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Conditional Mean Risk Sharing of Independent Discrete Losses in Large Pools

Michel Denuit¹ · Christian Y. Robert²

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Abstract

This paper considers a risk sharing scheme of independent discrete losses that combines risk retention at individual level, risk transfer for too expensive losses and risk pooling for the middle layer. This ensures that pooled losses can be considered as being uniformly bounded. We study the no-sabotage requirement and diversification effects when the conditional mean risk-sharing rule is applied to allocate pooled losses. The no-sabotage requirement is equivalent to Efron's monotonicity property for conditional expectations, which is known to hold under log-concavity. Elementary proofs of this result for discrete losses are provided for finite population pools. The no-sabotage requirement and diversification effects are then examined within large pools. It is shown that Efron's monotonicity property holds asymptotically and that risk can be eliminated under fairly general conditions which are fulfilled in applications.

Keywords Risk pooling · Peer-to-Peer (P2P) insurance · Conditional mean risk-sharing · Likelihood ratio order · Log-concavity

Mathematics Subject Classification (2010) 62P05

1 Introduction and Motivation

Peer-to-Peer (P2P) insurance, or crowdsurance refers to emerging, technology-based risk sharing networks where participants pool their resources to mutually protect against the financial consequences of a given peril. Rooted in the sharing economy, it revives early forms of mutual insurance. P2P insurance aims to mitigate the conflict inherent to commercial insurance (where insurers keep the part of premiums that is not paid out for claims) and to lower prices or extend coverability.

✉ Michel Denuit
michel.denuit@uclouvain.be

Christian Y. Robert
chrobert@ensae.fr

¹ Institute of Statistics, Biostatistics and Actuarial Science - ISBA, Louvain Institute of Data Analysis and Modeling - LIDAM, UCLouvain, Louvain-la-Neuve, Belgium

² Laboratory in Finance and Insurance - LFA, CREST - Center for Research in Economics and Statistics, ENSAE, Paris, France

Actuaries started to investigate the mathematics supporting P2P insurance quite recently. See e.g. Abdikerimova and Feng (2022) and Denuit (2019). The main challenge is to fairly distribute total losses among participants within P2P insurance funds aiming at standardization (that is, using general behavioral traits like risk aversion but not individual preferences). Appropriate risk-sharing rules are needed to that end. Inspired by properties that premium principles or risk measures may or may not satisfy, Denuit et al. (2022a) systematically studied the properties that risk-sharing rules could/should satisfy. This resulted in a list of reasonable (not necessarily independent) requirements that a risk-sharing scheme may fulfill. In particular, the no-sabotage (as termed by Carlier and Dana (2003) or comonotonicity property (as termed in Denuit et al. 2022a) appears to be a desirable property since it ensures that the interests of all participants are aligned, in the sense that they all have an interest in keeping their losses as small as possible. Throughout this paper, we consider the conditional mean risk-sharing rule where the contribution paid in arrear by each participant is the conditional expectation of the loss brought to the pool, given the total losses experienced by the entire pool.

The conditional mean risk-sharing rule has been proposed by Denuit and Dhaene (2012) building upon the work by Landsberger and Meilijson (1994). It has been axiomatized by Jiao et al. (2022). The reader is referred e.g. to Blier-Wong et al. (2022); Clemente et al. (2023, 2024); Denuit and Robert (2023); Ghossoub et al. (2024) and Levantesi and Piscopo (2022) for contributions to the study of the conditional mean risk-sharing rule, as well as to Feng (2023) for a general presentation of the topic.

Under pure P2P insurance, participants' contributions are theoretically unlimited. A more effective solution can be obtained by combining self-insurance, or retention by individual participants, P2P pooling of the middle risk layer at the level of a community and risk transfer of the upper layer to an insurance company. By partnering with an established insurance company, the community can benefit not only from its risk-bearing capacity, but also from its experience in claim settlement and the related administration. Denuit (2020) designed such a scheme under the conditional mean risk-sharing rule and Denuit et al. (2023) extended it to any aggregate rule satisfying the no-sabotage or comonotonicity property. In practice, this means that losses brought to the pool are bounded from above, either thanks to the transfer of the upper layer to an insurance company, or by limiting the intervention of the pool to a maximum. Consider for instance cyber risk. According to Faure and Nieuwesteeg (2018), cyber risk pooling offers promising opportunities. An example is the new cyber mutual insurer MIRIS (acronym for "Mutual Insurance and Reinsurance for Information Systems") licensed in 2022 by the Belgian regulator to underwrite direct cyber insurance on behalf of its owner-members in the European Union. MIRIS allocates up to 25 millions euros of capacity to each member for underwriting years 2023 and 2024. The same ceiling of indemnities happens with other types of insurance risk pooling. Thus, we can safely assumed that pooled losses are bounded from above.

In the present paper, we study the no-sabotage requirement and diversification effects obtained when the conditional mean risk-sharing rule is applied to allocate discrete independent losses among participants. This is in line with practice where insurance losses are generally discretized before performing actuarial calculations.

Under the conditional mean risk-sharing rule, the no-sabotage requirement is equivalent to Efron's monotonicity for conditional expectations given sums. The latter is known to hold true under log-concavity. We provide an elementary proof of this classical result for discrete distributions (since existing proofs rely on the basic composition formula and deal with probability density functions). The no-sabotage property is also carefully studied in large pools where log-concavity is recovered thanks to local limit theorems. We thus provide results

comparable to those obtained in Denuit and Robert (2021b) where the loss distributions have been assumed to be absolutely continuous.

We also study diversification effects obtained with the conditional mean risk-sharing rule. Diversification effects of this risk-sharing rule when the size of the pool becomes larger and larger have already been documented in Denuit and Robert (2021a) for absolutely continuous losses or when the losses have compound Poisson distributions for which the claim severities are positive and absolutely continuous. Results have also been provided when losses are given by independent Bernoulli random variables multiplied by a capital belonging to a predefined menu in Denuit et al. (2022b). But no results seem to be available for general discrete distributions. We show that insurance at fair price can be recovered within large pools, under fairly general conditions which are fulfilled in applications to P2P insurance.

The remainder of this paper is organized as follows. Section 2 presents the 3-layer scheme studied in this paper where risk pooling operates in the middle layer. This ensures that insurance losses to be pooled are uniformly bounded from above. It also recalls the definition of the conditional mean risk-sharing rule. Section 3 presents the no-sabotage property and then discusses the likelihood ratio order and the log-concavity concept. This section also provides the reader with an elementary proof of Efron's monotonicity property for counting random variables. Log-concavity appears to be a strong requirement since it constrains in particular the loss distribution to be light-tailed. Section 4 is dedicated to large pools. It is proved that the no-sabotage property is generally recovered within large pools thanks to local limit theorems. The latter are also exploited to obtain risk elimination under fairly general conditions. Section 5 applies these results to losses obeying Binomial distributions with varying success probabilities, recovering as particular cases results similar to some of those derived in Denuit et al. (2022b) for binary losses. Section 6 offers numerical illustrations within an hypothetical P2P insurance pool. The final Section 7 summarizes the main findings of the paper and discusses some of the results.

2 Risk Retention, Transfer and Sharing

2.1 Collaborative System and Pooled Losses

As explained in the introduction, this paper considers a collaborative system built on self-insurance, or retention by individual participants, P2P pooling of the middle risk layer at the level of a community and risk transfer of larger losses to an insurance company. Let us briefly explain how this 3-layer system operates.

Consider n individuals, numbered $i = 1, 2, \dots, n$. Each of them is exposed to some peril causing a random non-negative monetary loss at the end of a given observation period, taken to be the time interval $(0, 1)$. P2P insurance is not costless to operate so that running fees supplement participants' contributions. Therefore, some participants may consider retaining some risk, to save money. This may also be imposed by the sponsor, to guarantee that participants have interest to keep their losses under control. In practice, this is often achieved by applying a deductible or a quota-share arrangement. Let $Y_i \geq 0$ be the loss for participant i , in excess of individual retention.

In liability insurance for instance, some large losses may happen which exceed the risk-sharing capacities of the P2P insurance community. It is therefore useful to limit the maximum loss per participant, to some value m . This means that $\min\{Y_i, m\}$ is pooled and that $(Y_i - m)_+$ is transferred to the partnering insurer through an excess-of-loss cover and does not enter risk pooling. The corresponding premium must be supported directly by participant i , in addition to his or her own contribution to the pool.

From now on, let $X_i = \min\{Y_i, m\}$ be the amount of losses pooled by participant i . Thus, $X_i \leq m$ holds for all i , where m corresponds to the priority of the yearly excess-of-loss cover limiting the amount entering pooling. Moreover, we assume that $X_1, X_2, \dots$ are independent and valued in $\{0, 1, 2, \dots, m\}$.

2.2 Conditional Mean Risk-sharing Rule

Henceforth, we use the notation $S_n = \sum_{i=1}^n X_i$ for the total risk of the pool. In a risk pooling scheme, each participant in a pool of size n contributes an amount $h_{i,n}(s)$ where $s = \sum_{i=1}^n x_i$ is the sum of the observed realizations of $X_1, X_2, \dots, X_n$.

Denuit and Dhaene (2012) studied the conditional mean risk allocation $h_{i,n}^{\text{cmrs}}$ defined as

$$h_{i,n}^{\text{cmrs}}(S_n) = E[X_i | S_n], \quad i = 1, 2, \dots, n. \quad (2.1)$$

Clearly, the conditional mean risk sharing (2.1) allocates the full risk S_n as we obviously have

$$\sum_{i=1}^n h_{i,n}^{\text{cmrs}}(S_n) = \sum_{i=1}^n E[X_i | S_n] = S_n.$$

In words, participant i must contribute the expected value of the risk X_i brought to the pool, given the total loss S_n . This rule is obviously fair in the actuarial sense as the expected contribution is equal to the expected loss, that is, $E[h_{i,n}^{\text{cmrs}}(S_n)] = E[X_i]$ for all $i = 1, 2, \dots, n$.

3 No-sabotage Condition

In the context of risk sharing, it is important that the interests of all participants are aligned, in the sense that they all have an interest in keeping their losses as small as possible. Under the conditional mean risk-sharing rule, this means that all functions $s \mapsto h_{i,n}^{\text{cmrs}}(s)$ are non-decreasing. This is referred to as the no-sabotage condition, after Carlier and Dana (2003) and renders individual contributions $h_{i,n}^{\text{cmrs}}(S_n)$ perfectly positively dependent (or comonotonic) since they are all non-decreasing functions of the random variable S_n . It implies that no individual contribution $h_{i,n}^{\text{cmrs}}(S_n)$ is allowed to increase more than the total losses S_n .

3.1 Likelihood Ratio Order

The likelihood ratio order plays a central role in the study of the no-sabotage condition. Recall that given two counting random variables V and W , V is said to be smaller than W in the likelihood ratio order, which is henceforth denoted as $V \leq_{\text{LR}} W$, when the ratio of their respective probability mass functions $P[W = k]/P[V = k]$ is non-decreasing in $k \in \{0, 1, \dots\}$. Equivalently,

$$V \leq_{\text{LR}} W \Leftrightarrow P[V = k]P[W = k'] \geq P[V = k']P[W = k] \text{ for all } k \leq k' \in \{0, 1, \dots\}.$$

When V and W are independent, this means that it is more likely that W assumes the larger value k' and V the smaller k , expressing the fact that W tends to be larger than V . We refer the reader to Denuit et al. (2005, Chapter 3) for more details concerning this stochastic order relation expressing the idea of “being larger than” for random variables.

3.2 Log-concavity

Recall that Z is said to be log-concave when the probability ratio $P[Z = k + 1]/P[Z = k]$ is non-increasing in $k \in \{0, 1, \dots\}$. Clearly,

$$\begin{aligned} Z \text{ log-concave} &\Leftrightarrow \frac{P[Z = k - 1]}{P[Z = k]} \text{ non-decreasing in } k \in \{0, 1, \dots\} \\ &\Leftrightarrow \frac{P[Z + 1 = k]}{P[Z = k]} \text{ non-decreasing in } k \in \{0, 1, \dots\} \Leftrightarrow Z \leq_{\text{LR}} Z + 1. \end{aligned}$$

More generally,

$$Z \text{ log-concave} \Leftrightarrow Z \leq_{\text{LR}} Z + j \text{ for any } j \in \{0, 1, \dots\}. \quad (3.1)$$

Log-concavity implies that the distribution is light-tailed, as explained next. Define the discrete failure rate as

$$r_Z(k) = \frac{P[Z = k]}{P[Z \geq k]} \text{ provided } P[Z \geq k] > 0.$$

We refer the reader e.g. to Chapter 6 in Lai and Xie (2006) for an introduction to discrete failure time models. In particular, if Z is log-concave then it has an increasing failure rate, that is, $r_Z(0) \leq r_Z(1) \leq r_Z(2) \leq \dots$ holds true. Now,

$$\begin{aligned} 1 - r_Z(0) &= 1 - P[Z = 0] = P[Z > 0] \\ 1 - r_Z(1) &= \frac{P[Z \geq 1] - P[Z = 1]}{P[Z \geq 1]} = \frac{P[Z > 1]}{P[Z > 0]} \end{aligned}$$

and in general

$$1 - r_Z(k) = \frac{P[Z > k]}{P[Z > k - 1]}$$

so that

$$Z \text{ log-concave} \Rightarrow P[Z > k] = \prod_{j=0}^k (1 - r_Z(j)) \leq (1 - r_Z(0))^{k+1} = (P[Z > 0])^{k+1}$$

which expresses the light-tailed behavior of Z .

Notice that when the probability ratio $P[Z = k + 1]/P[Z = k]$ is constantly equal to $c > 0$, only three distributions are possible: the Geometric distribution for which $P[Z = k] = (1 - p)^k p$ for some $p \in (0, 1)$, the Uniform distribution and the distribution with probability mass function $P[Z = k] = \frac{c^k}{1 + c + c^2 + \dots + c^m}$ for $k \in \{0, 1, \dots, m\}$. The discrete failure rate is constantly equal to p for the Geometric distribution.

The ratio $P[Z = k + 1]/P[Z = k]$ can be used to order discrete distributions. Considering Proposition 3.4.10 in Denuit et al. (2005), if Z is log-concave then there is a unique crossing between $P[Z = k + 1]/P[Z = k]$ and the horizontal line at any height $1 - p \in (0, 1)$ corresponding to the Geometric distribution, so that the latter dominates Z in the convex order when p is selected so that expected values coincide. Hence, Z has uniformly smaller stop-loss premiums compared to those of the Geometric distribution with the same mean, as well as uniformly smaller moments of any order. This confirms that log-concavity implies that the distribution is light-tailed.

Let us show that $(j, k) \mapsto P[Z = j - k]$ is a TP2 function when Z is log-concave. This result is well known but generally stated for random variables with a probability density

function. Here, we provide a direct proof of it for counting random variables, deduced from the close relationship between log-concavity and likelihood ratio order recalled earlier.

Property 3.1 *If the counting random variable Z is log-concave then*

$$P[Z = j' - k']P[Z = j - k] \geq P[Z = j' - k]P[Z = j - k'] \text{ for any } j \leq j' \text{ and } k \leq k'.$$

Proof Clearly,

$$Z \text{ log-concave} \Rightarrow \frac{P[Z = l - k + 1]}{P[Z = l - k]} \text{ non-increasing in } l \geq k$$

so that if Z is log-concave then $Z + k$ also is for any $k \in \{0, 1, \dots\}$. Considering (3.1), we then see that

$$Z \text{ log-concave} \Rightarrow Z + k \preceq_{LR} Z + k' \text{ for any } k \leq k',$$

which is equivalent to the stated inequality. This ends the proof. $\square$

It is well documented that the likelihood ratio order is not preserved by convolution, in general. However, log-concavity ensures its stability under convolution. This result is generally established by applying the so-called basic composition formula to probability density functions. We propose here a direct proof of this result for integer-valued random variables, inspired from Hu and Zhu (2001). The direct reasoning given below will be adapted in Section 4 to deal with large pools.

Property 3.2 *Let U , V and W be independent random variables valued in $\{0, 1, 2, \dots\}$. If W is log-concave then $U \preceq_{LR} V \Rightarrow U + W \preceq_{LR} V + W$.*

Proof We have to show that

$$P[U + W = l]P[V + W = l'] \geq P[U + W = l']P[V + W = l] \text{ for all } l \leq l' \in \{0, 1, \dots\}.$$

This is equivalent to show that $\Delta \geq 0$ for all $l \leq l' \in \{0, 1, \dots\}$ where

$$\Delta = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} P[U = j]P[V = k] \left(P[W = l - j]P[W = l' - k] - P[W = l' - j]P[W = l - k] \right).$$

Let us rewrite Δ as

$$\begin{aligned} \Delta &= \sum_{j \neq k} P[U = j]P[V = k] \left(P[W = l - j]P[W = l' - k] - P[W = l' - j]P[W = l - k] \right) \\ &= \sum_{j > k} P[U = j]P[V = k] \left(P[W = l - j]P[W = l' - k] - P[W = l' - j]P[W = l - k] \right) \\ &\quad + \sum_{j < k} P[U = k]P[V = j] \left(P[W = l - k]P[W = l' - j] - P[W = l' - k]P[W = l - j] \right). \end{aligned}$$

Now, $U \preceq_{LR} V$ implies that $P[U = k]P[V = j] \geq P[U = j]P[V = k]$ for all $j > k$ and Property 3.1 shows that $P[W = l - k]P[W = l' - j] \geq P[W = l' - k]P[W = l - j]$ when $j > k$. Hence,

$$\begin{aligned} \Delta &\geq \sum_{j > k} P[U = j]P[V = k] \left(P[W = l - j]P[W = l' - k] - P[W = l' - j]P[W = l - k] \right) \\ &\quad + P[W = l - k]P[W = l' - j] \end{aligned}$$

$$= 0,$$

which ends the proof. $\square$

3.3 Efron's Monotonicity Property

Consider two independent, integer-valued random variables Y and Z . This section studies the monotonicity of the conditional expectation $E[Y|Y + Z = k]$ as a function of $k \in \{0, 1, \dots\}$. This property is referred to as Efron's monotonicity, after Efron (1965).

The next result shows that log-concavity of Z ensures that Efron's monotonicity property holds true. The proof proposed in this paper uses the size-biased transform, whose definition is recalled next. Given a random variable X valued in $\{0, 1, \dots\}$ with finite expectation, define $\tilde{X}$ valued in $\{1, 2, \dots\}$ with probability mass function

$$P[\tilde{X} = k] = \frac{kP[X = k]}{E[X]}, \quad k \in \{1, 2, \dots\};$$

the random variable $\tilde{X}$ is said to be a size-biased version of X . We refer the reader to Arratia et al. (2019) for a detailed account of the size-biased transform.

Proposition 3.3 *Consider independent random variables Y and Z valued in $\{0, 1, \dots\}$. If Z is log-concave then the conditional expectation $E[Y|Y + Z = k]$ is non-decreasing in $k \in \{0, 1, \dots\}$.*

Proof Using Proposition 2.2(ii) in Denuit (2019), we have

$$E[Y|Y + Z = k] = E[Y] \frac{P[\tilde{Y} + Z = k]}{P[Y + Z = k]}$$

where the size-biased version $\tilde{Y}$ is assumed to be independent of Y and Z . Now,

$$\frac{P[\tilde{Y} = k]}{P[Y = k]} = \frac{k}{E[X]} \text{ increases in } k \in \{0, 1, \dots\} \Rightarrow Y \leq_{\text{LR}} \tilde{Y}.$$

By Property 3.2, the log-concavity of Z ensures that $Y + Z \leq_{\text{LR}} \tilde{Y} + Z$ so that $\frac{P[\tilde{Y} + Z = k]}{P[Y + Z = k]}$ is non-decreasing in k . This ends the proof. $\square$

Compared to Efron (1965), Property 3.3 does not impose any log-concavity condition on Y . Only Z is assumed to be log-concave there. The result is known but generally stated for probability density functions. See e.g. Denuit and Robert (2021b) or Pellerey and Navarro (2022).

As a consequence of Property 3.3, we can conclude that $s \mapsto h_{i,n}^{\text{cmrs}}(s)$ is a non-decreasing function if $\sum_{j \neq i} X_j$ is log-concave. Log-concavity of losses is therefore a sufficient condition for no sabotage but it appears to be restrictive for applications to insurance. For a large pool however, log-concavity does not appear to be such a restrictive assumption as explained in the following section.

4 Large Pools

In this section, we let the size of the pool increase to infinity. The no-sabotage property has already been studied in large pools by Denuit and Robert (2021b) when the loss distributions

are absolutely continuous. This assumption however appears as being too strong for insurance applications since the probability that a participant does not report any loss is in general high. In our framework this probability will be assumed to be larger than the probabilities to have a loss with size $k \in \{1, 2, \dots\}$ for all participants. This is in line with empirical data where the no-loss probability largely exceeds 0.5 for individual insurance contracts.

Diversification effects of the conditional mean risk-sharing rule for large pools has been studied in Denuit and Robert (2021a) for absolutely continuous losses or when the losses have compound Poisson distributions for which the claim severities are positive and absolutely continuous. We show further in this section that insurance at fair price can also be recovered within large pools for discrete losses, under fairly general conditions.

4.1 Local Limit Theorem

Local limit theorems (LLTs) describe how the probability mass function of a sum of random variables follows the Normal curve. In probability theory, these theorems have often been considered as being of secondary importance when compared with Central limit theorems, as explained in the recent survey by Szewczak and Weber (2022). But they are essential for our results. Let $X_1, X_2, X_3, \dots$ be independent integer-valued random variables with

$$\mu_k = E[X_k], \quad a_n = E[S_n] = \sum_{k=1}^n \mu_k \text{ and } b_n^2 = \text{Var}[S_n] = \sum_{k=1}^n E[(X_k - \mu_k)^2].$$

LLTs provide conditions such that

$$\sup_k \left| b_n P[S_n = k] - \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(k - a_n)^2}{2b_n^2}\right) \right| \rightarrow 0 \quad (4.1)$$

as $n \rightarrow \infty$.

Let us assume that $P[X_i = 0] \geq P[X_i = j]$ for every participant i and all $j \in \{1, 2, \dots\}$. Such an assumption is generally satisfied in insurance applications, where the probability mass at zero dominates all other probabilities. We refer the reader e.g. to Klein et al. (2014) for empirical evidence in motor insurance as well as to Chapter 2 in Kaas et al. (2008) describing the individual risk model for insurance portfolios, specially designed to account for the large probability mass at 0, referred to as the no-claim probability.

As proved in Theorem 3 of Petrov (1975), a necessary and sufficient condition for the convergence in Eq. 4.1 hold when $X_1, X_2, X_3, \dots$ are uniformly bounded integer-valued random variables is that

$$\text{g.c.d} \left\{ l : \sum_{i=1}^{\infty} P[X_i = l] = \infty \right\} = 1,$$

where g.c.d is the greatest common divisor of the terms enclosed in parentheses. The previous condition means that the g.c.d. of the positive integers such that there is an infinite number of X_i that can take as values these positive integers must be equal to 1.

In this paper, we need stronger conditions to get asymptotic expansions and uniform estimates for the remainder term. We will use the following result given in Petrov (1975).

Proposition 4.1 (Theorem 12 of Chapter VII in Petrov (1975)) Assume that:

- (I) $P[X_k = 0] \geq P[X_k = j]$ for all k and j ;
- (II) there exists a constant $c > 0$ such that $\liminf_{n \rightarrow \infty} \frac{1}{n} b_n^2 \geq c$;

- (III) $\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \mathbb{E}[|X_k - \mu_k|^p] < \infty$ and $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \sum_{|j - \mu_k| > n^\tau} |j - \mu_k|^p = 0$ for some integer $p \geq 3$ and some positive $\tau < 1/2$;
- (IV)

$$g.c.d \left\{ l : \frac{1}{\log n} \sum_{k=1}^n \mathbb{P}[X_k = 0] \mathbb{P}[X_k = l] \rightarrow \infty \right\} = 1.$$

Then,

$$b_n \mathbb{P}[S_n = k] = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) \left(1 + \sum_{v=1}^{p-2} \frac{p_{vn}(x)}{n^{v/2}} + o\left(\frac{1}{n^{(p-2)/2}}\right)\right)$$

uniformly in k ($-\infty < k < \infty$), where $x = (k - a_n)/b_n$ and $p_{vn}(\cdot)$ is a polynomial of degree $3v$ with coefficients depending only on the cumulants of the random variables $X_1, \dots, X_n$ up to and including order $v + 2$.

LLTs are often studied by using specific indicators. For an integer-valued random variable X , define

$$\vartheta_X = \sum_{j=-\infty}^{\infty} \min\{\mathbb{P}[X = j], \mathbb{P}[X = j + 1]\}$$

and

$$\delta_X = \sum_{j=-\infty}^{\infty} |\mathbb{P}[X = j] - \mathbb{P}[X = j - 1]|.$$

The latter series is often used to measure smoothness. Let us recall some useful properties of these indicators, that can be found in Szewczak and Weber (2022). For any positive real values a and b , one has $a + b - |a - b| = 2 \min\{a, b\}$ so that

$$\begin{aligned} \vartheta_X &= \frac{1}{2} \sum_{j=-\infty}^{\infty} \left(\mathbb{P}[X = j] + \mathbb{P}[X = j + 1] - |\mathbb{P}[X = j] - \mathbb{P}[X = j + 1]| \right) \\ &= 1 - \frac{\delta_X}{2} \end{aligned}$$

or $\delta_X = 2(1 - \vartheta_X)$.

If the distribution of X is unimodal, in the sense that there exists an integer j_0 such that $\mathbb{P}[X = j - 1] \leq \mathbb{P}[X = j]$ for all $j \leq j_0$ and $\mathbb{P}[X = j + 1] \leq \mathbb{P}[X = j]$ for all $j \geq j_0$ then

$$\begin{aligned} \delta_X &= \sum_{j \leq j_0} (\mathbb{P}[X = j] - \mathbb{P}[X = j - 1]) + \sum_{j > j_0} (\mathbb{P}[X = j - 1] - \mathbb{P}[X = j]) \\ &= \sum_{j \leq j_0} \mathbb{P}[X = j] + \sum_{j \geq j_0} \mathbb{P}[X = j] - \left(\sum_{j < j_0} \mathbb{P}[X = j] + \sum_{j > j_0} \mathbb{P}[X = j] \right) \\ &= 2\mathbb{P}[X = j_0] = 2 \max_j \mathbb{P}[X = j]. \end{aligned}$$

This is for instance the case when X is log-concave since every log-concave distribution is unimodal.

With individual yearly insurance losses, it is common to have the largest probability mass at zero and then a unique mode on strictly positive values, that is,

$$\begin{aligned} P[X = 0] &\geq P[X = j] \text{ for all } j = 1, 2, \dots \\ P[X = j - 1] &\leq P[X = j] \text{ for all } j = 2, \dots, j_0 \\ P[X = j + 1] &\leq P[X = j] \text{ for all } j \geq j_0. \end{aligned}$$

Then,

$$\begin{aligned} \delta_X &= P[X = 0] + P[X = 0] - P[X = 1] + \sum_{j=2}^{j_0} (P[X = j] - P[X = j - 1]) \\ &\quad + \sum_{j > j_0} (P[X = j - 1] - P[X = j]) \\ &= P[X = 0] - 2P[X = 1] + \sum_{j \leq j_0} P[X = j] + \sum_{j \geq j_0} P[X = j] \\ &\quad - \left(\sum_{j=1}^{j_0-1} P[X = j] + \sum_{j > j_0} P[X = j] \right) \\ &= 2(P[X = 0] - P[X = 1] + P[X = j_0]). \end{aligned}$$

Sometimes, there is a local mode at the maximum value m , because of truncation, that is, $P[X = m - 1] < P[X = m]$. We then get

$$\begin{aligned} \delta_X &= P[X = 0] + P[X = 0] - P[X = 1] + \sum_{j=2}^{j_0} (P[X = j] - P[X = j - 1]) \\ &\quad + \sum_{j=j_0+1}^{m-1} (P[X = j - 1] - P[X = j]) + P[X = m] - P[X = m - 1] \\ &= 2(P[X = 0] - P[X = 1] + P[X = j_0] - P[X = m - 1] + P[X = m]). \end{aligned}$$

We will use the following LLT given in McDonald (1979) for establishing risk diversification of the conditional mean risk sharing rule.

Proposition 4.2 (Theorem 1 in McDonald (1979)). Assume that:

- (I) $\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n E[\exp(a|X_k|)] < \infty$ for some positive constant a ;
- (II) there exists a constant $c > 0$ such that $\liminf_{n \rightarrow \infty} \frac{1}{n} b_n^2 \geq c$;
- (III) $\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \vartheta_{X_k} > 0$.

Let $\omega(n)$ be a sequence such that $\lim_{n \rightarrow \infty} \omega(n) = \infty$. Then

$$P[S_n = k] = \frac{1}{b_n \sqrt{2\pi}} \exp \left(-\frac{(k - a_n)^2}{2b_n^2} + \frac{(k - a_n)^3}{n^2} \lambda_n \left(\frac{k - a_n}{n} \right) \right) \left(1 + O \left(\frac{k - a_n}{n} \right) \right)$$

uniformly for k in

$$1 \leq |k - a_n| \leq \frac{n}{\omega(n)}$$

where $\lambda_n(\tau)$ is a special power series converging uniformly with respect to n for sufficiently small τ .

Condition (III) of Proposition 4.2 requires that two consecutive integer values must be taken by a large number of random variables X_k . Indeed, it is easy to see that if for instance X_k only assumes even values then $\vartheta_{X_k} = 0$. If $\vartheta_{X_k} > 0$ then there is a positive integer j such that $P[X_k = j] > 0$ and $P[X_k = j - 1] > 0$ simultaneously hold, so that X_k assumes the two consecutive values $j - 1$ and j . Condition (III) expresses the fact that this must happen for sufficiently many indices k . This excludes the classical counter-example with $X_1 \in \{0, 1\}$ and all other $X_k \in \{0, 2\}$ where $E[X_1|S_n] = X_1$ and the conditional mean risk-sharing rule fails to diversify risk. Condition (III) of Proposition 4.2 generally holds true in the applications to risk sharing as it can be seen from the values of ϑ_X computed in the cases commonly encountered with insurance losses. Condition (III) of Proposition 4.2 is automatically fulfilled when all random variables X_k have the same support $\{0, 1, 2, \dots\}$, i.e. $P[X_k = j] > 0$ for any $j \geq 0$.

Conditions (I)-(II)-(III) of Proposition 4.2 generally hold true when individual losses X_i have been obtained by discretizing a continuous severity model over $(0, \infty)$, like Gamma or Inverse Gaussian for instance, supplemented with a probability mass at zero.

4.2 No-sabotage Condition for Large Pools

LLTs lead to show that the assumption that log-concavity for sums approximately holds true in a sufficiently large pool and for a sufficiently large interval around the mean of the sum, so that the no-sabotage requirement is almost valid for the conditional mean risk-sharing rule operated among many participants. The next result shows that the no-sabotage condition asymptotically holds for intervals larger than those where the Central limit theorem holds.

Proposition 4.3 *Let $X_1, X_2, X_3, \dots$ be independent integer-valued random variables such that there is an integer m such that $X_i \leq m$ holds for all i . Suppose also conditions (I), (II) and (IV) of Proposition 4.1 are fulfilled. Let $\mathcal{I}_n = \{l \in \mathbb{Z} : |l - a_n| \leq n^{3/4}/\omega(n)\}$ where $\omega(n)$ is a sequence such that $\lim_{n \rightarrow \infty} \omega(n) = \infty$ and $\limsup_{n \rightarrow \infty} n^{3/4}/\omega(n) \geq 1$. There exists n_0 such that for $n \geq n_0$*

$$E[X_i|S_n = k] \text{ is non-decreasing in } k \text{ over } \mathcal{I}_n.$$

Proof Without loss of generality we consider the case $i = 1$. Let us define $S_n^{(\wedge 1)} = \sum_{k=2}^n X_k$, $a_n^{(\wedge 1)} = \sum_{k=2}^n \mu_k$ and $(b_n^{(\wedge 1)})^2 = \sum_{k=2}^n E[(X_k - \mu_k)^2]$. The proof then proceeds as follows. Using Proposition 2.2(ii) in Denuit (2019), we have

$$E[X_1|X_1 + S_n^{(\wedge 1)} = k] = E[X_1] \frac{P[\widetilde{X}_1 + S_n^{(\wedge 1)} = k]}{P[X_1 + S_n^{(\wedge 1)} = k]}$$

where the size-biased version $\widetilde{X}_1$ is assumed to be independent of X_1 and $S_n^{(\wedge 1)}$ and satisfies

$$\frac{P[\widetilde{X}_1 = l]}{P[X_1 = l]} = \frac{l}{E[X_1]}, l \in \{0, 1, \dots\}.$$

Therefore $E[X_i|S_n = l]$ is non-decreasing in l over the range of integer values $\mathcal{I}_n$ if and only if

$$\frac{P[X_1 + S_n^{(\wedge 1)} = l]}{P[\widetilde{X}_1 + S_n^{(\wedge 1)} = l]} \text{ is non-increasing in } l \in \mathcal{I}_n,$$

or equivalently

$$P[X_1 + S_n^{(\wedge 1)} = l]P[\widetilde{X}_1 + S_n^{(\wedge 1)} = l + 1] \geq P[X_1 + S_n^{(\wedge 1)} = l + 1]P[\widetilde{X}_1 + S_n^{(\wedge 1)} = l], l \in \mathcal{I}_n.$$

This is equivalent to show that $\Delta_l \geq 0$ for all $l, l+1 \in \mathcal{I}_n$ where

$$\Delta_l = \sum_{j=0}^l \sum_{k=1}^l P[X_1 = j] P[\widetilde{X}_1 = k] \\ \left(P[S_n^{(\wedge 1)} = l-j] P[S_n^{(\wedge 1)} = l+1-k] - P[S_n^{(\wedge 1)} = l+1-j] P[S_n^{(\wedge 1)} = l-k] \right).$$

Let us assume that l is large enough such that $l > m$ and let us rewrite Δ_l as

$$\Delta_l = \frac{1}{E[X_1]} \sum_{j=0}^m \sum_{k=1}^m k P[X_1 = j] P[X_1 = k] \\ \left(P[S_n^{(\wedge 1)} = l-j] P[S_n^{(\wedge 1)} = l+1-k] - P[S_n^{(\wedge 1)} = l+1-j] P[S_n^{(\wedge 1)} = l-k] \right) \\ = \frac{1}{E[X_1]} \sum_{0 \leq j < k \leq m} k P[X_1 = j] P[X_1 = k] \\ \left(P[S_n^{(\wedge 1)} = l-j] P[S_n^{(\wedge 1)} = l+1-k] - P[S_n^{(\wedge 1)} = l+1-j] P[S_n^{(\wedge 1)} = l-k] \right) \\ + \frac{1}{E[X_1]} \sum_{1 \leq k < j \leq m} k P[X_1 = j] P[X_1 = k] \\ \left(P[S_n^{(\wedge 1)} = l-j] P[S_n^{(\wedge 1)} = l+1-k] - P[S_n^{(\wedge 1)} = l+1-j] P[S_n^{(\wedge 1)} = l-k] \right) \\ = \frac{1}{E[X_1]} \sum_{k=1}^m k P[X_1 = 0] P[X_1 = k] \\ \left(P[S_n^{(\wedge 1)} = l] P[S_n^{(\wedge 1)} = l+1-k] - P[S_n^{(\wedge 1)} = l+1] P[S_n^{(\wedge 1)} = l-k] \right) \\ + \frac{1}{E[X_1]} \sum_{1 \leq j < k \leq m} (k-j) P[X_1 = j] P[X_1 = k] \\ \left(P[S_n^{(\wedge 1)} = l-j] P[S_n^{(\wedge 1)} = l+1-k] P[S_n^{(\wedge 1)} = l+1-j] P[S_n^{(\wedge 1)} = l-k] \right) \\ = \frac{1}{E[X_1]} \sum_{0 \leq j < k \leq m} (k-j) P[X_1 = j] P[X_1 = k] \\ \left(P[S_n^{(\wedge 1)} = l-j] P[S_n^{(\wedge 1)} = l+1-k] - P[S_n^{(\wedge 1)} = l+1-j] P[S_n^{(\wedge 1)} = l-k] \right).$$

Then we have

$$P[S_n^{(\wedge 1)} = l-j] P[S_n^{(\wedge 1)} = l+1-k] - P[S_n^{(\wedge 1)} = l+1-j] P[S_n^{(\wedge 1)} = l-k] \\ = P[S_n^{(\wedge 1)} = l+1-j] P[S_n^{(\wedge 1)} = l-k] \left(\frac{P[S_n^{(\wedge 1)} = l-j]}{P[S_n^{(\wedge 1)} = l-(j-1)]} \frac{P[S_n^{(\wedge 1)} = l-(k-1)]}{P[S_n^{(\wedge 1)} = l-k]} - 1 \right).$$

Let us study the sign of

$$\ln \left(\frac{P[S_n^{(\wedge 1)} = l-j]}{P[S_n^{(\wedge 1)} = l-(j-1)]} \frac{P[S_n^{(\wedge 1)} = l-(k-1)]}{P[S_n^{(\wedge 1)} = l-k]} \right)$$

for $l, l + 1 \in \mathcal{I}_n$. Let

$$x_{lj} = \frac{l - j - a_n^{(\setminus 1)}}{b_n^{(\setminus 1)}} \quad \text{and} \quad x_{lk} = \frac{l - k - a_n^{(\setminus 1)}}{b_n^{(\setminus 1)}}.$$

By Proposition 4.1 with $p = 4$

$$\begin{aligned} & \ln \left(\frac{\mathbb{P}[S_n^{(\setminus 1)} = l - j]}{\mathbb{P}[S_n^{(\setminus 1)} = l - (j - 1)]} \frac{\mathbb{P}[S_n^{(\setminus 1)} = l - (k - 1)]}{\mathbb{P}[S_n^{(\setminus 1)} = l - k]} \right) \\ &= -\frac{1}{2} \left((x_{lj}^2 - x_{lj-1}^2) - (x_{lk}^2 - x_{lk-1}^2) \right) \\ &+ \sum_{v=1}^2 \frac{1}{n^{v/2}} \left((p_{vn}(x_{lj}) - p_{vn}(x_{lj-1})) - (p_{vn}(x_{lk}) - p_{vn}(x_{lk-1})) \right) + o\left(\frac{1}{n}\right). \end{aligned}$$

We have

$$\frac{1}{2} \left((x_{lj}^2 - x_{lj-1}^2) - (x_{lk}^2 - x_{lk-1}^2) \right) = \frac{k - j}{(b_n^{(\setminus 1)})^2}$$

and

$$(p_{vn}(x_{lj}) - p_{vn}(x_{lj-1})) - (p_{vn}(x_{lk}) - p_{vn}(x_{lk-1})) = O\left(\frac{(x_{l0})^{3v-2}}{(b_n^{(\setminus 1)})^2}\right).$$

We deduce that, if for $v = 1, 2$, $(x_{l0})^{3v-2} = o(n^{v/2})$, then the dominant term is $(k - j)/(b_n^{(\setminus 1)})^2$, which is equivalent to have

$$\frac{l - a_n^{(\setminus 1)}}{b_n^{(\setminus 1)}} = o(n^{1/4}).$$

This condition is satisfied for $l \in \mathcal{I}_n$. It follows that, for $l \in \mathcal{I}_n$,

$$\lim_{n \rightarrow \infty} b_n^2 \left(\frac{\mathbb{P}[S_n^{(\setminus 1)} = l - j]}{\mathbb{P}[S_n^{(\setminus 1)} = l - (j - 1)]} \frac{\mathbb{P}[S_n^{(\setminus 1)} = l - (k - 1)]}{\mathbb{P}[S_n^{(\setminus 1)} = l - k]} - 1 \right) = j - k < 0.$$

Finally, we can conclude that there exists n_0 such that for $n \geq n_0$,

$$\left(\frac{b_n}{\mathbb{P}[S_n^{(\setminus 1)} = l]} \right)^2 \Delta_l < 0$$

uniformly for $|l - a_n| \leq n^{3/4}/\omega(n)$. $\square$

The sequence $\omega(n)$ appearing in the statement of Proposition 4.3 should not increase too rapidly to ∞ , to ensure that there is a non-empty set where the conditional expectation is non-decreasing. For instance, we see that with $\omega(n) = 3n^{3/4}$ there is no integer value k fulfilling the condition $|k - a_n| \leq n^{3/4}/\omega(n) = 1/3$. The only relevant sequences $\omega(n)$ are those for which there is a non-empty set of integer values k fulfilling this constraint. The same comment applies to the sequence $\omega(n)$ involved in Proposition 4.2.

4.3 Risk-reducing Effect, Risk Elimination and Insurance at Fair Price

The conditional mean risk allocation is regarded as beneficial by all risk-averse economic agents, because

$$E[u_i(w_i - X_i)] \leq E[u_i(w_i - h_{i,n}^{\text{cmrs}}(S_n))] \text{ for all } i \in \{1, 2, \dots, n\}, \quad (4.2)$$

whatever the concave utility function u_i and the initial wealth level w_i . This is a direct application of Jensen inequality for conditional expectations. Inequality (4.2) shows that provided total losses are distributed among participants according to the conditional mean risk-sharing rule, every risk-averse agent prefers joining the pool and paying $h_{i,n}^{\text{cmrs}}(S_n)$ over the stand-alone position, keeping the individual loss X_i . Interestingly, this is true whatever the distribution of the other losses X_j , $j \neq i$, comprised in the pool.

Enlarging the pool is always beneficial, even with heterogeneous losses provided S_n is allocated among participants according to the conditional mean risk-sharing rule $h_{i,n}^{\text{cmrs}}$. Precisely, for any independent losses X_i , we have

$$E[u_i(w_i - h_{i,n}^{\text{cmrs}}(S_n))] \leq E[u_i(w_i - h_{i,n+1}^{\text{cmrs}}(S_{n+1}))] \text{ for all } n \text{ and } i \in \{1, 2, \dots, n\}, \quad (4.3)$$

for any concave utility function u_i and initial wealth w_i . See e.g. Denuit and Robert (2023) and the references therein.

Risk elimination may be obtained at the limit, when it can be shown that $h_{i,n}^{\text{cmrs}}(S_n)$ converges to the pure premium $E[X_i]$ as the number n of participants tends to infinity. Insurance at fair price is recovered at the limit when such a law of large numbers applies to the pool under consideration. We now provide sufficient conditions such that risk elimination becomes possible within infinitely large pools with independent integer-valued losses.

Proposition 4.4 *Let $X_1, X_2, X_3, \dots$ be independent integer-valued random variables. Assume that:*

- (I) *there exists an integer m such that $|X_i| \leq m$, for all $i \geq 1$;*
- (II) *there exists a constant $c > 0$ such that $\liminf_{n \rightarrow \infty} \frac{1}{n} b_n^2 \geq c$;*
- (III) *$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \vartheta_{X_k} > 0$.*

Then, for all i , $E[X_i|S_n] \xrightarrow{L_1} E[X_i]$ as $n \rightarrow \infty$.

Proof Without loss of generality we consider the case $i = 1$. Let us define $S_n^{(\setminus 1)} = \sum_{k=2}^n X_k$, $a_n^{(\setminus 1)} = \sum_{k=2}^n \mu_k$ and $(b_n^{(\setminus 1)})^2 = \sum_{k=2}^n E[(X_k - \mu_k)^2]$. The proof then proceeds as follows. First, notice that Lyapunov's condition holds for partial sums S_n , $n \geq 1$, i.e. for any $\delta > 0$, we have

$$\lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n E[|X_k - \mu_k|^{2+\delta}]}{b_n^{2+\delta}} = 0.$$

Therefore, for any $t \in \mathbb{R}$,

$$P\left[\frac{S_n - a_n}{b_n} \leq t\right] \rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t \exp(-u^2/2) du \text{ as } n \rightarrow \infty.$$

Second, notice that

$$\begin{aligned} E[X_1|S_n = s] &= \sum_{l=0}^m l P[X_1 = l|S_n = s] = \sum_{l=0}^m l \frac{P[X_1 = l]P[S_n = s|X_1 = l]}{P[S_n = s]} \\ &= \sum_{l=0}^m l \frac{P[X_1 = l]P[S_n^{(\setminus 1)} = s - l]}{\sum_{j=0}^m P[X_1 = j]P[S_n^{(\setminus 1)} = s - j]} \end{aligned}$$

and

$$\mathbb{E}[X_1 | S_n = s] - \mathbb{E}[X_1] = \sum_{l=0}^m l \mathbb{P}[X_1 = l] \sum_{j=0}^m d_{l,j,n}(s)$$

with

$$d_{l,j,n}(s) = \mathbb{P}[X_1 = j] \frac{\mathbb{P}[S_n^{(\wedge 1)} = s - l] - \mathbb{P}[S_n^{(\wedge 1)} = s - j]}{\sum_{r=0}^m \mathbb{P}[X_1 = r] \mathbb{P}[S_n^{(\wedge 1)} = s - r]}.$$

Then, we have, for $t > 0$,

$$\begin{aligned} \mathbb{E}[|\mathbb{E}[X_1 | S_n] - \mathbb{E}[X_1]|] &= \mathbb{E}\left[|\mathbb{E}[X_1 | S_n] - \mathbb{E}[X_1]| \mathbf{I}\left[\left|S_n - a_n^{(\wedge 1)}\right| \leq t b_n^{(\wedge 1)}\right]\right] \\ &\quad + \mathbb{E}\left[|\mathbb{E}[X_1 | S_n] - \mathbb{E}[X_1]| \mathbf{I}\left[\left|S_n - a_n^{(\wedge 1)}\right| > t b_n^{(\wedge 1)}\right]\right] \\ &= I_n(t) + II_n(t). \end{aligned}$$

The second part, $II_n(t)$, is bounded as follows

$$\begin{aligned} II_n(t) &\leq 2m \mathbb{P}\left[\left|\frac{S_n - a_n^{(\wedge 1)}}{b_n^{(\wedge 1)}}\right| > t\right] \\ &\leq 2m \mathbb{P}\left[\left|\frac{S_n - a_n}{b_n}\right| > \sqrt{\left(1 - \frac{\mathbb{E}[(X_1 - \mu_1)^2]}{b_n^2}\right)} \left(t - \frac{\mu_1}{b_n^{(\wedge 1)}}\right)\right]. \end{aligned}$$

and therefore

$$\limsup_{n \rightarrow \infty} II_n(t) \leq 2m \frac{1}{\sqrt{2\pi}} \int_t^\infty \exp(-u^2/2) du$$

so that

$$\lim_{t \rightarrow \infty} \limsup_{n \rightarrow \infty} II_n(t) = 0.$$

Let us now consider the first part, $I_n(t)$. We have

$$I_n(t) \leq \sum_{l=0}^m l \mathbb{P}[X_1 = l] \sum_{j=0}^m \sum_{s: 1 \leq |s - a_n^{(\wedge 1)}| \leq t b_n^{(\wedge 1)}} |d_{l,j,n}(s)| \mathbb{P}[S_n = s].$$

By Proposition 4.2 for which all the conditions are satisfied, we have

$$\begin{aligned} \mathbb{P}[S_n^{(\wedge 1)} = s] &= \frac{1}{b_n^{(\wedge 1)} \sqrt{2\pi}} \exp\left(-\frac{(s - a_n^{(\wedge 1)})^2}{2(b_n^{(\wedge 1)})^2} + \frac{(s - a_n^{(\wedge 1)})^3}{n^2} \lambda_n\left(\frac{s - a_n^{(\wedge 1)}}{n}\right)\right) \\ &\quad \times \left(1 + O\left(\frac{s - a_n^{(\wedge 1)}}{n}\right)\right) \end{aligned}$$

uniformly for s in

$$1 \leq |s - a_n^{(\wedge 1)}| \leq \frac{n}{\omega(n)}$$

where $\lambda_n(\tau)$ is a special power series converging uniformly with respect to n for sufficiently small τ , and $\lim_{n \rightarrow \infty} \omega(n) = \infty$. We easily deduce choosing $\omega(n) = n/(t b_n^{(\wedge 1)})$ that

$$\lim_{n \rightarrow \infty} \sup_{1 \leq |s - a_n^{(\wedge 1)}| \leq t b_n^{(\wedge 1)}} \sup_{0 \leq l, j \leq m} |d_{l,j,n}(s)| = 0$$

and therefore that, for all $t > 0$,

$$\lim_{n \rightarrow \infty} I_n(t) = 0.$$

This ends the proof. $\square$

Proposition 4.4 shows that insurance at fair price can be recovered at the limit, by pooling among a large number of participants. The added value of the insurance company remains for covering the upper layer, since the assumption $X_i \leq m$ in Condition (I) is needed to establish Proposition 4.4. The middle risk layer can be effectively pooled among participants, offering them insurance at fair price for medium losses.

5 Application to Binomial Distributions

Consider losses X_i obeying the Binomial distribution with common size parameter m and respective success probabilities $p_i, i = 1, 2, \dots$. We assume that the sequence of probabilities p_i is such that the inequalities $d < p_i < 1/(m+1)$ hold for all i , for some $0 < d < 1/(m+1)$. The particular case $m = 1$ has been studied by Denuit et al. (2022b).

We know from Proposition 3.3 in Denuit et al. (2022b) that the conditional expectations $E[X_i | S_n = k], i = 1, 2, \dots, n$, are non-decreasing in k when $m = 1$. This result may be extended when summing Bernoulli indicators to get Binomially distributed losses. Let us recover this result as an application of Proposition 4.3. The constraints imposed on the success probabilities p_i ensure that Conditions (I), (II) and (IV) of Proposition 4.1 are fulfilled. If the sequence $\omega(n)$ grows slowly towards infinity (e.g. as $\log(\log(n))$), then the length of the interval (around the expectation of S_n), where $E[X_i | S_n = k]$ is non-decreasing, is of order $n^{3/4} / \log(\log(n))$.

Moreover it is shown in Proposition 3.1 in Denuit et al. (2022b) that

$$\lim_{n \rightarrow \infty} \text{Var}[E[X_i | S_n]] = 0$$

and therefore that $E[X_i | S_n] \xrightarrow{L_1} E[X_i]$ for the Binomially distributed losses considered in this section. Conditions (I)-(II) of Proposition 4.4 are clearly fulfilled. Considering Condition (III), the decreasingness of the probability mass function over $\{0, 1, \dots, m\}$ ensures that

$$\vartheta_{X_k} = 1 - \frac{\delta_{X_k}}{2} = 1 - P[X_k = 0] = 1 - (1 - p_k)^m.$$

Hence, Condition (III) is valid, too. Proposition 4.4 then shows that $E[X_i | S_n] \xrightarrow{L_1} E[X_i]$ as $n \rightarrow \infty$.

6 Numerical Illustrations

Consider a P2P insurance community composed of three risk profiles. Each participant brings a loss $X_i = I_i C_i$ to the pool, where I_i is Bernoulli(q_i), that is $P[I_i = 1] = 1 - P[I_i = 0] = q_i$ and severities C_i are obtained by discretizing the Beta distribution with mean $a_i/(a_i + b_i)$ and variance $a_i b_i / ((a_i + b_i)^2 (a_i + b_i + 1))$, concentrating the probability mass of each interval $(j/10, (j+1)/10)$ on its upper right endpoint, $j = 0, 1, \dots, 9$. The corresponding probabilities are then assigned to severity values $10j, j = 1, 2, \dots, 10$. Hence, each C_i is valued in $\{1, 2, \dots, 10\}$ and each X_i is valued in $\{0, 1, \dots, 10\}$ so that $m = 10$. Precisely, the three groups are as follows:

Group 1 (low risks) with $n_1 = 0.5n$ individuals, $q_i = 5\%$, $a_i = 2$ and $b_i = 5$.

Group 2 (medium risks) with $n_2 = 0.3n$ individuals, $q_i = 10\%$, $a_i = 3$ and $b_i = 5$.

Group 2 (high risks) with $n_3 = 0.2n$ individuals, $q_i = 20\%$, $a_i = 4$ and $b_i = 5$.

The probability mass functions of the severities C_i are displayed in Fig. 1 for each of the three groups. We can see there that the probability mass progressively shifts towards larger values. Participants in Group 3 bring comparatively higher severities compared to participants in Group 2 who themselves bring comparatively higher severities compared to participants in Group 1. Groups are thus ranked in increasing magnitude of both claim probabilities and severities. Computing $(P[X_i = k])^2 - P[X_i = k - 1]P[X_i = k + 1]$ for $k \in \{0, \dots, 10\}$, we can see that these differences are positive, except for $k = 1$. Hence, individual risks X_i are not log-concave.

The R packages `distr` contributed by Ruckdeschel et al. (2006) is particularly effective to perform convolutions while the `actuar` package contributed by Dutang et al. (2008) implements Panjer algorithm. The latter has been used to derive the distribution of the total losses for each group, which obeys the compound Binomial distribution with parameters n_i , q_i and severity distribution C_i . Panjer recursion is performed using the function `aggregatedDist` included in `actuar`. The three compound Binomial distributions have finally been convoluted with `distr` to derive the distribution of S_n .

The functions $s \mapsto E[X_i|S_n = s]$ are displayed in Fig. 2 for each of the three groups when $n \in \{10, 100\}$. Within a small pool gathering $n = 10$ participants, the conditional expectations are not comonotonic. We have for instance $E[X_1|S_{10} = 2] = 0.1914391$, $E[X_1|S_{10} = 3] = 0.1897761$, $E[X_1|S_{10} = 4] = 0.1815142$, $E[X_1|S_{10} = 5] = 0.1812234$ and $E[X_1|S_{10} = 6] = 0.1976959$ so that this conditional expectation exhibits a U-shape over $\{2, 3, 4, 5, 6\}$. As guaranteed by Proposition 4.3, the conditional expectations become monotonic when n increases (from 10 to 100 in this example) as it can be seen from the lower panel of Fig. 2.

The assumptions of Proposition 4.4 are fulfilled in our example: (I) holds with $m = 10$, (II) is valid since $\frac{1}{n}b_n^2$ is a weighted average of $\text{Var}[X_i]$ in the three groups (with respective weights 0.5, 0.3 and 0.2) and $\frac{1}{n} \sum_{k=1}^n \vartheta_{X_k}$ appearing in (III) is also a weighted average of $2(P[X_i = 0] - P[X_i = 1] + \max_{j>0} P[X_i = j])$ in the three groups. Figure 3 displays the expected absolute differences $E[|E[X_i|S_n] - E[X_i]|]$ in function of n . The decrease to 0 guaranteed by Proposition 4.4 is clearly visible there, at rates (square root of n) as could be expected, but up to coefficients which depend on the size and variability of the risks: smaller risks are more rapidly diversified within the pool while larger risks require more participants to be diluted within the community.

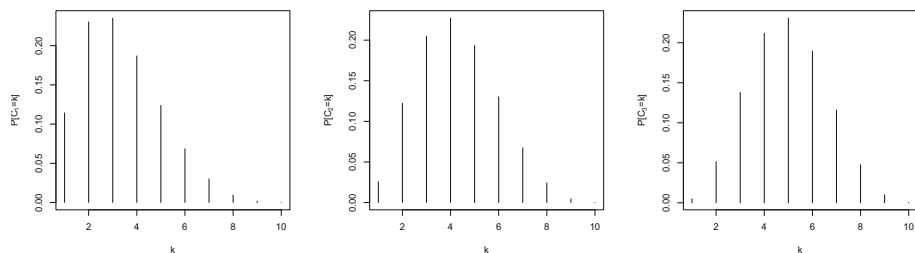


Fig. 1 Probability mass functions of the severities C_i in Group 1 (left panel), in Group 2 (middle panel), and in Group 3 (right panel)

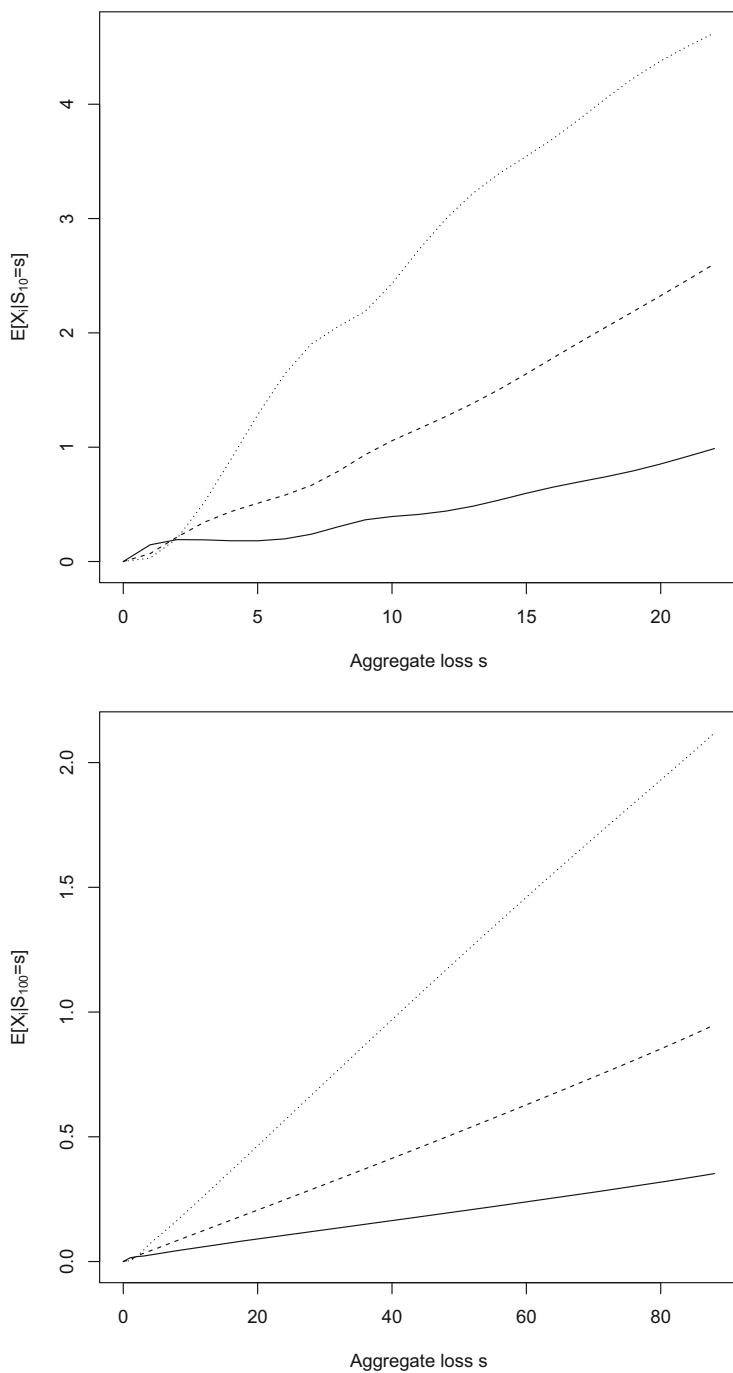


Fig. 2 Contributions $s \mapsto E[X_i | S = s]$ in Group 1 (solid line appearing at the bottom), in Group 2 (broken line appearing in the middle), and in Group 3 (dotted line appearing at the top). Pool size $n = 10$ (top panel) and 100 (bottom panel)

7 Discussion

This paper studied the no-sabotage requirement and investigated diversification effects within pools gathering discrete losses. The discrete case is of particular importance for applications because actuarial calculations are generally carried out after discretization, resulting in losses valued in an arithmetic grid. The analysis is restricted to bounded random variables. As explained in the introduction, this assumption is generally fulfilled in applications since insurance pools always limit the maximal indemnity paid to participants.

Section 3 considered pools with a finite number of participants. Log-concavity appears to provide the actuary with a sufficient condition for avoiding sabotage opportunities. This was expected since Efron (1965) related this concept to the monotonicity of conditional expectations. This paper nevertheless relaxes some results in Efron (1965) by imposing log-concavity on the random variable representing the aggregate loss of the rest of the pool, only. Section 4 then studied the same problem by letting the number of participants tend to infinity. The key argument is that log-concavity for sums approximately holds true in a sufficiently large pool under general conditions. Risk elimination at the limit is finally investigated and shown to hold under general conditions, too. Sections 5-6 illustrate the methodological results with some examples, so that the reader can figure out the practical applications of the mechanism studied in this paper.

Figure 2 shows that the risk-sharing functions are linear in s as the pool size becomes very large. This does not contradict Proposition 4.4, which shows that agent i 's contribution converges to the expectation of his/her risk brought to the pool. Indeed, when s is close to $E[S_n]$, his/her contribution (as a function of s) will be approximated by a straight line with slope tending towards 0 (as the pool size increases) and level $E[X_i]$. On the other hand, approximation by a straight line is not always possible, even for comonotonic risk-sharing rules. The following example shows that, in some cases, it will be necessary to approximate by a piecewise linear function (depending on whether s is larger or smaller than $E[S_n]$). The next example shows that some results derived in this paper under the conditional mean risk-sharing rule do not extend to general comonotonic ones.

Example 7.1 Assume that individual losses X_i are independent and identically distributed, and such that $X_i \leq m$. Consider an even number n of participants and define the following comonotonic risk-sharing rule:

$$h_{i,n}(s) = \begin{cases} \frac{s}{2n} \mathbf{I}[s < E[S_n]] + \left(\frac{3s}{2n} - \frac{E[S_n]}{n} \right) \mathbf{I}[s \geq E[S_n]] & \text{if } i \text{ is even} \\ \frac{3s}{2n} \mathbf{I}[s < E[S_n]] + \left(\frac{s}{2n} + \frac{E[S_n]}{n} \right) \mathbf{I}[s \geq E[S_n]] & \text{if } i \text{ is odd.} \end{cases}$$

It is easy to see that the risk-sharing functions $h_{i,n}(\cdot)$ are increasing, piecewise linear and that the full-allocation condition holds true as

$$\sum_{i=1}^n h_{i,n}(s) = s.$$

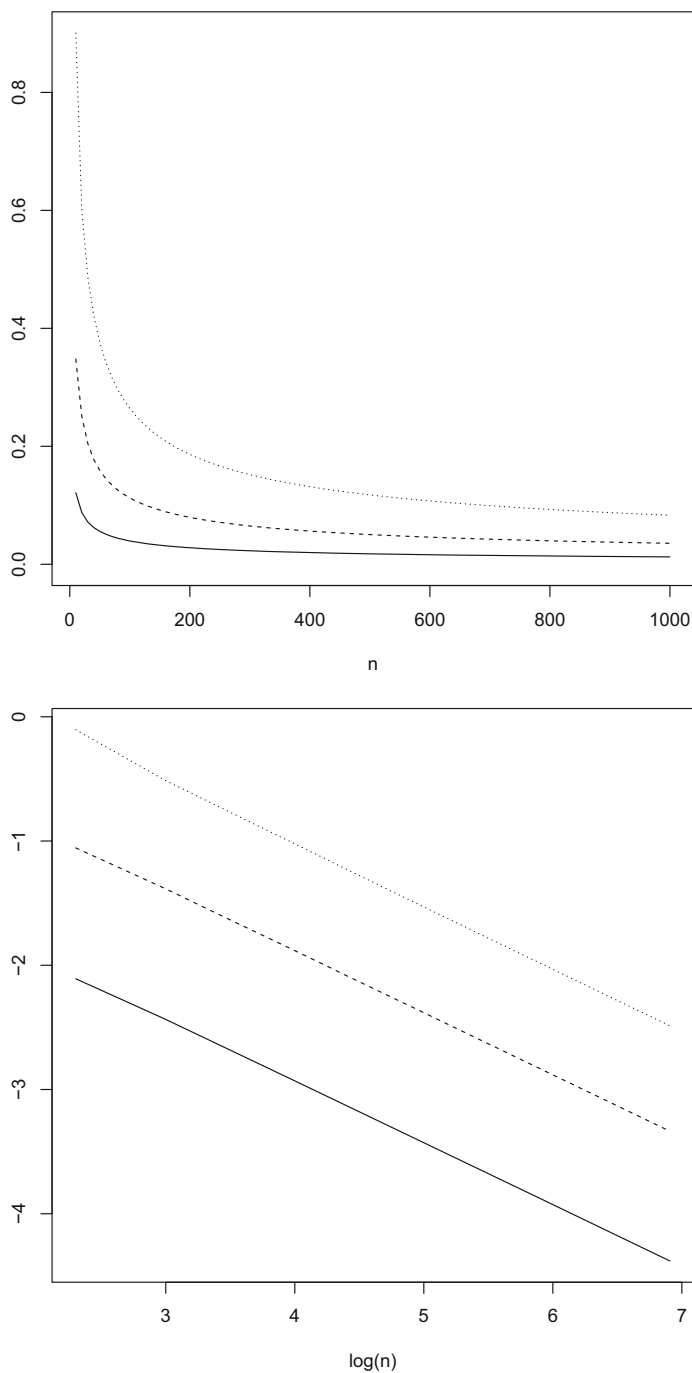


Fig. 3 Expected absolute differences $E[|E[X_i|S_n] - E[X_i]|]$ in function of n (original scale in top panel, log-log scale in bottom panel) for X_i in Group 1 (solid line appearing at the bottom), in Group 2 (broken line appearing in the middle), and in Group 3 (dotted line appearing at the top)

Let us show that the result of Proposition 4.4 does not hold for this comonotonic risk-sharing rule. If i is even, we have that

$$\begin{aligned} h_{i,n}(S_n) - \frac{E[X_i]}{2} &= \frac{1}{2} \frac{S_n}{n} \mathbb{I} \left[\frac{S_n}{n} < E[X_i] \right] + \left(\frac{3}{2} \frac{S_n}{n} - E[X_i] \right) \mathbb{I} \left[\frac{S_n}{n} \geq E[X_i] \right] - \frac{E[X_i]}{2} \\ &= \frac{1}{2} \left(\frac{S_n}{n} - E[X_i] \right) \mathbb{I} \left[\frac{S_n}{n} < E[X_i] \right] + \frac{3}{2} \left(\frac{S_n}{n} - E[X_i] \right) \mathbb{I} \left[\frac{S_n}{n} \geq E[X_i] \right]. \end{aligned}$$

Hence, we deduce that

$$\left| h_{i,n}(S_n) - \frac{E[X_i]}{2} \right| \leq 2 \left| \frac{S_n}{n} - E[X_i] \right|$$

and then, for $\varepsilon > 0$,

$$\mathbb{P} \left[\left| h_{i,n}(S_n) - \frac{E[X_i]}{2} \right| > \varepsilon \right] \leq \mathbb{P} \left[\left| \frac{S_n}{n} - E[X_i] \right| > \frac{\varepsilon}{2} \right].$$

By the weak law of large numbers, this implies that $h_{i,n}(S_n)$ converges in probability to $E[X_i]/2$. Proceeding in a similar way, we see that if i is odd then $h_{i,n}(S_n)$ converges in probability to $3E[X_i]/2$.

A systematic study of the properties investigated in this paper for general comonotonic risk-sharing rules is a relevant topic for future research.

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Declarations

Competing Interests The authors declare no competing interests.

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