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A TWO-POINT PROBLEM WITH NONLINEARITY DEPENDING ONLY ON THE DERIVATIVE*

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Abstract. We study the existence and multiplicity of solutions for a two-point boundary value problem at resonance in the first eigenvalue; the nonlinearity depends only on the first derivative and has finite limits at $\pm\infty$.

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1. Introduction and main result.

The two-point semilinear problem

$$\begin{gathered} u'' + u + g(u') = p(t), \\ u(0) = u(\pi) = 0, \end{gathered} \tag{1}$$

where g is a continuous function in $\mathbb{R}$ and p a continuous function in $[0, \pi]$, has been studied by Cañada and Drábek [3] and Kannan, Nagle, and Pothoven [4]. Related problems involving other boundary conditions are included in [3] and have also been the object of a paper by Mawhin [5].

It has been remarked (see [3], [2], and [4]) that conditions of the Landesman–Lazer type are not appropriated to yield the existence of solutions to (1). Let us denote by $\tilde{C}[0, \pi]$ the subspace of $C[0, \pi]$ consisting of functions $\tilde{u}(t)$ such that $\int_0^\pi \tilde{u}(t) \sin t \, dt = 0$. With respect to the direct sum $C[0, \pi] = \operatorname{span}\{\sin t\} \oplus \tilde{C}[0, \pi]$, every function $p \in C[0, \pi]$ has a decomposition

$$p(t) = \bar{p} \sin t + \tilde{p}(t), \quad \bar{p} \in \mathbb{R}, \ \tilde{p} \in \tilde{C}[0, \pi]. \tag{2}$$

In [3] the authors have used a result of Amann, Ambrosetti, and Mancini [1] to show that, given a bounded g, the solvability of (1) can be described in terms of the decomposition of p as follows: for each $\tilde{p}$ there exists a nonempty, bounded interval $I = I(\tilde{p})$ such that (1) has a solution if and only if (2) holds with $\bar{p} \in I$. They compute the upper bound of I in a case where

$$g(-\infty) := \lim_{u \to -\infty} g(u) = g(+\infty) := \lim_{u \to +\infty} g(u).$$

On the other hand, in [4], a contraction argument is used to show that, if $g(s) = \arctan s$, (1) is solvable for p sufficiently small.

In this note, we consider a nonlinearity based on the model studied in [4], and we give a new multiplicity result which in some sense completes the picture given in [3] and [4]; in particular, we prove that there are indeed two solutions of (1) when $\bar{p} \in \overset{\circ}{I}$ and $\bar{p} \neq \frac{2}{\pi}(g(-\infty) + g(+\infty))$. We now state precisely our main result.

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Theorem 1. *Let $g : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz continuous and such that the limits $g(-\infty)$ and $g(+\infty)$ exist and are finite. Let $p \in C[0,\pi]$ split according to* (2). *Then there exist real numbers $a = a(\tilde{p}, g) \le b = b(\tilde{p}, g)$ such that*

$$l := \frac{2}{\pi}(g(-\infty) + g(+\infty)) \in [a,b]; \tag{3}$$

and problem (1) *has*

(i) *no solution if $\bar{p} \notin [a,b]$;*
(ii) *at least one solution if $\bar{p} \in (a,b)$ or $\bar{p} \in \{a,b\} \setminus \{l\}$;*
(iii) *at least two solutions if $\bar{p} \in (a,b) \setminus \{l\}$.*

2. Proof of the theorem. We shall use the following lemma.

Lemma 2. *Let $m \in L^1(0,a)$ and $v \in W^{2,1}(0,a)$ satisfy the conditions*

(a) *$v'' + m(t)v' + v < 0$ in $(0,a)$,*
(b) *$v(0) = v'(0) = 0$.*

Then there exists $a_1 \in (0,a)$ such that $v(a_1) < 0$.

Proof. Suppose that $v \ge 0$ in $(0,a)$. Let $M(t) = \int_0^t m(s)\,ds$. Then assumption (a) can be written

$$((\exp M(t))v')' + (\exp M(t))v < 0 \quad \text{in } (0,a),$$

and in particular $(\exp M(t))v'(t)$ is strictly decreasing in $[0,a]$. From (b) we infer that $v < 0$ in $(0,a)$, a contradiction. ∎

Remark. The lemma shows that in fact v takes negative values arbitrarily near $t = 0$.

Proof of the theorem. First we note that without loss of generality we can suppose $l = 0$; i.e., $g(-\infty) + g(+\infty) = 0$. (We shall henceforth add this to our hypotheses.) In fact, letting

$$h(s) := g(s) - \frac{g(-\infty) + g(+\infty)}{2} = g(s) - \frac{\pi l}{4} \quad \text{and} \quad q(t) = p(t) - \frac{\pi l}{4},$$

we obtain an equivalent problem,

$$\begin{gathered} u'' + u + h(u') = q(t), \\ u(0) = u(\pi) = 0. \end{gathered}$$

We also have $a(\tilde{p}, g) - l = a(\tilde{q}, h)$ and $b(\tilde{p}, g) - l = b(\tilde{q}, h)$.

Proof of (i) *and* (ii). The existence of a and b satisfying (i) and (ii) is proved in [3], except possibly for the fact that (1) is solvable when $\bar{p} = b > 0 = l$ or $\bar{p} = a < l = 0$. However, this follows immediately from Claim 1 in the proof of (iii) below, by a standard approximation procedure. ∎

Proof of (3). Before proceeding to the proof, we shall introduce some notations. Let $C_0^1[0,\pi]$ be the subspace of $C^1[0,\pi]$ consisting of those functions that vanish at $t = 0$ or $t = \pi$ and $\tilde{C}_0^1[0,\pi] = C_0^1[0,\pi] \cap \tilde{C}[0,\pi]$. These are Banach spaces with the usual C^1 norm, $\|u\| := \max(\|u\|_\infty, \|u'\|_\infty)$. Consider the projector $Q : C[0,\pi] \to \tilde{C}[0,\pi]$ defined by

$$Qu(t) = \tilde{u}(t) = u(t) - \frac{2}{\pi}\left(\int_0^\pi u(s)\sin s\,ds\right)\sin t.$$

Denote by K the (linear, compact) inverse of the differential operator $L : \tilde{C}_0^1[0,\pi] \to \tilde{C}[0,\pi]$, $u \mapsto u'' + u$, with domain $\mathrm{Dom}L = C^2[0,\pi] \cap \tilde{C}_0^1[0,\pi]$. The problem

$$\begin{gathered} \tilde{u}'' + \tilde{u} = \tilde{p} - Qg(\bar{u}\cos t + \tilde{u}'), \\ \tilde{u}(0) = \tilde{u}(\pi) = 0 \end{gathered} \tag{4}$$

can be written

$$\tilde{u} = \tilde{T}\tilde{u} := K(\tilde{p} - Qg(\bar{u}\cos t + \tilde{u}')).$$

The operator $\tilde{T} : \tilde{C}_0^1[0,\pi] \to \tilde{C}_0^1[0,\pi]$ is completely continuous and bounded, whence Schauder's theorem applies. This proves that there is a $k > 0$ so that for any given $\bar{u} \in \mathbb{R}$, the problem (4) has a solution $\tilde{u} \in \tilde{C}_0^1[0,\pi]$ with $\|\tilde{u}\| < k$. It follows that (1) has a solution $u(t) = \bar{u}\sin t + \tilde{u}(t)$, for $\bar{p}\sin t = (I - Q)g(\bar{u}\cos t + \tilde{u}')$. By Lebesgue's theorem, we have

$$\lim_{\bar{u}\to\infty} \int_0^\pi g(\bar{u}\cos s + \tilde{u}')\sin s\, ds = 0.$$

Hence, choosing $\bar{u}$ sufficiently large, we obtain numbers

$$\bar{p} := \frac{2}{\pi}\left(\int_0^\pi g(\bar{u}\cos s + \tilde{u}'(s))\sin s\, ds\right)$$

arbitrarily small such that $\bar{p} \in [a,b]$ and, going to the limit, we have $l = 0 \in [a,b]$. $\square$

Proof of (iii). Setting $u(t) = \bar{u}\sin t + \tilde{u}(t)$, with $\bar{u} \in \mathbb{R}$ and $\tilde{u} \in \tilde{C}_0^1[0,\pi]$, it is easily seen that (1) can be rewritten as the system

$$\begin{gathered} \tilde{u} - K[\tilde{p} - Q(g(\bar{u}\cos t + \tilde{u}'))] = 0, \\ \frac{\pi}{2}\,\bar{p} - \int_0^\pi g(\bar{u}\cos s + \tilde{u}'(s))\sin s\, ds = 0. \end{gathered} \tag{5}$$

As a function of the pair $(\bar{u}, \tilde{u})$, the right-hand side of (5) is a compact perturbation of the identity in $\mathbb{R} \times \tilde{C}_0^1[0,\pi]$; we denote it by $T_{\bar{p}}$ to emphasize its dependence on $\bar{p}$.

We divide this proof into several steps.

Claim 1. Given $\varepsilon > 0$, there exists $R_0 > 0$ such that if u is a solution of (1) for some $\bar{p}$ with $|\bar{p}| \geq \varepsilon$, then $\|u\| < R_0$.

Proof. Assume that there exists a sequence $(\bar{p}_n)_n$ with $|\bar{p}_n| \geq \varepsilon$ and corresponding solutions u_n with $\|u_n\| \to \infty$. Split $u_n(t) = \bar{u}_n\sin t + \tilde{u}_n(t)$ according to (2). As in the above argument, it follows from (5) that the sequence $\|\tilde{u}_n\|$ is bounded. Hence $|\bar{u}_n| \to \infty$. Also, we obtain from (5)

$$\left|\int_0^\pi g(\bar{u}_n\cos s + \tilde{u}_n'(s))\sin s\, ds\right| = \frac{\pi}{2}|\bar{p}_n| \geq \frac{\pi}{2}\,\varepsilon,$$

and, by Lebesgue's theorem again, we reach the contradiction $0 \geq \frac{\pi}{2}\varepsilon$. $\square$

Claim 2. Set $B_R = \{(\bar{u},\tilde{u}) \mid u = \bar{u}\sin t + \tilde{u} \in C_0^1[0,\pi], \|u\| \leq R\}$. Given $\bar{p}_0 \neq 0$, there exists R_0 such that for every $R > R_0$,

$$\deg(T_{\bar{p}_0}, B_R, 0) = 0. \tag{6}$$

Proof. Suppose $\bar{p}_0 > 0$. As shown in Claim 1, there exists R_0 such that for all solutions u of (5) with $\bar{p} \geq \bar{p}_0$, we have $\|u\| < R_0$. On the other hand, by (i), we can fix $\bar p_1 > \bar p_0$ such that (5) (with $\bar p = \bar p_1$) has no solution at all. The homotopy and existence properties of Leray–Schauder degree allow us to conclude that for every $R > R_0$,

$$\deg(T_{\bar p_0}, B_R, 0) = \deg(T_{\bar p_1}, B_R, 0) = 0.$$

The same argument applies if $\bar p_0 < 0$. $\square$

Claim 3. If $0 < \bar p < b$ or $0 > \bar p > a$, there exists a bounded open set $\Omega \subset \mathbb{R} \times \tilde C_0^1[0,\pi]$ such that

$$|\deg(T_{\bar p}, \Omega, 0)| = 1.$$

Proof. Step 1*: Construction of the set* Ω. To fix ideas, suppose that $0 < \bar p < b$. Choose $\rho \in (\bar p, b)$. By the characterization of a and b, there exists a solution of the problem

$$\begin{gathered} \alpha'' + \alpha + g(\alpha') = \rho \sin t + \tilde p(t) > \bar p \sin t + \tilde p(t), \quad \text{on } (0,\pi), \\ \alpha(0) = \alpha(\pi) = 0. \end{gathered} \tag{7}$$

Fix $0 < \varepsilon < \frac{\pi}{4}\bar p$. Recall that there is $k > 0$ so that any solution of (5) is such that $\|\tilde u(t)\| \le k$. Let us then choose $\bar\beta_0$ large enough so that for any $\tilde u$ with $\|\tilde u(t)\| \le k$,

$$\bar\beta_0 \sin t > \alpha(t) + \tilde u(t) \quad \text{in } (0,\pi)$$

and

$$\begin{aligned} (I-Q)g(\bar\beta_0 \cos t + \tilde u') &= \frac{2}{\pi}\left(\int_0^\pi g(\bar\beta_0 \cos s + \tilde u'(s)) \sin s\, ds\right) \sin t \\ &< \left(\bar p - \frac{4\varepsilon}{\pi}\right) \sin t. \end{aligned}$$

Next, we choose a $\tilde\beta_0$ solution of

$$\tilde u - K[\tilde p - \tilde\varepsilon - Q(g(\bar\beta_0 \cos t + \tilde u'))] = 0,$$

where $\tilde\varepsilon = Q\varepsilon = \varepsilon(1 - \frac{4}{\pi}\sin t)$. The function $\beta_0(t) := \bar\beta_0 \sin t + \tilde\beta_0(t)$ satisfies

$$\begin{aligned} \beta_0'' + \beta_0 + g(\beta_0') &= \tilde p - \tilde\varepsilon + (I-Q)g(\bar\beta_0 \cos t + \tilde\beta_0') \\ &< \tilde p + \bar p \sin t - \varepsilon, \qquad 0 < t < \pi, \\ &\quad \beta_0(0) = \beta_0(\pi) = 0, \end{aligned}$$

and therefore $\beta(t) := \beta_0(t) + \varepsilon$ is a strict upper solution of (1) such that $\beta \ge \alpha + \varepsilon$ in $[0,\pi]$.

Choose $k > 0$ so large that

$$k(\beta - \alpha) > \beta'' - \alpha'' \quad \text{in } [0,\pi]. \tag{8}$$

Then let us consider the homotopy

$$\begin{gathered} u'' + \lambda(u + g(u') - p) - (1-\lambda)\left[k\left(u - \tfrac{\alpha+\beta}{2}\right) + \tfrac{\alpha''+\beta''}{2}\right] = 0, \\ u(0) = u(\pi) = 0, \qquad 0 \le \lambda \le 1, \end{gathered} \tag{9}$$

and the bounded open subset of $\mathbb{R} \times \tilde{C}_0^1[0,\pi]$,

$$\Omega = \{(\bar{u},\tilde{u}) \mid u = \bar{u}\sin t + \tilde{u} \in C_0^1[0,\pi], \alpha(t) < u(t) < \beta(t) \text{ in } (0,\pi),$$
$$\|u'\|_\infty < N,\ u'(0) > \alpha'(0) \text{ and } u'(\pi) < \alpha'(\pi)\},$$

where N is a bound for derivatives of solutions u of (9) such that $\alpha \le u \le \beta$. This bound clearly exists, since g is bounded.

Step 2: There exists no solution $u = \bar{u}\sin t + \tilde{u}$ of (9) such that $(\bar{u},\tilde{u}) \in \partial\Omega$. First, we prove that there exists no solution $u = \bar{u}\sin t + \tilde{u}$ of (9) on $\partial\Omega$ if $0 \le \lambda < 1$. Arguing by contradiction, suppose that for some $s \in [0,\pi]$ we have $u(s) = \alpha(s)$, $u'(s) = \alpha'(s)$, $u''(s) \ge \alpha''(s)$. Then we compute

$$\alpha''(s) \le u''(s) \le \lambda\alpha''(s) + (1-\lambda)\left[k\frac{\alpha(s)-\beta(s)}{2} + \frac{\alpha''(s)+\beta''(s)}{2}\right]$$

so that

$$\alpha''(s) \le k\frac{\alpha(s)-\beta(s)}{2} + \frac{\alpha''(s)+\beta''(s)}{2},$$

contradicting the choice of k in (8). An analogous computation shows that it is impossible to have $u(s) = \beta(s)$, $u'(s) = \beta'(s)$, $u''(s) \le \beta''(s)$ for some $s \in (0,\pi)$.

Next, we show that, if $\lambda = 1$, no solution $u \in \overline{\Omega}$ can satisfy $u(s) = \alpha(s)$, $u'(s) = \alpha'(s)$ or $u(s) = \beta(s)$, $u'(s) = \beta'(s)$. This is straightforward since, in case $0 < s < \pi$,

$$\alpha''(s) + \alpha(s) + g(\alpha'(s)) - p(s) > 0 > \beta''(s) + \beta(s) + g(\beta'(s)) - p(s).$$

Also, $\beta(0) = \beta(\pi) = \varepsilon > 0$. Hence, it remains to show that $u(s) = \alpha(s)$, $u'(s) = \alpha'(s)$ cannot hold with $(\bar{u},\tilde{u}) \in \overline{\Omega}$ and $s = 0$ or $s = \pi$. Let us consider the case $s = 0$, the other being similar. If $u(0) = \alpha(0)$, $u'(0) = \alpha'(0)$, the function $v(t) := u(t) - \alpha(t)$ would satisfy $v \ge 0$ in $[0,\pi]$ and also, on account of (7),

$$v'' + v + m(t)v' = (\bar{p} - \rho)\sin t < 0 \quad \text{in } [0,\pi],$$

where $m(t) := \frac{g(u'(t)) - g(\alpha'(t))}{u'(t) - \alpha'(t)}$ if $u'(t) \ne \alpha'(t)$ and $m(t) = 0$ otherwise. Since g is locally Lipschitz continuous, the function m is (measurable and) bounded, and we obtain a contradiction with Lemma 2.

Step 3: Proof of the claim. Note that (9) can be written in operator form as

$$\tilde{u} - KQ[\lambda N(\bar{u},\tilde{u}) + (1-\lambda)M(\bar{u},\tilde{u})] = 0,$$
$$\int_0^\pi [\lambda N(\bar{u},\tilde{u})(t) + (1-\lambda)M(\bar{u},\tilde{u})(t)]\sin t\,dt = 0,$$

where

$$N(\bar{u},\tilde{u})(t) = p - g(\bar{u}\cos t + \tilde{u}'(t)) \quad \text{and}$$
$$M(\bar{u},\tilde{u})(t) = k\left(\bar{u}\sin t + \tilde{u}(t) - \tfrac{\alpha(t)+\beta(t)}{2}\right) + \tfrac{\alpha''(t)+\beta''(t)}{2} - \bar{u}\sin t - \tilde{u}(t).$$

For $\lambda = 1$, this reduces to $T_{\bar{p}} = 0$. On the other hand, for $\lambda = 0$, it is clear that this is an equation of the form $L = 0$, where L is a linear, invertible compact perturbation of the identity. We derive from (9) that its unique solution is

$$u(t) = \frac{\alpha(t)+\beta(t)}{2} - \frac{\varepsilon}{2}\frac{\cosh\sqrt{k}(t-\pi/2)}{\cosh\sqrt{k}\pi/2} = \bar{u}\sin t + \tilde{u}(t),$$

and it is clear that $(\bar{u}, \tilde{u})$ belongs to Ω. By the invariance property of Leray–Schauder degree, we easily conclude the proof of Claim 3. $\square$

Claim 4. Assertion (iii) holds.

With $0 < \bar{p} < b$ fixed, we construct Ω as in Claim 3 and $R > R_0$ given by Claim 2 so that $\Omega \subset B_R$. Then the excision property of Leray–Schauder degree implies

$$|\deg(T_{\bar{p}}, B_R \setminus \Omega, 0)| = 1.$$

The existence of a solution of (5) (with $\bar{p} = \bar{p}_0$) in $B_R \setminus \Omega$ follows. There exists another solution in Ω by Claim 3. Similarly, we conclude that there exist two solutions if $a < \bar{p} < 0$, and the proof is complete. $\square$

3. Additional results. We do not know whether the interval $[a(\tilde{p}, g), b(\tilde{p}, g)]$ is not reduced to a point for some $\tilde{p}$. However, following the idea of [4], we can give simple conditions to ensure that, at least for small $\tilde{p}$, it is in fact nondegenerate.

PROPOSITION 3. *Assume that $g : \mathbb{R} \to \mathbb{R}$ is of class C^1, $g(0) = 0$, the limits $g(-\infty)$ and $g(+\infty)$ exist, and $g'(0) \neq 0$. Let $p \in C[0, \pi]$ split according to* (2). *Then there exists $\varepsilon > 0$ such that, if $\|p\|_\infty < \varepsilon$, we have $a(\tilde{p}, g) < 0 < b(\tilde{p}, g)$.*

Proof. The mapping $\mathcal{F} : C^2[0, \pi] \cap C_0^1[0, \pi] \to C[0, \pi]$, $u \mapsto u'' + u + g(u')$ is differentiable at $u = 0$. Since the problem

$$\begin{aligned} v'' + v + cv' &= 0, \\ v(0) = v(\pi) &= 0, \end{aligned}$$

where c is a nonzero constant, has only the trivial solution, it follows that $\mathcal{F}'(0)$ is an isomorphism. Hence, by the inverse mapping theorem, the range of $\mathcal{F}$ contains an open ball centered at the origin, and we can conclude. $\square$

Another interesting feature of problem (1) under the assumptions of the above proposition is that for p small and $\bar{p} \neq 0 = l$, one can indeed prove the existence of two *ordered* solutions.

PROPOSITION 4. *Assume that $g : \mathbb{R} \to \mathbb{R}$ is of class C^1, $g(0) = 0$, the limits $g(-\infty)$ and $g(+\infty)$ exist, and $g'(0) \neq 0$. Let $p \in C[0, \pi]$ split according to* (2) *and $\bar{p} \neq l = 0$. Then there exists $\varepsilon > 0$ such that, if $\|p\|_\infty < \varepsilon$, we have two solutions u_1 and u_2 of* (1) *with*

$$u_1(t) > u_2(t), \quad \text{in } (0, \pi).$$

Proof. Let $\|p\|_\infty$ be so small that the corresponding solution $u_1(t)$ given by the inverse mapping theorem has the property that the solution of the linear Cauchy problem

$$v'' + v + g(u_1')v' = 0, \quad v(0) = 0,\ v'(0) = 1 \tag{10}$$

is *positive* in $(0, \pi]$. Making the change of variable $u = u_1(t) + w$ in (1), we obtain a new problem

$$w'' + w + g(u_1'(t) + w') = g(u_1'(t)), \quad w(0) = w(\pi) = 0. \tag{11}$$

For definiteness, assume $\bar{p} > 0$. Consider the solution $z(t, \lambda)$ of the Cauchy problem associated with (11):

$$z'' + z + g(u_1'(t) + z') = g(u_1'(t)), \quad z(0) = 0,\ z'(0) = \lambda.$$

Since $v(t) := \frac{\partial z}{\partial \lambda}(t,0)$ is the solution of (10), we deduce that for $\lambda < 0$ and $|\lambda|$ sufficiently small, we have $z(t,\lambda) < 0$ for every $t \in (0,\pi]$. On the other hand, substituting $z(t,\lambda) = \lambda \sin t + x(t,\lambda)$, $x(t,\lambda)$ solves the problem

$$x'' + x + g(u_1'(t) + \lambda \cos t + x') = g(u_1'(t)), \quad x(0) = x'(0) = 0.$$

As $\lambda \to -\infty$, $x(t,\lambda)$ converges uniformly in $[0,\pi]$ to the solution $\hat{x}(t)$ of

$$x'' + x + \hat{g}(t) = g(u_1'(t)), \quad x(0) = x'(0) = 0,$$

where $\hat{g}(t) = g(-\infty)$ if $0 < t < \frac{\pi}{2}$ and $\hat{g}(t) = g(+\infty)$ if $\frac{\pi}{2} < t < \pi$. Now this is given explicitly by

$$\hat{x}(t) = \int_0^t \sin(t-s)[g(u_1'(s)) - \hat{g}(s)]\, ds.$$

As $\hat{x}(\pi) = 2\bar{p} > 0$, we have $x(\pi,\lambda) > 0$ for $|\lambda|$ large. We infer that for some (negative) value of λ, $z(.,\lambda)$ is a negative solution of (11). Hence, (1) has a second solution $u_2(t)$ such that $u_2(t) < u_1(t)$ in $(0,\pi)$. $\square$

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