

Chapter 3

Risk measure

3.1 Introduction

This chapter is a preliminary chapter where we briefly review the theory of risk measure in finite case. We present here the fundamental results in a finite probability space that we will later use on chapter 4 and on chapter 5. We recall some implications from a mathematical programming point of view (see Shapiro et al. [104])¹ and the extension in a multi-period framework. We additionally introduce the risk measures used later in Chapter 4 and Chapter 5 namely the conditional value at risk and the good-deal. We discuss the good-deal, a less-known risk measure introduced by Cochrane and Saá-Requejo [31], and elaborate on its interpretation in terms of standard corporate finance and portfolio optimization.

3.2 Convex risk measure in short

The quantification of how risk affects the value of a portfolio, and the comparison of uncertain future cash-flows, received a particular attention in the past decade². This led to the introduction of certain functionals, called *risk measures*, of the random pay-off. In their seminal paper, Artzner et al. [7] presented and justified a set of four desirable properties that define coherent risk measures. Föllmer and Schied [51] extended the notion to convex risk measures.

¹Note that [104] focus on losses and use a slightly different presentation of the axioms.

²Notably the value at risk, a measure widely used in the industry, receives many critics. Artzner et al. [6] pointed several deficiencies as encouraging the accumulation of risk in particular scenarios.

We work in a finite probability space (Ω, P) and denote a scenario and its probability by ω and $\text{prob}(\omega)$. Consider a two stage context $(t = 0, 1)$: a random variable of future net worth accruing in the second stage $(t = 1)$ is described by a function $Z : \Omega \rightarrow \mathbb{R}$. We work with discounted quantities, i.e. value every asset in terms of zero-coupon bonds. Let $\mathcal{Z}$ be the space of all bounded measurable functions (containing the constants) on the space Ω and $\mathcal{P}$ the set of all probability measures on Ω .

A *risk measure* is a mapping ρ from the space $\mathcal{Z}$ to the real line. The risk measure of the random pay-off Z is a number $\rho(Z)$ that can be interpreted as capital requirement, i.e. the minimal (certain) amount of capital which when added to the position makes its future value acceptable. If it is negative, $-\rho(Z)$ can be withdrawn from the position and the position $Z + \rho(Z)$ is still acceptable.

Definition 3.1 (Convex risk measure). A convex (monetary) risk measure is a function $\rho : \mathcal{Z} \rightarrow \mathbb{R}$ satisfying the following axioms.

- *Monotonicity*: $\forall Z_1, Z_2 \in \mathcal{Z} : \text{if } Z_1 \leq Z_2, \text{ then } \rho(Z_1) \geq \rho(Z_2)$
- *Cash invariance*: $\forall Z \in \mathcal{Z} : \text{if } a \in \mathbb{R} \text{ then } \rho(Z + a) = \rho(Z) - a$
- *Convexity*: $\forall Z_1, Z_2 \in \mathcal{Z}, \forall t \in [0, 1] :$

$$\rho(tZ_1 + (1-t)Z_2) \leq t\rho(Z_1) + (1-t)\rho(Z_2)$$

The monotonicity axiom is very natural: an uncertain cash-flow Z_2 which is higher than Z_1 in every state of nature, needs less capital injection to be acceptable for an investor. Cash invariance, also called translation property, is justified by the existence of a perfectly liquid risk-free asset. Finally, the axiom of convexity imposes that diversification does not increase the risk. As noted by Föllmer and Schied [52], the idea becomes clearer when one notes that, for a monetary risk measure, convexity is in fact equivalent to the weaker requirement of

- *Quasi-Convexity*: $\forall Z_1, Z_2 \in \mathcal{Z}, \forall t \in [0, 1] :$

$$\rho(tZ_1 + (1-t)Z_2) \leq \max(\rho(Z_1), \rho(Z_2))$$

Every convex risk measure ρ takes only finite values on (Ω, P) and is Lipschitz continuous with respect to the infinity norm for the space $\mathcal{Z}$.

Definition 3.2 (Coherent risk measure). A convex risk measure is called coherent if it also satisfies the axiom of positive homogeneity.

- *Positive homogeneity*: $\forall \lambda \in \mathbb{R}^+, \forall Z \in \mathcal{Z} : \rho(\lambda Z) = \lambda\rho(Z)$

The positive homogeneity is particularly suited for measuring risk for traded assets in perfect liquid markets. It states that the risk grows linearly in the volume of the portfolio Z . When liquidity cannot be assured, the exposure might grow faster than linear (the bigger the position, the more difficult to convert it into cash). This fact initially motivated the relaxed notion of convex risk measures.

Definition 3.3 (Acceptance set). The acceptance set for any risk measure $\rho : \mathcal{Z} \rightarrow \mathbb{R}$ is defined by

$$\mathcal{A}_\rho := \{Z \in \mathcal{Z} | \rho(Z) \leq 0\}$$

The acceptance set $\mathcal{A}_\rho$ is the set of positions which are acceptable in the sense that they do not require additional capital. Clearly this set is closed, convex and non-empty for convex risk measures.

Important representation theorems have been derived, originally by Artzner et al. [7] in the coherent case. We recall here the representation theorem for convex risk measures.

Theorem 3.1 (Representation theorem due to Föllmer and Schied [51]). *Any convex risk measure has a dual representation :*

$$\rho(Z) = \sup_{Q \in \mathcal{M}} \{\mathbb{E}_Q[-Z] - \alpha(Q)\}$$

where $\mathcal{M} = \{Q \in \mathcal{P} : \alpha(Q) < +\infty\}$ is a closed and convex set of probability measures. The functional $\alpha(\cdot) : \mathcal{P} \rightarrow \mathbb{R} \cup \{-\infty\}$ is a penalty function; it is convex and continuous on $\mathcal{P}$.

This representation theorem states that every convex risk measure ρ can be represented as an expectation taken with respect to a probability measure Q (selected from some convex set) and penalized by the function α .

Corollary 3.1 (Representation of Coherent Risk Measure). *The risk measure ρ is coherent if and only if the penalty function α in Theorem 3.1 is an indicator function of a non-empty, closed and convex set $\mathcal{Q}_\rho$:*

$$\alpha(Q) = \begin{cases} 0 & \text{if } Q \in \mathcal{Q}_\rho \subset \mathcal{P} \\ \infty & \text{otherwise} \end{cases}$$

Definition 3.4 (Entropic Risk Measure). The entropic risk measure, also called expected exponential risk measure, of a random profit Z is defined as follows:

$$\rho^E(Z) = \gamma \ln \mathbb{E}_P \left[e^{\frac{-Z}{\gamma}} \right]$$

It is convex and admits the representation:

$$\rho^{\mathbb{E}}(Z) = \sup_{Q \in \mathcal{P}} \mathbb{E}_Q[-Z] + \frac{1}{\gamma} \sum_{\omega \in \Omega} q(\omega) \ln \frac{q(\omega)}{\text{prob}(\omega)}$$

Definition 3.5. [CVaR] The conditional value at risk $\text{CVaR}_\alpha(Z)$ of a random profit Z is the conditional expected loss $(-Z)$ over the $(1 - \alpha)$ % worst scenarios. The CVaR is a coherent risk measure and has the following representation

$$\text{CVaR}_\alpha(Z) = \sup_{Q \in \mathcal{Q}_{\text{CVaR}}} \mathbb{E}_Q[-Z]$$

where $\mathcal{Q}_{\text{CVaR}} = \{Q \in \mathcal{P} : 0 \leq q(\omega) \leq \frac{\text{prob}(\omega)}{\alpha}\}$.

Rockafellar and Uryasev [96] have shown that it can be computed by solving the linear program:

$$\text{CVaR}_\alpha(Z) := \min_{t \in \mathbb{R}} \{t + \alpha^{-1} \mathbb{E}[-Z - t]_+\}$$

The concept is illustrated in Figure 3.1, where the shaded area measures the CVaR.

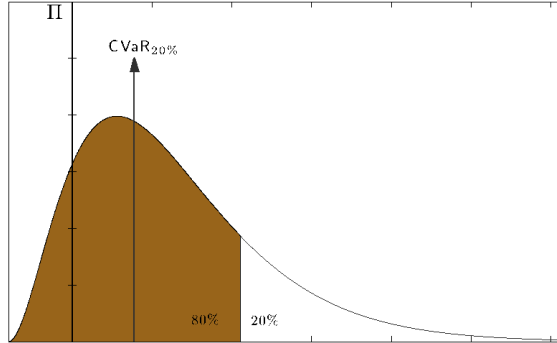


Figure 3.1: Illustration of the CVaR

We also want to emphasize that convex risk measures are not differentiable in general and may lead to non-smooth problem. We recall here the generic portfolio optimization's example studied by Lüthi and Doege [79].

Problem 3.1. [Generic Portfolio Optimization] An investor, having a convex risk measure as disutility function, optimizes its portfolio by choosing a decision vector $s \in S \subset \mathbb{R}^m$ which minimizes its risk:

$$\inf_{s \in S} \left\{ \sup_{Q \in \mathcal{M}} \{-\alpha(Q) + \mathbb{E}_Q[-As - b]\} \right\}$$

where $A \in \mathbb{R}^{|\Omega| \times m}$, $b \in \mathbb{R}^{|\Omega|}$ and S is a nonempty, convex and compact set. The vector b can be seen as an uncertain pay-off that the investor wants to hedge by a portfolio of financial products. The decision vector s gives the weights of the portfolio, composed of m different assets. The net value of this portfolio in the different scenarios is given by As .

The objective function of Problem 3.1 is the following:

$$f(s) = \sup_{Q \in \mathcal{M}} \{-\alpha(Q) + \mathbb{E}_Q[-As - b]\}$$

The function $f(s)$ is clearly non-smooth and the subgradients of $f(s)$ are given by :

$$\partial f(s) = -A^t \hat{Q}(s)$$

where $\hat{Q}(s)$ is an optimal probability measure for measuring the risk of the position s . It is yet differentiable in particular whenever $\alpha(\cdot)$ is continuous and strictly convex (i.e. the solution $\hat{Q}(s)$ is unique). This is for example the case for entropy based risk measures where $\alpha(Q) = \sum_{\omega \in \Omega} \mu(\omega) \frac{q(\omega)}{\text{prob}(\omega)} \ln \frac{q(\omega)}{\text{prob}(\omega)}$, as the entropic risk measure. Actually, the structure of the generic portfolio is the one of non smooth problem studied in Nesterov [84], for which efficient approximation algorithms based on smoothing techniques exist.

Finally, some risk measures can also be represented as mathematical programs with differentiable objective function. This is the case of the popular CVaR risk measure that can be represented as a linear program [96].

3.3 The good-deal risk measure

3.3.1 Framework and definition

Asset pricing (valuation of an uncertain pay-off) can be approached by two polar approaches: the arbitrage pricing and the equilibrium theory. The arbitrage pricing, also called relative pricing, is based on replication arguments. The absence of arbitrage opportunity allows one to uniquely value uncertain pay-offs in a complete market by perfect hedging. The prices are determined on the basis of underlying traded assets and without reference to agents' risk-aversion. The Black-Scholes option pricing and most financial engineering models are classical examples. This pricing methodology breaks down in incomplete markets. The price of an asset is no longer unambiguously determined by its correlation to other assets. It is still possible to bound the price by no arbitrage consideration (super replication), but it is well recognized that such bounds are too wide to be helpful in practice. In the equilibrium theory, the asset price results from an equilibrium among agents who maximize their utility. The prices are function of the assets exposure to fundamental sources of macroeconomic risk. The consumption-based and general equilibrium models are examples of this approach.

Cochrane and Saá-Requejo introduced the good-deal pricing as a mix of arbitrage valuation and equilibrium theory. It gives tight bounds on prices of uncertain pay-offs, that the authors call good-deal bounds. They impose that the pricing kernel rule out arbitrage opportunities (i.e. the price kernel should be positive) and assets with too high Sharpe ratio because, similarly to arbitrage opportunities, those "good-deals" would quickly disappear as investors would immediately grab them. It is common in finance to regard high Sharpe ratios as "good-deals" that are not sustainable at equilibrium³.

Problem 3.2. [Good-deal bounds from [31]] Consider an incomplete market with I traded assets, and a pay-off Z that one wants to value. The vector f_0 is formed by the I assets prices at $t = 0$: $f_0 \in \mathbb{R}^{|I|} = (f_{0,i})_{i \in I}$ and the vector $f_1(\omega)$ is the assets prices realizations at scenario ω : $f_1(\omega) \in \mathbb{R}^{|I|} = (f_{1,i}(\omega))_{i \in I}$. The good-deal theory give valuation bounds (noted $\underline{\pi}$ and $\bar{\pi}$) of the random pay-off Z , that are obtained by solving problems of the form:

$$\begin{aligned} \underline{\pi} = & \min_{m(\omega)} \mathbb{E}_P[m(\omega)Z(\omega)] \\ \text{s.t.} \quad & m(\omega) \geq 0 \\ & f_0 = \mathbb{E}_P[m(\omega)f_1(\omega)] \\ & \text{Var}[m] \leq \left(\frac{h}{R_f}\right)^2 \end{aligned}$$

³For example, the CAPM theory specifies that the market portfolio is mean-variance efficient, i.e. no asset can have a higher Sharpe ratio than the market.

According to the good-deal theory, the lower bound for the price of asset Z is given by $\mathbb{E}_P[m(\omega)Z(\omega)]$, where m is a stochastic discount factor, i.e. a stochastic vector that generates prices from pay-offs : $p = \mathbb{E}_P[m(\omega)x(\omega)]$, where p is the price of the pay-off x (we refer to [29] for a more complete discussion on stochastic discount factors). This discount factor is constrained to be positive and to price selected traded assets $i \in I$, which implies that there is no arbitrage opportunity (first and second constraints). Finally, it also imposes that no cash-flows priced by m can have a Sharpe ratio greater than h . This is imposed by the last constraint which is actually the famous bound of Hansen and Jagannathan [58]⁴. The measure thus rules out high Sharpe ratio (no "good deals") as well as pure arbitrage opportunities.

The upper good-deal bound is obtained by solving the same problem but replacing minimization by maximization.

Cerny and Hodges [27] extend the work of Cochrane and Saá-Requejo and present a theory of generalized arbitrage pricing that is based on the absence of attractive investment opportunities at equilibrium but does not necessarily invoke Sharpe ratio. Price bounds can then be derived in their theory by considering super-hedging and arbitrage arguments. Jaschke and Küchler [68] show that the mathematical structure behind coherent risk measures and good-deal bounds is exactly the same, i.e. to any generalized arbitrage bounds correspond an essentially equivalent coherent risk measure.

The coherent risk measure associated to the bounds of Cochrane and Saá-Requejo [31] based on Sharpe ratio is given below (Problem 3.3). It is obtained by reformulating Problem 3.2, using the variables $q(\omega) = m(\omega)\text{prob}(\omega)$ and focusing on capital requirement instead of price bounds.

Problem 3.3. [Good-deal risk measure] The good-deal risk measure is defined as

$$\rho^{\text{GD}}(Z) = \sup_{Q \in \mathcal{Q}_{\text{GD}}} \mathbb{E}_Q[-Z]$$

where $\mathcal{Q}_{\text{GD}}$ is the following convex and compact set:

$$\mathcal{Q}_{\text{GD}} = \left\{ Q \in \mathcal{P} : q(\omega) \geq 0; \mathbb{E}_Q[f_1] = f_0; \mathbb{E}_P \left[\left(\frac{dQ}{dP} \right)^2 \right] \leq A^2 \right\}$$

The interpretation of the good-deal risk measure is similar to the good-deal bounds. The capital requirement of Z is computed in a way that does not

⁴Indeed, all assets i priced by the discount factor m satisfy

$$\frac{(\mathbb{E}[R^i - R^f])^2}{\text{Var}[R^i]} \leq \frac{\text{Var}[m]}{(R^f)^2},$$

where R^i is the return of asset i and R^f the risk-free return

introduce arbitrage opportunities (i.e. accounts for the risk exposure of Z to the different assets $i \in I$ and does not ask more (or less) capital requirement for the risk that can be hedged), but also rules out Sharpe ratio greater than $h := \sqrt{A^2 - 1}$. In the context of risk measure, the scalar A is related to the investor view about what the market would consider as "good-deal" (i.e. he will accept to pay for an uncertain payoff not more than the price that would turn it into an opportunity to grab for the market).

The good-deal risk measure is coherent and already formulated in the "representation theorem" form (cf. Theorem 3.1) of coherent risk measure. It requires to solve a Second Order Cone Program (SOCP)⁵[11]). Finally, the formulation of the good-deal in continuous time and in term of equivalent martingale measure can be found in Carassus and Temam [23], Björk and Slinko [17]. They also provide very interesting results of option pricing in incomplete market.

As an illustration of the good-deal risk measure, Figure 3.2 shows the difference between the physical probability measure P and the one of the good-deal risk measure Q . One clearly sees that this probability is well risk averse and puts more probability weights on the bad events.

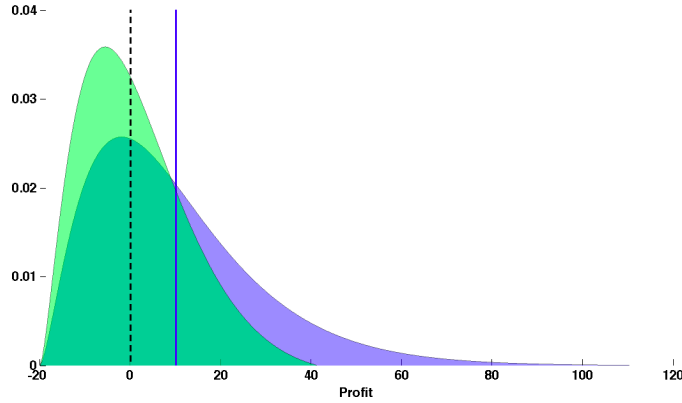


Figure 3.2: Cumulative distribution function of a pay-off under the true probability measure P (blue) and under the one defined by the good-deal (green).

⁵Recall that $\mathbb{E}_P[(dQ/dP)^2] = \sum_{\omega} \frac{q(\omega)^2}{\text{prob}(\omega)}$ and so the inequality $(\frac{q(\omega)}{\sqrt{\text{prob}(\omega)}}, A) \in \mathbb{L}^{\Omega+1}$, defines a Lorentz cone.

3.3.2 The dual representation of the good-deal

Coherent risk measures have both primal and dual representations. In a finite probability space, the good-deal risk measure admits the following Lagrangian dual formulation.

Problem 3.4. The dual of the conic problem 3.3 is the following conic problem:

$$\begin{aligned} \rho^{\text{GD}}(Z) &= \min_{\mathbf{w} \in \mathbb{R}^{|I|}, \eta \in \mathbb{R}_+^{|\Omega|}} A(\mathbb{E}_P[\eta^2])^{\frac{1}{2}} - f_0^T \mathbf{w} \\ \eta(\omega) &\geq f_1(\omega)^T \mathbf{w} - Z(\omega) \end{aligned}$$

It is known (see e.g [27, 17]) the problem of hedging is dual to the problem of pricing and it is thus not surprising that Problem (3.4) has a natural interpretation in terms of portfolio replication. The replicating portfolio is constituted in $t = 0$ and composed of $i \in I$ assets with weight $\mathbf{w}$. Because the market is incomplete, this portfolio cannot perfectly hedge the random cash-flow Z . The variable η in Problem (3.4) is a measure of the under performance. Indeed, the optimality conditions of (3.4) give :

$$\eta(\omega) = \max(0, f_1(\omega)^T \mathbf{w} - Z(\omega)) \quad (3.1)$$

The vector η is the one of the positive deviations from Z . In other words, when $\eta(\omega)$ is positive, the portfolio in the scenario ω gives a higher cash-flow than $Z(\omega)$ and thus badly hedges the possible loss of Z . The capital requirement for holding Z is then equal to the capital requirement of the portfolio (its cost) plus a measure of the under-performance equal to $\mathbb{E}[\eta^2]$. The expectation $\mathbb{E}[\eta^2]$ is called a regret in the literature (see e.g. Dembo and Rosen [37]). When $A \rightarrow \infty$, the appearance of the regret in the objective function of Problem 3.4 imposes that the replicating portfolio over-hedges Z and the good-deal value is equal to the bound computed under the sole absence of arbitrage⁶.

To sum up, the good-deal risk measure is a compensation in $t = 0$ for holding the uncertain payoff Z occurring in $t = 1$. It can be decomposed in the capital requirement in $t = 0$ of the portfolio which best hedges Z , plus an additional sum of money, to compensate one for a the un-hedged part of Z . Such additional value, can be as large up to make the compensation of holding Z a good-deal.

3.3.3 The good-deal and capital budgeting

The common practice in corporate finance is to use the Capital Asset Pricing Model (CAPM, from [105]) to discount cash-flows for investment decisions.

⁶Note that this interpretation can be directly derived from the primal problem (see Problem 3.3).

The primary implication of the CAPM is that there exists a positive linear relation between expected returns and the market portfolio return. However, over the last decades, many researchers find empirically that expected returns do not line up with CAPM beta, but rather with multifactors betas such as firm-size, value-to-price ratios, past returns and liquidity while market beta has no power (see [49, 30, 9] and the cited literature). This suggests to use the Arbitrage Pricing Theory (APT, from [98]) for capital budgeting. The different risk factors signify different types of systematic risk that cannot be neutralized by diversification. But the APT has also its own difficulties; no one knows the exact factors that constitute the systematic risk, the specification of a multifactors model is unsure and all the important parameters are rather poorly estimated. The good-deal recognizes that inability to accurately identify all the sources of systematic risk and accounts for the risk that is not captured by the factors. It values it by adding a safe measure⁷ of the possible losses associated to that risk, i.e. the expression $\sqrt{\mathbb{E}[\eta^2]}$ in Problem 3.4.

The assets $i \in I$ used in the good-deal play a role analogous to the risk factors in CAPM or APT models. In order to see it, consider the following complementarity condition obtained from the KKT conditions of Problem 3.3:

$$0 \leq q(\omega) \perp \frac{q(\omega)}{\text{prob}(\omega)} - \sum_{i \in I} \alpha_i f_{1,i}(\omega) - \alpha_z Z(\omega) \geq 0 \quad (3.2)$$

The corresponding strictly positive pricing kernel $m(\omega) = q(\omega)/\text{prob}(\omega)$ would be linear in the risk factors, as in the two cited theories. Alternatively one would immediately obtain a linear pricing kernel by dropping the non arbitrage conditions, as in the CAPM⁸ or APT.

⁷Note that both CAPM and APT would have discount that risk at risk-free rate.

⁸One can recover the CAPM formula by introducing the same assumptions of that theory. Suppose, as in the CAPM that the market portfolio is mean-variance efficient (i.e. no asset can have a higher Sharpe ratio) and consider only the risk-free asset (R_f) and the market portfolio (R_M) as risk factors; dropping the positivity constraint on the pricing kernel (that is, allowing for arbitrage opportunities), the good-deal risk measure of asset Z (in Problem 3.3) becomes

$$\begin{aligned} \rho^{\text{CAPM}}(Z) &= \sup_{\{\zeta \in \mathcal{Z}\}} \langle \zeta, Z \rangle \\ &\quad \langle \zeta, R_f \rangle = 1 \ ; \ \langle \zeta, R_M \rangle = 1 \\ &\quad \sigma^2 \left(\frac{\zeta}{\text{prob}} \right) \leq \frac{(\mathbb{E}[R_M] - R_f)^2}{\sigma_M^2 R_f^2} \end{aligned} \quad (3.3)$$

The optimality conditions of problem (3.3) give a pricing kernel $m(\omega) = \text{prob}(\omega)\zeta(\omega)$ equal to

$$m(\omega) = \frac{1}{R_f} - \frac{(\mathbb{E}[R_M] - R_f)}{\sigma_M^2 R_f^2} (R_M(\omega) - \mathbb{E}[R_M])$$

This pricing kernel is identical to the one of the CAPM (cf. [29]).

3.4 Multiperiod risk measure

In this section we discuss the extension of a static risk measure to a multistage setting, i.e. how to quantify the risk of a sequence of discounted⁹ future cash-flows $(Z_t)_{t=1,\dots,T}$ occurring at stages $t = 1, 2, \dots, T$. The interpretation of a multiperiod risk measure is a capital requirement that makes the whole process acceptable at every stage $t = 1, \dots, T$.

We work on a finite probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $(\mathcal{F}_t)_{t \in 1, \dots, T}$, i.e. a sequence of increasingly refined algebras of subsets of Ω . In other words, the sigma algebra $\mathcal{F}_t$ models the information available after each stage t . The pay-off incurred at stage t is represented by a function $Z_t \in \mathcal{Z}_t$, where $\mathcal{Z}_t$ denotes the space of measurable function on $(\Omega, \mathcal{F}_t)$.

Definition 3.6. A multiperiod risk measure is a function $\rho : \mathcal{Z} \rightarrow \mathbb{R}$, where $\mathcal{Z} = \mathcal{Z}_1 \times \mathcal{Z}_2 \times \dots \times \mathcal{Z}_T$.

The construction of multiperiod risk measures raises the question of time consistency. Judgements based on a risk measure may not be contradictory over time (i.e. a project preferred to another one in all states of the world in period t remains preferred in all states of the world in period $t - 1$). Time consistency has been formulated in different ways (see among [94, 8, 104]); we here focus on Shapiro's definition, which has been tailored to risk optimization but also applies to pure risk evaluation: a decision policy (or evaluation) should not involve states that cannot happen in the future [103]. We disregard risk models applying directly to the entire sum of Z , as they ignore the dynamic character of the problem and thus fail to be time consistent. We rather follow various authors [94, 99, 15, 104] and concentrate on dynamic risk measures.

Definition 3.7. A dynamic risk measure ρ consists of a composition mappings $\rho = (\rho_t)_{t=2,\dots,T}$ such that $\rho_t : \mathcal{Z}_t \rightarrow \mathcal{Z}_{t-1}$:

$$\begin{aligned} \rho(Z_1, \dots, Z_T) &= Z_1 + \rho_{2|\mathcal{F}_1} \left[Z_2 + \dots \right. \\ &\quad \left. + \rho_{T-1|\mathcal{F}_{T-2}} \left[Z_{T-1} + \rho_{T|\mathcal{F}_{T-1}} [Z_T] \right] \right] \end{aligned}$$

where the mappings $\rho_t : \mathcal{Z}_t \rightarrow \mathcal{Z}_{t-1}$ satisfy the following properties.

- *Monotonicity:* $\forall Z, Z' \in \mathcal{Z}_t$, if $Z \preceq Z'$, then $\rho_t(Z) \geq \rho_t(Z')$
- *Convexity:* $\forall Z, Z' \in \mathcal{Z}_t, \forall \alpha \in [0, 1]$:

$$\rho_t(\alpha Z + (1 - \alpha)Z') \leq \alpha \rho_t(Z) + (1 - \alpha) \rho_t(Z')$$

⁹We choose zero coupon bond with maturity T , the time of the last stage. The price of a bond at time t is given by the random variable $P_{t,T}$, where $P_{T,T} = 1$, that is, the bond is default free. As noted in Artzner et al. [8], it is only after this choice of numéraire, that risk measure can be qualified as monetary.

- *Translation invariance:* $\forall Z \in \mathcal{Z}_t, Y \in \mathcal{Z}_{t-1}: \rho_t(Z + Y) = \rho_t(Z) - Y$

Lemma 3.1 (from Shapiro [103]). *A dynamic risk measure is convex and time-consistent.*

Note that the operation generally suffers from two important defects. First the recursive formulation is not directly amenable to a general-purpose solver. A restatement of the recursive formulation into a single static risk function optimization problem would be useful for computation. Second the multistage and single stage formulation are not interpretable in the same terms: the single stage risk function is lost in the recursive formulation (i.e. the CVaR of a CVaR is not a CVaR).

3.4.1 Multiperiod good-deal

Cochrane and Saá-Requejo[31] extended the good-deal measure to multiperiod and continuous time environments. We limit ourselves to the discrete form and consider their recursive definition.

Definition 3.8. The good-deal of a sequence $(Z_1, \dots, Z_T)$ of $\mathcal{F}_t$ measurable pay-off Z_t is given by :

$$\begin{aligned} \rho^{\text{GD}}(Z_1, \dots, Z_T) = \max_{\zeta_2(\omega), \dots, \zeta_T(\omega)} & -Z_1 + \mathbb{E}_P \left[\zeta_2(\omega) \left(-Z_2 + \zeta_3(\omega) \dots \right. \right. \\ & \left. \left. \dots + \zeta_{T-1}(\omega) \left(-Z_{T-1} + \zeta_T(\omega) (-Z_T) \right) \right) \middle| \mathcal{F}_0 \right] \\ \text{s.t. } & \zeta_2(\omega) \in \mathcal{U}_2^{\text{GD}}(\omega), \dots, \zeta_T(\omega) \in \mathcal{U}_T^{\text{GD}}(\omega) \end{aligned}$$

where the sets $\mathcal{U}^{\text{GD}}$ are defined by

$$\mathcal{U}_{t+1}^{\text{GD}}(\omega) = \left\{ \zeta_{t+1}(\omega) \middle| \begin{aligned} & f_{t,i}(\omega) = \mathbb{E}_P[\zeta_{t+1}(\omega) f_{t+1,i}(\omega) | \mathcal{F}_t]; \\ & \mathbb{E}_P[\zeta_{t+1}(\omega)^2 | \mathcal{F}_t] \leq A_t^2; \\ & \zeta_{t+1}(\omega) \geq 0 \end{aligned} \right\} \quad (3.4)$$

One shall note that the measures appearing are conditional: the corresponding pricing kernel ζ_t is a function of the conditioning information available at time t . One can again recover a conditional CAPM formula as in the static case (3.3). Because the corresponding static good deal measure is coherent it is also straightforward to show that the dynamic good-deal is coherent.

Chapter 4

Liquidity risks on power exchange

4.1 Introduction

Electricity is a derivative product: its price depends on the evolution of other energy forms such as oil, natural gas, CO₂ or coal. These impacts are non linear and the prices processes of the underlying energy forms often difficult to construct. Electricity is thus per se a difficult product. Limited storage possibilities also make it an unusual commodity. The electricity system is subject to forced outages that can suddenly change the price pattern of the product without storage being able to smooth out the shocks. This introduces a mix of seasonality and spike effects that existing commodity price models do not easily accommodate. Last electricity is difficult to transport and the grid is subject to defaults and bottlenecks. Prices are thus geographically differentiated whether at a quite disaggregated level (as in the nodal systems of the restructured US markets) or by zones (as in the European Market Coupling, the Nordic market or Australia). As if these idiosyncrasies were not sufficient, electricity markets and their derivatives are often poorly liquid¹; which adds a substantive complexity. All this renders the understanding of electricity prices difficult. It also makes the pricing of its derivatives particularly challenging.

¹Insufficient liquidity was already evident in the early days of the restructured electricity markets [86, 85]. Since then, the volume of spot and derivative contracts increased significantly but electricity still remains considerably less liquid than other energy commodities (Table C.1). As an example PJM [91] recognizes that mature energy markets will require increased forward trading in order to reduce risk and provide clear price signals to support investment and hedging. Nowadays, market operators or power exchanges regularly publish technical reports on trade volumes and numbers of active participants [e.g. 81].

We consider a spatial market where the electricity prices can be hedged by standard financial contracts such as futures, options and swaps. In order to increase liquidity these products are generally only traded at some hubs of the grid and for long delivery periods². Because the sole trading of these products leaves agents (generators and consumers) exposed to congestion risks that result from bottlenecks in the grid, the Independent System Operator (ISO) runs a market of locational derivatives often called Financial Transmission Rights (FTRs)³. In contrast with energy hedging products the FTRs market is organized through auctions conducted by the ISO, which for network feasibility reason also imposes restrictions⁴ on the volume of FTRs traded. Possibly because of these idiosyncrasies the FTRs market received a particular attention in the past years. Siddiqui et al. [107], Adamson and Englander [2] and [3] empirically point to the inefficiency of the market and the high risk premium embedded in FTR prices. They suggest that it may be due to the low liquidity of the product. Deng et al. [38] argue that the market is inefficient because "quantity limits on the FTR bids may cause the auction clearing prices to differ from the bid prices and this phenomenon is inherent in the theoretical properties of the optimization algorithm used to clear the auction". All this justifies paying a particular attention to liquidity problems in electricity market.

Our analysis focuses on the liquidity of both energy and transmission financial markets. The chapter illustrates and quantifies the effect of illiquidity in the market of power derivatives. To our knowledge this problem has not been explored before. Liquidity plays a crucial role in financial markets. It enlarges the capacity of the market to accommodate order flows and guarantees the ability to quickly buy/sell sufficient quantities of an asset without significantly affecting its price. Liquidity is among the most important market characteristics for asset managers who want to easily convert their portfolio into cash. Insufficient liquidity, on the other hand, creates new risks and frictions. The literature provides ample empirical evidence that liquidity is an important state variable for asset pricing and that investors demand a higher return from less liquid securities (see Pàstor and Stambaugh [89], Amihud [4] and the cited literature).

Two important issues arise when dealing with liquidity. One is the empirical proxy used to measure it. Liquidity is an unobserved variable that embeds several dimensions such as volume, depth, resiliency and tightness⁵. The volume of exchange and the bid-ask spread are the most commonly mentioned

²e.g. PJM AEP Dayton Hub Peak LMP Swap Futures (traded on NYMEX), or German Month Futures (traded on the EEX)

³Other names (e.g. Transmission Congestion Contracts) are sometimes given.

⁴There exists a secondary market, not regulated by the ISO, where one can trade those contracts through bilateral transactions

⁵for definitions of those concepts, see among O'Hara [87], Kyle [75]

measures but it is now generally recognized that they are not fully appropriate. Several other measures that relate the volume of the trade and the size of the price movements have been proposed and explored. Hasbrouck [62] provides a comprehensive discussion of these questions. A second issue is the effect of insufficient liquidity on the pricing of financial contracts. Most pricing models rely on the assumption of absence of arbitrage, which is only sustainable in a very liquid market where arbitrageurs can instantaneously exploit all possible mispricings. Illiquid markets do not satisfy this assumption and hence such models might not be applicable.

We analyze liquidity through a two stage stochastic equilibrium model of power derivatives in a market that satisfies the usual assumptions of perfect competition (price taking agents) except for liquidity that is limited. We define liquidity on the basis of the volume exchanged and examine the impact of restricting the amount of available derivative contracts. In this set up agents cannot hedge up to their desired level because their strategy sets are restricted by the actions of the others players. Mathematically, this situation is formulated as a Generalized Nash Equilibrium (GNE) problem for which different solutions may exist. The chapter is organized as follows. Section 2 introduces an equilibrium model in a liquid financial market where agents value their portfolio according to a convex risk measure (in the sense of Föllmer and Schied [51]). We show that such an approach allows one to derive the usual no arbitrage property without imposing it *ex ante* in the reasoning. We treat illiquidity in section 3 where we explicitly introduce volume constraints on financial transactions in the equilibrium model. We illustrate it on a 6-node example taken from Chao et al. [28] on section 4. Simulation results are presented for liquid and insufficiently liquid energy markets. We introduce two liquidity constraints, one on energy futures, the other on FTRs. We analyze a range of equilibria with the view of quantifying the effect of illiquidity on the agents hedging strategies, profits volatility and risk-premiums. We also study the impact of illiquidity on the energy market as well as the implication of illiquidity in one market (here FTRs) on the other market (here energy). As found in Deng et al. [38], we observe that such restrictions importantly impact the contract prices. We also show that illiquid FTRs can drastically decrease the incentive to hedge in the energy market.

4.2 Forward equilibrium in a liquid market

4.2.1 Framework and notation

The literature offers two main methodologies for pricing derivative products. The most common approach is to resort to risk neutral valuation. The method

is based on stochastic processes that capture the spot price (e.g. [55, 114]) or the forward curve (e.g. [12, 63]) dynamics and are used to value contingent claims on this price. The other stream of the literature relies on economic models of power generation and consumption. Bessembinder and Lemmon [13] were the first to introduce a two stage equilibrium model of the power futures market where market participants hedge their profit by entering futures positions before bidding in the spot market. Their methodology has subsequently been used by several authors. Cavallo and Termini [25] study the benefits of introducing a market of standardized derivatives. Notably, they showed that this increases electricity purchases through the spot market and diminishes the share of bilateral contracts. Willems and Morbee [116] quantify the impact of power derivatives on welfare and investment incentives. Their computational results indicate that aggregate welfare increases with the number of derivatives offered and that investment decisions improve with market completeness because of decoupling of investment and speculation. Bülher and Müller-Merbach [21] extend the model to a dynamic equilibrium and derive an endogenous term structure of electricity futures prices. Our model is part of this latter stream of literature.

We study the financial contract market through a two stage equilibrium model, where hedging through financial products takes place in the first stage ($t = 0$), before the spot prices (and hence the spot profits) are revealed. We assume a (finite) probability space (Ω, P) and denote a scenario and its probability by ω and $\text{prob}(\omega)$. A random variable is described by a function $Z : \Omega \rightarrow \mathbb{R}$. We denote by $\mathcal{Z}$ the space of all bounded measurable functions (containing the constants) on the space Ω and $\mathcal{P}$ the set of all probability measures on Ω .

There are N players aiming at hedging their spot profit by contracting financial derivatives. The financial market comprises a risk free asset and C different contracts. We discount all prices at the risk-free rate, i.e. we take the risk free asset as numéraire. Each financial contract $c = 1, \dots, C$ has price p_c^f and gives a (finite-valued) stochastic discounted pay-off $p_c^s(\omega)$ at $t = 1$. We assume w.l.o.g. that the contracts payoffs are linearly independent (which expresses that contracts are not redundant). For each player $\nu = 1, \dots, N$, we let $\pi_\nu^s(\omega)$ be its (finite valued) random spot profit ($t = 1$) and $x_\nu := (x_{\nu,c})_{c=1}^C \in \mathbb{R}^C$ be the vector of its positions (the strategy in game parlance) in the different contracts. A positive $x_{\nu,c}$ corresponds to a long position in the financial contract c and a negative $x_{\nu,c}$ corresponds to a short position. The vector of all these financial positions is denoted $x := (x_\nu)_{\nu=1}^N \in \mathbb{R}^{N \times C}$, and the vector of all player's positions except those of player ν is denoted by $x_{-\nu} := (x_{\nu'})_{\nu'=1, \nu' \neq \nu}^N \in \mathbb{R}^{(N-1) \times C}$. Finally, p^f is the vector of all contract prices $p^f := (p_c^f)_{c=1}^C \in \mathbb{R}^C$.

4.2.2 Players hedging problem

Players trade financial contracts to hedge the random profit earned in the spot market. They are price takers in a perfectly competitive market. They can influence neither the spot price, nor the price of the financial contracts in order to earn extra profit from trading. The outcome of the spot market (π_ν^s) is independent of the financial portfolios of agents⁶. We model the player's risk aversion in the optimization problem by the modern approach of risk measure⁷ (cf. Chapter 3). A player ν optimizes its portfolio by choosing a strategy that minimizes the risk measure ρ_ν (a disutility function) of its profit distribution $\Pi_\nu(\omega)$. This problem is stated as follow:

Problem 4.1. Under the usual assumption of no transaction cost and portfolio restriction, an agent ν hedges its stochastic spot profit with financial contracts and solves the following stochastic program :

$$\begin{aligned} \mathcal{P}^\nu(p^f) &\equiv \min_{x_\nu \in \mathbb{R}^C} \rho_\nu(\Pi_\nu) \\ \Pi_\nu(\omega) &= \pi_\nu^s(\omega) + \sum_{c=1}^C x_{\nu,c} (p_c^s(\omega) - p_c^f) \end{aligned}$$

We specialize the agents hedging problem by assuming that they quantify their risk with a convex risk measure (see chapter 3.1). Using the representation Theorem 3.1, it is possible to restate the player's hedging Problem 4.1 as follows:

Problem 4.2. An agent ν having a convex risk measure as disutility function, hedges its stochastic spot profit by solving:

$$\mathcal{P}^\nu(p^f) \equiv \min_{x_\nu \in \mathbb{R}^C} \left\{ \sup_{Q \in \mathcal{M}_\nu} \mathbb{E}_Q[-\pi_\nu^s] - \sum_{c=1}^C x_{\nu,c} (\mathbb{E}_Q[p_c^s] - p_c^f) - \alpha_\nu(Q) \right\}$$

where $\mathcal{M}_\nu = \{Q \in \mathcal{P} : \alpha_\nu(Q) < +\infty\}$ is a compact set of probability measures. It is a characteristics of the risk measure ρ_ν .

⁶This implies that one can first find the equilibrium of the spot market and then solve for the forward market on the basis of the obtained equilibrium spot prices. Note that this simplification does not hold in an imperfectly competitive environment because forward decisions can influence the outcome of the spot market. Zhang et al. [118] propose a stochastic equilibrium model with equilibrium constraints (SEPEC) to characterize the interaction between the two markets in a Cournot game. It also breaks down when considering more general risk management that combines hedging decisions on financial contract and on the dispatch of a hydro pump storage facility (see [80, 40]).

⁷Mean-risk models are the best known examples of risk measure. An agent minimizes its expected loss $\mathbb{E}[-\Pi]$ accounting for a measure $\mathbb{D}[\Pi]$ of the profit dispersion. This risk measure is expressed as $\rho(\Pi) = \mathbb{E}[-\Pi] + \kappa \mathbb{D}[\Pi]$.

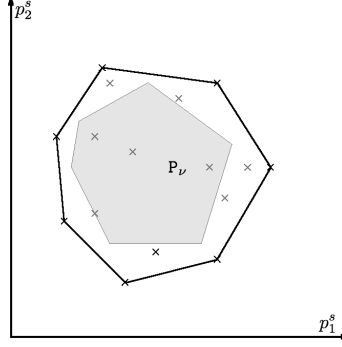


Figure 4.1: Illustration of the set P_ν in a market with two financial contracts. The crosses represent the pay-offs realizations of the financial contracts. The set P_ν is included in the convex hull of those pay-offs.

The hedging problem of agent ν defines the following convex set P_ν , which is an linear transformation of $\mathcal{M}_\nu$ (see Figure 4.1).

$$P_\nu = \left\{ p \in \mathbb{R}^C \mid \exists Q \in \mathcal{M}_\nu, \forall c = 1, \dots, C : p_c = \mathbb{E}_Q[p_c^s] \right\} \quad (4.1)$$

Obviously, Problem 4.2 is unbounded if $p^f \notin P_\nu$. Note also that the boundedness of the payoffs $p_c^s(\omega)$ ensures that P_ν is compact. When $p^f \in P_\nu$, one can formulate the agent's hedging Problem 4.2 as follows

Problem 4.3. Let $p^f \in P_\nu$, the agent's hedging problem becomes

$$\begin{aligned} \mathcal{P}^\nu(p^f) \equiv & \inf_Q \mathbb{E}_Q[-\pi_\nu^s] - \alpha_\nu(Q) \\ \text{s.t. } & Q \in \mathcal{M}_\nu = \{Q \in \mathcal{P} : \alpha_\nu(Q) < \infty\} \\ & p_c^f = \mathbb{E}_Q[p_c^s] \end{aligned} \quad (x_{\nu,c})$$

where the $x_{\nu,c}$ are the dual variables of the equality constraints $p_c^f = \mathbb{E}_Q[p_c^s]$.

Problem 4.3 looks similar to the representation Theorem 3.1. The difference is that it imposes that the probability measure Q , used to value the agent's profit, must also price⁸ the derivatives contracts.

Proposition 4.1. The set of solutions to Problem 4.3 is non-empty, compact and convex iff $p^f \in \text{int } P_\nu$

⁸This is commonly referred to in financial parlance by saying that the Radon-Nikodym derivative of the measure Q with respect to the measure P is a price kernel or a state price [e.g. 106].

Proof. The constraint set of Problem 4.3 satisfies the strong Slater condition (e.g. Hiriart-Urruty and Lemaréchal [64, chap. VII]) when $p^f \in \text{int } P_\nu$. The objective function of Problem 4.3 is finite value $\forall Q \in \mathcal{M}_\nu$ because the $\pi_\nu^s(\omega)$ are assumed to be finite. Those are necessary and sufficient conditions for the set of multipliers x_ν to be a non-empty compact and convex set. $\square$

4.2.3 Equilibrium: definition and properties

Definition 4.1 (Equilibrium). An allocation x^* and a price vector $p^{f,*}$ constitute a competitive equilibrium in a liquid financial market if :

- (i) For every player $\nu = 1, \dots, N$, x_ν^* is an optimal hedging strategy, $\forall x_\nu \in \mathbb{R}^C$:

$$\rho_\nu(x_\nu^*, p^{f,*}) \leq \rho_\nu(x_\nu, p^{f,*})$$

- (ii) For each financial contract $c = 1, \dots, C$, the following market clearing condition holds:

$$\sum_{\nu=1}^N x_{\nu,c}^* = 0$$

Throughout this section, we make the assumption that the agents risk measures are convex (in the sense of Definition 3.1) but not necessarily differentiable⁹ (cf. 3.2). We define the following sets $\bar{\mathcal{M}} := \bigcap_{\nu=1}^N \mathcal{M}_\nu$ and $\bar{P} = \bigcap_{\nu=1}^N P_\nu$, which are compact and convex (see Figure 4.2). We assume that $\bar{\mathcal{M}}$ has a non empty interior. Accordingly the space $\bar{P}$ has also a non empty interior. The economic interpretation of this condition will be explained later.

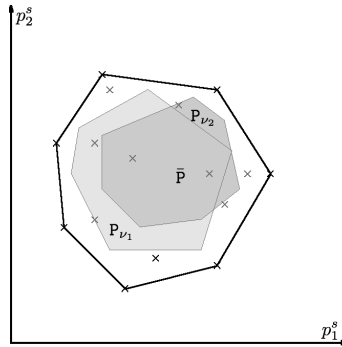


Figure 4.2: Illustration of the set $\bar{P}$ in a market with two financial contracts.

⁹Notice that, because of this lack of differentiability, we will not study the Nash equilibrium by the theory of variational inequality (VI) as is common in practice [see e.g. 48].

Before proving an existence result for the competitive equilibrium in a perfectly liquid market, we introduce the corresponding Nash game involving the following market agent.

Problem 4.4 (Market agent). The market agent chooses a financial price vector p^f that solves:

$$\mathcal{P}^M(x_{\nu,c}) \equiv \min_{p^f \in \mathbb{R}^C} \sum_{c=0}^C -p_c^f \left(\sum_{\nu=0}^N x_{\nu,c} \right) \quad (4.2)$$

$\sum_{\nu=0}^N -x_{\nu,c}$ is referred to as the the total demand shortfall of financial contract c (remember that $x_{\nu,c}$ is positive for a long position, and negative for a short position). The market agent selects prices that minimize the total value of demand shortfalls.

Definition 4.2 (Nash game). An allocation x^* and a price vector $p^{f,*}$ constitute an equilibrium of the Nash game with the market agent and the N players if :

- (i) For every player $\nu = 1, \dots, N$, x_ν^* is an optimal hedging strategy, $\forall x_\nu \in \mathbb{R}^C$:

$$\rho_\nu(x_\nu^*, p^{f,*}) \leq \rho_\nu(x_\nu, p^{f,*})$$

- (ii) For the market agent, the price $p^{f,*}$ minimizes the value of demand shortfall, $\forall p^f \in \mathbb{R}^C$:

$$-\sum_{c=0}^C p_c^{f,*} \sum_{\nu=0}^N x_{\nu,c}^* \leq -\sum_{c=0}^C p_c^f \sum_{\nu=0}^N x_{\nu,c}^*$$

The relation between the competitive and Nash equilibrium is stated in the following result.

Proposition 4.2. A Nash equilibrium of the game 4.2, if it exists, is a competitive equilibrium in a liquid market and conversely.

Proof. Note that players $\nu = 1, \dots, N$ optimize their hedging strategy in both the Nash game and the competitive equilibrium. Note also that the market clearing condition implies that the price of a competitive equilibrium is a solution to the market agent problem in the Nash game. A competitive equilibrium is thus a Nash equilibrium of the game 4.2.

In order to prove the converse proposition one only needs to show that an equilibrium of the Nash game satisfies the market clearing condition. Suppose that the market does not clear for contract c ; the contract price p_c^f is then unbounded as this is optimal for the market agent. Take $p_c^f = +\infty$ (respectively $-\infty$), the optimal hedge for all players $\nu = 1, \dots, N$ is then an infinite short (resp. long) position on this contract: $\sum_{\nu=1}^N x_{\nu,c} = -\infty$ (resp. $+\infty$). The objective function value of the market agent is then $+\infty$ which is clearly not its optimum in the Nash game. We have a contradiction and conclude that every equilibrium of the Nash game satisfies the market clearing conditions for all contracts. $\square$

Proposition 4.3. Suppose (x, p^f) is a Nash equilibrium of the game 4.2 ; then the price vector p^f belongs to the set $\bar{\mathbf{P}}$ defined by

$$\bar{\mathbf{P}} = \left\{ p \in \mathbb{R}^C \mid \exists Q \in \bar{\mathcal{M}}, \forall c = 1, \dots, C : p_c = \mathbb{E}_Q[p_c^s] \right\} \quad (4.3)$$

Proof. Suppose that $p^f \notin \bar{\mathbf{P}}$ for the Nash equilibrium (x, p^f) . Applying the strict separation hyperplane theorem between a point and a compact convex set (e.g. Boyd and Vandenberghe [20]), there exists a strategy $y \in \mathbb{R}^C$ that we can normalize to $\|y\|_2 = 1$ and some $\beta \in \mathbb{R}$ such that $\forall Q \in \bar{\mathcal{M}}, \sum_{c=1}^C y_c \mathbb{E}_Q[p_c^s] > \beta$ and $\sum_{c=1}^C y_c p_c^f = \beta$. Recalling that $\bar{\mathcal{M}} = \bigcap_{\nu=1}^N \mathcal{M}_\nu$, the condition becomes:

$$\exists \nu : \forall Q \in \mathcal{M}_\nu, \sum_{c=1}^C y_c \mathbb{E}_Q[p_c^s] > \beta \text{ and } \sum_{c=1}^C y_c p_c^f = \beta \quad (4.4)$$

Such strategy y is a strict arbitrage opportunity for player ν , meaning that the optimal hedging position of player ν is unbounded (abusing language, $x_\nu = +\infty y$ is an optimal hedging strategy for player ν 's Problem 4.2, leading to a infinite negative value of its risk measure) and hence (x, p^f) is not a Nash equilibrium. $\square$

Proposition 4.4. Suppose (x, p^f) is a Nash equilibrium of the game 4.2 ; then the financial hedging positions x_ν are bounded for every player $\nu = 1, \dots, N$.

Proof. The result is obvious if $p^f \in \text{int } \bar{\mathbf{P}}$ (cf. Proposition 4.1). Suppose that $p^f \in \text{bd } \bar{\mathbf{P}}$ and there exists a player ν with an unbounded set of optimal positions. Let x_ν^k be an unbounded sequence of optimal positions of the player. The unboundedness¹⁰ of the x_ν^k implies that $p^f \in \text{bd } \mathbf{P}_\nu$ and

$$\forall Q \in \mathcal{M}_\nu : \sum_{c=1}^C -x_{\nu,c}^k (\mathbb{E}_Q[p_c^s] - p_c^f) \geq 0 \quad (4.5)$$

¹⁰If $p^f \in \text{int } \mathbf{P}_\nu$, the set of optimal position of ν would be bounded. If the condition 4.14 was not satisfied, an unbounded x_ν^k would lead to an infinite positive value of the risk measure. Note also that this condition is feasible because $p^f \in \text{bd } \mathbf{P}_\nu$ and the existence of supporting hyperplane.

or equivalently, in order to work in a compact set

$$\forall Q \in \mathcal{M}_\nu : \sum_{c=1}^C \frac{-x_{\nu,c}^k}{\|x_\nu^k\|_2} (\mathbb{E}_Q[p_c^s] - p_c^f) \geq 0 \quad (4.6)$$

This proves that there exists a limit point $y_\nu \neq 0$ of a subsequence of $\frac{x_{\nu,c}^k}{\|x_\nu^k\|_2}$ such that

$$\forall Q \in \mathcal{M}_\nu : \sum_{c=1}^C y_{\nu,c} p_c^f \geq \sum_{c=1}^C y_{\nu,c} \mathbb{E}_Q[p_c^s] \quad (4.7)$$

Because the market clears (for every k in the sequence) there exists a set of players $\mathcal{I} \subseteq \{1, \dots, N\} \setminus \{\nu\}$ also taking unbounded positions and hence for some subsequence

$$\sum_{i \in \mathcal{I}} x_{i,c}^k = -x_{\nu,c}^k \text{ or } \sum_{i \in \mathcal{I}} \frac{x_{i,c}^k}{\|x_\nu^k\|_2} \rightarrow -y_\nu \quad (4.8)$$

There thus exists a (non empty) subset of player $i \in \mathcal{I}' \subseteq \mathcal{I}$ such that

$$\frac{x_i^k}{\|x_\nu^k\|_2} \rightarrow y_i \neq 0 \quad (4.9)$$

$$\sum_{i \in \mathcal{I}'} y_{i,c} = -y_{\nu,c} \quad (4.10)$$

$$\forall Q_i \in \mathcal{M}_i : \sum_{c=1}^C y_{i,c} p_c^f \geq \sum_{c=1}^C y_{i,c} \mathbb{E}_{Q_i}[p_c^s] \quad (4.11)$$

where conditions 4.11 are required by the optimality of the unbounded positions x_i^k . These hold simultaneously $\forall Q \in \mathcal{M}_{\mathcal{I}'} = \bigcap_{i \in \mathcal{I}'} \mathcal{M}_i$. Summing these conditions on $i \in \mathcal{I}'$, one gets

$$\forall Q \in \mathcal{M}_{\mathcal{I}'} : \sum_{c=1}^C \left(\sum_{i \in \mathcal{I}'} y_{i,c} \right) (p_c^f - \mathbb{E}_Q[p_c^s]) = \sum_{c=1}^C (-y_{\nu,c}) (p_c^f - \mathbb{E}_Q[p_c^s]) \geq 0 \quad (4.12)$$

This latter condition combined with 4.4 means that y_ν is a separating hyperplane between $\mathbf{int} \mathbf{P}_\nu$ and $\mathbf{int} \mathbf{P}_{\mathcal{I}'} = \bigcap_{i \in \mathcal{I}'} \mathbf{P}_i$ and hence $\mathbf{int} \mathbf{P}_\nu \cap \mathbf{int} \mathbf{P}_{\mathcal{I}'} = \emptyset$. This violates the hypothesis that the set $\overline{\mathcal{M}}$ has a non empty interior. $\square$

We recall a fundamental theorem that will be used to prove the existence of an equilibrium in the financial market.

Theorem 4.1 (Debreu [36]). *Consider a game in strategic form whose strategy space X_ν are non-empty compact convex subsets of an euclidean space. If the disutility functions $\theta_\nu(x_\nu, x_{-\nu})$ are continuous in x_ν and convex in $x_{-\nu}$, there exists a pure strategy Nash equilibrium.*

We can state the following existence result.

Theorem 4.2. *Suppose players risk functions are convex and $\overline{\mathcal{M}}$ has an non empty interior, then there exists a competitive equilibrium in a liquid financial market*

Proof. We first consider a Nash game in a truncated economy defined as follows. We first restrict the strategy sets of the players to belong to $\widehat{X}_\nu := \{x_{\nu,c} : |x_{\nu,c}| \leq r\}$. We then impose that the market agent minimizes the value of demand shortfall by choosing financial contract prices $p^f \in \Delta$, with $\Delta = \prod_{c=1}^C \Delta_c$ and $\Delta_c = [\underline{\delta}_c, \overline{\delta}_c]$ where :

$$\underline{\delta}_c = \inf_{\omega \in \Omega} (p_c^s(\omega)) - \epsilon ; \quad \overline{\delta}_c = \sup_{\omega \in \Omega} (p_c^s(\omega)) + \epsilon \quad (4.13)$$

and $\epsilon > 0$.

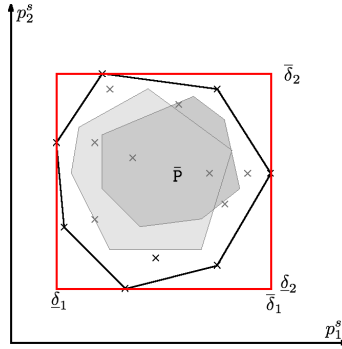


Figure 4.3: Illustration of the set Δ in a market with two financial contracts.

Note that $\overline{\mathcal{P}} \subset \Delta^{11}$. It is obvious that the objective function in $\mathcal{P}^\nu(p^f)$ is convex in p^f . Similarly the market agent objective function is convex in the x_ν . Following Theorem 4.1, every Nash game in a truncated economy has a (at least one) equilibrium $(x^r, p^{f,r})$.

We first prove that this equilibrium satisfies the market clearing condition. Suppose not and let c be the contract that violates the market clearing condition. The market agent selects a price $p_c^f = \underline{\delta}_c$ (resp. $p_c^f = \overline{\delta}_c$ depending on the sign of the violation). The only optimal response of all agents $\nu = 1, \dots, N$ to this price is to take a position $x_{\nu,c} = r$ (resp. $x_{\nu,c} = -r$). This in turn implies an objective function value of $-rN\underline{\delta}_c$ (resp. $rN\overline{\delta}_c$) for the market agent, which

¹¹By definition, $\forall p \in \overline{\mathcal{P}}, \exists Q \in \overline{\mathcal{M}}, \forall c = 1, \dots, C : p_c = \mathbb{E}_Q[p_c^s]$. One also has that $\underline{\delta}_c < \mathbb{E}_Q[p_c^s] < \overline{\delta}_c, \forall Q \in \mathcal{P}$. This implies that $\overline{\mathcal{P}} \subset \Delta$.

is clearly not optimal for this agent and hence contradicts the definition of a Nash equilibrium.

Consider now a sequence $(x^r, p^{f,r})$ of Nash equilibrium, obtained by letting $r \rightarrow \infty$ in the truncated economy. The sequence of p_c^r is obviously bounded because the prices belong to the compact set Δ . We show that the sequence of x^r is also bounded. Suppose not. Then, there exists at least one player ν with an unbounded sequence of optimal positions x_ν^r . As $r \rightarrow \infty$, it must be the case that

$$\forall Q \in \mathcal{M}_\nu : \sum_{c=1}^C -x_{\nu,c}^r (\mathbb{E}_Q[p_c^s] - p_c^{f,r}) \geq 0 \quad (4.14)$$

Because the market clears for every game in the truncated economy, there exists a (non empty) subset of players who also take unbounded positions that globally clear the position of player ν . One can show, similarly to the proof of Proposition 4.4, that this implies the existence of a separating hyperplane between the interior of some agent sets $\mathbf{P}_\nu$. This violates the hypothesis $\mathbf{int} \bar{\mathcal{M}} \neq \emptyset$.

There thus exists a r' in the sequence of truncated Nash games such that the equilibrium in the truncated economy satisfies $x_\nu^{r'} \in \mathbf{int} \hat{X}_\nu$ for all $\nu = 1, \dots, N$. Note also that $x_\nu^{r'} \in \mathbf{int} \hat{X}_\nu$ implies¹² $p^{f,r'} \in \bar{\mathbf{P}}$. Because the players hedges their profit according to a convex program, every optimal position in the truncated economy satisfying $x_\nu^{r'} \in \mathbf{int} \hat{X}_\nu$ is also optimal in the economy of the initial game. A Nash equilibrium in a truncated economy satisfying $x_\nu^{r'} \in \mathbf{int} \hat{X}_\nu$ is thus a Nash equilibrium of the initial game 4.2 and hence a competitive equilibrium in a liquid financial market. $\square$

Remark 4.1. The proof highlights the key role of the set $\bar{\mathcal{M}}$ and its interior in the existence of an equilibrium solution. We also pointed in Proposition 4.3 that $\bar{\mathcal{M}} = \emptyset$ is not amenable to an equilibrium. Indeed in this particular situation, there always exists (at least) a player which hedging problem is unbounded. The final case is when $\bar{\mathcal{M}} = \mathbf{bd} \bar{\mathcal{M}}$. The Proposition 4.4 shows that the hedging positions might be unbounded in that case. A famous example of this kind of situation is when players are risk neutral (i.e. $\rho_\nu(Z) = \mathbb{E}_P[Z]$) which leads to a set $\bar{\mathcal{M}}$ equal to the singleton $\{P\}$. It is well-known that in this risk-neutral case, $p^f = \mathbb{E}_P[p_c^s]$ and any hedging position is optimal for the players.

4.2.4 Discussion on arbitrage opportunities

In contrast with the risk-neutral valuation approach, equilibrium models do not rely on the assumption of no arbitrage but rather suppose that prices are

¹²Otherwise, one can show that the strategy x_ν^r is not optimal for an agent as in Proposition 4.3

determined by the simultaneous minimization of agent's risk measures. If this equilibrium contains arbitrage opportunities, outside speculators in a liquid market will massively enter and trade them away. We do not model speculators and hence do not make the usual assumption of no arbitrage¹³. It is however possible to infer the absence of arbitrage as a result of the equilibrium by modeling the agents risk aversion by convex and equivalent (Definition 4.3) risk measures.

Definition 4.3. A convex risk measure is called equivalent if the probabilities $Q \in \mathcal{M}$ in the dual representation 3.1 and the probability measure P share the same set of zero measure events.

Lemma 4.1. *Suppose players risk measures are convex and equivalent, then a competitive equilibrium in a liquid market is arbitrage free.*

Proof. Proposition 4.3 shows that equilibrium prices satisfy $p^f \in \bar{P}$ at equilibrium. When the risk measures are convex and equivalent, the contract prices can be written as $p_c^f = \mathbb{E}_Q[p_c^s]$, where Q is a probability measure equivalent to P . By a well known theorem of finance [e.g. 106] this latter condition ensures that the market has no arbitrage. $\square$

Modeling risk aversion through non convex risk measure may lead to equilibrium with arbitrage opportunities. For example, mean-variance, as used in Bessembinder and Lemmon [13] or Willems and Morbee [116], violates the positive homogeneity and monotonicity axioms, and hence is not coherent. The absence of arbitrage in the equilibrium solution can be checked ex-post, by verifying that the prices of the derivatives strictly lie inside the convex hull of the spot payoffs of these derivatives. We give an illustration of an equilibrium leading to arbitrage opportunities in appendix C.3.

4.3 Forward equilibrium in an illiquid market

4.3.1 Modeling illiquidity

As explained in the introduction we model illiquidity through constraints on transaction volumes. We first measure trade by the absolute value of agents hedging positions and characterize illiquidity by a bound on the sum of these absolute values. This is represented through the constraints

$$\sum_{\nu} |x_c^{\nu}| \leq L_c \quad (4.15)$$

¹³The market contains an arbitrage opportunity, if there exists a portfolio x_c such that: $\sum_{c=1}^C x_c p_c^f = 0$, $\forall \omega \in \Omega : \sum_{c=1}^C x_c p_c^s(\omega) \geq 0$ and $\sum_{c=1}^C x_c p_c^s(\omega) > 0$ for some ω .

where the constant L_c can be adapted to represent different degrees of liquidity. Constraints 4.15 are shared by all players. In game parlance this implies that the set of hedging strategies of a player is restricted by the hedging actions of the others (noted $x_{-\nu}$). This suggests reformulating the hedging problem of player ν as follows:

Problem 4.5. A player ν hedges its stochastic profit with illiquid contracts by solving the following program:

$$\begin{aligned} \mathcal{P}^\nu(p^f, x_{-\nu}) \equiv & \min_{x_\nu \in \mathbb{R}^C} \sup_{Q_\nu \in \mathcal{M}_\nu} \left\{ \mathbb{E}_Q[-\pi_\nu^s] - \sum_{c=1}^C x_{\nu,c} (p_c^f - \mathbb{E}_Q[p_c^s]) - \alpha_\nu(Q) \right\} \\ \text{s.t. } & \Pi_\nu(\omega) = \sum_c x_{\nu,c} (p_c^s(\omega) - p_c^f) + \pi_\nu^s(\omega) \\ & x_{\nu,c} \leq L_c - \sum_{-\nu} |x_{-\nu,c}| \quad (\lambda_{\nu,c}) \\ & x_{\nu,c} \geq -L_c + \sum_{-\nu} |x_{-\nu,c}| \quad (\mu_{\nu,c}) \end{aligned}$$

The problem is the same as Problem 4.2, except that the player hedging strategy $x_{\nu,c}$ is now restricted by the other players positions so that it satisfies the liquidity bounds L_c .

4.3.2 Equilibrium: definition and properties

The inclusion of liquidity constraints (equation 4.15) transforms the perfect competitive equilibrium problem into a Generalized Nash Equilibrium problem. The goal is now to find a hedge tuple $(x^*, p^{f,*})$ such that $\mathbf{x}_\nu^*$ minimizes the risk measure of each agent's portfolio taking the hedging strategies of the other agents as given. Any such tuple is called a Generalized Nash Equilibrium (GNE). This is formalized as follows:

Definition 4.4. An allocation x^* and a price vector $p^{f,*}$ constitute an equilibrium in an illiquid financial market if :

- (i) For every player $\nu = 1, \dots, N$, x_ν^* is an optimal hedging strategy:

$$\rho_\nu(x_\nu^*, p^{f,*}) \leq \rho_\nu(x_\nu, p^{f,*}) \quad \forall x_\nu \in \mathbb{R}^C : |x_{\nu,c}| \leq L_c - \sum_{-\nu} |x_{-\nu,c}^*|$$

- (ii) For each financial contract $c = 1, \dots, C$, the following market clearing condition holds:

$$\sum_{\nu=1}^N x_\nu^* = 0$$

The Generalized Nash Equilibrium problem was formally introduced by Arrow and Debreu [5] in their paper on abstract economies. It is well known (e.g. Harker [60], Facchinei and Kanzow [46]) that General Nash Equilibrium problems (GNE) can be formulated as Quasi Variational Inequality problems (QVI),

when utility functions are differentiable. These are known to have multiple, possibly infinitely many solutions (often a manifold). This lack of uniqueness is often interpreted as a difficulty that limits the usefulness of the concept. For this reason GNEs have often been criticized by economists because they do not offer a plausible solution concept of a meaningful game.

We take a quite different view and note that the multiplicity of solutions reflects a fundamental feature of a market affected by liquidity problems: illiquidity is a market failure and the indeterminate outcome of the market is a consequence of that market failure. With this remark in mind and given our practical objective of illustrating the impact of the liquidity constraints on the equilibrium, we aim at finding a large set of GNEs in order to assess the degree of inefficiency that illiquidity can lead to. From a mathematical point of view, our problem is a GNE problem with shared constraints [97, 53], meaning that the liquidity constraint bears on all market agents. This special class of problems has received increasing attention in recent years.

Unlike Nash Equilibrium problems, there are only few methods available to solve GNE problems (see Facchinei and Kanzow [46] for a recent survey on GNEs). Existing algorithms are usually based on parametrized variational inequality problems¹⁴[83] or on control penalty methods[47, 53]. As for liquid markets, we analyze the existence of equilibria by resorting to an associated Generalized Nash game constructed by introducing a market agent. The N players solve their hedging Problem 4.5 and the market agent minimizes, as in the liquid case, the demand shortfall value (cf. Problem 4.4).

Definition 4.5 (Corresponding Generalized Nash game). An allocation x^* and a price vector $p^{f,*}$ constitute a GNE of the game if:

- (i) For every player $\nu = 1, \dots, N$, x_ν^* is an optimal hedging strategy, $\forall x_\nu \in \mathbb{R}^C : |x_{\nu,c}| \leq L_c - \sum_{-\nu} |x_{-\nu,c}^*|$:

$$\rho_\nu(x_\nu^*, p^{f,*}) \leq \rho_\nu(x_\nu, p^{f,*})$$

- (ii) For the market agent, the price $p^{f,*}$ minimizes the value of demand shortfall :

$$-\sum_{c=1}^C p_c^{f,*} \sum_{\nu=1}^N x_{\nu,c}^* \leq -\sum_{c=1}^C p_c^f \sum_{\nu=1}^N x_{\nu,c}^*$$

Proposition 4.5. A Generalized Nash Equilibrium of the game 4.5, if it exists, is an equilibrium in an illiquid financial market, and conversely.

¹⁴which, under a mild constraint qualification, allow one to find all solutions of the GNEP.

Proof. It is clear that an equilibrium in an illiquid market is a GNE of the game 4.5. To prove the converse, one shows that every GNE of the game satisfies the market clearing condition. Suppose not and let c be a contract where the condition is violated. The optimal strategy of the market agent is to select an unbounded price $p_c^f = +\infty$ (respectively $p_c^f = -\infty$). With such price, the optimal hedge of any agent is to take a lower (resp. upper) position in contract c : $x_{\nu,c} = -L_c + \sum_{-\nu} |x_{-\nu,c}^*|$ (resp. $x_{\nu,c} = L_c - \sum_{-\nu} |x_{-\nu,c}^*|$). This implies $\sum_{\nu} x_{\nu,c} = -L_c$ (resp. L_c) which gives an objective value for the market agent equal to $+\infty$ and hence cannot be its optimal position. $\square$

Proposition 4.6. Suppose (x, p^f) is a GNE of the game 4.5, then

$$p_c^f \in \left[\inf_{\omega \in \Omega} (p_c^s(\omega)) \ ; \ \sup_{\omega \in \Omega} (p_c^s(\omega)) \right]$$

Proof. Writing the KKT conditions of player ν 's hedging Problem 4.5, we get

$$(p_c^f - \mathbb{E}_{Q_\nu^*}[p_c^s]) = \mu_{\nu,c} - \lambda_{\nu,c} \quad (4.16)$$

Suppose that $p_c^f < \inf_{\omega \in \Omega} (p_c^s(\omega))$ for contract c (respectively $p_c^f > \sup_{\omega \in \Omega} (p_c^s(\omega))$). One then has that

$$\forall Q_\nu \in \mathcal{M}_\nu : \ p_c^f - \mathbb{E}_{Q_\nu}[p_c^s] < 0 \quad (\text{or respectively } > 0) \quad \forall \nu : 1, \dots, N. \quad (4.17)$$

This implies $\lambda_{\nu,c} > 0$ (resp. $\mu_{\nu,c} < 0$), meaning that all players positions are positive (resp. negative) and constrained by the upper (lower) liquidity bounds : $\sum_{\nu} x_{\nu,c} = L_c$ (resp. $\sum_{\nu} x_{\nu,c} = -L_c$). The market clearing conditions do not hold and hence (x, p^f) is not a GNE of the game 4.5. $\square$

Invoking the following result of Arrow and Debreu [5]

Theorem 4.3 (Arrow and Debreu [5]). *If, for each ν , the disutility function $\theta_\nu(x_\nu, x_{-\nu})$ is continuous and quasi-convex in x_ν , for every $x_{-\nu}$, $X_\nu(x_{-\nu})$ is a convex compact and non-empty set, then the abstract economy has an equilibrium point.*

we can state

Theorem 4.4. *If players risk measures are convex, there exists an equilibrium in an illiquid market.*

Proof. The strategy set of player ν $X_\nu(x_{-\nu}) = \{x_{\nu,c} : -L_c + \sum_{-\nu} |x_{-\nu,c}| \leq x_{\nu,c} \leq L_c - \sum_{-\nu} |x_{-\nu,c}|\}$ is convex, compact and non-empty. Following the proof of Proposition 4.2, one can also bound the strategy set of the market agent to the compact, convex and non empty set Δ (cf. equation 4.13). Theorem 4.3 applies directly and proves the existence of the equilibrium. $\square$

4.3.3 Illiquidity and remaining arbitrage

Arbitrage opportunities are more likely to persist over time in illiquid markets. These result from the difficulties for arbitrageurs to exploit mispricing. For example, Deville and Riva [39] show the existence of temporary arbitrage in option markets and note that the speed of reversion to the no arbitrage situation is critically impacted by liquidity-linked variables. Perfect liquidity and unconstrained portfolio formation are key hypothesis to justify the fundamental no-arbitrage assumption that most asset pricing theories rely upon¹⁵. These are far from satisfied in restructured power markets.

The theory of Generalized Nash Equilibrium suggests that modeling illiquidity by shared constraints on tradable volumes implies that the obtained equilibrium solutions may contain arbitrage opportunities, regardless of the valuation criterion used to model agents' risk aversion. Indeed, the optimality conditions of an agent restricted by shared constraints on its positions¹⁶ lead to the condition :

$$p_c^f = \mathbb{E}_Q[p_c^s] + \mu_{\nu,c} - \lambda_{c,\nu} \quad (4.18)$$

where $(\mu_{\nu,c}, \lambda_{\nu,c})$ are agent ν 's shadow prices of the illiquidity constraints on contract c . In a GNE, those shadow prices can differ by agents, which implies an important difference with respect to the equilibrium of the liquid market. While the pricing kernels associated with Q_ν in a liquid market can differ among agents, they still give the same price of the futures contracts. In contrast the appearance of the shadow prices in 4.18 implies that different agents ν no longer value contract at the same level. Arbitrage opportunities may thus exist at equilibrium when the volume constraints are tight and agents are not able to hedge up to the desired level. Insufficient liquidity prevents traders from exploiting these arbitrage opportunities. This is a market failure induced by illiquidity.

4.4 A stylized power market example

The restructuring of electricity markets has led to many, sometimes quite different, market designs. Congestion management plays a key element in these

¹⁵Recently, the theory have been adapted to tackle assets which dynamics changes with the order balance. Çetin et al. [26] extend the fundamental theorems of finance and show that the no-arbitrage condition still implies the existence of a risk neutral measure, but that the perfect hedging, even for complete market, does not hold anymore. One difficulty of applying the model is to quantify correctly the asset's dynamics with respect to the level liquidity.

¹⁶i.e. relation 4.15 ; other quantitative restrictions on hedging positions would lead to the same result

designs. Following up on a general trend in the literature, we focus on a particular design where the market is geographically segmented into several nodes of supply or demand connected by transmission lines. The prices are defined at each node of the network reflecting that only feasible bids, i.e. bids that comply with the limited capacity of the network, can be accepted at the nodes in the day ahead auction¹⁷. The nodal prices are called Locational Based Marginal Prices (LBMP) and are calculated for each generation and load zone by an Independent System Operator (ISO). In such system, the buyers pay the LBMP calculated at the node in which they take delivery of electricity and the sellers receive the LBMP at the bus to which they supply. Besides the price volatility due to factors such as fuel prices and forced outages, market participants are also exposed to the risk of congestion rents, which are highly volatile [3]. In order to offer risk mitigation capabilities on transmission, the ISO organizes a forward market of congestion revenue rights. We focus here on financial transmission rights (FTRs, initially proposed by Hogan [65]) which pay the price difference between two specified point of the network.

4.4.1 The day ahead market model

We consider a restructured electricity system where the forward market trades contracts that matures on the day-ahead. We develop the argument on Chao et al. [28]’s six nodes network (Figure 4.4) and use both their model and numerical assumptions (see the original paper for more discussion of that example).

The description of the market is standard. The power grid contains N busses and L transmission lines. Each line $\ell \in L$ is characterized by its impedance and a thermal capacity K_ℓ . Using the DC approximation of the AC load flow equations, every MW injected (retrieved) at a generating (load) bus n and withdrawn (injected) at some reference hub node creates a power flow $\text{PTDF}_{n,\ell}$ ($-\text{PTDF}_{n,\ell}$) on the line $\ell \in L$. The ISO clears the energy and transmission markets (as well as other markets that are not represented here, e.g. reserve). At each bus of the network, there is a single economic agent $\nu \in N$, which can be a generator ($\nu \in N_p$) or a retailer ($\nu \in N_r$).

Definition 4.6 (Generators). Generators have unlimited capacity. Each generator $\nu \in N_p$ bids its marginal cost of supply C_ν in the day-ahead market. Generators have no fixed cost and their total and marginal cost functions, noted

¹⁷The day-ahead market trades power for physical delivery the next day. It is based on an hourly auction with bids for purchase and sale.

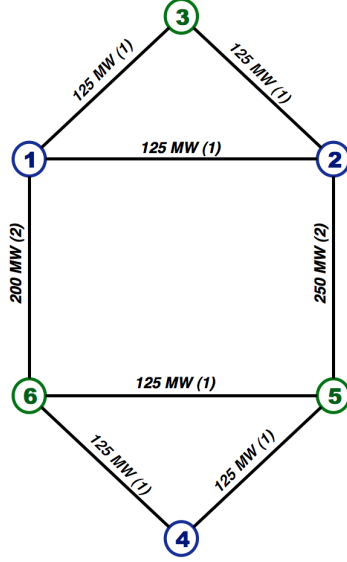


Figure 4.4: 6-nodes network from Chao et al. [28]

C_ν^T and C_ν , take the following form¹⁸:

$$C_\nu^T(q_\nu) = a_\nu q_\nu + b_\nu \frac{q_\nu^2}{2}; C_\nu(q_\nu) = a_\nu + b_\nu q_\nu$$

Definition 4.7 (Retailers). Each retailer $\nu \in N_r$ serves final consumers at its bus. It sells power at a fixed retail price p_ν^r . It bids its inverse demand function, which is also assumed to be linear¹⁹ in the day-ahead market.

$$p_\nu(q_\nu) = a_\nu - b_\nu q_\nu$$

Table 4.1 recalls the bids of the different economic agents assumed in Chao et al. [28].

¹⁸The model can easily be extended to more complex production functions and production sets restricted by a limited capacity.

¹⁹One can easily extend the model to inelastic demand, which is probably a better representation of actual markets. The computation of the spot equilibrium by maximizing the welfare becomes then the minimization of the total cost for meeting the inelastic demand.

Bus-ID	Supply bids: $C_\nu(q_\nu)$	Bus-ID	Load bids: $p_\nu(q_\nu)$	p_ν^r
1	$10 + 0.05 q$	3	$37.5 - 0.05 q$	33.6
2	$15 + 0.05 q$	5	$75 - 0.1 q$	61.7
4	$42.5 + 0.025 q$	6	$80 - 0.1 q$	62.6

Table 4.1: Bid functions of generation and load

Definition 4.8 (System Operator). The ISO collects the bids in the day-ahead market and maximizes the total welfare subject to network constraints, leading to the following mathematical programming problem.:

$$\begin{aligned}
 \max_{q_\nu \in \mathbb{R}_+^N} \quad & \left[\sum_{\nu \in N_r} \int_0^{q_\nu} p_\nu(\xi_\nu) d\xi_\nu - \sum_{\nu \in N_p} \int_0^{q_\nu} C_\nu(\xi_\nu) d\xi_\nu \right] \\
 \text{s.t.} \quad & \sum_{\nu \in N} q_\nu = 0 \\
 & -K_\ell \leq \sum_{\nu \in N} \text{PTDF}_{\nu,\ell} q_\nu \leq K_\ell
 \end{aligned}$$

The day-ahead price p_ν^s at each node ν is given by the marginal cost at a generating bus, or by the inverse demand function at a load bus. The ISO makes a spot profit π_{so}^{spot} (the "merchandising surplus") by collecting transmission rents:

$$\pi_{so}^s = \sum_{\ell \in L} tr_\ell(p_{\ell_2}^s - p_{\ell_1}^s) \quad (4.19)$$

In this expression, tr_ℓ is the power flow on line ℓ . It can be derived from the injection/withdrawal q_ν and the power distribution factor of the line. $(p_{\ell_2}^s - p_{\ell_1}^s)$ is the difference of electricity prices at the end nodes of line ℓ . The profits of generators/retailers on the day-ahead market are given by:

$$\pi_\nu^s = \begin{cases} q_\nu p_\nu^s - C_\nu^T(q_\nu) & \text{if } \nu \in N_p \\ q_\nu (p_\nu^r - p_\nu^s) & \text{if } \nu \in N_r \end{cases} \quad (4.20)$$

4.4.2 A risk analysis of the day-ahead market

We assume two risk factors affecting the day-ahead market. One is the weather variation : it influences the final consumer and induces a stochastic demand at the retail side. We model this risk factor through scenarios for the parameters a_ν of the demand function at the load busses. Specifically the load parameters

4.4. A STYLIZED POWER MARKET EXAMPLE

a_ν vary from -25% to +25% (by step of 12.5%). Following Bessembinder and Lemmon [13] we suppose that the retail price p_ν^r is fixed in the short term. It is here set at 120% of the expected nodal price. The network availability is the second risk factor. Line outages can seriously impact the spot equilibrium. We consider two contingencies: there is no default with 90% probability and the line linking nodes 1 and 6 is down with 10% probability. The realization of the two risk factors is known at the time of the day-ahead market but unknown in the forward market. Combining both risk factors leads to a total of 250 scenarios of equal probability. Solving the equilibrium of the day-ahead market (cf. Problem 4.8) for the different scenarios we obtain the results summarized in Table 4.2.

Bus-ID:	1	2	3	4	5	6
Nodal prices:						
- $\mathbb{E}[p_\nu^s]$	24.72	26.2	28.13	45.98	51.46	53.05
- $\text{Var}[p_\nu^s]$	3.08	4.56	17.96	1.71	43.31	76.69
Market Agents:						
- $\mathbb{E}[\pi_\nu^{spot}]$	2197	1300	652	345	1646	1979
- $\text{vol}(\pi_\nu^{spot})$	11%	37%	75%	56%	76%	87%
- $-\text{CVaR}_{75\%}(\pi_\nu^{spot})$	1517	702	-83	86	78	-309

Table 4.2: Nodal prices and agents profit statistics

$\mathbb{E}$ and Var in the table respectively denote expectation and variance. The statistics "vol" measures the volatility of the profit; it is the ratio between the standard deviation and the expectation. The definition of the conditional value at risk CVaR_α is recalled in chapter 3.5. This measure is growing in popularity in the literature on electricity restructuring [10, 93, 56] and risk management in power markets[80]. In all tables of results of this section, the statistics $-\text{CVaR}$ represents the negative of the capital requirement. In other words a negative value in those tables represents a capital requirement for the agent to insure again its losses and remain in the business.

Table 4.2 shows that retailers profits are much more volatile than those of generators. Retailers face price and volume risks in meeting their load, and both are highly correlated. More importantly, their $-\text{CVaR}_{75\%}$ (i.e. the conditional expectation of their profit in the 63 worst scenarios) are very low compared to their expected profits. They are even negative²⁰ at busses 3 and 6. The revenues collected by the ISO are given in Table C.2, appendix A. They result from congestion on a line (price difference at its two extremity busses). The transmission prices are more volatile than the nodal energy prices, reflecting

²⁰indicating that they require a capital injection in their business

the fact that the demands in the three load busses are independent. Table C.2 also shows the first and second moments of those prices. Not surprisingly, the transmission price between the nodes 1 and 6 is the most expensive and volatile, because those two nodes are linked by a transmission line subject to outages.

4.4.3 The financial market

We consider two types of financial derivatives namely on energy and transmission. Energy futures maturing in the day-ahead market are the most widely traded in real markets. An agent holding a long futures position receives the difference between the day-ahead price at maturity and the futures price. Energy futures do not exist for all nodes. In order to enhance liquidity, they are often restricted to a few busses (the hubs). Agents remain exposed to the congestion charge between their home node and the hub. In order to allow agents to mitigate that risk, the ISO periodically auctions financial transmission rights (FTRs) that are effectively futures on congestion charges. We therefore complement the 6-nodes day-ahead market by introducing financial energy and transmission markets. There is one energy futures on node 6 day-ahead price and FTRs from all other nodes to node 6 (point to hub FTRs contracts that span all node to node transmission risks).

Generators and retailers participating in the financial market are exposed to the risk incurred because of fluctuating day-ahead prices, demand shocks and network congestion. They mitigate this risk by taking positions in the financial contracts leading to the following profit formula (adapted from equations 4.19 and 4.20) for retailers, generators and ISO :

$$\begin{aligned} \Pi_\nu(\omega) &= \begin{cases} \sum_c x_c^\nu (p_c^s(\omega) - p_c^f) + q_\nu(\omega) p_\nu^s(\omega) - C_\nu^T(q_\nu(\omega)) & \text{if } \nu \in N_p \\ \sum_c x_c^\nu (p_c^s(\omega) - p_c^f) + q_\nu(\omega) (p_\nu^r - p_\nu^s(\omega)) & \text{if } \nu \in N_r \end{cases} \\ \Pi_{so,\omega} &= \sum_c x_c^{so} (p_c^s(\omega) - p_c^f) + \sum_{\ell \in L} tr_\ell(\omega) (p_{\ell_2}^s(\omega) - p_{\ell_1}^s(\omega)) \end{aligned} \quad (4.21)$$

Producers and retailers hedge their stochastic profit according to Problem 4.1 . We model risk aversion through a E-CVaR $_{\alpha,\beta}$ defined as follows:

Definition 4.9. The risk measure E-CVaR $_{\alpha,\beta}$ is a weighted sum of the expectation of the losses and a CVaR $_{\alpha}$.

$$\text{E-CVaR}_{\alpha,\beta}(\Pi_\nu) = (1 - \beta) \mathbb{E}[-\Pi_\nu] + \beta \text{CVaR}_{\alpha}(\Pi_\nu)$$

The $\text{E-CVaR}_{\alpha,\beta}$ is a coherent risk measure; it is also equivalent in the sense of Definition 4.3.

We set the parameters β to 0.9 and α to 20% in the computation. The corresponding hedging problem for the agents (retailers and producers) is given in appendix C.1.

The ISO problem is slightly more complicated. The ISO is the ultimate counter-party in the transmission market. For reasons of "revenue adequacy", the ISO initially auctions FTR contracts that it restricts to be feasible for a given network topology subject to reliability constraints²¹. Also, the ISO only trades FTRs and does not take positions in energy futures on the hub (here node 6).

Problem 4.6. The ISO sells FTRs so as to minimize the risk measure of its profit, subject to the N-1 simultaneous feasibility constraints²². The ISO problem is then stated as follows.

$$\begin{aligned} \mathcal{P}^{so} \equiv \quad & \min_{x_{so,c}} \quad \rho_{so}(\Pi_{so}) \\ & -K_\ell \leq \sum_{\text{ftr}} \text{PTDF}_{\text{ftr},\ell}^{N-1} x_{so,\text{ftr}} \leq K_\ell \\ & x_{so,\text{energy}} = 0 \end{aligned}$$

The complete formulation of the equilibrium is given in appendix C.2. We prove that the set of equilibrium solutions is convex (cf. Proposition C.2). We also describe the fixed point algorithm used to compute the equilibrium.

4.4.4 Simulation results: the liquid case

Table 4.3 reports the futures prices p_c^f and volumes²³ of the different derivatives at equilibrium. $p_c^f - \mathbb{E}[p_c^s]$ is the risk premium embedded in the futures price. One first observes that the traded volume is high as energy futures traded at node 6 amount to 82% of the expected day-ahead trade at that node. One also observes that the contracts with the highest risk premium capture the highest share of the market. Those contracts are both most risky in terms of pay-off and most effective for hedging purposes.

²¹This is normally expressed through the "N-1" requirement that imposes that flows of FTRs remain feasible in case of default of one equipment.

²²Note that this formulation of the problem with a risk measure goes beyond the "N-1" requirement of revenue adequacy. It is known that the revenue adequacy is not fully guaranteed if contingencies occurs that have not been foreseen in the auction organized by the ISO. The recourse to a risk measure and its interpretation of capital requirement allows one to overcome that problem (assuming the TSO finds a counter party for accepting the risk of revenue adequacy in exchange of some capital payment). We also note that the contracts of the ISO problem guarantee the simultaneous physical feasibility of the FTRs.

²³The volume refers to the total quantity of MWH sold/bought on the market.

	p_c^f	$p_c^f - \mathbb{E}[p_c^s]$	Volume (MW)
FUTURE 6	53.5	0.45	564
FTR 1→6	28.9	0.58	462
FTR 2→6	27.4	0.55	696
FTR 3→6	25.53	0.55	359
FTR 4→6	7.59	0.46	242
FTR 5→6	1.57	-0.01	94

Table 4.3: Equilibrium prices, risk premium and volumes of the derivatives contract

A comparison of Table 4.4 and Table 4.2 shows the benefits of the derivative contracts for the players. Profit volatility decreases considerably compared to the situation with a full exposition to the day-ahead market. The $\text{CVaR}_{75\%}$ are also closer to the expected profits than without financial market. Forward positions dramatically reduce the risk exposure of all agents at the cost of only a small change of expected profit. Specifically, all agents remain in business without injection of capital. This effective impact of derivatives is illustrated in the figure 4.5, which compares the cumulative distribution function (cdf) of the day-ahead and hedged profits (π_6^s and Π_6 respectively) for the retailer located at node 6 (the hub).

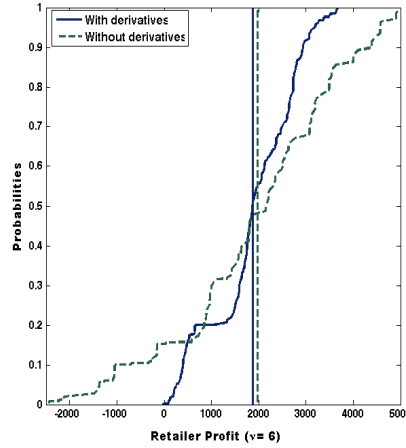


Figure 4.5: The cdf ²⁴ for the retailer at node 6

²⁴Cumulative distribution function of the profit : $\text{cdf}(\pi_\nu) = \text{Prob}(\Pi_\nu \leq \pi_\nu)$

	$\mathbb{E}[\Pi_\nu]$	$\text{vol}(\Pi_\nu)$	$-\text{CVaR}_{75\%}(\Pi_\nu)$
1	2198	1.8%	1517
2	1283	4.9%	1209
3	657	33%	329
4	349	17%	277
5	1602	49%	600
6	1890	48%	604
SO	11138	35%	6624

Table 4.4: Statistics of the market players total profit and

These figures, which reveal an important trading volume are in line with those precedingly obtained by various authors. Our treatment adds to that literature by introducing a transmission market and convex risk measures that guarantee the absence of arbitrage and hence tie in with the risk-neutral approach.

4.4.5 Simulation results : insufficient liquidity

The literature on contingent claims in electricity pays comparatively little attention to the impact of illiquidity. Pricing models do not include illiquidity as a state variable and hence cannot quantify its impact. Commodity portfolio models sometimes explicitly include bounds to represent illiquidity but prices remain exogenously given and independent of the positions²⁵ [54]. This implicitly assumes that modifying the bounds that reflect illiquidity does not change prices, which is a logical contradiction. Equilibrium models commonly predict high hedge ratio, even with agents with low risk aversion. For example, Bessembinder and Lemmon [13] find optimal hedge ratios²⁶ roughly ranging from 0.8 to 1.2, depending on the market parameters. In Willems and Morbee [116]’s computation, the total optimal number of futures goes up to 68GW when the expected demand is only 60 GW. All studies based on this type of methodology conclude to similar quantitative results. Producers and retailers massively buy/sell financial contracts in order to minimize their risks. Our model uses a different risk measure but concludes similarly (Table 4.5).

These results contradict observation as volumes such as those predicted by equilibrium models have never been observed on any real power exchange: illiquidity constrains agents’ strategies and makes such high hedging positions impossible. This justifies modifying the model to explicitly insert illiquidity.

²⁵where the latter assumption only holds in very liquid market

²⁶the hedge ratio is the ratio between the volume of future contracts and the expected production

	1	2	3	4	5	6
Hedge ratio	0.92	0.82	0.26	0.63	0.6	1.4

Table 4.5: Market player's future position divided by its expected day-ahead quantity

As explained in the section 4.3.1 we represent illiquidity by upper bounds on the volume of each traded energy contract (see equation 4.15). Because the ISO does not trade energy, these liquidity constraints only apply to generators and retailers.

We already mentioned that the "revenue adequacy" condition imposed by the auction on the FTRs imposes specific liquidity constraints on the ISO. These are expressed in the formulation of the ISO problem stated in (4.6). But, the practice of the FTR auction in the transmission market reveals other liquidity bounds. Indeed, besides the $N - 1$ rule, the ISO also restricts the number of auctioned FTRs to some share of the total transmission capacity. For example, New-England [see 67, p. 88-90] offers 50% of its capacity in the annual process, while California ISO [see 22, p. 10-19] delivers up to 75 %. Accordingly, the retailers and the generators are restricted by those bounds in their hedging problem.

The complete formulation of the equilibrium problem, accounting for these liquidity constraints, is given in appendix C.2. As discussed in section 3, the solution of the model is in term of Generalized Nash Equilibrium. We compute them using the method presented in Nabetani et al. [83] based on price-directed parametrization. Because we are interested in assessing how illiquidity can create inefficiencies, we randomly sample on the players shadow prices in order to heuristically span a subset of the solution space. We compute up to 4000 equilibria for each case presented in the analysis.

4.4.6 Simulation results: the illiquid case

We set the liquidity constraints on energy futures as a fraction of the expected total power traded in the day-ahead market. This reflects the observation that the volume effectively traded in the market is generally lower than the one coming out of equilibrium models. We similarly fix the liquidity constraint on FTRs to a share of the total capacity of the transmission lines (sum of thermal capacities over all lines of the grid). We construct a first scenario (noted LIQ_{66%}), where the total futures volume at the hub cannot exceed 66% of the expected total day-ahead energy and the FTR's volume is limited to 100% of the total transmission capacity. We fix these bounds at 33% and 50%

4.4. A STYLIZED POWER MARKET EXAMPLE

respectively for a second scenario (noted LIQ_{33%}).

	LIQ _{66%}		LIQ _{33%}	
	p_c^f	Volume	p_c^f	Volume
FUTURE 6	[51.2 , 53.7]	[33 , 457]	[50.1 , 54.3]	[17 , 228]
FTR 1→6	[26.8 , 29.1]	[34 , 307]	[25.4 , 29.2]	[0 , 28]
FTR 2→6	[25.2 , 27.5]	[2 , 262]	[23.8 , 27.6]	[0 , 232]
FTR 3→6	[23.3 , 25.8]	[21 , 196]	[21.9 , 26.2]	[0 , 26]
FTR 4→6	[5.5 , 7.6]	[125 , 140]	[4.5 , 7.8]	[0 , 179]
FTR 5→6	[1.4 , 2.1]	[75 , 136]	[0.9 , 3.2]	[0 , 25]

Table 4.6: Computed intervals for equilibria prices and volume of derivatives contracts)

Table 4.6 shows the range of contract prices and volumes at equilibrium. As found in Deng et al. [38] for the FTR market, the model reveals that volume limits may cause drastic changes in the contract prices compared to the perfectly liquid case and may even lead to equilibria with different signs of risk premium. Tables 4.7 and 4.8 show important profit statistics in the different equilibria. One observes that the agents' profit can be severely impacted by insufficient liquidity. It may even happen that some agent is excluded from financial trading. This corresponds to the worst hedging situation. Also, as liquidity decreases, the intervals of the $\text{CVaR}_{75\%}(\Pi)$ and $\text{vol}(\Pi)$ (and obviously the volume) shrink to a value closer to the day-ahead. Figures of the profit distribution in Table C.3 indicate how illiquidity impacts the profit of retailer at the node 6.

	$\mathbb{E}[\Pi_\nu]$	$\text{vol}(\Pi_\nu)$	$-\text{CVaR}_{75\%}(\Pi_\nu)$	Volume
1	[2116 , 2226]	[1.8% , 22%]	[1509 , 2160]	[1 , 712]
2	[1220 , 1315]	[3.1% , 37%]	[697 , 1260]	[0 , 551]
3	[628 , 743]	[27% , 75%]	[-83 , 444]	[0 , 656]
4	[334 , 441]	[14% , 79%]	[-33 , 316]	[17 , 363]
5	[1583 , 1814]	[44% , 76%]	[78 , 711]	[0 , 590]
6	[1860 , 2502]	[41% , 48%]	[599 , 1003]	[113 , 677]

Table 4.7: Producers and retailers profits for LIQ_{66%}

Even though insufficient liquidity limits trading, one also observes that the constraint expressing illiquidity of the energy futures market is not always binding. This may be surprising but is easily explainable. Indeed, the limited

	$\mathbb{E}[\Pi_\nu]$	$\text{vol}(\Pi_\nu)$	$-\text{CVaR}_{75\%}(\Pi_\nu)$	Volume
1	[2197 , 2960]	[5% , 24%]	[1480 , 2775]	[0 , 546]
2	[1300 , 1973]	[2% , 48%]	[614 , 1905]	[0 , 485]
3	[652 , 1173]	[37% , 76%]	[−83 , 418]	[0 , 198]
4	[332 , 921]	[13% , 250%]	[−145 , 370]	[0 , 426]
5	[1577 , 2210]	[42% , 76%]	[78 , 838]	[0 , 571]
6	[1853 , 2655]	[38% , 87%]	[−308 , 1254]	[0 , 541]

Table 4.8: Producers and retailers profits for LIQ_{33%}

number of FTR's prevents players from effectively hedging nodal profits, which essentially depend on the day-ahead price that they receive at their home node while energy futures only allow to hedge at the hub. Being unable to purchase FTRs, they have less incentive to take futures positions at the hub. The extremely low volume of 17MW at the hub is achieved when all Northern players are unable to buy any FTRs.

Volume FTRs	Volume Futures	$p_c^f(\text{FUTURE } 6)$
960	570 (92%) ²⁷	[52.9, 53.6]
720	478 (69%)	[52.9, 53.7]
480	320 (56%)	[52.9, 53.8]
240	269 (39%)	[53.5, 54.0]
0	152 (22%)	54.8

Table 4.9: Induced energy futures volume for a given liquidity bound on FTRs

Table 4.9 illustrates the interdependence between the energy and transmission markets in a remarkable way, by showing the maximum (over all computed GNEs) volume of energy futures as a function of the bound (illiquidity) on FTRs trading. One clearly sees the energy futures positions decreasing with the liquidity of the FTR market. In the extreme situation, with no FTR market, the volume of energy futures drops to 152 MW. One also notes that the maximal energy futures price tends to be higher. This reflects the fact that producers, not located at the hub, demand higher expected returns on the energy futures, as they can less perfectly hedge their profits (which highly depend on the congestion costs because of the lack of FTRs).

²⁷The percentage of volume with respect to the expected quantities contracted in the day-ahead market

4.4. A STYLIZED POWER MARKET EXAMPLE

Chapter 5

Stochastic capacity expansion models

5.1 Introduction

Generation capacity expansion models have a long tradition in the power industry. Designed as optimization problems for the regulated monopoly industry, they can be interpreted in terms of equilibrium in a competitive environment. This transposition is straightforward for a deterministic model representing a risk free economy: investment and operations levels found by the optimization model are equilibrium quantities and dual variables are equilibrium prices. The property remains valid in a perfectly competitive economy in a risky world but its interpretation can be much more complex.

Standard stochastic optimization capacity expansion models, minimizing expected cost or maximizing expected profits bypass risk aversion. Their interpretation as a perfectly competitive economy is similar to the one of the deterministic model, where all agents of the economy are risk-neutral. Investment and operations levels at the optimum of the stochastic capacity expansion model are equilibrium quantities in different states of the world. The dual variables of the model play a crucial role as they represent prices in these states of the world. We focus on those models in the first section of this chapter and particularly on two-stage models where uncertain parameters are modeled by random variables. We notably show that the Lagrange multipliers of the nonanticipativity constraints are the profit margins of the different technologies. We give a procedure for estimating the distribution of these profit margins (which allows one to compute accurately statistics of their distribution). We illustrate our findings on a simple example. We particularly show that the uncertainty

about wind penetration leads to high risk than cannot be hedged in an usual manner.

In the second section we cast the standard multiperiod stochastic capacity expansion problem in a risk measure context. The good deal introduced by Cochrane and Saá Requejo in [31] is the chosen risk measure. We explain that this formulation has several advantages. From the point of view of industrial practice, Chapter 3.3 show that the good-deal can be seen as an extension of the stochastic discount factor constructed from standard corporate finance theories such as the CAPM and the APT. In term of optimization of risk measure, the multiperiod good-deal benefits from the two very desirable properties of coherence and time consistency. The risk measure offers also an interesting economic interpretation and computational advantages. The multi-period capacity expansion model under Good-Deal is a second order cone program and hence amenable to a treatment of large size.

The last section of this chapter extends the stochastic risk averse capacity expansion problem to an equilibrium version, considering risk averse agent that value their profit according to a good-deal risk measure. This extension departs from the optimization paradigm (cost minimization) and requires a full equilibrium model. We focus on the two-stage problem and solve the problem for the very specific case of symmetric generators.

5.2 Stochastic capacity expansion problem

The restructuring of the electricity system provides the contextual background of this chapter. Changes occurred in this sector from at least two sides: the sector moved from a monopoly to a competitive regime; it also went from an almost risk-free environment to one assailed by uncertainties from all sides. We only treat the first problem in this section and assume that all agents are risk neutral, or more realistically value their investment according to a common exogenous discount factor (e.g. coming from the CAPM theory).

We adopt the following standard formulation of stochastic programs from [104, 16] and consider a two-stage¹ framework where some investment decisions x need to be made in the first stage and other operations decisions y take place in a second stage after uncertain data $\xi := (q, h, T, W)$ has been revealed. We work on a probability space $(\Omega, \mathcal{F}, P)$ equipped with a sigma algebra $\mathcal{F}$ and a probability distribution P . The vector ξ is a random vector whose probability distribution P is supported on the set $\Xi \subset \mathbb{R}^d$.

¹Capacity expansion problems in electricity are typically multi-stage, but for the sake of simplification, we start with two-stage problems.

Both x and y may be subject to stage specific constraints but the second stage operations variables y are also constrained by investment decisions x , implying a relation between x and y . Mathematically, all operations constraints on y are expressed by the linear relations $Tx + Wy = h$. These relations always embed constraints representing the satisfaction of the demand, the production set of the generators and relations linking the operations variables y to the level of investment x . The constraint set can also embed more general constraints as the electric grid limited capacity due to the thermal limits of electric lines. All parameters of the models (e.g. demand, fuel cost, availability of older power plant) can be taken as uncertain.

Problem 5.1. The general capacity expansion problem is the following two stage stochastic linear programming problem

$$\begin{aligned} \min_{x \in \mathcal{X}} \quad & \iota^t x + \mathbb{E}[Q(x, \xi)] \\ \text{s.t.} \quad & \mathcal{X} := \{x \in \mathbb{R}^n \mid a_i^T x \leq b_i, i = 1, \dots, p\} \\ & Q(x, \xi) = \min_y c^t y \\ & \text{s.t.} \quad Tx + Wy = h \\ & y \geq 0 \end{aligned}$$

where the vector a_i are the rows of a matrix $A \in \mathbb{R}^{n \times p}$ and $y \in \mathbb{R}^m$. The vector ι is the one of annual investment costs of the different technologies computed from overnight construction and fixed operating cost using a standard annualisation procedure. The vector c represents the operations cost in scenario.

We start the analysis by recalling well-known results about linear two stage problems.

Definition 5.1. Let us define the following sets that appear in the analysis.

- ◊ $\Pi(c) := \{\pi : W^T \pi \leq c\}$
- ◊ The positive hull of a matrix W is defined as:

$$\text{pos}W := \{\chi : \chi = Wy, y \geq 0\}$$

- ◊ $\text{dom}\phi := \{x \in \mathbb{R}^n : h - Tx \in \text{pos } W \text{ w.p. } 1\}$
- ◊ $\mathfrak{D}(x, \xi) := \arg \max_{\pi \in \Pi(c)} \pi^T (h - Tx)$

The recourse cost $Q(\cdot, \xi)$ is convex in x and lower semicontinuous for every $\xi \in \Xi$ and its differentiability structure can be described by the following proposition.

Proposition 5.1. Suppose that for given $x = x_0$ and $\xi \in \Xi$, the value of $Q(x_0, \xi)$ is finite. Then $Q(\cdot, \xi)$ is subdifferentiable at x_0 and

$$\partial Q(x_0, \xi) = -T^T \mathfrak{D}(x_0, \xi).$$

One can further specialize the property of the recourse function when the recourse is fixed, i.e. the matrix W is deterministic.

Theorem 5.1 (e.g. [16, 71]). *For a stochastic program with fixed recourse where ξ has finite second moments and an absolutely continuous distribution, the recourse function $Q(x, \xi)$ is differentiable in x .*

We can now formulate the optimality conditions and duality relations for the linear two-stage capacity expansion problem.

Theorem 5.2 (from [104]). *Let $\bar{x}$ be a feasible solution of the Problem 5.1. Suppose that: (i) the matrix W is deterministic², (ii) for a.e. c the set $\Pi(c)$ is non empty, (iii) the two following conditions are satisfied*

$$\mathbb{E}[\|c\| \|h\|] < +\infty \text{ and } \mathbb{E}[\|c\| \|T\|] < +\infty,$$

*(iv) **int** $\text{dom}\phi \cap X$ is non empty and (v) the matrix T is deterministic. Then $\bar{x}$ is an optimal solution of Problem 5.1 iff there exist a measurable function $\pi \in \mathfrak{D}(x, \xi)$; and a vector $\lambda \in \mathbb{R}_+^p$ such that*

$$T^T \mathbb{E}[\pi] + A^T \lambda \leq c, \tag{5.1}$$

$$\bar{x}^T (c - T^T \mathbb{E}[\pi] - A^T \lambda) = 0. \tag{5.2}$$

²In other words, the two-stage Problem 5.1 has fixed recourse

5.2.1 A basic model

We elaborate on a basic model taken from [42] and [16, section 1.3]. The industry initially invests in K different technology (e.g. nuclear, coal-fired, CCGT,...). Each equipment k has both an investment cost ι_k and operating cost c_k per MWh . The industry faces a price insensitive demand which is described by a load duration curve³ decomposed in L different time segments. The duration of a time segment ℓ is denoted τ_ℓ and its level d_ℓ (see picture 5.1).

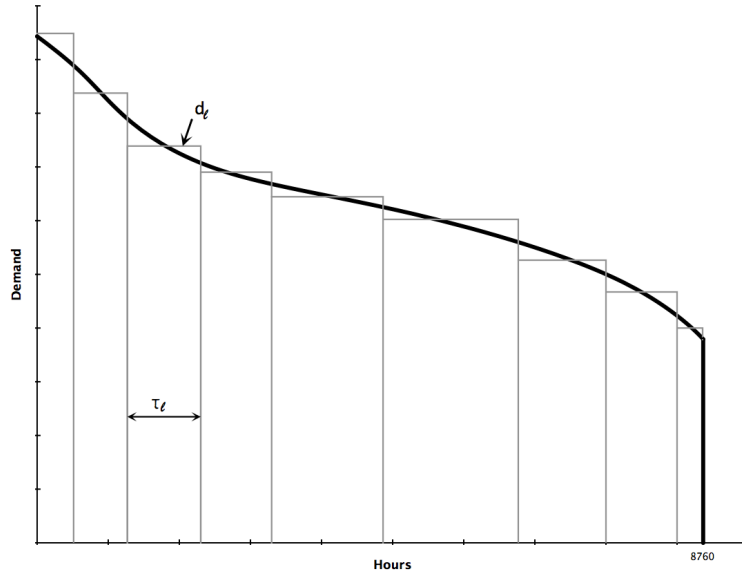


Figure 5.1: Segmentation of the load duration curve

Finally, the industry has the possibility not to satisfy a demand segment at the cost of PC . We denote the investment in capacity of technology k by x_k , its operating level in time segment ℓ by $y_{k,\ell}$ and the unsatisfied demand during time segment ℓ by z_ℓ . All data's in the second period are possibly random. We denote by $D = (D_1, \dots, D_L)$ the random demand and $C = (C_1, \dots, C_K)$ the random cost (d and c denote a particular realization of those random vector).

³The load duration curve measures the number of hours per year the total load is at or above any given level of demand (definition taken from [109])

Problem 5.2 (The basic capacity expansion problem).

$$\min_{x \geq 0} \sum_{k=1}^K v_k x_k + \mathbb{E}[Q(x, D, C)]$$

where $Q(x, d, c)$ is the optimal value of the second stage problem

$$\begin{aligned} \min_{y \geq 0, z \geq 0} \quad & \sum_{\ell=1}^L \tau_{\ell} \left(\sum_{k=1}^K c_k y_{k,\ell} + PC z_{\ell} \right) \\ \text{s.t.} \quad & 0 \leq x_k - y_{k,\ell} \quad (\tau_{\ell} \mu_{k,\ell}) \\ & 0 \leq \sum_{k=1}^K y_{k,\ell} + z_{\ell} - d_{\ell} \quad (\tau_{\ell} \pi_{\ell}) \end{aligned}$$

One recovers the interpretation of the competitive equilibrium for the operations stage. Indeed, the KKT conditions of the second stage can be stated as follows:

$$0 \leq x_k - y_{k,\ell} \perp \mu_{k,\ell} \geq 0 \quad (5.3)$$

$$0 \leq \sum_{k=1}^K y_{k,\ell} + z_{\ell} - d_{\ell} \perp \pi_{\ell} \geq 0 \quad (5.4)$$

$$0 \leq c_k + \mu_{k,\ell} - \pi_{\ell} \perp y_{k,\ell} \geq 0 \quad (5.5)$$

$$0 \leq PC - \pi_{\ell} \perp z_{\ell} \geq 0 \quad (5.6)$$

Given a particular realization of the load duration curve $(d_1, \dots, d_L)$ and technology costs $(c_1, \dots, c_K)$, complementarity conditions (5.3) to (5.6) are interpretable in terms of a perfect competitive equilibrium where price taking agents face a price insensitive demand (e.g. [42] and the cited literature). It complies with market organization of the "energy only" type (see [109, 69] and the note in 2.2.1 for a discussion) where PC is the Value of Lost load (VOLL), i.e. the economic value of unsatisfied demand. The dual variable π_{ℓ} is the price in time segment ℓ and is equal to the variable cost of the marginal generating unit or, when load is curtailed, is equal to PC . The variable $\mu_{k,\ell}$ gives the hourly marginal value (profit) on capacity k in time segment ℓ . It is nonnegative when the capacity is saturated.

The basic Problem 5.2 has the following properties

- The corresponding matrices W and T of the problem are fixed (not random).
- The problem has relatively complete recourse, meaning that $Q(x, d, c) < +\infty$ holds true for every $x \geq 0$, for all $c \in C$ and $d \in D$.
- The set $\Pi(c) := \{\pi \in \mathbb{R}_+^L, \mu \in \mathbb{R}_+^{K \times L} : \pi_{\ell} \leq c_k + \mu_{k,\ell}; \pi_{\ell} \leq PC\}$ is non-empty.

- **int** $\text{dom}\phi \cap X$ is non empty.

$\Pi(c)$ can be interpreted as the set of acceptable price π and units profit $\mu_{k,\ell}$ for the units. Indeed the first constraint specifies that the price π_ℓ can not be higher than the operating cost c_k without making that unit profitable (i.e. $\mu_{k,\ell} > 0$). It constraints also that price to be positive and less than the price cap PC .

The set $\mathfrak{D}(x, d, c)$ involves the dual of the second stage problem which can be stated as follows

Problem 5.3. For fixed x , the dual problem of the second stage of 5.2 is the following LP problem

$$\begin{aligned} \mathfrak{D}(x, d, c) \quad &\equiv \quad \arg \max_{\mu \in \mathbb{R}_+^{K \times L}, \pi \in \mathbb{R}_+^L} && \sum_{\ell=1}^L \tau_\ell \pi_\ell d_\ell - \sum_{k=1}^K \left(\sum_{\ell=1}^L \tau_\ell \mu_{k,\ell} \right) x_k \\ \text{s.t.} &&& c_k + \mu_{k,\ell} - \pi_\ell \geq 0 && k = 1, \dots, K; \ell = 1, \dots, L \\ &&& PC - \pi_\ell \geq 0 && \ell = 1, \dots, L \end{aligned}$$

Proposition 5.2. Suppose that the condition (iii) of Theorem 5.2 is satisfied. A feasible solution $\bar{x}$ is optimal iff there exist measurable functions $(\pi_\ell, \mu_{k,\ell}) \in \mathfrak{D}(x, D, C)$ such that

$$0 \leq x_k \perp \iota_k - \sum_{\ell=1}^L \tau_\ell \mathbb{E}[\mu_{k,\ell}] \geq 0. \quad (5.7)$$

The interpretation of condition (5.7) as investment criterion is straightforward. One invests in technology k if the investment cost ι_k is compensated by the (discounted) expected gross margin of the plant k , i.e. the marginal value summed over the year. It reveals in a striking way the importance of the measurable function $\mu_{k,\ell}(D, C)$ in the investment choice. We discuss in the next section how estimate this function accurately. Finally, as noted previously and pointed by various authors, the whole model (i.e. the condition (5.7) combined with the operations relations (5.3) to (5.6)) describes a perfectly competitive equilibrium with risk-neutral agents.

5.2.2 Nonanticipativity constraints

General discussion

The concept of nonanticipativity has been introduced since the beginning of stochastic programming (e.g. Rockafellar and Wets [95]). It permits to gain additional insight into the structure and properties of stochastic program but also to develop algorithms. We recall here from [104] the fundamental analysis of the nonanticipativity constraints and their dualizations. We thank Alexander Shapiro for his additional developments that permitted us to tackle the

problem of estimating the distribution of the Lagrange multipliers.

Consider the two-stage Problem 5.1 and write it in the following equivalent form

$$\min_{x \in \mathbb{R}^n} \{\bar{f}(x) := \mathbb{E}[\bar{F}(x, \xi)]\} \quad (5.8)$$

where $\bar{F}(x, \xi) := \iota^T x + Q(x, \xi) + \mathbb{I}_{\mathcal{X}}$ and $\mathbb{I}_{\mathcal{X}}$ is an indicator function, meaning that :

$$\mathbb{I}_{\mathcal{X}} = \begin{cases} 0 & \text{if } x \in \mathcal{X} \\ +\infty & \text{if } x \notin \mathcal{X} \end{cases}.$$

Note that the function $\iota^T x + Q(x, \xi)$ is a random lower semicontinuous function and $Q(\cdot, \xi)$ is convex.

Let $\mathfrak{X}$ be the space of measurable mappings⁴ from Ω to $\mathbb{R}^n$ where, for every $\mathbf{x} \in \mathfrak{X}$ the expectation $\mathbb{E}[\bar{F}(\mathbf{x}, \xi)]$ is well defined. The dual space is denoted by⁵ $\mathfrak{X}^*$. We can write the Problem 5.8 as

$$\min_{\mathbf{x} \in \mathfrak{L}} \mathbb{E}[\bar{F}(\mathbf{x}(\xi), \xi)] \quad (5.9)$$

where $\mathfrak{L}$ is a linear subspace of $\mathfrak{X}$ formed by the mapping $\mathbf{x} : \Omega \rightarrow \mathbb{R}^n$ which are constant almost everywhere, i.e.

$$\mathfrak{L} := \{\mathbf{x} \in \mathfrak{X} : \mathbf{x}(\xi) = x \text{ for some } x \in \mathbb{R}^n\},$$

where $\mathbf{x}(\xi) = x$ holds true for a.e. $\xi \in \Xi$. After some developments (see [104]), the associated Lagrangian of problem (5.9) can be stated as follows :

$$L(\mathbf{x}, \lambda) = \mathbb{E}[\bar{F}(\mathbf{x}(\xi), \xi) + \lambda(\xi)^T \mathbf{x}(\xi)]. \quad (5.10)$$

This leads to the following dual of problem (5.9)

$$\begin{aligned} \max_{\lambda \in \mathfrak{X}^*} \quad & \{D(\lambda) := \inf_{\mathbf{x} \in \mathfrak{X}} L(\mathbf{x}, \lambda)\} \\ \text{s.t.} \quad & \mathbb{E}[\lambda] = 0. \end{aligned} \quad (5.11)$$

We can now state the following theorem that gives the optimality condition for the two stage Problem 5.1.

Theorem 5.3. *Suppose that $\mathcal{X}$ is convex and closed. There is no duality gap between the problems (5.9) and (5.11), and both problems have optimal solutions iff there exists $\bar{x} \in \mathbb{R}^n$ satisfying the following condition*

$$0 \in \mathbb{E}[\partial_x \bar{F}(\bar{x}, \xi)].$$

⁴We actually use $\mathfrak{X} := \mathcal{L}_p(\Omega, \mathcal{F}, P; \mathbb{R}^n)$ for some $p \in [1, +\infty]$, i.e. the set of all $\mathcal{F}$ -measurable functions $\phi : \Omega \rightarrow \mathbb{R}^n$ such that $\int_{\Omega} \|\phi\|^p dP < +\infty$.

⁵Recall that for the dual of the space $\mathfrak{X} := \mathcal{L}_p(\Omega, \mathcal{F}, P; \mathbb{R}^n)$ for some $p \in [1, +\infty]$, is the $\mathfrak{X}^* = \mathcal{L}_q(\Omega, \mathcal{F}, P; \mathbb{R}^n)$ where $1/p + 1/q = 1$.

In that case $\bar{x}$ is an optimal solution of the primal problem (5.9) and a measurable function $\bar{\lambda}(\xi) \in \partial_x \bar{F}(\bar{x})$ such that $\mathbb{E}[\bar{\lambda}] = 0$ is an optimal solution of the dual problem (5.11).

Estimation of distribution of $\lambda(\xi)$

Let $\bar{x} \in \mathcal{X}$ be an optimal solution of problem (5.8). Then a measurable function $\lambda(\xi) \in -\partial \bar{F}(\bar{x}, \xi)$ such that $\mathbb{E}[\lambda(\xi)] = 0$, is an optimal solution of (5.11) (cf. Theorem 5.3). As we consider the two-stage Problem 5.1, we can further specialize

$$\partial_x \bar{F}(\bar{x}, \xi) = \iota + \partial_x Q(\bar{x}, \xi) + \mathcal{N}_{\mathcal{X}}(\bar{x}),$$

where the normal cone $\mathcal{N}_{\mathcal{X}}$ is generated by vectors $\{a_i\}_{i \in \mathcal{I}(\bar{x})}$, with

$$\mathcal{I}(\bar{x}) := \{i : a_i^T \bar{x} = b_i, i = 1, \dots, p\}$$

being the set of active constraints.

Suppose now that the function $Q(\cdot, x)$ is differentiable at $x = \bar{x}$ with probability 1, i.e. $\partial_x Q(\bar{x}, \xi) = \{\nabla_x Q(\bar{x}, \xi)\}$ is a singleton for a.e. ξ . This typically happens with continuous distributions (cf. Theorem 5.1). Then Lagrange multiplier functions can be any measurable function of the form

$$\lambda(\xi) = -\iota - \nabla_x Q(\bar{x}, \xi) - \sum_{i \in \mathcal{I}(\bar{x})} \gamma_i(\xi) a_i \quad (5.12)$$

such that $\mathbb{E}[\lambda(\xi)] = 0$ and $\gamma_i(\xi) \geq 0, i \in \mathcal{I}(\bar{x}), \xi \in \Xi$. Since $Q(\cdot, \xi)$ is supposed to be differentiable at $x = \bar{x}$, the expected function $f(x) = \mathbb{E}[\iota^T x + Q(x, \xi)]$ is also differentiable and $\nabla f(\bar{x}) = \mathbb{E}[\iota + \nabla_x Q(\bar{x}, \xi)]$. Consequently (cf. Theorem 5.3), by optimality of $\bar{x}$ we have that there exists $\bar{\gamma}_i \geq 0, i \in \mathcal{I}(\bar{x})$, such that:

$$\nabla f(\bar{x}) + \sum_{i \in \mathcal{I}(\bar{x})} \bar{\gamma}_i a_i = 0. \quad (5.13)$$

Moreover, the multipliers $\bar{\gamma}_i$ are defined uniquely if the vector $a_i, i \in \mathcal{I}(\bar{x})$, are linearly independent. In that case, combining (5.12) and (5.13), leads to $\bar{\gamma}_i = \mathbb{E}[\gamma_i(\xi)], i \in \mathcal{I}(\bar{x})$. It seems to be very natural (as it directly implies $\mathbb{E}[\lambda(\xi)] = 0$) to choose $\lambda(\xi)$ in (5.12) such that coefficients $\gamma_i(\xi)$ are the constants in (5.13), i.e., to take

$$\lambda(\xi) = -\iota - \nabla_x Q(\bar{x}, \xi) + \nabla f(\bar{x})$$

In order to estimate the distribution of $\lambda(\xi)$ we can proceed as follows. First compute an estimate $\hat{x} \in \mathcal{X}$ of the optimal solution $\bar{x}$, by applying the SAA method. Then generate a sample $\xi^1, \dots, \xi^N$ of ξ and set

$$\hat{\lambda}^j = -\nabla_x Q(\hat{x}, \xi^j) + \frac{1}{N} \sum_{k=1}^N \nabla_x Q(\hat{x}, \xi^k), \quad j = 1, \dots, N. \quad (5.14)$$

One can then use the $\hat{\lambda}^j$, $j = 1, \dots, N$ to construct an approximation of the distribution $\bar{\lambda}(\xi)$.

Remark 5.1. Unless $\nabla f(\bar{x}) = 0$, there is more than one solution $\lambda(\xi)$ satisfying (5.12). Let c be a vector such that $c^T a_i = 0$, $i \in \mathcal{I}(\bar{x})$. Then $c^T \lambda(\xi)$ is defined uniquely,

$$c^T \lambda(\xi) = -c^T \iota - c^T \nabla_x Q(\bar{x}, \xi)$$

Estimation of the profit margins distribution for the basic stochastic capacity expansion problem

Let us apply those results to the basic capacity expansion Problem 5.2. We have the $\mathcal{X} = \mathbb{R}_+^K$, $F(x, d, c) = \sum_{k=1}^K \iota_k x_k + Q(x, d, c)$. The Lagrangian of the second (operation) stage of Problem 5.2 is:

$$L(x, y, z, \mu, \pi) = \sum_{\ell=1}^L \left(\sum_{k=1}^K (c_k + \mu_{k,\ell} - \pi_\ell) y_{k,\ell} + (PC - \pi_\ell) z_\ell \right) - \sum_{\ell=1}^L \sum_{k=1}^K \tau_\ell \mu_{k,\ell} x_k + \sum_{\ell=1}^L \tau_\ell d_\ell \pi_\ell$$

According to Proposition 5.1, we have that the subdifferential $\partial_x Q(x, d, c)$ is given by

$$\partial_x Q(x, c, d) = \left(\sum_{\ell=1}^L \tau_\ell \bar{\mu}_{1,\ell}, \dots, \sum_{\ell=1}^L \tau_\ell \bar{\mu}_{K,\ell} \right),$$

where $(\bar{\mu}, \bar{\pi}) = (\bar{\mu}(d, c), \bar{\pi}(d, c))$ is an optimal solution of (5.3).

If the random vectors D and C satisfy the hypothesis of Theorem 5.1, the recourse Q is differentiable. In that case, the Lagrange multipliers of nonanticipativity constraints of the basic capacity expansion problem are given by

$$\bar{\lambda}_k(d, c) = \sum_{\ell=1}^L \tau_\ell \bar{\mu}_{k,\ell}(d, c) - \mathbb{E} \left[\sum_{\ell=1}^L \tau_\ell \bar{\mu}_{k,\ell} \right].$$

Combining this result with the optimality condition (5.7), one has that the Lagrange multipliers of the nonanticipativity constraint represents the net margin of the investment for a particular realization of D and C .

To estimate the distribution of the gross margins we can proceed now as follows. Let $\hat{x}$ be an estimate of optimal solution of the first stage Problem 5.2. Suppose that $\hat{x} > 0$, i.e. all components of $\hat{x}$ are strictly positive. Generate i.i.d. sample $d^1, \dots, d^N$ and $c^1, \dots, c^N$ of random vectors D and C . For $x = \hat{x}$ and each $(d, c) = (d^j, c^j)$, $j = 1, \dots, N$, solve the problem in Proposition 5.1.

Let $(\mu^j, \pi^j), j = 1, \dots, N$ be the respective optimal solutions. Then vectors $\hat{\lambda}^j, j = 1, \dots, N$ computes as (5.14) with

$$\nabla_x Q(\hat{x}, c^j, d^j) = \left(\sum_{\ell=1}^L \tau_{\ell} \mu_{1,\ell}^j, \dots, \sum_{\ell=1}^L \tau_{\ell} \mu_{K,\ell}^j \right), \quad j = 1, \dots, N$$

can be used to construct an empirical distribution of the Lagrange multipliers of the nonanticipativity constraints.

Remark 5.2. Adapting Remark 5.1 to the basic stochastic capacity expansion problem, one deduces that Lagrange multipliers of the nonanticipativity constraints are not uniquely defined for the components of $\bar{x}$ equal to zeros.

5.2.3 A case study: the wind penetration

Description of the problem

We illustrate our findings on a simple example of the basic capacity expansion problem. The simple example is meant to represent a market subject to 3 types of uncertainty, namely wind penetration, demand growth and the gas price. We consider $K = 4$ capacity types: nuclear, coal, Combined Cycle Gas Turbine (CCGT) and Open Cycle Gas turbine (OCGT). The investment costs ι_k are annualized and the operating costs c_k are derived from fuel prices, carbon prices, the heat rate (thermal efficiency) and the carbon emission rate η_k of the technology. The gas price is a uncertain. We model it through the random variable ξ_1 by setting $p^{\text{gas}} = 28\xi_1$.

$$c_k = \text{HR}_k \times p^{\text{fuel}} + \eta_k \times p^{\text{CO}_2}$$

Finally we set PC to 500[€/MW].

For the sake of simplicity, we only take gas prices as uncertain. The price of CO_2 remains at 15[€/tonne $_{\text{CO}_2}$] while the variable cost of nuclear unit is set to 9[€/MW] and to 47.48[€/MW] for coal unit. Table 5.1 summarizes the cost structure for the different technologies. This example is inspired by data found in Du and Parsons [41].

	Nuclear	Coal	CCGT	OCGT
ι_k : [k€/MW]	499.3	280.3	96.36	57.8
c_k : [€/MW]	9	47.48	$1.89 \times p^{\text{gas}} + 6.04$	$2.91 \times p^{\text{gas}} + 9.29$

Table 5.1: Fixed annual cost and operation cost structure

We consider initially a load duration curve which is a straight line between $[\underline{d}, \bar{d}]$ that we decompose in $L = 50$ different segments, each of them of duration

$\tau_\ell = 175.2$ h. The load duration curve is impacted by the growth and the wind penetration. Our model does not account for endogenous wind production but we capture the effects of exogenous wind penetration by considering a load duration curve that is only satisfied by the thermal generators (i.e. we subtract the wind production from the load to obtain a net load satisfied by thermal generators). The obtained load duration curve exhibits more peaky behavior (see [1]). We accordingly stylize this finding by considering that wind penetration impacts the slope of the load duration curve with load now varying in the interval $[\underline{d} - (\xi_2 - 1)\bar{d}, \xi_2\bar{d}]$. The second risk factor is the global demand growth (i.e. the area under the load duration curve). We model it by the random variable ξ_3 and set the total yearly demand to $\frac{\xi_3}{2}(\underline{d} + \bar{d})$. To summarize, for particular realizations (ξ_2, ξ_3) , the demand d_ℓ is given by the following expression:

$$d_\ell = \xi_2\bar{d} - \frac{1}{L} \left[\frac{1}{2} + (\ell - 1) \right] ((2\xi_2 - \xi_3)\bar{d} - \xi_3\underline{d}).$$

We assume the following distributions⁶ for the random variables ξ_1, ξ_2, ξ_3 :

$$\xi_1 \sim N(0.04; 0.0009) , \quad \xi_2 \sim U(1; 1.1) , \quad \xi_3 \sim U(1; 1.03)$$

Results

We solve the problem using the sample average approximation (SAA) method, i.e. by generating an i.i.d. sample of N realizations of the vector $\xi := (\xi_1, \xi_2, \xi_3)$. We get the corresponding realizations of the demand $d^1, \dots, d^N$ and cost $c^1, \dots, c^N$. This leads to the following approximation of the "true" Problem 5.2.

$$\min_{x \geq 0} \left\{ \hat{f}_N(x) = \sum_{k=1}^K \iota_k x_k + \frac{1}{N} \sum_{j=1}^N Q(x, d^j, c^j) \right\}. \quad (5.15)$$

By the Law of Large Numbers, we know that, under some regularity conditions $\hat{f}_N(x)$ converges pointwise w.p. 1 to $f(x)$ as $N \rightarrow \infty$. Let (θ^*, x^*) and $(\hat{\theta}^N, \hat{x}^N)$ denote the optimal value and optimal solutions of respectively the true problem and the SAA problem. Table 5.2 reports the solution of the problem obtained with 2000 realizations of (d, c) .

⁶ $N(\mu, \sigma^2)$ is a normal distribution of mean μ and variance σ^2 and $U(a, b)$ is the continuous uniform distribution supported on the interval $[a, b]$.

5.2. STOCHASTIC CAPACITY EXPANSION PROBLEM

Installed capacity	$\hat{x}^N$ (N=2000) [MW]
Nuclear	4795.1
Coal	0.0
CCGT	4818.0
OCGT	901.5
$\hat{\theta}^N$ [G€/year]	4655.0

Table 5.2: Solution of the basic stochastic capacity expansion problem

The total installed capacity is 10514.6MW and it is optimal not to invest in coal units. The next table 5.3 shows some confidence intervals for the true optimal solution⁷ (we refer to chapter 4 of Shapiro et al. [104] for the statistical inference of the SAA method). One sees that the problem was solved with enough accuracy.

	θ^* [G€/year]
@ 95 %	[4656.06 ,4659.94]
@ 99 %	[4655.98 ,4661.67]
@ 99.9 %	[4655.91 ,4662.29]

Table 5.3: Confidence interval for the optimal value

Net margins distribution

Applying the procedures described previously and generating up to 30,000 scenarios, we obtain the following histogram for the distribution of the profit margins for each installed capacity.

The distribution patterns are certainly quite unusual. They take the form of multimodal distributions for the nuclear technology and the two other distributions do not resemble at any known distribution. Each distribution is positively skewed indicating, for each technology, a higher probability of unprofitable investment.

One also notes that the distributions are subdivided into 4 different regions (which particularly remarkable for the OCGT). Those correspond to the number of time periods where demand is unsatisfied (e.g. the least profitable scenario are obtained when demand is always satisfied).

⁷The bounds were obtained by solving $M = 100$ times the problem (with different realizations of the random vector ξ). The approximation of the value of $f(\hat{x}^N)$ (which differs of $\hat{\theta}^N$) was obtained with 5000 scenarios.

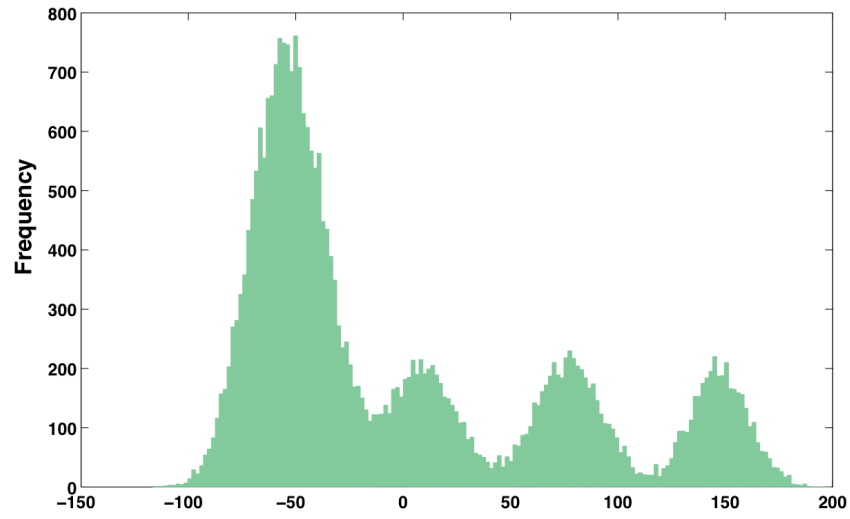


Figure 5.2: Histogram of the profit margins for the Nuclear technology.

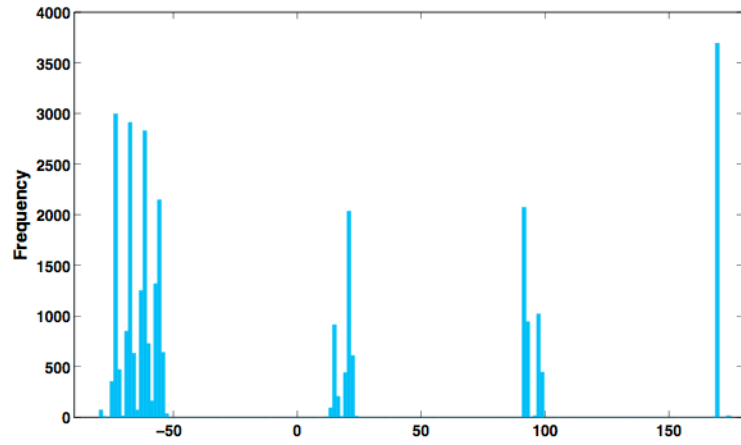


Figure 5.3: Histogram of the profit margins for the CCGT technology.

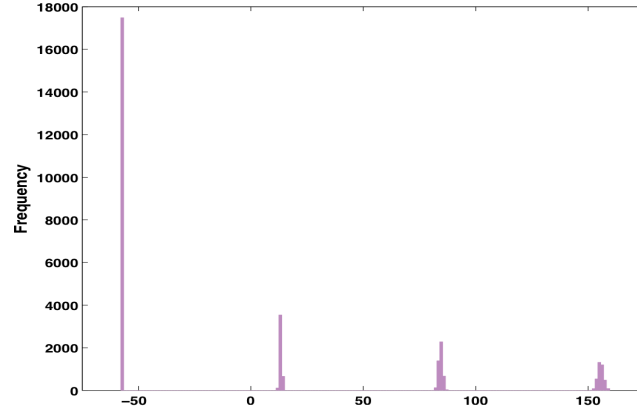


Figure 5.4: Histogram of the profit margins for the OCGT technology.

	Prob
$\{\#\ell : \pi_\ell = PC\} = 0$	58.27 %
$\{\#\ell : \pi_\ell = PC\} = 1$	14.38 %
$\{\#\ell : \pi_\ell = PC\} = 2$	14.98 %
$\{\#\ell : \pi_\ell = PC\} = 3$	12.37 %

Table 5.4: Probability of time period where demand is unsatisfied

Table 5.4 reports the probability of the period's occurrence of unsatisfied demand. One can interpret the high probability of curtailment as resulting from the low price cap PC used in this example. Alternatively it can also be interpreted as resulting from a high penetration of demand response. Load curtailment (and the associated price jump to PC) is actually the main driving factor of the margin. Correspondingly, the correlation coefficient between the profit margins of different technologies is very close to one (see Table 5.5). There is thus no diversification/hedging effect: either the demand is low and the whole industry suffers from over-investment, or the demand is high compared to the installed capacity and all technologies earn large profits.

	CCGT	OCGT
Nuclear	0.9736	0.9750
CCGT	-	0.9979

Table 5.5: Correlation coefficient of marginal profits

As mentioned before, wind penetration acts both on the global demand level and on the shape of the load duration curve. We highlight the fact that uncertainty on the shape of the curve induces new risks that are particularly important. In order to see this consider the two problems respectively labelled "wind" and "growth" cases where ξ_2 and ξ_3 are the sole random variables and the other risk factors are taken at their average. The optimal investment levels are given in Table 5.6.

Installed capacity	$\hat{x}^N$ [MW] "Wind"	$\hat{x}^N$ [MW] "Growth"
Nuclear	4797.7	5048.1
Coal	0.0	0.0
CCGT	4811.0	4269.5
OCGT	897.4	659.46
$\bar{\theta}^N$ [G€/year]	4656.5	4568.6

Table 5.6: Solution of the basic stochastic capacity expansion problem : the "Wind" and "Growth" cases.

5.2. STOCHASTIC CAPACITY EXPANSION PROBLEM

The investment profile differs between the two cases. Clearly, in the "wind" case it is optimal to invest more in "peak" (less capital incentives) units. The corresponding histograms of net margins are given in Figures 5.5, 5.6 and 5.7.

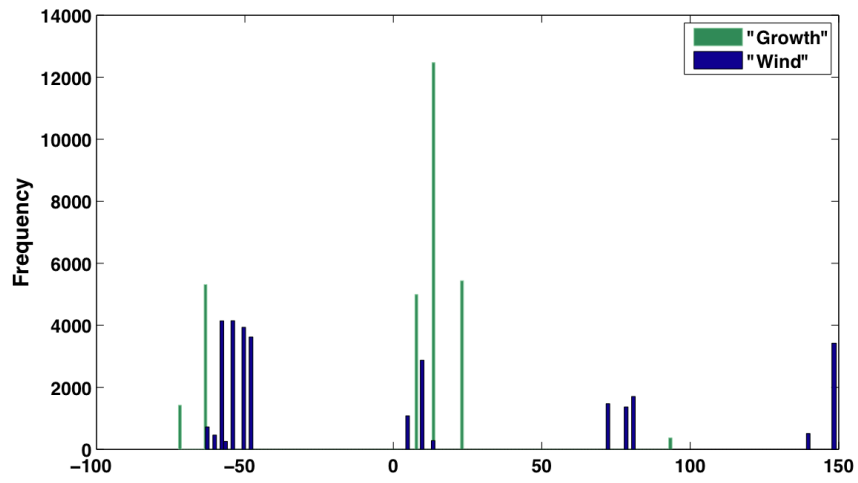


Figure 5.5: Nuclear technology ("Wind" and "Growth" case).

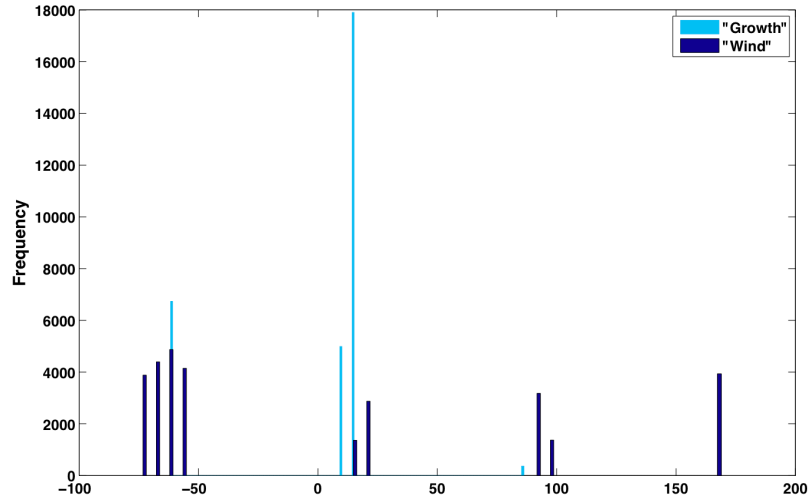


Figure 5.6: CCGT technology ("Wind" and "Growth" cases).

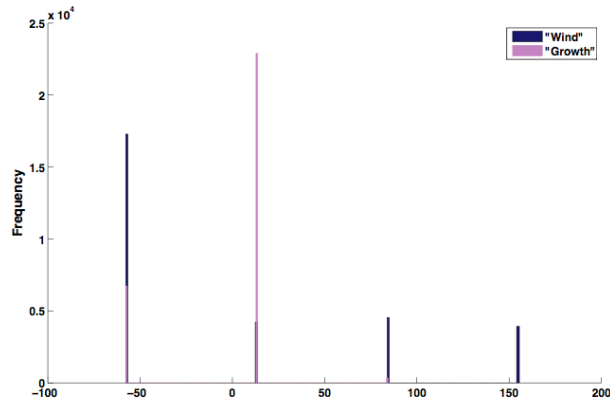


Figure 5.7: OCGT technology ("Wind" and "Growth" case).

5.2. STOCHASTIC CAPACITY EXPANSION PROBLEM

One also sees that the margin distribution in the "Wind" case involves more scenarios with negative profit. To measure and compare the risks, we show for each technology in Table 5.7 the $\text{CVaR}_{30\%}$ of the profit margins and their standard deviation σ for the different cases.

	3 uncertainties	"Growth"	"Wind"
Nuclear : - $\text{CVaR}_{30\%}$	43.75	10.63	44.07
- σ	74.25	35.22	73.78
CCGT : - $\text{CVaR}_{30\%}$	50.70	8.68	51.63
- σ	85.73	33.6	86.82
OCGT : - $\text{CVaR}_{30\%}$	46.05	7.71	46.99
- σ	77.7	31.14	78.74

Table 5.7: $\text{CVaR}_{30\%}$ [k€/MW] and standard deviation σ [k€/MW] of the profit margins

In case of risk aversion (which was not assumed in this section), uncertainty about future wind penetration clearly exacerbates the need of capital requirement of an investment (here measured by the CVaR). Those figures motivate the next two sections where we consider risk aversion in the analysis.

5.3 Risk averse stochastic capacity expansion

This section discusses risk averse optimization capacity expansion models in an environment affected by different uncertainties. It is well known that the NPVs of nuclear, coal, or gas plants drastically change with the discount rate adopted in a capacity expansion problem. The discount factor, which is directly derived from considerations of cost of equity and debt is a crucial and commonly used parameter in investment evaluation in general and capacity expansion models in particular. We rather use here the modern approach of risk averse stochastic optimization based on risk measures. We particularly focus on the good-deal risk measure (presented in Chapter 3.3) which can be interpreted in corporate finance terms. Also this risk measure is both time consistent and exhibits more satisfactory hedging behavior than the CVaR. Indeed, even though the CVaR is gaining in popularity as a risk measure that overcomes some defects of the well-established VaR, it still suffers from some drawbacks. One is that it is not time consistent⁸ in the sense of Artzner et al. [7], the other is that it may induce inappropriate hedging behaviors as pointed out in Eisenberg [44].

Throughout this section, we work on a finite probability space (Ω, P) , i.e. we only consider random variable with finite distribution. A scenario is denoted by ω and its probability by $\text{prob}(\omega)$. We recall in the next subsection the move from a risk neutral optimization to a risk averse formulation with a good-deal risk measure and its interpretation in terms of discount factor.

5.3.1 Stochastic discount factor

The Problem 5.1 is formulated by assuming an exogenous discount factor (e.g. coming from the CAPM theory). The question of the appropriate discounting of $\mathbb{E}[Q(x, \omega)]$ or equivalently of the computation of the annual investment cost is minor from a stochastic programming point of view, but quite relevant in capacity expansion whether formulated as optimization or equilibrium models. As a first departure from the standard analysis, we modify the objective function in Problem 5.1 into

$$\min_{x \in \mathcal{X}} \quad \iota^T x + \mathbb{E}[\zeta(\omega)Q(x, \omega)]. \quad (5.16)$$

This expression can be interpreted as follows. We still value every asset in terms of zero-coupon bonds⁹. The vector ι of annualized cost is thus computed

⁸One can easily get a time consistent problem by taking a nested formulation, but at the price of loosing the traditional interpretation of the CVaR.

⁹We choose zero coupon bond with maturity T , the final time of the capacity expansion problem. The price of a bond at time t is given by the random variable $P_{t,T}$, where $P_{T,T} = 1$, that is, the bond is default free. As noted in Artzner et al. [8], it is only after this choice of numéraire, that risk measure can be qualified as monetary.

from the overnight investment cost using the risk free rate R^f . The variable $\zeta(\omega)$ is a stochastic vector that satisfies $\mathbb{E}[\zeta(\omega)] = 1$. All other elements of the problem (5.1) remain unchanged. The stochastic vector $\zeta(\omega)$ is referred to either a state price or a stochastic discount rate. Models of type (5.16) allow one to deal with CAPM or APT formulations by choosing a stochastic discount factor compatible with these theories (Ehrenmann and Smeers [42]).

As a further step in the analysis, we suppose that $\zeta(\omega)$ takes its values in a convex set of probabilities $\mathcal{U}$ and replace the expression (5.16) by

$$\min_{x \in \mathcal{X}} i^T x + \max_{\zeta \in \mathcal{U}} \mathbb{E}[\zeta(\omega) Q(x, \omega)] \quad (5.17)$$

Denoting $Q(x, \omega)$ as $-Z(\omega)$, the expression introduces a new function $\rho(Z)$ of the type

$$\rho(Z) = \max_{\zeta \in \mathcal{U}} \mathbb{E}[-\zeta(\omega) Z(\omega)]. \quad (5.18)$$

In the context of the quantification of the risk of financial positions, $\rho(Z)$ is referred to as a static coherent risk measure (cf. Chapter 3). Stochastic optimization models with risk measures of the form

$$\min_{x \in \mathcal{X}} i^T x + \rho(Q(x, \omega)), \quad (5.19)$$

have been extensively treated in [104]. An extension of this notion to an equilibrium model of the investment capacity type can be found in Ehrenmann and Smeers [42].

This section deals with problem (5.19) with a particular focus on the good deal risk measure introduced in the Section 3.3 of this thesis. We can reformulate (5.19) as a full minimization by using the dual formulation of the good-deal measure. The good-deal capacity expansion problem can be restated as the following conic two stages problem:

$$\min_{x \in \mathcal{X}, \mathbf{w}} i^T x + \mathbf{w}^T f_0 + \mathbb{E}[Q(x, \mathbf{w}, \omega)]^{\frac{1}{2}}, \quad (5.20)$$

where $Q(x, \mathbf{w}, \omega)$ is the optimal value of the second stage problem

$$\begin{aligned} Q(x, \mathbf{w}, \omega) = & \min_{y_\omega, \eta_\omega} A^2 \eta_\omega^2 \\ \text{s.t.} & f_{1,\omega}^T \mathbf{w} + \eta_\omega \geq c_\omega^T y_\omega \\ & W_\omega y = h_\omega - T_\omega y; y \geq 0 \end{aligned} \quad (5.21)$$

Remember from Section 3.3.1 that the good-deal risk measure involves a hedging portfolio composed of $i \in I$ assets. The variable vector $\mathbf{w}$ is composed of the weights of the assets in the portfolio, the vector f_0 is formed by the assets

prices at $t = 0$ ($f_0 \in \mathbb{R}^{|I|} = (f_{0,i})_{i \in I}$) and the vector $f_{1,\omega}$ is the assets prices realization at scenario ω . We also show in the previous section that the assets play a role analogous to the risk factors in the standard CAPM or APT models. The variable $\eta(\omega)$ represents the deviation and is positive if the portfolio in the scenario ω gives a smaller cash-flow than the operating cost $c_\omega^T y_\omega$. It thus badly hedges the possible loss. The capital requirement for investing is then equal to the investment cost $v^T x$ plus the capital requirement of the portfolio (its cost) and plus a measure of the under-performance equal to $(\mathbb{E}[\eta^2])^{\frac{1}{2}}$.

5.3.2 The multiperiod problem

These notions extend to the multiperiod capacity expansion problem. Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with a filtration $(\mathcal{F}_t)_{t \in 1, \dots, T}$ (i.e a sequence of increasingly refined algebras of subsets of Ω). Because investment and operations variables occur in each period, they can no longer be associated to specific stages as in the two stage model (respectively investment in stage 0 and operations in stage 1). To simplify notation, we collect the investment vector x_t , the total installed capacity k_t and the operations variable y_t in a vector $u_t = (x_t, k_t, y_t)$ for each t . The value taken by u_t is a function of the available information (i.e. fuels and investment cost, load level,...) up to time t . In other words u_t is $\mathcal{F}_t$ measurable.

Because operations variables y_t are now constrained by the total capacity k_t , we modify the stage specific constraints so that they relate y_t and k_t . This total installed capacity is obtained by the following recursive formula $k_t = k_{t-1} + x_{t-1}$. The time t contribution to the objective function and constraint of the problem are as follows:

$$\begin{aligned} g_t(u_t, \omega) &\equiv c_t(\omega)^T y_t(\omega) + v_t(\omega)^T x_t(\omega) \\ u_t \in \chi_t(u_{t-1}(\omega), \omega) &\equiv \begin{cases} k_t(\omega) = k_{t-1}(\omega) + x_{t-1}(\omega) \\ x_t \in X_{t,\omega} \\ T_{t,\omega} k_{t-1}(\omega) + W_t(\omega) y_t(\omega) = h_t(\omega) \\ y_t(\omega) \geq 0 \end{cases} \end{aligned} \quad (5.22)$$

The expression is slightly different in stage 1, which only involves the investment variables, i.e. $g_1(u_1) = I^T x_1$ as no investment has been committed before ($k_1 = 0$). The situation is again different in the final stage, which only involves operations costs, therefore leading to $g_T(u_T(\omega), \omega) = c(\omega)^T x_T(\omega)$.

Applying the multi-period good-deal measure to our problem, the multi-

period risk averse capacity expansion problem can be stated as

$$\begin{aligned} \min_{u_1 \in \chi_1} \quad & g_1(u_1) + \rho_{2|\mathcal{F}_1}^{GD} \left[\inf_{u_2(\omega) \in \chi_2(u_1, \omega)} g_2(u_2(\omega), \omega) + \dots \right. \\ & \left. + \rho_{T|\mathcal{F}_{T-1}}^{GD} \left[\inf_{u_T(\omega) \in \chi_T(u_{T-1}(\omega), \omega)} g_T(u_T(\omega), \omega) \right] \right] \end{aligned} \quad (5.23)$$

This recursive derivation leads to a time consistent risk averse multistage stochastic programming [103]. Also, due to the good-deal characteristics, the multistage model can be reformulated a single static optimization model that can be submitted to a general-purpose code. Second the multistage and single stage problem can both be interpreted in terms of optimal hedging in incomplete markets. In order to see this consider first a restatement of (5.23) obtained by recursively calling upon the dual formulation of the Good-Deal. This leads to the problem:

$$\begin{aligned} \min_{\mathbf{u}, \mathbf{w}, \eta} \quad & g_1(u_1) + f_1^T \mathbf{w}_1 + A\mathbb{E}[\eta_2^2 | \mathcal{F}_1]^{\frac{1}{2}} \\ \text{s.t.} \quad & u_1 \in \chi_1, \mathbf{u}_t(\omega) \in \chi_t(\mathbf{u}_{t-1}(\omega), \omega), t = 2, \dots, T \\ & f_t(\omega)^T (\mathbf{w}_{t-1}(\omega) - \mathbf{w}_t(\omega)) + \eta_t(\omega) \dots \\ & \geq g_t(u_t(\omega), \omega) + A\mathbb{E}[\eta_{t+1}(\omega)^2 | \mathcal{F}_t]^{\frac{1}{2}} \end{aligned} \quad (5.24)$$

The obtained problem is a SOCP and hence amenable to a treatment of very large size (using interior point method). One can also see that the obtained model takes the form of a dynamic hedging problem in an incomplete market.

5.3.3 Stylized example

Our model was developed as an approach to overcome shortcomings arising from the use of standard stochastic programming models in real industrial conditions. We argued in the introduction that multistage stochastic planning models have a natural interpretation of competitive markets. These models, while affected by usual numerical difficulties due to their size, also raise their own problems when used in this economic context. They rely on risk-adjusted discount rates that are difficult to estimate these days because of rapidly changing economic conditions. Standard stochastic programming models also reveals plants with quite different risk exposures that investors maybe reluctant to value on the sole basis of expectation. The Good Deal function remedies those two difficulties. It bypasses the question of the risk adjusted discount rate by using a pricing kernel that prices the free asset ($\mathbb{E}[\zeta] = 1$ in (5.16)). It also allows investors to directly insert a market interpretable parameter of risk aversion (the Sharpe ratio) in the evaluation of their plants. The full implementation of this type of model in an industrial context is a subject for further work¹⁰.

¹⁰We believe that it overcomes two major difficulties encountered with existing stochastic programming models.

We rather illustrate the approach on a simple generation capacity expansion problem with two risks factors, namely demand growth and gas price. The problem is inspired by [43]. Consider a 3-stage scenario tree representing the evolution of the corresponding data process. Stage (time) $t = 0$ consists of a single root node denoted ω_0 . Stage $t = 1$ comprises 25 nodes corresponding to different realizations of the demand and gas price. Each of them is connected to the root node. We have 625 different scenarios at stage $t = 2$. A generic node is denoted ω_1 at stage $t = 1$ and ω_2 at $t = 2$. Each node ω_1 has 25 successors nodes. We denote Ω_t the set of all possible nodes at stage t .

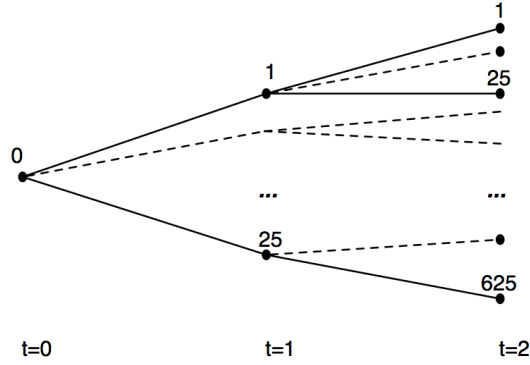


Figure 5.8: The tree representation

Because we are dealing with an investment problem, these periods are multiyear and can be interpreted as five years long. Multiyear periods imply some discount technicalities in the calculation of investment and operating costs. These are standard and not discussed here.

The industry initially invests at stage $t = 0$. Depending on the realization of data at $t = 1$, the industry operates those capacities and possibly reinvests in new ones. Finally, at $t = 2$, the industry operates the total capacity installed according to the realization of demand and gas prices. The number K of capacity types consists of Coal, Combined Cycle Gas Turbine (CCGT) and Open Cycle Gas Turbine (OCGT). Each equipment has both an investment cost and operating cost. The investment costs are annualized costs in thousand euro per MW that affect existing capacities in each year of a period. They are computed from overnight construction and fixed operating cost using a standard annualisation procedure (see Table 5.12). The operating costs are derived from fuel prices and the heat rate (thermal efficiency) of the technology: $c_{t,k}(\omega_t) = \text{HR}(k) \times p_t^{\text{fuel}}(\omega_t)$. For the sake of simplicity we assume that

5.3. RISK AVERSE STOCHASTIC CAPACITY EXPANSION

the price of the coal remains constant at 12 €/MW_{th} and do not consider CO₂ regulation or emission market.

	Coal	CCGT	OCGT
$i(k)$: Investment cost [k€/MW]	160	80	60
HR(k): Heat rate [MW _{th} /Mw]	2.78	1.89	2.91

Table 5.8: Fixed annual cost and heat rate

Demand is price insensitive and described by a load duration curve decomposed in L different time segments as depicted in Figure 1. The shape of the load duration curve is given in a reference scenario noted d_0 . Table 5.9 reports the decomposition of this reference load duration curve in time segments of duration τ_ℓ and level $d_{0,\ell}$.

	ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6
d_0 : demand [GW]	86	83	80	60	40	20
τ : duration [k hours]	0.01	0.04	0.31	4.4	3	1

Table 5.9: Reference load duration curve

The evolution of demand is random. We model this risk factor in stage $t = 1$ through 5 different scenarios determined by the growth α of the load duration curve¹¹: $d_{1,\ell}(\omega_1) = \alpha_1(\omega_1) \times d_{0,\ell}$. The load duration curve at stage $t = 2$ is similarly obtained using the same stochastic growth rate α and the realization of the load duration curve at $t = 1$: $d_{2,\ell}(\omega_2) = \alpha(\omega_2) \times d_{1,\ell}(\omega_1)$. Table 5.11 represents the different scenarios and their probabilities for the demand growth rate α .

α : demand growth	- 5%	0%	5%	10%	15%
Probability	10%	15%	30 %	25 %	20 %

Table 5.10: Demand growth

Demand growth in Table 5.11 is stated for five years periods. The corresponding annual growth rate therefore ranges from less than -1% to less than $+3\%$.

¹¹We assume that all demand blocks varies with the same growth rate α

The price of the gas fuel is the second source of uncertainty. It impacts the operations cost of the CCGT and OCGT technologies. Similarly to the load, we model the gas evolution by assuming a random growth rate β . The reference gas price is set at 14 €/MW_{th}. The two growth rates are stochastically independent, generating 25 scenarios at the stage $t = 1$ and 625 scenarios at $t = 2$. Again one should interpret these growth rates as referring to five year periods.

β : Gas price growth	0%	12%	24%	36%	48%
Probability	10%	40%	20 %	20 %	10 %

Table 5.11: Gas price growth

We introduce the following notation: $x_{t,k}(\omega_t)$ is the capacity (in MW) of technology k in stage t ; this capacity is operated at level $y_{t,k,\ell}(\omega_t)$ (in MW) in time segment ℓ . The variable PC can be interpreted as the Value of Lost Load (VOLL), that is, the economic value (set to 500 €/MWh) of unsatisfied electricity demand. The VOLL notion has been around since several decades but has so far escaped any precise evaluation. 500 €/MWh is definitively a low value inspired by regulation concerns. It does not provide a sufficient incentive to invest. We accordingly introduce $z_{t,\ell}(\omega_t)$, the unsatisfied demand in demand segment ℓ (in MW) in time t for scenario ω_t and demand segment ℓ .

Finally we assume a risk free rate R^f equal to 1.02 and set the scalar A to $\sqrt{1.5}$ in the good-deal risk measure. The model (5.24) adapted to this simplified example leads to the following conic problem.

$$\begin{aligned}
 \min_{\mathbf{u}, \mathbf{w}, \eta} \quad & \sum_{k=1}^K \iota(k) x_{0,k}(\omega_0) + \mathbf{1} \mathbf{w}_0(\omega_0) + A \mathbb{E}[\eta_1(\omega_1)^2 | \mathcal{F}_0]^{\frac{1}{2}} \\
 \text{s.t.} \quad & x_{0,k}(\omega_0) \geq 0 \quad ; \quad x_{1,k}(\omega_1) \geq x_0(k, \omega_0) \\
 & y_{t,k,\ell}(\omega_t) \geq 0 \quad ; \quad z_{t,\ell}(\omega_t) \geq 0 \quad t = 1, 2 \\
 & x_{t-1,k}(\omega_{t-1}) \geq y_{t,k,\ell}(\omega_t) \quad t = 1, 2 \\
 & \sum_{k \in K} y_{t,k,\ell}(\omega_t) + z_{t,\ell}(\omega_t) - d_{t,\ell}(\omega_t) \geq 0 \quad t = 1, 2 \\
 & (R_f)(\mathbf{w}_0(\omega_0) - \mathbf{w}_1(\omega_1)) + \eta_1(\omega_1) \geq \dots \\
 & \sum_{\ell=1}^L \tau_\ell \left(\sum_{k=1}^K c_{1,k}(\omega_1) y_{1,k,\ell}(\omega_1) + \text{PC } z_{1,\ell}(\omega_1) \right) \\
 & + \sum_{k=1}^K \iota_k x_{1,k}(\omega_1) + A \mathbb{E}[\eta_2(\omega_2)^2 | \mathcal{F}_1]^{\frac{1}{2}} \\
 & (R_f)^2 \mathbf{w}_1(\omega_1) + \eta_2(\omega_2) \geq \dots \\
 & \sum_{\ell=1}^L \tau_\ell \left(\sum_{k=1}^K c_{2,k}(\omega_2) y_{2,k,\ell}(\omega) + \text{PC } z_{2,\ell}(\omega_2) \right)
 \end{aligned} \tag{5.25}$$

Problem (5.25) has 16354 variables. The expressions $\mathbb{E}[\eta_{t+1}^2(\omega) | \mathcal{F}_t]$ define a total of 26 Lorentz cones, each of dimension 25. We solve the problem using yalmip [77] and the MOSEK solver. The solver running time is approximatively 3 seconds.

Simulation results

The following table reports the investments in the different types of plants in stage 0 (operated in stage 1 and 2) and the expected investment in stage 1 (operated in stage 2). Recall that there is no initial exogenous capacity in this model and hence that investment in $t = 0$ have to make up for the bulk of d_1 . The industry invests principally in coal and OCGT technologies, mainly due to the "peaky" characteristic of the load duration curve (interpreted as resulting from a significant wind penetration).

As expected the investment in stage $t = 1$ strongly depends on the realization of the demand growth and the gas price. The total level of investment is more impacted by demand growth and the optimal technology mix depends more on the gas price. This is highlighted in Figure 5.9 and 5.10 showing the

Investment [GW]	Coal	CCGT	OCGT
$x_0(k, \omega_0)$	66	3	23
$\mathbb{E}[x_1(k, \omega_1) \mathcal{F}_0]$	4.4	0.6	1.4

Table 5.12: Investment

investment in $t = 1$ in Coal and CCGT. These are standard stochastic programming phenomena.

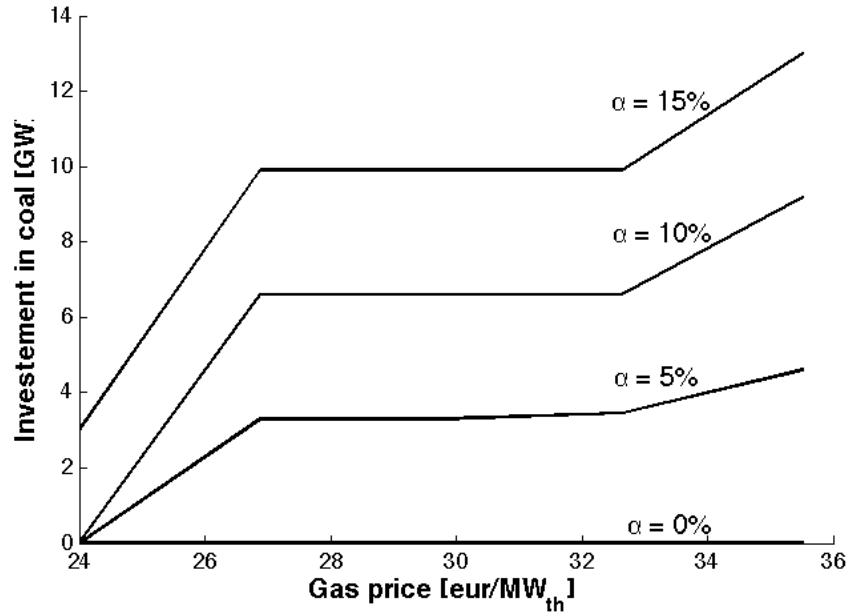


Figure 5.9: Investment in Coal at $t = 1$

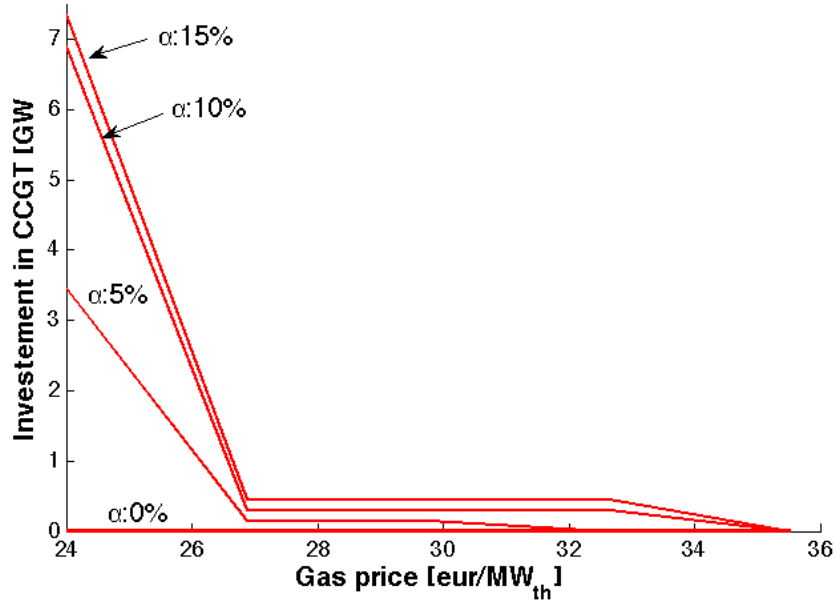


Figure 5.10: Investment in CCGT at $t = 1$

Less standard, Table 5.13 shows the optimal mix of technology at $t = 0$ varying with the risk aversion parameter A . The more the industry is risk averse, the more it invests in capacities and also neglects the CCGT technology. The higher investment is meant to avoid curtailments and hence VOLL costs. The second effect is intended to avoid a lower utilization of relatively high capital investment plants (CCGT) when demand is low.

	$A=1.05$	$A=1.22$	$A=2.23$
Coal	63	65	69
CCGT	6	3	0
OCGT	19	23	23

Table 5.13: Initial investment ($t = 0$) for different sharpe ratio

5.4 Stochastic capacity expansion equilibrium

A theoretically convenient way to represent a perfect competition equilibrium under uncertainty is to consider an economy that trades all risks at some price. Risk trading takes place through so called Arrow Debreu securities that pay one currency unit in a given state of the world and zero otherwise. The price of risk is found as the price of these Arrow Debreu securities at equilibrium. Applying this concept to standard stochastic programming implies that exogenously given probabilities of the stochastic capacity expansion model are exogenously given risk neutral probabilities or alternatively, prices of those Arrow Debreu securities that are not represented explicitly in the model. A stochastic capacity expansion model can thus be interpreted as a partial equilibrium model where the price of risk is given. While partial equilibrium models are common in practice, this one embeds an internal contradiction: it determines endogenous electricity prices but considers that the probability of occurrence of these prices is exogenously given. The easy interpretation of the model and the implicit contradiction that it contains disappear when one dispenses with risk neutral agents, as one may wish to do in the very risky environment where utilities are operating today. The risk neutral probabilities are no longer given to the models as exogenous probabilities of the stochastic capacity expansion model; in contrast their derivation should be made part of the problem.

Ehrenmann and Smeers [42] address that problem that they cast in the context of stochastic discount rates (see e.g. Cochrane [29] for the concept) where risk neutral probabilities are made endogenous to the problem. They consider two different cases. One assumes a stochastic discount factor that is linear in some risk factors. This is a reinterpretation of the standard CAPM or APT methodologies of corporate finance. We study this case in the following section by representing the behavior of agents with respect to risk through the good-deal risk measure. This extension however departs from the optimization paradigm (cost minimization) and requires a full equilibrium model. The work is related to [42, 43] where the authors represent the behavior of agents with respect to idiosyncratic risk through CVaRs. Also Ralph and Smeers [92] consider a CVaR based capacity expansion model of the optimization type and elaborate on its interpretation in terms of equilibrium in physical and financial assets.

5.4.1 The general two stage equilibrium

We work on a probability space $(\Omega, \mathcal{F}, P)$ with a finite number of scenario $|\Omega|$. We denote by ω and $\text{prob}(\omega)$ a scenario and its probability. We adapt the basic capacity expansion problem to a two stage equilibrium problem. There are N

risk averse players on the market aiming to minimize the capital requirement of their investment. They are price taker and value the capital requirement according to a risk measure ρ_ν . At the first stage $t = 0$, each player ν invests an amount $x^\nu := (x_k^\nu)_{k=1}^K$ in K different technologies, and operates its plants at level $y_{k,\ell}^\nu(\omega)$ after the scenario ω (and so the operating cost $c_k(\omega)$ and price $\pi_\ell(\omega)$) has been revealed. Define the following vectors :

- ◇ $\mathbf{x}^\nu := (x_k^\nu)_{k=1}^K \in \mathbb{R}^K$, i.e. the investments of all players in the different technologies
- ◇ $\mathbf{y}^\nu := \left(y_{k,\ell}^\nu(\omega) \right)_{k=1, \ell=1, \omega=1}^{K, L, |\Omega|} \in \mathbb{R}^{K \times L \times |\Omega|}$, i.e. the productions of the units for all the players (for the different load time segment and realization of ω).
- ◇ $\boldsymbol{\pi} := (\pi_\ell(\omega))_{\ell=1, \omega=1}^{L, |\Omega|} \in \mathbb{R}^{L \times |\Omega|}$, i.e. the electricity prices for the different load time segment and realization of ω .
- ◇ $\mathbf{d} := (d_\ell(\omega))_{\ell=1, \omega=1}^{L, |\Omega|} \in \mathbb{R}^{L \times |\Omega|}$, i.e. the demand parameter for the different load time segment and realization of ω .

Problem 5.4 (Player's investment problem). A risk averse player ν chooses its optimal level of investment $\mathbf{x}^\nu$ and its production $\mathbf{y}^\nu$ by solving

$$\mathcal{P}^\nu(\boldsymbol{\pi}) \equiv \min_{\mathbf{x}^\nu} \iota^T x_\nu + \rho_\nu(\Pi_\nu(\mathbf{x}^\nu, \boldsymbol{\pi})),$$

where Π_ν is the optimal players profit at the second stage and

$$\begin{aligned} \Pi_\nu(\mathbf{x}^\nu, \pi_\ell(\omega)) = & \max_y \sum_{\ell=1}^L \sum_{k=1}^K \tau_\ell y_{k,\ell}^\nu(\omega) (\pi_\ell(\omega) - c_k(\omega)) \\ \text{s.t.} \quad & 0 \leq y_{k,\ell}^\nu(\omega) \leq x_k^\nu \end{aligned}$$

We are now ready to give the definition of the competitive stochastic capacity expansion equilibrium problem with risk averse agents facing a stochastic price insensitive demand $\mathbf{d}$.

Definition 5.2. An investment plan $(\mathbf{x}^{\nu,*})_{\nu=1}$, generations $(\mathbf{y}^{\nu,*})_{\nu=1}^N$ and a price vector $\boldsymbol{\pi}^*$ constitute a perfect competitive equilibrium in a market with a price insensitive demand if :

- (i) For every player $\nu = 1, \dots, N$, $(\mathbf{x}^{\nu,*}, \mathbf{y}^{\nu,*})$ is an optimal solution of $\mathcal{P}^\nu(\boldsymbol{\pi}^*)$
- (ii) For each scenario $\omega \in \Omega$, and each demand segment $\ell = 1, \dots, L$, the following market clearing conditions hold:

$$\begin{aligned} 0 \leq \pi_\ell^*(\omega) & \perp \sum_{\nu=1}^N \sum_{k=1}^K y_{k,\ell}^{\nu,*}(\omega) + z_\ell(\omega) - d_\ell(\omega) \geq 0 \\ 0 \leq z_\ell(\omega) & \perp PC - \pi_\ell^*(\omega) \geq 0 \end{aligned}$$

Remark 5.3. As usual in equilibrium analysis, the corresponding Nash game is obtained by replacing the market clearing conditions by introducing a market agent, who minimizes the excess demand :

$$\begin{aligned} \min_{\pi} \quad & \sum_{\omega \in \Omega} \sum_{\ell=1}^L \pi_{\ell}(\omega) \left(d_{\ell}(\omega) - \sum_{k=1, \nu=1}^{K,N} y_{k,\ell}^{\nu}(\omega) \right) \\ \text{s.t.} \quad & 0 \leq \pi_{\ell}(\omega) \leq PC \end{aligned}$$

5.4.2 Investment valuation using good-deal

When the player optimizes the good-deal of its profit, its investment problem becomes the following SOCP.

Problem 5.5. [Investment good-deal valuation] A player ν valuing its investment according to the good-deal measure, chooses its investment $\mathbf{x}^{\nu}$ and production $\mathbf{y}^{\nu}$ so as to solve

$$\begin{aligned} \mathcal{P}_{\text{GD}}^{\nu}(\pi) \equiv \quad & \min_{\mathbf{x}^{\nu}, \mathbf{y}^{\nu}, \mathbf{w}, \eta^{\nu}} A^{\nu} \left(\mathbb{E}[(\eta^{\nu})^2] \right)^{\frac{1}{2}} + \mathbf{i}^T \mathbf{x}^{\nu} - f_0^T \mathbf{w}^{\nu} \\ \text{s.t.} \quad & 0 \leq y_{k,\ell}^{\nu}(\omega) \\ & y_{k,\ell}^{\nu}(\omega) \leq x_k^{\nu} \quad (\mu_{k,\ell}^{\nu}(\omega)) \\ & \eta^{\nu}(\omega) \geq \sum_{\ell=1}^L \sum_{k=1}^K \tau_{\ell} (c_k(\omega) - \pi_{\ell}(\omega)) y_{k,\ell}^{\nu}(\omega) + f_1^T(\omega) \mathbf{w}^{\nu} \quad (\phi^{\nu}(\omega)) \end{aligned}$$

The KKT conditions of the optimization Problem 5.5, in their nonlinear complementarity form (i.e. considering only $\mathbb{R}_+^{\eta}$ cone), are the following :

$$\begin{aligned} f_{0,i} &= \sum_{\omega \in \Omega} \phi^{\nu}(\omega) f_{1,i}(\omega) \\ \phi^{\nu}(\omega) - A \frac{\text{prob}_{\omega} \eta^{\nu}(\omega)^{\frac{1}{2}}}{(\mathbb{E}[(\eta^{\nu})^2])^{\frac{1}{2}}} &= 0 \\ 0 \leq x_k^{\nu} \quad \perp \quad & \mathbf{i}_k - \sum_{\omega \in \Omega} \phi^{\nu}(\omega) \left(\sum_{\ell=1}^L \tau_{\ell} \mu_{k,\ell}(\omega) \right) \geq 0 \\ 0 \leq \phi^{\nu}(\omega) \quad \perp \quad & \eta^{\nu}(\omega) - \sum_{\ell=1}^L \sum_{k=1}^K \tau_{\ell} (c_k(\omega) - \pi_{\ell}(\omega)) y_{k,\ell}^{\nu}(\omega) - f_1^T(\omega) \mathbf{w}^{\nu} \geq 0 \\ 0 \leq y_{k,\ell}^{\nu}(\omega) \quad \perp \quad & c_k(\omega) - \pi_{\ell}(\omega) + \mu_{k,\ell}(\omega) \geq 0 \end{aligned} \tag{5.26}$$

Taking into account the conic structure of the problem, we present in appendix the KKT conditions written in a conic form (cf. Problem D.1) and give the corresponding generalized complementarity problem's formulation (cf. Problem D.3).

When players value their investment according to the good-deal measure, a tuple $((\mathbf{x}^{\nu,*})_{\nu=1}^N, (\mathbf{y}^{\nu,*})_{\nu=1}^N, \boldsymbol{\pi}^*)$ constitute a competitive equilibrium (definition 5.2) iff

- (i) For every player $\nu = 1, \dots, N$, $((\mathbf{x}^{\nu,*})_{\nu=1}^N, (\mathbf{y}^{\nu,*})_{\nu=1}^N, \boldsymbol{\pi}^*)$ solve the complementarity conditions 5.26 (or equivalently Problem D.1 or Problem D.3)
- (ii) For each scenario $\omega \in \Omega$, and each demand segment $\ell = 1, \dots, L$, the market clearing conditions hold.

We give in appendix the formulation of the equilibrium problem in term of a variational inequalities problem, denoted as $VI(F, K)$. The obtained mapping is not monotone and the set is not compact. Accordingly, we do not have any results about the existence of solution [59]. The problem is computationally hard to solve. Its size depends on the number of scenarios (which is large for any reasonable application), and efficient algorithm for solving VI requires monotonicity.

The symmetric case

In order to make the problem tractable, we assume in this section that the players are symmetric, i.e. they value their investment according to the same risk measure and they all invest an equal amounts of the total investment. In that case, $(\mathbf{x}^\nu, \mathbf{y}^\nu, \boldsymbol{\mu}^\nu, \mathbf{w}^\nu, \boldsymbol{\eta}^\nu, \boldsymbol{\phi}^\nu)$ are equal among all $\nu = 1, \dots, N$. Also, as the good-deal risk measure is coherent, the variable $\boldsymbol{\phi}^\nu$ do not depend on the number of players¹² The total investment level do not thus depends on the number of player. We describe in the Appendix 5.2 a simple heuristic that permit us to obtain a solution.

We illustrate this discussion on a finite version of the example described in Section 5.2.3, i.e. we consider 1500 scenarios with equal probability, obtained by sampling the random variable $\xi := (\xi_1, \xi_2, \xi_3)$. We take the risk-free asset as factor in the good-deal measure (cf. chapter 3.3.3). Table 5.14 displays the installed capacity for different risk aversion parameter A .

The more the players are risk averse, the less they invest and the more those investments are profitable (cf. Table 5.15). When comparing those results with the risk-neutral case 5.2, one observes that players invest less in the CCGT and OCGT technologies. Those result are in line with the statistics of

¹²Indeed, if Π denote the total profit of the industry, the good deal measure of the profit, for agent ν , is given by

$$\rho_\nu^{GD}(\frac{\Pi}{N}) = \frac{1}{N} \rho_\nu^{GD}(\Pi).$$

The probability measure $\boldsymbol{\phi}^\nu \in \mathcal{M}_\nu^{GD}$, characterizing the good-deal measure, does not depend thus on the number of players (because of the positive homogeneity).

5.4. STOCHASTIC CAPACITY EXPANSION EQUILIBRIUM

Installed capacity [MW]	A=1.22	A=1.8	A=2.5
Nuclear	4933.4	5046.5	5080.0
CCGT	4539.9	4314.3	4240.6
OCGT	769.7	670.8	649.3
$\sum_{k=1}^K x_k$	10,243.0	10,031.6	9969.9

Table 5.14: Total installed capacity

the margin profit in Table 5.7. The generation is now less risky and, the capital requirement measured by a $\text{CVaR}_{30\%}$ is now negative.

	A=1.22	A=1.8	A=2.5
Nuclear : - $\mathbb{E}$	83.0	175.6	210.6
- $\text{CVaR}_{30\%}$	-18.46	-108.4	-144.34
- σ	145	221.1	249
CCGT : - $\mathbb{E}$	92.11	192.7	230.28
- $\text{CVaR}_{30\%}$	-21.65	-119.8	-158.8
- σ	161	242.6	272.2
OCGT : - $\mathbb{E}$	86	181.5	217.26
- $\text{CVaR}_{30\%}$	-19.9	-112.8	-149.8
- σ	150	228.5	256.9

Table 5.15: Esperance[k€/MW] , $\text{CVaR}_{30\%}$ [k€/MW] and standard deviation σ [k€/MW] of the profit margins for different risk aversion parameter A

We display the histogram of the net margin for each technology in figures 5.11, 5.12 and 5.13.

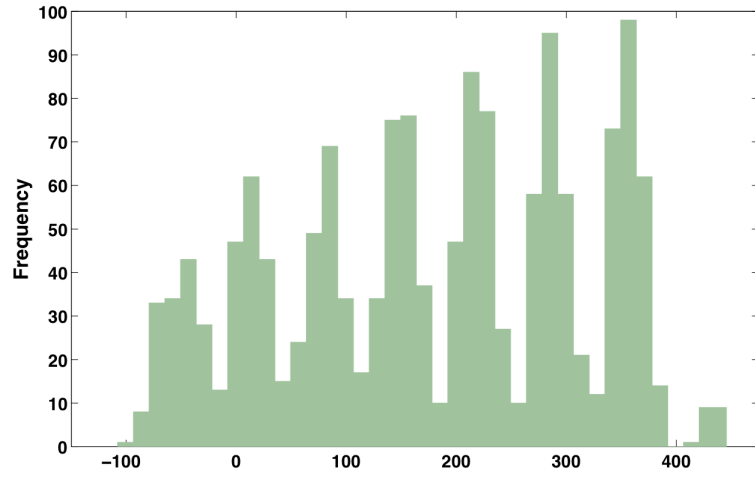


Figure 5.11: Profit margins for the Nuclear technology ($A=1.8$)

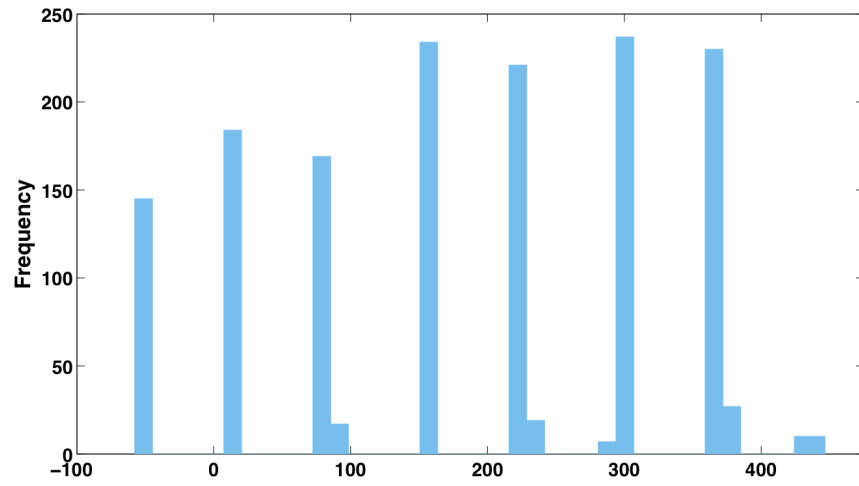


Figure 5.12: Profit margins for the CCGT technology ($A=1.8$)

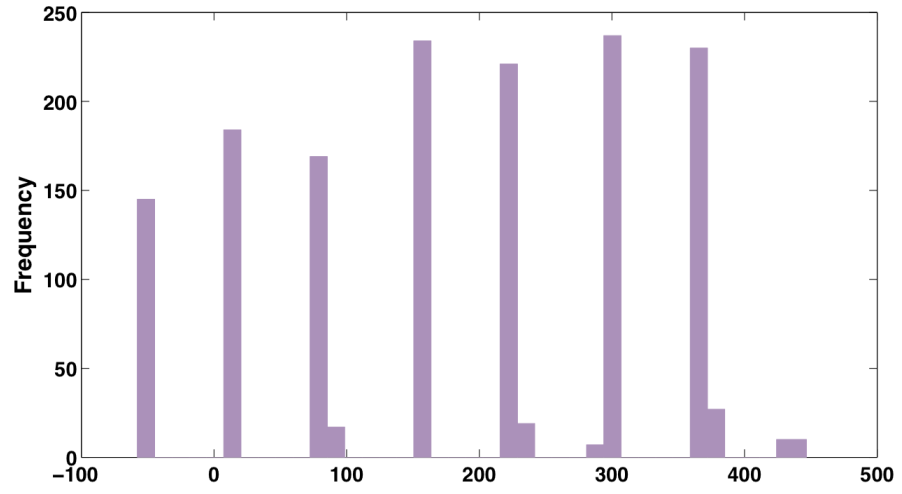


Figure 5.13: Profit margins for the OCGT technology ($A=1.8$)

Chapter 6

Conclusion

Summary and Further research

We present in chapter 2 a new model for pricing power futures. It is based on the main fundamental factors driving electricity generation. We assume a perfectly competitive market and perfectly dispatchable generators. In that case the price of electricity is given by the marginal cost of producing the last unit of power. The main factors driving this short run marginal cost are the demand, the fuel prices and the carbon price in European countries. We extend the Pirrong-Jermakyan framework to all those variables and finally get a partial differential equation that any power contingent claim must satisfy. Examining the special structure of marginal cost, we simplify this equation for the futures valuation problem in order to make it computationally tractable. Also, as demand is not a traded asset, this valuation problem contains a market price of risk that has to be estimated from market data. We test the model and extract the market price of risk of BaseLoad and PeakLoad contracts on the German market for June and September 2008. We find that the modeled prices fit market data quite well and that the marginal cost hypothesis can well reproduce the power futures dynamics resulting from fuel and carbon prices. Nevertheless, we observe two different market prices of risk for Baseload and Peakload. The market price of risk is significant for PeakLoad contracts but is almost non-existent for Baseload contracts. A possible explanation of this contradiction can be found in the computation of the marginal cost that links the state variables to electricity prices. Indeed we have evidence that spot prices can somewhat differ from computed marginal costs. The reason for this is that we are neglecting important constraints of electricity generation, like minimum output and start-up cost. From a practical point of view, a possible way to obtain a better fit to observed prices is to use the following spot price model from Langregé, Campi, and Aïd [76].

CONCLUSION

$$S_t = g(K_t^{\max} - Q_t) \mathbf{MC}(Q_t, f_t^g, f_t^c, f_t^{CO_2})$$

where $K_t^{\max}$ is the total available capacity and the term $(K_t^{\max} - Q_t)$ represents the margin capacity of the system. The function g measure the scarcity of the system and is a bounded real-valued given by

$$g(x) = \min\left(M, \frac{\alpha}{x^\nu}\right) \mathbb{I}_{\{x>0\}} + M \mathbb{I}_{\{x\leq 0\}}$$

The authors [76] give a clear methodology to estimate this scarcity function and obtain better fit to historical spot prices. Note that including this scarcity function will only change the initial condition in the PDE and the model will still be computationally tractable. For a fundamental point of view, adding a better representation of the generators cost function is difficult because it makes the unit commitment problem harder to solve, and, more importantly, because it is not anymore possible to define an equilibrium price vector. Future work could also try to incorporate the discrete nature of the unit commitment in order to improve the matching between computed and observed spot prices. In this perspective, pricing schemes presented by Gribik, Hogan, and Pope [57] seem a good starting point.

In chapter 3, we recall briefly the basic theory of risk measure in a finite probability space which is used in the two following chapter. We present in this chapter the good-deal, a risk measure introduced by Cochrane and Saá-Requejo [31]. We discuss its attractiveness and derive its dual representation which is interpretable in terms of portfolio replication. Further work should extend the work to a general probability space $(\Omega, \mathcal{F}, P)$ and continuous random variables.

The chapter 4 studies the impact of insufficient liquidity in financial energy and transmission markets on the pricing of these financial products through an equilibrium model. The analysis is inspired by the empirical findings of [107, 3, 38]. We model both liquid and illiquid markets, the former as a competitive equilibrium (equivalent to a Nash equilibrium), the latter as a social equilibrium (equivalent to a Generalized Nash Equilibrium problem). Taking up the liquid market first we show that equilibrium models without speculators may leave residual arbitrage opportunities. The reason is technical and due to the modeling of the players risk aversion by non-convex risk measures. The no arbitrage property can be restored either by modeling risk aversion by convex risk measures, or more traditionally by allowing speculators in the market. Considering illiquid markets we show that arbitrage opportunities may remain even with convex risk measures and with speculators: these are intrinsic to illiquidity and correspond to a market failure. An important consequence of illiquidity is that there exist several equilibria in the market. Computing a large panel of equilibria, we show that insufficient liquidity can dramatically

impact the distribution of agents profits and that the risk premium may drastically increase in illiquid markets. We also detect that an illiquid FTR market can notably impact the financial energy market: lack of hedging possibilities on transmission may render energy hedging ineffective. Through this study, we rely on a definition of illiquidity based on the volume. We leave to future research to explore the impact of other measures such as the bid-ask spread on the equilibrium.

The last chapter performs some risk analysis of investment in generation capacity expansion models. The first part of the chapter is dedicated to stochastic capacity expansion models which are interpretable in terms of a perfectly competitive economy with risk-neutral agents. We give a procedure to estimate the distribution of the Lagrange multipliers of the nonanticipativity constraints. In a restructured electricity markets operating under perfect competition, those are the profit margins of the investments. In the second part of this chapter, we cast the standard multiperiod capacity expansion planning in a risk measure context. The good deal introduced in chapter 3 is the chosen risk measure. We explain that this formulation has several advantages. From the point of view of industrial practice, the good-deal can be seen as an extension of the stochastic discount factor constructed from standard corporate finance theories. In term of optimization of risk measure, the multiperiod good-deal benefits from the two very desirable properties of coherence and time consistency. The risk measure has also economic interpretation and computational advantages. The multi-period capacity expansion model under good-deal is a SOCP and hence amenable to a treatment of very large size. From an economic point of view, one can interpret the dual of the capacity expansion model in terms of portfolio replication in an incomplete market. In the last part of the chapter, we briefly present a full equilibrium model of generation capacity expansion where agents are risk averse and value their investments according to the good-deal. We give a heuristic for solving the simple symmetric case. Further work should involve more general cases and extend the work to a multiperiod equilibrium.

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Appendix

APPENDIX

Appendix A

A.1 The functions $\mathcal{H}$ in the Futures valuation PDE

We develop in this appendix how to obtain the partial differential equation (2.20) on the function g . We compute all the term of the initial PDE on F (2.13) by using the decomposition:

$$F_{t,\tau} = g(q_t, \tau, \gamma_t) (f_{t,\tau}^c + c_c f_{t,\tau}^{CO_2})$$

Let us first compute the different derivatives of F in terms of g , γ and ζ . The first derivative of F with respect to the fuels and carbon prices are given by

$$\begin{aligned} \frac{\partial F}{\partial(f^g)} &= \frac{\partial (f^c + c_c f^{CO_2}) g}{\partial(f^g)} \\ &= (f^c + c_c f^{CO_2}) \frac{\partial g}{\partial \gamma} \frac{\partial \gamma}{\partial f^g} \\ &= \frac{\partial g}{\partial \gamma} \\ \frac{\partial F}{\partial(f^c)} &= \frac{\partial (f^c + c_c f^{CO_2}) g}{\partial(f^c)} \\ &= g - \gamma \frac{\partial g}{\partial \gamma} \\ \frac{\partial F}{\partial(f^{CO_2})} &= \frac{\partial (f^c + c_c f^{CO_2}) g}{\partial(f^{CO_2})} \\ &= c_c g + \frac{(c_g f^c - c_c f^g)}{(f_c + c_c f^{CO_2})} \frac{\partial g}{\partial \gamma} \end{aligned} \tag{A.1}$$

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The second derivatives are given by

$$\begin{aligned}
\frac{\partial^2 F}{\partial(f^g)^2} &= \frac{\partial^2 g}{\partial\gamma^2} \frac{1}{(f^c + c_c f^{CO_2})} \\
\frac{\partial^2 F}{\partial(f^c)^2} &= \frac{\partial g}{\partial\gamma} \frac{\partial(f^c)}{\partial\gamma} - \frac{\partial g}{\partial\gamma} \frac{\partial(f^c)}{\partial\gamma} \\
&\quad + \gamma^2 \frac{\partial^2 g}{\partial\gamma^2} \frac{1}{(f^c + c_c f^{CO_2})} \\
\frac{\partial^2 F}{\partial(f^{CO_2})^2} &= c_c \frac{\partial g}{\partial\gamma} \frac{(c_g f^c - c_c f^g)}{(f_c + c_c f^{CO_2})^2} - c_c \frac{(c_g f^c - c_c f^g)}{(f_c + c_c f^{CO_2})^2} \frac{\partial g}{\partial\gamma} \\
&\quad + \frac{(c_g f^c - c_c f^g)^2}{(f_c + c_c f^{CO_2})^3} \frac{\partial^2 g}{\partial\gamma^2}
\end{aligned} \tag{A.2}$$

The cross derivatives of $F = (f_c + c_c f^{CO_2}) g$ are given by

$$\begin{aligned}
\frac{\partial^2 F}{\partial(f^g)\partial(f^c)} &= -\gamma \frac{1}{(f^c + c_c f^{CO_2})} \frac{\partial^2 g}{\partial\gamma^2} \\
\frac{\partial^2 F}{\partial(f^g)\partial(f^{CO_2})} &= c_g \left(1 - \frac{c_c}{c_g} \gamma\right) \frac{1}{(f^c + c_c f^{CO_2})} \frac{\partial^2 g}{\partial\gamma^2} \\
\frac{\partial^2 F}{\partial(f^c)\partial(f^{CO_2})} &= \frac{\partial g}{\partial\gamma} \frac{\partial(f^{CO_2})}{\partial\gamma} - \frac{\partial g}{\partial\gamma} \frac{\partial(f^{CO_2})}{\partial\gamma} \\
&\quad - \gamma \frac{\partial^2 g}{\partial\gamma^2} c_g \left(1 - \frac{c_c}{c_g} \gamma\right) \frac{1}{(f^c + c_c f^{CO_2})}
\end{aligned} \tag{A.3}$$

The derivatives with respect to the logarithm of the load q are given by:

$$\begin{aligned}
\frac{\partial F}{\partial q} &= (f^c + c_c f^{CO_2}) \frac{\partial g}{\partial q} \\
\frac{\partial^2 F}{\partial q^2} &= (f^c + c_c f^{CO_2}) \frac{\partial^2 g}{\partial q^2} \\
\frac{\partial^2 F}{\partial q \partial(f^g)} &= \frac{\partial^2 g}{\partial q \partial\gamma} \\
\frac{\partial^2 F}{\partial q \partial(f^c)} &= \frac{\partial g}{\partial q} - \gamma \frac{\partial^2 g}{\partial q \partial\gamma} \\
\frac{\partial^2 F}{\partial q \partial(f^{CO_2})} &= c_c \frac{\partial g}{\partial q} - c_g \left(1 - \frac{c_c}{c_g} \gamma\right) \frac{\partial^2 g}{\partial q \partial\gamma}
\end{aligned} \tag{A.4}$$

This leads to the following terms of the partial differential equation (2.13):

$$\begin{aligned}
(k(q - \theta_q(t)) - \sigma_q \lambda(q, t)) \frac{\partial F}{\partial q} &= (f_c + c_c f^g) \left(q (k(q - \theta_q(t)) - \sigma_q \lambda(q, t)) \frac{\partial g}{\partial q} \right) \\
\frac{\sigma_q^2}{2} \frac{\partial F}{\partial q^2} &= (f_c + c_c f^g) \left(\frac{\sigma_q^2}{2} \frac{\partial^2 g}{\partial q^2} \right) \\
\rho_{q,g} \sigma_q \sigma_g (f^g) \frac{\partial^2 F}{\partial q \partial (f^g)} &= (f_c + c_c f^g) \left(\rho_{q,g} \sigma_q \sigma_g (\gamma - \zeta) \frac{\partial^2 g}{\partial q \partial \gamma} \right) \\
\rho_{q,c} \sigma_q \sigma_c (f^g) \frac{\partial^2 F}{\partial q \partial (f^g)} &= (f_c + c_c f^g) \left(\rho_{q,c} \sigma_q \sigma_c \left(1 - \frac{c_c}{c_g} \zeta\right) \frac{\partial g}{\partial q} \right) \\
&\quad - (f_c + c_c f^g) \left(\rho_{q,c} \sigma_q \sigma_c \left(1 - \frac{c_c}{c_g} \zeta\right) \gamma \frac{\partial^2 g}{\partial q \partial \gamma} \right) \\
\rho_{q,CO_2} \sigma_q \sigma_{CO_2} (f^{CO_2}) \frac{\partial^2 F}{\partial q \partial (f^{CO_2})} &= (f_c + c_c f^g) \left(\rho_{q,CO_2} \sigma_q \sigma_{CO_2} \frac{c_c}{c_g} \zeta \frac{\partial g}{\partial q} \right) \\
&\quad - (f_c + c_c f^g) \left(\rho_{q,CO_2} \sigma_q \sigma_{CO_2} \left(1 - \frac{c_c}{c_g} \gamma\right) \zeta \frac{\partial^2 g}{\partial q \partial \gamma} \right) \\
\frac{\sigma_g^2}{2} (f^g)^2 \frac{\partial F}{\partial (f^g)^2} &= (f_c + c_c f^g) \left(\frac{\sigma_g^2}{2} (\gamma - \zeta)^2 \frac{\partial^2 g}{\partial \gamma^2} \right) \\
\frac{\sigma_c^2}{2} (f^g)^2 \frac{\partial F}{\partial (f^c)^2} &= (f_c + c_c f^g) \left(\frac{\sigma_c^2}{2} \gamma^2 \left(1 - \frac{c_c}{c_g} \zeta\right)^2 \frac{\partial^2 g}{\partial \gamma^2} \right) \\
\frac{\sigma_{CO_2}^2}{2} (f^{CO_2})^2 \frac{\partial F}{\partial (f^{CO_2})^2} &= (f_c + c_c f^g) \left(\frac{\sigma_{CO_2}^2}{2} \zeta^2 \left(1 - \frac{c_c}{c_g} \gamma\right)^2 \frac{\partial^2 g}{\partial \gamma^2} \right) \\
\rho_{g,c} \sigma_g \sigma_c (f^c f^g) \frac{\partial^2 F}{\partial (f^c) \partial (f^g)} &= (f_c + c_c f^g) \left(\rho_{g,c} \sigma_g \sigma_c (\gamma - \zeta) \left(1 - \frac{c_c}{c_g} \zeta\right) \frac{\partial^2 g}{\partial \gamma^2} \right) \\
\rho_{g,CO_2} \sigma_g \sigma_{CO_2} (f^{CO_2} f^g) \frac{\partial^2 F}{\partial (f^g) \partial (f^{CO_2})} &= (f_c + c_c f^g) \\
&\quad \times \left(\rho_{g,CO_2} \sigma_g \sigma_{CO_2} (\gamma - \zeta) \left(1 - \frac{c_c}{c_g} \gamma\right) \zeta \frac{\partial^2 g}{\partial \gamma^2} \right) \\
\rho_{c,CO_2} \sigma_c \sigma_{CO_2} (f^{CO_2} f^c) \frac{\partial^2 F}{\partial (f^c) \partial (f^{CO_2})} &= (f_c + c_c f^g) \\
&\quad \times \left(\rho_{c,CO_2} \sigma_c \sigma_{CO_2} \left(1 - \frac{c_c}{c_g} \zeta\right) \left(1 - \frac{c_c}{c_g} \gamma\right) \zeta \gamma \frac{\partial^2 g}{\partial \gamma^2} \right)
\end{aligned} \tag{A.5}$$

Combining all this terms, one gets the PDE (2.20) on g, γ and ζ .

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Appendix B

B.1 Estimation of the load process

We estimate the stochastic process of the load Q using non-parametric techniques, as Pirrong-Jermakyan [90]. We collect the data $(t_i, Q_i)_{i=1}^n$ for the daily peak load value¹ (at 12 am) for 2006-2007 during working days. Ignoring the local time processes, the SDE to calibrate is the following exponential mean-reverting process.

$$\frac{dQ_t}{Q_t} = k (\ln(Q_t) - \theta_q(t)) dt + \sigma_q du_t \quad (\text{B.1})$$

In this SDE, the parameter $\theta_q(t)$ is the expected value of the logarithm of the series. We estimate it by fitting a local polynomial to the series for each day t_j of the year. The expected value of the log-load $\theta_q(t_j)$ at t_j is the value of the local polynomial. This local polynomial is obtained by minimizing the following function:

$$\mathcal{R}(\theta_q(t_j), b_1; t) = \sum_{i=1}^n \left[\ln(Q_i) - \theta_q(t_j) - b_1 \text{dist}(t_i, t_j) \right]^2 K \left(\frac{\text{dist}(t_i, t_j)}{h} \right) \quad (\text{B.2})$$

The symbol $\text{dist}(t_i, t_j)$ represents the distance between 2 working days. It takes care of the calendar effects of the date. $K(\dots)$ is a Gaussian probability density. It gives the weight of the different observations for the least square problem. Only the data that are within a bandwidth h of t_j have a consequent weight in the minimization problem (B.2). Finally, once the conditional mean is known, k and σ_q can be obtained by OLS regressing $\frac{\Delta Q_t}{Q_t}$ on $(\ln(Q_t) - \theta_q(t))$.

¹The net load data in Germany comes from the Union of Co-ordination of Transmission of Electricity (www.ucte.org). For the import-export flows, we use data from ETSO (www.etsovista.org). The wind production data are given by the Association of German network operators (www.vdn-berlin.de)

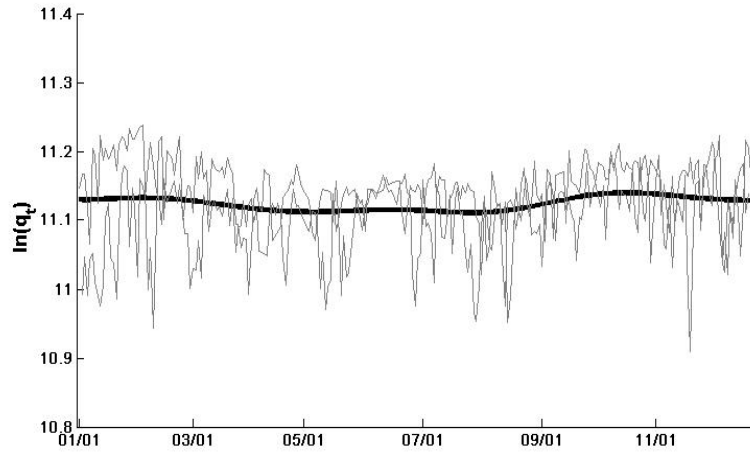


Figure B.1: Log load time series and estimated $\theta_q(t)$

Appendix C

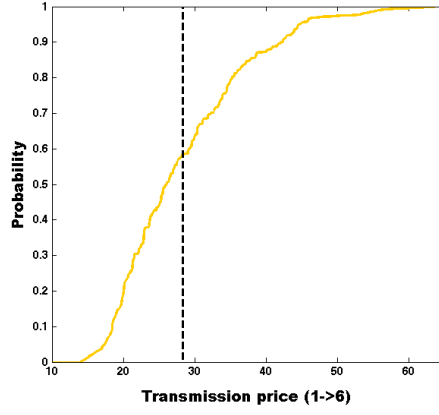
C.1 Additional figures and tables

Category	Oil	Natural Gas	Coal	Electricity
Physical Suppliers	7730	7730	1990	1197
Physical Wholesale Buyers	4836 + 3216 ¹	2376	313	72 + 7325 ²⁰
Volume (Physical)	7.6B barrels (\$22B)	22B MMBtu (\$152B)	1.1B short tons (\$22 B)	3.8B MWh (\$152 B)
Price Volatility	11%	29%	6%	66%

Table C.1: Commodity Comparison (from PJM [91])

¹non-bulk data

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	$\mathbb{E}[P_\ell^s]$	$\mathbb{V}\text{ar}[P_\ell^s]$
1→3	3.4	20.4
1→2	1.5	5.9
1→6	28.3	85.2
2→5	25.3	47.9
2→3	1.9	6.0
4→6	7.1	65.4
4→5	5.5	32.4
5→6	1.6	27.7

$$\mathbb{E}[\pi_{so}^s] = 10,965$$

$$-\text{CVaR}_{75\%}(\pi_{so}^s) = 6,624$$

Table C.2: Statistics of transmission day-ahead prices and cumulative distribution function for 1→6

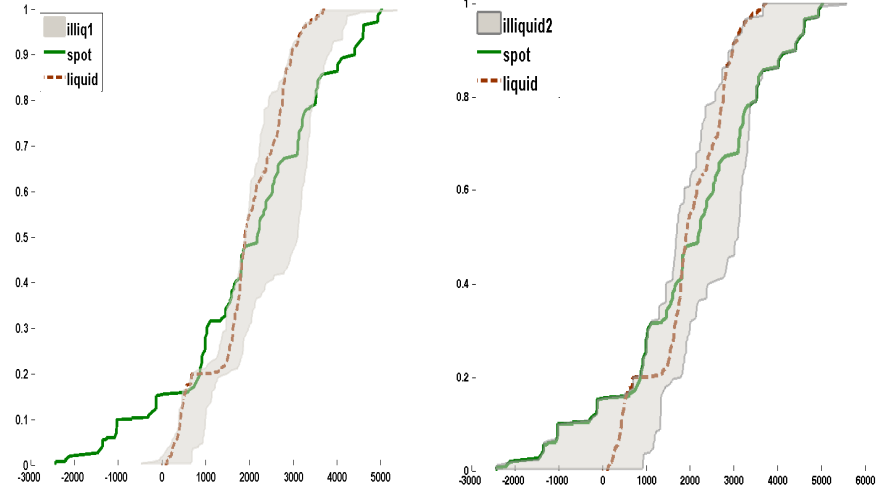


Table C.3: Range of distribution functions of the profit of the hub retailer (left : LIQ_{66%}, right: LIQ_{33%})

C.2 Formulation and computation of the futures equilibrium in a liquid market

Problem C.1. The problem of an agent minimizing a E-CVaR risk measure in a liquid market can be formulated as:

$$\begin{aligned} \mathcal{P}^\nu \equiv & \max_{\mathbf{t}_\nu, \mathbf{x}_\nu, \mathbf{U}_\nu, \omega} \left\{ \beta_\nu \mathbf{t}_\nu + \sum_{\omega} \text{prob}(\omega) [(1 - \beta_\nu) \Pi_\nu(\omega) - \beta_\nu \alpha_\nu^{-1} \mathbf{U}_\nu(\omega)] \right\} \\ \text{s.t: } & \mathbf{U}_\nu(\omega) \geq 0 \\ & \mathbf{U}_\nu(\omega) \geq \mathbf{t}_\nu - \Pi_\nu(\omega) \\ & \Pi_\nu(\omega) = \sum_c x_c^\nu (p_c^s(\omega) - p_c^f) + \pi_\nu^s(\omega) \end{aligned} \quad (\phi_\nu(\omega))$$

Problem C.2 (Nash Equilibrium). Applying standard duality theory, one can convert the utility optimization problem of the agent into the following optimality conditions for all agents (i.e. retailers, producers and ISO). An equilibrium is a solution of the complementarity system given by conditions C.1 - C.8.

$$0 \leq \mathbf{U}_\nu(\omega) \perp \beta_\nu \alpha_\nu^{-1} \text{prob}(\omega) - \beta \phi_\nu(\omega) \geq 0 \quad (\text{C.1})$$

$$0 \leq \phi_\nu(\omega) \perp \beta \left(\mathbf{U}^\nu(\omega) \mathbf{t}_\nu + \sum_c x_c^\nu (p_c^s(\omega) - p_c^f) + \pi_\nu^s(\omega) \right) \geq 0 \quad (\text{C.2})$$

$$\sum_{\omega \in \Omega} \phi_\nu(\omega) = 1 \quad (\text{C.3})$$

Equations C.4-C.5 impose that the FTRs auctioned by the ISO satisfy the N-1 rule.

$$0 \leq \mu_\ell^+ \perp K_\ell - \sum_c \text{PTDF}_{\text{ftr}, \ell}^{N-1} x_{so, \text{ftr}} \geq 0 \quad (\text{C.4})$$

$$0 \leq \mu_\ell^- \perp \sum_c \text{PTDF}_{\text{ftr}, \ell}^{N-1} x_{so, \text{ftr}} - K_\ell \geq 0 \quad (\text{C.5})$$

Finally, the following equations give the contract prices at equilibrium and the market clearing conditions.

$$p_c^f = (1 - \beta) \sum_{\omega \in \Omega} \text{prob}(\omega) p_c^s(\omega) + \beta \sum_{\omega \in \Omega} \phi_\nu(\omega) p_c^s(\omega) \quad (\text{C.6})$$

$$\begin{aligned} p_c^f = & (1 - \beta) \sum_{\omega} \text{prob}(\omega) p_c^s(\omega) + \beta \sum_{\omega \in \Omega} \phi_\nu(\omega) p_c^s(\omega) \\ & + \sum_\ell (\mu_\ell^- - \mu_\ell^+) \text{PTDF}_{c, \ell}^{N-1} \end{aligned} \quad (\text{C.7})$$

$$\sum_\nu x_{\nu, c} + x_{so, c} = 0 \quad (\text{C.8})$$

Problem C.3 (Variational Inequalities formulation). One can transform this Karush-Kuhn-Tucker system into a variational inequality problem $VI(F, K)$ where the goal is to find a $y \in K$ such that

$$(z - y)^T F(y) \geq 0, \quad \forall z \in K$$

The vector y is composed of $y := (y_1, \dots, y_N, y_{so}, y_{MK})^T$, where $y_\nu = (\mathbf{t}_\nu; x_{\nu,c}; \mathbf{U}_\nu(\omega); \phi_\nu(\omega))$, $y_{so} = (\mathbf{t}_{so}; x_{so,c}; \mathbf{U}_{so}(\omega); \phi_{so}(\omega))$ and $y_{MK} := (p_c^f)$. The functional F is given by $F = (F_1, \dots, F_N, F_{so}, F_{MK})$ where

$$F_\nu(\mathbf{t}_\nu; x_{\nu,c}; \mathbf{U}_\nu(\omega); \phi_\nu(\omega)) = \begin{pmatrix} 0 \\ p_c^f - (1 - \beta) \sum \text{prob}(\omega) p_c^s(\omega)^s - \beta \sum \phi_\nu(\omega) p_c^s(\omega) \\ \beta(\alpha^{-1} \text{prob}(\omega) - \phi_\nu(\omega)) \\ \beta(\mathbf{U}_\nu(\omega) - \mathbf{t}_\nu + \sum_{c=1}^C x_{\nu,c} (p_c^s(\omega) - p_c^f) + \pi_\nu^s(\omega)) \end{pmatrix}$$

$$F_{MK}(p_c^f) = \left(-\sum_{\nu=1}^N x_{\nu,c} - x_{so,c} \right)$$

The space $K = K_1 \times \dots \times K_N \times K_{so} \times K_{MK}$ has the following description.

$$K_\nu := \left\{ \begin{array}{l} \sum_\omega \phi_\nu(\omega) = 1 ; \phi_\nu(\omega) \geq 0 ; \alpha^{-1} \text{prob}(\omega) - \phi_\nu(\omega) \geq 0 \\ \mathbf{U}_\nu(\omega) \geq 0 ; \mathbf{U}_\nu(\omega) \geq \mathbf{t}_\nu - \sum_{c=1}^C x_{\nu,c} (p_c^s(\omega) - p_c^f) - \pi_\nu^s(\omega) \end{array} \right\}$$

$$K_{so} := K_\nu \cap \left\{ K_\ell \geq \sum_c \text{PTDF}_{\text{ftr}, \ell}^{N-1} x_{so, \text{ftr}} \geq -K_\ell \right\}$$

$$K_{MK} := \bar{\mathbb{P}}$$

Proposition C.1. The mapping F of problem D.4 is monotone on K .

Proof. By definition, one has to show that $(F(y^1) - F(y^2))^T(y^1 - y^2) \geq 0$, $\forall y^1, y^2 \in K$. Let us first analyze the contribution of all agents (including the ISO):

$$\begin{aligned}
(F_\nu(y_\nu^1) - F_\nu(y_\nu^2))^T(y_\nu^1 - y_\nu^2) &= (0 - 0) \times (\mathbf{t}_\nu^1 - \mathbf{t}_\nu^2) \\
&+ \sum_{c=1}^C \left((p_c^{f,1} - p_c^{f,2}) \times (x_{\nu,c}^1 - x_{\nu,c}^2) \right. \\
&\quad \left. - \sum_{\omega \in \Omega} \beta p_c^s(\omega) (\phi_\nu^1(\omega) - \phi_\nu^2(\omega)) \times (x_{\nu,c}^1 - x_{\nu,c}^2) \right) \\
&- \sum_{\omega \in \Omega} \beta (\phi_\nu^1(\omega) - \phi_\nu^2(\omega)) \times (\mathbf{U}_\nu^1(\omega) - \mathbf{U}_\nu^2(\omega)) \\
&+ \beta \left((\mathbf{t}_\nu^2 - \mathbf{t}_\nu^1) + \sum_{c=1}^C (x_{\nu,c}^2 p_c^{f,2} - x_{\nu,c}^1 p_c^{f,1}) \right) \times \sum_{\omega \in \Omega} (\phi_\nu^1(\omega) - \phi_\nu^2(\omega)) \\
&+ \sum_{\omega} \beta \left(\sum_{c=1}^C p_c^s(\omega) (x_{\nu,c}^1 - x_{\nu,c}^2) \times (\phi_\nu^1(\omega) - \phi_\nu^2(\omega)) \right. \\
&\quad \left. + \beta (\mathbf{U}_\nu^1(\omega) - \mathbf{U}_\nu^2(\omega)) \times (\phi_\nu^1(\omega) - \phi_\nu^2(\omega)) \right)
\end{aligned}$$

Recalling that for $y \in K : \sum_{\omega} (\phi_\nu^1(\omega) - \phi_\nu^2(\omega)) = (1 - 1)$, one gets

$$(F_\nu(y_\nu^1) - F_\nu(y_\nu^2))^T(y_\nu^1 - y_\nu^2) = \sum_{c=1}^C (p_c^{f,1} - p_c^{f,2}) \times (x_{\nu,c}^1 - x_{\nu,c}^2)$$

The contribution of the market agent is given by

$$\begin{aligned}
(F_{MK}(y_{MK}^1) - F_{MK}(y_{MK}^2))^T(y_{MK}^1 - y_{MK}^2) &= \\
&- \sum_{c=1}^C \left(\sum_{\nu=1}^N (x_{\nu,c}^1 - x_{\nu,c}^2) + (x_{so,c}^1 - x_{so,c}^2) \right) \times (p_c^{f,1} - p_c^{f,2})
\end{aligned}$$

One concludes that $(F(y^1) - F(y^2))^T(y^1 - y^2) = 0$ and hence the mapping is monotone. $\square$

Proposition C.2. The Nash Equilibrium problem C.2 has a convex non-empty solution set.

Proof. First, according to theorem 4.2, the problem has a solution. We argued in the proof of the latter theorem that one can restrict the financial position x_ν to belong to some compact and convex set strategy set $\hat{X}_\nu$, without perturbing the equilibrium. Correspondingly, one can restrict the set K to some compact and convex set $\hat{K}$ in the variational inequality problem D.4. By the theorem of the VI with pseudo-monotone functional [e.g. 59, proposition 3.1], the problem $VI(F, \hat{K})$ has a convex solution set. $\square$

C.2.1 Computation and algorithm

There exists several algorithms [see 48, chapter 12] for solving monotone variational inequalities problem. In our context, the variational inequalities problem has a large size depending on the number of scenario. We rather propose a heuristic method to find the equilibria. The fixed point method used requires a sequence of maximization problems and is well known in macroeconomics (Negishi like algorithm) community for solving large scale general equilibrium problem.

Suppose a tentative equilibrium prices $\mathbf{p}_c^f$. One can compute the optimal positions of an agent by solving the following linear programming problem (where the $\mathbf{p}_c^f$ are fixed).

$$\begin{aligned} \mathcal{E}_{lp} := \max \quad & \sum_{\nu \in \{N_r, N_p, SO\}} \left\{ \beta_\nu \mathbf{t}_\nu + \sum_{\omega \in \Omega} \text{prob}(\omega) ((1 - \beta_\nu) \Pi_\nu(\omega) - \beta_\nu \alpha_\nu^{-1} \mathbf{U}_\nu(\omega)) \right\} \\ & \mathbf{U}_\nu(\omega) \geq 0 \\ & \mathbf{U}_\nu(\omega) \geq \mathbf{t}_\nu - \Pi_\nu(\omega) \\ & \Pi_\nu(\omega) = \sum_c x_c^\nu (p_c^s(\omega) - \mathbf{p}_c^f) + \pi_\nu^s(\omega) \\ & -K_\ell \leq \sum_c \text{PTDF}_{c,\ell}^{N-1} x_c^{so} \leq K_\ell \quad (\mu_\ell^-, \mu_\ell^+) \\ & \sum_\nu x_c^\nu = 0 \quad (\eta_c) \end{aligned} \quad (\text{C.9})$$

The Karush-Kuhn-Tucker optimality conditions of C.9 are similar to the original NEP except that p_c^f is replaced by $\mathbf{p}_c^f$ and that equality (C.6) and (C.7) are changed to:

$$\mathbf{p}_c^f + \eta_c = (1 - \beta) \sum_{\omega \in \Omega} \text{prob}(\omega) p_c^s(\omega) + \beta \sum_{\omega \in \Omega} \phi_\nu(\omega) p_{c,\omega}^s \quad (\text{C.10})$$

$$\begin{aligned} \mathbf{p}_c^f + \eta_c = \quad & (1 - \beta) \sum_{\omega \in \Omega} \text{prob}(\omega) p_c^s(\omega) \\ & + \beta \sum_{\omega \in \Omega} \phi_\nu(\omega) p_{c,\omega}^s + \sum_\ell (\mu_\ell^- - \mu_\ell^+) \text{PTDF}_{c,\ell}^{N-1} \end{aligned} \quad (\text{C.11})$$

One see that when the dual variables η_c are all equal to zero (i.e. given the derivatives prices, no agent has an incentive to modify its portfolio), then the solution of $\mathcal{E}_{lp}$ is also a Nash Equilibrium. This lead to the following fixed-point algorithm.

Require: $\delta > 0, \mathbf{P}_c^f \in \mathbb{R}^c$

- 1: **while** $\epsilon \geq \delta$ **do**
 - 2: Solve $\mathcal{E}_{lp}$ using lp
 - 3: $\mathbf{P}_c^f \leftarrow \mathbf{P}_c^f + \eta_c$
 - 4: $\epsilon = \|\eta_c\|$
 - 5: **end while**
-

This fixed-point algorithm is a heuristic and do not guarantee any convergence. Nevertheless, we implemented it in GAMS using CPLEX as solver and in all the cases solved in this thesis, it performed rather well (usually less than ten iterations) and could solve every cases.

C.3 Non convex risk measure

While mean-variance has probably been the most used risk function for modeling risk aversion, it is not convex in the sense of Föllmer and Schied [51] and, in the context of an equilibrium model, it may lead to solutions with arbitrage opportunities. This is highlighted in the following example. Consider a market with 2 goods. Their prices are denoted (p_1^s, p_2^s) and depend on the state s of the world. We consider 3 possible states, each having a probability ϕ_s to occur. There is a financial market where futures contracts on those goods are traded. Their futures prices are (P_1^f, P_2^f) .

	$s = sc1$	$s = sc2$	$s = sc3$
ϕ_s	0.3	0.3	0.4
p_1^s	1	1	2
p_2^s	1	2	3.5

There are 2 agents yielding a stochastic profit (π_a^s, π_b^s) depending on s . The profit of agent a is positively affected by good 1 and negatively by good 2. The profit of agent b is inversely impacted.

	$s = sc1$	$s = sc2$	$s = sc3$
π_a^s	45	31.67	50
π_b^s	30	110	50

$\text{Cov}(.,.)$	p_1^s	p_2^s
π_a^s	2.8	3.6
π_b^s	-4.8	2.4

Agents are risk averse and their utility is modeled by a mean-variance with a parameter $\alpha = 0.1$.

$$\rho(\Pi) = -\mathbb{E}[\Pi] + \alpha \text{Var}[\Pi] \quad (\text{C.12})$$

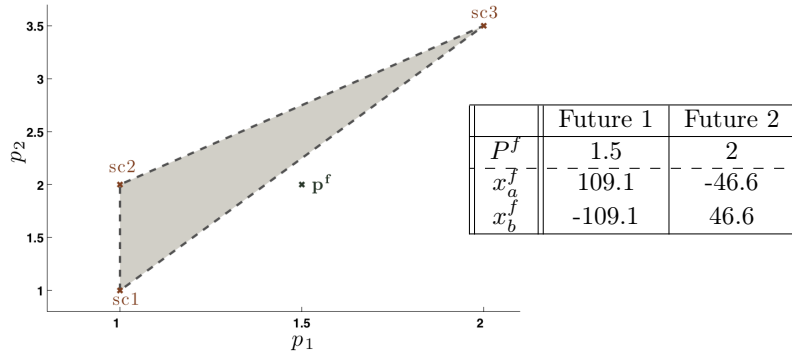
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They trade futures in order to minimize their risk measures. Intuitively, a , because of his stochastic profit, has an incentive to sell futures 1 and buy futures 2 and conversely for agent b .

The equilibrium conditions are, as derived in Bessembinder and Lemmon [13]:

$$\begin{aligned} p^f &= \mathbb{E}[p_s] + \frac{\alpha_a \alpha_b}{\alpha_a + \alpha_b} \sum_{i=a,b} \text{Cov}(\pi_i^s, p_s) \\ x_i &= \frac{\Sigma^{-1}}{\alpha_i} (p^f - \mathbb{E}p_s) - \Sigma^{-1} \text{Cov}(\pi_i^s, p_s) \end{aligned} \quad (\text{C.13})$$

The equilibrium solution is reported in the next Table. One can see that it contains arbitrage opportunities². Indeed, there exist no equivalent risk-neutral probability measure. Graphically, futures prices do not strictly belong to the convex hull of the day-ahead prices.



²Numerically, the strategy $q^f = (-1, 0.5)$ is an arbitrage opportunity. Notice that, as financial contracts are futures, the payments are due at maturity.

C.4 Formulation and computation of the future equilibrium with liquidity constraints

Problem C.4. The problem of an agent minimizing a E-CVaR risk measure in an illiquid market is:

$$\begin{aligned} \mathcal{P}^\nu(\mathbf{x}^{-\nu}) \equiv & \max_{\mathbf{t}_\nu, x_{\nu,c}, \mathbf{U}_\nu(\omega)} \left\{ \beta_\nu \mathbf{t}_\nu + \sum_{\omega \in \Omega} \text{prob}(\omega) \left((1 - \beta_\nu) \Pi_\nu(\omega) - \beta_\nu \alpha_\nu^{-1} \mathbf{U}_\nu(\omega) \right) \right\} \\ \text{s.t: } & \mathbf{U}_\nu(\omega) \geq 0 \\ & \mathbf{U}_\nu(\omega) \geq \mathbf{t}_\nu - \Pi_\nu(\omega) \\ & \Pi_\nu(\omega) = \sum_c x_{\nu,c} (p_c^s(\omega) - p_c^f) + \pi_\nu^s(\omega) \\ & x_{\nu,c} \leq L_c - \sum_{-\nu} |x_{-\nu,c}| \\ & x_{\nu,c} \geq -L_c + \sum_{-\nu} |x_{-\nu,c}| \end{aligned} \quad \begin{array}{l} (\lambda_{\nu,c}) \\ (\mu_{\nu,c}) \end{array}$$

The complementary conditions of this problem are quite similar to the previous problem. Complementary conditions C.1 - C.5 are unchanged. One adds the following conditions to represent the volume constraints.

$$0 \leq \lambda_{\nu,c} \perp L_c - \sum_{-\nu} |x_{-\nu,c}| - x_{\nu,c} \geq 0 \quad (\text{C.14})$$

$$0 \leq \mu_{\nu,c} \perp x_{\nu,c} + L_c - \sum_{-\nu} |x_{-\nu,c}| \geq 0 \quad (\text{C.15})$$

$$(\text{C.16})$$

The equilibrium derivative prices are also changed into :

$$p_c^f = (1 - \beta) \sum_{\omega \in \Omega} \text{prob}(\omega) p_c^s(\omega) + \beta \sum_{\omega \in \Omega} \phi_\nu(\omega) p_c^s(\omega) + \mu_{\nu,c} - \lambda_{c,\nu} \quad (\text{C.17})$$

C.4.1 Computation

As the shared constraints are separable, we compute the different GNE using the parametrized Variational Inequality approaches described in Nabetani et al. [83]. We construct a family of VIs that contains all the equilibria of the initial GNEP. We perturb the initial objective function of agent by penalizing it by its financial volume with a positive weight n^ν :

$$\tilde{\rho}_\nu(x_c^\nu, y_c^\nu) = \rho_\nu(x^\nu) + \sum_c n_c^\nu |x_{\nu,c}| \quad (\text{C.18})$$

We then compute the associate NEP, using the heuristic developed in appendix C.2. According to Nabetani et al. [83], the equilibrium found is a solution of the initial GNEP if:

$$\sum_{\nu,c} n_c^\nu (L_c - \sum_\nu |x_{\nu,c}|) = 0 \quad (\text{C.19})$$

APPENDIX

Appendix D

D.1 Formulations of the optimal investment problem

We present in this appendix different formulations of the investment valuation problem using a good-deal measure.

Problem D.1 (KKT conditions). Given a price vector $\boldsymbol{\pi}$, an investment plan $\mathbf{x}^\nu$ and the corresponding operations decision $\mathbf{y}^\nu$ are an optimal strategy for a player ν minimizing the good-deal measure of its profit (problem 5.5), iff it satisfies the KKT conditions:

$$\begin{aligned}
 (\eta_0^\nu, \eta^\nu(\omega)_{\omega \in \Omega}) &\in \mathbb{L}^{|\Omega|+1} & \left(\left(\frac{\phi^\nu(\omega)}{\sqrt{\text{prob}(\omega)}} \right)_{\omega \in \Omega}, A^\nu \right) &\in \mathbb{L}^{|\Omega|+1} \\
 \mathbf{w}^\nu &\in \mathbb{R}^{|I|} & f_0 - \sum_{\omega \in \Omega} \phi^\nu(\omega) f_1(\omega) &= 0 \\
 \mathbf{x}^\nu &\in \mathbb{R}_+^K & i_k - \sum_{\omega \in \Omega} \sum_{\ell=1}^L \tau_\ell \mu_{k,\ell}^\nu(\omega) &\in \mathbb{R}_+^K \\
 \mathbf{y}^\nu &\in \mathbb{R}_+^{K \times L \times |\Omega|} & \mu_{k,\ell}^\nu(\omega) + \phi^\nu(\omega) (c_k(\omega) - \pi_\ell(\omega)) &\in \mathbb{R}_+^{K \times L \times |\Omega|} \\
 \phi^\nu &\in \mathbb{R}_+^{|\Omega|} & \left(\begin{array}{c} \frac{\eta^\nu(\omega)}{\sqrt{\text{prob}(\omega)}} - f_1^T \mathbf{w}^\nu - \dots \\ \sum_{\ell=1}^L \sum_{k=1}^K \tau_\ell (c_k(\omega) - \pi_\ell(\omega)) y_{k,\ell}^\nu(\omega) \end{array} \right) &\in \mathbb{R}_+^{|\Omega|} \\
 \boldsymbol{\mu}^\nu &\in \mathbb{R}_+^{K \times L \times |\Omega|} & x_k^\nu - y_{k,\ell}^\nu(\omega) &\in \mathbb{R}_+^{K \times L \times |\Omega|}
 \end{aligned}$$

$$\begin{aligned}
& \sum_{i \in I} \mathbf{w}_i^\nu \left(f_{0,i} - \sum_{\omega \in \Omega} \phi^\nu(\omega) f_{1,i}(\omega) \right) + A\eta_0 - \sum_{\omega \in \Omega} \frac{\eta^\nu(\omega)}{\sqrt{\text{prob}(\omega)}} \phi^\nu(\omega) + \dots \\
& \sum_{k=1}^K x_k^\nu \left(\imath_k - \sum_{\ell=1}^L \sum_{\omega \in \Omega} \tau_\ell \mu_{k,\ell}^\nu(\omega) \right) + \dots \\
& \sum_{\ell=1}^L \sum_{k=1}^K \sum_{\omega \in \Omega} y_{k,\ell}^\nu(\omega) (\mu_{k,\ell}^\nu(\omega) + \phi^\nu(\omega) (c_k(\omega) - \pi_\ell(\omega))) = 0
\end{aligned}$$

Problem D.2 (Dual formulation). The dual problem of 5.5 is the following SOCP. It involves two decision variables $\boldsymbol{\mu}^\nu \in \mathbb{R}^{K \times L \times |\Omega|}$ and $\boldsymbol{\phi}^\nu \in \mathbb{R}^{|\Omega|}$.

$$\begin{aligned}
\mathcal{P}_{\text{GD}}^\nu(\boldsymbol{\pi}) \equiv & \max_{\boldsymbol{\mu}^\nu, \boldsymbol{\phi}^\nu} 0^T \boldsymbol{\alpha}^\nu + 0^T \boldsymbol{\phi}^\nu \\
\text{s.t.} \quad & f_{0,i} = \sum_{\omega \in \Omega} \phi^\nu(\omega) f_{1,i}(\omega) \quad (w_i^\nu) \\
& \sum_{\omega \in \Omega} \frac{\phi^\nu(\omega)^2}{\text{prob}(\omega)} \leq (A^\nu)^2 \\
& \imath_k - \sum_{\omega \in \Omega} \sum_{\ell=1}^L \tau_\ell \mu_{k,\ell}^\nu(\omega) \geq 0 \quad (x_k^\nu) \\
& \mu_{k,\ell}^\nu(\omega) \geq \phi^\nu(\omega) (\pi_\ell(\omega) - c_k(\omega)) \quad (y_{k,\ell}(\omega))
\end{aligned}$$

One recovers that the probability distribution $\boldsymbol{\phi}^\nu$ belongs to the set $\mathcal{Q}^{GD}$ characterizing the good-deal measure (cf. section 3.3.1).

Problem D.3. One can formulate the problem of satisfying the KKT conditions (i.e. problem D.1) into a generalized complementarity problem, denoted $GCP(K_\nu, F_\nu)$, where the goal is to find a vector $u_\nu^* \in K^\nu$ such that¹

$$F_\nu(u_\nu^*) \in K^* \quad \text{and} \quad F_\nu(u_\nu^*)^T u_\nu^* = 0$$

where the variable u^ν is the following vector :

$$u_\nu = (\mathbf{w}^\nu, \boldsymbol{\eta}^\nu; \mathbf{x}^\nu; \mathbf{y}^\nu; \boldsymbol{\phi}^\nu; \boldsymbol{\mu}^\nu)$$

¹We denote by K^* the dual space of the set K , i.e. $K^* = \{y | y^T x \geq 0 \forall x \in K\}$

the mapping F_ν is defined as :

$$F_\nu \begin{pmatrix} \mathbf{w}_i^\nu \\ \eta_0^\nu \\ \eta^\nu(\omega) \\ x_k^\nu \\ y_{k,\ell}^\nu(\omega) \\ \phi^\nu(\omega) \\ \mu_{k,\ell}^\nu(\omega) \end{pmatrix} = \begin{pmatrix} f_{0,i} - \sum_{\omega \in \Omega} \phi^\nu(\omega) f_{1,i}(\omega) \\ A^\nu \\ \frac{-\phi^\nu(\omega)}{\sqrt{\text{prob}(\omega)}} \\ \imath_k - \sum_{\omega \in \Omega} \sum_{\ell=1}^L \tau_\ell \mu_{k,\ell}^\nu(\omega) \\ \mu_{k,\ell}^\nu(\omega) + \phi^\nu(\omega) (c_k(\omega) - \pi_\ell(\omega)) \\ 0 \\ 0 \end{pmatrix}$$

and finally the set K_ν is the convex set:

$$K_\nu = \left\{ \begin{array}{l} \mathbf{w}^\nu \in \mathbb{R}^{|\mathcal{I}|} \\ (\eta^\nu(\omega)_{\omega \in \Omega}, \eta_0^\nu) \in \mathbb{L}^{|\Omega|+1} \\ \mathbf{x}^\nu \in \mathbb{R}_+^K \\ \mathbf{y}^\nu \in \mathbb{R}_+^{K \times L \times |\Omega|} \\ \phi^\nu \in \mathbb{R}_+^{|\Omega|} \\ \boldsymbol{\mu}^\nu \in \mathbb{R}_+^{K \times L \times |\Omega|} \\ x_k^\nu - y_{k,\ell}^\nu(\omega) \geq 0 \\ \sum_{\ell=1}^L \sum_{k=1}^K \tau_\ell (c_k(\omega) - \pi_\ell(\omega)) y_{k,\ell}^\nu(\omega) + f_1(\omega)^T \mathbf{w}^\nu \leq \frac{\eta^\nu(\omega)}{\sqrt{\text{prob}(\omega)}} \end{array} \right\}$$

D.2 Variational inequalities formulation of the equilibrium problem

Karamardian [72] was the first to establish the direct relationship between the generalized complementarity problem and the variational equality problem. This allows us to formulate the equilibrium problem as the following variational inequality problem

Problem D.4 (Variational Inequalities formulation). One can transform the complementarity conditions defining the competitive equilibrium (definition 5.2) into a variational inequality problem $VI(F, K)$ where the goal is to find a vector $v^* \in K$ such that

$$(v - v^*)^T F(v^*) \geq 0, \quad \forall v \in K$$

The vector v is composed of $v := (u_1, \dots, u_N, u_{MK})^T$, where u_ν is defined in problem D.3 and $u_{MK} := (\boldsymbol{\pi}; \mathbf{z})$. The functional F is given by $F = (F_1, \dots, F_N, F_{MK})$ where F_ν is defined in problem D.3 and

$$F_{MK}(u_{MK}) = \left(\begin{array}{c} \sum_{\nu=1}^N \sum_{k=1}^K y_{k,\ell}^{\nu}(\omega) + z_{\ell}(\omega) - d_{\ell}(\omega) \\ PC - \pi_{\ell}(\omega) \end{array} \right)$$

The space $K = K_1 \times \dots \times K_N \times K_{MK}$ is composed of K_{ν} defined in problem D.3 and

$$K_{MK} := \{ \pi_{\ell}(\omega) \geq 0 ; z_{\ell}(\omega) \geq 0 \}$$

D.2.1 The symmetric case

Symmetric producers have the same risk measure and hence share the total investment (proportionally to their number). We already argue that the global outcome of the equilibrium problem do not depend of the number of players. Computationally, we treat the problem as there was a unique producer. We develop a heuristic to find an equilibrium which is very similar to the one exposed in section C.2.1 . It is based on the observation that, once the variable ϕ is known, it can be considered as an exogenous discounted factor and thus leads to an equilibrium amenable to a stochastic capacity expansion optimization problem such as studied in section 5.2. Correspondingly, when the investments levels $\mathbf{x}$ are determined, one can compute the good-deal value and the variable ϕ associated with it. The heuristic seek a fixed point by sequentially solving the two problems and stop when convergence has been reached.