A BAYESIAN APPROACH TO THE ECONOMETRICS OF FIRST-PRICE AUCTIONS *

Gian Luigi Albano[1]   Frédéric Jouneau[2]

[1]*C.O.R.E., Université Catholique de Louvain*
[2]*GREMARS, Université de Lille 3*
and
C.O.R.E., Université Catholique de Louvain

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Abstract

We propose a Bayesian approach to empirical auction models. We argue that the Bayesian paradigm is more suitable to the study of empirical strategic models than its frequentist counterpart. We perform an estimation of our model on an auction of hand-made miniature sculptures organized by Christie's in London.

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Corresponding author: Frédéric Jouneau, GREMARS, Université de Lille 3, France, Email:**jouneau@univ-lille3.fr**

1 Introduction

Since Vickrey's (1961) seminal paper there has been an increasing interest in the theory of auctions. Particularly in the last decade the development of refined game theoretical tools has enhanced the understanding of auction mechanisms.[1]

From an econometric viewpoint auction models are also appealing because they are "fully" structural. Indeed, as Laffont *et al.* (1991)[2] point out

> "A noticeable property of the econometric model is that randomness enters naturally without adding stochastic error terms as it is usually done in econometric modelling."

Indeed the economic and the econometric models coincide exactly. We take this property as a starting point and we call "fully structural" a couple of economic model and econometric model which enjoys this property. We will stress extensively that the structural feature concerns *both* the economic and the econometric frameworks.

In this paper we argue that the econometric models considered in the existing literature on empirical auction games are not "fully structural" in the sense defined above. We develop a Bayesian approach to the econometric analysis of first-price auction. We explain in details that the Bayesian paradigm is a more appropriate framework than the frequentist conterpart. In particular the econometric model we propose coincides exactly with the underlying economic model.

In the existing literature on empirical auction models the statistical treatment of the available data has not been as structural as it may appear at a first glance. The frequentist approach to inference relies on asymptotic theory. More precisely, the proposed estimators are defined on a product space. Each element of this space is a random variable whose realisation is an observed experiment. Thus, if the econometric model is to be "fully structural" the underlying economic model should admit some asymptotic dimension. In the non-strategic framework this dimension is usually the number of agents and/or the number of periods. Both these choices have been considered in

[1]See for instance, McAfee and McMillan (1987), Wilson (1992) and Wolfstetter (1996) for extensive surveys.

[2]This quotation appears in the mimeo version of Laffont *et al.* (1995).

the existing literature on empirical auction games. The relevant choice seems to depend mostly on the nature of available data.

In the case of sealed-bid auctions[3] all bids are observed. When the bidders are numerous, it is tempting to choose their number as the asymptotic dimension. When only the winning bid is observed[4] the econometric analysis typically considers more than one auction. In this context, the number of repetition is the natural asymptotic dimension.

The paper is organized as follows. In section 2 we examine more carefully the asymptotic approaches proposed in the existing literature. In particular we show that the econometric treatments proposed so far fail at being "fully structural". The rest of the paper describes a Bayesian approach specifically designed for the study of real empirical auction games. In section 3 we briefly recall the main assumptions and the equilibrium strategies of an IPV first-price auction. In section 4 we describe our econometric modelling of a single auction and perform some explanatory simulations. In section 5 we propose an econometric model of repetead auctions and perform an estimation on a real dataset. Section 6 summarizes some concluding remarks.

2 A methodological overview

In this section we examine some econometric models proposed in the literature of empirical auction games. We pay particular attention to the coincidence between the econometric and the economic models involved. We first consider the case where the number of players is the asymptotic index. This assumption is explicit in Florens and Protopopescu (1997)[5], and more implicit in other papers (see, for instance, Pesendorfer (1995)). Second we consider the case of repeated auction.

[3]Sealed-bid auctions are selling mechanisms in which participants tender secret offers. The object is awarded to the participant who has submitted the highest bid. The price equals the highest bid if the auction is first-price or the second highest bid if the auction is second-price.

[4]This is typically the case of the Dutch auction. The Dutch auction is an oral first-price auction in which the auctioneer first announces a very high price. Then, he lowers the price continuously until one participant says "stop" and takes the object at that price.

[5]See, e.g., example 1 and theorem 2.

2.1 Asymptotics on the number of players

We have emphasized in the introduction that, when the asymptotics is the number of agents, the underlying economy involves an infinite number of players. If this is not the case, the econometric model cannot be "fully structural". However, an economy which consists of an infinite number of agents generally differs from the case a finite number of agents. The analysis of strategic behavior emphasizes the difference between economies with finite and infinite number of agents. It is widely known that some outcomes of a finite economy may disappear in the "limit" economy. In this respect the (implicit) asymptotic approach chosen by Pesendorfer (1995) seems particularly unsatisfactory since it deals with collusion mechanisms in a first-price auction.

The previous argument stresses an economic difficulty. But some econometric problems are also at hand. In game-theoretical frameworks the goal of inference is usually agents' private information. In the Independent Private Value first-price auction models, for instance, [6] the goal is the cumulative density function of the agents' signals. If these signals are observed, the estimation of their c.d.f. would cause no problem. However, agents' strategic behaviour implies some "shading" of their private information. Consequently, the observed bids are functions of the private signals. The econometric complexity of the problem comes from the fact that the optimal bid function depends on the unknown cumulative density function. If the economy were to consist of an infinite number of agents the optimal bid function would involve fully revelation of private signals. The "structural" formula which provides the optimal bidding strategy as function of private signals converges uniformly to the identity function. Thus as far as inference is concerned, the estimator of the c.d.f. may be computed directly by using the observed bids. In the limit, the complexity of the econometric problem disappears.

This last difficulty is specific to the game theoretical framework. Instead, consider the case where we observe the consumptions levels of some price-taking agents for some given commodity. These agents are drawn out of an infinite number of agents (since it is the usual justification for the price-taking behaviour). In this setting, the argument we just described does not apply. Indeed what we observe is a "piece" of a limit economy with an infinite number of agents. In this limit economy, the behaviour of the agents we

[6] See section 3 below for a more complete presentation.

observed is exactly the same. The underlying economy and the econometric framework both involve a continuum of traders.

2.2 Asymptotics on the number of repetitions

To avoid the difficulty implied by the asymptotics on the number of agents, some authors (see, for instance Paarsch (1992,1994) and Laffont *et al.* (1991), Vuong *et al.* (1997)) have considered the case of repeated auctions. However, as Laffont *et al.* (1991) underline, there is a serious theoretical problem in considering successive auctions. The strategic behaviour in repeated auctions can be a much more complex than the one in a static auction [7].

Once again, we shall consider the underlying economic model. If the asymptotic approach is to be "fully structural" the number of repetitions has be infinite. Since we only observe a finite number of them the "fully structural" feature of this model depends largely on the specification of the strategies.

Suppose first we specify the strategies as simple repetitions of a one-shot auction game (this is basically what Laffont *et al.* and Vuong *et al.* do). Then the model is "fully structural" if and only if the repetition of the one-shot strategy is an Nash equilibrium of the limit economy. In a sense, showing that the Laffont *et al.* model is "fully structural" seems a feasible task. In particular it does not require the study of a game with a finite number of repetitions. Of course, it would be totally convincing to show that it is the only equilibrium, but this seems more doubtfull...

Now suppose that we specify the strategies in a genuine dynamic way. For instance, we would like to test whether the number of remaining auctions has any influence on the bidding behaviour. Indeed, the literature on the *declining price anomaly* conveys the intuition that potential buyers do not play a repetition of a one-shot strategy (see however Ginsburgh (1997) for an institutional explanation of this phenomenon for wine auctions, and the references therein). Then the asymptotic model (where asymptotic means that we consider an infinite number of repetitions) cannot be "fully structural". What the asymptotic approach can capture in a "structural" way is a static behaviour.

More generally, relying on asymptotic theory seems to cause recurrent

[7] See Bikhchandani (1988) for a formal treatment of repeated auctions.

problems in the econometric analysis of strategic behavior. Indeed, the usual "sources" of asymptotics, i.e., the number of agents and/or the number of periods, are inadequate. Consequently, if the underlying economy consists of a finite number of agents and repetitions, only the finite sample properties of the estimators should matter. However, given the high degree of nonlinearity of auction models, assessing the finite sample properties of the estimators proposed in the existing literature seems a formidable task.

Instead, the Bayesian paradigm allows the derivation of estimators whose exact properties are known. It is in fact a very intriguing feature of the existing literature that the econometrics of auctions rests on asymptotic theory whereas theoretical economic models use the Nash-Bayes equilibrium concept to derive the optimal strategies. Of course the main goal of inference differs from individual behavior in auction games. Nevertheless, we feel that in the auction framework, the Bayesian approach will not be considered a dogmatic choice.

3 The Independent Private Value model

We briefly recall the essential feature of a first-price auction model. An object has to be auctioned to a set of bidders. Each bidder $i, i = 1, \ldots, N$, is assumed to know exactly how much the item is worth to him, say s_i. He does not know anyone else's valuation of the object; instead, he perceives any other bidder's valuation as a draw from some probability distribution, $F(\cdot)$, which is assumed to be common knowledge. Similarly, he knows that the other bidders (and the seller) regard his own valuation as being drawn from the same probability distribution. Any one bidder's valuation is statistically independent from any other bidder's valuation. This is called the *independent private value model*. Riley and Samuelson (1981) among others characterized the symmetric Bayesian Nash equilibrium of this game.

To summarize, each bidder knows his own evaluation s_i, the distribution $F(\cdot)$, the number of potential buyers N and the seller's reservation price v. It is assumed that $F(\cdot)$ is absolutely continuous with respect to Lebesgue measure with support $[a, a+h]$. It is also assumed that each buyer is risk neutral.

The symmetric Bayesian Nash equilibrium gives for every $i = 1, \ldots, N$ the optimal bid for the i-th potential buyer

$$b(s_i) = s_i - \frac{1}{(F(s_i))^{N-1}} \int_v^{s_i} (F(x))^{N-1} dx \tag{1}$$

if $s_i \geq v$ (see, e.g., Riley and Samuelson (1981)). If $s_i < v$, then the i-th bid can take any value strictly less than the reservation price v. One important property of equation (1) is that $b(\cdot)$ is a strictly increasing function of s_i on $[\max(v, a); a + h]$.

4 Inference for a single auction

In order to highlight the possibilities of the Bayesian approach we start our analysis by a single auction whose number of participant N is known. We then turn our attention to repeated auctions in the case where the sets of participants have two-by-two empty intersections. That simply means that each auction is really independent from the others and players do adopt optimal strategies required by a one-shot auction game. Furthermore, we consider several degrees of heterogeneity such as the number of participants and the bounds of the support of private values. We investigate the performance of our model on a real data set.

4.1 Econometric modelling

Our basic assumptions are the following:

- N participants;
- a vector of random variables representing the individual private values, i.e.,
$$\tilde{s}_i \longrightarrow \text{i.i.d. } U_{[a,a+h]};$$
- prior density functions on the parameters defining the support $[a, a+h]$

$$\pi(a) \propto \mathbf{I}_{[\underline{a},\overline{a}]} \tag{2}$$

$$\pi(h) \propto \mathbf{I}_{[\underline{h},\overline{h}]} \tag{3}$$

In economic models, the distribution of private values is assumed common knowledge among the participants. Instead, from an econometric viewpoint the parameters of the model are unknown. We are naturally led to consider two cases depending on whether the number of participants N is known to the econometrician or not.

The estimation of the parameters has a practical interest. Indeed, in our simple setting the seller does not know the distribution of private values. He could be interested in the estimation of, say, the largest possible selling price, i.e., $a + h$ and/or the number of active participants N if it is unknown.

The econometrician observes the selling price which is a random variable since

$$\tilde{p} \equiv \max_i [b(\tilde{s}_i)].$$

In our setting, the bidding function is easely derived. Assuming that the reserve price v is strictly smaller than a, we get

$$b(s_i) = \frac{N-1}{N} s_i + \frac{a}{N} \tag{4}$$

for $s_i \in [a, a+h]$.

4.2 The number of participants is known

By using expression (4) we can compute the conditional density function of p, that is

$$f(p \mid a, h, N) = \frac{1}{h^N} \frac{N^{N+1}}{(N-1)^N} (p-a)^{N-1} \mathbf{I}_{p \in [a, a+\frac{N-1}{N}h]}. \tag{5}$$

Bayes' rule allows us to combine expression (5) and priors on a and h in order to derive the posterior conditional distributions

$$f(a \mid p, h, N) \propto (p-a)^{N-1} \times \mathbf{I}_{a \in [\underline{a}, \overline{a}]} \times \mathbf{I}_{a \in [p - \frac{N-1}{N}h, p]} \tag{6}$$

and

$$f(h \mid p, a, N) \propto \frac{1}{h^N} \times \mathbf{I}_{h \in [\underline{h}, \overline{h}]} \times \mathbf{I}_{h > (p-a)\frac{N}{N-1}} \tag{7}$$

As usual the usefullness of the observations is captured by comparing the prior and the posterior distributions. In our setting there is only one observation which consists of the couple: price, number of participants. So the content of the observation may look very small. Note however that the posterior distribution may be very concentrated compared to the prior. First, consider the support of the posterior distribution. The sets $[\underline{a}, \overline{a}] \cap [p - \frac{N-1}{N} h, p]$ and $[\underline{h}, \overline{h}] \cap [(p-a)\frac{N}{N-1}, +\infty]$ are represented below (see figure 1) for $N = 2$. It is seen that the reduction induced by a single observation in the support of a may be large. It is only in the case represented in figure 1c that no reduction occurs. In other terms if we know *a priori* that the signals are not too dispersed then the data may be very informative. Second, if we now look at the conditional distributions it is seen that the role of N is crucial. Indeed, if N is large, then the observed price will be close to the maximum of the support so the conditional distribution of both h and a are heavily concentrated. This last point is very intuitive since $a + h$ is the maximum value for the signal. Thus, given a (resp. h), observing the price is very informative for h (resp. a) so that inference may be reasonably precise even with a single observation.

In our setting, the posterior distributions can be tabulated. Note, however, that this table would involve six parameters, that is $\underline{a}, \overline{a}, \underline{h}, \overline{h}, p, N$. So we restrict the analysis to some particular cases. Figure 2.a and 2.b represent the posterior distributions for a and h respectively in the case $N = 2, \underline{a} = 0, \overline{a} = 100, \underline{h} = 5, \overline{h} = 10, p = 25$.

The numerical example corresponds to the case described in figure 1.a. These densities have been obtained by Monte Carlo Markov Chain techniques. Indeed equations (6) and (7) allow for a Gibbs Sampling algorithm described in appendix 1. Furthermore, figure 2.c shows the joint conditional posterior p.d.f. of a and h. It is worth pointing out that the "observation" of one price can provide information about the lowest possible valuation of the object, that is, the parameter a. Indeed, the posterior conditional p.d.f. of a is highly concentrated between 22 and 25.

4.3 The number of participants is unknown

Throughout the previous subsection, we have explicitly assumed that the econometrician knows the number of participants. In pratice this is almost never the case. It may be argued that the number of people attending the

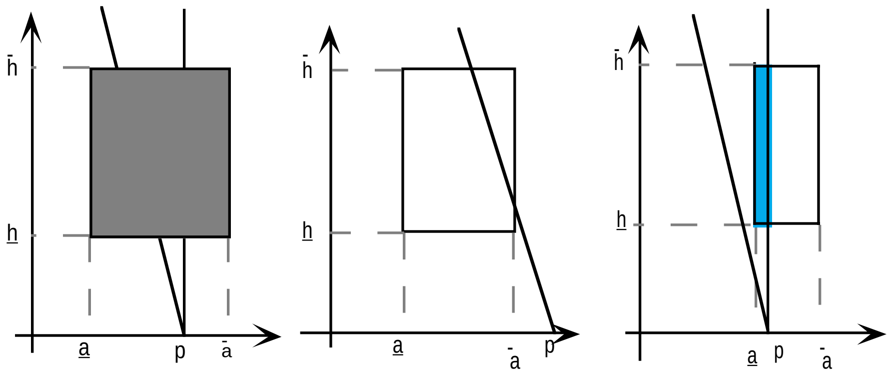

Figure 1: Supports of posterior distributions

auctions is directly linked with the number of participants. This is not so obvious as it looks at first glance. Indeed the auctions are usually grouped so some people attending a given auction may be interested in another item. In this respect, the number of people attending the auctions is larger than the number of participants. On the other hand, some active participants may be absent (see Ginsburgh (1997) for such a case in the context of wine auction).

According to our general approach, we treat N as a parameter. Choosing a prior distribution for N should take into account the characteristics of real auctions. The auction we consider in the empirical part of the paper is an art auction. So people attending the auction are typically interested in a few items. Suppose for the moment that L items are to be sold in L auctions and that M people are attending the auctions. We assume that the probability that a potential buyer is interested in a given item is equal to $1/L$. This corresponds to the case where each potential buyer is interested in exactly one item, but the econometrician ignores which participant is interested in which item. Thus the econometrician is bound to assign equal probabilities to the event that a given potential buyer is willing to buy a specific object.

If potential buyers behave "independently"[8], then the number of active participants to a given auction is distributed as $\mathcal{B}(M, 1/L)$. If M and L are

[8]We role out the case where a given potential buyer is interested in a given item for the mere reason that another participant is willing to buy it.

large, then we approximately have $\mathcal{B}(M, 1/L) \simeq \mathcal{P}(M/L)$. Recall that the auction model requires the number of participant to be at least equal to 2. Then, we are naturally led to choose a Poisson distribution as a prior for N with a lower truncation at 2.

The only conditional posterior distribution we need to compute is that of N :

$$f(N \mid p, a, h) \propto \left(\frac{\lambda}{h}\right)^{N-1} \left(\frac{N}{N-1}\right)^{N} \frac{(p-a)^{N-1}}{(N-1)!} \mathbf{I}_{N \in [2,+\infty]} \times \mathbf{I}_{N \geq \frac{h}{a+h-p}}, \quad (8)$$

where $\lambda = M/L$, and where the density f has to be understood w.r.t the counting measure on $\mathbb{N}$. In appendix 1 we describe how to draw in the previous distribution.

For what concerns the precision of the estimation, it is worth pointing out that we still use one observation, i.e., the selling price. It is clear that the estimations cannot be more precise that in the previous subsection, where N is known. Nevertheless, the usefullness of the data has to be evaluated form two viewpoints : the supports of the posterior distributions and their shapes. Concerning the first point, the effect of a random N is minor since it only affects the slope of the line $a + \frac{N}{N-1}h = p$ in the (a, h) plane (see again, figure 1). Concerning the shapes, the effect is much stronger. Indeed a high value of N produces more concentrated posterior conditional distributions for both a and h.

5 Inference in repeated auctions

The previous section has a slight provocative flavour. As a matter of fact, very little inference can be made with a single observation. We are naturally led to examine the case of repeated auctions. Fullfill the requirement of a genuine structural econometric estimation is more difficult in this context.

5.1 Description of the mechanism

We consider the situation where L items are to be sold in L first-price auctions. The first problem is to derive bidders' strategies. This point definitely calls for a repeated game theory framework which raises two related complications.

First, we are not aware of close form equilibrium strategies in the context of repeated first price auctions. Consequently, we cannot directly use an approach similar to that of the previous section. Second, a general result of repeated game theory is that strategies involve a dynamic behaviour. It is only in the case where the L items are sold in different L auctions with no common bidder across the auctions that the bidding strategies are simple repetitions of the one-shot case. In this last case the joint distribution of the selling prices would be i.i.d. This distribution would write:

$$f(p_1,\ldots,p_L \mid a,h,N) = \frac{1}{h^{LN}}\frac{N^{LN+L}}{(N-1)^{LN}}\prod_{i=1}^{L}(p_i-a)^{N-1}\mathbf{I}_{p_i\in[a,a+\frac{N-1}{N}h]}.$$

In addition to the theoretical economic problems discussed above, we could face severe statistical difficulties with this formulation. The hardest one is that the support of the posterior distributions of a and h may be empty. If we look again at figure 1, we see that this could be the case if the selling prices are "sufficiently" dispersed. Of course, we could avoid this difficulty by enlarging the prior distribution for h. But this would violate the Bayesian paradigm, since the prior cannot be adjusted to data.

From an econometric viewpoint, one appealing solution to introduce dynamic patterns would be to allow for some dependency across prices of the different repetitions in a fashion similar to time series modelling. At each repetition we allow the distribution of the selling prices to depend on the previous selling prices. Then, our repeated auction framework is built around two main points:

1. We assume that each repetition is independent from the others so that individual strategies correspond to the one-shot auction case;

2. We drop the assumption of identically distributed prices by allowing some heterogeneity across the different repetitions.

In the following subsection we address the problem of heterogeneity.

5.2 Heterogeneity problem

There are various possibilities to introduce some heterogeneity in the simple framework we have just described.

The first possibility is to let the number of participant vary across repetitions. Then, the distribution of the selling prices would be :

$$f(p_1,\ldots,p_L \mid a,h,N) = h^{-\sum_i N_i} \prod_{i=1}^{L} \frac{N_i^{N_i+1}}{(N_i-1)^{N_i}} (p_i-a)^{N_i-1} \mathbf{I}_{p_i \in [a, a+\frac{N_i-1}{N_i}h]}.$$

This framework could be used, for instance, in the case of auctions for highly homogenous items.

The heterogeneity may also come from the support of the signals. One possible formulation could be the following model :

$$f(p_1,\ldots,p_L \mid \alpha,\beta,h,N_1,\ldots,N_L,x_1,\ldots,x_L) =$$
$$h^{-\sum_i N_i} \prod_{i=1}^{L} \frac{N_i^{N_i+1}}{(N_i-1)^{N_i}} (p_i-a_i)^{N_i-1} \mathbf{I}_{p_i \in [a_i, a_i+\frac{N_i-1}{N_i}h]}, \tag{9}$$

where $a_i = \alpha + \beta' x_i$.

This last formulation allows for some dynamics in the sense described above. Indeed, the explanatory variables may include some past realisations of the selling prices. This is exactly what we have done for the empirical analysis presented below. More precisely, we choose :

$$x_1 = 0, x_2 = p_1, \ldots x_L = p_{L-1}.$$

The estimation of this model may done thanks to an algorithm using described in appendix 2.

An interesting problem is raised by the introduction of the past prices in the distribution of the signals. Suppose N is constant for sake of simplicity. The conditional distribution of a given selling price is

$$f(p_i \mid \alpha,\beta,p_{i-1},h,N) =$$
$$\tfrac{1}{h^N} \tfrac{N^{N+1}}{(N-1)^N} (p_i-\alpha-\beta p_{i-1})^{N-1} \mathbf{I}_{p_i \in [\alpha-\beta p_{i-1}, \alpha-\beta p_{i-1}+\frac{N-1}{N}h]},$$

Thus, the selling price process is clearly markovian. We will now assess the problem of existence of an invariant distribution for the prices. The following proposition makes rigorous this last point.

Proposition 1 *If $\alpha + p_1 > 0$, then $\mid \beta \mid < 1$ is a necessary and sufficent condition for the existence of an invariant distribution.*

Proof :
First, assume that $\beta \neq 1$.
Consider :

$$q_i = p_i - \frac{\alpha}{1-\beta}.$$

By direct computation, we have :

$$f(q_i \mid q_{i-1}) = \frac{n}{\eta^N}(q_i - \beta q_{i-1})^{N-1}\mathbf{I}_{q_i \in [\beta q_{i-1}; \beta q_{i-1}+\eta]},$$

where $\eta = \frac{N-1}{N}h$.

Again, by direct computation, we get :

$$f(q_i - \beta q_{i-1} = x \mid q_{i-1}) = \frac{N}{\eta^N}x^{N-1}\mathbf{I}_{x \in [0;\eta]},$$

so $q_i - \beta q_{i-1}$ is an i.i.d process. More precisely, q_i appears as an AR(1) process whose innovations are distributed as the maximum of N independant drawings in a uniform distribution in $[0, h]$. Thus we deduce that a sufficient condition for the existence of an invariance measure is $\mid \beta \mid < 1$. Moreover, in such a case, this measure is unique. This also shows that no invariant measure exists if $\mid \beta \mid > 1$.

Finally in the case $\beta = 1$ it is directly seen that the sequence of the minimum values of the price process is bounded from below by the following strictly increasing sequence $\alpha + p_1, 2(\alpha + p_1), 3(\alpha + p_1), \ldots$. It follows that no invariant measure may exists. ■

In the case where the invariant measure exists, it is possible to compute some “long term” characteristic of the price distribution. For instance, the maximum of the long term selling prices is $\frac{\alpha+\eta}{1-\beta}$. Moreover, if p_1 is strictly smaller than this value, this maximum provides a uniform upper bound of the price process. In a similar fashion, the (marginal) expected selling price is, in the long term :

$$\frac{N}{N+1}\frac{\eta}{1-\beta} + \frac{\alpha}{1-\beta} = \frac{N-1}{N+1}\frac{h}{1-\beta} + \frac{\alpha}{1-\beta}$$

5.3 Inference on a real data set

We estimate the model described in equation (9) thanks to an algorithm described in the appendix 2. The data come from a real auction that took place in London on the 6th december 1993. Ten individually hand-made miniature sculptures by Peter Fagan were sold at prices ranging from 900 to 1050 British pounds. The auction was organized by Christie's[9]

The heterogeneity of the prices may come from the fact that the statuettes are not copies one of the other. Thus it seems reasonable that the number of active participants is not constant accross the auctions. However the serie of the selling prices presented in table 1 below shows that the selling prices varies much more in the five first auctions than in the last ones. As we explained in the previous subsection this type of heterogeneity may be captured by the β parameter. The dynamic feature of the price pattern appears clearly in the figure 3.b.

Table 1 : Selling prices

auction number	1	2	3	4	5	6	7	8	9	10
prices	950	900	900	1050	900	950	900	900	900	900

The shape of the posterior density of the β parameter is easily explained by looking at the price dynamics. Indeed, the smallest mode corresponds to the fact that prices are "broadly" decreasing. The second mode around zero correspond to the small variance episode of the five last auctions.

It also appears in figure 3.d that the number of participant is likely to be larger in the forth auction (the highest selling price). Contrary to the previous one, this type of heterogeneity is mainly static.

It is difficult to find a clear interpretation of the α parameter. First, it appears as the smallest possible selling price for the first auction. However, since β appears to be different from zero, this interpretation is incorrect for the other auctions. The discussion of the previous section shows that α also appears in the expressions the marginal distribution of the prices. So α also plays a role in the "long term" behaviour of the selling prices.

[9]We are grateful to V. Ginsburg for providing us these data.

6 Conclusion

In this paper we have proposed a Bayesian approach for the econometrics of first-price auctions. We have argued that the Bayesian paradigm is more suitable to a structural approach in empirical analyses of strategic behaviours than its frequentist counterpart. Our main argument is that there is no clear asymptotics to rely on.

We have developed some Bayesian inference methods for the Independent Private Value model. In discussing the single auction case we have chosen a particular class of priors, that is, flat priors. The main reason relies on computational tractability to derive the posterior distribution of the selling price.

We have largely underlined that a repeated auction scheme definitely calls for a dynamic game framework. Given the complexity and the large number of possible equilibrium strategies in repeated auctions, we have elaborated a reasonable "compromise". The assumption of independence across repetitions has allowed the use of the one-shot equilibrium strategy. Instead, the introduction af explanatory variables for selling prices has introduced a true dynamic behavior in the sequence of prices. Finally, we have implemented our model on a real data set.

The Bayesian approach may easely apply to many other parametric auction models, such as the Common Value model. A test between the two paradigms (IPV vs CV) may be performed as in Paarsch (1992). This study is indeed the object of a companion paper[10].

[10] See Albano and Jouneau (1998).

Appendices

The aim of this section is to describe the algorithms used. The first appendix deals with the single auction case. The second appendix sketches the approach for the repeated auction case. It will be clear that we can mainly rely on the single auction case, so we only sketch the main technics. An algorithm written in Gauss is available upon request (a downloadable version is available on `www.univ-lille3.fr\gremars\jouneau\gauss\prg`).

Appendix 1

The problem is to draw random variables from the three following density functions:

$$\begin{aligned}
\pi_h(h;\theta,h_m,h_M) &\propto h^{-\theta}\,\mathbb{I}_{h\in[h_m;h_M]};\theta\geq 2\\
\pi_a(a;\theta,a_m,a_M,p) &\propto (p-a)^{\theta-1}\,\mathbb{I}_{a\in[a_m;a_M]};\theta\geq 2\\
\pi_N(N;\alpha,N_m) &\propto \left(\frac{N}{N-1}\right)^N \frac{\alpha^{N-1}}{(N-1)!}\,\mathbb{I}_{N\in\{N_m,\ldots,+\infty\}}.
\end{aligned}$$

Simulation algorithm for density π_h

Note that the c.d.f. of this p.d.f is easily computed. Moreover the corresponding p.d.f. can be explicitely inverted. Thus we can rely on a inversion algorithm.

Indeed we have :

$$\int_{h_m}^{h} x^{-\theta}dx = \frac{1}{\theta-1}\left(h_m^{(1-\theta)} - h^{(1-\theta)}\right).$$

Thus, the cumulative function Π_h is

$$\Pi_h(h;\theta,h_m,h_M) = \frac{h_m^{(1-\theta)} - h^{(1-\theta)}}{h_m^{(1-\theta)} - h_M^{(1-\theta)}}\mathbb{I}_{h\in[h_m;h_M]}.$$

Finally, we get

$$\Pi_h^{-1}(u;\theta,h_m,h_M) = \left\{h_m^{1-\theta} - \left(h_m^{1-\theta} - h_M^{1-\theta}\right)u\right\}^{\frac{1}{1-\theta}}.$$

Then, $\tilde{h}$ can be drawn by using the following algorithm:

$[\mathrm{A}_1](\theta, h_m, h_M)$

`Step 1` : draw $\tilde{u} \sim U_{[0,1]}$
`Step 2` : let $\tilde{h}$ be $\Pi_h^{-1}(\tilde{u};\theta,h_m,h_M)$.

Simulation algorithm for density π_a

For the second p.d.f we have

$$\int_{a_m}^{a} (p-x)^{\theta-1}dx = \frac{1}{\theta}\left((p-a_m)^\theta - (p-a)^\theta\right).$$

Thus, the cumulative function Π_a is

$$\Pi_a(a;\theta,a_m,a_M,p) = \frac{(p-a_m)^\theta - (p-a)^\theta}{(p-a_m)^\theta - (p-a_M)^\theta}\mathbf{I}_{a\in[a_m;a_M]}.$$

Finally, we get

$$\Pi_a^{-1}(u;\theta,a_m,a_M,p) = p - \left\{(p-a_m)^\theta - \left((p-a_m)^\theta - (p-a_M)^\theta\right)u\right\}^{\frac{1}{\theta}}.$$

Then, $\tilde{a}$ can be drawn by using the following algorithm:

$[\mathrm{A}_2](\theta, a_m, a_M, p)$

`Step 1` : draw $\tilde{u} \sim U_{[0,1]}$
`Step 2` : let $\tilde{h}$ be $\Pi_a^{-1}(\tilde{u};\theta,a_m,a_M,p)$.

Simulation algorithm for density π_N

Since $\tilde{N}$ is an integer, the last p.d.f. is a bit more involving.

First note that this distribution is very close to that of a truncated Poisson distribution. Indeed we have :

$$4 \geq \left(\frac{N}{N-1}\right)^N \geq \exp(1).$$

Thus $\pi_N(N;\alpha,N_m)$ is bounded from above and below by two left-truncated Poisson distribution densities. Powerfull simulation algorithms for Poisson distribution are provided by Devroye (1985) or Atkinson (1979). An overview of this simulation techniques is given in Robert (1997). Since the truncation parameter is small in practice, we may rely on a simple acceptance rejection algorithm for π_N.

Let $\pi_{\mathcal{P}(\alpha)}$ be the p.d.f of the Poisson distribution with parameter α. The algorithm is then:

$[\mathrm{A}_3](\alpha, N_m)$

`Step 1` : draw $\tilde{n} \sim \mathcal{P}(\alpha)$, $\tilde{u} \sim U_{[0,1]}$

`Step 2` : Accept $\tilde{N} = \tilde{n}+1$ if $4\tilde{u} \times \pi_{\mathcal{P}(\alpha)}(\tilde{n}+1) \leq \pi_N(\tilde{n}+1;\alpha,N_m)$ and $\tilde{n} \geq N_m - 1$, otherwise go back to Step 1.

6.1 Complete algorithm

Thus in the single auction case when the selling price is p and the number of bidders N is known, the tth step of the Gibbs-sampling algorithm is simply :

`Step a` : Let

$$\begin{array}{ll} a_m(t) & = \max\left\{\underline{a}, p - \tilde{h}(t-1)\frac{N-1}{N}\right\}; \\ a_M(t) & = \min\left\{\overline{a}, p\right\}; \end{array}$$

draw $\tilde{a}(t) \sim [\mathrm{A}_2](N, a_m(t), a_M(t), p)$.

`Step h` : Let

$$\begin{array}{ll} h_m(t) & = \max\left\{\underline{h}, \frac{N}{N-1}(p - \tilde{a}(t))\right\}; \\ h_M(t) & = \overline{h}; \end{array}$$

draw $\tilde{h}(t) \sim [\mathrm{A}_1](N, h_m(t), h_M(t))$.

When the number of bidders is unkown this algorithm becomes

`Step a` : Let

$$\begin{aligned} a_m(t) &= \max\left\{\underline{a}, p - \tilde{h}(t-1)\tfrac{\tilde{N}(t-1)-1}{\tilde{N}(t-1)}\right\}; \\ a_M(t) &= \min\left\{\overline{a}, p\right\}; \end{aligned}$$

draw $\tilde{a}(t) \sim [\mathrm{A}_2](\tilde{N}(t-1), a_m(t), a_M(t), p)$.

`Step h` : Let

$$\begin{aligned} h_m(t) &= \max\left\{\underline{h}, \tfrac{\tilde{N}(t-1)}{\tilde{N}(t-1)-1}(p - \tilde{a}(t))\right\}; \\ h_M(t) &= \overline{h}; \end{aligned}$$

draw $\tilde{h}(t) \sim [\mathrm{A}_1](\tilde{N}(t-1), h_m(t), h_M(t))$.

`Step N` : Let

$$N_m(t) = \max\left\{2, \frac{\tilde{h}(t)}{\tilde{a}(t) + \tilde{h}(t) - p}\right\};$$

draw $\tilde{N}(t) \sim [\mathrm{A}_3]\left(\frac{\lambda(p-\tilde{a}(t))}{\tilde{h}(t)}, N_m(t)\right)$.

Appendix 2

In the repeated auction case we observe prices $p_1, \ldots, p_L$. Recall that the conditional distribution of the observed prices is :

$$f(p_1, \ldots, p_L \mid \gamma, h, N_1, \ldots, N_L, y_1, \ldots, y_L) =$$

$$h^{-\sum_i N_i} \prod_{i=1}^{L} \frac{N_i^{N_i+1}}{(N_i-1)^{N_i}} (p_i - \gamma' y_i)^{N_i - 1} \mathbf{I}_{p_i \in [\gamma' y_i, \gamma' y_i + \frac{N_i - 1}{N_i} h]},$$

where $y_1, \ldots, y_L$ are some observed explanatory variables (this formulation includes the constant).

It is clear that the posterior density for h is of the same type as in the single auction case. Moreover, it is easily seen that the random variables $N_1, \ldots, N_L$ are *ex post* independent. As a matter of fact, the posterior density of $(N_1, \ldots, N_L)$ may be written as a product of densities of the same type as π_N. Thus the drawings in the posterior density of $(N_1, \dots, N_L)$ may be done independently thanks to the algorithm [A$_3$]. The only new problem comes from the simulation of the γ parameter. We choose for this parameter a flat prior. The posterior is always proper thanks to the truncations induced by the index functions $\mathbf{I}_{p_i \in [\gamma' y_i, \gamma' y_i + \frac{N_i - 1}{N_i} h]}$. We propose a componentwise algorithm, so the problem is to draw in a distribution of the following type :

$$f(\gamma_k \mid h, q_1, \dots, q_L, y_{1,k}, \dots, y_{L,k}, N_1, \dots, N_L) =$$

$$\prod_{i=1}^{L} (q_i - \gamma_k y_{i,k})^{N_i - 1} \mathbf{I}_{q_i \in [\gamma_k y_{i,k}, \gamma_k y_{i,k} + \frac{N_i - 1}{N_i} h]}.$$

Now this function of γ_k is log-concave, whatever the signs of the $y_{i,k}$'s. So we may apply a universal algorithm proposed by Gilks and Wild (1992).

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