $dt = \frac{d\theta}{\rho(k(t))}$ and obtain the following Hamiltonian ¹:

$$\mathcal{H}_\Delta = \frac{c^{1-\sigma} e^{-\theta}}{(1-\sigma)\rho(k)} + \lambda \left(\frac{f(k) - c - \delta k}{\rho(k)} \right) \quad (1.7)$$

The first order condition for the control variable is given by:

$$c^{-\sigma} e^{-\theta} = \lambda \quad (1.8)$$

Let's rewrite $c = \left(\frac{e^{-\theta}}{\lambda}\right)^{1/\sigma}$ and substitute this in the Hamiltonian in order to obtain the Hamiltonian along the optimal path. Now let's take second-order conditions with respect to k to see if the Hamiltonian is indeed concave along the optimal path. The first-order condition is given by

$$\begin{aligned} \frac{\partial \mathcal{H}_\Delta}{\partial k} &= \lambda \left(\frac{f'(k) - \delta}{\rho(k)} \right) \\ &\quad - \left\{ \frac{c^{1-\sigma}}{1-\sigma} e^{-\theta} + \lambda [f(k) - \left(\frac{e^{-\theta}}{\lambda}\right)^{1/\sigma} - \delta k] \right\} \frac{\rho'(k)}{\rho(k)^2} \end{aligned} \quad (1.9)$$

while the second-order condition is given by

$$\begin{aligned} \frac{\partial^2 \mathcal{H}_\Delta}{\partial k^2} &= -\mathcal{H} \left(\frac{\rho''(k)}{\rho(k)} - 2 \frac{\rho'(k)^2}{\rho(k)^2} \right) + \lambda \frac{f''(k)}{\rho(k)} \\ &\quad - 2\rho'(k)\lambda \left(\frac{f'(k) - \delta}{\rho(k)^2} \right) \end{aligned} \quad (1.10)$$

Since $\mathcal{H}_\Delta > 0$, the Hamiltonian is concave if the following two conditions are respected:

$$\begin{aligned} \rho''(k) &> 2 \frac{\rho'(k)^2}{\rho(k)} \\ f''(k) &< 2\rho'(k) \frac{f'(k) - \delta}{\rho(k)} \end{aligned}$$

¹From now on, we drop the time dependency for convenience.

In order to see the implications of these two conditions we apply them to the functional forms chosen by Strulik (2012). At first Strulik proposes an AK production function such that $f(k) = Ak$ coupled with the following discount function $\rho(k) = \bar{\rho} + \rho_0 k^{-\eta}$. In this case, the first sufficiency condition is equivalent to:

$$\eta(\eta + 1)\rho_0 k^{-\eta-2} > 2 \frac{(\eta\rho_0)^2 k^{-2\eta-2}}{\bar{\rho} + \rho_0 k^{-\eta}} \quad (1.11)$$

$$k > \left[\frac{(\eta - 1)\rho_0}{(\eta + 1)\bar{\rho}} \right]^{\frac{1}{\eta}} \quad (1.12)$$

such that if $\eta < 1$ this condition is always satisfied. This is indeed the case in Strulik's paper since for his numerical simulations η is set to 0.33 or 0.35.

The second condition is equivalent to:

$$2\eta\rho_0 k^{-\eta-1}(A - \delta) < 0 \quad (1.13)$$

which is never satisfied given the value of the parameters. The second condition in fact requires a sufficient degree of concavity concerning the production function such that we cannot adopt an AK production function which is linear in capital and still be sure to respect the second-order conditions in the case of positive utility.

Strulik then studies what he calls the full model where the production function becomes $f(k) = Ak + Bk^\alpha$ and the discount function is still the same. This production function is drawn from Jones and Manuelli (1990) and delivers both a decreasing interest rate and endogenous growth. The first condition is not affected and the second becomes:

$$\bar{\rho} + \rho_0 k^{-\eta} > \frac{2\eta\rho_0 k^{-\eta-1}}{B\alpha(1-\alpha)k^{\alpha-2}}(A + B\alpha k^{\alpha-1} - \delta) \quad (1.14)$$

$$\frac{(1 - \alpha - 2\eta)\rho_0}{k^\eta} > \frac{(A - \delta)2\eta\rho_0 k^{1-\eta-\alpha}}{B\alpha} - (1 - \alpha)\bar{\rho} \quad (1.15)$$

Given that $1 - \alpha - \eta > 0$, since α is set around 0.33 and η around 0.33 or 0.35 in Strulik's numerical simulations, for large values of k , this condition will not be fulfilled and we cannot ensure that the second-order conditions are indeed satisfied if we consider a utility function taking only positive values. It thus seems necessary to modify the production function or the discount function. In the present case, we will focus on the discount function since our objective is to keep a relatively simple production function exhibiting a decreasing interest rate and delivering endogenous growth.

We propose a discount function of the following form $\rho(k) = \bar{\rho} + \rho_0 e^{-\beta k^\gamma}$. In this case the sufficiency conditions are given by:

$$(\rho_0 + \bar{\rho}e^{\beta k^\gamma}) \left(1 + \frac{1 - \gamma}{\gamma\beta k^\gamma} \right) > 2\rho_0 \quad (1.16)$$

$$\frac{B\alpha[(1 - \alpha)(\rho_0 + \bar{\rho}e^{\beta k^\gamma}) - 2\beta\gamma\rho_0 k^\gamma]}{k^{1+\gamma-\alpha}} > 2\beta\gamma\rho_0(A - \delta) \quad (1.17)$$

Concerning the first condition, we can see that the limit of the left hand side of expression (1.16) when $k \rightarrow 0$ is ∞ if $\gamma < 1$ while when $k \rightarrow \infty$, the limit is ∞ . We thus choose to impose $\gamma < 1$ in our discount function such that the condition is satisfied for low and high values of the capital stock. However, we wish to ensure that the condition is satisfied for all values of the capital stock. A sufficient condition for this requirement to be satisfied is that the ratio $(1 - \gamma)\bar{\rho}e^{\beta k^\gamma} / \gamma\beta k^\gamma$ is increasing in k . This is indeed the case if

$$k > \left(\frac{1}{\beta}\right)^{\frac{1}{\gamma}} \quad (1.18)$$

such that we will restrict the forthcoming analysis to values of k which respect this constraint. This implies that β should be large enough in order to avoid large restrictions on the capital stock. We thus impose that $\beta > 1$.

Concerning expression (1.17), the limit of the left hand side is ∞ both when $k \rightarrow 0$ and when $k \rightarrow \infty$. The latter result can be obtained by applying several times l'Hospital's rule. We can also see that the numerator of expression (1.17) is actually increasing in k such that the second condition will be more easily satisfied for larger values of the capital stock. These results allows us to use the discount function suggested above so that we can proceed with the analysis in this case.

Assumption 2:

As stated before, the production function is the same as in the full model of Strulik (2012) and drawn from Jones and Manuelli (1990). It takes the following form:

$$F(K, L) = AK + BK^\alpha L^{1-\alpha}$$

with an AK as well as a Cobb-Douglas part. In intensive form, $f(k) = Ak + Bk^\alpha$ and this function follows all the standard assumptions on concavity, such that $f(0) = 0$, $f'(k) > 0$, $f''(k) < 0$.

The production function is not neoclassical only because it violates one of the Inada conditions: $\lim_{k \rightarrow \infty} f'(k) = A > 0$, in the long run, the marginal productivity of capital is bounded away from zero. Moreover this production function still exhibits a decreasing interest rate.

The discount function is given by

$$\rho(k) = \bar{\rho} + \rho_0 e^{-\beta k^\gamma}$$

with $\gamma < 1$ and $\bar{\rho} \geq 0$. Concerning the minimal discount rate $\bar{\rho}$, both a zero or a positive value can be considered. We can imagine that very rich agents can afford not to discount future utilities as well as that individuals will always discount the future because of reasons such as the difficulty to value future enjoyments (see for example Becker and Mulligan 1997), the fact that there is always a relative preference for the present or the possibility of death or extinction (Stern 2007).

We can now proceed with the analysis of the model. Given that we want to compare

the implications of our model with stylized facts, we recast the problem as an equilibrium one. Individuals supply one unit of labor and receive a wage w and capital income net of depreciation $(r - \delta)k$ where r is the interest rate. Income is spent on consumption and investment such that we obtain the following budget constraint:

$$\dot{k}_t = (r - \delta)k + w - c - \delta k \quad (1.19)$$

Given our production function, the interest rate and the wage are given by $r = A + B\alpha k^{\alpha-1}$ and $w = (1 - \alpha)Bk^\alpha$. Moreover by taking the derivative of equation (1.2) with respect to time we obtain:

$$\dot{\theta} = \bar{\rho} + \rho_0 e^{-\beta k^\gamma} \quad (1.20)$$

which can be used as a second state variable concerning the evolution of the discount rate. The representative agent then solves the following problem:

$$\max_{c,k} \int_0^\infty u(c) e^{-\theta} dt \quad (1.21)$$

subject to

$$\begin{cases} \dot{k} = (r - \delta)k + w - c & \forall t \\ \dot{\theta} = \bar{\rho} + \rho_0 e^{-\beta k^\gamma} & \forall t \\ k(0) > \left(\frac{1}{\beta}\right)^{\frac{1}{\gamma}}, c \geq 0 & \forall t \\ \text{with } k(0) \text{ given} \end{cases}$$

We write the present value Hamiltonian of the above system:

$$\mathcal{H} = \frac{c^{1-\sigma}}{1-\sigma} e^{-\theta} + \lambda[(r - \delta)k + w - c] - \mu(\bar{\rho} + \rho_0 e^{-\beta k^\gamma}) \quad (1.22)$$

The first order necessary conditions for optimality are:

$$c^{-\sigma} e^{-\theta} = \lambda \quad (1.23)$$

$$\lambda(r - \delta)k + \beta\mu\gamma k^{\gamma-1} \rho_0 e^{-\beta k^\gamma} = -\dot{\lambda} \quad (1.24)$$

$$-\frac{c^{1-\sigma}}{1-\sigma} e^{-\theta} = \dot{\mu} \quad (1.25)$$

$$\lim_{t \rightarrow \infty} \mathcal{H}(t) = 0 \quad (1.26)$$

As explained in Michel (1982), the transversality condition is modified in infinite horizon problems and the appropriate one is given by expression (1.26).

By solving the model and using the equilibrium values for r and w , we obtain the following dynamical system:

$$\frac{\dot{c}}{c} = \frac{A + B\alpha k^{\alpha-1} - \delta - \bar{\rho} - \rho_0 e^{-\beta k^\gamma}}{\sigma} + \frac{\mu\gamma\beta c^\sigma k^{\gamma-1} \rho_0 e^{-\beta k^\gamma}}{\sigma e^{-\theta}} \quad (1.27)$$

$$\dot{k} = Ak + Bk^\alpha - c - \delta k \quad (1.28)$$

$$\dot{\mu} = -\frac{c^{1-\sigma}}{1-\sigma} e^{-\theta} \quad (1.29)$$

Lemma 1:

The previous dynamical system can be reduced to the following planar system:

$$\frac{\dot{c}}{c} = \frac{A + B\alpha k^{\alpha-1} - \delta - \bar{\rho} - \rho_0 e^{-\beta k^\gamma}}{\sigma} + \frac{[(1-\sigma)(Ak + Bk^\alpha - \delta k) + \sigma c]\gamma\beta k^{\gamma-1}\rho_0 e^{-\beta k^\gamma}}{\sigma(1-\sigma)(\bar{\rho} + \rho_0 e^{-\beta k^\gamma})} \quad (1.30)$$

$$\frac{\dot{k}}{k} = A + Bk^{\alpha-1} - \frac{c}{k} - \delta \quad (1.31)$$

Proof. First note that the Hamiltonian is autonomous. Differentiating the Hamiltonian with respect to time we obtain:

$$\frac{d\mathcal{H}}{dt} = \frac{\partial\mathcal{H}}{\partial t} + \frac{\partial\mathcal{H}}{\partial c}\dot{c} + \frac{\partial\mathcal{H}}{\partial k}\dot{k} + \frac{\partial\mathcal{H}}{\partial\theta}\dot{\theta} + \frac{\partial\mathcal{H}}{\partial\lambda}\dot{\lambda} - \frac{\partial\mathcal{H}}{\partial\mu}\dot{\mu} \quad (1.32)$$

Using the first order conditions for optimality:

$$\frac{d\mathcal{H}}{dt} = \frac{\partial\mathcal{H}}{\partial t} - \dot{\lambda}\dot{k} + \dot{k}\dot{\lambda} + \dot{\mu}\dot{\theta} - \dot{\theta}\dot{\mu} \quad (1.33)$$

$$\frac{d\mathcal{H}}{dt} = \frac{\partial\mathcal{H}}{\partial t} \quad (1.34)$$

Since the Hamiltonian is autonomous, $\frac{\partial\mathcal{H}}{\partial t} = 0$. Combining this result with the transversality condition, $\lim_{t \rightarrow \infty} \mathcal{H}(t) = 0$, implies that the Hamiltonian takes the value zero along the optimal trajectory.

We now have a solution for μ by transforming the Hamiltonian:

$$\mu = \frac{c^{-\sigma} e^{-\theta} [c + (1-\sigma)\dot{k}]}{(1-\sigma)(\bar{\rho} + \rho_0 e^{-\beta k^\gamma})} \quad (1.35)$$

Replacing μ by its value in the differential equation for consumption and dividing the capital equation by k yields the desired result. $\square$

Equations (1.30) and (1.31) capture the dynamics of our economy. The capital accumulation equation is standard while the consumption equation can be separated into two terms. The first one is similar to a standard neoclassical accumulation equation where the depreciation and the discount rates are subtracted from the marginal productivity of capital. The second is a positive term taking into account the effect of a decreasing discount rate on the growth rate of consumption through an expression involving the shadow value of the discount rate.

1.3 Equilibria

We have two kind of equilibria in this economy. The first one is the balanced growth path equilibrium (BGP) where consumption, output and capital grow without bound at a common positive constant rate while the other is the steady-state equilibrium

where the growth rates of consumption, output and capital are equal to zero. We first focus on the BGP equilibrium. As it has been proved in Palivos, Wang and Zhang (1997), a necessary and sufficient condition for the existence of a BGP is that the elasticity of marginal utility and the discount rate be constant along this path. This is not the case in our model since the discount rate is endogenous. Nevertheless, we can apply the concept of an asymptotically balanced growth path (ABGP) to our problem.

Definition 1:

An ABGP equilibrium of this economy is a solution (c_t, k_t) to equations (1.30) and (1.31) given $k(0)$, such that $\lim_{c \rightarrow \infty} g_c = \lim_{k \rightarrow \infty} g_k = \lim_{k \rightarrow \infty} g_f > 0$, where g represents the constant growth rate of the respective endogenous variables.

In order to draw some conclusions from the ABGP, the usual method is to operate on variables which are constant along the ABGP. From equation (1.31), we know that a constant growth rate for capital obtains if $z = \frac{f(k)}{k} = A + Bk^{\alpha-1}$ and $x = \frac{c}{k}$ are constant. x is the consumption to capital ratio while z is the output to capital ratio. Equation (1.31) also implies the equality of the limiting growth rates given in the definition of the ABGP.

We first determine the differential equation for z :

$$z = A + Bk^{\alpha-1} \quad (1.36)$$

$$\dot{z} = (\alpha - 1)Bk^{\alpha-1} \frac{\dot{k}}{k} \quad (1.37)$$

$$\dot{z} = (\alpha - 1)(z - A)(z - x - \delta) \quad (1.38)$$

with $z(0)$ given.

We now determine the differential equation for x :

$$\frac{\dot{x}}{x} = \frac{\dot{c}}{c} - \frac{\dot{k}}{k} \quad (1.39)$$

$$\begin{aligned} \frac{\dot{x}}{x} = & \frac{1 - \sigma}{\sigma}(A - \delta) + (z - A) \frac{\alpha - \sigma}{\sigma} + x - \frac{\bar{\rho} + \rho_0 e^{-\beta k^\gamma}}{\sigma} \\ & + \frac{[(1 - \sigma)(z - \delta) + \sigma x] \gamma \beta k^\gamma \rho_0 e^{-\beta k^\gamma}}{\sigma(1 - \sigma)(\bar{\rho} + \rho_0 e^{-\beta k^\gamma})} \end{aligned} \quad (1.40)$$

Proposition 1:

An ABGP such that $\lim_{k \rightarrow \infty} \dot{z} = \lim_{k \rightarrow \infty} \dot{x} = 0$ exists if $\bar{\rho} > 0$ and if and only if $1 > \sigma > 1 - \frac{\bar{\rho}}{A - \delta}$. Furthermore, this equilibrium is saddle-path stable.

Proof. Concerning z , we can see from equation (1.36) that $z - A$ converges to zero as capital grows unboundedly thus $\lim_{k \rightarrow \infty} \dot{z} = 0$.

Concerning x , the limit of equation (1.40) is not defined since k^γ grows unboundedly

while $e^{-\beta k^\gamma}$ converges to zero. A slight modification of the numerator of the last term gives:

$$\lim_{k \rightarrow \infty} [(1 - \sigma)(z - \delta) + \sigma x] \gamma \beta k^\gamma \rho_0 e^{-\beta k} = \lim_{k \rightarrow \infty} F \frac{k^\gamma}{e^{\beta k^\gamma}} \quad (1.41)$$

where F is a constant. Making use of l'Hospital's rule:

$$\lim_{k \rightarrow \infty} \frac{k^\gamma}{e^{\beta k^\gamma}} = \lim_{k \rightarrow \infty} \frac{1}{\beta e^{\beta k^\gamma}} = 0 \quad (1.42)$$

$$\lim_{k \rightarrow \infty} F \frac{k^\gamma}{e^{\beta k^\gamma}} = 0 \quad (1.43)$$

Since the limit of the denominator of the last term in expression (1.40) is a constant equal to $\sigma(1 - \sigma)\bar{\rho}$, we can conclude that $\lim_{k \rightarrow \infty} \dot{x}$ exists and is finite.

From equations (1.38) and (1.40), we obtain:

$$\lim_{k \rightarrow \infty} \dot{z} = \lim_{k \rightarrow \infty} (\alpha - 1)(z - A)(z - x - \delta) \quad (1.44)$$

$$\lim_{k \rightarrow \infty} \dot{x} = \left\{ \frac{1 - \sigma}{\sigma}(A - \delta) + \lim_{k \rightarrow \infty} [(z - A) \frac{\alpha - \sigma}{\sigma} + x] - \frac{\bar{\rho}}{\sigma} \right\} \lim_{k \rightarrow \infty} x \quad (1.45)$$

Setting the two previous equations to zero such that the limiting growth rates of consumption, capital and output are equal, we obtain:

$$\lim_{k \rightarrow \infty} x = \frac{\bar{\rho} + (\sigma - 1)(A - \delta)}{\sigma} \quad (1.46)$$

$$\lim_{k \rightarrow \infty} z = A \quad (1.47)$$

Expression (1.46) must be positive for an ABGP to exist. This condition is equivalent to:

$$1 > \sigma > 1 - \frac{\bar{\rho}}{A - \delta} \quad (1.48)$$

The elasticity of marginal utility is thus bounded above by one by assumption and below by the preceding condition. From this same condition, we can directly conclude that $\bar{\rho} > 0$ is a necessary condition for the existence of an ABGP since by assumption $\sigma < 1$.

Assuming that condition (1.48) is satisfied, we study the local stability of the system around the asymptotic steady-state. In order to realize this task, we compute the Jacobian matrix at $(\lim_{k \rightarrow \infty} x, \lim_{k \rightarrow \infty} z)$. As can be seen from equations (1.38) and (1.40), the Jacobian matrix will be continuous in x and z such that we are able to take first the limit and then the derivative in order to get rid of the terms depending on k in equation (1.40). We thus use equations (1.44) and (1.45) to compute the Jacobian matrix and obtain:

$$J = \begin{bmatrix} \frac{\bar{\rho} + (\sigma - 1)(A - \delta)}{\sigma} & \frac{(\alpha - \sigma)(\bar{\rho} + (\sigma - 1)(A - \delta))}{\sigma^2} \\ 0 & (\alpha - 1) \frac{A - \delta - \bar{\rho}}{\sigma} \end{bmatrix} \quad (1.49)$$

The system is saddle-path stable if we have one positive and one negative eigenvalue. We denote λ_1 and λ_2 the two eigenvalues and we have to prove that the determinant is negative. In this case, the computation is straightforward since we have a triangular matrix:

$$\text{Det}(J) = \frac{\bar{\rho} + (\sigma - 1)(A - \delta)}{\sigma}(\alpha - 1)\frac{A - \delta - \bar{\rho}}{\sigma} \quad (1.50)$$

We can see from expression (1.46) that the existence of the ABGP implies that the first eigenvalue is positive. We can also impose that $A - \delta - \bar{\rho} > 0$ which is a standard assumption in the endogenous growth literature (this implies that the long term interest rate is higher than the sum of capital depreciation and the minimal discount rate) in order to ensure long-term growth. Moreover, the share of capital in the neoclassical part of the production function is lower than one such that the second eigenvalue is negative. The determinant is thus negative and the system is saddle-path stable. This completes the second part of the proof. $\square$

Before proceeding, the reader should notice that contrary to the case with a negative utility function used by Strulik (2012), the restriction needed to ensure the existence of a ABGP plays an important role by fixing boundaries for the elasticity of marginal utility and showing the need of a positive minimal discount rate. In the case, of negative utility, since $\sigma > 1$, the parameter restrictions do not play any role concerning the existence of an ABGP equilibrium.

From the previous results we can now compute the asymptotically growth rates of $c_t, k_t, f(k_t)$ and confirm that they are equal without any parameter restriction.

$$\lim_{k \rightarrow \infty} \frac{\dot{c}}{c} = \lim_{k \rightarrow \infty} \frac{A + B\alpha k^{\alpha-1} - \delta - \bar{\rho} - \rho_0 e^{-\beta k^\gamma}}{\sigma} \quad (1.51)$$

$$+ \lim_{k \rightarrow \infty} \frac{[c + (1 - \sigma)\dot{k}] \gamma \beta k^{\gamma-1} \rho_0 e^{-\beta k^\gamma}}{\sigma(1 - \sigma)(\bar{\rho} + \rho_0 e^{-\beta k^\gamma})}$$

$$\lim_{k \rightarrow \infty} \frac{\dot{c}}{c} = \frac{A - \delta - \bar{\rho}}{\sigma} \quad (1.52)$$

The change from equation (1.51) to (1.52) is a direct consequence of the proof of Proposition 1 where we see that the second term in equation (1.51) converges to zero.

$$\lim_{k \rightarrow \infty} \frac{\dot{k}}{k} = \lim_{t \rightarrow \infty} z - \lim_{t \rightarrow \infty} x - \delta \quad (1.53)$$

$$\lim_{k \rightarrow \infty} \frac{\dot{k}}{k} = \frac{A - \delta - \bar{\rho}}{\sigma} \quad (1.54)$$

The result is obtained by replacing expressions (1.46) and (1.47) in equation (1.53).

$$\frac{f(\dot{k})}{f(k)} = \frac{A + B\alpha k^{\alpha-1} \dot{k}}{A + Bk^{\alpha-1} k} \quad (1.55)$$

$$\lim_{k \rightarrow \infty} \frac{f(\dot{k})}{f(k)} = \lim_{k \rightarrow \infty} \frac{\dot{k}}{k} = \frac{A - \delta - \bar{\rho}}{\sigma} \quad (1.56)$$

The gross saving rate is also constant along the ABGP:

$$s = 1 - \frac{c}{y} = 1 - \frac{x}{z} \quad (1.57)$$

$$\lim_{k \rightarrow \infty} s = \frac{A - \bar{\rho} - (1 - \sigma)\delta}{\sigma A} \quad (1.58)$$

It is important to note that the conditions for the existence of an ABGP imply an upper bound on the growth and saving rates preventing explosive or unstable behavior.

We now proceed with some comparative statics on the constant variables x and s (we exclude z since it is equal to A). The saving rate is increasing in A while the converse is true for the consumption to capital ratio:

$$\frac{\partial}{\partial A} \lim_{k \rightarrow \infty} s = \frac{\bar{\rho} + (1 - \sigma)\delta}{\sigma A^2} > 0 \quad (1.59)$$

$$\frac{\partial}{\partial A} \lim_{k \rightarrow \infty} x = \frac{\sigma - 1}{\sigma} < 0 \quad (1.60)$$

The positive effect on the saving rate comes from the fact that after an increase in the productivity parameter, a share of this increase is devoted to savings. The decrease in the consumption to capital ratio is due to our assumption on positive utility ($\sigma < 1$). This reflects the opposite effects in case of positive and negative utility. In the latter case, the introduction of a discount rate decreasing in capital would create an incentive towards consumption over capital accumulation.

Concerning the minimal discount rate and the elasticity of marginal utility, they both increase the consumption to capital ratio and decrease the saving rate.

$$\frac{\partial}{\partial \sigma} \lim_{k \rightarrow \infty} x = \frac{1}{\sigma} \frac{A - \delta - \bar{\rho}}{\sigma} > 0 \quad (1.61)$$

$$\frac{\partial}{\partial \bar{\rho}} \lim_{k \rightarrow \infty} x = \frac{1}{\sigma} > 0 \quad (1.62)$$

$$\frac{\partial}{\partial \sigma} \lim_{k \rightarrow \infty} s = -\frac{A - \bar{\rho} - (1 - \sigma)\delta}{A\sigma^2} < 0 \quad (1.63)$$

$$\frac{\partial}{\partial \bar{\rho}} \lim_{k \rightarrow \infty} s = -\frac{1}{\sigma A} < 0 \quad (1.64)$$

A higher minimal discount rate implies a higher discount factor at each point in time thus reducing the incentives for accumulation. The decrease in the intertemporal elasticity of substitution works in a similar way. These two effects tend to deteriorate the saving rate and thus increase consumption as a share of capital. This concludes the study of the ABGP. We can now focus on the possible existence of steady-states equilibria.

Definition 2:

A steady-state equilibrium of this economy is a solution (c_t, k_t) to equations (1.30) and (1.31) given $k(0)$, such that $g_c = g_k = g_y = 0$.

Setting $\dot{c} = \dot{k} = 0$ in equations (1.30) and (1.31) we obtain:

$$c = \frac{(A + B\alpha k^{\alpha-1} - \delta - \bar{\rho} - \rho_0 e^{-\beta k^\gamma})(\sigma - 1)(\bar{\rho} + \rho_0 e^{-\beta k^\gamma})}{\gamma\beta k^{\gamma-1}\rho_0 e^{-\beta k^\gamma}} \quad (1.65)$$

$$c = Ak + Bk^\alpha - \delta k \quad (1.66)$$

This system of equations is particularly difficult to solve due to expression (1.65) such that we will rely on a phase diagram analysis.

Proposition 2:

In this economy, either there is no steady-state equilibrium either there are two of them (k_1 and k_2 with $k_1 < k_2$). In the latter case, the first one (k_1) characterized by a lower capital stock is saddle-path stable while the second one (k_2) is unstable. The first steady-state can thus be seen as a poverty trap. Moreover, if $\gamma > \alpha$, $k_1 > 0$ while if $\gamma < \alpha$, $k_1 = 0$.

Proof. We first plug equation (1.66) into (1.65) in order to obtain the following polynomial of order two in the variable $\rho(k) = \bar{\rho} + \rho_0 e^{-\beta k^\gamma}$.

$$\begin{aligned} \mathcal{P}(k) &= -(1 - \sigma)\rho(k)^2 - \bar{\rho}\gamma\beta k^\gamma(A + Bk^{\alpha-1} - \delta) \\ &\quad + [(A + Bk^{\alpha-1} - \delta)\gamma\beta k^\gamma + (1 - \sigma)(A + B\alpha k^{\alpha-1} - \delta)]\rho(k) \\ &= 0 \end{aligned} \quad (1.67)$$

The discriminant is the following:

$$\begin{aligned} \Delta(k) &= [(A + Bk^{\alpha-1} - \delta)\gamma\beta k^\gamma + (1 - \sigma)(A + B\alpha k^{\alpha-1} - \delta)]^2 \\ &\quad - 4(1 - \sigma)\bar{\rho}\gamma\beta k^\gamma(A + Bk^{\alpha-1} - \delta) \end{aligned} \quad (1.68)$$

If the previous expression is negative, there are no positive real solutions for $\rho(k)$ implying that there is no steady-state in this case and the only equilibrium is the ABGP equilibrium. Moreover, since the product and the sum of the roots are positive given that $\sigma < 1$, in the case of a positive discriminant, we are in the presence of two positive real roots and thus two steady-state equilibria. We should proceed with the analysis in this case.

The steady-state curves have the following shape. Equation (1.66) is standard in endogenous growth models and goes from zero to infinity for both capital and consumption. Equation (1.65) goes from $(c, k) = (-\infty, 0)$ if $\gamma > \alpha$ or $(c, k) = (0, 0)$ if $\gamma < \alpha$ to $(c, k) = (-\infty, +\infty)$.

This can be observed by rewriting equation (1.65) as

$$c = \frac{1}{\gamma\beta\rho_0}(A + B\alpha k^{\alpha-1} - \delta - \bar{\rho} - \rho_0 e^{-\beta k^\gamma})(\sigma - 1)(\rho_0 + \bar{\rho}e^{\beta k^\gamma})k^{1-\gamma} \quad (1.69)$$

and taking the limit when $k \rightarrow 0$ and $k \rightarrow \infty$:

$$\lim_{k \rightarrow 0} = \frac{B\alpha}{\gamma\beta\rho_0} \lim_{k \rightarrow 0} k^{\alpha-\gamma} \quad (1.70)$$

$$\lim_{k \rightarrow \infty} = -\infty \quad (1.71)$$

As can be noticed, the limit as $k \rightarrow 0$ depends on the sign of $\alpha - \gamma$. In order to compute the second limit we used the fact that $\sigma < 1$ and $A - \delta - \bar{\rho} > 0$.

We now take the derivative of equations (1.65) and (1.66) with respect to capital to obtain:

$$\begin{aligned} \frac{\partial c}{\partial k} = & \left(\frac{\sigma - 1}{\gamma \beta \rho_0} \right) \{ [B\alpha(\alpha - 1)k^{\alpha-1-\gamma} + \gamma\beta\rho_0 e^{-\beta k^\gamma}] (\rho_0 + \bar{\rho} e^{\beta k^\gamma}) \\ & + \bar{\rho} e^{\beta k^\gamma} (A + B\alpha k^{\alpha-1} - \delta - \bar{\rho} - \rho_0 e^{-\beta k^\gamma}) \left(\frac{\sigma - 1}{\rho_0} \right) \end{aligned} \quad (1.72)$$

$$\begin{aligned} & + \left(\frac{\sigma - 1}{\gamma \beta \rho_0} \right) (1 - \gamma) k^{-\gamma} (A + B\alpha k^{\alpha-1} - \delta - \bar{\rho} - \rho_0 e^{-\beta k^\gamma}) (\rho_0 + \bar{\rho} e^{\beta k^\gamma}) \\ \frac{\partial c}{\partial k} = & A + B\alpha k^{\alpha-1} - \delta \end{aligned} \quad (1.73)$$

Equation (1.66) is increasing and concave with a constant slope equal to $A - \delta$ in the limit as k goes to infinity. The result is not immediate for equation (1.65) so that we compute the limits of this derivative.

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{\partial c}{\partial k} = & \left(\frac{\sigma - 1}{\gamma \beta \rho_0} \right) (A - \delta - \bar{\rho}) \bar{\rho} \lim_{k \rightarrow \infty} e^{\beta k^\gamma} \\ & + \left(\frac{\sigma - 1}{\gamma \beta \rho_0} \right) (1 - \gamma) (A - \delta - \bar{\rho}) \lim_{k \rightarrow \infty} \frac{e^{\beta k^\gamma}}{k^\gamma} \end{aligned} \quad (1.74)$$

$$\lim_{k \rightarrow \infty} \frac{\partial c}{\partial k} = -\infty \quad (1.75)$$

To obtain the result we make use of l'Hospital's rule and the fact that by assumption $\sigma < 1$.

We now compute the derivative when k converges toward zero.

$$\begin{aligned} \lim_{k \rightarrow 0} \frac{\partial c}{\partial k} = & (\sigma - 1)(\bar{\rho} + \rho_0) \\ & + \left(\frac{\sigma - 1}{\gamma \rho_0 \beta} \right) (A - \delta - \bar{\rho} - \rho_0) [\gamma \beta \bar{\rho} + (1 - \gamma)(\rho_0 + \bar{\rho}) \lim_{k \rightarrow 0} k^{-\gamma}] \\ & + \left\{ \frac{(\sigma - 1) \lim_{k \rightarrow 0} k^{\alpha-1-\gamma}}{\gamma \rho_0 \beta} [\bar{\rho} \beta \alpha \lim_{k \rightarrow 0} k^\gamma + B\alpha(\rho_0 + \bar{\rho})(\alpha - \gamma)] \right\} \end{aligned} \quad (1.76)$$

$$\lim_{k \rightarrow 0} \frac{\partial c}{\partial k} = +\infty \quad (1.77)$$

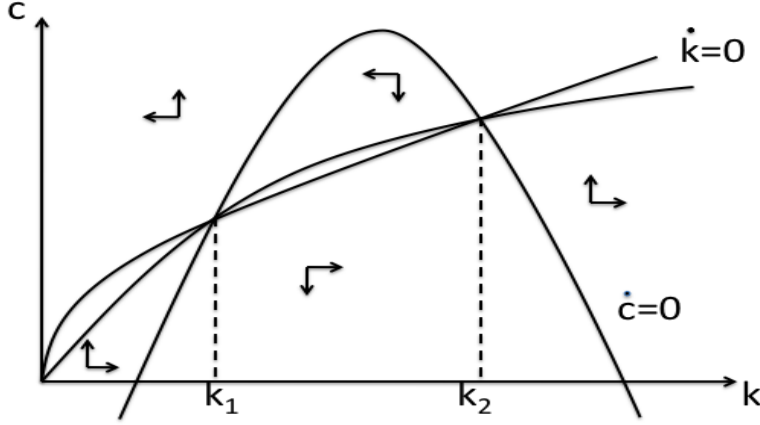
since $\sigma < 1$ and $\alpha < 1$. Equation (1.65) is thus first increasing as k converges to zero and decreasing when k tends toward infinity. Finally, in order to obtain the arrows of motion we compute the following derivatives.

$$\begin{aligned} \frac{\partial \dot{c}}{\partial c} = & \frac{A + B\alpha k^{\alpha-1} - \delta - \bar{\rho} - \rho_0 e^{-\beta k^\gamma}}{\sigma} \\ & + \frac{[(1 - \sigma)(Ak + Bk^\alpha - \delta k) + \sigma 2c] \gamma \beta k^{\gamma-1} \rho_0 e^{-\beta k^\gamma}}{\sigma(1 - \sigma)(\bar{\rho} + \rho_0 e^{-\beta k^\gamma})} > 0 \end{aligned} \quad (1.78)$$

$$\frac{\partial \dot{k}}{\partial c} = -1 < 0 \quad (1.79)$$

Combining the different elements, we obtain the phase diagram represented in Figure 1 which delivers a first saddle-path stable steady-state of stagnation and a second unstable one. It should be noticed that if $\gamma < \alpha$, the poverty trap is characterized by capital and consumption levels equal to zero. $\square$

Figure 1: Phase diagram



If there is no steady-state equilibrium, the economy will simply converge toward the asymptotic balanced growth path. We should come back to this case when we will study the transitional dynamics of the model. The previous proposition shows that the poverty trap case can be characterized by positive or zero capital and consumption levels depending on the sign of $\alpha - \gamma$. In the following we should assume that $\alpha < \gamma$ and focus on the case where capital and consumption take positive values at the poverty trap equilibrium. As can be seen from Figure 1, the first steady-state is saddle-path stable while the second could seem a strange attractor given the arrows of motion that could imply the existence of a periodic solution. However, this possibility can be ruled out by the use of the Bendixson's criterion. In the present case, the criterion states that if the sum

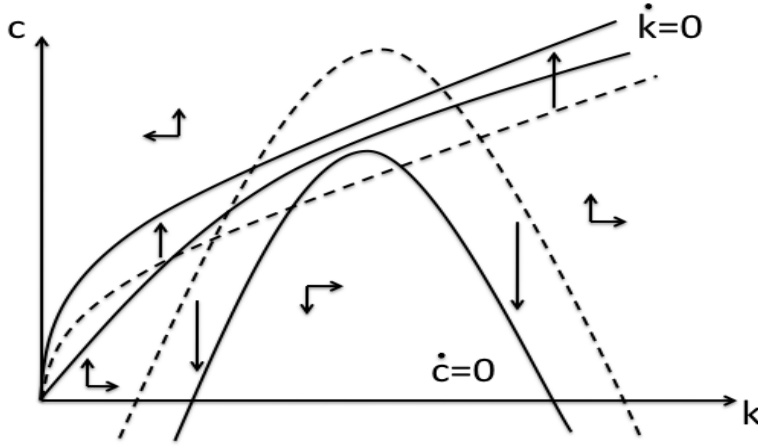
$$\frac{\partial \dot{c}}{\partial c} + \frac{\partial \dot{k}}{\partial k} \quad (1.80)$$

has the same sign ($\neq 0$) almost everywhere in a simply connected region, then the planar autonomous system consisting of equations (1.16) and (1.17) has no periodic solutions. This is indeed the case since our model is developed over a simply connected region and the sum of partial derivatives is always positive.

From the phase diagram, we can see that if the economy starts with a level of capital lower than k_2 , it will necessarily converge toward k_1 which is a steady-state of stagnation that can be interpreted as a poverty trap. This equilibrium will be

reached by positive or negative growth respectively if $k < k_1$ and $k_1 < k < k_2$. The existence of a poverty trap is directly related to a low initial capital stock level and its influence on the discount rate. The latter being relatively high pushes agents to consume in order to satisfy present needs. As a consequence, agents not willing or being unable to save enough, the economy converges toward a steady-state of stagnation. On the contrary, if the initial level of capital is higher than k_2 , the economy will converge toward the ABGP. In this case, the discount rate is sufficiently low to incentivize agents to increase savings ensuring positive growth in the long run.

Figure 2: Effect of productivity shocks



There is however a possibility to escape this poverty trap. A positive shock (or a series of positive shocks) on the productivity parameters of the production function might lead to the disappearance of both the poverty trap and the unstable steady-state by shifting upward the steady-state curve of capital and downward the one of consumption. The following partial derivatives show the effect of an increase in the productivity parameter A on the steady-state equations. The first concerns the $\dot{k} = 0$ equation while the second concerns the $\dot{c} = 0$ one.

$$\frac{\partial c}{\partial A} = k > 0 \quad (1.81)$$

$$\frac{\partial c}{\partial A} = \frac{(\sigma - 1)(\bar{\rho} + \rho_0 e^{-\beta k^\gamma})}{\gamma \beta k^{\gamma-1} \rho_0 e^{-\beta k^\gamma}} < 0 \quad (1.82)$$

Provided that the shock is sufficiently strong, the economy will converge toward the ABGP. This situation is depicted in Figure 2 with the corresponding movements of the steady-state curves.

Note that the result depends on the fact that $\sigma < 1$ which implies that the second

partial derivative is negative. The mechanism behind this positive result is the following: as A increase, capital becomes more productive thus creating an incentive towards accumulation and decreasing the discount rate. This in turn reduce the incentives for immediate consumption and engage the economy on the dynamics of the ABGP equilibrium. The inverse phenomenon is however totally possible such that a negative shock affecting the productivity parameters might take the economy to the dynamic path of the poverty trap. In this simplified model, this is the only way to escape from the poverty trap. However, in a more realistic framework, any element affecting positively the productivity of the economy could play a similar role. We can think about productive public spending, knowledge acquisition through international trade or policies related to education for example.

1.4 Transitional dynamics

This section focuses on the transitional dynamics of our model. Depending on the value of the capital stock, the results will be different. We can distinguish three cases.

Proposition 3 (Part 1):

If $k_0 > k_2$ or if the only competitive equilibrium is the ABGP equilibrium:

- 1) $g_k > g_y > g_c > 0$
- 2) the saving rate s increases along the transition
- 3) g_k , g_y and g_c increase along the transition

Proof. Let's start by showing that $g_k > g_y$. By computing g_y , we obtain

$$g_y = \left(\frac{A + B\alpha k^{\alpha-1}}{A + Bk^{\alpha-1}} \right) g_k < g_k \quad (1.83)$$

since $\alpha < 1$. We also know that for $k > k_2$, $g_k > 0$ since the economy converges toward the ABGP. Since we reach the balanced growth path only asymptotically, it is not possible to have $g_c > g_y$ along the transition. By contradiction, we know that $g_y > g_c$. This completes the first part of the proof. A direct implication is that g_z and g_x are both negative such that the average product of capital as well as the consumption to capital ratio decrease along the transition. Moreover we also can conclude that $g_x < g_z < 0$

Concerning the second part of the proposition, the gross saving rate is given by $s = 1 - \frac{c}{y}$. From the first part of the proposition we know that $g_y > g_c$ implying that the saving rate increases along the transition.

For the third part, we take the derivative of the growth rate of capital and output with respect to time obtaining:

$$\frac{dg_k}{dt} = \dot{z} - \dot{x} > 0 \quad (1.84)$$

since $\dot{z} \frac{x}{z} > \dot{x}$ and $\frac{x}{z} < 1$.

$$\frac{dg_y}{dt} = \frac{ABk^{\alpha-1}[(\alpha-1)g_k]^2}{(A+Bk^{\alpha-1})^2} + \left(\frac{A+B\alpha k^{\alpha-1}}{A+Bk^{\alpha-1}} \right) \frac{dg_k}{dt} > 0 \quad (1.85)$$

from equation (1.84).

Concerning the growth rate of consumption, the sign of $\frac{dg_c}{dt}$ is ambiguous. However, we know from the first part of the proposition that $g_k > g_c$. Given that g_k increases along the transition and $\lim_{k \rightarrow \infty} g_k = \lim_{k \rightarrow \infty} g_c > 0$, we can conclude that g_c will also increase along the transition toward the ABGP. $\square$

The acceleration of the growth rate of output results from two different effects. The first one is the increase in the growth rate of capital while the second is the convergence of the marginal product of capital with the average product of capital. In the limit, both variables are equal to A . The increase in the growth rates of capital and consumption is due to the continuous decrease in the discount rate which more than compensate for the decrease in the marginal productivity of capital. Given that the saving rate is also increasing since output grows faster than consumption along the transition, we obtain the desired positive correlation between increasing growth and saving rates for economies converging toward the ABGP. The increasing patience of the representative agent is the key to understand the dynamics of endogenous variables in the present framework.

Proposition 3 (Part 2):

If $k_0 < k_1$:

- 1) $g_k > g_y > 0$ and $g_c > g_y$
- 2) the saving rate s decreases
- 3) g_k , g_y and g_c will decrease along the transition

Proof. Growth of capital is positive during the transition ($g_k > 0$) but must come to an end in order to reach the steady-state of stagnation k_1 . This implies that $g_c > g_y$ is a necessary condition for convergence. The fact that $g_k > g_y$ comes again from equation (1.83).

From the first part of the proposition we can directly conclude that the saving rate decreases.

Since $g_k > 0$ during the transition and $\lim_{k \rightarrow k_1} g_k = 0$, we know that g_k has to decrease along the transition toward k_1 . Given that $g_k > g_y$, the same is true for g_y . Concerning the growth rate of consumption, we know that $g_c > 0$ and $\lim_{c \rightarrow c_1} g_c = 0$ such that g_c must also decrease along the transition. $\square$

The economy faces two opposite effects with relatively high interest and discount rates which respectively favor capital accumulation and immediate consumption. As before, the discounting effect dominates such that despite the fact that accumulation proceeds, the path is unsustainable in the long run since consumption grows faster than output, thus driving the economy toward a steady-state of stagnation.

Proposition 3 (Part 3):

If $k_1 < k_0 < k_2$:

- 1) $g_k < g_y < 0$ and $g_c < g_y$
- 2) the saving rate s increases
- 3) g_k , g_y and g_c will increase along the transition

Proof. This situation is the exact opposite of the previous one and the arguments can be easily reverted. In this case, growth of capital is negative but must reach zero implying that $g_y > g_c$ and again from equation (1.83) we know that $g_k < g_y$.

The saving rate is negative increases since $g_y > g_c$.

Since all growth rates are negative and must be equal to zero in the limit, we can apply the same methodology as before and conclude that the growth rates of capital, output and consumption will increase along the transition. $\square$

The interest rate and discount rate effects are still present. However, the second is still dominant in this case and induce the representative agent to consume the entire output and a share of the capital stock at each period of time, thus enhancing negative savings and growth.

1.5 Confrontation to empirical evidence

This section shows that the characteristics of the present model are in accordance with the empirical results found in the literature. These are decreasing and then stabilizing interest and discount rates, increasing and then stabilizing growth and saving rates as well as the tendency toward a twin peak distribution of the world income with growth miracles and disasters.

- Clark (2007) shows that the interest rate has been falling if we focus on a sufficiently large period starting in the thirteen century with interest rates close to 11 percent in Europe while they had fallen to a 5 percent average for the last century (in fact interest rates stayed roughly constants until the seventeen century). In the long run however, we observe a roughly constant interest rate as documented by Kaldor (1957).
- The tendency for the discount rate to decrease is empirically studied in Lawrence (1991) who argues that the discount rate of poor households is 5 percent higher than for rich households or in Samwick (1998) which observes that the discount rate decreases with the income level. Clark (2007) also argues that the observed decrease in the real interest rate can only be due to a similar decrease in the discount rate. In our model, this is made possible through the accumulation of capital.
- Concerning the increasing growth rate of endogenous variables, Galor (2005) documented an increasing growth rate of income per capita in Europe from 0,15 percent per year before 1820 to 0,95 percent between 1820 and 1870, finally reaching 2 percent during the last century. On average, this growth rate has been stable from then on.
- The saving rate has been increasing from 11,7 percent in 1831 (Galor 2005) to around 23 percent in the seventies for developed countries. A share that has then stabilized.

- Jones (1997) finds that the world distribution is changing from a somewhat normal distribution with thick tails toward a bimodal distribution (usually denominated twin peaks in the literature). Moreover Azariadis and Stachurski (2005) documented that poor countries tend to stagnate while rich countries tend to become richer. This feature is captured in our model by the existence of both a low income saddle-path stable steady-state and an ABGP. The unstable steady-state might be interpreted as the unstable attractor of the normal distribution.
- Jones (1997) also identifies a series of growth miracles such as some well known east-Asian countries but also Botswana and Romania and growth disasters as in several sub-Saharan African countries and Venezuela to cite a few examples. These reversal of fortune are explained in our model by the effect of changes in the productivity parameters on the existence of steady-state equilibria. We don't want to make the claim that growth miracles and disasters can be explained only by productivity shocks but rather that public policies affecting the productivity of the economy might explain the movements that we observe within the world distribution of income.

1.6 Conclusion

The present paper had two main objectives: the first one was to argue that a model where the discount rate is decreasing in capital accumulation should impose a utility function that takes only positive values following the argumentation of Schumacher (2011). The second was to show that in this case a model respecting appropriate second-order sufficiency conditions can still shed light on some key stylized facts concerning the growth process.

In order to do so, we derived necessary and sufficient conditions for the existence of an ABGP equilibrium which impose a lower bound on the elasticity of marginal utility. We also showed that this economy may or not exhibit multiple equilibria. In the latter case, we proved the existence of a saddle-path stable steady-state which can be interpreted as a poverty trap. The combination of these different elements allows the model to generate a positive correlation between increasing saving and growth rates as well as a bimodal distribution of income around the ABGP equilibrium and the poverty trap. The possibility of escaping the poverty trap has been explored through exogenous shocks on the productivity parameters. However, further research should focus on the cost of implementing public policies which actually increase those productivity parameters allowing to escape the poverty trap.

Chapter 2

Habits in consumption, environmental quality and intergenerational inequalities

2.1 Introduction

The present paper focuses on the role of habits in consumption concerning environmental degradation and intergenerational inequalities in an overlapping generation setup. It has recently become common in economics to define sustainable development as growth paths ensuring non-decreasing utility levels or at least utility levels above a certain reference point (Stavins, Wagner and Wagner 2003, Pezzey 2004). The choice of these criteria can potentially rule out any growth path characterized by fluctuations of endogenous variables. Related to this issue is the observation that economists have started to devote attention to the intergenerational aspects of environmental degradation (see for example Howarth 1997, Pezzey and Toman 2002, John and Pecchenino 1994, Seegmuller and Verchere 2004, Cao, Wang and Wang 2011). Most of the current models devoted to sustainability problems in fact take into account environmental quality as an important element of the analysis. These models without habits and augmented with an environmental constraint are however mostly characterized by monotonic dynamics except the works of Seegmuller and Verchere (2004) and Cao, Wang and Wang (2011). In the first case, the authors prove the existence of a Flip bifurcation using a utility function linear in consumption. In the second, by using a nonlinear pollution accumulation equation, the authors also prove the existence of a Flip bifurcation and the breakdown of several sustainability criteria.

Our particular interest on habits derives from the fact that as shown by de la Croix and Michel (1999) or Artige, Camacho and de la Croix (2004), habits can have severe effects on transitional dynamics such as local oscillations or limit cycles. For a long time, these kind of "extended preferences", following Becker's (1997) terminology, have been disconnected from environmental concerns. However recently, a certain number of authors have introduced habits in pollution or in environmental

quality in otherwise standard overlapping generation frameworks. Schumacher and Zou (2008) study habits in pollution and identify a large set of possible behaviors for their dynamical system including local oscillations and bifurcations giving rise to intergenerational inequalities during the transition and in the long run. Chen and Li (2011) analyze habits in environmental quality and derive results concerning possible chaotic behavior for their one dimensional system. These models however only focus on the competitive equilibrium and do not try to design an adequate optimal policy. Our objective is to study an overlapping generations framework along the lines of John and Pecchenino (1994) and Seegmuller and Verchere (2004) where agents possess habits in consumption and are subject to an environmental constraint in which consumption from both young and old generations exerts a negative impact on environmental quality which can be at least partially compensated by maintenance investment. We wish to focus on both the competitive and the optimal outcome in order to be able to address policy issues. Habits in consumption have up to now been modelled in different ways in the overlapping generation literature. While de la Croix (1996) and de la Croix and Michel (1999) study a framework where children inherit the consumption habits of their parents reflecting the idea of family capital, Wendner (2002) proposes a framework where higher adult's consumption has a negative impact on old age utility for a given level of consumption when old reflecting the idea of personal capital. Given our particular interest concerning intergenerational inequalities, we choose to focus on the first approach where habits transmission can act as an important intergenerational externality. One implication of this formulation is that reported satisfaction levels do not necessarily rise with economic development which is in accordance with empirical evidence provided by Easterlin (1995). As expressed by Lucas (1988), if a generation of individuals is twice as rich as a previous one, this does not imply that they will be twice as satisfied. The model thus exhibits two kind of intergenerational inequalities related respectively to habits and pollution accumulation.

An account of the results is as follows. In the present case, the competitive steady-state capital stock level can be higher or lower than in the standard Diamond economy. The combination of habits in consumption and potential environmental degradation generates non-monotonic dynamic behavior such as local oscillations and limit-cycles. The consequence being that some important sustainability criteria such as non-decreasing utility levels can be violated. We also study the optimal economy by solving the social planner's problem and derive conclusions concerning its decentralization. In order to reach the optimal outcome, investment should be taxed or subsidized depending on the parameters of interest and maintenance investment should always be subsidized. If capital needs to be taxed, the optimal tax policy goes against the one applied in the case without environmental quality studied by de la Croix and Michel (1999) where subsidies to capital accumulation are always needed. The optimal solution can however still be subject to local oscillations even in the no discounting case implying that if a standard social welfare function is used without an appropriate sustainability constraint, we are able to solve for the environmental externalities but not for the sustainability issue.

The remainder of this paper is organized as follows. Section 2 introduces the model, derives the first order conditions and argues that these are also sufficient to ensure the existence of a maximum. The intertemporal equilibrium is studied in section 3. Section 4 focuses on the local dynamics of our model and contains some core results of the paper. The optimal solution is presented in section 5 while the decentralization of the first best solution is addressed in section 6. The last section is devoted to the conclusion.

2.2 The model

We consider a competitive overlapping generations model where agents live for three periods and have perfect foresight. Population is constant and normalized to one. The young generation does not take any decision and inherits consumption habits from their parents, h_t . In adulthood, the representative agent supplies inelastically one unit of labor and earns in exchange the real wage w_t . This wage is split between present consumption c_t , savings s_t and maintenance investment m_t .

$$w_t = c_t + s_t + m_t \quad (2.1)$$

In old age, the agent retires and earns the gross return R_{t+1} on his savings from which he consumes d_{t+1} .

$$d_{t+1} = R_{t+1}s_t \quad (2.2)$$

Environmental quality is a public good that evolves according to:

$$E_{t+1} = (1 - b)E_t - \chi(c_t + d_t) + \xi m_t \quad (2.3)$$

where $b \in [0, 1]$ measures the speed at which environmental quality returns to its natural level which in this case is zero, $\chi > 0$ measures the impact of total consumption on environmental quality and $\xi > 0$ the effectiveness of maintenance investment. The fact that present decisions only affect future levels of environmental quality creates an intergenerational externality, which is justified by the fact that the evolution of environmental quality is a long term process. The possible negative externalities due to consumption can in general arise several years after their creation.

The life-cycle utility function of the representative agent is defined over present and future consumption, the future environmental quality level, habits in consumption and takes the following form:

$$U(c_t, h_t, d_{t+1}, P_{t+1}) = \theta \ln(c_t - \gamma h_t) + \delta \ln(d_{t+1}) + \eta \ln(E_{t+1}) \quad (2.4)$$

Habits can be seen as a frame of reference against which present consumption is evaluated and γ represents the intensity of the habit effect. The reason for not including present environmental quality in adulthood utility is that agents are unable to influence present quality when making their decisions about consumption and maintenance investment. We thus assume that present quality has no effect on

adulthood utility. The parameters θ , δ and η are the weights associated respectively to present consumption, future consumption and environmental quality. Furthermore, for technical reasons we should assume that $\delta + \theta + \eta = 1$.

We will also consider that habits are equivalent to the consumption of the previous adult generation such that children get used to particular consumption standards when living with their parents.

$$h_t = c_{t-1} \quad (2.5)$$

Concerning preferences, we assume that the depreciation rate of habits is particularly high since the old generation is not affected by them. This assumption can be justified by empirical evidence showing that aspirations are less important for older persons. Clark and al. (1996) show for example that reported satisfaction levels increase with age. Older persons putting less weight on comparisons in their welfare evaluation.

In the present framework, the economy faces two kind of intergenerational externalities. The first one is due to habits as a frame of reference originating in the consumption of the previous young generation. The second is due to old age consumption, which will affect the level of environmental quality faced by the following generations.

Concerning production, there is a representative firm which produces an homogeneous good with a Cobb-Douglas production function, $y_t = Ak_t^\alpha$ where α is the share of capital in the production process and we assume complete depreciation after one period. The firm then maximizes profits in a competitive market that clears:

$$R_t = \alpha Ak_t^{\alpha-1} \quad (2.6)$$

$$w_t = (1 - \alpha)Ak_t^\alpha \quad (2.7)$$

$$s_t = k_{t+1} \quad (2.8)$$

A representative adult faces the following problem:

$$\max_{c_t, d_{t+1}, s_t, m_t} \theta \ln(c_t - \gamma h_t) + \delta \ln(d_{t+1}) + \eta \ln(E_{t+1}) \quad (2.9)$$

subject to

$$\begin{cases} w_t = c_t + s_t + m_t \\ d_{t+1} = R_{t+1}s_t \\ E_{t+1} = (1 - b)E_t - \chi(c_t + d_t) + \xi m_t \\ c_t, d_{t+1}, s_t, m_t \geq 0 \end{cases}$$

given w_t , R_{t+1} , E_t and d_t .

Substituting for c_t , d_{t+1} and E_{t+1} in expression (2.9) and taking the derivative with respect to s_t and m_t we obtain the following first order conditions:

$$\frac{\partial U}{\partial s_t} = \frac{-\theta}{c_t - \gamma h_t} + \frac{\delta R_{t+1}}{d_{t+1}} + \frac{\eta \chi}{E_{t+1}} = 0 \quad (2.10)$$

$$\frac{\partial U}{\partial m_t} = \frac{-\theta}{c_t - \gamma h_t} + \frac{\eta(\chi + \xi)}{E_{t+1}} = 0 \quad (2.11)$$

Concerning second order conditions, we can see that our constraints are linear. We thus only need to prove that our utility function is concave. The computation of the Hessian matrix concerning the utility function gives:

$$H = \begin{bmatrix} -\frac{\theta}{(c_t - \gamma h_t)^2} & 0 & 0 \\ 0 & -\frac{\delta}{(d_{t+1})^2} & 0 \\ 0 & 0 & -\frac{\eta}{(E_{t+1})^2} \end{bmatrix} \quad (2.12)$$

As can be seen immediately, the leading principal minors alternate in sign such that the Hessian matrix is negative definite and our objective function is indeed concave. We can proceed with the first order conditions.

$$\frac{\delta R_{t+1}}{d_{t+1}} = \frac{\eta \xi}{E_{t+1}} \quad (2.13)$$

$$\frac{\theta}{c_t - \gamma h_t} = \frac{\eta(\chi + \xi)}{E_{t+1}} \quad (2.14)$$

Equation (2.13) represents the equalization between the marginal benefit of increasing savings and the marginal cost of reducing present consumption. In order to identify the effect of the different parameters, we use the implicit function theorem and obtain

$$s_\delta = \frac{s}{\delta} > 0, s_\eta = -\frac{\xi s^2}{\delta E_{t+1}} < 0, s_\xi = -\frac{\eta s^2}{\delta E_{t+1}} < 0$$

From these results, we can see that a higher relative preference for future consumption, δ , will imply higher savings. A higher relative preference towards environmental quality, η , will push the representative agent to save less and invest more in maintenance investment in order to increase future environmental quality. Finally, if agents consider that maintenance effectiveness, ξ , is quite low, they will prefer to save more in order to consume more in the future and to avoid environmental degradation through immediate consumption. Equation (2.14) represents the equalization between the marginal benefit of increasing future environmental quality through maintenance investment and the marginal cost of decreasing actual consumption. As before, we use the implicit function theorem and obtain

$$c_\theta = \frac{(c_t - \gamma h_t)}{\theta} > 0, c_\gamma = h_t > 0, c_\eta = -\frac{(\chi + \xi)(c_t - \gamma h_t)^2}{\theta E_{t+1}} < 0$$

$$c_\chi = -\frac{\eta(c_t - \gamma h_t)^2}{\theta E_{t+1}} < 0, c_\xi = -\frac{\eta(c_t - \gamma h_t)^2}{\theta E_{t+1}} < 0$$

A higher relative preference for present consumption, θ , as well as a higher effect of habits, γ , imply higher present consumption and thus a decrease in future environmental quality. A higher relative preference towards environmental quality, η , will imply lower present consumption in order to avoid an important environmental degradation. Similarly, a higher impact of total consumption on environmental quality, χ , and of maintenance effectiveness, ξ , will imply a substitution towards savings or maintenance investment.

2.3 Intertemporal equilibrium

By using the first order conditions, the market clearing condition for capital, the feasibility constraint and the fact that $E_{t+1} = \frac{\xi\eta}{\delta}k_{t+1}$ at equilibrium, we are able to define the intertemporal equilibrium of this economy.

Definition 1:

An intertemporal equilibrium of this economy is a sequence $\{k_t, h_t\}_0^\infty$ with initial conditions $\{k_0, h_0\}$ that satisfies the following difference equations:

$$k_{t+1} = \eta(1-b)k_t + \frac{\delta}{\xi}[\xi - (\chi + \xi)\alpha]Ak_t^\alpha - \frac{\delta}{\xi}(\chi + \xi)\gamma h_t \quad (2.15)$$

$$h_{t+1} = \frac{\theta\xi\eta(1-b)}{\delta(\chi + \xi)}k_t + \frac{\theta[\xi - (\chi + \xi)\alpha]}{(\chi + \xi)}Ak_t^\alpha + (1-\theta)\gamma h_t \quad (2.16)$$

By setting $k_{t+1} = k_t = \bar{k}$ and $h_{t+1} = h_t = \bar{h} = \bar{c}$, we derive the steady-states of this economy. There exist two steady-states, the first one is trivial with $\{\bar{k}, \bar{c}\} = (0, 0)$. The other steady-state is given by

$$\bar{k} = \left\{ \frac{A\delta[\xi - (\chi + \xi)\alpha](1-\gamma)}{\xi(1-\gamma)[1 - \eta(1-b)] + \theta\xi\gamma} \right\}^{\frac{1}{1-\alpha}} \quad (2.17)$$

$$\bar{c} = \frac{\theta\xi}{\delta(\chi + \xi)(1-\gamma)}\bar{k} \quad (2.18)$$

The condition for the existence of a positive steady-state is that the numerator and the denominator of equation (2.17) have the same sign. The denominator is always positive given our assumption concerning the parameters of the model. The numerator is also positive only if the following condition is met:

$$\frac{\chi}{\xi} < \frac{1-\alpha}{\alpha} \quad (2.19)$$

This allows for the possibility that it takes less effort to pollute than to clean up but that this difference cannot be too large and is bounded above by an ratio involving the elasticity of capital in the production function. Moreover, if this condition is met, output has a positive impact on capital growth as can be seen from equation (2.15).

We can now study the influence of the parameters of interest on the steady-state values. We should focus on parameters which are not present in the standard Diamond framework. In order to do so, we take the derivative of equations (2.17) with respect to γ , ξ , χ , η and b .

$$\frac{\partial \bar{k}}{\partial \gamma} = -\frac{A\theta\delta[\xi - (\chi + \xi)\alpha]}{\xi^2(1-\alpha)\{(1-\gamma)[1 - \eta(1-b)] + \theta\gamma\}^2}\bar{k}^\alpha < 0 \quad (2.20)$$

The negative impact of γ on the steady-state capital stock can be explained by the deterring effect that habits play on savings and thus on capital accumulation.

$$\frac{\partial \bar{k}}{\partial \xi} = \frac{\{(1-\gamma)[1 - \eta(1-b)] + \theta\gamma\}A\delta(1-\gamma)\chi^\alpha}{\xi^2(1-\alpha)\{(1-\gamma)[1 - \eta(1-b)] + \theta\gamma\}^2}\bar{k}^\alpha > 0 \quad (2.21)$$

In this case, the impact is positive since a high maintenance effectiveness allows the representative agent to devote more resources to savings.

$$\frac{\partial \bar{k}}{\partial \chi} = -\frac{A\delta\alpha(1-\gamma)}{(1-\alpha)\xi\{(1-\gamma)[1-\eta(1-b)] + \theta\gamma\}}\bar{k}^\alpha < 0 \quad (2.22)$$

Intuitively, a large impact of consumption on environmental quality pushes the agents to invest in environmental maintenance at the expenses of the capital stock.

$$\frac{\partial \bar{k}}{\partial \eta} = \frac{A\delta\xi(1-b)[\xi - (\chi + \xi)\alpha](1-\gamma)^2}{(1-\alpha)\{\xi(1-\gamma)[1-\eta(1-b)] + \theta\xi\gamma\}^2}\bar{k}^\alpha > 0 \quad (2.23)$$

A larger relative preference for future environmental quality induces agents to save more in order to avoid environmental degradation through immediate consumption.

$$\frac{\partial \bar{k}}{\partial b} = -\frac{A\delta\xi\eta[\xi - (\chi + \xi)\alpha](1-\gamma)^2}{(1-\alpha)\{\xi(1-\gamma)[1-\eta(1-b)] + \theta\xi\gamma\}^2}\bar{k}^\alpha < 0 \quad (2.24)$$

Finally, a larger speed of recovery of the environment has a negative impact on the steady-state capital stock since agents can devote resources to present consumption without large consequences on environmental quality thus depressing savings.

The opposite effects that we have identified imply that the steady-state capital stock might be lower or higher than in the standard Diamond model. We can now proceed with the dynamics of the model in the following section.

2.4 The dynamics

We now compute the Jacobian matrix around the non-trivial steady-state $(\bar{k}, \bar{c})$:

$$J = \begin{bmatrix} \frac{\eta(1-b)(1-\alpha)(1-\gamma) + \alpha[1-\gamma(1-\theta)]}{1-\gamma} & -\frac{\delta(\chi+\xi)\gamma}{\xi} \\ \frac{\theta\xi}{\delta(\chi+\xi)} \left\{ \frac{\eta(1-b)(1-\alpha)(1-\gamma) + \alpha[1-\gamma(1-\theta)]}{1-\gamma} \right\} & (1-\theta)\gamma \end{bmatrix} \quad (2.25)$$

The characteristic function $P(\lambda)$ is given by:

$$P(\lambda) = \lambda^2 - Tr(J)\lambda + Det(J)$$

where

$$Det(J) = \frac{\eta(1-b)(1-\alpha)(1-\gamma) + \alpha[1-\gamma(1-\theta)]}{1-\gamma} \quad (2.26)$$

$$Tr(J) = Det(J) + (1-\theta)\gamma \quad (2.27)$$

It is usefull to notice that given our assumption on parameters, both the determinant and the trace can only take positive values. The following proposition assesses the possible dynamic behavior of our planar system.

Proposition 1:

Consider an interior competitive equilibria:

(i) The fixed point $(\bar{k}(\gamma), \bar{h}(\gamma))$ is hyperbolic if $\gamma \neq \hat{\gamma}$ where:

$$\hat{\gamma} = \frac{1 - \eta(1 - b)(1 - \alpha) - \alpha}{1 - \eta(1 - b)(1 - \alpha) - \alpha(1 - \theta)} \quad (2.28)$$

(ii) If the combination of parameters is such that $Tr(J)^2 - 4Det(J) < 0$, it is possible that the orbit around the fixed point $(\bar{k}(\gamma), \bar{h}(\gamma))$ is oscillatory on an interval $\gamma_1 < \gamma < \gamma_2$. Moreover, for this complex eigenvalues case, the fixed point is asymptotically stable if $\gamma < \hat{\gamma}$ and unstable otherwise.

(iii) In the case of real eigenvalues, that is when $Tr(J)^2 - 4Det(J) > 0$, the fixed point is asymptotically stable if $\gamma < \hat{\gamma}$ and unstable otherwise.

(iiii) Neimark-Sacker bifurcation: Let $(\hat{k}, \hat{h})$ be the fixed point associated to $\gamma = \hat{\gamma}$. There is a neighborhood U of $\hat{\gamma}$ for which there is, either for $\gamma < \hat{\gamma}$ or for $\gamma > \hat{\gamma}$, a closed invariant curve Γ which encircles $(\hat{k}, \hat{h})$.

Proof. In planar maps, non-hyperbolicity may only arise if there is an eigenvalue equal to 1, if there is an eigenvalue equal to -1, or if the two eigenvalues are complex conjugates with modulus 1.

In the present case, we can see that the first condition is equivalent to $\gamma = 1/(1 - \theta)$ since $Tr(J) = Det(J) + (1 - \theta)\gamma$. This case can be excluded given our assumptions on the parameters.

The second condition is equivalent to $1 + Tr(J) + Det(J) = 0$ which can also be excluded given that $Tr(J) > 0$ and $Det(J) > 0$.

The third condition is equivalent to $Det(J) = 1$ and $Tr(J) \in [-2, 2]$. Notice first that since $Tr(J) = Det(J) + (1 - \theta)\gamma$, the restriction on the determinant immediately implies that the necessary restriction on the trace is met. We can then obtain the corresponding value for our habit parameter:

$$\hat{\gamma} = \frac{1 - \eta(1 - b)(1 - \alpha) - \alpha}{1 - \eta(1 - b)(1 - \alpha) - \alpha(1 - \theta)} \quad (2.29)$$

This concludes the first part of the proof.

Concerning the second part of the proposition, by computing the condition for complex eigenvalues, we obtain the following cubic function:

$$\begin{aligned} \mathcal{P}(\gamma) &= 2(1 - \theta)[\eta(1 - b)(1 - \alpha) + (1 - \theta)\alpha]\gamma^3 \\ &\quad + \{\eta(1 - b)(1 - \alpha)[\eta(1 - b)(1 - \alpha) + 2(1 - \theta)\alpha - 4(1 - \theta)]\}\gamma^2 \\ &\quad + (1 - \theta)\alpha[(1 - \theta)\alpha - 2(2 - \theta)]\gamma^2 + 2(1 - \theta)\alpha(1 - \alpha)\gamma \\ &\quad + 2\eta(1 - b)(1 - \alpha)[\eta(1 - b)(1 - \alpha) - (2 - \theta)\alpha + 1 - \theta]\gamma \\ &\quad + \eta(1 - b)(1 - \alpha)[\eta(1 - b)(1 - \alpha) + 2\alpha] + \alpha^2 \\ &< 0 \end{aligned} \quad (2.30)$$

Given that the polynomial is of order three and involves an important number of parameters in the coefficients, we choose to not compute the roots but to check if it is possible to obtain an interval on which the orbit around the fixed point is

oscillatory. In order to do so, we rely on Descartes' rule of sign and check the sign of the different elements of the polynomial. Assuming first that $\gamma > 0$, it can be proved that the signs alternate in the following way starting from the cubic coefficient: $> 0, < 0, > 0, > 0$. There are thus two changes of sign implying that the number of positive real roots is equal to two or zero. Assuming now that $\gamma < 0$, we can determine the number of negative real roots. In this case, there is only one change of sign implying that we have a unique negative real root. If the two positive roots are not complex conjugates, they constitute an interval $\gamma_1 < \gamma < \gamma_2$ on which the orbits around the fixed point are oscillatory. Moreover, we know that in the case of complex eigenvalues, the condition for asymptotic stability is that $Det(J) < 1$ which is equivalent to $\gamma < \hat{\gamma}$.

In the case of real eigenvalues, that is $\gamma \notin [\gamma_1; \gamma_2]$, the fixed point is asymptotically stable if $Det(J) < 1$ and $1 - Tr(J) + Det(J) > 0$. These conditions are respectively equivalent to $\gamma < \hat{\gamma}$ and $\gamma < 1/(1 - \theta)$. The second condition being always satisfied given our assumptions on the parameters, we are left with the first one.

Concerning the last part of the proposition, it can be checked that when $\gamma = \hat{\gamma}$, the two eigenvalues are complex conjugates, they cross the unit circle at non-zero speed when γ changes around $\hat{\gamma}$ and none of them may be of the first four roots of unity. The fulfillment of these conditions implies the existence of a Neimark-Sacker bifurcation. $\square$

As can be seen from the first proposition, the critical values of γ only depend on a limited number of parameters such as the share of capital in the production function α , the relative preference for present consumption θ and environmental quality δ and the natural regeneration rate b . The productivity parameter A , the maintenance effectiveness χ and the pollution impact of total consumption ξ have no influence on hyperbolicity, stability or on convergence (monotonic or oscillatory). The interactions between the four parameters (α, θ, η, b) are thus the key elements explaining the dynamic behavior of our dynamical system.

The appearance of a Neimark-Sacker bifurcation ensures the existence of a limit-cycle (the invariant curve Γ) implying constant fluctuations of endogenous variables around the steady-state. It is important to explain the mechanism by which cyclical behavior can appear in the present model. At the intertemporal equilibrium, savings finance the capital stock that is used to produce and to pay the wages of young workers. This process exhibits decreasing returns given our assumption on the production function. On the other hand, habits create an incentive for increasing present consumption which in turn decreases environmental quality. At some point, this negative effect on environmental quality generates an incentive for maintenance investment. The combination of habits in consumption and maintenance investment has a depressing effect on savings inducing a recession. As a consequence, the capital stock decreases followed by consumption. Maintenance investment and the decrease in consumption having a positive effect on environmental quality. Once the decrease is sufficiently strong, a rise in savings occurs together with the start of an expansion period. This process can converge or not to the steady-state with the possibility of everlasting fluctuations.

The relatively simple expression for $\hat{\gamma}$ allows us to identify the role that the different parameters can play in the appearance of a limit-cycle. We can see that $\partial\hat{\gamma}/\partial\theta < 0$, $\partial\hat{\gamma}/\partial\eta < 0$ and $\partial\hat{\gamma}/\partial b > 0$. A higher θ pushes present consumption upward such that a lower level of the habit parameter is needed in order to obtain cyclical behavior. Similarly, a higher η pushes maintenance investment upwards and acts in a similar way by depressing savings such that a lower level of the habit parameter is compatible with the existence of a limit-cycle. On the contrary, a higher b implies that environmental quality converges faster to its natural level such that maintenance investment might be less needed and a larger habit parameter is required for a Neimark-Sacker bifurcation to appear.

Given the possibility of different kind of endogenous fluctuations in the present model, successive generations might not enjoy the same level of capital, consumption and environmental quality in the short as well as in the long run. It is thus important to rewrite utility as a function of our state variables in order to assess the behavior of welfare. This is the objective of the next proposition.

Proposition 2:

The equilibrium level of utility of the representative generation can be expressed as a function of the equilibrium future capital stock. In particular, utility is always pro-cyclical.

Proof. The utility of a particular generation is $U(.) = \theta \ln(c_t - \gamma h_t) + \delta \ln(d_{t+1}) + \eta \ln(E_{t+1})$. By using the FOC, the constraints $d_{t+1} = R_{t+1}s_t$ and $s_t = k_{t+1}$, we obtain $U(.) = \theta \ln(\frac{\theta \xi k_{t+1}}{\delta(\chi + \xi)}) + \delta \ln(A\alpha k_{t+1}^\alpha) + \eta \ln(\frac{\eta \xi k_{t+1}}{\delta})$. Taking the derivative with respect to k_{t+1} :

$$\frac{\partial U(.)}{\partial k_{t+1}} = \frac{\theta}{k_{t+1}} + \frac{\delta \alpha}{k_{t+1}} + \frac{\eta}{k_{t+1}} > 0 \quad (2.31)$$

□

Welfare follows the motion of capital and this result combined with endogenous fluctuations imply that generations born at different periods of time might suffer from welfare inequalities. The existing literature on sustainable development identifies intergenerational inequalities as one of the factors giving rise to what can be considered as unsustainable development paths. It has become common in economic theory to define sustainable development as the necessity to ensure non-decreasing utility for future generations or a least to ensure that utility will not decrease under a given utility level (Stavins, Wagner and Wagner 2003, Pezzey 2004). In the present case, both criteria might not be respected since fluctuating welfare in the short and the long run is a possible outcome of the model. Our next goal is to study the optimal allocation of this economy.

2.5 Optimal solution

In this section, we consider the case of a central planner who chooses the allocation of resources in order to maximize the discounted welfare of current and future generations. Contrary to the representative agent in the competitive equilibrium case, the planner takes into account the effect of habits on present consumption and the effect of old age consumption on environmental quality. The social discount factor is given by β and the planner solves the following problem:

$$\max_{\{c_t, d_{t+1}, m_t, h_{t+1}, k_{t+1}, E_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t [\theta \ln(c_t - \gamma h_t) + \delta \ln(d_{t+1}) + \eta \ln(E_{t+1})] \quad (2.32)$$

subject to:

$$\begin{cases} Ak_t^\alpha = c_t + d_t + k_{t+1} + m_t \\ h_t = c_{t-1} \\ E_{t+1} = (1 - b)E_t - \chi(c_t + d_t) + \xi m_t \end{cases}$$

given initial conditions $\{k_0, E_0, h_0\}$.

The Lagrangian function is the following:

$$\begin{aligned} \mathcal{L} = & \sum_{t=0}^{\infty} \beta^t [\theta \ln(c_t - \gamma h_t) + \delta \ln(d_{t+1}) + \eta \ln(E_{t+1})] \\ & + \sum_{t=0}^{\infty} \beta^t \lambda_t (Ak_t^\alpha - c_t - d_t - m_t - k_{t+1}) \\ & + \sum_{t=0}^{\infty} \beta^t \mu_t (E_{t+1} - (1 - b)E_t + \chi(c_t + d_t) - \xi m_t) \\ & + \sum_{t=0}^{\infty} \beta^t \nu_t (h_t - c_{t-1}) \end{aligned} \quad (2.33)$$

The first order conditions of the maximization problem are

$$\frac{\partial \mathcal{L}}{\partial c_t} = \frac{\beta^t \theta}{c_t - \gamma h_t} - \beta^t \lambda_t + \beta^t \mu_t \chi - \beta^{t+1} \nu_{t+1} = 0 \quad (2.34)$$

$$\frac{\partial \mathcal{L}}{\partial d_{t+1}} = \frac{\beta^t \delta}{d_{t+1}} - \beta^{t+1} \lambda_{t+1} + \beta^{t+1} \mu_{t+1} \chi = 0 \quad (2.35)$$

$$\frac{\partial \mathcal{L}}{\partial m_t} = -\beta^t \lambda_t - \beta^t \mu_t \xi = 0 \quad (2.36)$$

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} = -\beta^t \lambda_t + \beta^{t+1} \lambda_{t+1} A \alpha k_{t+1}^{\alpha-1} = 0 \quad (2.37)$$

$$\frac{\partial \mathcal{L}}{\partial h_{t+1}} = -\frac{\beta^{t+1} \theta \gamma}{c_{t+1} - \gamma c_t} + \beta^{t+1} \nu_{t+1} = 0 \quad (2.38)$$

$$\frac{\partial \mathcal{L}}{\partial E_{t+1}} = \frac{\beta^t \eta}{E_{t+1}} + \beta^t \mu_t - \beta^{t+1} \mu_{t+1} (1 - b) = 0 \quad (2.39)$$

$$\lim_{t \rightarrow \infty} \beta^t \lambda_t k_t = 0 \quad (2.40)$$

$$\lim_{t \rightarrow \infty} \beta^t \mu_t P_t = 0 \quad (2.41)$$

$$\lim_{t \rightarrow \infty} \beta^t \nu_t h_t = 0 \quad (2.42)$$

As before, given that our objective function is concave and our constraints are concave or linear, we know that the first order conditions are also sufficient for optimality and we obtain:

$$\frac{\delta}{\beta d_t} = \frac{\theta}{c_t - \gamma h_t} - \frac{\beta \gamma \theta}{c_{t+1} - \gamma c_t} \quad (2.43)$$

$$d_{t+1} = \beta d_t A \alpha k_{t+1}^{\alpha-1} \quad (2.44)$$

$$\frac{\eta(\chi + \xi)}{E_{t+1}} = \frac{\delta}{\beta d_t} - \frac{\delta(1-b)}{d_{t+1}} \quad (2.45)$$

Equation (2.43) describes the allocation of consumption between generations alive at the same time. The marginal utility of consumption is corrected to internalize the effect of habits and is equalized to the marginal utility of consumption of the old. Equation (2.44) is standard and describes the intertemporal allocation of consumption. Finally, equation (2.45) describes the allocation between consumption and maintenance at a point in time. The marginal benefit of maintenance investment is equalized to the marginal utility of consumption corrected for the effect of habits as well as for the effect of old-age consumption on environmental quality.

Definition 2:

An intertemporal optimal allocation of this economy is a sequence $\{c_t, d_t, m_t, k_t, h_t, E_t\}_0^\infty$ with initial conditions $\{k_0, h_0, E_0\}$ that satisfies the following difference equations:

$$c_{t+1} = \frac{\beta^2 \theta \gamma (c_t - \gamma h_t) d_t}{\beta \theta d_t - \delta (c_t - \gamma h_t)} + \gamma c_t \quad (2.46)$$

$$d_{t+1} = \frac{\beta^2 \delta (1-b) d_t E_{t+1}}{\delta E_{t+1} - \eta \beta (\chi + \xi) d_t} \quad (2.47)$$

$$d_{t+1} = \beta d_t A \alpha k_{t+1}^{\alpha-1} \quad (2.48)$$

$$k_{t+1} = A k_t^\alpha - c_t - d_t - m_t \quad (2.49)$$

$$E_{t+1} = (1-b)E_t - \chi(c_t + d_t) + \xi m_t \quad (2.50)$$

$$h_{t+1} = c_t \quad (2.51)$$

The steady-state of this optimal allocation is given by

$$k^* = (A \alpha \beta)^{\frac{1}{1-\alpha}} \quad (2.52)$$

$$d^* = \left(\frac{1 - \alpha \beta}{\alpha \beta} \right) k^* \times \quad (2.53)$$

$$c^* = \frac{\xi \delta (1-\gamma) [1 - \beta(1-b)]}{(\chi + \xi) \{ [1 - \beta(1-b)] [\theta \beta (1 - \beta \gamma) + \delta (1 - \gamma)] + \beta b \eta (1 - \gamma) \}} \frac{\theta \beta (1 - \beta \gamma)}{\delta (1 - \gamma)} d^* \quad (2.54)$$

$$E^* = \frac{\eta \beta (\chi + \xi)}{\delta [1 - \beta(1-b)]} d^* \quad (2.55)$$

$$m^* = \left(\frac{1 - \alpha\beta}{\alpha\beta} \right) k^* - \frac{\theta\beta(1 - \beta\gamma) + \delta(1 - \gamma)}{\delta(1 - \gamma)} d \quad (2.56)$$

where starred variables denote the optimal outcome. Equation (2.52) is the modified golden rule such that the optimal steady-state capital stock level is the same as in the standard overlapping generation model. The allocation of consumption between young and old and the level of environmental quality at the steady-state are however strongly influenced by the main parameters of the model. It can be seen that the steady-state capital stock level is not necessarily higher in the optimal case. This can be translated as a condition on the social discount factor. In fact, $k^* > \bar{k}$ if and only if:

$$\beta > \frac{\delta[\xi - (\chi + \xi)\alpha](1 - \gamma)}{\xi(1 - \gamma)[1 - \eta(1 - b)] + \theta\xi\gamma} \quad (2.57)$$

If the previous condition is not fulfilled, it would be necessary to tax capital contrary to the case without pollution studied by de la Croix and Michel (1999) where investment always needs to be subsidied. This is due to the fact that in the competitive case, agents have an incentive to increase savings in order to avoid a relatively low environmental quality when old. However, by not taking into account the externality that old-age consumption exerts on environmental quality, agents might push the steady-state capital stock level away from the modified golden rule.

It is also interesting to compare this solution with the no discounting case equivalent to $\beta = 1$. In this case, the steady-state optimal solution does not depend on the habit parameter γ as can be seen from the following expressions:

$$k^* = (A\alpha)^{\frac{1}{1-\alpha}} \quad (2.58)$$

$$d^* = \frac{\xi\delta(1 - \alpha)}{\alpha(\chi + \xi)} k^* \quad (2.59)$$

$$c^* = \frac{\xi\theta(1 - \alpha)}{\alpha(\chi + \xi)} k^* \quad (2.60)$$

$$E^* = \frac{\xi\eta(1 - \alpha)}{\alpha b} k^* \quad (2.61)$$

$$m^* = \left(\frac{1 - \alpha}{\alpha} \right) \left[1 - \frac{\xi(1 - \eta)}{\chi + \xi} \right] k^* \quad (2.62)$$

This result implies that if generations are treated equally by the planner, habits will have no influence on the steady-state allocation of consumption between young and old generations and on the environmental quality level. The steady-state capital stock being equal to the golden rule. In this case, the elimination of the discount factor compensates for the distortions generated by habits such as the overconsumption of the young as well as environmental degradation. This is however only true for the steady-state case and not for the transitional dynamics as can be seen from the first order conditions (2.43), (2.44) and (2.45).

An important question related to the previous result is how could our optimal solution converge to this outcome which partly eliminates the distortions linked to

habits. This is not possible if we don't modify our objective function given that the latter would not be bounded anymore and convergence could not be guaranteed. A potential solution has been suggested by Ramsey (1928) and extended by Michel (1990) more recently. The idea is to fix the rate of time preference at a rate equal to the growth rate of population. In this case, the planner does not discount future utilities and treat all generations equally. In the present case, with no population growth, the rate of time preference should be equal to zero. Michel (1990) shows that a formulation allowing to obtain this result corresponds to the original Ramsey problem. In the present case, we obtain the following welfare function:

$$\sum_{t=0}^{\infty} [\theta \ln(c_t - \gamma h_t) + \delta \ln(d_{t+1}) + \eta \ln(E_{t+1}) - \hat{U}] \quad (2.63)$$

where $\hat{U}$ is the maximal utility achievable by a generation and is defined by:

$$\hat{U} = \max U(c, d, E) \quad (2.64)$$

subject to

$$Ak^\alpha = c + d + m + k \quad (2.65)$$

$$Eb = \xi m - \chi(c + d) \quad (2.66)$$

$$h = c \quad (2.67)$$

We should then replace our discounted welfare function by the objective function (2.63) which allows to obtain the solution where $\beta = 1$.

Turning now to the dynamics of the optimal solution, the number of endogenous variables implies that the dimension of the corresponding Jacobian matrix does not allow us to solve analytically for the eigenvalues in order to assess the stability of the system so that we have decided to rely on numerical methods. However, our objective here is not to describe completely the dynamics of the model but to show that for realistic values of the parameters, endogenous fluctuations can still be present implying the breakdown of sustainability criteria even in the optimal case. In general, optimal solutions are characterized by monotonic convergence in the one-sector overlapping generation model. This is however only true if the utility function is separable accross generations and periods of life. When the utility function is non-separable accross periods of life, Michel and Venditti (1996) have shown that optimal paths can be characterized by endogenous fluctuations under the form of optimal two-cycles. Concerning, non-separability accross generations which is also the case in the present framework, de la Croix and Michel (1999) have shown that local converging oscillations are a possible outcome in the optimal case. We will now proceed with our numerical simulations in two cases: with and without a social discount factor.

Table 1: Value for the parameters

Parameter	Value
A	10
α	0.33
β	0.8 or 1
θ	0.6
δ	0.25
η	0.15
ξ	0.2
χ	0.4
b	0.2
γ	0.65

Most of the parameters do not need further comments. The reader should just notice that in order to respect the necessary condition for a positive steady-state in the competitive equilibrium case, the ratio χ/ξ should be lower than 2.33 given that $\alpha = 0.33$ so that we choose a ratio equal to 2 with $\xi = 0.2$ and $\chi = 0.4$. We also choose to focus on a quite high levels of habits with $\gamma = 0.65$ which increases the probability to obtain endogenous fluctuations. We simulate our model during 200 periods in each case. The results concerning the eigenvalues for the case where $\beta = 0.8$ are given in Table 2.

Table 2: Eigenvalues $\beta = 0.8$

Modulus	Real	Imaginary
1.038e-16	-1.038e-16	0
0.6036	0.5763	0.1795
0.6036	0.5763	-0.1795
2.071	1.977	0.6159
2.071	1.977	-0.6159

In this case the sytem exhibits damped oscillations and thus convergence to the steady-state implying that intergenerational inequalities are present in the short run. We will now simulate the model when $\beta = 1$ in order to see if the no discounting case can generate monotonic convergence toward the steady-state. The results are given in Table 3.

Table 3: Eigenvalues $\beta = 1$

Modulus	Real	Imaginary
3.987e-15	-3.987e-15	0
0.6504	0.6385	0.124
0.6504	0.6385	-0.124
1.538	1.509	0.2932
1.538	1.509	-0.2932

As before, we are in the presence of damped oscillations such that even in the no discounting case, the optimal solution can exhibit intergenerational inequalities in the short run implying the breakdown of several sustainability criteria such as non-decreasing utility.

2.6 Decentralizing the optimal solution

A policy intervention should focus on three objectives: adjust adult's consumption correcting for habits, adjust savings in order to reach the modified golden rule and adjusting old age consumption correcting for pollution accumulation. As we will see shortly, these objectives can be reached by subsidizing or taxing savings and by subsidizing maintenance investment. A different policy based for example on a value-added tax could be possible, however, in order to be able to compare our policy to the case without pollution studied in de la Croix and Michel (1999) we choose to rely on a similar policy. Let's denote the gross investment subsidy by i_t such that $i_t - 1$ is the net subsidy. A negative net subsidy would imply a taxation on savings. The gross maintenance investment subsidy is l_t such that as before the net one is $l_t - 1$. We also use lump-sum transfers to the adult, g_t , and to the old, n_t . The maximization problem of the individual becomes

$$\max_{c_t, d_{t+1}, s_t, m_t} \theta \ln(c_t - \gamma h_t) + \delta \ln(d_{t+1}) + \eta \ln(E_{t+1}) \quad (2.68)$$

subject to

$$\begin{cases} w_t + g_t = c_t + s_t + m_t \\ d_{t+1} = R_{t+1}i_{t+1}s_t + n_{t+1} \\ E_{t+1} = (1 - b)E_t - \chi(c_t + d_t) + \xi l_t m_t \\ c_t, d_{t+1}, s_t, m_t \geq 0 \end{cases}$$

We then obtain the following first order conditions:

$$\frac{\delta R_{t+1}}{d_{t+1}} i_{t+1} = \frac{\theta}{c_t - \gamma h_t} - \frac{\eta \chi}{E_{t+1}} \quad (2.69)$$

$$\frac{\eta(\chi + \xi l_t)}{E_{t+1}} = \frac{\theta}{c_t - \gamma h_t} \quad (2.70)$$

Proposition 3:

In order to decentralize the first best solution, investment should be taxed or subsidized while maintenance investment should always be subsidized.

Proof. The gross investment subsidy should satisfy

$$i_{t+1} = 1 + \frac{\beta d_t^*}{\delta} \left(\frac{\beta \theta \gamma}{c_{t+1}^* - \gamma c_t^*} - \frac{\eta \chi}{E_{t+1}^*} \right) \quad (2.71)$$

We cannot ensure that this expression is higher than one such that it might be necessary to tax savings in order to decentralize the optimal solution.

The gross maintenance investment subsidy is

$$l_t = 1 + \frac{E_{t+1}^*}{\eta\xi} \left[\frac{\beta\theta\gamma}{c_{t+1}^* - \gamma c_t^*} + \frac{\delta(1-b)}{d_{t+1}^*} \right] > 1 \quad (2.72)$$

This result imply that the net maintenance investment subsidy $l_t - 1$ is always positive no matter the value of parameters. Contrary to what could be thought the policy is not forward looking since c_{t+1}^* , d_{t+1}^* and E_{t+1}^* can be defined from the first order conditions and the law of motion for environmental quality. $\square$

The decentralization of the first best solution is completed by the following lump-sum transfers:

$$g_t = k_{t+1}^* + c_t^* + m_t^* - w_t^* \quad (2.73)$$

$$n_t = -g_t - (i_t - 1)R_t^*k_t^* - (l_t - 1)m_t^* \quad (2.74)$$

Equation (2.73) ensures that the capital stock is set at the level of the modified golden rule while equation (2.74) is the planner's budget constraint.

The possibility of taxing capital is closely related to the influence of environmental quality on the investment subsidy. As can be seen from equation (2.72) if the future level of environmental quality is too low or if either the relative preference for environmental quality η or the parameter governing the impact of total consumption on environmental quality χ are too large investment should be taxed.

Concerning the social discount factor, a relatively high value for β implies that we discount less the utility of future generations. This induces an increase in the investment subsidy in order to accumulate more capital for future generations. This is coupled with an increase in the maintenance investment subsidy to compensate for future environmental degradation. In any case, the two instruments are always implemented together in order to correct the intergenerational externality linked to old age consumption and adult's overconsumption implied by habits.

The fact that both instruments are implemented together provides an economic intuition concerning the existence of oscillations. The necessary increase in maintenance investment in the optimal case coupled with the existence of habits exerts a negative effect on capital accumulation which can generate cyclical behavior. This is even more the case if the optimal policy suggests that investment should be taxed.

The possibility of taxing investment in the present model is also closely related to the fact that capital is indirectly responsible environmental degradation. Here we do not take into account the possibility of having a clean sector from which capital could also be accumulated. Concerning the behavior of actual economies, the policy would only imply taxing investment devoted to polluting activities such that the message conveyed might be less pessimistic concerning capital accumulation if clean technologies become available.

2.7 Conclusion

In this paper, we have extended the overlapping generations literature by introducing habits in consumption in a model with environmental quality and maintenance

investment. By reducing the initial three dimensional system to a planar one, we were able to greatly simplify the dynamic analysis. In the present case, the steady-state capital stock might be higher or lower compared with the standard Diamond framework depending on the value of the parameters.

Concerning the dynamic behavior of the model, we were able to identify both short and long term fluctuations implying the breakdown of several sustainability criteria which value intergenerational equality. The short term fluctuations arise due to the possibility of converging local oscillations around the steady-state while the long term ones arise through the possibility of limit-cycles with the appearance of a Neimark-Sacker bifurcation.

We also studied the optimal solution which converges to the modified golden rule for the capital stock and is clearly different from the competitive outcome in terms of consumption and environmental quality allocation. Moreover, in the no discounting case, the steady-state solution is independent of the habit parameter. The latter can be reached by using the original Ramsey formulation as the objective of the planner. However, we found evidence suggesting that the optimal outcome can still be characterized by short-run fluctuations even in this case which led us to conclude that if we use a standard welfare function without any sustainability constraint, we are able to solve for the environmental externalities but not for the sustainability issue. We then proceed to derive an optimal policy allowing for the decentralization of the first best solution. This one is characterized by a tax or a subsidy on investment and by a subsidy to maintenance investment.

Further research could try to introduce in this kind of models an appropriate sustainability constraint (or to modify the objective function of the planner) in order to avoid an outcome with persistent intergenerational inequalities.

Other important issues are the possible combination of habits in both consumption and environmental quality and the inclusion of a non-polluting sector from where capital could also be accumulated.

Chapter 3

Discounting, consumption externalities and growth

3.1 Introduction

The present paper explores the implication that individual time preferences can be affected by individual wealth as well as by social forces which are not under the control of the agent. A related idea has already been exploited in the endogenous discounting literature where the discount rate is not considered as constant anymore but can depend on individual consumption (Epstein 1987, Obsfeld 1990, Drugeon 1996, 1998, Das 2003) or individual capital (Schumacher 2009, 2011, Strulik 2012) to cite some examples. The debate concerning the link between wealth and discounting is an old one and can be related to the work of Fisher (1930) who conceived a positive link between individual wealth and the valuation of the future. Empirical evidence has by now confirmed that the discount rate is in fact decreasing in wealth accumulation (Haussman 1979, Lawrance 1991, Samwick 1998, Frederick and al. 2002). There are several possible explanations for this but important ones are the fact that higher wealth is correlated with lower mortality rates (for example through better health standards) allowing for a higher valuation of the future (Fieldling and al. 2009, Grossman 2003, Richards and Barry 1998) and that the ability to enjoy possible future utility gains might be restricted for poorer agents which need to focus on present issues (Becker and Mulligan 1997). The empirical evidence thus suggests that the discount rate should depend on some variable reflecting this wealth effect. Along this line, Schumacher (2009, 2011) builds an optimal growth model where the discount rate is a decreasing function of the individual capital level. Our objective is to build on this idea by introducing external effects and study the implications on both the competitive equilibrium and the optimal outcome. The discount rate might indeed not only depend on variables under the control of the agent. The effect of social forces on individual time preferences was already emphasized by economist such as Rae (1834) and Fisher (1930) who highlighted the potential connections between capital accumulation and culture. On the empirical side, Easterlin (1995) shows that reported satisfaction does not necessarily increase

with economic development suggesting that agents might be affected by their relative position concerning consumption levels. Clark, Oswald and Warr (1996) also present empirical evidence concerning British workers, showing that their satisfaction is inversely related to comparison wage rates suggesting a similar mechanism. Evidence thus indicates that comparison of income or consumption might play an important role in the growth process. Authors such as Shi (1999), Dupor and Liu (2003) or Meng (2006) have built theoretical models where average consumption decreases the agent's lifetime utility, an idea related to jealousy or fashion effects. Our idea is similar and is based on a discount rate increasing in average consumption inducing a lower valuation of the future. However, in our opinion, the influence of average consumption might not be limited to time preferences. In the present article, we would like to introduce a specific type of production externality which is also related to average consumption. The idea being that living standards (reflected through average consumption) determine at least partially individuals' productive efficiency. Concrete examples can be related to the covering of basic needs and better access to health and nutrition standards. The effect is then to increase individual productivities which will have a positive impact on the aggregate production function. This possibility has only been explored in the literature in two contributions by Kehoe, Levine and Romer (1991) and Drugeon (1998). In this way, we allow consumption to play a different role than in standard models where it always acts as an impediment in the growth process. In the present framework, it is actually possible that consumption enhances capital accumulation.

The paper will mostly focus on the implications concerning equilibrium dynamics. In the present case the equilibrium might not be unique (a feature already present in the case without externalities) but moreover we face the possibility of local indeterminacies and local bifurcations. As shown by Woodford (1986), local indeterminacies imply the existence of stationary sunspot equilibria which represent an alternative way to explain economic fluctuations (see the early works of Shell 1977, Azariadis 1981, Azariadis and Guesnerie 1981, Cass and Shell 1983). In the indeterminacy case, two economies with an identical initial capital stock will converge to the same steady-state equilibrium, however, the transitory growth rates depend on the initial value of consumption which can be chosen freely and might be influenced by cultural, social or historical factors which are not related to the economy's fundamentals. This implies the existence of a continuum of equilibrium growth paths leading to the same steady-state equilibrium in the long run. A large part of the indeterminacy literature is based on the assumption of increasing returns or production externalities as can be seen in the works of Benhabib and Farmer (1994, 1999). In our framework, indeterminacy can only arise if discounting externalities are present while the existence of a unique indeterminate steady-state is only possible when both kind of externalities are present and are sufficiently large. Given the presence of production externalities, the model is also able to generate endogenous growth with a unique balanced-growth path which is always indeterminate. Concerning local bifurcations, we study the possible existence of limit-cycles as well as sudden changes in the number of steady-states or the exchange of stability

properties between two steady-states. While limit-cycles can be associated to long run fluctuations around the steady-state, the other two types of bifurcations can be associated to sudden economic take-offs or depressions (Azariadis 1993).

We then proceed with the study of the optimal outcome which shows that on the dynamic side saddle-path stability prevails suggesting that local indeterminacies and endogenous fluctuations entail a welfare loss in the competitive case. Moreover, the comparison between the optimal and the competitive case allows us to design a potential policy. It is shown that the marginal cost of a unit of consumption is higher in the optimal case if production externalities are not too large. This result is directly related to the effect of discounting externalities and implies that the representative agent does not save enough in the competitive case. If the effect of production externalities decreases along the development process, an adequate policy might need to implement a consumption subsidy at low stages of development in order to take advantage of the potential large externalities in the production function and move gradually toward an investment subsidy or a consumption tax policy as the economy develops.

The structure of the paper is the following. Section 2 presents the competitive framework and provides a necessary and sufficient condition for the existence of multiple steady-states. The model with only one type of externality at a time is presented in section 3 while the full model with both type of externalities is at the heart of section 4. Section 5 explores the possibility of endogenous growth while the optimal solution with its steady-state and dynamics properties are presented in section 6. The last section is then devoted to the conclusion.

3.2 The model

The model is a direct extension of the one-sector neoclassical growth model where the economy is composed by a large number of identical individuals, normalized to one and seeking to maximize their intertemporal discounted utility. The discount factor is however endogenous and depends on the historical paths of capital and average consumption. The intertemporal discounted utility function is the following:

$$U(c, k) = \int_0^\infty u(c_t) e^{-\int_0^t \rho(k_\tau, \bar{c}_\tau) d\tau} dt \quad (3.1)$$

where c_t is the level of consumption of the representative agent at time t , $u(c_t)$ is the felicity function, k_t is the individual level of capital, $\bar{c}_t$ is the average level of consumption and $\rho(k, \bar{c})$ is the subjective discount rate function.

Assumption 1:

(i) The utility function $u(c)$ is twice continuously differentiable and has the following properties: $u(c) > 0$, $u'(c) > 0$, $u''(c) < 0$ and $\lim_{c \rightarrow 0} u'(c) = \infty$.

(ii) The discount rate $\rho(k, \bar{c})$ is twice continuously differentiable and has the following properties: $\rho(k, \bar{c}) > 0$, $\rho_k(k, \bar{c}) < 0$, $\rho_{kk}(k, \bar{c}) > 0$ and $\rho_{\bar{c}}(k, \bar{c}) > 0$.

Assumption 1 together with specification (3.1) imply that the intertemporal utility is positively affected by individual capital accumulation. Following the idea of Fisher (1930), an increase in wealth implies a higher valuation of the future. On the contrary, in a way related to Shi (1999) and Dupor and Liu (2003), average consumption affects negatively intertemporal utility by implying a lower valuation of the future. The discount rate function is moreover convex in individual capital. The representative agent accumulates wealth by renting the amount of capital he owns at time t at the rate r_t and by supplying inelastically one unit of labor at the wage rate w_t . Given an initial level of capital k_0 , the representative agent maximizes (1) subject to the following budget constraint:

$$\dot{k}_t = (r_t - \delta)k_t + w_t - c_t \quad (3.2)$$

where δ is the depreciation rate of capital. The single good is produced by a production function $F(k, l, \bar{c})$ which uses capital and labor as inputs and is influenced by average consumption.

Assumption 2:

The production function $F(k, l, \bar{c})$ exhibits constant returns to scale in k and l , is increasing and concave in k , l and increasing in $\bar{c}$.

We can rewrite the production function in intensive form as $f(k_t, \bar{c}_t)$ and we also have $f_k(k_t, \bar{c}_t) > 0$, $f_{\bar{c}}(k_t, \bar{c}_t) > 0$ and $f_{kk}(k_t, \bar{c}_t) < 0$ implying that average consumption affects positively individual productivities. We moreover assume that $f_{k\bar{c}}(k_t, \bar{c}_t) = f_{\bar{c}k}(k_t, \bar{c}_t) > 0$. This assumption is intuitive if we believe that consumption externalities might play an important role in the production process.

Since we assume perfect competition in factor markets, we know that $r_t = f_k(k_t, \bar{c}_t)$ and $w_t = f(k_t, \bar{c}_t) - f_k(k_t, \bar{c}_t)k_t$ so that the budget constraint can be replaced by:

$$\dot{k} = f(k_t, \bar{c}_t) - c_t - \delta k_t \quad (3.3)$$

The fact that the discount rate $\theta_t = \int_0^t \rho(k_\tau, \bar{c}_\tau) d\tau$ is not constant implies that we have to rely on Uzawa's virtual time method. In order to do so we define an implicit function $t = p(\theta)$ where $p' = \frac{1}{\rho(k_\theta, \bar{c}_\theta)}$ and use θ as an alternative independent variable (a virtual time) so that

$$dt = \frac{d\theta}{\rho(k_\theta, \bar{c}_\theta)} \quad (3.4)$$

The new problem is the following:

$$\max_{c_\theta, k_\theta} \int_0^\infty \frac{u(c_\theta) e^{-\theta}}{\rho(k_\theta, \bar{c}_\theta)} d\theta \quad (3.5)$$

subject to:

$$\begin{cases} \dot{k} = \frac{dk_\theta}{d\theta} = \frac{f(k_\theta, \bar{c}_\theta) - c_\theta - \delta k_\theta}{\rho(k_\theta, \bar{c}_\theta)} \\ 0 < c_\theta \leq f(k_\theta) \\ \text{with } k_0 > 0 \text{ given} \end{cases}$$

The present-value Hamiltonian is:

$$\mathcal{H} = \frac{1}{\rho(k_\theta, \bar{c}_\theta)} \left\{ u(c_\theta) e^{-\theta} + \lambda_\theta [f(k_\theta, \bar{c}_\theta) - c_\theta - \delta k_\theta] \right\} \quad (3.6)$$

The first-order necessary conditions are:

$$u'(c_\theta) e^{-\theta} = \lambda_\theta \quad (3.7)$$

$$-\dot{\lambda} = -\frac{\rho_k(k_\theta, \bar{c}_\theta)}{\rho(k_\theta, \bar{c}_\theta)} \mathcal{H} + \lambda_\theta \left[\frac{f_k(k_\theta, \bar{c}_\theta) - \delta}{\rho(k_\theta, \bar{c}_\theta)} \right] \quad (3.8)$$

$$\lim_{\theta \rightarrow \infty} \lambda_\theta k_\theta = 0 \quad (3.9)$$

To eliminate λ_θ , one proceeds by taking the derivative of equation (3.7) with respect to the virtual time variable θ and obtains:

$$u'(c_\theta) e^{-\theta} - u''(c_\theta) e^{-\theta} \dot{c} = -\dot{\lambda} \quad (3.10)$$

By noting that $\dot{c} = \dot{c}/\rho(k_t, \bar{c}_t)$ and $\dot{k} = \dot{k}/\rho(k_t, \bar{c}_t)$ and rearranging, we obtain the following dynamical system at the competitive equilibrium (dropping time subscripts for convenience):

$$\dot{c} = -\frac{u'(c)}{u''(c)} \left[f_k(k, c) - \delta - \rho(k, c) - \frac{\rho_k(k, c)}{\rho(k, c)} \left(\frac{u(c)}{u'(c)} + \dot{k} \right) \right] \quad (3.11)$$

$$\dot{k} = f(k, c) - c - \delta k \quad (3.12)$$

These two equations are similar to the ones obtained by Schumacher (2009, 2011) and Strulik (2012) except that in our case the discount rate and the production function now depend directly on consumption. This will of course affect the steady-state solution and the dynamics of the model.

The steady-state equations are given by:

$$f_k(k, c) - \delta - \rho(k, c) = \frac{\rho_k(k, c)}{\rho(k, c)} \frac{u(c)}{u'(c)} \quad (3.13)$$

$$f(k, c) - \delta k = c \quad (3.14)$$

As can be seen from the last two expressions, both type of consumption externalities play a role in determining the steady-state. Our next step will now consist in studying the existence of positive steady-state equilibria. We first let the equality $f(k, c) - \delta k = c$ define for $f_c - 1 \neq 0$ an implicit function $c = g(k)$ where $g' = -\frac{f_k - \delta}{f_c - 1}$. In the following, we drop the variables on which each function depends, but these remain the same as before.

Proposition 1:

A sufficient condition for the existence of a unique positive competitive equilibrium steady-state is:

$$\begin{aligned} f_{kk} &< \rho_k + \left(\frac{\rho_{kk}}{\rho_k} - \frac{\rho_k}{\rho} \right) (f_k - \delta - \rho) \\ &+ \left(\frac{f_k - \delta}{f_c - 1} \right) \left[f_{kc} - \rho_c - \frac{\rho_k}{\rho} + (f_k - \delta - \rho) \left(\frac{u''}{u'} + \frac{\rho_c}{\rho} - \frac{\rho_{kc}}{\rho_k} \right) \right] \end{aligned} \quad (3.15)$$

$\forall k$ and where $c = g(k)$.

A necessary and sufficient condition for the existence of multiple competitive equilibrium steady-states is that $\exists k > 0$ which solves $f_k - \delta - \rho_k = \frac{\rho_k}{\rho} \frac{u}{u'}$ where $c = g(k)$ and for this value of k , we have:

$$\begin{aligned} f_{kk} &> \rho_k + \left(\frac{\rho_{kk}}{\rho_k} - \frac{\rho_k}{\rho} \right) (f_k - \delta - \rho) \\ &+ \left(\frac{f_k - \delta}{f_c - 1} \right) \left[f_{kc} - \rho_c - \frac{\rho_k}{\rho} + (f_k - \delta - \rho) \left(\frac{u''}{u'} + \frac{\rho_c}{\rho} - \frac{\rho_{kc}}{\rho_k} \right) \right] \end{aligned} \quad (3.16)$$

Proof. We use the steady-state equations (3.13) and (3.14). We are concerned with values of $k < \bar{k}$ where $\bar{k}$ solves $f = \delta k$, so that we can focus on the interval $k \in]0, \bar{k}[$. We then define $G(k) = A(k) - B(k)$ where $A(k) = f_k - \delta - \rho$ and $B(k) = \frac{\rho_k}{\rho} \frac{u}{u'}$ with $c = g(k)$.

We know that $\lim_{k \rightarrow 0} G(k) = \infty$ and $\lim_{k \rightarrow \bar{k}} G(k) = z < 0$ where z is a negative but finite number. Moreover:

$$\begin{aligned} G'(k) &= f_{kk} - \rho_k - \left(\frac{\rho_{kk}}{\rho_k} - \frac{\rho_k}{\rho} \right) (f_k - \delta - \rho) \\ &- \left(\frac{f_k - \delta}{f_c - 1} \right) \left[f_{kc} - \rho_c - \frac{\rho_k}{\rho} + (f_k - \delta - \rho) \left(\frac{u''}{u'} + \frac{\rho_c}{\rho} - \frac{\rho_{kc}}{\rho_k} \right) \right] \end{aligned} \quad (3.17)$$

where $c = g(k)$. A sufficient condition for a unique steady-state is that $G(k) = 0$ only for one k . This is the case when $G'(k) < 0, \forall k$. Multiple steady-state equilibria can only arise if $G(k)$ changes sign after being negative. A necessary and sufficient condition is then that $\exists k$ such that for this value of k , $G(k) = 0$ and $G'(k) > 0$ which gives the desired result. $\square$

Given the characteristics of the function $G(k)$, our model will always feature an odd number of steady-states. The possibility of multiple steady-state equilibria implies that we might face a situation where initial conditions (reflected in the initial level of capital k_0) play a crucial role concerning the steady-state to which the economy will converge in the long run. An economy lagging behind might never be able to catch up implying the existence of what is now known as a poverty trap. This can still be true even if the lagging economy is experimenting faster growth for some time during the transition due to the intrinsic unsustainable nature of its equilibrium path. In order to ensure that the possibility of multiple equilibria is not just a theoretical curiosity, we now give a relatively general example where multiple equilibria can arise.

Example 1: The production function is of the CES type and is given by $F(K, L, c) = [\alpha K^{-\epsilon} + (1 - \alpha)(cL)^{-\epsilon}]^{-1/\epsilon}$ so that the effect of labor on the production function also depends on the consumption level. In intensive form, we obtain $f(k, c) = [\alpha k^{-\epsilon} + (1 - \alpha)c^{-\epsilon}]^{-1/\epsilon}$ where $0 \leq \alpha \leq 1$ and $\epsilon \geq -1$. The utility function is given by $u(c) = c^{1-\sigma}/(1-\sigma)$ where $\sigma < 1$ while the discount function is of the Cobb-Douglas form and is given by $\rho(k, c) = k^{-\beta} c^\eta$ where $\beta \geq 0$ and $\eta \geq 0$. Finally we assume

that the depreciation rate of capital is equal to zero. A solution satisfying both steady-state equations is $k = [\alpha + \beta/(1 - \sigma)]^{1/(\eta - \beta)}$. In this case, condition (3.16) is equivalent to $(\beta - \eta)[\alpha + \beta/(1 - \sigma)]^{(\eta - \beta - 1)/(\eta - \beta)} > 0$ which is always satisfied if $\beta > \eta$. In this specific example, if the elasticity of the discount function with respect to capital is larger than the one with respect to consumption, we obtain multiple steady-state equilibria.

In order to see the influence of both type of externalities concerning existence and dynamic behavior, we will first study the model with one externality at a time.

3.3 The model with one externality at a time

3.3.1 The $\rho_c = 0$ and $f_c > 0$ case

We start by considering the sufficient condition (3.15) in the case without externalities affecting the discount rate. The condition becomes:

$$\begin{aligned} f_{kk} < \rho_k + \left(\frac{\rho_{kk}}{\rho_k} - \frac{\rho_k}{\rho} \right) (f_k - \delta - \rho) \\ + \left(\frac{f_k - \delta}{f_c - 1} \right) \left[f_{kc} - \frac{\rho_k}{\rho} + (f_k - \delta - \rho) \frac{u''}{u'} \right] \end{aligned} \quad (3.18)$$

We can notice that the last term between brackets on the right hand side of condition (3.18) is positive so that if $f_c - 1 > 0$, consumption externalities in the production function increase the possibility of facing a unique equilibrium since $f_{kk} < 0$ and $\rho_k < 0$. The economic interpretation of $f_c - 1 > 0$ is that an increase in consumption will enhance capital accumulation contrary to the standard case where consumption acts as an impediment in the growth process. This implies that relatively large production externalities might help economies to escape the poverty trap induced by endogenous discounting. Schumacher (2009) has shown that an exogenous increase in productivity might allow economies to escape poverty traps when the discount rate is decreasing in capital accumulation. By increasing individual productivities, average consumption plays a similar role in the present framework. The increase in production due to average consumption pushes the capital stock upwards and by extension reduces the discount rate. This in turn creates an incentive toward further capital accumulation.

Now, if $f_c - 1 < 0$, the conclusion is reversed and relatively low consumption externalities in the production function coupled with a discount rate decreasing in capital accumulation might result more easily in multiple steady-state equilibria. In this case, an increase in consumption still exerts a negative impact on capital accumulation and thus does not imply a decrease in the discount rate. Notice that the result would be equivalent if $f_c < 0$ such that the presence of negative production externalities would also increase the probability of facing multiple equilibria.

We will now proceed with the study of the dynamic behavior of the model in this case. In order to do so, we linearize our dynamical system around the steady-state

(c^*, k^*) . We then obtain:

$$\begin{pmatrix} \dot{c} \\ \dot{k} \end{pmatrix} = \begin{pmatrix} \frac{\partial \dot{c}}{\partial c} & \frac{\partial \dot{c}}{\partial k} \\ \frac{\partial \dot{k}}{\partial c} & \frac{\partial \dot{k}}{\partial k} \end{pmatrix} \Big|_{\dot{c}=0, \dot{k}=0} \begin{pmatrix} c - c^* \\ k - k^* \end{pmatrix}$$

where

$$\frac{\partial \dot{c}}{\partial c} \Big|_{\dot{c}=0, \dot{k}=0} = \rho + \delta - f_k - \frac{u'}{u''} \left(f_{kc} - \frac{\rho_k}{\rho} f_c \right) \quad (3.19)$$

$$\frac{\partial \dot{c}}{\partial k} \Big|_{\dot{c}=0, \dot{k}=0} = -\frac{u'}{u''} \left[f_{kk} - \rho_k - \frac{\rho_k}{\rho} (f_k - \delta) - \left(\frac{\rho_{kk}\rho - \rho_k^2}{\rho^2} \right) \frac{u}{u'} \right] \quad (3.20)$$

$$\frac{\partial \dot{k}}{\partial c} \Big|_{\dot{c}=0, \dot{k}=0} = f_c - 1 \quad (3.21)$$

$$\frac{\partial \dot{k}}{\partial k} \Big|_{\dot{c}=0, \dot{k}=0} = f_k - \delta \quad (3.22)$$

In this case, the trace and the determinant of the Jacobian matrix are given by:

$$Tr(J) = \rho - \frac{u'}{u''} \left(f_{kc} - \frac{\rho_k}{\rho} f_c \right) \quad (3.23)$$

$$\begin{aligned} Det(J) &= (f_k - \delta) \left[\rho + \delta - f_k - \frac{u'}{u''} \left(f_{kc} - \frac{\rho_k}{\rho} f_c \right) \right] \\ &\quad + \frac{u'}{u''} (f_c - 1) \left[f_{kk} - \rho_k - \left(\frac{\rho_{kk}\rho - \rho_k^2}{\rho^2} \right) \frac{u}{u'} \right] \end{aligned} \quad (3.24)$$

A first thing to be noticed is that if f_c and f_{kc} take positive values, indeterminacy is not a possible outcome of the model since the trace can only take positive values. Positive production externalities and an increasing interest rate in consumption exert a destabilizing effect on the dynamical system by pushing upward the trace of the Jacobian matrix and thus not allowing for indeterminacy in this version of the model. We thus obtain the following proposition.

Proposition 2.1:

(i) Consider the case where $f_c - 1 > 0$, if at the steady-state:

$$\begin{aligned} f_{kk} &> \rho_k + \left(\frac{\rho_{kk}}{\rho_k} - \frac{\rho_k}{\rho} \right) (f_k - \delta - \rho) \\ &\quad + \left(\frac{f_k - \delta}{f_c - 1} \right) \left[f_{kc} - \frac{\rho_k}{\rho} + (f_k - \delta - \rho) \frac{u''}{u'} \right] \end{aligned} \quad (3.25)$$

Then, the competitive equilibrium is saddle-path stable and unstable otherwise.

(ii) Consider the case where $f_c - 1 < 0$, if at the steady-state:

$$\begin{aligned} f_{kk} &< \rho_k + \left(\frac{\rho_{kk}}{\rho_k} - \frac{\rho_k}{\rho} \right) (f_k - \delta - \rho) \\ &\quad + \left(\frac{f_k - \delta}{f_c - 1} \right) \left[f_{kc} - \frac{\rho_k}{\rho} + (f_k - \delta - \rho) \frac{u''}{u'} \right] \end{aligned} \quad (3.26)$$

Then, the competitive equilibrium is saddle-path stable and unstable otherwise.

Remembering the results from proposition 1, we can see that in the case of a unique steady-state, if $f_c - 1 > 0$, this unique equilibrium will be unstable while in the case of multiple equilibria, instability will alternate with saddle-path stability. In the opposite case, that is when $f_c - 1 < 0$, if we are in the presence of a unique steady-state, this equilibrium will be saddle-path stable while in the multiple equilibria case, saddle-path stability will alternate with instability. When there are no discounting externalities it thus seems necessary to assume moderate production externalities (implying that consumption still acts as an impediment in the growth process) in order to obtain a unique steady-state which is saddle-path stable.

The reasoning concerning this result goes as follows, we start with the situation where $f_c - 1 > 0$. Intuitively, if everyone's consumption (average consumption) is higher on any given date, production will also be higher and since $f_c - 1 > 0$, consumption exerts a positive effect on capital accumulation which decreases the discount rate. This will in turn induce further savings and higher average consumption in the future. In this case, no equilibrium path starting away from the steady-state can locally converge. If $f_c - 1 < 0$, the positive effect of consumption on capital accumulation is not present and the economy will converge to a steady-state.

We will now focus in the next subsection on the case without production externalities.

3.3.2 The $\rho_c > 0$ and $f_c = 0$ case

We now consider condition (3.15) without production externalities. The condition then becomes:

$$f_{kk} < \rho_k + \left(\frac{\rho_{kk}}{\rho_k} - \frac{\rho_k}{\rho} \right) (f_k - \delta - \rho) \quad (3.27)$$

$$+ [f_k - \delta] \left[\rho_c + \frac{\rho_k}{\rho} + (\rho + \delta - f_k) \left(\frac{u''}{u'} + \frac{\rho_c}{\rho} - \frac{\rho_{kc}}{\rho_k} \right) \right]$$

As can be seen from the previous condition, an increase in ρ_c , corresponding to higher discounting externalities will increase the right hand side of expression (3.27) implying that a unique steady-state equilibrium is more probable. By increasing the discount rate, average consumption offsets at least partially the effect of individual capital on the discount rate. Economies with different initial capital levels will have more similar discount rates increasing the possibility of reaching a unique positive steady-state equilibrium.

We can now proceed as before by linearizing the model around the steady-state. We obtain the following expressions for the elements of the Jacobian matrix:

$$\frac{\partial \dot{c}}{\partial c} \Big|_{\dot{c}=0, k=0} = \rho + \delta - f_k + \frac{u'}{u''} \rho_c + \left(\frac{\rho_{kc} \rho - \rho_k \rho_c}{\rho^2} \right) \frac{u}{u''} \quad (3.28)$$

$$\frac{\partial \dot{c}}{\partial k} \Big|_{\dot{c}=0, k=0} = -\frac{u'}{u''} \left[f_{kk} - \rho_k - \frac{\rho_k}{\rho} (f_k - \delta) - \left(\frac{\rho_{kk} \rho - \rho_k^2}{\rho^2} \right) \frac{u}{u'} \right] \quad (3.29)$$

$$\frac{\partial \dot{k}}{\partial c} \Big|_{\dot{c}=0, k=0} = -1 \quad (3.30)$$

$$\frac{\partial \dot{k}}{\partial k} \Big|_{\dot{c}=0, k=0} = f_k - \delta \quad (3.31)$$

In this case, the trace and the determinant are given by:

$$Tr(J) = \rho + \frac{u'}{u''} \rho_c \left(1 - \frac{\rho_k u}{\rho u'} \right) + \frac{u \rho_{kc}}{u'' \rho^2} \quad (3.32)$$

$$\begin{aligned} Det(J) = & -\frac{u'}{u''} \left[f_{kk} - \rho_k - \frac{u}{u'} \left(\frac{\rho_{kk} \rho - \rho_k^2}{\rho^2} \right) \right] \\ & + (f_k - \delta) \left[\rho + \delta - f_k + \frac{u'}{u''} \left(\rho_c + \frac{\rho_k}{\rho} \right) + \frac{u}{u''} \left(\frac{\rho_{kc} \rho - \rho_k \rho_c}{\rho^2} \right) \right] \end{aligned} \quad (3.33)$$

We can now formulate a proposition concerning the dynamic behavior of the model in this case.

Proposition 2.2:

If at the steady-state:

$$\rho + \frac{u \rho_{kc}}{u'' \rho^2} < -\frac{u'}{u''} \rho_c \left(1 - \frac{\rho_k u}{\rho u'} \right) \quad (3.34)$$

$$\begin{aligned} f_{kk} & > \rho_k + \left(\frac{\rho_{kk}}{\rho_k} - \frac{\rho_k}{\rho} \right) (f_k - \delta - \rho) \\ & + [f_k - \delta] \left[\rho_c + \frac{\rho_k}{\rho} + (\rho + \delta - f_k) \left(\frac{u''}{u'} + \frac{\rho_c}{\rho} - \frac{\rho_{kc}}{\rho_k} \right) \right] \end{aligned} \quad (3.35)$$

Then, the competitive equilibrium is locally indeterminate and there is a continuum of equilibrium growth paths converging to the same steady-state.

If at the steady-state:

$$\begin{aligned} f_{kk} & < \rho_k + \left(\frac{\rho_{kk}}{\rho_k} - \frac{\rho_k}{\rho} \right) (f_k - \delta - \rho) \\ & + [f_k - \delta] \left[\rho_c + \frac{\rho_k}{\rho} + (\rho + \delta - f_k) \left(\frac{u''}{u'} + \frac{\rho_c}{\rho} - \frac{\rho_{kc}}{\rho_k} \right) \right] \end{aligned} \quad (3.36)$$

Then, the competitive equilibrium is saddle-path stable.

In any other configuration, the steady-state is unstable.

Using once again, the results from proposition 1, we can observe that in the case of a unique steady-state, the equilibrium will be saddle-path stable while if we have multiple equilibria saddle-path stability will alternate with indeterminacy or instability depending on the sign of the trace of the Jacobian matrix. As can be seen from expression (3.32), indeterminacy can only arise if discounting externalities are relatively large implying a negative trace. In this case, discounting externalities play

a stabilizing role by pushing the trace toward negative values.

To see more clearly how indeterminacy works in the present specification, let's consider that we start from an equilibrium growth path. Suppose that when an agent is optimistic about the fact that permanent income will be higher (since the discount rate is decreasing in capital accumulation), he decides to increase its initial consumption level thus jumping onto a new equilibrium path. In our representative agent framework, all agents will behave in the same way thus increasing average consumption. Since $\rho_c > 0$, the discount rate applied to future utility will be higher inducing a lower valuation of the future. If the discounting externality is sufficiently strong, accumulation will shrink and the new equilibrium path can be a convergent one.

In this version of the model however, indeterminacy cannot arise in the case of a unique steady-state, since higher discounting externalities tend to push the determinant toward negative values and thus saddle-path stability. As we will see in the next section, it is the combination of both type of externalities that is able to generate a unique steady-state which is locally indeterminate.

3.4 The full model

We can now proceed with the dynamic analysis of the model when both type of externalities are present by linearizing the system around the steady-state. We obtain the following expressions for the elements of the Jacobian matrix:

$$\begin{aligned} \frac{\partial \dot{c}}{\partial c}|_{\dot{c}=0, \dot{k}=0} &= \rho + \delta - f_k - \frac{u'}{u''} \left(f_{kc} - \rho_c - \frac{\rho_k}{\rho} f_c \right) \\ &\quad + \frac{u}{u''} \left(\frac{\rho_{kc}\rho - \rho_k\rho_c}{\rho^2} \right) \end{aligned} \quad (3.37)$$

$$\frac{\partial \dot{c}}{\partial k}|_{\dot{c}=0, \dot{k}=0} = -\frac{u'}{u''} \left[f_{kk} - \rho_k - \frac{\rho_k}{\rho} \left(f_k - \delta - \frac{u\rho_k}{u'\rho} \right) - \frac{u\rho_{kk}}{u'\rho} \right] \quad (3.38)$$

$$\frac{\partial \dot{k}}{\partial c}|_{\dot{c}=0, \dot{k}=0} = f_c - 1 \quad (3.39)$$

$$\frac{\partial \dot{k}}{\partial k}|_{\dot{c}=0, \dot{k}=0} = f_k - \delta \quad (3.40)$$

In this case, the trace and the determinant are given by:

$$Tr(J) = \rho - \frac{u'}{u''} \left(f_{kc} - \rho_c - \frac{\rho_k}{\rho} f_c \right) + \frac{u}{u''} \left(\frac{\rho_{kc}\rho - \rho_k\rho_c}{\rho^2} \right) \quad (3.41)$$

$$\begin{aligned} Det(J) &= -\frac{u'}{u''} (f_c - 1) \left[f_{kk} - \rho_k - \frac{u}{u'} \left(\frac{\rho_{kk}\rho - \rho_k^2}{\rho^2} \right) \right] \\ &\quad + (f_k - \delta) \left[\rho + \delta - f_k - \frac{u'}{u''} \left(f_{kc} - \rho_c - \frac{\rho_k}{\rho} \right) \right] \\ &\quad + (f_k - \delta) \left(\frac{\rho_{kc}\rho - \rho_k\rho_c}{\rho^2} \right) \frac{u}{u''} \end{aligned} \quad (3.42)$$

We now have a proposition concerning the dynamics of the full-model. We will distinguish between the cases where $f_c - 1 > 0$ and $f_c - 1 < 0$.

Proposition 2.3:

(i) Consider the case where $f_c - 1 > 0$, if at the steady-state:

$$u'' > \frac{u'}{\rho} \left[f_{kc} - \rho_c - \frac{\rho_k}{\rho} f_c - \frac{u}{u'} \left(\frac{\rho_{kc}\rho - \rho_k\rho_c}{\rho^2} \right) \right] \quad (3.43)$$

$$\begin{aligned} f_{kk} < & \rho_k + \left(\frac{\rho_{kk}}{\rho_k} - \frac{\rho_k}{\rho} \right) (f_k - \delta - \rho) \\ & + \left(\frac{f_k - \delta}{f_c - 1} \right) \left[f_{kc} - \rho_c - \frac{\rho_k}{\rho} + (f_k - \delta - \rho) \left(\frac{u''}{u'} + \frac{\rho_c}{\rho} - \frac{\rho_{kc}}{\rho_k} \right) \right] \end{aligned} \quad (3.44)$$

Then, the equilibrium is locally indeterminate and there is a continuum of equilibrium growth paths converging to the same steady-state.

If

$$\begin{aligned} f_{kk} > & \rho_k + \left(\frac{\rho_{kk}}{\rho_k} - \frac{\rho_k}{\rho} \right) (f_k - \delta - \rho) \\ & + \left(\frac{f_k - \delta}{f_c - 1} \right) \left[f_{kc} - \rho_c - \frac{\rho_k}{\rho} + (f_k - \delta - \rho) \left(\frac{u''}{u'} + \frac{\rho_c}{\rho} - \frac{\rho_{kc}}{\rho_k} \right) \right] \end{aligned} \quad (3.45)$$

Then, the steady-state is saddle-path stable and there is a unique equilibrium growth path converging to the steady-state.

In any other case, the equilibrium is unstable.

(ii) Consider the case where $f_c - 1 < 0$, if at the steady-state

$$u'' > \frac{u'}{\rho} \left[f_{kc} - \rho_c - \frac{\rho_k}{\rho} f_c - \frac{u}{u'} \left(\frac{\rho_{kc}\rho - \rho_k\rho_c}{\rho^2} \right) \right] \quad (3.46)$$

$$\begin{aligned} f_{kk} > & \rho_k + \left(\frac{\rho_{kk}}{\rho_k} - \frac{\rho_k}{\rho} \right) (f_k - \delta - \rho) \\ & + \left(\frac{f_k - \delta}{f_c - 1} \right) \left[f_{kc} - \rho_c - \frac{\rho_k}{\rho} + (f_k - \delta - \rho) \left(\frac{u''}{u'} + \frac{\rho_c}{\rho} - \frac{\rho_{kc}}{\rho_k} \right) \right] \end{aligned} \quad (3.47)$$

Then, the equilibrium is locally indeterminate.

If

$$\begin{aligned} f_{kk} < & \rho_k + \left(\frac{\rho_{kk}}{\rho_k} - \frac{\rho_k}{\rho} \right) (f_k - \delta - \rho) \\ & + \left(\frac{f_k - \delta}{f_c - 1} \right) \left[f_{kc} - \rho_c - \frac{\rho_k}{\rho} + (f_k - \delta - \rho) \left(\frac{u''}{u'} + \frac{\rho_c}{\rho} - \frac{\rho_{kc}}{\rho_k} \right) \right] \end{aligned} \quad (3.48)$$

Then, the equilibrium is saddle-path stable.

In any other case, the equilibrium is unstable.

As can be seen immediately, if $f_c - 1 > 0$, the condition for a positive determinant is exactly the same as the condition for the existence of a unique steady-state equilibrium. If we only have one steady-state equilibrium, that the trace of the Jacobian matrix is negative (the reader should notice that this is only possible if discounting externalities are large enough), then it is locally indeterminate since the determinant is positive in this case. In the case of multiple steady-states, the equilibria will alternate between local indeterminacy or instability depending on the sign of the trace and saddle-path stability. From both inequalities we can notice that the condition on the trace is equivalent to a restriction on the concavity of the utility function while the condition on the determinant is equivalent to a restriction on the concavity of the production function. While the utility function should not be too concave in order to ensure a negative trace, the production function should be concave enough in capital in order to obtain multiple equilibria coupled with local indeterminacies.

If $f_c - 1 < 0$, we are back to a more standard result where in the case of a unique steady-state, saddle-path stability will prevail. Concerning multiple steady-states, saddle-path stability will alternate with indeterminacy or instability. Production and discounting externalities thus play a fundamental role concerning the dynamic behavior of the model. If production externalities imply that an increase in consumption has a positive effect on capital accumulation, indeterminacy might appear for odd steady-states while in the opposite case indeterminacy might appear for even steady-states provided that discounting externalities are large enough so that the trace of the Jacobian matrix takes negative values. The existence of a unique indeterminate steady-state thus requires relatively large external effects concerning both the production function and the discount rate.

In order to see how indeterminacy works in the full model, we start as before from an equilibrium path. Suppose that an optimistic agent about the fact that is permanent income will be higher (since $\rho_k < 0$) decides to increase its initial consumption level thus jumping onto a new equilibrium growth path. In our representative agent model, this will increase production as well as the discount rate through the average consumption channel. Without the influence of discounting externalities, this equilibrium would be unstable due to over-accumulation implied by a decreasing discount rate and an increasing production level. However, in this case, discounting externalities play a stabilizing role by inducing the agent to value less future consumption and increasing present one. This will lead to a reversal of over-accumulation and the new equilibrium path can be a convergent one. Indeterminacy is thus the result of two opposing forces: while ρ_k and f_c act as a destabilizing force, ρ_c acts as a stabilizing one. This also explains why without the inclusion of discounting externalities, the model can only generate saddle-path stable or unstable steady-states.

As can be noticed from our previous results, the importance of production externalities plays a crucial role concerning the possibility of indeterminacy. Large production externalities (such that $f_c - 1 > 0$) might in fact be compatible only with increasing returns to scale. In order to tackle this issue we will work with a CES production

function with increasing returns. Our objective is to observe in which case increasing returns are needed in order to ensure that $f_c - 1 > 0$ at the steady-state. The production function is the following:

$$F(K, L, c) = [\alpha K^{-\epsilon} + (1 - \alpha)(Lc)^{-\epsilon}]^{-\frac{\tau}{\epsilon}}$$

with $0 \leq \alpha \leq 1$, $\epsilon \geq -1$ and $\tau \geq 1$ is the degree of increasing returns. Once again, in accordance with our assumption, the effect of labour on the production function also depends on the consumption level. In intensive form, we obtain:

$$f(k, c) = [\alpha k^{-\epsilon} + (1 - \alpha)c^{-\epsilon}]^{-\frac{\tau}{\epsilon}}$$

Using the second steady-state equation (3.14), we know that at the steady-state:

$$\alpha k^{-\epsilon} + (1 - \alpha)c^{-\epsilon} = (c + \delta k)^{-\frac{\epsilon}{\tau}} \quad (3.49)$$

Concerning the marginal productivity of consumption, it is given by:

$$f_c(k, c) = \tau[\alpha k^{-\epsilon} + (1 - \alpha)c^{-\epsilon}]^{-\frac{(\tau+\epsilon)}{\epsilon}} (1 - \alpha)c^{-(\epsilon+1)} \quad (3.50)$$

$$f_c(k, c) = \tau(c + \delta k)^{\frac{\tau+\epsilon}{\tau}} (1 - \alpha)c^{-(\epsilon+1)} \quad (3.51)$$

The elasticity of substitution is given by $\mu = 1/(1 + \epsilon)$ and we start with $\epsilon = -1$ so that k and c are perfect substitutes. In this case, $f_c > 1$ if:

$$\tau(c + \delta k)^{\frac{\tau-1}{\tau}} (1 - \alpha) > 1 \quad (3.52)$$

The previous condition implies that increasing returns are necessary in this case in order to obtain large production externalities. This can be confirmed by rewriting the previous condition without increasing returns such that $\tau = 1$. In this case the condition is equivalent to $1 - \alpha > 1$ which is ruled out by assumption. We now set $\epsilon = +\infty$ so that k and c are perfect complements and compute a similar condition:

$$\tau(1 - \alpha) \lim_{\epsilon \rightarrow +\infty} (c + \delta k)^{\frac{\tau+\epsilon}{\tau}} > \lim_{\epsilon \rightarrow +\infty} c^{(\epsilon+1)} \quad (3.53)$$

In this case, increasing returns are not necessary to obtain large production externalities. Once again we can confirm this by rewriting the condition without increasing returns. The condition becomes:

$$(1 - \alpha) \lim_{\epsilon \rightarrow +\infty} (1 + \frac{\delta k}{c})^{(\epsilon+1)} > 1 \quad (3.54)$$

which is always true if $\delta k/c > 0$. We can thus conclude that in order to avoid large increasing returns and still obtain large consumption externalities, a certain degree of complementarity between k and c is necessary in the present framework. The result is particularly intuitive since in the case of complementarity, consumption plays an important role in the production process implying that its marginal impact can be large while it is actually the contrary when capital and consumption are closed substitutes.

A last point concerning transitional dynamics in the present model is that they might not exhibit a monotonic behavior but possibly an oscillatory one. Complex dynamics of this sort will only occur if we are in the presence of a pair of complex eigenvalues. A direct implication is that the following condition should be fulfilled: $Tr(J)^2 < 4Det(J)$. Since the determinant should be positive, we can only observe this situation in the case of indeterminacy or instability. The dynamical system will then exhibit local oscillations converging or not to the steady-state equilibrium.

The dynamic analysis concerning hyperbolic steady-state equilibria is now complete. However, in the present model, the steady-state might not always be hyperbolic. In this particular case, we cannot use a linear approximation in order to establish the stability of our dynamical system. Non-hyperbolic fixed points are nonetheless natural candidates for the detection of local bifurcations which is the objective of the forthcoming analysis. In fact, the present model is able to give rise to different kind of bifurcations. We should first focus on the possible existence of a limit-cycle through the appearance of a Hopf bifurcation. The dynamical system will then not converge to a steady-state but undergo permanent oscillations around the steady-state. Mathematically, a Hopf bifurcation appears if we are in the presence of a pair of complex eigenvalues with zero real part and the respective crossing condition of the imaginary axis at non-zero speed is fulfilled. The previous conditions imply that the trace of our Jacobian matrix should be equal to zero and the determinant should be positive. The following proposition applies the Hopf bifurcation theorem to the present dynamical system.

Proposition 3.1:

If $f_c - 1 > 0$, assume that the dynamical system is parametrized by γ where $\hat{\gamma}$ is defined from

$$\rho = \frac{u'}{u''} \left(f_{kc} - \rho_c - \frac{\rho_k}{\rho} f_c \right) - \frac{u}{u''} \left(\frac{\rho_{kc}\rho - \rho_k\rho_c}{\rho^2} \right) \quad (3.55)$$

$$\begin{aligned} f_{kk} < & \rho_k + \left(\frac{\rho_{kk}}{\rho_k} - \frac{\rho_k}{\rho} \right) (f_k - \delta - \rho) \\ & + \left(\frac{f_k - \delta}{f_c - 1} \right) \left[f_{kc} - \rho_c - \frac{\rho_k}{\rho} + (f_k - \delta - \rho) \left(\frac{u''}{u'} + \frac{\rho_c}{\rho} - \frac{\rho_{kc}}{\rho_k} \right) \right] \end{aligned} \quad (3.56)$$

Let $(\hat{c}, \hat{k})$ be the fixed point defined by $\hat{\gamma}$. At $\hat{\gamma}$, the dynamical system admits a pair of complex eigenvalues $\{\mu, \bar{\mu}\}$ and for $d\mathcal{R}(\mu)/d\gamma|_{\gamma=\hat{\gamma}} \neq 0$, there is a neighborhood U of $\hat{\gamma}$ for which either for $\gamma < \hat{\gamma}$ or $\gamma > \hat{\gamma}$ a closed invariant curve Γ encircles $(\hat{c}, \hat{k})$. If $f_c - 1 < 0$, the second condition becomes

$$\begin{aligned} f_{kk} > & \rho_k + \left(\frac{\rho_{kk}}{\rho_k} - \frac{\rho_k}{\rho} \right) (f_k - \delta - \rho) \\ & + \left(\frac{f_k - \delta}{f_c - 1} \right) \left[f_{kc} - \rho_c - \frac{\rho_k}{\rho} + (f_k - \delta - \rho) \left(\frac{u''}{u'} + \frac{\rho_c}{\rho} - \frac{\rho_{kc}}{\rho_k} \right) \right] \end{aligned} \quad (3.57)$$

and the rest of the proposition follows.

In order to understand how both short run fluctuations and limit-cycles can appear in the present model, let's assume that for a given level of capital k_t , accumulation of capital proceeds. This increase in the capital stock will reduce the discount rate inducing a higher valuation of the future and further capital accumulation. This process will ultimately increase average consumption and push the discount rate and production upwards. If the discounting externalities are sufficiently large, the following increase in consumption will induce a lower valuation of the future and a further increase in average consumption. At a certain point in time, accumulation comes to an end inducing a recession. Once the subsequent reduction in capital and average consumption is strong enough, the incentives for accumulation appear once again and an expansion period can start. This process can be temporary as in the case of converging oscillations or permanent as in the case of a limit-cycle. The opposite effects of capital and average consumption on the discount rate explain the result. The presence of permanent fluctuations is particularly interesting since it is only made possible by the introduction of discounting externalities. Comparisons of consumption levels among agents seem thus to be able to play an important role concerning endogenous fluctuations in the growth process. This fact was already highlighted for example in an overlapping generation setup by de la Croix and Michel (1999) where young agents evaluate their consumption against the one of their parents. In both cases, the mechanism is similar and is based on overconsumption implied by externalities having a depressing effect on capital accumulation. We now present an example compatible with the existence of a Hopf bifurcation.

Example 2: we take once again the functional forms of our first example. In this case, $f_c - 1 < 0$ and the condition concerning the determinant of the Jacobian matrix is equivalent to $(\beta - \eta)[\alpha + \beta/(1 - \sigma)]^{(\eta - \beta - 1)/(\eta - \beta)} > 0$ which is always satisfied if $\beta > \eta$. Concerning the condition on the trace we obtain:

$$(\eta - \sigma) \left(\alpha + \frac{\beta}{1 - \sigma} \right) = (1 - \alpha)[\beta + \alpha(1 + \epsilon)] \quad (3.58)$$

A necessary condition for the last expression to make sense is $\eta > \sigma$ so that the elasticity of the discount function with respect to consumption should be larger than the elasticity of marginal utility (remember that we have assumed a utility function defined over the positive domain so that $\sigma < 1$). From the last expression we can define a value for η so that the condition is satisfied:

$$\eta = \frac{(1 - \sigma)(1 - \alpha)[\beta + \alpha(1 + \epsilon)]}{\alpha(1 - \sigma) + \beta} + \sigma \quad (3.59)$$

This combination of parameters together with the restriction that $\beta > \eta > \sigma$ opens the door to a Hopf bifurcation provided that the respective crossing condition is satisfied.

The present framework can also give rise to a different kind of bifurcation. If one of the eigenvalues of the Jacobian matrix is equal to zero, the system might undergo a saddle-node or a transcritical bifurcation. In the first case, two fixed points collide and annihilate each other such that we observe a change in the number of

steady-states. In the second case, the number of steady-states is constant but when two fixed points collide, they exchange their stability properties. The following proposition applies the saddle-node and the transcritical bifurcation theorems to the present dynamical system. In order to do so we first define the Hessian matrix at the steady-state as H and the matrix M which elements are the partial derivatives of the elements of the Jacobian matrix with respect to γ (where γ parametrizes the dynamical system).

Proposition 3.2:

Assume that the dynamical system is once again parametrized by γ where $\bar{\gamma}$ is defined from

$$\begin{aligned} f_{kk} = & \rho_k + \left(\frac{\rho_{kk}}{\rho_k} - \frac{\rho_k}{\rho} \right) (f_k - \delta - \rho) \\ & + \left(\frac{f_k - \delta}{f_c - 1} \right) \left[f_{kc} - \rho_c - \frac{\rho_k}{\rho} + (f_k - \delta - \rho) \left(\frac{u''}{u'} + \frac{\rho_c}{\rho} - \frac{\rho_{kc}}{\rho_k} \right) \right] \end{aligned} \quad (3.60)$$

- (i) Let $(\bar{c}, \bar{k})$ be the fixed point defined by $\bar{\gamma}$. At $\bar{\gamma}$, the dynamical system admits one eigenvalue $\mu = 0$ and for $\left(\frac{\partial \dot{c}}{\partial \gamma} \right) \big|_{\dot{c}=0, \dot{k}=0, \gamma=\bar{\gamma}} \neq 0$ and $Det(H)|_{\gamma=\bar{\gamma}} \neq 0$, two fixed-points (one saddle and one node) collide and disappear.
- (ii) At $\bar{\gamma}$, the dynamical system admits one eigenvalue $\mu = 0$ and for $\left(\frac{\partial \dot{c}}{\partial \gamma} \right) \big|_{\dot{c}=0, \dot{k}=0, \gamma=\bar{\gamma}} = 0$, $Det(H)|_{\gamma=\bar{\gamma}} \neq 0$ and $Det(M)|_{\gamma=\bar{\gamma}} \neq 0$, two fixed-points collide and exchange their stability properties.

An abrupt change in the number of steady-states or in stability behavior following a smooth change in γ can be an interesting feature in the present framework. This implies that our dynamical system can in a certain way overreact following an external change in environmental conditions. As explained in Azariadis (1993), these kind of phenomena can be associated to abrupt changes such as economic take-offs or depressions. In the present framework where we always have an odd-number of steady-states, we can suppose for a moment that we face a situation with three steady-states. In this case, the presence of a saddle-node bifurcation can allow countries which were stuck in a poverty trap to converge to a higher unique steady-state after the bifurcation takes place. Similarly, a transcritical bifurcation can imply that a poverty trap becomes suddenly unstable thus generating an economic take-off. Using the functional forms of our two examples, we can observe that if the elasticities of the discount function with respect to capital (β) and consumption (η) are equal, then the determinant of the Jacobian matrix is equal to zero so that one of these two bifurcations can potentially arise.

These observations end up the analysis of the dynamics of the model in the stationary case. However, the presence of production externalities also generates the possibility of endogenous growth in the present framework. The study of this potential outcome is the subject of the next section.

3.5 Competitive balanced growth path

By introducing a second reproducible factor of production, consumption externalities generate the possibility of endogenous growth in the present framework.

Definition 1:

A competitive balanced growth path equilibrium of this economy is a solution (c_t, k_t) to equations (3.11) and (3.12) given k_0 , such that consumption and capital grow at the common rate $g > 0$.

In order to ensure the existence of a balanced growth path (BGP), we need to impose some restrictions on our functional forms. We first rewrite our dynamical system in a slightly different form.

$$\frac{\dot{c}}{c} = -\frac{u'(c)}{u''(c)c} \left[f_k(k, c) - \delta - \rho(k, c) - \frac{\rho_k(k, c)k}{\rho(k, c)} \left(\frac{c}{(1-\sigma)k} + \frac{\dot{k}}{k} \right) \right] \quad (3.61)$$

$$\frac{\dot{k}}{k} = \frac{f(k, c)}{k} - \frac{c}{k} - \delta \quad (3.62)$$

From the last two expressions we can derive conditions compatible with the existence of a competitive BGP.

Proposition 4.1:

A set of sufficient restrictions on preferences and technology compatible with the existence of a competitive BGP are:

- (i) $u(c) = (1-\sigma)^{-1}c^{1-\sigma}$ for $\sigma < 1$.
- (ii) $f(k, c)$ is homogenous of degree one in k and c .
- (iii) $\rho(k, c)$ is homogenous of degree zero in k and c .

Proof. From the capital accumulation equation, we notice that since $g_k = g_c = g$ along a BGP, $f(k, c)/k$ should also be constant. This is possible if $f(k, c)$ is homogenous of degree one in k and c leading to

$$\frac{\dot{k}}{k} = f\left(1, \frac{c}{k}\right) - \frac{c}{k} - \delta \quad (3.63)$$

Considering now the consumption accumulation equation, we first notice that since $f(k, c)$ is homogenous of degree one, $f_k(k, c)$ is homogenous of degree zero ensuring a constant marginal productivity of capital along a BGP. $\rho(k, c)$ should also be constant along a BGP implying that $\rho(k, c)$ is restricted to be homogenous of degree zero. The last term between brackets is constant since $g_k = g_c$ and since $\rho_k(k, c)$ is homogenous of degree minus one. Finally, the intertemporal elasticity of substitution should also be constant in order to obtain a constant growth rate for consumption such that the utility function is of the CIES form. We then obtain the following law of motion for consumption.

$$\frac{\dot{c}}{c} = -\frac{u'(c)}{u''(c)c} \left[f_k\left(1, \frac{c}{k}\right) - \delta - \rho\left(1, \frac{c}{k}\right) - \frac{\rho_k\left(1, \frac{c}{k}\right)}{\rho\left(1, \frac{c}{k}\right)} \left(\frac{c}{(1-\sigma)k} + \frac{\dot{k}}{k} \right) \right] \quad (3.64)$$

□

While condition (i) restricts the utility function to the positive domain and ensures a constant intertemporal elasticity of substitution, condition (ii) implies that there are constant returns to scale concerning both reproducible inputs which is a necessary condition in standard endogenous growth models. Finally, condition (iii) implies that the increasing effect of average consumption on the discount rate is totally compensated by the individual capital accumulation effect along the BGP so that the discount rate is constant.

We now define $x = c/k$ such that the balanced growth paths are solutions to $\dot{x}/x = F(x) = 0$ where from expressions (3.61) and (3.62)

$$F(x) = \frac{1}{\sigma} \left[f_k(x) - \delta - \rho(x) - \frac{\rho_k(x)}{\rho(x)} \left(\frac{x}{1-\sigma} + f(x) - x - \delta \right) \right] - (f(x) - x - \delta) \quad (3.65)$$

$F(x) = 0$ can be rewritten as a polynomial of order two in the variable $\rho(x)$:

$$\begin{aligned} F(x) &= -\rho(x)^2 + [f_k(x) - \delta - \sigma(f(x) - x - \delta)]\rho(x) \\ &\quad - \rho_k(x) \left(\frac{x}{1-\sigma} + f(x) - x - \delta \right) \\ &= 0 \end{aligned} \quad (3.66)$$

The previous equation describes an inverted U-shaped parabola for which we have two steady-state solutions since the discriminant is given by:

$$[f_k(x) + \delta - \sigma(f(x) - x - \delta)]^2 - 4\rho_k(x) \left(\frac{x}{1-\sigma} + f(x) - x - \delta \right) > 0 \quad (3.67)$$

The discriminant is always positive given that $\rho_k < 0$ and $g_k > 0$ along a BGP equilibrium. However since $\rho_k < 0$, the product of the roots is negative and these are of opposite sign. We are thus left with a unique BGP solution for which $x > 0$. The dynamic behavior of the BGP equilibrium is given by the sign of $F'(x)$ along the BGP leading to the following proposition:

Proposition 4.2:

Consider the unique BGP solution x^* such that $F(x^*) = 0$:

- (i) if $F'(x^*) > 0$ the BGP equilibrium is locally unstable (determinate)
- (ii) if $F'(x^*) < 0$ the BGP equilibrium is locally stable (indeterminate)
- (iii) if $F'(x^*) = 0$, depending on higher order terms, the BGP equilibrium is either locally stable or unstable.

The BGP is unstable or determinate in the sense that any initial value $x_0 \neq x^*$ will diverge from x^* . For every k_0 , there is a unique choice for c_0 such that $c_0/k_0 = x^*$ and along this unique equilibrium path, consumption and capital grow at the same constant rate g . Now, if the BGP is indeterminate, for a given k_0 , any choice for c_0

will generate an equilibrium path converging to x^* . In this case there is a continuum of equilibrium trajectories converging to the same BGP which is thus stable or indeterminate.

However, in the present case, given that $F(x)$ is an inverted U-shaped parabola, we know that at the BGP equilibrium which corresponds to the second root of the polynomial, $F'(x) < 0$ such that the equilibrium is always indeterminate. In this case, economies with similar fundamentals can save and grow at different rates during the transition and will converge toward the same BGP in the long run.

3.6 The optimal solution

In the competitive case, the representative agent was taking average consumption as given. We now turn to the problem faced by the social planner which can directly influence average consumption and maximizes the discounted welfare function subject to the feasibility constraint. In this case, the discount rate $\theta_t = \int_0^t \rho(k_\tau, c_\tau) d\tau$ is still not constant. However, here we will use dynamic programming approach to solve the problem since this will result in easier computations (notably for the dynamics).

We define the value function

$$V(k_t) = \max_{c_t} U(c_t, k_t) \quad (3.68)$$

subject to:

$$\begin{cases} \dot{k}_t = f(k_t, c_t) - c_t - \delta k_t \\ \text{with } k_0 \text{ given} \end{cases}$$

which leads to the following Hamilton-Jacobi-Bellman equation

$$0 = \max_{c_t} \left\{ u(c_t) - \rho(k_t, c_t)V(k) + V'(k_t)\dot{k} \right\} \quad (3.69)$$

In order to obtain equation (3.69), we first write the value function in the initial period as

$$V(k_0) = \max_{c_t} \int_0^t u(c_s) e^{-\int_0^s \rho(k_\tau, c_\tau) d\tau} ds + V(k_t) \quad (3.70)$$

Taking the derivative of equation (3.70) with respect to t , making use of Leibniz's rule and the differentiability of the value function we obtain our Hamilton-Jacobi-Bellman equation. From now on, we drop the time subscripts for convenience.

The first order necessary condition is

$$u'(c) - \rho_c(k, c)V(k) + V'(k)(f_c(k, c) - 1) = 0 \quad (3.71)$$

while the second order sufficient condition is

$$u''(c) - \rho_{cc}(k, c)V(k) + V'(k)f_{cc}(k, c) < 0 \quad (3.72)$$

Notice that the second-order condition is always satisfied if the discount function is linear in consumption and the production function is concave in consumption.

Differentiating the Bellman equation with respect to k and using the envelope theorem, we obtain

$$-\rho_k(k, c)V(k) + V''(k)\dot{k} + V'(k)(f_k(k, c) - \delta - \rho(k, c)) = 0 \quad (3.73)$$

To obtain the law of motion of consumption in the optimal case we differentiate the first order condition with respect to time, use expression (3.73) and obtain the following dynamical system:

$$\dot{c} = \frac{(1 - f_c(k, c))[\rho_k(k, c)V(k) - V'(k)(f_k(k, c) - \delta - \rho(k, c))]}{u''(c) - \rho_{cc}(k, c)V(k) + V'(k)f_{cc}(k, c)} \quad (3.74)$$

$$+ \frac{[\rho_{ck}(k, c)V(k) + (\rho_c(k, c) - f_{ck}(k, c))V'(k)]\dot{k}}{u''(c) - \rho_{cc}(k, c)V(k) + V'(k)f_{cc}(k, c)}$$

$$\dot{k} = f(k, c) - c - \delta k \quad (3.75)$$

where $V'(k) = u'(c) - \rho_c(k, c)V(k)/(1 - f_c(k, c))$.

The steady-state equilibrium is given by

$$f_k(k, c) - \delta - \rho(k, c) = \frac{\rho_k(k, c)u(c)(1 - f_c(k, c))}{u'(c)\rho(k, c) - u(c)\rho_c(k, c)} \quad (3.76)$$

$$c = f(k, c) - \delta k \quad (3.77)$$

The first steady-state equation is derived by using the value of $V'(k)$ and noting that $V(k) = u(c)/\rho(k, c)$ at the steady-state. We are now able to compare the steady-state equations in the competitive and optimal case. We will restrict our attention to the case of a unique steady-state but distinguish between $f_c - 1 < 0$ and $f_c - 1 > 0$.

Proposition 5.1:

Consider the case where $f_c - 1 < 0$, the steady-state optimal capital stock level k^* is higher than its equilibrium counterpart k if $u(c^*)\rho_c(k^*, c^*) > u'(c^*)\rho(k^*, c^*)f_c(k^*, c^*)$. Consider the case where $f_c - 1 > 0$, the steady-state optimal capital stock level k^* is higher than its equilibrium counterpart k if $u(c^*)\rho_c(k^*, c^*) < u'(c^*)\rho(k^*, c^*)f_c(k^*, c^*)$.

Proof. We start with the first part of the proposition. From the optimal case, we know that

$$\delta = f_k(k^*, c^*) - \rho(k^*, c^*) - \frac{\rho_k(k^*, c^*)u(c^*)(1 - f_c(k^*, c^*))}{u'(c^*)\rho(k^*, c^*) - u(c^*)\rho_c(k^*, c^*)} \quad (3.78)$$

A direct implication of the previous equation is that

$$\delta > f_k(k^*, c^*) - \rho(k^*, c^*) - \frac{\rho_k(k^*, c^*)u(c^*)}{u'(c^*)\rho(k^*, c^*)} \quad (3.79)$$

if $u(c^*)\rho_c(k^*, c^*) > u'(c^*)\rho(k^*, c^*)f_c(k^*, c^*)$. We also know from the steady-state equation of the competitive case that

$$\delta = f_k(k, c) - \rho(k, c) - \frac{\rho_k(k, c)}{\rho(k, c)} \frac{u(c)}{u'(c)} \quad (3.80)$$

If we differentiate equation (3.80) with respect to capital, we obtain our function $G'(k)$ from proposition 1. Remember that $G'(k) < 0$ in the case of a unique steady-state. The implication is that k should be lower than k^* in order to respect the equality between equations (3.78) and (3.80) if $u(c^*)\rho_c(k^*, c^*) > u'(c^*)\rho(k^*, c^*)f_c(k^*, c^*)$. Concerning the second part of the proposition, we proceed in the same way except that now

$$\delta > f_k(k^*, c^*) - \rho(k^*, c^*) - \frac{\rho_k(k^*, c^*)u(c^*)}{u'(c^*)\rho(k^*, c^*)} \quad (3.81)$$

if $u(c^*)\rho_c(k^*, c^*) < u'(c^*)\rho(k^*, c^*)f_c(k^*, c^*)$. This concludes the proof. $\square$

If the inequalities given in the previous proposition are satisfied, the representative agent does not save enough in order to reach the highest possible capital stock. An intuitive policy would then be to subsidize investment. We will however see by comparing first order conditions in both cases that this is not necessarily the end of the story and that along an equilibrium growth path, it might be necessary to subsidize consumption if production externalities are large. But first let's focus on the dynamic characteristics of the model in the optimal case. In order to do so, we proceed as before and linearize the dynamical system around the steady-state (c^*, k^*) .

$$\begin{pmatrix} \dot{c} \\ \dot{k} \end{pmatrix} = \begin{pmatrix} \frac{\partial \dot{c}}{\partial c} & \frac{\partial \dot{c}}{\partial k} \\ \frac{\partial \dot{k}}{\partial c} & \frac{\partial \dot{k}}{\partial k} \end{pmatrix} \Big|_{\dot{c}=0, \dot{k}=0} \begin{pmatrix} c - c^* \\ k - k^* \end{pmatrix}$$

where

$$\frac{\partial \dot{c}}{\partial c} \Big|_{\dot{c}=0, \dot{k}=0} = 0 \quad (3.82)$$

$$\begin{aligned} \frac{\partial \dot{c}}{\partial k} \Big|_{\dot{c}=0, \dot{k}=0} &= \frac{(f_k - \delta)[\rho_{ck}V + (\rho_c - f_{ck})V']}{u'' - \rho_{cc}V + V'f_{cc}} \\ &\quad + \frac{(1 - f_c)[\rho_{kk}V - V''(f_k - \delta - \rho) - V'(f_{kk} - 2\rho_k)]}{u'' - \rho_{cc}V + V'f_{cc}} \end{aligned} \quad (3.83)$$

$$\frac{\partial \dot{k}}{\partial c} \Big|_{\dot{c}=0, \dot{k}=0} = f_c - 1 \quad (3.84)$$

$$\frac{\partial \dot{k}}{\partial k} \Big|_{\dot{c}=0, \dot{k}=0} = f_k - \delta \quad (3.85)$$

As can be seen directly, the trace of the Jacobian matrix is always positive and equal to $f_k - \delta$. The following proposition characterizes the dynamics at the optimal solution.

Proposition 5.2:

Consider a steady-state of the planning economy, if

$$\begin{aligned} f_{kk} &< \frac{\rho_{kk}V - (f_k - \delta - \rho)V'' + 2\rho_kV'}{V'} \\ &\quad - \frac{f_k - \delta}{V'(f_c - 1)}[\rho_{ck}V + (\rho_c - f_{ck})V'] \end{aligned} \quad (3.86)$$

Then, the equilibrium is saddle-path stable and unstable otherwise.

Proof. First notice that $Tr(J) = f_k - \delta > 0$ such that the equilibrium can only be saddle-path stable or unstable.

Concerning the determinant, we obtain

$$\begin{aligned} Det(J) = & -\frac{(f_c - 1)(f_k - \delta)[\rho_{ck}V + (\rho_c - f_{ck})V']}{u'' - \rho_{cc}V + V'f_{cc}} \\ & + \frac{(f_c - 1)^2[\rho_{kk}V - V''(f_k - \delta - \rho) - V'(f_{kk} - 2\rho_k)]}{u'' - \rho_{cc}V + V'f_{cc}} \end{aligned} \quad (3.87)$$

Saddle-path stability implies that the determinant of the Jacobian matrix is negative. Since $u'' - \rho_{cc}V + V'f_{cc} < 0$ from the second order condition, the equilibrium is saddle-path stable if

$$\begin{aligned} & (f_c - 1)(f_c - 1)[\rho_{kk}V - V''(f_k - \delta - \rho) - V'(f_{kk} - 2\rho_k)] \\ & - (f_c - 1)(f_k - \delta)[\rho_{ck}V + (\rho_c - f_{ck})V'] > 0 \end{aligned} \quad (3.88)$$

The previous expression is equivalent to expression (3.85) for both $f_c - 1 > 0$ and $f_c - 1 < 0$. $\square$

This result imply that the inefficiency present in the competitive case can partially be related to the possibility of facing local indeterminacies and endogenous fluctuations (including limit cycles) since the optimal path is unique and characterized by monotonic behavior. The presence of our two types of consumption externalities thus entails a possible welfare loss in the competitive case.

As stated before, we will now compare the first order conditions in both situations in order to highlight the differences between the competitive and the optimal outcome and design a potential policy for the present case. In the competitive case, the first order condition derived from dynamic programming is:

$$u'(c) = V'(k) \quad (3.89)$$

We can compare this last equation with the first order condition in the optimal case given by equation (3.71). Let's first assume that $f_c \rightarrow 0$ so that consumption externalities in the production function are negligible. In this case, the first order condition in the optimal case becomes:

$$u'(c) - \rho_c(k, c)V(k) = V'(k) \quad (3.90)$$

The marginal cost of a unit of consumption is higher due to the effect that consumption exerts on the discount rate implying that agents don't save enough in the competitive case. The solution might then be to subsidize investment in order to foster capital accumulation. In the case in which $f_c > 0$, this observation is not necessarily valid anymore. As we saw before, the first order condition in the optimal case is given by:

$$u'(c) - \rho_c(k, c)V(k) + V'(k)(f_c(k, c) - 1) = 0$$

This implies that the marginal cost of a unit of consumption is higher in the optimal case if

$$\rho_c(k, c)V(k) - f_c(c, k)V'(k) > 0 \quad (3.91)$$

and lower if the previous expression is negative. We can directly see that if f_c is sufficiently large, the expression is negative implying that consumption should be subsidized given its positive impact on the production function. On the contrary, if ρ_c is relatively large, the reverse is true and investment should be subsidized. As argued by Drugeon (1998), it should be expected that production externalities are stronger in developing countries where individual productivities can be more affected by frequent diseases, a lack of health services or problems related to basic needs. In this case, an adequate policy might be to subsidize consumption at low levels of development and gradually move toward an investment subsidy or a consumption tax policy as the economy becomes richer. A public intervention is thus necessary given that the decentralized economy is unable to take advantage of large production externalities at low levels of development while having a tendency to overconsume at later stages of development. It is however necessary to stress some important points concerning the implementation of the public policy. In our simplified model, the presence of a unique consumption good does not allow to distinguish between goods which might increase individual productivities and others that are not so effective in doing so. A potential subsidy should only be directed toward consumption goods which ensure large externalities in production. At low levels of development we can think as before about goods related to health access or basic needs. A related argument can be made concerning the discount rate given that the goods sustaining the jealousy effect might not be the same as the ones affecting the production function. The policy implementation might then be more complicated and costly than what the model seems to suggest given that it is necessary to identify the specific consumption goods that should be taxed or subsidized.

3.7 Conclusion

The present paper has extended the one sector neoclassical growth model by introducing a discount rate decreasing in individual capital and average consumption as an externality increasing both the discount rate and the production function. By studying one externality at a time we have shown that the model without discounting externalities and with large production externalities favors the existence of a unique unstable steady-state. When only discounting externalities are taken into account, we have shown that indeterminacy can only arise in the case of multiple steady-states and for even ones while large discounting externalities favor the existence of a unique steady-state. The study of the full model with both type of externalities however shows that the combination of large production and discounting externalities is necessary in order to obtain a unique steady-state which is indeterminate. Furthermore, we proved that the model is able to generate local bifurcations inducing the possible existence of limit-cycles as well as the possibility of

a sudden economic take-off or depression. Concerning non-stationary environments, we derived conditions compatible with the existence of a unique balanced growth path which in the present framework is always indeterminate. We also established that the differences between the competitive and the optimal outcome are driven by two elements. First, the possibility of local indeterminacies and local bifurcations which are ruled out in the optimal case entail a welfare loss in the competitive case. Second, the competitive equilibrium can be characterized either by a too low or a too high level of consumption depending on the magnitude of the external effects implying the need of an appropriate policy intervention.

As expressed in the last part of the paper, further research could focus on multiple consumption goods which don't affect the discount rate and the production function to the same extent.

Bibliography

Artige L., Camacho C., de la Croix D. (2004). Wealth breeds decline: reversals of leadership and consumption habits. *Journal of Economic Growth* 9, 423-449.

Ayong Le Kama A. and Schubert K. (2007). A note on the consequences of an endogenous discounting depending on the environmental quality. *Macroeconomic Dynamics* 11(2), 272-289.

Azariadis C. (1981). Self-fulfilling Prophecies. *Journal of Economic Theory* 25, 380-396

Azariadis C. (1993). Intertemporal Macroeconomics, Blackwell, Oxford.

Azariadis C. and Guesnerie R. (1981). Sunspots and Cycles. *Review of Economic Studies* 53, 725-736.

Azariadis C. and Starchurski J. (2005). Poverty traps. In Aghion P., Durlauf S. (Eds.), *Handbook of Economic Growth*, vol. 1, Elsevier, North-Holland, Amsterdam.

Barro R. J. and Sala-i-Martin X. (2003). *Economic growth*, second edition, M.I.T. press, Cambridge, Massachusetts.

Becker G. and Mulligan C.B. (1997). The Endogenous Determination of Time Preference. *Quarterly Journal of Economics* 112, 729-758.

Benhabib, J. and Farmer R.E. (1994). Indeterminacy and increasing returns. *Journal of Economic Theory* 63, 1941.

Benhabib, J. and Farmer R.E. (1999). Indeterminacy and sunspots in macroeconomics. In: Taylor, J.B., Woodford, M. (Eds.), *Handbook of Macroeconomics*, vol. 1A. North-Holland, Amsterdam.

Cao D., Wang L. and Wang Y. (2011). Endogenous fluctuations induced by non-linear pollution accumulation in an OLG economy and the bifurcation control. *Economic Modelling* 28, 2528-2531.

Cass, D. and Shell, K. (1983). Do sunspots matter? *Journal of Political Economy* 91, 193-227.

- Chen H.J. and Li M.C. (2011). Environmental tax policy, habit formation and nonlinear dynamics. *Nonlinear Analysis: Real World Applications* 12, 246-253.
- Clark G. (2007). A Farewell to Alms: A brief economic history of the world. Princeton University press, Princeton, New Jersey.
- Clarck A., Oswald A. and Warr P. (1996). Is job satisfaction U-shaped in age? *Journal of Occupational and Organizational Psychology* 69 (1), 57-81.
- Das M. (2003). Optimal growth with decreasing marginal impatience. *Journal of Economics Dynamics and Control* 27, 1881-1898.
- Dasgupta P. and Heal G. (1979). Economic theory and exhaustible resources. Cambridge University Press, Cambridge.
- de la Croix D. (1996). The dynamics of bequeathed tastes, *Economic Letters* 53, 89-96.
- de la Croix D., Michel P. (2002). A Theory of Economic Growth. Cambridge University Press, Cambridge.
- de la Croix D., Michel P. (1999). Optimal growth when tastes are inherited. *Journal of Economic Dynamics and Control* 23, 519-537.
- Drugeon J. P. (1998). A model with endogenously determined cycles, discounting and growth. *Economic Theory* 12, 349-369.
- Drugeon J. P. (1996a). On discount as an engine for growth. Manuscript, C.n.r.s. - M.a.d.
- Drugeon J.P. (1996b). Impatience and long-run growth. *Journal of Economics Dynamics and Control* 20, 281-313.
- Dupor B. and Liu W. F. (2003). Jealousy and equilibrium overconsumption. *American Economic Review* 93, 423-428.
- Easterlin R. (1995). Will raising the incomes of all increase the happiness of all? *Journal of Economic Behavior and Organization* 27(1), 35-46.
- Epstein L.G. (1987). A simple dynamic general equilibrium model. *Journal of Economic Theory* 41, 6895.
- Fielding D., Torres S. and Zealand D. N. (2009). Health, wealth, fertility, education, and inequality. *Review of Development Economics* 13(1), 39-55.
- Fisher I. (1930). The theory of interest. Macmillan, New-York.
- Frederick S., Loewenstein G., and O'Donoghue T. (2002). Time discounting and time preference: a critical review. *Journal of Economic Literature* 40(2), 351-401.

- Galor O. (2005). From stagnation to growth: unified growth theory. In Aghion P., Durlauf S (Eds.), *Handbook of Economic Growth*
- Harrod R. (1948). *Towards a dynamic economy*. Macmillan, London.
- Hausman J.A. (1979). Individual discount rates and the purchase and utilization of energy-using durables. *Bell Journal of Economics* 10, 3354.
- Howarth R. (1997). Sustainability as opportunity. *Land Economics* 73(4) 569-579.
- John A. and Pecchenino R. (1994). An overlapping generations model of growth and the environment. *Economic Journal* 104, 1393-1410.
- Jones C. (1997). On the evolution of the world income distribution. *Journal of Economic Perspectives*, 11, 19-36.
- Jones, L. E. and Manuelli R.E. (1990). A Convex Model of Equilibrium Growth: Theory and Policy Implications. *Journal of Political Economy*, 98, 1008-1038.
- Kaldor N. (1957). A model of economic growth, *The Economic Journal* 67, 591-624.
- Kehoe T., Levine D. and Romer P. (1991). On characterizing equilibria of economies with externalities and taxes as solutions to optimization problems. *Economic Theory* 2, 43-68.
- Lawrence, E.C. (1991). Poverty and the rate of time preference: evidence from panel data. *Journal of Political Economy* 99, 54-75.
- Lucas R. (2002). *Lectures on economic growth*, Harvard University Press, Cambridge, MA.
- Lucas R. (1988). On the mechanics of economic development. *Journal of Monetary Economics* 22, 3-42.
- Meng Q. (2006). Impatience and equilibrium indeterminacy. *Journal of Economics Dynamics and Control* 30, 2671-2692.
- Michel P. (1982). On the Transversality Conditions in Infinite Horizon Problems. *Econometrica* 50, 975-985.
- Michel P. (1990). Criticism of the social time-preference hypothesis in optimal growth. *CORE Discussion Papers* 1990039, Universit catholique de Louvain, Center for Operations Research and Econometrics (CORE).
- Michel P. and Venditti A. (1997). Optimal growth and cycles in overlapping generation models. *Economic Theory* 9(3), 511-528.
- Obstfeld M. (1990). Intertemporal dependence, impatience and dynamics. *Journal of Monetary Economics* 26, 45-75.

- Ono T. (2002). Is habitual consumption harmful to the environment?. *Economics Bulletin* 17, 1-10.
- Palivos T., Wang P. and Zhang J. (1997). On the existence of balanced growth equilibrium. *International Economic Review*, 38(1), 205-223.
- Pezzey J. (2004). Sustainability and environmental policy. *Scandinavian Journal of Economics* 106(2), 339-359.
- Pezzey J. and Toman M. (Eds.) (2002). *The Economics of Sustainability*, Ashgate Publishing.
- Pittel K. (2002). *Sustainability and endogenous growth*, Edward Elgar Publishing, United Kingdom.
- Quah D. (1996). Twin Peaks: Growth and convergence in models of distribution dynamics. *Economic Journal* 106(437), 1045-1055.
- Rae J. (1834). *The Sociology Theory of Capital*, Macmillan, London.
- Ramsey F. (1928). A mathematical theory of savings. *Economic Journal* 38, 543-59.
- Samwick A. (1998). Discount rate homogeneity and social security reform. *Journal of Development Economics* 57, 117-146.
- Shell K. (1997). Monnaie et allocation intertemporelle. Mimeo, CNRS Sminaire Roy-Malinvaut, Paris.
- Schmitt-Grohe S. and Uribe M. (2003). Closing small open economy models. *Journal of International Economics* 61, 163-185.
- Schumacher I. (2011). Endogenous discounting and the domain of the felicity function. *Economic Modelling* 28, 574-581.
- Schumacher I. (2009). Endogenous discounting via wealth, twin-peaks and the role of technology. *Economic Letters* 103, 78-80.
- Schumacher I. and Zou B. (2008). Pollution perception: A challenge for intergenerational equity. *Journal of Environmental Economics and Management* 55, 296-309.
- Seegmuller T. and Verchere A. (2004). Pollution as a source of endogenous fluctuations and periodic welfare in OLG economies. *Economic Letters* 84, 363-369.
- Shi S. (1999). Fashion and wealth accumulation. *Economic Theory* 14, 439-461.
- Stavins R., Wagner A. and Wagner G. (2003). Interpreting sustainability in economic terms: dynamic efficiency plus intergenerational equity. *Economic Letters* 79, 339-343.
- Stern N. (2007). *The economics of climate change: the Stern review*. Cambridge University press, Cambridge.

Strulik H. (2012). Patience and prosperity. *Journal of Economic Theory* 147, 336-352.

Uzawa H. (1968). Time preference, the consumption function, and optimum asset holdings. In J. N. Wolfe (Ed.), *Value, Capital, and Growth*, Papers in honor of Sir John Hicks, Edinburgh, University Press, 485-504.

Wendner R. (2002). Capital accumulation and habit formation. *Economics Bulletin* 4, 1-10.

Woodford M. (1986). Stationary Sunspot Equilibria: the case of small fluctuations around a steady-state, unpublished.

Yanase A. (2011). Impatience, pollution and indeterminacy. *Journal of Economics Dynamics and Control* 35, 1789-1799.

Zhang J. (1999). Environmental sustainability, nonlinear dynamics and chaos. *Economic Theory* 14, 489-500.