

Non-local Linearization of Nonlinear Differential Equations via *Polyflows*

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Abstract—Motivated by the mathematics literature on the algebraic properties of so-called “polynomial vector flows”, we propose a technique for approximating nonlinear differential equations by linear differential equations. Although the idea of approximating nonlinear differential equations with linear ones is not new, we propose a new approximation scheme that captures both local as well as global properties. This is achieved via a hierarchy of approximations, where the N th degree of the hierarchy is a linear differential equation obtained by *globally* approximating the N th Lie derivatives of the trajectories.

We show how the proposed approximation scheme has good approximating capabilities both with theoretical results and empirical observations. In particular, we show that our approximation has convergence range at least as large as a Taylor approximation while, at the same time, being able to account for asymptotic stability (a nonlocal behavior). We also compare the proposed approach with recent and classical work in the literature.¹

I. INTRODUCTION

The constantly increasing scale and complexity of modern engineering systems has led to a renewed interest in the development of analysis techniques for nonlinear systems. There is a great variety of techniques, all of which arguably come with their advantages and weaknesses. Let us mention: **i)** exact methods, that leverage particular algebraic structures in the system equations such as feedback linearizability or flatness (e.g., [2]); **ii)** optimization methods, such as Lyapunov or sum-of-squares, that can produce satisfactory answers, but rarely come with guarantees of efficiency because of nonconvexity of most problems/systems (e.g., [3]); **iii)** other methods relying on the approximation of nonlinear systems by linear ones with the goal of applying powerful techniques from linear systems theory. In this last family, let us mention linearization (e.g., nearby an equilibrium point) or infinite-dimensional approaches like the Carleman linearization or the Koopman-operator approach [4], [5], [6]. This list is far from exhaustive, but as the reader will see, the ideas developed in this paper bear similarities with these three approaches, while at the same time trying to cope with their disadvantages.

Our goal is to approximate the solutions of a nonlinear

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¹A full version of this work, containing all the proofs, can be found in [1]

differential equation of the form:

$$\dot{x} = f(x), \quad (1)$$

where f is a given function from \mathbb{R}^n to \mathbb{R}^n . We denote by $\psi(x_0, t)$ the solution of (1) with initial condition x_0 at time $t = 0$. We assume that f is C^∞ (i.e., infinitely differentiable). Our goal is to find an approximation that leads to a closed-form expression for $\psi(x_0, t)$. However, rather than approximating the trajectory $\psi(x_0, t)$, we wish to *approximate the differential equation* (1) with another one, which we can analyze with exact techniques. More precisely, we will approximate (1) with (a projection of) a linear differential equation:

$$\dot{z} = Az, \quad (2)$$

where $z \in \mathbb{R}^N$ and $A \in \mathbb{R}^{N \times N}$ for some $N \in \mathbb{N}$. The goal is to account for non-local properties of the trajectory, as for instance, stability, limit cycles, etc. As we will see in Section VI, our approximated systems are able to reproduce such global behaviours, and we are thus hopeful that on a longer term, one could extend this technique to approximate feedback-linearization of nonlinear systems, or even approximate optimal control, by applying classical linear optimal control techniques on the approximating linear system (2) rather than on the initial system.

A. Related work

The literature on nonlinear systems is large (see [7] for a general account) and techniques to analyze nonlinear dynamical systems are countless. Closer to our setting are techniques that approximate nonlinear systems by linear, but infinite dimensional systems. Among them, Carleman Linearization [4], [8], or the Koopman approach [5], [9], [10], have recently attracted attention while dating back to the 1930s. The main idea in these methods is to represent the dynamical system as a linear operator acting on an infinite dimensional Hilbert space (the space of functions). This functional view brings linearity but requires working on infinite dimensional spaces. In practice, a particular basis for the infinite-dimensional Hilbert space is chosen, and then the linear operator describing the dynamics is truncated for numerical analysis purposes. Even though these techniques bear some similarity with ours, we do not perform truncation but rather a more detailed approximation directly resulting in a finite dimensional linear differential equation. For instance, in [5], the approximation relies on the choice of a basis of polynomials in order to approximate the nonlinear behavior, which is not the case in our method. However we believe

that there are interesting connections to be made with these approaches, and we leave for further work a more detailed analysis of these connections.

Other approaches rely on linearization nearby a particular point in the state space [11], or more generally aiming at generating a linear system by identifying coefficients with the Taylor expansion of the vector field f [12]. Compared with these works, our approach differs in that one of our goals is to approximate the system globally and not at a particular point in the state space.

Finally, several works have analyzed particular nonlinear systems that are *actually equivalent* to linear systems modulo some change of coordinates (e.g., [13] and [14]). Further extensions of this line of work considered the embedding of nonlinear systems into higher-dimensional linear systems [15]. While the motivation is the same, our work aims at analyzing *arbitrary* nonlinear systems. It can be seen as an approximate version, projecting in some suitable sense arbitrary systems onto such a nice one enjoying exact linearizability properties.

II. LIE DERIVATIVES AND TAYLOR APPROXIMATIONS

The main tool we will use in order to generate a suitable linear system (2) is the concept of Lie derivative, which we now review.

Definition 1 (Lie derivative): Consider the differential equation (1) and let $g : \mathbb{R}^n \rightarrow \mathbb{R}$ be a C^∞ function. For $N \in \mathbb{N}$, the N -th Lie derivative of g with respect to (1) is defined as follows:

$$\begin{aligned} \mathcal{L}_f^0 g &= g & N = 0, \\ \mathcal{L}_f^N g &= \frac{\partial \mathcal{L}_f^{N-1} g}{\partial x} f & \forall N \geq 1. \end{aligned} \quad (3)$$

We denote the evaluation of $\mathcal{L}_f^N g$ at x_0 by $\mathcal{L}_f^N g(x_0)$. If g is a vector valued function, $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$, we still denote by $\mathcal{L}_f^N g(x)$ the vector of the m Lie derivatives, i.e., the vector v such that $v_i = \mathcal{L}_f^N g_i(x)$.

Let us also review two very classical ways of approximating the solution of a differential equation. In doing so we denote by 1_K the identity function on a set K .

Definition 2 (Taylor approximation): The Taylor approximation τ_N at $t = 0$, and of order $N \in \mathbb{N}$, of the solution $\psi(x_0, t)$ of (1) is the polynomial:

$$\tau_N(x_0, t) = \sum_{i=0}^N \mathcal{L}_f^i 1_{\mathbb{R}^n}(x_0) \frac{t^i}{i!}. \quad (4)$$

By increasing N , the Taylor approximation τ_N will converge to $\psi(x_0, t)$ as long as t belongs to the *radius of convergence* which is defined as the largest number $R \in \mathbb{R}_{+0}$ such that for every $t \in]-R, R[$ the series:

$$\sum_{i=0}^{\infty} \mathcal{L}_f^i 1_{\mathbb{R}^n}(x_0) \frac{t^i}{i!},$$

converges.

We recall the Hadamard formula for the radius of convergence.

Theorem 1 (Hadamard): The radius of convergence of the Taylor approximation $\tau_N(x_0, t)$ is given by:

$$R^{-1} = \limsup \left(\frac{1}{i!} |\mathcal{L}_f^i 1_{\mathbb{R}^n}(x_0)| \right)^{1/i}. \quad (5)$$

In case the system has dimension n larger than one, the notation $|\cdot|$ above denotes the 1-norm (sum of the absolute values of the entries of a vector).

A simpler linearization technique consists in retaining the linear part of (1) and eliminating all the nonlinear terms.

Definition 3 (Linearization): Denoting the Jacobian of f at $x^* \in \mathbb{R}^n$ by:

$$T_{x^*} f = \left. \frac{\partial f}{\partial x} \right|_{x=x^*},$$

we define the *linearization at $x^* \in \mathbb{R}^n$* of (1) as the linear differential equation:

$$(\dot{x} - x^*) = T_{x^*} f(x - x^*). \quad (6)$$

III. INTRODUCING POLYFLOWS

Our work was first motivated by the mathematics literature where a great effort has been devoted to understand the properties of differential equations (1) for which the solution $\psi(x_0, t)$ is a polynomial function of the initial condition x_0 , hence the name *polyflow*, a portmanteau obtained by blending the terms *polynomial* and *flow* (see, e.g., [16], [15], [17], [18], [19]). We now provide an alternative definition of polyflows that makes the connection to our approximation problem clearer.

Definition 4: The solution $\psi(x_0, t)$ of the differential equation (1) is a *polyflow* if there exist $\ell \in \mathbb{N}$ and an injective smooth map $\xi : \mathbb{R}^n \rightarrow \mathbb{R}^\ell$ such that $\xi \circ \psi(x_0, t)$ is the solution of a linear differential equation on \mathbb{R}^ℓ with initial condition $\xi(x_0)$.

The map ξ embeds the nonlinear differential equation into a linear one on a typically higher-dimensional state space. If the linear differential equation is denoted as in (2), the previous definition requires the existence of a matrix $A \in \mathbb{R}^{\ell \times \ell}$ such that $T_x \xi \cdot f = A\xi$. When this type of equality holds, the vector fields $f(x)$ and Az are said to be ξ -related (see Def. 4.2.2 in [20]). The theorem below provides a more constructive description of this equality:

Theorem 2: [18] The solution $\psi(x_0, t)$ of the differential equation (1) is a polyflow if and only if there exists $N \in \mathbb{N}$ such that for every $n' \geq N$ and $k \in \{1, \dots, n\}$ there exist $\lambda_{i,j} \in \mathbb{R}$ satisfying:

$$\mathcal{L}_f^{n'} f_k = \sum_{i \leq N, j \leq n} \lambda_{i,j} \mathcal{L}_f^i f_j. \quad (7)$$

That is, polyflows are dynamical systems endowed with a nilpotency property: after a finite number of Lie derivatives, further differentiation only produces functions that are trapped in a finite dimensional vector space. This very same idea appeared in the control literature (e.g., [15]) related to the problem of embedding nonlinear systems into, possibly higher dimensional, linear systems.

Corollary 1: If the solution $\psi(x_0, t)$ of the differential equation (1) is a polyflow, then, there exist $N \in \mathbb{N}$,

coefficients $\Lambda_0, \dots, \Lambda_{N-1} \in \mathbb{R}^{n \times n}$, and $N - 1$ functions $y_i : \mathbb{R}_+ \rightarrow \mathbb{R}^n$, $i \in \{1, \dots, N - 1\}$, such that:

$$\begin{bmatrix} \dot{\psi} \\ \dot{y}_1 \\ \vdots \\ \dot{y}_{N-1} \end{bmatrix} = \begin{bmatrix} 0 & I & \dots & 0 \\ 0 & \dots & \dots & 0 \\ 0 & \dots & \dots & I \\ \Lambda_0 & \Lambda_1 & \dots & \Lambda_{N-1} \end{bmatrix} \begin{bmatrix} \psi \\ y_1 \\ \vdots \\ y_{N-1} \end{bmatrix}, \quad (8)$$

with initial conditions:

$$\psi(x_0, 0) = x_0, \text{ and } y_i(0) = \mathcal{L}_f^i 1_{\mathbb{R}^n}(x_0), i \in \{1, \dots, N-1\}. \quad (9)$$

The integer N will play an important role in the remainder of the paper and, for this reason, we call a polyflow satisfying the conditions of Corollary 1 a N -polyflow.

Using the language of Definition 4 we see that:

$$\xi = \begin{bmatrix} 1_{\mathbb{R}^n} \\ \mathcal{L}_f^1 1_{\mathbb{R}^n} \\ \vdots \\ \mathcal{L}_f^{N-1} 1_{\mathbb{R}^n} \end{bmatrix}, \quad T_x \xi \cdot f(x) = A \xi(x),$$

where A is the matrix defining the right hand side of the linear differential equation (8).

Corollary 1 may seem to imply that the polyflow property is very restrictive: the trajectory of a dynamical system which is a polyflow must be the projection of the trajectory of a linear system. Therefore, nonlinear behavior is only encoded in the initial conditions. However, we will give below both theoretical and empirical evidence that an arbitrary nonlinear differential equation is “close” to a polyflow (provided that the degree N is taken large enough).

IV. THE TECHNIQUE:

POLYFLOW APPROXIMATION OF NONLINEAR SYSTEMS

The Taylor approximation (4) proceeds by approximating the solution through the computation of its first derivatives. Rather than approximating the solution of the ODE, we directly approximate the differential equation itself, thereby hoping to obtain a global description of its solutions.

Our main idea leverages the observation that, when the right hand side f of (1) is a polynomial of degree d , its solution is a N -polyflow if and only if it belongs to some lower dimensional manifold in the space of polynomials of degree d . Indeed, one can see that in the characterization (7), one can restrict the equalities to $n' = N$ (see Proposition 2 below), and these are algebraic conditions on the coefficients of the polynomial f . Since polyflows have the nice property that they allow for a closed-form formula for their solutions, a natural idea is to “project” our system on the closest polyflow. We formalize this in the next two definitions:

Definition 5 (Polyflow Approximation): Consider the differential equation (1) and a compact set $K \subset \mathbb{R}^n$. We define a N -th polyflow approximation of (1) on K , to be any linear differential equation, as in (8), where the matrices $\Lambda_i \in \mathbb{R}^{n \times n}$ are obtained by approximating $\mathcal{L}_f^N 1_K$ with:

$$\sum_{i=0}^{N-1} \Lambda_i \mathcal{L}_f^i 1_K. \quad (10)$$

In the above definition, nothing is said on how to compute the parameters Λ_i nor the initial conditions even though the quality of the approximation will depend on these choices. In fact, it is not clear to the authors, at the moment, how to best define them. Most probably, there is no unique “right” choice, but the wisest strategy will depend on what precisely is the final objective, as is often the case when one resorts to approximations. We present here some natural strategies, even though others could be introduced.

Definition 6 (Numerical Strategies): Consider the differential equation (1) and a compact set $K \subset \mathbb{R}^n$. The coefficients Λ_i in Definition 5 can be defined in the following ways:

- I. For any given norm $|\cdot|$ define:

$$\tilde{f}_N := \operatorname{argmin}_{g \in S} \{|g - \mathcal{L}_f^N 1_K|\}, \quad (11)$$

where $S = \operatorname{span}_{\mathbb{R}}\{\mathcal{L}_f^i 1_K : i \leq N - 1\}$ and Λ_i is such that $\tilde{f}_N = \sum \Lambda_i \mathcal{L}_f^i 1_K$.

- II. For any given norm $|\cdot|$ define:

$$\tilde{g} := \operatorname{argmin}_{g \in S} \{|\mathcal{L}_g^N 1_K - \mathcal{L}_f^N 1_K|\}$$

where S is the set of vector fields for which the solution of the corresponding differential equations are polyflows and Λ_i is such that $\mathcal{L}_{\tilde{g}}^N 1_K = \sum \Lambda_i \mathcal{L}_f^i 1_K$.

If one represents a polyflow approximation like in (8), one has the following alternative definitions for the initial conditions:

- III. For any $i \in \{1, \dots, N - 1\}$, $y_i(0) = \mathcal{L}_f^i 1_K(x_0)$;
- IV. (In case Item 2 was chosen above) Denoting \tilde{g} as above, $y_i(0) = \mathcal{L}_{\tilde{g}}^i 1_K(x_0)$.

In Item 2 above, we pre-specify the degree of the vector field \tilde{g} , restrict it to be a polyflow, and then find such an optimum \tilde{g} . In Item 1, the approach is bolder: we only look at the Lie derivatives of the actual field f , and directly project the N th Lie derivative onto the space generated by the previous ones. Unless specified otherwise, in our numerical experiments, we take Items 1 and 3, together with the infinity norm:

$$|f| := \sup_{x \in K} \{f(x)\}.$$

In our numerical experiments, we solve the approximation problem by discretizing the compact set K , and solving the approximation problem on the obtained finite set of points. This can be done with standard Linear Programming. The approach scales well, as at step N , it only requires to project n polynomial scalar functions on a subspace spanned by $n(N - 1)$ polynomial functions. The number of necessary discretization points in order to reach a specified accuracy grows exponentially with the dimension of the state-space, though. Also, note that the degrees of the polynomial functions obtained at step N grow exponentially with N .

We finish this section by providing a detailed analysis of two examples. The first one is the trivial case where f is linear.

Example 1: Consider the scalar linear system:

$$\dot{x} = -\lambda x.$$

For any N , there are infinitely many solutions to the approximation problem (11) and the obtained approximation is exact (because all the Lie Derivatives are linear functions). However, there is one which is strictly better than the others: the one such that the polyflow has only one eigenvalue, equal to the eigenvalue of the local linearization, namely $-\lambda$. With this choice, the coefficients of the polyflow in (8) are ²:

$$\Lambda_i = -|\lambda|^{N-i} \binom{N}{i},$$

and all the successive polyflow approximations lead to the exact trajectory.

Example 2: We consider the two-dimensional system:

$$\dot{x} = \begin{bmatrix} x_1 + 2x_2^2 + x_2^3 - x_2^4 \\ -x_2 \end{bmatrix}. \quad (12)$$

The system is nonlinear, but as it turns out, it is a 4-polyflow. It is thus equivalent to a linear system, and, as a consequence, our approximation becomes exact after finitely many steps. We show this on Figure 1 (for $x_0 = (1, 1)$).

Let us detail the procedure for $N = 1$. First, the forms $x_1 + 2x_2^2 + x_2^3 - x_2^4$ and $-x_2$ are projected³ on $V = \text{span}_{\mathbb{R}}\{x_1, x_2\}$. We retrieve the coefficients defining the projection on the linear space V and obtain the approximating linear system:

$$\dot{x} = \begin{bmatrix} x_1 + 3.3143x_2 \\ -x_2 \end{bmatrix}. \quad (13)$$

Observe that, x_2 being a linear function in the true system, the projection is exact for this variable. Finally we compute the initial conditions $\mathcal{L}_f^i 1_K(x_0)$, $i < N$, and we obtain the approximate trajectory, represented in magenta in Figure 1. We do similarly for $N = 2$ and $N = 3$.

Now, since the true system is a 4-polyflow, the projection is exact for both variables at the fourth step ($N=4$), and the approximation is perfect, as one can see on Figure 1.

V. THEORETICAL RESULTS

In this section we provide theoretical arguments showing that our procedure comes with good approximation properties. We show that our approximation strategy can have performance guarantees that are at least as good as Taylor's approximation, while at the same time encapsulating asymptotic stability. Thus, our solution recovers an asymptotic property (i.e. asymptotic stability), but also satisfies the local properties of the Taylor approximation, *even if the point where the Taylor approximation is done is different from the equilibrium*. Our proof is an existence proof: we exhibit a theoretical construction for an approximating linear differential equation, which has the above-mentioned good properties. We emphasize that the construction in the proof is taken as simple as possible and, in particular, does not

²The notation $\binom{N}{i}$ denotes the binomial coefficient.

³We project numerically by discretizing the state space on $[0, 2]^2$ with discretization step 0.2. We minimize the 1-norm of the error vector as previously stated. Since the projection is based on the whole compact set of interest ($[0, 2]^2$ in this case) it captures global information about the system.

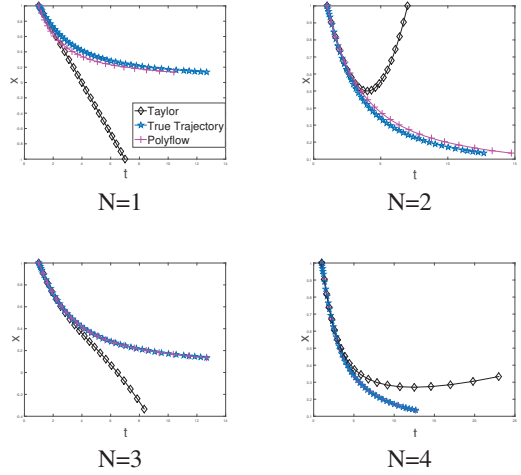


Fig. 1. Example 2: The approximation scheme (with $N = 1, 2, 3, 4$) applied on a two-dimensional 4-polyflow (only the x_1 -coordinate is represented). The fourth step provides a zero error, because the system is a 4-polyflow. One can see that even for $N = 1$, our approximation retrieves stability, contrary to the Taylor approximation. As we observed in all our experiments, the approximation performance is already very good for small N , and the error is barely noticeable for $N = 3$.

fully exploit the knowledge of the righthand side f in (1) as proposed in Definition 6. Thus, in practice, an optimized choice for the approximating linear differential equation can have an even better behavior than the one theoretically proven here, and indeed we do observe that our approximation outperforms the Taylor approximation in practice (see the examples above, and in Section VI). At the end of this section we discuss possible extensions.

A. Main result

Theorem 3: Let $\psi(x_0, t)$ be the solution of (1) with initial condition x_0 at time $t = 0$, let $\tau_N(x_0, t)$ be the Taylor approximation of order N of $\psi(x_0, t)$, and let $R(x_0)$ be its radius of convergence. There exists a sequence of N -polyflow approximations π_N satisfying:

$$\forall N \in \mathbb{N}, \quad \lim_{t \rightarrow \infty} \pi_N(x_0, t) = 0, \quad (14)$$

and such that for any initial condition $x_0 \in \mathbb{R}^n$ and for any $t \in]-R(x_0), R(x_0)[$ the successive approximations of the solution tend pointwise to the true solution, that is:

$$\forall t \in]-R(x_0), R(x_0)[, \quad \lim_{N \rightarrow \infty} \pi_N(x_0, t) = \psi(x_0, t). \quad (15)$$

Although Theorem 3 does not require the existence of an asymptotically stable equilibrium, this is one of the scenarios where it will be most useful since the approximating polyflows π_N are guaranteed to satisfy (14).

In the proof of the theorem, we express the approximating trajectory, solution of the polyflow approximation, as a Laurent series. Then, since the first N derivatives of this trajectory at $t = 0$ are equal to those of the Taylor polynomial, one observes that convergence of the polyflow approximations to the true value in the radius of convergence, is equivalent to convergence of the remainder to zero. Then,

one only needs to bound the larger derivatives at $t = 0$ of the polyflow approximation (i.e., of order larger than N); this is done by expressing these derivatives as the solution of a recurrence equation. We start by recalling a few technical results.

Proposition 1: Given a recurrence equation in \mathbb{R}^m :

$$X_n = \sum_{0 \leq i \leq N-1} k_i X_{n-N+i} \quad (16)$$

where $k_i \in \mathbb{R}^{m \times m}$, such that every entry in the matrices k_i has absolute value bounded by $K \in \mathbb{R}_+$, one has⁴:

$$\forall n \geq N-1, |X_n| \leq (mK+1)^{n-N+1} \sum_{0 \leq i \leq N-1} |X_i|.$$

Our second ingredient is an elementary property of linear differential equations:

Proposition 2: Given a linear differential equation, as in (8), for any $t \geq 0$, and for any $i \geq N$ the i th derivative $\phi^{(i)}(x_0, t)$ of the solution $\phi(x_0, t)$ satisfies:

$$\phi^{(i)}(x_0, t) = \sum_{0 \leq j \leq N-1} \Lambda_j \phi^{(i-N+j)}(x_0, t).$$

The proof is immediate and we omit it. We are now in position to prove Theorem 3:

B. Discussion

In fact, in the above theorem, at each step N , we place the N eigenvalues of the polyflow arbitrarily, without leveraging our knowledge of the system. Obviously, there should be an optimal way of choosing the coefficients, so that the mid-term approximation (i.e., for $R(x_0) \leq t < \infty$) performance is optimized while ensuring that the asymptotic behavior is stable and the successive approximations converge on $[0, R]$. We conjecture that such a procedure would enable convergence, not only on the Taylor radius of convergence, but on the *entire* interval of time $t \in [0, \infty]$:

Conjecture 1: Consider the system (1), and suppose that $x = 0$ is an asymptotically stable equilibrium. There exists a way of computing the polyflow coefficients from f , on some neighborhood K of 0 such that, for all $x_0 \in K$, the polyflow approximations satisfy

$$\forall t \geq 0, \lim_{N \rightarrow \infty} \pi_N(x_0, t) = \psi(x_0, t).$$

We suspect that these coefficients should be based on the Taylor coefficients of $f(x)$, in a similar way as in [12]. As a matter of fact, it is well known from approximation theory that linear combinations of exponentials are dense in the sets of functions on a compact [21], and then one can indeed approximate arbitrarily well any trajectory with a solution of a linear system. However, firstly, in our setting, we do not allow arbitrary coefficients for the linear combinations (these coefficients are determined by the initial conditions as in (9)) and secondly, we would like to compute the polyflow coefficients *implicitly*, that is, from the sole knowledge of f and not from the knowledge of the trajectories themselves.

⁴If $X_n \in \mathbb{R}^m$ are vectors (i.e., $m > 1$) we use $|X| := \sum_{1 \leq j \leq m} |X(j)|$.

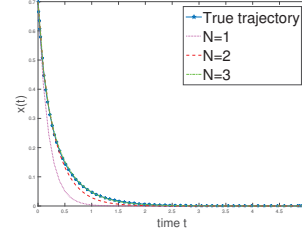


Fig. 2. Successive approximations of the trajectories (as a function of time) of the 1D polynomial system given by $f = -x^3 - 3x^2 - 2x$ for $N = 1, \dots, 3$. The projection is made on a discretization of the interval $[-0.1, 1]$ with a discretization step equal to 0.1. Already for $N = 3$, the difference between the trajectory and the approximation is barely noticeable.

VI. NUMERICAL EXAMPLES

In this section we report numerical examples on one- and two-dimensional systems.

A. One-dimensional systems

We observe in practice, for all the stable polynomial systems that we tried, that indeed all the polyflow approximations are stable, provided that we approximate the vector field in a compact set contained in the interior of the basin of convergence. This tends to confirm Conjecture 1. As an example, we represent in Figure 2 a one dimensional nonlinear (cubic) system and its successive polyflow approximations. One can see that the approximation works remarkably well, even for small values of N .

B. Limit cycles in dimension larger than one

Our long-term goal is to achieve more complex control tasks with the same kind of approximations, and as a first step beyond stability analysis, we investigate here the approximability of limit cycles. This property is quite challenging to reproduce with a linear system, as (nontrivial) limit cycles are numerically unstable for linear systems. However, we observed that for relatively short time-horizons, the approximations exhibit a cyclic behavior which tends to simulate the limit cycle. We also observed numerically that, in this case, some eigenvalues of the approximating linear differential equation have a real part significantly close to zero, which is encouraging. We leave a formalization of this observation for further research.

In Figures 3 and 4, we approximated the Lotka-Volterra system with our N -polyflow approximation for $N = 1, 2, 5, 6$. More precisely the system is

$$\begin{aligned} \dot{x} &= \alpha(x+1) - \beta(x+1)(y+1/2), \\ \dot{y} &= \gamma(x+1)(y+1/2) - \delta(y+1/2), \end{aligned} \quad (17)$$

with $\alpha = 2/3; \beta = 4/3; \delta = 1; \gamma = 1$. The numerical values are chosen such that a limit cycle exists, which is represented in blue in Figure 4. We observe that the polyflow approximations indeed have an approximate limit cycle (for $n = 7$, probably due to numerical errors, asymptotically the approximation diverges from this cycle).

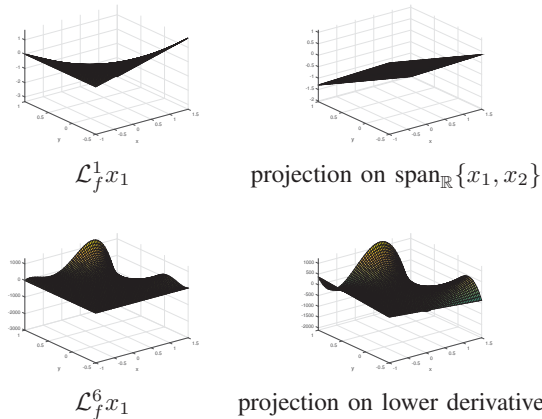


Fig. 3. The approximation of the N th Lie derivative of one particular coordinate, for $N = 1$ and $N = 6$.

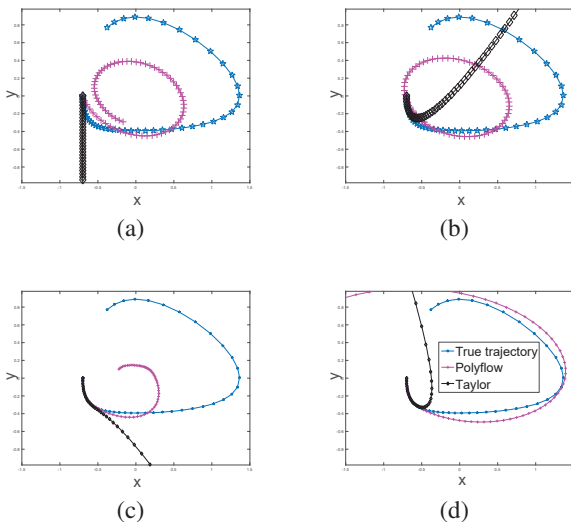


Fig. 4. The Lotka-Volterra limit cycle and its successive approximations ($N = 1, 2, 6, 7$). Comparison of the true trajectory, the Taylor approximation, and the polyflow approximation for several values of N . (The projection for the polyflow approximation is made with respect to the infinity norm on a grid $x \in [-1, 1.5]$; $y \in [-0.5, 1]$ with discretization step equal to 0.1.)

VII. CONCLUSION

In this paper we introduced a natural approach for analyzing nonlinear differential equations, by leveraging the notion of ‘polyflow’ from Algebra. Our technique is nonlocal in two aspects: it is nonlocal in the statespace, in that we approximate the system over a compact set (i.e. not at a particular point), and nonlocal in time, in that one of our goals is to account for asymptotic properties (e.g. asymptotic stability).

Despite its simplicity, we demonstrate that this scheme exhibits very good convergence properties, some of which being theoretically provable. We hope that such idea of ‘polyflow approximation’ can be pushed further in the future: First, we leave several questions open about the theoretical convergence properties. Second, we would like to use this

technique for more involved objectives than pure stability analysis. For instance, instead of projecting the nonlinear system on a linear one, one could take the same idea to define an approximate notion of feedback-linearization of nonlinear systems. Also, the notion of flatness of a nonlinear system, being intrinsically related to the nilpotence of Lie-differential operators, seems well suited for the same type of ideas. Finally, we plan to study the performance of our approach on other systems, and compare it with other techniques, like the Koopman approach.

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