P-I Boundary Control of Countercurrent Heat Exchanger

Jacques Kadima Kazaku *,** **Denis Dochain** * **Moïse Mukepe Kahilu** **
Jimmy Kalenga Kaunde Kasongo **

* *Department of Mathematical Engineering, ICTEAM, UCLouvain, 4-6 avenue G. Lemaître, B-1348,Louvain-La-Neuve, Belgium (e-mail: jacques.kadima, denis.dochain@ uclouvain.be)*
** *Department of Electromechanics and Mine, Université de Lubumbashi, route Kasapa, 1825, D.R. Congo (e-mail: jacqueskazaku,moisemukepe,kalenga.kaunde@polytechunilu.ac.cd)*

Abstract: Heat exchangers are generally described by two coupled partial differential equations (PDEs), and belong to the class of systems of conservation laws. In this paper we propose a result on the regulation of heat exchangers, in which we are choosing a proportional-integral (PI)-controller implemented on an output feedback from a boundary condition. First, a study on the existence and uniqueness of solutions of uncontrolled system is carried out, and we give a condition on the parameters of the system that ensures that the control-observation problem is well-posedness. Secondly we successively study the stability of the system controlled by a proportional and integral action, when the gains are adjusted following respectively a classic Lyapunov approach, and using a recent result that combines Lyapunov's approach and operator perturbation theory.

Keywords: Well-posedness, Boundary control, Lyapunov analysis, heat exchangers, distributed parameter systems, PI-control.

1. INTRODUCTION

This paper is motivated by a major application: heat exchangers. Heat exchangers are devices that allow to exchange the heat between two fluids. The choice of this system is based on the fact that it is widely used in energy applications in industry (chemical, biochemical, food industry, etc). The dynamics of this system is described by hyperbolic or parabolic partial differential equations (PDEs) according to the assumptions made on the Peclet number Burns and Cliff (2014).

There are several control strategies for these devices. A majority is based on the notion of relative degree, which consists in differentiating the output variable until the control variable explicitly appears Maidi et al. (2009). These techniques provide interesting simulation results; yet their implementation requires at least two sensors (due to the discretization of the transport operator), depending on the assumptions made on the observed output. In general the proposed solutions consist in coupling the control laws to a state observer Kazaku et al. (2020), Kazaku et al. (2022b). In this paper we propose a boundary control scheme. It appears that the control action is precisely one of the boundary condition of the PDE system. The control action that is derived is indeed a PI-controller (so as to cancel the closed loop static error). The boundary control strategy has the advantage of simplicity of implementation since it only requires the measurement of the state variables at the boundaries.

The mathematical formulation of the PI regulation problem was established by Pohjolainen (1982) for systems evolving in a Banach space. One of the assumptions of Pohjolainen (1982) is the analyticity of the semigroup. When the state space is a Hilbert space, this constraint was lifted by Xu and Jerbi (1995), by an application of Huang's theorem Huang (1985). In Xu and Jerbi (1995), the semigroup analyticity assumption is replaced by exponential stability. Nevertheless the authors consider bounded input and output operators. This does not correspond to the case of the hyperbolic model that we will describe in the following. For systems described with unbounded input and output operators, Xu and Sallet (2014) shows that the results of Xu and Jerbi (1995) can be applied provided that these operators are admissible for the semigroup and by using a Yosida approximation of output operator (see Weiss (1994), Dehaye and Winkin (2016)), so as to guarantee the regularity of the control-observation system. However, the expression for the integral gain of the controller given in Xu and Jerbi (1995) is a function of the norm of the resolvent of an unbounded operator, which is not easy to compute. In this paper we use a recent result of Terrand-Jeanne et al. (2020) which results from a combination of Lyapunov's approach and operator perturbation theory. The proportional gain is designed following a classic Lyapunov approach, as described in (Bastin and Coron, 2016, Chapter 5), Prieur and Winkin (2018).

The paper is organized as follows. In Section 2, we introduce the PDEs of the hyperbolic model of countercurrent heat exchangers, and we give some results concerning the existence, the uniqueness of solutions as well as the stability of the uncontrolled and controlled system. In section 3, we formulate the PI-control problem in the Hilbert space. Section 4 discusses the exponential stability and regulation of the closed-loop system.

$\star$ This research was conducted with the financial support of CAI (Conseil de l'Action International). Jacques Kadima Kazaku is a CAI Research Fellow.

Notation

I_n denotes, the identity matrix of order n, $\mathbb{R}^{n\times n}$ (or $\mathbb{C}^{n\times n}$) and $\mathbb{R}^n$ (or $\mathbb{C}^n$) denote the set real (or complex) of $n-$order matrices, respectively. $L^2(0,1)=L^2((0,1);\mathbb{C})$ denotes the Hilbert space of measurable square-integrable function with values in $\mathbb{R}$. $H^1(0,1)=H^1((0,1);\mathbb{R})$ is the Sobolev space of absolutely continuous $\mathbb{C}-$valued functions whose derivatives are in $L^2(0,1)$. $Z=H^1(0,1)\oplus H^1(0,1)$ and $X=L^2(0,1)\oplus L^2(0,1)$ are Hilbert spaces such that the injection $Z\subset X$ is continuous. We denote by $\rho(A)$ the resolvent set of A a linear operator, and, for any $\lambda\in\rho(A)$, the domain $\mathscr{D}(A)\subset X$ equipped with the norm $\|g\|_1=\|(\lambda I-A)g\|_X$ will be denoted by X_1. The completion of X equipped with the norm $\|g\|_{-1}=\|(\lambda I-A)^{-1}g\|_X$ will be denoted by X_{-1}.

2. SYSTEM MODEL AND WELL-POSEDNESS

Assuming that the thermal diffusion phenomenon is negligible and that the flows are plug type, i.e the transport velocities v_1 and v_2 of each fluid are respectively identical at all points of the heat exchanger, we lead to a model described by two hyperbolic PDEs:

$$\partial_t x=\underbrace{\begin{pmatrix}-v_1 & 0\\ 0 & v_2\end{pmatrix}}_{\Lambda}\partial_z x+\underbrace{\begin{pmatrix}-\alpha_1 & \alpha_1\\ \alpha_2 & -\alpha_2\end{pmatrix}}_{M}x(z,t) \quad (1)$$

where $x_i(z,t)[^\circ C]$ and $\alpha_i[s^{-1}]=\frac{h_iS}{\rho_iS_iC_{p_i}}$ represent the distributed temperature and the heat exchange coefficient of the fluid i respectively. h_i,ρ_i,C_{p_i},S_i represents the overall transfer coefficient, density, heat capacity, section of the tube of fluid i, and S the area used for heat transfer. The system (1) is completed by the following boundary conditions:

$$x_1(0,t)=x_{1,in}(t),\quad x_2(1,t)=x_{2,in}(t) \quad (2)$$

with $x_{1,in}$ and $x_{2,in}$ are the inlet temperatures of the heat exchanger. We consider $x_{2,in}=u\in\mathbb{R}$ as a control variable and $x_{1,in}$ as is a constant disturbance that we shall consider zero in the abstract formulation of the system.

2.1 *Definition of the abstract system*

Let $L\in\mathscr{L}(Z,X)$ and $G\in\mathscr{L}(Z,\mathbb{R})$ be the operators of the abstract boundary control system in the sense of (Tucsnak and Weiss, Boston, 2009, Chapter 8), defined from (1)-(2) as follows: $\dot{\xi}=L\xi,\ G\xi=u$, where

$$\begin{cases} L\xi=\Lambda\dfrac{d\xi}{dz}+M\xi\\ G\xi=\xi_2(1).\end{cases} \quad (3)$$

The state operator A is defined by

$$\begin{aligned} A\ &=\begin{pmatrix}-v_1\dfrac{d}{dz}-\alpha_1 & \alpha_1\\ \alpha_2 & v_2\dfrac{d}{dz}-\alpha_2\end{pmatrix}=\begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}\\ &:=L_{|X_1}\in\mathscr{L}(X_1,X)\end{aligned}$$

on its the domain $\mathscr{D}(A)=\mathscr{D}(A_{11})\oplus\mathscr{D}(A_{22})=\{\xi=(\xi_1\ \xi_2)^\top\in X:$ ξ absolutely continuous (a.c.), $\dfrac{d\xi}{dz}\in X$ and $\xi_1(0)=0=\xi_2(1)\}$.

2.2 *Analysis of the dynamic operator*

Lemma 2.1. The linear operator A generates a C_0-semigroup $(\mathbb{T}(t))_{t\geq0}$ on X.

Proof. We establish the proof in two steps:

a) The operator $\tilde{A}=A-\tilde{M}-\omega I$, where $\tilde{M}=\begin{pmatrix}0 & \alpha_1 I\\ \alpha_2 I & 0\end{pmatrix}$ and $\omega=-\min(\alpha_1,\alpha_2)$, is dissipative. For all $\xi\in\mathscr{D}(\tilde{A})=\mathscr{D}(A)$,

$$\begin{aligned}\mathfrak{Re}\langle(\tilde{A}\xi(z))^\top,\xi(z)\rangle_X\ &\leq\frac{1}{2}\left[-v_1\xi_1^2(z)+v_2\xi_2^2(z)\right]_0^1\\ &-\alpha_1|\xi_1|^2_{L^2(0,1)}-\alpha_2|\xi_2|^2_{L^2(0,1)}\\ &\leq-\omega\|\xi\|_X^2.\end{aligned}$$

b) $\mathrm{Ran}(I-\tilde{A})=X$.
The inclusion $\mathrm{Im}(I-\tilde{A})\subset X$ is directly deduced from the definition of the domain of $\tilde{A}$. The inclusion $\mathrm{Im}(I-\tilde{A})\supset X$ means that $(I-\tilde{A})$ is surjective, i.e., for all $g\in X$ there exists $\xi\in\mathscr{D}(\tilde{A})$ such that $(I-\tilde{A})\xi(z)=g(z)$. A straightforward computation yield, the following expressions for the component function of ξ:

$$\xi_1(z)=\frac{1}{v_1}\int_0^z e^{-\frac{1+\omega+\alpha_1}{v_1}(z-\eta)}g_1(\eta)d\eta,$$

and

$$\xi_2(z)=\frac{1}{v_2}\int_z^1 e^{-\frac{1+\omega+\alpha_2}{v_2}(z-\eta)}g_2(\eta)d\eta.$$

c) The operator A is the (infinitesimal) generator of a C_0-semigroup on X. Indeed, by using Lumer-Phillips theorem, it follows, from assertion a) that the operator $\tilde{A}+\omega I$ generates an exponentially stable C_0-semigroup $(\tilde{\mathbb{T}}(t))_{t\geq0}$ such that $\|\tilde{\mathbb{T}}(t)\|\leq e^{\omega t}$. Consequently, by (Curtain and Zwart, 2020, Theorem 5.3.1), the operator A generates a C_0-semigroup $(\mathbb{T}(t))_{t\geq0}$ such that $\|\mathbb{T}(t)\|\leq e^{(\omega+\nu)t}$, where $\nu=\|\tilde{M}\|=\max(\alpha_1,\alpha_2)$ and $\omega=-\min(\alpha_1,\alpha_2)$. In particular, $(\mathbb{T}(t))_{t\geq0}$ is a C_0-semigroup of contraction whenever $\alpha_1=\alpha_2$. ■

Lemma 2.2. The operator A has a compact resolvent.

Proof. It suffices to observe that the resolvent operator associated with A is an integral operator of Hilbert-Schmidt type. This operator is given for any $s\in\rho(A)$ and $y=(y_1,y_2)^\top\in X$, by

$$\begin{aligned}\xi(z)\ &=\begin{pmatrix}\xi_1(z)\\ \xi_2(z)\end{pmatrix}=(sI-A)^{-1}y(z)\\ &=e^{\tilde{A}(s)z}\begin{pmatrix}0\\ \xi_2(0)\end{pmatrix}+\int_0^z e^{\tilde{A}(s)(z-\eta)}y(\eta)d\eta,\end{aligned}$$

with

$$\xi_2(0)=\left[e^{-\tilde{A}(s)}\right]_{2\cdot}\left\{\begin{pmatrix}\xi_1(1)\\ 0\end{pmatrix}-\int_0^1 e^{\tilde{A}(s)(1-\eta)}y(\eta)d\eta\right\}$$

where $\tilde{A}(s)=\begin{pmatrix}-\dfrac{s+\alpha_1}{v_1} & -\dfrac{\alpha_1}{v_1}\\ \dfrac{\alpha_2}{v_2} & \dfrac{s+\alpha_2}{v_2}\end{pmatrix}$, and $y(z)=\left[\frac{y_1(z)}{v_1}\ -\frac{y_2(z)}{v_2}\right]^\top$.

$[M]_{2\cdot}$ denotes the second row of a 2×2 matrix M. The components of the transition matrix $e^{\tilde{A}(s)z}$ are:

$$\phi_{11}(z,s)=\frac{e^{\alpha(s)z}}{2\beta(s)}\left(2\beta(s)\cosh(\beta(s)z)-\gamma\sinh(\beta(s)z)\right),$$

$$\phi_{12}(z,s)=\frac{\beta_1}{\beta(s)}e^{\alpha(s)z}\sinh(\beta(s)z),$$

$$\phi_{21}(z,s)=-\frac{\beta_2}{\beta(s)}e^{\alpha(s)z}\sinh(\beta(s)z),$$

$$\phi_{22}(z,s)=\frac{e^{\alpha(s)z}}{2\beta(s)}\left(2\beta(s)\cosh(\beta(s)z)+\gamma\sinh(\beta(s)z)\right),$$

with: $\beta_1=\frac{\alpha_1}{v_1}$, $\beta_2=\frac{\alpha_2}{v_2}$, $\eta_{11}=\frac{s+\alpha_1}{v_1}$, $\eta_{22}=\frac{s+\alpha_2}{v_2}$, $\alpha(s)=\frac{1}{2}(\eta_{22}-\eta_{11})$, $\beta(s)=\frac{\sqrt{\Delta(s)}}{2}$, $\Delta(s)=\frac{1}{2}(\eta_{11}+\eta_{22})^2-4\beta_1\beta_2$, $\gamma=\eta_{11}(s)+\eta_{22}(s)$. ■

Moreover, it follows from Proposition 2.1, Lemma 2.1 and C_0-semigroup theory Curtain and Zwart (1995), that the following corollary, holds.

Corollary 2.1. For $\xi(0) = \xi_0 \in \mathscr{D}(A)$, the abstract system (3) admits a unique solution $\xi \in C^0([0,+\infty),\mathscr{D}(A)) \cap C^1([0,+\infty),X)$. Furthermore,

$$\xi(t) = \mathbb{T}(t)\xi_0, \qquad \forall\, t \in [0,+\infty).$$

The positivity of $\mathbb{T}(t)$ is established in Kazaku et al. (2022b). Let us start by recalling the definition of stability in the Lyapunov sense of infinite dimensional systems.

Definition 2.1. (Curtain and Zwart, 1995, Theorem 5.1.3) Let A be the generator of C_0-semigroup $\mathbb{T}(t)$ on the Hilbert space X. Then $\mathbb{T}(t)$ is exponetially stable iff there exists a positive operator $P \in \mathscr{L}(X)$ such that the operator Lyapunov equation

$$\langle A\xi, P\xi\rangle + \langle P\xi, A\xi\rangle = -\langle \xi,\xi\rangle \quad \text{for all } \xi \in \mathscr{D}(A) \qquad (4)$$

is satisfied.

To apply this definition we need the following result whose proof can be found in Grabowski (2007).

Lemma 2.3. Let $A \in \mathscr{D}(A)$. For all $\zeta \in X$, adjoint operator of A is defined by

$$A^*\zeta = -\Lambda\frac{d\zeta}{dz} + M^\top\zeta,$$

with his domain $\mathscr{D}(A^*) = \{\zeta = (\zeta_1,\zeta_2)^\top \in X : \ \zeta \text{ a.c., } \dfrac{d\zeta}{dz} \in X$ and $\zeta_1(1) = 0 = \zeta_2(0)\}$.

Theorem 2.1. For all $\xi \in \mathscr{D}(A)$ the operator $A \in \mathscr{D}(A)$ generates an exponentially stable C_0-semigroup $(\mathbb{T}(t))_{t\geq 0}$ on $L^2(0,1)$. Moreover, the Lyapunov operator is given by

$$P(z) = \begin{pmatrix} \dfrac{1}{2\alpha_1}\left(1-e^{-2\beta_1(1-z)}\right) & 0 \\ 0 & \dfrac{1}{2\alpha_2}\left(1-e^{-2\beta_2 z}\right) \end{pmatrix}. \qquad (5)$$

Proof. By (Curtain and Zwart, 1995, Lemma 4.1.24) we know that the positive self-adjoint operator $P \in \mathscr{L}(X)$ is a solution of (4) if and only if $P\mathscr{D}(A) \subset \mathscr{D}(A^*)$ and

$$A^*P\xi + PA\xi = -\xi, \quad \text{for } \xi \in \mathscr{D}(A). \qquad (6)$$

Let $P = \mathrm{diag}(p_1,p_2)$. By developing equation (6) we obtain:

$$\begin{cases} (A_{11}^*p_1 + p_1A_{11} + 1)\,\xi_1 = 0 \\ (A_{12}^*p_2 + p_1A_{12})\,\xi_2 = 0 \\ (A_{21}^*p_1 + p_2A_{21})\,\xi_1 = 0 \\ (A_{22}^*p_2 + p_2A_{22} + 1)\,\xi_2 = 0 \end{cases} \qquad (7)$$

where $A_{11}^* = v_1\frac{d}{dz} - \alpha_1$ defined on $\mathscr{D}(A_{11}^*) = \{\xi_1 \in L^2(0,1) :$ ξ_1 a.c., $\frac{d\xi_1}{dz} \in L^2(0,1)$ and $\xi_1(1) = 0\}$, and $A_{22}^* = -v_2\frac{d}{dz} - \alpha_2 \in \mathscr{D}(A_{22}^*) = \{\xi_2 \in L^2(0,1) : \xi_2$ a.c., $\frac{d\xi_2}{dz} \in L^2(0,1)$ and $\xi_2(0) = 0\}$. Using the same arguments as those found in the framework of optimal control (see Aksikas et al. (2009), Kazaku et al. (2022b)), one obtains that p_1 and p_2 are solution of the following differential system:

$$\begin{cases} v_1\dfrac{dp_1}{dz} - 2\alpha_1p_1(z) + 1 = 0, \quad p_1(1) = 0 \\ v_2\dfrac{dp_2}{dz} + 2\alpha_2p_2(z) - 1 = 0, \quad p_2(0) = 0 \end{cases} \qquad (8)$$

We can see that for $z \in [0,1]$, $p_1(z) \geq 0$ and $p_2(z) \geq 0$. ■

2.3 Well-posedness of boundary control-observation system

In the following we show that under certain conditions on the parameters, there exists a unique solution when the heat exchanger model (1)-(2) is controlled and observed at the boundary. For that, let us consider the stationary solution $\bar{x}(z) = \exp(-\Lambda^{-1}Mz)x(0)$ of the system (1)-(2). Taking into account the configuration of the system, the boundary condition is expressed by

$$\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ (0\ \ 1)\cdot\exp(-\Lambda^{-1}M) \end{pmatrix}^{-1} \begin{pmatrix} x_{1,in} \\ x_{2,in} \end{pmatrix}. \qquad (9)$$

The above matrix is invertible iff $\beta_1 \neq \beta_2$. Then by (Prieur and Winkin, 2018, Lemma1)

Proposition 2.1. If $\beta_1 \neq \beta_2$, then the heat exchanger model (1)-(2) is a boundary control system. The control operator is given by $B = (0 \quad v_2\delta_1)^\top \in \mathscr{L}(U,X_{-1})$, with δ_1 the Dirac function at $z = 1$.

Proof. First, according to (Prieur and Winkin, 2018, Lemma1) it is easy to verify that the determinant of the matrix (9) given by $\det := \beta_1 - \beta_2e^{\beta_2-\beta_1}$ is non-zero if and only if $\beta_1 \neq \beta_2$ [1]. Let us now show that the abstract problem $\dot{\xi} = L\xi$ with $G\xi = u$ is equivalent to $\dot{\xi}(t) = A\xi(t) + Bu(t)$. Note that the model (1)-(2) fits into the general framework of linear hyperbolic systems described by Prieur and Winkin (2018), for which the well-posedness of the boundary control system is established based on (Tucsnak and Weiss, Boston, 2009, Chapter 10). Then for all $\zeta \in \mathscr{D}(A^*)$ and by using (Tucsnak and Weiss, Boston, 2009, Remark 10.1.6) we can verify that

$$B^*\zeta = v_2\zeta_2(1), \quad \text{i.e.,} \quad B = (0, \quad v_2\delta_1)^\top. \qquad (10)$$

■

It follows that the following corollary, holds.

Corollary 2.2. For $\xi(0) = \xi_0 \in X_1$, the abstract problem (L,G) associated with the initial condition ξ_0 admits a unique solution $\xi \in C^0([0,+\infty),X_1) \cap C^1([0,+\infty),X_{-1})$. Furthermore,

$$\xi(t) = \mathbb{T}(t)\xi_0 + \int_0^t \mathbb{T}(t-\tau)Bu(\tau)d\tau, \quad \forall t \in [0,+\infty).$$

Let us consider the output equation

$$y = C\xi = \xi_1(1), \qquad (11)$$

with $C \in \mathscr{L}(X_1,\mathbb{R})$ the observation operator that we consider admissible for the semigroup $\mathbb{T}(t)$ by duality with B^* (see (Tucsnak and Weiss, Boston, 2009, Theorem 4.4.3)). In order to guarantee the regularity of the control-observation system, we shall use a Yosida approximation (Dehaye and Winkin (2016)) of the output operator C:

$$C_\Lambda\xi = \lim_{\lambda\to+\infty} \lambda C(\lambda I - A)^{-1}\xi = \xi_1(1). \qquad (12)$$

Theorem 2.2. The system (1)-(2) associated with (11) constitutes a regular system in the sense of Tucsnak and Weiss (2014), and is written on state space X as follows:

$$\begin{cases} \dfrac{d\xi}{dt} = A\xi(t) + Bu(t), \qquad \forall\, t \in [0,+\infty) \\ y(t) = C_\Lambda x(t). \end{cases} \qquad (13)$$

Remark 2.1. The advantage of using C_Λ is due to the fact that unlike C it is bounded. Indeed, as the observation is made on ξ_1, we can consider:

[1] An thermodynamics interpretation of the condition $\beta_1 \neq \beta_2$ based on the energy transfer is given in Kazaku et al. (2022a).

$$\lambda\left(\lambda I-A_{11}\right)^{-1}\xi_1(z)=\frac{\lambda}{v_1}\int_0^1 e^{-\frac{\lambda+\alpha_1}{v_1}(1-z)}\xi_1(z)dz.$$

By integrating by part on $[0,1]$ we have:

$$\begin{aligned}\lambda\left(\lambda I-A_{11}\right)^{-1}\xi_1(z) &= \frac{\lambda}{\lambda+v_1}\left[\xi_1(1)-e^{-\frac{\lambda+\alpha_1}{v_1}}\xi_1(0)\right.\\ &\left.-\int_0^1 e^{-\frac{\lambda+\alpha_1}{v_1}(1-z)}\frac{d\xi_1}{dz}dz\right],\end{aligned}$$

which is bounded according to the Cauchy-Schwartz inequality. At the same time, one can verify that:

$$\lim_{\lambda\to+\infty}\lambda\left(\lambda I-A_{11}\right)^{-1}\xi_1(z)=\xi_1(1).$$

3. PROBLEM STATEMENT

We propose to implement on the output feedback a proportional-integral controller defined by:

$$\begin{cases} u(t)=K_p(C_\Lambda\xi(t)-y_d(t))+k_iK_I\chi(t)\\ \dfrac{d\chi}{dt}=C_\Lambda\xi(t)-y_d(t),\end{cases}\tag{14}$$

where $K_p\in\mathbb{R}$ is the proportional gain of the regulator, $k_i\in\mathbb{R}$ the turning parameter and $K_I\in\mathbb{R}^{2\times1}$ the constant matrix. $y_d(t)\in\mathbb{R}$ is the reference and $\chi(t)\in\mathbb{R}$ the new state variable which corresponds to the integral of $C_\Lambda\xi(t)-y_d(t)$. The closed-loop system is defined by

$$\begin{cases} \dfrac{d}{dt}\begin{pmatrix}\xi\\ \chi\end{pmatrix}=\begin{pmatrix}A+BK_pC_\Lambda & Bk_iK_I\\ C_\Lambda & 0\end{pmatrix}\begin{pmatrix}\xi\\ \chi\end{pmatrix}-\begin{pmatrix}BK_p\\ 1\end{pmatrix}y_d(t)\\ y(t)=C_\Lambda\xi(t).\end{cases}\tag{15}$$

Let us consider the state space $\mathbb{X}=L^2(0,1)\times L^2(0,1)\times\mathbb{R}$, which is another Hilbert space (greater than X) equipped with the inner product

$$\langle(\xi,\chi),(\phi,\varphi)\rangle_{\mathbb{X}}=\langle\xi,\phi\rangle_X+\chi^\top\varphi.$$

Lemma 3.1. If $\beta_1\neq\beta_2$, then for all $(\xi_0,\chi_0)\in\mathbb{X}$ the closed-loop system (15) is well-posed on $\mathbb{X}$ for the $L^2(0,1)-$norm.

Proof. According to Theorem 2.2 we know that the space of solution is $\xi\in C^0([0,+\infty),\mathscr{D}(A))\cap C^1([0,+\infty);X)$. Thus we deduce that the space of boundary conditions is $C^1([0,+\infty),\mathbb{R})$. Since $\chi(t)$ is continuously depends on the $\int_0^t C_\Lambda\xi(\tau)d\tau$, its addition to $u(t)$ will not modify the state space $\mathbb{X}$. ■

The aim is to choose K_p, k_i, and K_I such that:

(1) The equilibrium state of the system (15) is exponentially stable. In other words, there are two positive scalars μ and ν such that $\forall\, t\in[0,+\infty)$ and $\xi_0\in\mathscr{D}(A)$
$$\|\xi(t)\|_X\leq\nu e^{-\mu t}\|\xi_0\|.$$
(2) The output of the system converges $y_d(t)$, i.e.
$$\lim_{t\to+\infty}\|C_\Lambda\xi(t)-y_d(t)\|=0,\quad t\in[0,\infty).$$
(3) Stability and regulation must be guaranteed for all $\xi_0\in\mathscr{D}(A)$, and this despite the constant disturbances acting on the system.

4. MAIN RESULTS

The open-loop system being exponentially stable, we can treat separately the choice of the integral part of the controller and that of the proportional part. In the following, the proportional gain will be adjusted using a classic Lyapunov approach (see (Bastin and Coron, 2016, Chapter 5)), and the integral part using recent results that combine operator parturbation theory and Lyapunov operators Terrand-Jeanne et al. (2020).

4.1 P-controller

Consider the control law

$$u(t)=K_p\xi_1(1,t).\tag{16}$$

The closed-loop system is can then be written in the form:

$$\begin{cases}\partial_t\xi(z,t)=\Lambda\partial_z\xi(z,t)+M\xi(z,t)\\ \xi_1(0,t)=0,\\ \xi_2(1,t)=K_p\xi_1(1,t).\end{cases}\tag{17}$$

It is important to note that the component p_1 of the operator $P(z)$ defined in (5) vanishes at the point where the control acts, i.e. at the point $z=1$. Therefore it is not interesting to use it in the analysis of the controlled system.

Let us introduce the following assumptions.

Assumption 4.1. Let $P(z)=\mathrm{diag}(p_1e^{-\mu z},p_2e^{\mu z})\in\mathbb{R}^{2\times2}$ be a diagonal positive definite matrix, where the signs of the exponential functions depend on the direction of the transport terms of system equations (1). There exists a positive constant $\mu>0$ such that

$$PM+M^\top P\leq-2\mu P\tag{18}$$

Assumption 4.2. There are two positive constants γ_1,γ_2, and $\lambda_{\min}(P)\geq\gamma_1$, $\lambda_{\max}(P)\leq\gamma_2$ a minimum and maximum eigenvalues of P respectively, such that for all $\xi\in X$ the following inequality is satisfied:

$$\lambda_{\min}(P)\|\xi\|_X^2\leq\xi^\top P\xi\leq\lambda_{\max}(P)\|\xi\|_X^2.\tag{19}$$

Remark 4.1. Condition (18) is equivalent to the existence of a Lyapunov functional for the open-loop system (1)-(2). It is found in (Bastin and Coron, 2016, Chapter 5) within the framework of hyperbolic systems of conservation laws. The inequality (19) is a derivability condition of the Lyapunov functional.

Proposition 4.1. There exists a positive constant K_p such that the closed-loop system (17) is an exponentially stable for the $L^2(0,1)-$norm.

Proof. Consider in $L^2(0,1)$ the following Lyapunov functional

$$V(t)=\int_0^1\left(\xi^\top(z,t)P(z)\xi(z,t)\right)dz.$$

By integrating by part on $[0,1]$ the derivative of $V(t)$ along the trajectories of (17) gives us:

$$\begin{aligned}\dot V(t) &= \int_0^1\xi^\top(z,t)P(z)\left(\Lambda\frac{\partial\xi}{\partial z}+M\xi(z,t)\right)dz\\ &+\int_0^1\left(\frac{\partial\xi^\top}{\partial z}\Lambda+\xi^\top(z,t)M^\top\right)P(z)\xi(z,t)dz\\ &=\xi^\top(1,t)P(1)\Lambda\xi(1,t)-\xi^\top(0,t)P(0)\Lambda\xi(0,t)\\ &\underbrace{+\int_0^1\xi^\top(z,t)\left(P(z)M+M^\top P(z)\right)\xi(z,t)dz}_{\overset{\text{Eq.(18)}}{\leq}-2\mu V(t)}\end{aligned}$$

The boundary conditions lead to

$$I(t)=-\xi_\partial^\top(t)\underbrace{\begin{pmatrix}v_1p_1e^{-\mu}-v_2p_2K_p^2e^{\mu} & 0\\ 0 & v_2p_2\end{pmatrix}}_{N}\xi_\partial(t),$$

with $\xi_\partial(t) = (\xi_1(1,t), \xi_2(0,t))^\top$. The matrix N is positive definite iff

$$K_p \leq e^{-\mu}\sqrt{\frac{v_1 p_1}{v_2 p_2}}. \tag{20}$$

Therefore we obtain $V(t) \leq e^{-2\mu t}V(0)$. Finally, using the inequality (19), we get

$$\begin{aligned}\|\xi(\cdot,t)\|^2 &\leq \frac{\lambda_{\max(P(z))}}{\lambda_{\min(P(z))}} e^{-2\mu t}\|\xi_0(z)\|^2\\ &\leq \frac{\gamma_2}{\gamma_1}e^{-2\mu t}\|\xi_0(z)\|^2,\end{aligned}$$

that proves the exponentially stability of closed-loop system (17). ■

4.2 I-controller

Consider the control law:

$$\begin{cases} u(t) = k_i K_I \chi(t)\\ \dfrac{d\chi}{dt} = C_\Lambda \xi(t) - y_d(t). \end{cases} \tag{21}$$

The closed-loop system (15) becomes:

$$\begin{cases} \dfrac{d}{dt}\begin{pmatrix}\xi\\ \chi\end{pmatrix} = \underbrace{\begin{pmatrix} A & Bk_iK_I\\ C_\Lambda & 0\end{pmatrix}}_{A_e}\begin{pmatrix}\xi\\ \chi\end{pmatrix} - \begin{pmatrix}0\\ 1\end{pmatrix} y_d(t)\\ y(t) = C_\Lambda \xi(t), \end{cases} \tag{22}$$

with $A_e \in \mathscr{D}(A_e) = \{\psi \in \mathbb{X}$ and $\frac{d\xi}{dz} \in X$ a.c., $\xi_1(0) = 0; \xi_2(1) = 0\}$. The model being well posed, it remains to determine the expression of the integral gain allowing to guarantee the stability of the closed-loop system (20).

Note that the operator A_e defined in (21) can be written as follows: $A_e = A_1 + k_i A_2$, where $A_1 = \begin{pmatrix} A & 0\\ C_\Lambda & 0\end{pmatrix}$ generates a C_0-semigroup and represents the open-loop system ($k_i = 0$) of (22), and $A_2 = \begin{pmatrix} 0 & BK_I\\ 0 & 0\end{pmatrix}$ bounded in $\mathbb{X}$. It comes from the form of A_1 that its spectrum is written as: $\sigma(A_1) = \sigma(A)\cup\{0\}$. Since A generates an exponentially stable semigroup (see Theorem 2.1), then a vertical line can separate $\sigma(A)$ and the isolated point $\{0\}$. Pohjolainen (1982) shows that this separation remains valid for a certain value of k_i lower than a limit value that we note k_i^*. To find the expression of k_i^* we shall use the solution based on the theory of the perturbation of operators. In this context we consider that the operator of the closed-loop system A_e is a disturbance of the operator of the open-loop system.

Under the assumption that A generates an analytical semigroup (meromorphic), B and C_Λ are admissible operators and $C_\Lambda A^{-1}B \neq 0$, Xu and Sallet (2014) showed that the integral gain which stabilizes the system is equal to: $k_iK_I = k_i\left(C_\Lambda A^{-1}B\right)^{-1}$, with k_i such that $0 < k_i < k_i^*$, and

$$k_i^* = \min_{\lambda\in\rho(A_1)}\left(\max\|A_2\|\|\mathrm{R}(\lambda, A_1)\| + 1\right)^{-1}.$$

To avoid having to calculate the norms of operator A_2 and resolvent $\mathrm{R}(\lambda, A_1)$, which are not easy to calculate, we use the recent result based on Lyapunov's approach (Terrand-Jeanne et al., 2020, Theorem 2).

Lemma 4.1. If B and C_Λ are admissible operators for $\mathbb{T}(t)$, with $\mathbb{T}(t)$ analytical C_0-semigroup, then there exist $\mu > 0$ and a positive self adjoint operator $P \in \mathscr{L}(X)$ such that

$$k_i^* = \frac{\nu}{2\|C_\Lambda A^{-1}\|\left\|PB\left(C_\Lambda A^{-1}B\right)^{-1}\right\|}. \tag{23}$$

The $\left(C_\Lambda A^{-1}B\right)^{-1}$ can be deduced from the expression of the transfer function given by:

$$\begin{aligned} G_{12}(s) &= \frac{\phi_{12}(1,s)}{\phi_{22}(1,s)}\\ &= \frac{\beta_1\sinh(\beta(s))}{\beta(s)\cosh(\beta(s)) + \gamma(s)\sinh(\beta(s))}. \end{aligned} \tag{24}$$

Thus,

$$K_I = \frac{(\beta_2 - 2\beta_1)e^{\beta_2-\beta_1} - \beta_1}{\beta_1\left(e^{\beta_2-\beta_1} - 1\right)}. \tag{25}$$

The self-adjoint positive operator P considered here is the Lyapunov operator whose expression is given by Theorem 2.1. A^{-1} is mapping $u \to y$ defined by the following ordinary differential equation:

$$\begin{cases} \Lambda\dfrac{d\xi}{dz} + M\xi = 0,\\ (0,\ \xi_2(1))^\top = u,\\ y = (\xi_1(1)\ \ \xi_2(0))^\top. \end{cases}$$

So,

$$C_\Lambda A^{-1} = \begin{pmatrix} \dfrac{\beta_1\beta_2(e^{(\beta_2-\beta_1)} - 1)}{\beta_2-\beta_1}h - \dfrac{\beta_2 - \beta_1 e^{(\beta_2-\beta_1)}}{\beta_2-\beta_1}\\ \beta_1 h \end{pmatrix}^\top, \tag{26}$$

with $h = \dfrac{1 - e^{(\beta_2-\beta_1)}}{(\beta_2 - 2\beta_1)e^{(\beta_2-\beta_1)} - \beta_1}$. Since A_e is exponentially stable, then the closed-loop system has robustness against external disturbances and modelling errors. In particular when the disturbance is assumed to be constant, we have $\lim\limits_{t\to+\infty}\|y(t) - y_d\| = 0$ (see Pohjolainen (1982) for the proof).

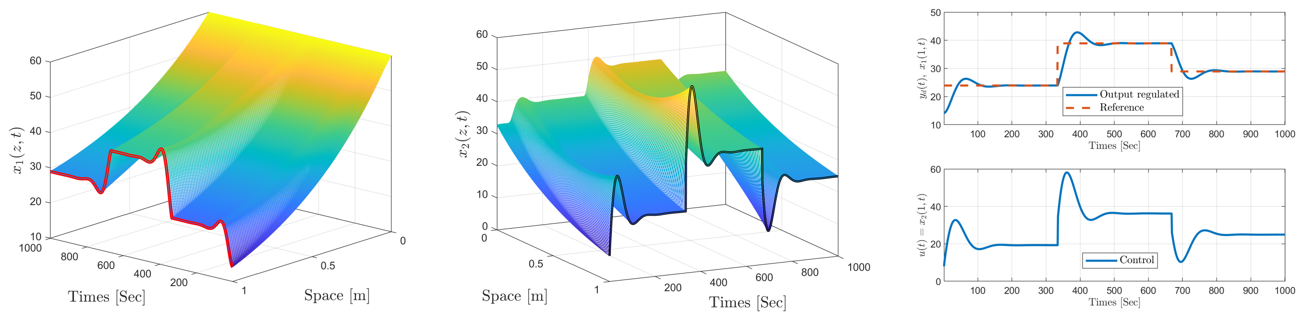
5. NUMERICAL SIMULATION

The control strategy proposed in the previous section has been implemented using the method of lines, with model parameters values obtained by identification [2]: $v_1 = 0.9749, v_2 = 1.6513, \alpha_1 = 3.2389$ and $\alpha_2 = 2.9030$; The inlet temperature $x_1(1,t) = 60°C$ is considered constant for the duration of the simulation. The initial state is chosen as in Maidi et al. (2009). By solving the LMI (18) using the YALMIP toolbox proposed by Lofberg (2004), we find: $P(z) = \mathrm{diag}(3.7646e^{-0.0012z},\ 4.9098e^{0.0012z})$. From Eq. (20) the numerical value of proportional gain is $K_p = e^{-\mu}\sqrt{\frac{v_1p_1}{v_2p_2}} = 0.8787$. The numerical value of $K_i = 0.9k_i^*K_I = 0.1319$ is obtained from expressions (5),(23),(25) and (26), for a value of $k_i^* = 0.0886$. Figure 1a and 1b show the distribution temperatures $x_1(z,t)$ and $x_2(z,t)$. Figure 1c compares the controlled variable $x_1(1,t)$ to the reference $y_d(t)$. We observe that the values of the controller parameters give conclusive results in terms of variation of the reference.

6. CONCLUSION

As we have seen during this work, the boundary control strategy offers an interesting alternative for the control of systems of conservation laws, when observation is done at the boundary of the system. This method fits perfectly into the framework

[2] The model parameter used in this paper were identified from laboratory data from the system described in Kazaku et al. (Jul 2018).



(a) Distributed state $x_1(z,t)$.

(b) Distributed state $x_2(z,t)$.

(c) Reference, Output and control variables.

Fig. 1. Closed-loop simulation

of heat exchangers. The study was carried out using an infinite dimensional model of countercurrent heat exchangers to which we associate one of the boundary conditions a controller of type PI. First, a study on the existence and uniqueness of solutions of uncontrolled system has been carried out, and we have give a condition on the parameters of the system that ensures the regularity of control-observation problem. Next, using a recent result arising from the combination of a Lyapunov approach and operator perturbation theory, we found conditions on the gains of the regulator, which guarantee the stability and the regulation of the controlled system. The simulation results show the relevance of the proposed control strategy.

REFERENCES

Aksikas, I., Fuxman, A., Forbes, J.F., and Winkin, J.J. (2009). Lq control design of a class of hyperbolic pde systems: Application to fixed-bed reactor. *Automatica*, 45(6), 1542–1548. doi:https://doi.org/10.1016/j.automatica.2009.02.017.

Bastin, G. and Coron, J.M. (2016). *Stability and boundary stabilization of 1–D hyperbolic systems*. volume 88 of Progress in Nonlinear Differential Equations and Their Applications Subseries in Control.

Burns, J. and Cliff, E. (2014). Numerical methods for optimal control of heat exchangers. In *American control conference*, 14467821. Portland, USA.

Curtain, R. and Zwart, H. (1995). *An introduction to infinite-dimensional linear systems theory*. New York: Springer-Verlag.

Curtain, R. and Zwart, H. (2020). *Introduction to Infinite-Dimensional Systems Theory: A State-Space Approach*, volume 71 of *Texts in Applied Mathematics book series*. Springer New York, United States. doi:10.1007/978-1-0716-0590-5.

Dehaye, J.R. and Winkin, J.J. (2016). Lq-optimal boundary control of infinite-dimensional systems with yosida-type approximate boundary observation. *Automatica*, 67, 94–106. doi:https://doi.org/10.1016/j.automatica.2015.12.033.

Grabowski, P. (2007). Stability of a heat exchanger feedback control system using the circle criterion. *International Journal of Control*, 80(9), 1388–1403. doi:10.1080/00207170701291744. URL `https://doi.org/10.1080/00207170701291744`.

Huang, F. (1985). Characteristic conditions for exponential stability of linear dynamical systems in hilbert spaces. *Ann. of Diff. Eqs.*, 1, 43–56.

Kazaku, J.K., Dochain, D., Kahilu, M.M., and Kaunde Kasongo, J.K. (2022a). Modelling and thermodynamics analysis of counterflow heat exchanger. In *In preparation*.

Kazaku, J.K., Dochain, D., Kahilu, M.M., and Kaunde Kasongo, J.K. (2022b). Optimal observer-based lq-feedback regulation for hyperbolic model of a countercurrent heat exchanger. In *to appear in proc. ACC*.

Kazaku, J.K., Dochain, D., Winkin, J., Kahilu, M.M., and Kaunde Kasongo, J.K. (2020). Port-hamiltonian sliding mode observer design for a counter-current heat exchanger. *IFAC-PapersOnLine*, 53(2), 4910–4915. doi: https://doi.org/10.1016/j.ifacol.2020.12.1066. 21th IFAC World Congress.

Kazaku, J., Kasongo, J., Banka, R., and Kadinda, F. (Jul 2018). Contribution à l'identification par logique floue d'un échangeur de chaleur à tube concentrique. In *4ème conférence sur les applications pratiques de l'intelligence artificielle*, pp:33 – 38. Nancy, France.

Lofberg, J. (2004). Yalmip : a toolbox for modeling and optimization in matlab. In *2004 IEEE International Conference on Robotics and Automation (IEEE Cat. No.04CH37508)*, 284–289. doi:10.1109/CACSD.2004.1393890.

Maidi, A., Diaf, M., and Corriou, J. (2009). Boundary geometric control of a counter-current heat exchanger. *Journal of process control*, 19, 297 – 313.

Pohjolainen, S. (1982). Robust multivariable pi-controller for infinite dimensional systems. *IEEE Transactions on Automatic Control*, 27(1), 17–30. doi: 10.1109/TAC.1982.1102887.

Prieur, C. and Winkin, J.J. (2018). Boundary feedback control of linear hyperbolic systems: Application to the saint-venant–exner equations. *Automatica*, 89, 44 – 51.

Terrand-Jeanne, A., Andrieu, V., Martins, V.D.S., and Xu, C.Z. (2020). Adding integral action for open-loop exponentially stable semigroups and application to boundary control of pde systems. *IEEE Transactions on Automatic Control*, 65(11), 4481–4492. doi:10.1109/TAC.2019.2957349.

Tucsnak, M. and Weiss, G. (Boston, 2009). *Observation and Control for Operator Semigroups*. Birkhäuser Basel.

Tucsnak, M. and Weiss, G. (2014). Well-posed systems—the lti case and beyond. *Automatica*, 50(7), 1757 – 1779.

Weiss, G. (1994). Transfer functions of regular linear systems. i. characterizations of regularity. *Transactions of the American Mathematical Society*, 342(2), 827–854.

Xu, C.Z. and Jerbi, H. (1995). A robust pi-controller for infinite-dimensional systems. *International Journal of Control*, 61(1), 33–45. doi:10.1080/00207179508921891.

Xu, C.Z. and Sallet, G. (2014). Multivariable boundary pi control and regulation of a fluid flow system. *Mathematical Control and Related Fields*, 4(4), 501–520.