Conclusion

This thesis is devoted to the derivation of lower and upper bounds on quantities of interest in actuarial science and finance and of the form $\mathbb{E}[\phi(X)]$, for some given ϕ, when X belongs to a class of random variables satisfying certain moment conditions (i.e. based on the knowledge of some partial information about X). The majority of the papers in this vein are devoted to random variables with support in a specified interval. In this thesis, we mainly deal with structured supports, as the set of the integers for instance. We mainly focus on the ways to find the best bounds for example on distribution functions, stop-loss premiums, adjustment coefficient, ruin probabilities and prices of contingent claims when the risks in presence are discrete and with known first $s-1$ first moments.

In this conclusion, we will concentrate on the possible extensions of the results derived in this thesis and on future research.

First, the scope of our study is the *reduced moment spaces* of order $s = 1, 2, 3, \ldots$, i.e. classes of probability laws sharing the same first $s-1$ moments (among others, mean, variance and skewness for $s = 4$). More generally, we could have worked with *moment spaces* of order $s = 1, 2, 3, \ldots$ where expectations of a set of functions ϕ_k $(k \leq s-1)$ would be fixed rather than expectations of the monomials x^k $(k \leq s)$. However, this is only possible when these function satisfy some conditions (for example, forming a Tchebycheff system).

Another possible extension that should be easy to achieve is to generalize the derivation of moment bounds on discrete stop-loss premiums to variables valued in an arbitrary grid $\{e_0, e_1, \ldots, e_n\}$ of points. It can also be extended to any number of moments superior to three.

In view of the dynamic setting part, there is an obvious connection between the papers devoted to extremal distributions that appeared in the actuarial literature and financial pricing in incomplete markets. This bridge between actuarial risk theory and financial mathematics seems to be a promising approach to the use of extremal distributions in finance. This promising approach also leaves some open questions. It is well-known that unimodality is often satisfied under the physical probability measure. Moreover, some improvements of the extrema with respect to $\preceq_{\mathrm{cx}}$ are possible when the underlying distributions are unimodal. In case where an equilibrium asset-pricing framework is specified, it can be seen that it is possible to derive relations between the physical distributions and the risk-neutral ones (see, e.g. Lo (1987) and De Schepper & Heijnen (2007)). Consequently, one challenging problem for future research is to investigate the possible transmission of unimodality to the class of risk-neutral distributions when such a relation is not known. The same problem could be investigated with ageing notions like "DFR".

Another interesting topic of research in the context of "deterministic immunization theory" could be to let the assets and liabilities be random variables instead of real numbers. In such a case, the post-shift change of the portfolio becomes a random variable and the immunization problem consists in finding conditions under which the expectation of the change in the portfolio value remains non-negative. To that end, a promising path to follow is to define the asset and liability risks as mixture of distributions with mixing variables being the random assets and liabilities. Then, immunization of the portfolio can be linked to an s-convex relation between the asset and liability risks and it remains thus to see how this relation can be used to find reliable immunization strategies.

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