Characterization of the ordering of path-complete stability certificates with addition-closed templates

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Abstract

As part of the development of Lyapunov techniques for cyber-physical systems, we study and compare graph-based stability certificates with respect to their conservatism. Previous work have highlighted the dependence of this ordering with respect to the properties of the chosen *template* of candidate Lyapunov functions. We extend here previous results from the literature to the case of templates closed under addition, as for instance the set of quadratic functions. In this context, we provide a characterization of the ordering, using an approach based on abstract operations on graphs, called lifts, which encode in a combinatorial way the algebraic properties of the chosen template. We finally provide a numerical method to algorithmically check the ordering relation.

1 Introduction

In this manuscript, we are interested in discrete-time switched dynamical systems described by

$$x(k+1) = f_{\sigma(k)}(x(k)) \tag{1}$$

where the state x lies in $\mathbb{R}^n$ at each time $k \in \mathbb{N}$, and the switching signal $\sigma : \mathbb{N} \to \langle M \rangle$ regulates the switching between the M subsystems of $F := \{f_1, \cdots, f_M\} \subset \mathcal{C}^0(\mathbb{R}^n, \mathbb{R}^n)$. In our setting, $\sigma : \mathbb{N} \to \{1, \ldots, M\}$ is supposed to be an exogenous and unpredictable input, and thus behaviors of (1) are defined and studied over all the possible switching signals. For an overview of this setting, we refer to [15, 16]. In this framework, a common tool to analyze stability/stabilizability of (1) is provided by Lyapunov theory. Indeed, it has been proven that the system (1) is stable if and only if a common Lyapunov function (i.e. a function decreasing along trajectories of *any* possible subsystem) exists, see [14]. On the other hand, the hybrid nature of (1) has led, in the past decades, to the study of stability criteria involving several positive definite functions, whose joint behavior implies the convergence/stability of trajectories. These results, collected under the name of *multiple Lyapunov criteria*, provide more flexible conditions in studying systems (1), both from a theoretical and numerical point of view. As a non-exhaustive list of remarkable results in this setting, we recall [5, 15, 10, 16].

More recently (for example in [2, 20, 19, 1, 9, 6]), multiple Lyapunov framework has been extended to the case in which inequalities involving the candidate functions are encoded in labeled and directed graphs. For studying stability in the arbitrary switching signals case, the graph defining the structure of the inequalities has to recognize (in an automata theory sense) every possible switching sequence, in which case it is usually called a *path-complete graph.* A connection between this graph framework (also called *path-complete Lyapunov functions setting*) and the general multiple Lyapunov functions approach is provided in [13]: a set of inequalities involving multiple Lyapunov functions is a valid certificate for stability if and only if the corresponding graph describing the inequalities is path-complete.

In the path-complete Lyapunov functions framework, the stability conditions strongly depend on the underlying combinatorial structure (the chosen path-complete graph), both from the numerical complexity and theoretical conservatism point of view. A natural problem is then to study and characterize, in a graph-theory setting, the conservatism level of (the conditions arising from) a given graph, in proving stability of (1). This indeed provides formal guarantees which can be used to properly choose a suitable stability criterion, given a particular switched system (1). This problem was first introduced in [22] and has been recently tackled in [7, 8]. In particular, the following fact is highlighted in [7]: the conservatism level of a path-complete criterion does not depend only on the combinatorial structure (the graph) but also on the *template*, i.e. the set of candidate Lyapunov functions in which the solutions of the problem are searched. More specifically, the relations between properties of the chosen template and the conservatism level are studied, providing a purely combinatorial characterization in the case of templates closed under pointwise minimum and maximum.

In this paper, we continue the analysis and provide a complete characterization of templates closed under addition. The main technical tools in our proofs are the notion of *sum lift* of graphs, already introduced in [7], and the concept of simulation between graphs already used, for comparison of path-complete criteria, in [20]. By the help of these concepts we provide the following equivalence result: a graph is more conservative than another for all the templates closed under addition, if and only if the *sum lift* of the first *simulates* the second graph. While the "if" part was already sketched in [7], the "only if" part, which was only conjectured, is more challenging, in that it amounts to demonstrate that whichever template is used, the algebraic property of being closed under addition is completely expressed by the sum lift operation, which is a purely combinatorial operation on graphs. As a second challenge in our proofs, the sum lift of a given graph is a graph with infinitely many nodes. However, we manage to circumvent this difficulty, and provide a proof that one may restrict oneself to a finite truncation of this graph without loss of generality, providing a finite procedure for the decision procedure. As it turns out, this finite procedure can even be made to run in polynomial time.

This result provides a step forward for the analysis and taxonomy of multiple *quadratic or SOS* Lyapunov criteria, since both these sets are closed under addition. Moreover, we provide a numerical appealing method (in the form of *linear programming*) to compare different path-complete criteria (in term of conservatism) in the case of templates closed under addition.

The rest of this manuscript is organised as follows: We start by reminding the main concepts of the path-complete Lyapunov framework in Section 2, and we motivate the restriction of the comparison of path-complete graphs to templates closed under addition. In Section 3, we draw up our major result which provides a simulation-based characterization of template-dependent ordering of graphs for this class of templates. We derive from this theorem a numerical algorithm to check the simulation relation in Section 4, and we present a numerical example in Section 5. Finally, we summarize our work and we present possible further extensions in Section 6.

Notation: Given $M, n \in \mathbb{N}$, we denote by $\langle M \rangle := \{1, \ldots, M\}$, $\mathcal{C}^0(\mathbb{R}^n, \mathbb{R}^n)$ the set of continuous vector fields on $\mathbb{R}^n$ and $\mathcal{C}^0_+(\mathbb{R}^n, \mathbb{R})$ the set of continuous, positive definite and radially unbounded functions. Given two sets A and B, A^B denotes the set of functions defined on B with values in A. $\mathcal{P}(A)$ denotes the power set of A, i.e. the set of all subsets of A, and $\mathcal{P}_0(A) := \mathcal{P}(A) \setminus \emptyset$.

2 Preliminaries

In this section, we start by reminding the main concepts of the path-complete Lyapunov formalism, and we formally define the conservatism-based comparison of path-complete graphs.

2.1 Graph theory and path-complete stability criteria

A directed and labeled graph on the alphabet $\langle M \rangle$ is a couple $\mathcal{G} = (S, E)$ where S denotes the set of nodes, and $E \subseteq S \times S \times \langle M \rangle$ is the set of directed edges labeled by an element of the alphabet. The following graph property of *path-completeness* introduced in [2] is the key concept of the path-complete Lyapunov function theory.

Definition 1 (Path-complete graph)**.** *Given $M \in \mathbb{N}$, a graph $\mathcal{G} = (S, E)$ on the alphabet $\langle M \rangle$ is* path-complete *if, for any $K \geq 1$ and any sequence $\hat{\jmath} = (j_1 \ldots j_K) \in \langle M \rangle^K$, there exists a* path $\{(a_k, a_{k+1}, j_k)\}_{k=1,\ldots,K}$

such that $(a_k, a_{k+1}, j_k) \in E$, *for each* $1 \le k \le K$.

In the context of stability analysis of (1), path-complete graphs are used to define multiple Lyapunov functions structures, as formally defined in what follows. Since we will associate a candidate function to any node of the considered graph, we need to recall the concept of candidate Lyapunov function associated to a discrete set.

Definition 2 (Candidate Lyapunov functions)**.** *Given a finite set* S*, a* candidate Lyapunov function (associated to S) *is an indexed set* $V_S \in \left(\mathcal{C}^0_+(\mathbb{R}^n, \mathbb{R})\right)^S$, *i.e. a collection of elements* $\{V_s : s \in S\}$ *such that*

$$\forall s \in S: \ V_s \in \mathcal{C}^0_+(\mathbb{R}^n, \mathbb{R}).$$

Definition 3 (Path-complete Lyapunov Function)**.** *Given a switching system* $F = \{f_1, \ldots, f_M\} \subset \mathcal{C}^0(\mathbb{R}^n, \mathbb{R}^n)$ *of dimension* $n \in \mathbb{N}$*, a* path-complete Lyapunov function *(PCLF) for* F *is a pair* $(\mathcal{G}, V_S)$ *where* $\mathcal{G} = (S, E)$ *is a path-complete graph, and* $V_S := \{V_s \mid s \in S\} \in \left(\mathcal{C}^0_+(\mathbb{R}^n, \mathbb{R})\right)^S$ *is a candidate Lyapunov function such that the following inequalities are satisfied:*

$$\forall\, (a, b, i) \in E,\ \forall x \in \mathbb{R}^n:\ V_b(f_i(x)) \ \le\ V_a(x). \tag{2}$$

If this is the case, we say that V_S *is* admissible for $\mathcal{G}$ and F*, and we denote it by* $V_S \in PCLF(\mathcal{G}, F)$.

Given a switched system (1), the existence of a path-complete Lyapunov function is a sufficient condition for stability (see [2, Theorem 2.4] for the linear case and [22, Theorem 2.5] for the general nonlinear setting). Conversely, using the construction in [13, Theorem 3], it can be shown that a set of inequalities involving multiple candidate Lyapunov functions provides a sufficient condition for stability only if the corresponding graph is path-complete. Thus, the PCLF formalism completely characterizes the family of multiple Lyapunov criteria for switched systems.

2.2 Conservatism-based comparison of graphs

In practice, in any multiple Lyapunov approach, one usually focuses on a subset $\mathcal{V}$ of candidate Lyapunov functions, called a *template*, for which algorithms have been developed to numerically check the existence of a solution. For example, the set of quadratic functions

$$\mathcal{Q} := \{V(x) := x^\top P x \mid P \succ 0, x \in \mathbb{R}^n, n \in \mathbb{N}\}. \tag{3}$$

Therefore, the template, i.e. the set of functions in which the Lyapunov functions are searched, is one of the two structural components of any path-complete stability criterion, together with the path-complete graph which defines the structure of the inequalities composing the stability criterion.

In the following definition, we introduce a notion of comparison of path-complete graphs, with respect to a template and according to their conservatism.

Definition 4 (Template-dependent ordering)**.** *Consider two path-complete graphs* $\mathcal{G} = (S, E)$ *and* $\widetilde{\mathcal{G}} = (\widetilde{S}, \widetilde{E})$ *on the same alphabet* $\langle M \rangle$ *and a template* $\mathcal{V}$*. The graph* $\widetilde{\mathcal{G}}$ is $\mathcal{V}$-greater *than* $\mathcal{G}$*, denoted by* $\mathcal{G} \le_{\mathcal{V}} \widetilde{\mathcal{G}}$*, if and only if*

$$\begin{aligned} &\forall n \in \mathbb{N}, \forall F \in \mathcal{C}^0(\mathbb{R}^n, \mathbb{R}^n)^M : \Big[\exists U_S \in \mathcal{V}^S \ s.t. \\ &U_S \in PCLF(\mathcal{G}, F)\Big] \Rightarrow \Big[\exists W_{\widetilde{S}} \in \mathcal{V}^{\widetilde{S}} \ s.t.\ W_{\widetilde{S}} \in PCLF(\widetilde{\mathcal{G}}, F)\Big]. \end{aligned} \tag{4}$$

If the expression (4) is satisfied for any template of functions, the graph $\widetilde{\mathcal{G}}$ is said *greater than* the graph $\mathcal{G}$, denoted by $\mathcal{G} \le \widetilde{\mathcal{G}}$, which means that the graph $\widetilde{\mathcal{G}}$ is less conservative than the graph $\mathcal{G}$ (no matter the chosen template). This latter ordering relation[1] has already been studied and characterized in [22] by a graph property called the *simulation*.

[1]We do not require the relations $\le_{\mathcal{V}}, \le$ between graphs to be antisymmetric; for this reason and for the sake of rigor, we refer to $\le_{\mathcal{V}}, \le$ as *ordering relations* rather than order relations.

Definition 5 (Simulation)**.** *Given two graphs $\mathcal{G} = (S, E)$ and $\widetilde{\mathcal{G}} = (\widetilde{S}, \widetilde{E})$ on the same alphabet $\langle M \rangle$. The graph $\mathcal{G}$* simulates *$\widetilde{\mathcal{G}}$ if and only if there exists a function $R : \widetilde{S} \to S$ such that*

$$\forall (a, b, i) \in \widetilde{E} : (R(a), R(b), i) \in E.$$

Theorem 3.5 in [22] states that, given two path-complete graphs $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ on the same alphabet, the general ordering $\mathcal{G} \leq \widetilde{\mathcal{G}}$ is equivalent to the fact that $\mathcal{G}$ simulates $\widetilde{\mathcal{G}}$. In practice, regardless of the switched system F, a simulation relation R provides an explicit admissible solution to $\widetilde{\mathcal{G}}$ and F defined by

$$W_{\widetilde{S}} \;:=\; \{W_s(x) := V_{R(s)}(x) \mid s \in \widetilde{S}, x \in \mathbb{R}^n\},$$

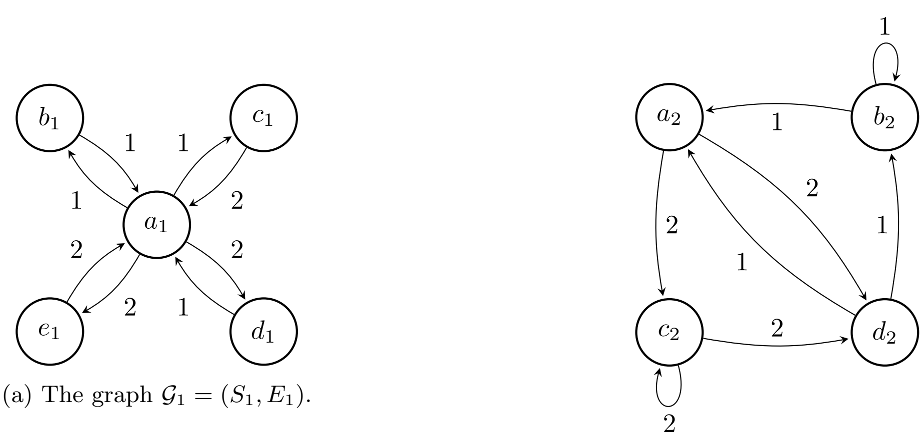
provided that the set $V_S := \{V_p \mid p \in S\}$ is admissible for $\mathcal{G}$ and F (see the proof of Theorem 3.5 in [22]). However, the general ordering of graphs is not sufficient to express template-dependent ordering as pointed out in [22, Example 3.9.]. In this example, the authors introduced two path-complete graphs for which no simulation relation holds. Still, it is shown in [22] that one of the two graph is structurally more conservative than the second one, when considering the template of quadratic functions. In what follows, we introduce a similar example underlying the dependence of the ordering relation on the template properties.

Example 1 (Motivating example)**.** *Consider two path-complete graphs $\mathcal{G}_1 = (S_1, E_1)$ and $\mathcal{G}_2 = (S_2, E_2)$ in Figure 1a and 1b respectively. One can prove that $\mathcal{G}_1$ does not simulate $\mathcal{G}_2$; Indeed, the graph $\mathcal{G}_2$ admits the loop $(b_2, b_2, 1)$ but $\mathcal{G}_1$ does not admit any loop. Therefore, it is impossible to associate the node b_2 to a node $R(b_2) \in S_1$ such that the loop $(R(b_2), R(b_2), 1) \in E_1$. We suppose now that the template $\mathcal{V}$ is closed under addition, i.e. for any $g_1, g_2 \in \mathcal{V}, g_1 + g_2 \in \mathcal{V}$. Given a switched system F with 2 modes and a solution $V_{S_1} := \{V_{a_1}, V_{b_1}, V_{c_1}, V_{d_1}, V_{e_1}\} \in \mathcal{V}^{S_1}$ admissible for $\mathcal{G}_1$ and F, one can build a solution $W_{S_2} := \{W_{a_2}, W_{b_2}, W_{c_2}, W_{d_2}\} \in \mathcal{V}^{S_2}$ admissible for $\mathcal{G}_2$ and F by defining*

$$W_{S_2} \;:=\; \begin{cases} W_{a_2} & := & V_{a_1} + V_{c_1}, \\ W_{b_2} & := & V_{a_1} + V_{b_1}, \\ W_{c_2} & := & V_{a_1} + V_{e_1}, \\ W_{d_2} & := & V_{a_1} + V_{d_1}. \end{cases} \tag{5}$$

Let us take for instance the edge $(a_2, d_2, 2) \in E_2$; The inequality encoded by this edge, i.e.

$$\forall x \in \mathbb{R}^n,\; \underbrace{V_{a_1}(f_2(x)) \;+\; V_{d_1}(f_2(x))}_{:=\; W_{d_2}(f_2(x))} \;\leq\; \underbrace{V_{a_1}(x) \;+\; V_{c_1}(x)}_{:=\; W_{a_2}(x)},$$



(a) The graph $\mathcal{G}_1 = (S_1, E_1)$.

(b) The graph $\mathcal{G}_2 = (S_2, E_2)$.

Figure 1: The path-complete graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ in Example 1. Even though the relation $\mathcal{G}_1 \leq \mathcal{G}_2$ does not hold, $\mathcal{G}_1 \leq_{\mathcal{V}} \mathcal{G}_2$ for any template $\mathcal{V}$ closed under addition.

is satisfied since $(a_1, d_1, 2)$ *and* $(c_1, a_1, 2) \in E_1$*. In terms of template-dependent ordering introduced in Definition 4, this means that the graph* $\mathcal{G}_2$ *is* $\mathcal{V}$*-greater than* $\mathcal{G}_1$ *for any template* $\mathcal{V}$ *closed under addition, i.e.* $\mathcal{G}_1 \leq_{\mathcal{V}} \mathcal{G}_2$*.*

3 Characterization for addition-closed templates

In [7] and [8], the authors highlight how the relation (4) strongly depends on the *closure property* of the considered template. Some common binary operations in the literature include the maximum, the minimum and the addition, for instance. This is explained by the fact that these operations conserve the positive definiteness and the radially unboundedness which are required to be a candidate Lyapunov function. In this paper, we focus on the binary operation of addition, and thus on the addition closure of a template, defined in what follows.

Definition 6 (Addition closure of a set of functions)**.** *Consider* $n, T \in \mathbb{N}$ *and a set of functions* $V \subseteq \mathcal{C}^0_+(\mathbb{R}^n, \mathbb{R})$*. The* T-addition closure *of* V*, denoted by* $V^{\oplus T}$*, is the set of functions defined by*

$$V^{\oplus T} \;:=\; \{g_1 + \cdots + g_T \mid g_j \in V, j = 1, \cdots, T\}\,.$$

The addition closure *of* V*, denoted by* $V^{\oplus}$*, is the union of the* T*-addition closures over* $\mathbb{N}$*, i.e.*

$$V^{\oplus} \;:=\; \bigcup_{T\in\mathbb{N}} V^{\oplus T}.$$

Definition 7 (Template closed under addition)**.** *Consider a template of candidate Lyapunov functions* $\mathcal{V}$*. The template* $\mathcal{V}$ *is* closed under addition *if it contains its addition closure, i.e.*

$$\mathcal{V}^{\oplus} \;\subseteq\; \mathcal{V}.$$

The comparison problem of path-complete stability criteria has been first tackled for templates closed under min and max operations in [8]. This analysis was simplified by an important property of max and min operators on space of functions: the idempotence. More formally, one has that, for any function f, $\max\{f, f\} = \min\{f, f\} = f$. This in particular implies that, given a finite set of functions V, the min closure of V (which can be defined by extension of Definition 6 for the pointwise minimum operation) is finite as well. Thanks to this feature, the authors in [7] managed to prove a simulation-based characterization of the ordering (4) for the family of templates closed under minimum and maximum (see [7, Theorems 3 and 4]).

In this section, we provide the complete characterization of the template-dependent ordering relation in Definition 4 in the context of template closed under addition. As opposed to the minimum and maximum, the addition is not idempotent and therefore involves infinite templates. Similarly to the min and max lifts used in [7, 8], we use the T*-sum* lift introduced in [8] as the central tool of our proof. The *sum lift*, defined as the union over $\mathbb{N}$ of all the T-sum lifts, in turn exploits the addition property, regardless of the number of terms in the addition. To this aim, $Multi^T(S)$ will refer to the set of *multi-sets*, i.e. sets with possible repetitions, with elements in the set S of cardinality $T \in \mathbb{N}$. The number of repetitions of an element s in a multi-set P is called the *multiplicity* of this element, and it is denoted by $m_P(s)$.

Definition 8 (Definition 10 in [8])**.** *Given* $T \in \mathbb{N}$ *and a graph* $\mathcal{G} = (S, E)$ *on the alphabet* $\langle M \rangle$*.*

(a) The T-sum lift *of* $\mathcal{G}$*, denoted by* $\mathcal{G}^{\oplus T} = (S^{\oplus T}, E^{\oplus T})$*, is defined as follows :*

(1) The set of nodes $S^{\oplus T}$ *is defined by*

$$S^{\oplus T} \;:=\; Multi^T(S).$$

(2) For each $i \in \langle M \rangle$ *and each multi-set of edges of* E *of the form* $\{(a_1, b_1, i), \ldots, (a_T, b_T, i)\}$ *such that* $\{a_1, \ldots, a_T\}$ *and* $\{b_1, \ldots, b_T\} \in S^{\oplus T}$*, the edge* $(\{a_1, \ldots, a_T\}, \{b_1, \ldots, b_T\}, i) \in E^{\oplus T}$*.*

(b) *The* sum lift *of $\mathcal{G}$, denoted by $\mathcal{G}^{\oplus} = (S^{\oplus}, E^{\oplus})$, is defined as the infinite disjoint union of the T-sum lifts, i.e.*

$$\mathcal{G}^{\oplus} := \bigcup_{T \in \mathbb{N}} \mathcal{G}^{\oplus T}. \tag{6}$$

Remark 1. *It is proven in [7, Theorem 3] that given a path-complete graph $\mathcal{G}$, for any value of $T \in \mathbb{N}$, both its T-sum lift and its sum lift are $\mathcal{V}$-greater than $\mathcal{G}$ for any template $\mathcal{V}$ closed under addition. This means that for such a template, one has that*

$$\mathcal{G} \leq_{\mathcal{V}} \mathcal{G}^{\oplus}, \tag{7}$$

and

$$\mathcal{G} \leq_{\mathcal{V}} \mathcal{G}^{\oplus T}$$

for any $T \in \mathbb{N}$. Indeed, if $\mathcal{V}$ denotes a template closed under addition and $V_S := \{V_s \mid s \in S\} \in \mathcal{V}^S$ is an admissible multiple Lyapunov function for a graph $\mathcal{G} = (S, E)$ and a switched system F, then the following candidate Lyapunov function $W_{S^{\oplus}}$ defined by

$$W_{S^{\oplus}} := \left\{ W_P := \sum_{p \in P} V_p \mid P \in S^{\oplus} \right\} \in \mathcal{V}^{S^{\oplus}}$$

is admissible for $\mathcal{G}^{\oplus}$ and F.

We are now able to state the main theorem of this work which provides a combinatorial characterization of the template-dependent ordering of graphs for the family of templates closed under addition.

Theorem 1. *Consider two path-complete graphs $\mathcal{G} = (S, E)$ and $\widetilde{\mathcal{G}} = (\widetilde{S}, \widetilde{E})$. The following statements are equivalent:*

(1) $\mathcal{G}^{\oplus}$ simulates $\widetilde{\mathcal{G}}$.

(2) $\mathcal{G} \leq_{\mathcal{V}} \widetilde{\mathcal{G}}$ for any template $\mathcal{V}$ closed under addition.

In order to prove this theorem, we need an auxiliary technical result which provides, given a path-complete graph $\mathcal{G}$, a switched system and a candidate Lyapunov function which satisfies a finite number of Lyapunov inequalities (encoded by the sum lift of $\mathcal{G}$) but violates a countable infinite number of Lyapunov inequalities as well. A crucial result is therefore provided be the following lemma.

Lemma 1. *For any path-complete graph $\mathcal{G} = (S, E)$ on the alphabet $\langle M \rangle$, there exist an integer $n \geq 1$, a switched system $F := \{f_i \mid i \in \langle M \rangle\}$ on M modes in dimension n and a candidate Lyapunov function $V_S := \{V_s \mid s \in S\}$ for which*

$$\forall (p, q, i) \in E,\ \forall x \in \mathbb{R}^n :\ V_q(f_i(x)) \leq V_p(x), \tag{8}$$

$$\begin{aligned} &\forall T \in \mathbb{N}, \forall (P, Q, i) \in \overline{E^{\oplus T}},\ \exists \tilde{x} \in \mathbb{R}^n : \\ &\sum_{q \in Q} V_q(f_i(\tilde{x})) > \sum_{p \in P} V_p(\tilde{x}), \end{aligned} \tag{9}$$

where $\overline{E^{\oplus T}} = (S^{\oplus T} \times S^{\oplus T} \times \langle M \rangle) \setminus E^{\oplus T}$ refers to the set of edges of the complement graph of $\mathcal{G}^{\oplus T}$.

The proof of this lemma relies in particular on the following observation regarding the definition of the edges of the T-sum lifts.

Proposition 1. *Consider a path-complete graph $\mathcal{G} = (S, E)$ on the alphabet $\langle M \rangle$, an integer $T \in \mathbb{N}$, two multi-sets P and Q of S of cardinality T and $i \in \langle M \rangle$. The edge (P, Q, i) is an element of $E^{\oplus T}$ if and only if there exists a perfect matching in the bipartite graph[2] $(P, Q, E_i(P, Q))$ where $E_i(P, Q) := \{(p, q, i) \in E \mid p \in P, q \in Q\}$ refers to the restriction of E to the edges of label i starting in P and ending in Q.*

[2]A *bipartite graph* $\mathcal{G} = (S, E)$ is a graph whose nodes S can be divided into two disjoint sets U and V such that each edge admits an extremity in U and the other in V. One often writes $\mathcal{G} = (U, V, E)$ to underline the partition of the set of nodes. A *perfect matching* of a bipartite graph is a set of edges without common vertices which covers every node of the graph.

Proof. Recalling Item (a) (2) of Definition 8, the edge (P, Q, i) is an element of $E^{\oplus T}$ if and only if there exists a multi-set of edges of E, denoted by $D := \{(a_1, b_1, i), \ldots, (a_T, b_T, i)\} \subseteq E$ such that $a_1, \ldots, a_T \in P$ and $b_1, \ldots, b_T \in Q$. Thus, D provides a perfect matching between the multi-sets P and Q composed by edges of $\mathcal{G}$ labeled by i. □

Hall's Marriage Theorem recalled hereafter provides a necessary and sufficient condition for the existence of a perfect matching.

Proposition 2 (Hall's Marriage Theorem [11])**.** *Let G be a finite bipartite graph with bipartite sets X and Y. There is an X-perfect matching, if and only if for every subset W of X,*

$$|W| \;\leq\; |N_G(W)|$$

where $N_G(W)$ denotes the neighborhood of W in G, i.e., the set of all vertices in Y adjacent to some element of W.

We are now able to prove Lemma 1.

Proof of Lemma 1. Given the graph $\mathcal{G} = (S, E)$ on the alphabet $\langle M \rangle$, we define a set of M block-diagonal $\{0, 1\}$-matrices $\{A_i \mid i \in \langle M \rangle\}$ of dimension $n := 2 \times M \times (2^{|S|} - 1)$, where $2^{|S|} - 1$ is the cardinality of $\mathcal{P}_0(S)$. Each 2×2 block is associated to a pair $(W, i) \in \mathcal{P}_0(S) \times \langle M \rangle$, and is defined by

$$A_j[W, i] \;:=\; \begin{cases} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \text{if } j = i, \\ \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \text{otherwise} \end{cases} \tag{10}$$

so that each matrix only acts on the blocks associated to a pair of the same label. We consider the template of primal copositive norms where $V_s(x) := {v_s}^\top x$ for $s \in S$ and $x \in \mathbb{R}^n_{\geq 0}$, and for which the vectors $\{v_s\}_{s \in S}$ are defined blockwise as well. In this context, satisfying the Lyapunov inequality associated to an edge $(p, q, i) \in S \times S \times \langle M \rangle$ amounts to satisfying a set of $2^{|S|} - 1$ scalar inequalities since

$$\begin{aligned} & \forall x \in \mathbb{R}^n_{\geq 0},\; V_q(f_i(x)) \;\leq\; V_p(x), \\ \Leftrightarrow \quad & \qquad {A_i}^\top v_q \;\leq_c\; v_p, \\ \Leftrightarrow \quad & \forall W \in \mathcal{P}_0(S),\; v_q[W, i]_2 \;\leq\; v_p[W, i]_1, \end{aligned} \tag{11}$$

where $\leq_c$ depicts the componentwise inequality. We define the blocks $v_s[W, i]$ for $s \in S$ by

$$v_s[W, i]_1 \;:=\; \begin{cases} 2 & \text{if } s \in PRE(W, i), \\ 1 & \text{otherwise,} \end{cases} \tag{12}$$

$$v_s[W, i]_2 \;:=\; \begin{cases} 2 & \text{if } s \in W, \\ 1 & \text{otherwise,} \end{cases} \tag{13}$$

where $PRE(W, i) = \{s \in S \mid \exists w \in W : (s, w, i) \in E\}$. We show now that this construction satisfies the expressions (8) and (9).

Let us start with the expression (8). Consider an edge $(a, b, i) \in E$. The condition (11) holds except if there exists a pair (W, i) such that $v_b[W, i]_2 = 2$ and $v_p[W, i]_2 = 1$. This happens if and only if $b \in W$ and $a \notin PRE(W, i)$. However $(a, b, i) \in E$ by assumption, which proves the contradiction. In all the other configurations, the inequality holds.

Let us now consider the expression (9), $T \in \mathbb{N}$ and $(P, Q, i) \in \overline{E^{\oplus T}}$. By the Hall's marriage theorem in Proposition 2, there exists a multi-set $W \subseteq Q$ such that $|W| > |N_{P,i}(W)|$, where the multi-set $N_{P,i}(W) := \{p \in P \mid \exists s \in W : (p, s, i) \in E\}$. Therefore, if we denote $\widetilde{W}$ as the underlying set of W formed from its distinct elements, we have

$$\sum_{q \in Q} v_q[\widetilde{W}, i]_2 \;:=\; \underbrace{\sum_{q \in W} v_q[\widetilde{W}, i]_2}_{:=\,2|W|} \;+\; \underbrace{\sum_{q \in Q \setminus W} v_q[\widetilde{W}, i]_2}_{:=\,T-|W|} \;=\; T + |W|$$

since for all $q \in W$, $q \in \widetilde{W}$ and $v_q[\widetilde{W}, i]_2 = 2$. Similarly,

$$\begin{aligned}\sum_{p \in P} v_p[\widetilde{W}, i]_1 \;&:=\; \underbrace{\sum_{p \in N_{P,i}(W)} v_p[\widetilde{W}, i]_1}_{:=\,2|N_{P,i}(W)|} \;+\; \underbrace{\sum_{p \in P \setminus N_{P,i}(W)} v_p[\widetilde{W}, i]_1}_{:=\,T-|N_{P,i}(W)|} \\ &=\; T + |N_{P,i}(W)|.\end{aligned}$$

Since $|W| > |N_{P,i}(W)|$, the inequality encoded by (P, Q, i) is violated. □

In order to to prove Theorem 1 we need an additional result, stated in the following Lemma 2. The proof is provided below and follows the notation introduced in the proof of Lemma 1.

Lemma 2. *Consider two path-complete graphs $\mathcal{G} = (S, E)$ and $\widetilde{\mathcal{G}} = (\widetilde{S}, \widetilde{E})$ on the alphabet $\langle M \rangle$. Assume that the switched system F and the candidate Lyapunov function $V_S := \{V_s \mid s \in S\}$ are constructed in the proof of Lemma 1 applied to $\mathcal{G}$. The following statement holds:*

$$\forall W_{\widetilde{S}} \in \left(V_S{}^{\oplus}\right)^{\widetilde{S}} \;\; s.t. \;\; W_{\widetilde{S}} \in PCLF(\widetilde{\mathcal{G}}, F), \;\; \exists T \in \mathbb{N} : W_{\widetilde{S}} \in \left(V_S{}^{\oplus T}\right)^{\widetilde{S}}, \tag{14}$$

where $V_S{}^{\oplus}$ and $V_S{}^{\oplus T}$ refer to the addition closure and the T-addition closure of V_S respectively.

Proof of Lemma 2. Without loss of generality, we assume that $\widetilde{\mathcal{G}}$ is strongly connected. Otherwise, the argument is valid for each strongly connected component of the graph. More precisely, each strongly connected component $\widetilde{\mathcal{H}} \subseteq \widetilde{\mathcal{G}}$ can be associated to a value of $T(\widetilde{\mathcal{H}})$ for which condition (14) holds. Moreover, any integer multiple of $T(\widetilde{\mathcal{H}})$ also satisfies the statement (14) for $\widetilde{\mathcal{H}}$. Thus, taking the least common multiple of the $T(\widetilde{\mathcal{H}})$, for all the strongly connected components $\widetilde{\mathcal{H}}$ of $\widetilde{\mathcal{G}}$, we can conclude.

Consider a path-complete graph $\widetilde{\mathcal{G}} = (\widetilde{S}, \widetilde{E})$ on $\langle M \rangle$ and $W_{\widetilde{S}} \in PCLF(\widetilde{\mathcal{G}}, F)$ an admissible solution in the template $\mathcal{V}$ defined as the sum-closure of $\{V_s\}$, i.e.

$$\mathcal{V} := \{V_{i_1} + \cdots + V_{i_T} \mid (i_1, \ldots, i_T) \in S^{\oplus T}, T \in \mathbb{N}\}.$$

Each node $\tilde{s}$ of $\widetilde{S}$ is associated to a multi-set of S denoted by $R(\tilde{s})$. Let us prove that for all $\tilde{s}_1, \tilde{s}_2 \in \widetilde{S}$, $|R(\tilde{s}_1)| = |R(\tilde{s}_2)|$. Assume by contradiction that there exists $i \in \langle M \rangle$, $\tilde{s}_1$ and $\tilde{s}_2 \in \widetilde{S}$ such that $(\tilde{s}_1, \tilde{s}_2, i) \in \widetilde{E}$ and $|R(\tilde{s}_2)| \neq |R(\tilde{s}_1)|$, and $|R(\tilde{s}_2)| > |R(\tilde{s}_1)|$ without loss of generality. Since $(\tilde{s}_1, \tilde{s}_2, i) \in \widetilde{E}$, it means that

$$\forall x \in \mathbb{R}^n_{\geq 0}, \;\; \sum_{q \in R(\tilde{s}_2)} V_q(A_i x) \;\leq\; \sum_{p \in R(\tilde{s}_1)} V_p(x), \tag{15}$$

$$\Leftrightarrow \;\; \forall (W, j) \in \mathcal{P}_0(S) \times \langle M \rangle, \; \sum_{q \in R(\tilde{s}_2)} v_q[W, j]_2 \;\leq\; \sum_{p \in R(\tilde{s}_1)} v_p[W, j]_1. \tag{16}$$

Consider W as the underlying set of $R(\tilde{s}_2)$ and $j = i$. Then by construction, $v_q[W, i]_2 = 2$ for all $q \in R(\tilde{s}_2)$. Therefore the sum over $R(\tilde{s}_2)$ is equal to $2|R(\tilde{s}_2)|$. By expression (16), we have that

$$\sum_{q \in R(\tilde{s}_2)} v_q[W, i]_2 \;=\; 2|R(\tilde{s}_2)| \;\leq\; 2|R(\tilde{s}_1)|$$

since for any $p \in S_1$, $v_p[W, j]_1 \leq 2$. However, we know by assumption that $|R(\tilde{s}_2)| > |R(\tilde{s}_1)|$ which contradicts the previous expression. It means that $|R(\tilde{s}_2)| = |R(\tilde{s}_1)|$. Since the graph $\widetilde{\mathcal{G}}$ is strongly connected, the inequality holds between all the nodes. □

We can now prove Theorem 1.

Proof of Theorem 1. (1) ⇒ (2) : By Theorem 3 in [7], the inequality $\mathcal{G} \leq_{\mathcal{V}} \mathcal{G}^{\oplus}$ is satisfied for any template $\mathcal{V}$ closed under addition. By assumption and recalling the simulation-based characterisation Theorem 3.5 in [22], $\mathcal{G} \leq \widetilde{\mathcal{G}}$. Then, by transitivity (see [8] for more details) of the ordering, we have

$$\mathcal{G} \leq_{\mathcal{V}} \widetilde{\mathcal{G}}$$

for this class of templates.

(2) ⇒ (1) : Consider two graphs $\mathcal{G} = (S, E)$ and $\widetilde{\mathcal{G}} = (\widetilde{S}, \widetilde{E})$ such that statement (2) in Theorem 1 is satisfied. Let us first apply Lemma 1 to $\mathcal{G}$. It provides a switched system $F := \{f_i \mid i \in \langle M \rangle\}$ and a candidate graph Lyapunov function $V_S := \{V_s \mid s \in S\}$ such that expressions (8) and (9) hold. Let us define the template $\mathcal{V}$ as the addition-closure of V_S. By construction, $V_S \subseteq \mathcal{V}$. This implies by hypothesis that there exists $W_{\widetilde{S}}$ in the template $\mathcal{V}$ such that $W_{\widetilde{S}} \in PCLF(\widetilde{\mathcal{G}}, F)$. Since $W_{\widetilde{S}} \subseteq \mathcal{V}$, we can associate a multi-set of S to each node of $\widetilde{\mathcal{G}}$, i.e. we can define a function $R : \widetilde{S} \to S^{\oplus}$ such that

$$W_s(x) \ := \sum_{p \in R(s)} V_p(x).$$

By Lemma 2, there exists $T \in \mathbb{N}$ such that $R : \widetilde{S} \to S^{\oplus T}$. Suppose by contradiction that the function R is not a simulation relation. This means that there exists an edge $(p, q, i) \in \widetilde{E}$ such that $(R(p), R(q), i) \notin E^{\oplus T}$. By the expression (9) in Lemma 1, there exists $\tilde{x} \in \mathbb{R}^n$ such that

$$W_q(x) := \sum_{s \in R(q)} V_s(f_i(\tilde{x})) \ > \sum_{d \in R(p)} V_d(\tilde{x}) := W_p(x),$$

i.e. the set $\{W_s \mid s \in \widetilde{S}\}$ is not admissible. But it is by construction, here is the contradiction. □

4 Algorithmic verification of the ordering for addition-closed templates

Although Theorem 1 provides a simulation-based characterization of the template-ordering (4) of graphs for the family of templates closed under addition, the simulation relation involves an infinite graph and then cannot be checked easily. This section aims to provide both algebraic and numerical methods to verify statement (a) in Theorem 1.

The following lemma points out that when a path-complete graph $\widetilde{\mathcal{G}}$ is simulated by the sum lift of another path-complete graph $\mathcal{G}$, we can restrict the simulation to a T-level of the sum lift for which the number of terms in the sum is fixed at $T \in \mathbb{N}$.

Lemma 3. *Consider two path-complete graphs $\mathcal{G} = (S, E)$ and $\widetilde{\mathcal{G}} = (\widetilde{S}, \widetilde{E})$. The following statements are equivalent:*

(1) $\mathcal{G}^{\oplus}$ simulates $\widetilde{\mathcal{G}}$.

(2) $\exists T \in \mathbb{N}$ such that $\mathcal{G}^{\oplus T}$ simulates $\widetilde{\mathcal{G}}$.

Proof. Without loss of generality, we assume that $\widetilde{\mathcal{G}}$ is strongly connected. The general case follows from a reasoning similar to the one in proof of Lemma 2.

The implication (2) $\Rightarrow$ (1) is direct since $S^{\oplus T} \subseteq S^{\oplus}$. For the reverse implication (1) $\Rightarrow$ (2), first we note that the sum lift, introduced in Definition 9, is the union of a countable number of strongly connected components, since it is the union of all the T-sum lifts, for any $T \in \mathbb{N}$. We now suppose that $\mathcal{G}^{\oplus}$ simulates $\widetilde{\mathcal{G}}$ via a function $R : \widetilde{S} \to S^{\oplus}$, as illustrated in Definition 5. Since by hypothesis $\widetilde{\mathcal{G}}$ is strongly connected, by the properties of simulation, the nodes $\{R(\tilde{s})\}_{\tilde{s}\in\widetilde{S}}$ are strongly connected in $\mathcal{G}^{\oplus}$. Thus, $\{R(\tilde{s})\}_{\tilde{s}\in\widetilde{S}}$ lie in the same strongly connected component of $\mathcal{G}^{\oplus}$ i.e. there exists a $T \in \mathbb{N}$ such that $\{R(\tilde{s})\}_{\tilde{s}\in\widetilde{S}}$ are nodes of $\mathcal{G}^{\oplus T}$. Thus, we have shown that $\mathcal{G}^{\oplus T}$ simulates $\widetilde{\mathcal{G}}$, concluding the proof. □

Thus, Lemma 3 provides us with a semi-algorithm in order to check the simulation by the sum lift. By iteratively checking condition (2) above for increasing $T \in N$, we get a sufficient condition for the simulation of $\widetilde{\mathcal{G}}$ by the sum lift. Though, a numerically appealing characterisation is still possible thanks to the definition introduced in [20, Definition IV.2] and recalled hereafter.

Definition 9. *Consider two path-complete graphs $\mathcal{G} = (S, E)$ and $\widetilde{\mathcal{G}} = (\widetilde{S}, \widetilde{E})$ on the same alphabet $\langle M\rangle$. We write*

$$\mathcal{G} \leq_{\sum} \widetilde{\mathcal{G}} \tag{17}$$

if there is a matrix $C \in \mathbb{R}_{\geq 0}^{|\widetilde{S}|\times|S|}$, satisfying $\forall \tilde{s} \in \widetilde{S} : \sum_{s\in S} C_{\tilde{s},s} > 0$, such that for any $n \in \mathbb{N}$, switched system $F := \{f_i \mid i \in \langle M\rangle\} \subset \mathcal{C}^0(\mathbb{R}^n, \mathbb{R}^n)$ and set of Lyapunov functions $V_S := \{V_s \mid s \in S\} \in PCLF(\mathcal{G}, F)$, the set of Lyapunov functions $U_{\widetilde{S}} := \{U_{\tilde{s}} \mid \tilde{s} \in \widetilde{S}\}$ such that

$$\forall \tilde{s} \in \widetilde{S}, \forall x \in \mathbb{R}^n : U_{\tilde{s}}(x) \ := \ \sum_{s\in S} C_{\tilde{s},s} V_s(x), \tag{18}$$

satisfies $U_{\widetilde{S}} \in PCLF(\widetilde{\mathcal{G}}, F)$. In this case, we say $\mathcal{G} \leq_{\sum} \widetilde{\mathcal{G}}$ through C.

In [20, Theorem IV.4.], it is shown that the existence of a matrix C satisfying the inequality (17) can be checked via a *linear program (LP)* with integer coefficients. Thus, if the problem is feasible, there is at least a solution to this LP with rational elements. Then, an integer-valued solution can be derived by multiplying a rational-valued solution by the least common multiple of the denominators and dividing by the greatest common divisor of the products. We summarize this discussion in the following statement.

Lemma 4. *Given two path-complete graphs $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ on the same alphabet. If $\mathcal{G} \leq_{\sum} \widetilde{\mathcal{G}}$, then there exists an integer matrix $C \in \mathbb{N}^{|\widetilde{S}|\times|S|}$ such that $\mathcal{G} \leq_{\sum} \widetilde{\mathcal{G}}$ through C.*

Before to state the main theorem in this section, let us provide a connection between the sum lift, the ordering $\leq_{\sum}$ and the simulation relation.

Lemma 5. *Consider two path-complete graphs $\mathcal{G} = (S, E)$ and $\widetilde{\mathcal{G}} = (\widetilde{S}, \widetilde{E})$ on the same alphabet. The following holds:*

1. *If $\mathcal{G}$ simulates $\widetilde{\mathcal{G}}$, then*
$$\mathcal{G} \leq_{\sum} \widetilde{\mathcal{G}}$$
through a matrix $C \in \{0, 1\}^{|\widetilde{S}|\times|S|}$.

2. *For any $T \in \mathbb{N}$,*
$$\mathcal{G} \leq_{\sum} \mathcal{G}^{\oplus T}$$
through a matrix $C \in \{0, 1, \ldots, T\}^{|S^{\oplus T}|\times|S|}$.

Proof. (1): Consider $R : \widetilde{S} \to S$ a simulation relation. Given a switched system $F := \{f_i \mid i \in \langle M\rangle\}$ and $V_S := \{V_s \mid s \in S\}$ an admissible set of Lyapunov functions for $\mathcal{G}$ and F, we know that the set

$$W_{\widetilde{S}} \ := \ \{W_s := V_{R(s)} \mid s \in \widetilde{S}\}$$

is admissible for $\widetilde{\mathcal{G}}$ and F. Therefore, if we define $C \in \{0,1\}^{|\widetilde{S}|\times|S|}$ such that for all $p \in \widetilde{S}$ and $q \in S$,

$$C_{p,q} = \begin{cases} 1, & \text{if } q = R(s), \\ 0, & \text{otherwise,} \end{cases}$$

we have that CV_S is equal to $W_{\widetilde{S}}$. Then, regardless of the switched system and the solution V_S admissible for $\mathcal{G}$, CV_S is admissible for $\widetilde{\mathcal{G}}$.

(2): Consider $s \in S$ and a multi-set $P := \{p_1, \cdots, p_T\} \in S^{\oplus T}$ of cardinality T. We define the matrix $C \in \{0, 1, \ldots, T\}^{|S^{\oplus T}|\times|S|}$ by

$$C_{P,s} \;=\; m_P(s).$$

Given a switched system $F := \{f_i \mid i \in \langle M\rangle\}$ and a set of Lyapunov functions V_S admissible for $\mathcal{G}$ and F. If we define $W_{\widetilde{S}} := CV_S$,

$$W_P := \sum_{p_i \in P} V_{p_i}$$

Therefore, $W_{\widetilde{S}}$ is admissible for $\mathcal{G}^{\oplus T}$ by construction of the T-sum lift (see the proof of [7, Theorem 3] for more details). □

We now show that the relation $\leq_{\sum}$ is transitive.

Lemma 6. *Consider three path-complete graphs $\mathcal{G} = (S, E)$, $\mathcal{G}^\star = (S^\star, E^\star)$ and $\widetilde{\mathcal{G}} = (\widetilde{S}, \widetilde{E})$ on the same alphabet $\langle M\rangle$. The following expression holds:*

$$\left(\mathcal{G} \leq_{\sum} \mathcal{G}^\star \;\wedge\; \mathcal{G}^\star \leq_{\sum} \widetilde{\mathcal{G}}\right) \;\Rightarrow\; \left(\mathcal{G} \leq_{\sum} \widetilde{\mathcal{G}}\right).$$

Proof. Consider $n \in \mathbb{N}$, a switched system $F := \{f_i \mid i \in \langle M\rangle\}$ on M modes and V_S a set of candidate Lyapunov functions. We assume that the graphs $\mathcal{G}$ and $\mathcal{G}^\star$ and the graphs $\mathcal{G}^\star$ and $\widetilde{\mathcal{G}}$ satisfy the inequality (17) through the matrices C_1 and C_2 respectively. Then, the following implications hold

$$V_S \in PCLF(\mathcal{G}, F) \;\Rightarrow\; C_1 V_S \in PCLF(\mathcal{G}^\star, F)$$
$$\Rightarrow\; C_1 C_2 V_S \in PCLF(\widetilde{\mathcal{G}}, F).$$

Therefore, $\mathcal{G} \leq_{\sum} \widetilde{\mathcal{G}}$ through the matrix $C := C_1\, C_2$ by definition. □

We are now able to prove the following characterization theorem between the sum lift simulation and the inequality (17) thanks to Theorem 1.

Theorem 2. *Given two graphs $\mathcal{G} = (S, E)$ and $\widetilde{\mathcal{G}} = (\widetilde{S}, \widetilde{E})$ on the same alphabet $\langle M\rangle$, the following statements are equivalent.*

(1) $\mathcal{G}^{\oplus}$ simulates $\widetilde{\mathcal{G}}$.

(2) $\mathcal{G} \leq_{\sum} \widetilde{\mathcal{G}}$.

Proof. (2) ⇒ (1) : Suppose that $\mathcal{G} \leq_{\sum} \widetilde{\mathcal{G}}$ and consider any template $\mathcal{V}$ closed under addition. By Lemma 4, we can assume without loss of generality that $\mathcal{G} \leq_{\sum} \widetilde{\mathcal{G}}$ through an integer matrix C. By Definition 9, for any switched system F and any admissible solution V_S to $\mathcal{G}$ and F, the candidate Lyapunov function $U_{\widetilde{S}} \in \mathcal{V}^{\widetilde{S}}$ defined by equation (18) is admissible for $\widetilde{\mathcal{G}}$ and F. Observe that this function is defined as the sum of functions in $\mathcal{V}$ (possibly with repetitions). Thus $\mathcal{G} \leq_{\mathcal{V}} \widetilde{\mathcal{G}}$ and by the characterization Theorem 1, this in particular implies that $\mathcal{G}^{\oplus}$ simulates $\widetilde{\mathcal{G}}$.

(1) ⇒ (2) : By Lemma 3, statement (1) implies that there exists $T \in \mathbb{N}$ and $R : \widetilde{S} \to S^{\oplus T}$ a simulation relation. By Item (1) of Lemma 5, $\mathcal{G}^{\oplus T} \leq_{\sum} \widetilde{\mathcal{G}}$ through a matrix $C_1 \in \{0,1\}^{|\widetilde{S}|\times|S^{\oplus T}|}$. Moreover, by Item (2) of Lemma 5, $\mathcal{G} \leq_{\sum} \mathcal{G}^{\oplus T}$ through a matrix $C_2 \in \langle T\rangle^{|S^{\oplus T}|\times|S|}$. By Lemma 6, we can conclude. □

Combining the results in Theorem 1 and Theorem 2, we can finally derive the following corollary which summarizes the algorithmic contribution of our work.

Corollary 1. *Given two graphs $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ on the same alphabet, the following statements are equivalent:*

(1) $\mathcal{G} \leq_{\mathcal{V}} \widetilde{\mathcal{G}}$ for any template $\mathcal{V}$ closed under addition.

(2) $\mathcal{G} \leq_{\sum} \widetilde{\mathcal{G}}$.

Therefore, since Item (2) can be verified by solving a linear program, so is Item (1).

Thanks to this result, one can use the LP characterization in [21] to check the existence of a simulation relation. We illustrate this result in the following example.

Example 2. *Consider the graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ in Figure 1. On one hand, we have proven in Example 1 that $\mathcal{G}_1 \leq_{\mathcal{V}} \mathcal{G}_2$ for any template closed under addition. By Theorem 1 and Lemma 3, this means that the 2-sum lift of $\mathcal{G}_1$ simulates $\mathcal{G}_2$. From Lemma 5, we can derive the matrices $C^1_{sum} \in \{0,1,2\}^{|S_1{}^{\oplus 2}| \times |S_1|}$ and $C^1_{sim} \in \{0,1\}^{|S_2| \times |S_1{}^{\oplus 2}|}$ such that*

$$\mathcal{G}_1 \leq_{\sum} \mathcal{G}_1{}^{\oplus 2} \text{ and } \mathcal{G}_1{}^{\oplus 2} \leq_{\sum} \mathcal{G}_2$$

through C^1_{sum} and C^1_{sim} respectively. Therefore, by Lemma 6, $\mathcal{G}_1 \leq_{\sum} \mathcal{G}_2$ through the matrix

$$C^1 \ := \ C^1_{sim} \times C^1_{sum} \ := \ \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

On the other hand, thanks to Theorem 2, we can solve the LP criterion to verify the conditions of Definition 9. We implemented the LP in Matlab *with the toolbox Yalmip [17] and the solver Mosek [4]. Due to the solver, we obtain a floating number solution; to obtain a rational solution, we thus round it and we check a posteriori the feasability, which is indeed still satisfied. Eventually, we get the following matrix*

$$C^1_{LP} \ := \ \begin{pmatrix} \frac{1373}{1408} & 0 & \frac{1373}{1408} & 0 & 0 \\ \frac{1373}{1408} & \frac{1373}{1408} & 0 & 0 & 0 \\ \frac{1373}{1408} & 0 & 0 & 0 & \frac{1373}{1408} \\ \frac{1373}{1408} & 0 & 0 & \frac{1373}{1408} & 0 \end{pmatrix}$$

which provides the same simulation relation by multiplying by $\frac{1408}{1373}$.

Consider now the graph $\mathcal{G}_3 = (S_3, E_3)$ in Figure 2. One can prove that the 8-sum lift of $\mathcal{G}_3$ simulates $\mathcal{G}_0 = (\{s_0\}, \{(s_0, S_0, 1), (s_0, S_0, 2)\})$, the common Lyapunov function graph with 2 modes. Indeed, given a switched system F, if the set of functions $V_{S_3} := \{V_{a_3}, V_{b_3}, V_{c_3}, V_{d_3}, V_{e_3}\}$ is admissible for $\mathcal{G}_3$ and F, the function

$$W \ := \ 2V_{a_3} + V_{b_3} + V_{c_3} + 2V_{d_3} + 2V_{e_3}$$

is a common Lyapunov function for F. By Lemma 5 and using the same construction as the previous example, one can derive the following matrix

$$C^3_{sim} \ := \ \begin{pmatrix} 2 & 1 & 1 & 2 & 2 \end{pmatrix},$$

through which the graphs $\mathcal{G}_3$ and $\mathcal{G}_0$ satisfy Definition 9. Furthermore, if we use the LP criterion provided in [21], we find (after rounding) the matrix $C^3_{LP} = \frac{1}{4} \times C^3_{sim}$ which provides the same simulation relation.

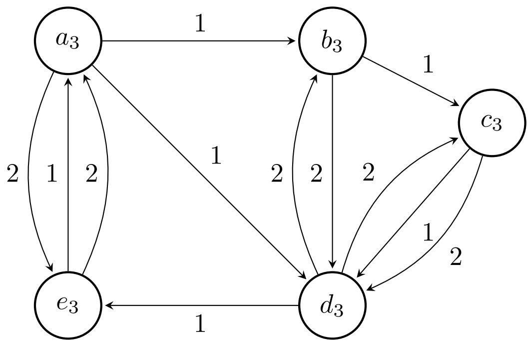


Figure 2: The graph $\mathcal{G}_3 = (S_3, E_3)$ of Example 2.

5 Numerical example

In this section, we consider the path-complete graph $\mathcal{G}_4 = (S_4, E_4)$, illustrated in Figure 3, and the common Lyapunov function graph $\mathcal{G}_0 = (\{s_0\}, \{(s_0, s_0, 1), (s_0, s_0, 2)\})$ on two modes. First, by [7, Proposition 3] and [8, Theorems 3 & 4], one can show that the graphs $\mathcal{G}_{4\min}$ and $\mathcal{G}_{4\max}$ both simulate $\mathcal{G}_0$. In this sense, $\mathcal{G}_4$ seems to be a very inefficient graph. Indeed, despite the fact that it defines a multiple Lyapunov function criterion with five node-functions, it is not more efficient than $\mathcal{G}_0$ for the class of templates closed under minimum or maximum. Thus, one might wonder whether it is more efficient than $\mathcal{G}_0$ with a template such as the quadratic Lyapunov functions, which is not closed under minimum nor maximum while being closed under addition. Thanks to the LP characterization in Theorem 2, we can prove that there does not exist any value of $T \in \mathbb{N}$ such that $\mathcal{G}_4^{\oplus T}$ simulates $\mathcal{G}_0$. By Theorem 1 and by Definition 4, this means that there exists at least a template $\mathcal{V}$ closed under addition for which

$$\begin{aligned} &\exists n \in \mathbb{N}, \exists F := \{f_1, f_2\} \subset \mathcal{C}^0(\mathbb{R}^n, \mathbb{R}^n) : \Big[\exists V_{S_4} \in \mathcal{V}^{|S_4|} \text{ s.t.} \\ &V_{S_4} \in PCLF(\mathcal{G}_4, F)\Big] \wedge \Big[\forall W_{S_0} \in \mathcal{V}^{|S_0|}, W_{S_0} \notin PCLF(\mathcal{G}_0, F)\Big]. \end{aligned} \tag{19}$$

In order to numerically verify this statement, we consider the template of quadratic functions $\mathcal{Q}$ defined in (3) and we sample randomly 10 000 2×2 linear switched systems with 2 modes of the form

$$x(k+1) \;=\; A_{\sigma(k)} x(k),$$

where $x(k) \in \mathbb{R}^2$ for any $k \in \mathbb{N}$, and $\mathcal{A} = \{A_1,\, A_2\} \subset \mathbb{R}^{2\times 2}_{\geq 0}$. For each system $\mathcal{A}$ and for any path-complete graph $\mathcal{G} = (S, E)$, we can compute the graph-based upper bound of the *joint spectral radius* (JSR) (see [12]

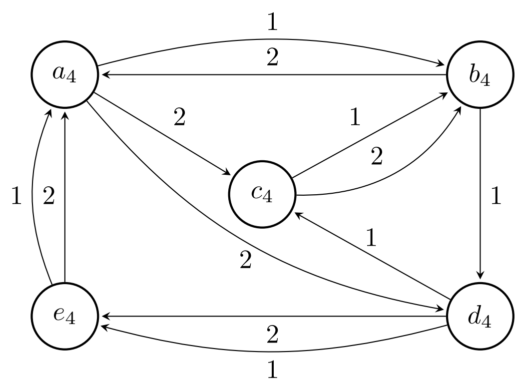


Figure 3: The graph $\mathcal{G}_4 = (S_4, E_4)$ of the numerical example in Section 5.

for more details), denoted by $\gamma_{\mathcal{G},\mathcal{V}}(\mathcal{A})$ and defined by

$$\gamma_{\mathcal{G},\mathcal{V}}(\mathcal{A}) \ := \ \min_{\gamma>0} \left\{ \exists V_S \in \mathcal{V}^{|S|} : V_S \in PCLF(\mathcal{G}, \mathcal{A}_\gamma) \right\}, \tag{20}$$

where $\mathcal{A}_\gamma$ refers to the scaled system

$$\mathcal{A}_\gamma \ := \ \left\{ \frac{1}{\gamma} A_i \mid i \in \langle M \rangle \right\}.$$

Therefore, if a switched system $\mathcal{A}$ satisfies statement (19) for the quadratic template, $\mathcal{G}_4$ will provide a strictly smaller upper bound than $\mathcal{G}_0$, i.e. $\gamma_{\mathcal{G}_4,\mathcal{Q}}(\mathcal{A}) < \gamma_{\mathcal{G}_0,\mathcal{Q}}(\mathcal{A})$. For each system $\mathcal{A}_\ell$, we compute $\gamma_{\mathcal{G}_0,\mathcal{Q}}(\mathcal{A}_\ell)$ and $\gamma_{\mathcal{G}_4,\mathcal{Q}}(\mathcal{A}_\ell)$ respectively for $\ell = 1, \ldots, 10\,000$, and we compare them by defining the following index:

$$I(\ell) \ = \ \log \left(\frac{\gamma_{\mathcal{G}_0,\mathcal{Q}}(\mathcal{A}_\ell)}{\gamma_{\mathcal{G}_4,\mathcal{Q}}(\mathcal{A}_\ell)} \right). \tag{21}$$

The distribution of this index is illustrated in Figure 4. As expected, the index I is non-negative for all the switched systems since it can be proven that $\mathcal{G}_0 \leq_{\mathcal{Q}} \mathcal{G}_4$ (it is easy to see that the common Lyapunov function graph $\mathcal{G}_0$ is more conservative than any other path-complete graph). On the other hand, the sampled systems also provide instances for proving the non-relation $\mathcal{G}_4 \not\leq_{\mathcal{Q}} \mathcal{G}_0$, i.e. satisfying (19). More specifically, the difference between the output of $\mathcal{G}_4$ and $\mathcal{G}_0$ is significant for 1668 systems out of 10 000, and thus these systems satisfy (19).

On the other hand, for 8332 out of the 10 000 sampled systems, $\mathcal{G}_4$ and $\mathcal{G}_0$ provide the exactly same approximation[3] of the JSR: i.e. the graph $\mathcal{G}_4$ provide the same JSR estimation as the classical common quadratic approach, which can be considered as the "most conservative" one. This numerical result is consistent with the theoretical results since we know that both graphs $\mathcal{G}_4$ and $\mathcal{G}_0$ provide the same approximation of the JSR if we consider a template closed under minimum or maximum. We highlight that the goal of this

[3] We assume that all the values of I smaller than 10^{-6} are due to numerical errors, and are then set at 0.

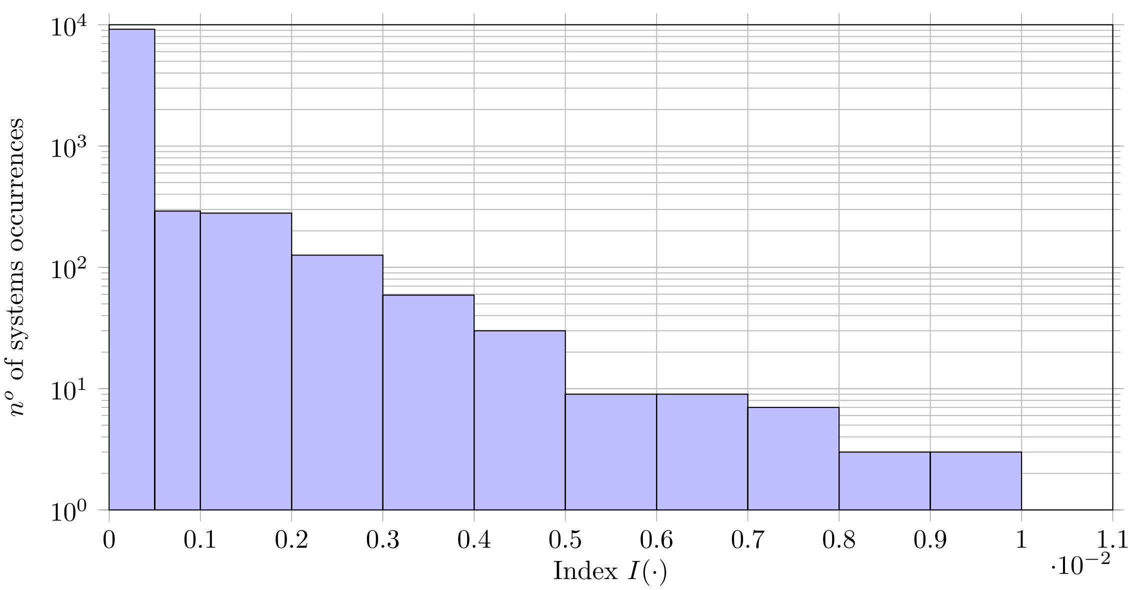


Figure 4: Histogram of the index $I(\cdot)$ in equation (21) for 10 000 linear switched systems with 2 matrices of dimension 2. Note that in the first column of the histogram, among the 9182 systems composing it, for 8332 of them, $I(i)$ is 0: for these systems, $\mathcal{G}_0$ and $\mathcal{G}_4$ provide exactly the same estimation of the JSR.

$\gamma_{\mathcal{G},\mathcal{V}}(\mathcal{A})$	$\mathcal{G}_0$	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$
$\mathcal{L}$	9.2696	9.1712	8.7019	9.2696	9.2696
$\mathcal{Q}$	9.5868	9.0161	8.6881	9.5868	9.4886

Table 1: Graph-based approximations of the JSR of system (22) with $\mathcal{G}_0$, $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$ and $\mathcal{G}_4$ for the templates of quadratic functions and primal copositive norms.

example was to validate numerically the "conservatism relations" between $\mathcal{G}_0$ and $\mathcal{G}_4$ (proven in Theorem 1), and *not* to provide numerically appealing approximation of the JSR: in this case the hierarchy of De Bruijn path-complete graphs (described in [7]) is more relevant.

To further analyse the relations between the graphs introduced in this paper, we focus on the switched system for which the index I is maximal, given by the matrices

$$A_1 = \begin{bmatrix} 1.5519 & 0.4474 \\ 7.6412 & 7.4716 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 0.4750 & 9.1755 \\ 1.8955 & 0.1850 \end{bmatrix}. \tag{22}$$

The graph-based results are reflected in the approximations of the JSR of system (22) in Table 1 provided by $\mathcal{G}_0$ and $\mathcal{G}_4$ and two different templates; First the template of linear primal copositive norms $\mathcal{L}$ which is closed under minimum and addition (see [7] for the details) and the template of quadratic functions $\mathcal{Q}$ which is closed under addition. As expected, both $\mathcal{G}_0$ and $\mathcal{G}_4$ provide the same approximation of the JSR with the copositive norms ($\gamma_{\mathcal{G}_0,\mathcal{L}}(\mathcal{A}) = \gamma_{\mathcal{G}_4,\mathcal{L}}(\mathcal{A}) = 9.2696$), while $\mathcal{G}_4$ provides a better approximation than $\mathcal{G}_0$ when we use the quadratic template ($\gamma_{\mathcal{G}_4,\mathcal{Q}}(\mathcal{A}) = 9.4886 < 9.5868 = \gamma_{\mathcal{G}_0,\mathcal{Q}}(\mathcal{A})$). Note also that for both graphs, the copositive norms provide a better approximation than the quadratic functions whereas they are easier to solve numerically.

We have also computed the approximations provided by the graphs $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ in Figures 1 and 2 for system (22). The results provided in Table 1 highlight the comparison relations that we have proved in previous examples: $\mathcal{G}_2$ provides a better approximation than $\mathcal{G}_1$ for both templates while $\mathcal{G}_3$ is as bad as the common Lyapunov function graph $\mathcal{G}_0$. The code can be found in [18].

To conclude this section, we want to notice that the "statistical" approach applied here provides a new route for open research: given two path-complete graphs, we would like to provide a *probabilistic conservatism-based relation* between them. More precisely, given two graphs, we want to compute/approximate the probability, given a random (with respect to a certain probability distribution) switched system, that the estimation provided by the first graph is better than the one provided by the second. This approach has already been addressed in [3] for instance.

6 Conclusion

In this paper, in the context of discrete-time switched systems, we provided a complete characterization of the conservatism-degree of stability conditions arising from graph-based structures. This characterization, already tackled in the past for the general case, is here proven for all the sets of candidate Lyapunov functions closed under addition. This theoretical result provides as a by-product a (polynomial time) decision procedure allowing to decide the ordering relation. These results provided a crucial step in the problem of classify stability criteria based on multiple quadratic Lyapunov functions, even if the complete characterization of the ordering for this specific template is still an open question. Future research will tackle the complete characterization for more general templates, and a probabilistic counterpart of the comparison problem studied here.

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