Odd Khovanov Homology and Higher Representation Theory

Léo Schelstraete   Pedro Vaz

Abstract

We define a supercategorification of the q-Schur algebra of level two and an odd analogue of $\mathfrak{gl}_2$-foams. Using these constructions, we define a homological invariant of tangles, and show that it coincides with odd Khovanov homology when restricted to links. This gives a representation theoretic construction of odd Khovanov homology. In the process, we define a tensor product on the category of chain complexes in super-2-categories which is compatible with homotopies. This could be of independent interest.

Contents

1 Introduction

Odd Khovanov homology [47] is a homological invariant of links categorifying the Jones polynomial. It agrees with Khovanov homology [34] modulo 2, but the two homologies are distinct in the sense that one can find pairs of knots distinguished by one but not the other, and vice-versa [55]. The appearance of odd Khovanov homology sparked interests in finding odd analogues to known categorified and geometric structures [6, 7, 8, 20, 22, 27, 28, 29, 31, 32, 38, 45], motivated by their relation to Khovanov homology. In particular, Khovanov homology is known to be related to higher representation theory by the work of Webster [58] using categorification of tensor products and the work of Lauda, Queffelec and Rose [37] using categorical skew Howe duality. In this article, we give an odd analogue of the latter approach, initiated in [56], leading to a representation theoretic construction of odd Khovanov homology.

In a nutshell, Lauda–Queffelec–Rose's construction can be described as follows. Recall that the q-Schur algebra of level two is the algebra of fundamental representations of $U_q(\mathfrak{gl}_2)$. Lifting skew Howe duality [10] to the categorical level, one can define a categorification of the latter as a cyclotomic quotient of categorified $U_q(\mathfrak{gl}_n)$ [42]; in particular, this categorifies the R-matrix as a length-two complex. The topological (or combinatorial) counterpart of this construction is given by $\mathfrak{gl}_2$-*foams* [3, 35], singular surfaces providing a suitable notion of cobordisms between webs. Mapping categorified intertwiners to $\mathfrak{gl}_2$-foams defines a so-called *foamation 2-functor*:

$$\left\{\begin{array}{c}\text{categorification}\\ \text{of the } q\text{-Schur algebra}\\ \text{of level two}\end{array}\right\} \xrightarrow{\text{foamation 2-functor}} \left\{\begin{array}{c}\text{2-category}\\ \text{of } \mathfrak{gl}_2\text{-foams}\end{array}\right\}$$

Finally, using the foamation 2-functor one can identify the invariant arising from the categorified R-matrix with Khovanov homology, or rather with its extension to tangles defined using $\mathfrak{gl}_2$-foams [3].

The odd analogue of a 2-category is a *super-2-category* [6]. In [56], the second author defined a superalgebra which can be seen as an odd analogue of the KLR algebra [36, 51] of level two for the A_n quiver. Taking a cyclotomic quotient of this construction leads to a supercategorification of the negative half of the q-Schur algebra of level two. We extend it to a supercategorification of the q-Schur algebra of the level two (Section 3), giving an odd analogue of [42] in the level two case. We also introduce *super* $\mathfrak{gl}_2$-*foams* as some singular surfaces equipped with a suitable Morse function (Section 2) and explain how they fit into a super-2-category. This gives an odd analogue of the 2-category of $\mathfrak{gl}_2$-foams, together with an odd analogue of the foamation 2-functor:

$$\left\{\begin{array}{c}\textit{super}\text{categorification}\\ \text{of the } q\text{-Schur algebra}\\ \text{of level two}\end{array}\right\} \xrightarrow{\textit{super}\text{foamation 2-functor}} \left\{\begin{array}{c}\textit{super}\text{-2-category}\\ \text{of } \textit{super } \mathfrak{gl}_2\text{-foams}\end{array}\right\}$$

In Section 4, we use super $\mathfrak{gl}_2$-foams to define a homological invariant of tangles. We then show that it coincides with odd Khovanov homology when restricted to links. As our invariant coincides with *not even Khovanov homology* defined in [56] through the super foamation 2-functor, not even Khovanov homology coincides with odd Khovanov homology when restricted to links, as conjectured in [56]. Finally, constructing our invariant requires a suitable notion of tensor product on chain complexes in super-2-categories: this is done in Section 5. Such a tensor product was first constructed in the first author's Master thesis [54]. To the authors' knowledge, this has not yet appeared elsewhere in the literature and could be of independent interest. As this last section is more technical (although not difficult), the reader will find a minimal version in Subsection 4.1, sufficient for the purpose of Section 4.

Our invariant also gives a relatively simple extension to tangles of odd Khovanov homology. In [44], Naisse and Putyra gave another extension of odd Khovanov homology to tangles based on arc algebras, building on previous work of Putyra [48] and Naisse–Vaz [45]. They conjectured that their construction coincides with the construction of the second author in [56]. By construction, our tangle invariant coincides with the tangle invariant define by the second author in [56]. Following their conjecture, Naisse and Putyra's construction should coincide with ours.

This remains an open problem.

Supercategorification is known to be related to super Lie theory. For instance, consider the case of $\mathfrak{sl}_2$, associated to the Cartan datum consisting in a single vertex. The Lie superalgebra $\mathfrak{osp}_{1|2}$ similarly arises from the Cartan super datum consisting in a single odd vertex. Hill and Wang introduced in [30] (see also the series of papers [12, 15, 16, 17, 18, 19]) a *covering quantum group* $U_{q,\pi}(\mathfrak{sl}_2)$ defined over $\mathbb{Q}(q)[\pi](\pi^2-1)$, such that setting $\pi=1$ recovers $U_q(\mathfrak{sl}_2)$ while setting $\pi=-1$ recovers $U_q(\mathfrak{osp}_{1|2})$. On the other hand, Ellis and Lauda [28] constructed a supercategorification of $U_{q,\pi}(\mathfrak{sl}_2)$, later reformulated (and extended to other super Cartan data) by Brundan and Ellis [7] (see also the beginning of Section 3 for further references). Given those interactions, it was conjectured that an odd homology should correspond to a covering quantum group (resp. a Lie superalgebra), with odd Khovanov homology corresponding to $U_{q,\pi}(\mathfrak{sl}_2)$ (resp. $\mathfrak{osp}_{1|2}$). However, an explicit connection between odd Khovanov homology and $\mathfrak{osp}_{1|2}$ remains an open problem (see however [14, 21]). We conjecture that a further careful study of our construction can lead to such a connection.

Let us also note that under some assumptions [30], the only Lie superalgebras in finite type are the $\mathfrak{osp}_{1|2n}$ Lie superalgebras. This suggests that an odd $\mathfrak{so}_{2n+1}$-link homology should exists. This is further corroborated by the work of Mikhaylov and Witten [43] on link homologies associated to $\mathfrak{so}_{2n+1}$, where the $\mathfrak{sl}_2\cong\mathfrak{so}_3$ case is conjectured to coincide with odd Khovanov homology. See also [28] for a discussion.

In order to simplify the exposition, we restricted this introduction to the super, or $\mathbb{Z}/2\mathbb{Z}$-graded, case. In order to encompass both even and odd Khovanov homology, the rest of the article considers the more general setting of *graded-2-categories*. This includes the tensor product on chain complexes in graded-2-categories.

Also, note that in this article we do not address the problem of showing that our super-2-category of $\mathfrak{gl}_2$-foams is sufficiently non-degenerate (see Theorem 2.23 for a precise statement), on which most of our results rely. This is addressed in [53] (in preparation).

The end of this introduction provides an extended summary of the paper, with extra historical and motivation notes. We suggest the casual reader to start there.

1.1 Acknowledgments

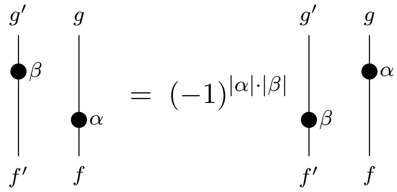
We thank James MacPherson, Grégoire Naisse, Kris Putyra, Louis-Hadrien Robert and Paul Wedrich for interesting discussions. L.S. was supported by the Fonds de la Recherche Scientifique – FNRS under the Aspirant Fellowship FC 38559. P.V. was supported by the Fonds de la Recherche Scientifique – FNRS under Grants no. MIS-F.4536.19 and CDR-J.0189.23.

1.2 Extended summary

1.2.1 Super-2-categories

A *super-2-category* [6] (or 2-supercategory) is a certain categorical structure akin to a linear 2-category where the interchange law is twisted by the data of a $\mathbb{Z}/2\mathbb{Z}$-grading, or *parity*, on the

2Hom-spaces. Diagrammatically, this is pictured as follows:



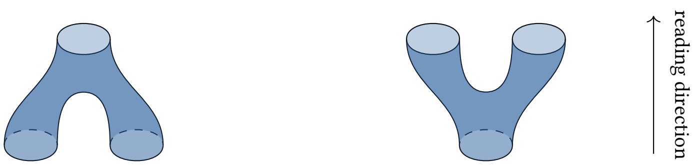
Here $|\alpha|$ and $|\beta|$ denote the respective parities of the 2-morphisms α and β. Note that in particular, a super-2-category is *not* a 2-category endowed with extra structure.

1.2.2 Odd Khovanov homology

The original construction of Khovanov homology consists in introducing a *hypercube of resolutions* associated to every link diagram, using a suitable 2-dimensional TQFT to algebrize the hypercube, and turning the hypercube into a chain complex by assigning signs to its edges following the Koszul rule. In that case, the commutative Frobenius algebra associated to the TQFT is the algebra $\mathbb{Z}[x]/x^2$.

Odd Khovanov homology is constructed in a similar fashion, only with the exterior algebra $\wedge(x_1,\ldots,x_n)$ taking the role of $\mathbb{Z}[x_1,\ldots,x_n]/(x_1^2,\ldots,x_n^2)$. Of course, the exterior algebra is not a commutative Frobenius algebra, and so the associated TQFT is only a *projective* TQFT in the sense that it is only functorial up to signs. Somehow miraculously, it was shown in [47] that this defect in functoriality can be balanced out when assigning signs to the hypercube. This however requires a much more intricate sign assignment than the Koszul rule, based on a case-by-case analysis of possible squares in the hypercube.

As the anti-commutativity in the exterior algebra is controlled by a $\mathbb{Z}/2\mathbb{Z}$-grading, it is natural to wonder whether one could give a construction of odd Khovanov homology using a super-2-category. Ideally, the superstructure would control all signs appearing in odd Khovanov homology, that is, all interchanges of saddles. One solution, pursued by Putyra [48], is to pull-back the TQFT to linear relations on cobordisms. This provides an analogue for odd Khovanov homology of Bar-Natan's "picture-world" construction of (even) Khovanov homology. In this context, a merge is even and a split is odd:



even *odd*

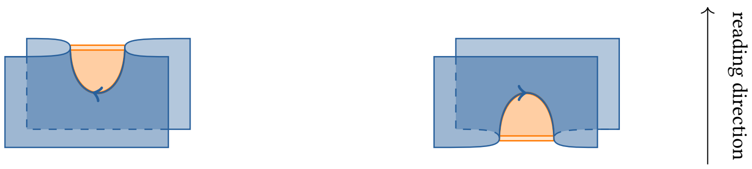
However, the superstructure only partially controls interchanges of saddles, and one still needs to use the intricate sign assignment from the original construction. Also, it does not generalize in an obvious way to tangles, as whether a saddle is a split or a merge is a global property. See however [44] for an answer to this.

This suggests that one should look further away from the original construction. Heuristically, it is plausible that different choices of TQFT's, identical up to signs, lead to the same invariant. After all, this is the take-home message of odd Khovanov homology: sign issues on the level of the TQFT can be balanced out when choosing a sign assignment on the hypercube. In this article, we show that a solution is given by super $\mathfrak{gl}_2$-foams.

1.2.3 Super $\mathfrak{gl}_2$-foams

An important early-day problem on Khovanov homology was also about signs. Namely, Khovanov's original construction is not properly functorial under link cobordisms, but only so up to signs. Solutions to this problem were provided by numerous authors [9, 13, 52, 57]. A solution introduced by Blanchet [3] used *foams*, certain decorated singular surfaces first introduced by Khovanov in his definition of $\mathfrak{sl}_3$-Khovanov homology [35] and generalized to $\mathfrak{gl}_n$ for $n \geq 3$ in [41]. Adapting this construction to the $n = 2$ case, Blanchet defined a functorial version of Khovanov homology using $\mathfrak{gl}_2$*-foams.*[1] The proof of functoriality was later generalized to all $\mathfrak{gl}_n$-link homologies in [25]. In practice, working with $\mathfrak{gl}_2$ instead of $\mathfrak{sl}_2$ leads to better-behaved constructions. This comes down to the fact that in the former case, the fundamental representations $\bigwedge^0(\mathbb{C}(q)^2)$ and $\bigwedge^2(\mathbb{C}(q)^2)$ are not isomorphic, and keeping track of this distinction leads to a better control on signs (see [37, 1F] for a discussion).

In this article, we show that the same heuristic gives control on signs in odd Khovanov homology. That is, we define a super-2-category of $\mathfrak{gl}_2$-foams (see Section 2) and use it to define a homological invariant of oriented tangles that coincides with odd Khovanov homology when restricted to links. Note that by using $\mathfrak{gl}_2$-foams instead of cobordisms with corners, saddles come in two kinds, either *zip* or *unzip*:



even *odd*

In this way, we have an *even saddle* and an *odd saddle*. This is the main ingredient that leads to fundamentally different properties in the super-2-category of $\mathfrak{gl}_2$-foams, compared to its non-super counterpart.

1.2.4 Odd Khovanov homology is a super tensor product of chain complexes

Once given the super-2-category of $\mathfrak{gl}_2$-foams, defining a homological invariant of oriented tangles is straightforward in most aspects. It follows the usual scheme of a categorified Kauffman bracket: we assign length-two complexes to crossings and take an appropriate tensor product (more precisely, a horizontal composition). This assigns a complex to every sliced oriented tangle diagram, whose homotopy type is shown to be an invariant of the associated oriented tangle.

The only step requiring extra work is the last step: taking an appropriate tensor product. This is done in Section 5 for a subclass of chain complexes called *polyhomogeneous complexes.* A *homogeneous complex* is a chain complex in a super-2-category such that each differential is homogeneous (although the parity can differ at distinct homological degrees), and a polyhomogeneous complexes is a tensor product of homogeneous complexes. This tensor product is coherent with homotopies in the following sense:

Main theorem A (Theorem 5.10)**.** *In any super-2-category, there exists a well-defined tensor product on polyhomogeneous complexes such that if $A_1^\bullet$ and $A_2^\bullet$ (resp. $B_1^\bullet$ and $B_2^\bullet$) are homotopic polyhomogeneous complexes, then so are $A_1^\bullet \otimes B_1^\bullet$ and $A_2^\bullet \otimes B_2^\bullet$.*

[1] Blanchet called them *enhanced* $\mathfrak{sl}_2$*-foams*, but as they were later understood to be related to $\mathfrak{gl}_2$ rather than $\mathfrak{sl}_2$, we call them $\mathfrak{gl}_2$-foams.

If all differentials are even, this recovers the usual Koszul rule for chain complexes in linear 2-categories.

In Section 4, we describe the construction of a tangle invariant using super $\mathfrak{gl}_2$-foams. We then show the following theorem, which is the main result of this paper:

Main theorem B (Theorem 4.3)**.** *Our construction coincides with odd Khovanov homology when restricted to links.*

1.2.5 A supercategorification of the q-Schur algebra of level two

As explained already in the beginning of the introduction, we construct in Section 3 a supercategorification of the q-Schur algebra of level two, together with a super foamation 2-functor into the super-2-category of $\mathfrak{gl}_2$-foams. This gives in particular a supercategorification of the R-matrix, whose image under the super foamation 2-functor recovers our previously constructed invariant of tangles. In particular:

Main theorem C. *The supercategorification of the q-Schur algebra of level two together with the super foamation 2-functor provide a representation theoretic construction of odd Khovanov homology.*

See the beginning of Section 3 for more references.

2 A Graded-2-category of $\mathfrak{gl}_2$-foams

In this section, we define a graded-2-category of $\mathfrak{gl}_2$-foams $\mathbf{GFoam}_d$ that categorifies the category of $\mathfrak{gl}_2$-webs $\mathbf{Web}_d$. This can be viewed as a graded analogue of $\mathfrak{gl}_2$-foams *à la* Blanchet [3]. See also [2, 23, 24] for further studies of $\mathfrak{gl}_2$-foams. More precisely, the graded-2-category $\mathbf{GFoam}_d$ gives a graded analogue of the category of *directed* $\mathfrak{gl}_2$-foams as defined in [49] (see Remark 2.16).

We define graded-2-categories in Subsection 2.1. Let us emphasize that a graded-2-category is *not* a 2-category with extra structure. Subsection 2.2 recalls the definition of $\mathbf{Web}_d$ and Subsection 2.3 defines $\mathbf{GFoam}_d$. We then introduce in Subsection 2.4 a string diagrammatics specific to graded $\mathfrak{gl}_2$-foams: this is necessary for computation, given that isotopies now depend on the underlying Morse function, contrary to the non-graded case. Finally, Subsection 2.5 shows that $\mathbf{GFoam}_d$ categorifies $\mathbf{Web}_d$.

2.1 Graded-2-categories

We define the notion of graded-2-categories, which generalizes the notion of super-2-categories [7] to allow gradings with generic abelian groups. Taking this abelian group to be $\mathbb{Z}/2\mathbb{Z}$ recovers the notion of super-2-categories. Graded-2-categories appeared in [5].[1] A similar definition also appeared in [48] in the context of algebras, under the name of *chronological algebras*. See also [6] for an in-depth study of superstructures, including non-strict versions. Our exposition is a direct generalization to the graded case of the exposition given in [7] for super-2-categories (which they call *2-supercategories*).

Throughout the section we fix G an abelian group and R a unital commutative ring. We denote $R^\times$ the abelian group of invertible elements in R equipped with the multiplicative structure. We also fix an $R^\times$-valued bilinear form μ on G, that is a $\mathbb{Z}$-bilinear map

$$\mu\colon G \times G \to R^\times .$$

[1] We thank James MacPherson for pointing out this reference to us.

Recall that a R-module V is said to be G-*graded* if it is equipped with the data of a direct sum decomposition $V = \bigoplus_{g\in G} V_g$. We write $\deg(v) = g$ whenever $v \in V_g$. An R-linear map $f\colon \bigoplus_{g\in G} V_g \to \bigoplus_{g\in G} W_g$ between two G-graded R-modules is said to be *graded* if there exists $h \in G$ such that $f(V_g) \subset W_{g+h}$ for all $g \in G$. In that case, we say that f is of degree $\deg f = h$. If $\deg f = 0$, we say that f is *degree-preserving*. Note that this equips the R-module $\mathrm{Hom}_R(V, W)$ with a G-grading.

Denote by R-Mod_G the category of G-graded R-modules, and by R-Mod_G^0 the wide subcategory of R-Mod_G consisting only of degree-preserving R-linear maps. The category R-Mod_G admits a standard monoidal structure, defined on objects $V = \bigoplus_{g\in G} V_g$ and $W = \bigoplus_{g\in G} W_g$ by the formula

$$(V \otimes W)_g = \bigoplus_{\substack{(g_1,g_2)\in G\times G\\ g_1g_2=g}} V_{g_1} \otimes W_{g_2},$$

and on morphisms f and g by the formula $(f \otimes g)(v \otimes w) = f(v) \otimes f(w)$. However, one can give an alternative definition using the data of the grading and the bilinear form μ:

Definition 2.1. *The* (G, μ)-graded tensor product *is defined on objects as* $V \otimes_{G,\mu} W := V \otimes W$ *and on morphisms as*

$$(f \otimes_{G,\mu} g)(v \otimes_{G,\mu} w) = \mu(\deg v, \deg g) f(v) \otimes_{G,\mu} g(w).$$

Equipped with this graded tensor product, R-Mod_G is *not* in general a monoidal category. Indeed, morphisms respect the following *graded interchange relation*:

$$(f \otimes_{G,\mu} g) \circ (h \otimes_{G,\mu} k) = \mu(\deg g, \deg h)(f \circ h) \otimes_{G,\mu} (g \circ k). \tag{1}$$

We now define the proper categorical structures that encompass this behaviour.

Definition 2.2. *A* G-graded R-linear category *is a* $(R\text{-}\mathrm{Mod}_G^0, \otimes)$*-enriched category. A* G-graded R-linear functor *is a* $(R\text{-}\mathrm{Mod}_G^0, \otimes)$*-enriched functor.*

In other words, A G-graded R-linear category is a category such that each Hom is a G-graded R-module, and such that composition is R-bilinear and preserves the grading in the sense that $\deg(f \circ g) = \deg f + \deg g$. A G-graded R-linear functor is a functor between two G-graded R-linear categories that restricts to a degree-preserving G-graded R-linear map on Homs.

Denote by R-$\mathcal{C}at_G$ the category of small G-graded R-linear categories and G-graded R-linear functors. For $\mathcal{A}$ and $\mathcal{B}$ two G-graded R-linear categories, their (G, μ)-*graded-cartesian tensor product* is the G-graded R-linear category $\mathcal{A} \boxtimes_{G,\mu} \mathcal{B}$ such that $\mathrm{ob}(\mathcal{A} \boxtimes_{G,\mu} \mathcal{B}) := \mathrm{ob}(\mathcal{A}) \times \mathrm{ob}(\mathcal{B})$ and

$$\mathrm{Hom}_{\mathcal{A}\boxtimes_{G,\mu}\mathcal{B}}((a, b), (c, d)) := \mathrm{Hom}_{\mathcal{A}}(a, c) \otimes_{G,\mu} \mathrm{Hom}_{\mathcal{B}}(b, d).$$

Most importantly, the composition in $\mathcal{A}\boxtimes_{G,\mu}\mathcal{B}$ is given by the graded interchange relation (1). This makes $\mathcal{A} \boxtimes_{G,\mu} \mathcal{B}$ a G-graded R-linear category. Moreover, one can check that $\boxtimes_{G,\mu}$ is associative and unital, giving R-$\mathcal{C}at_G$ the structure of a monoidal category.

Definition 2.3. *A* (G, μ)-graded-2-category *is a* R-$\mathcal{C}at_G$*-enriched category. A* (G, μ)-graded-2-functor *is a* R-$\mathcal{C}at_G$*-enriched functor.*

In particular:

Definition 2.4. *A* (G, μ)-graded-monoidal category *is a G-graded R-linear category $\mathcal{C}$ together with the data of a unit object and a G-graded R-linear functor $\otimes_{G,\mu}\colon \mathcal{C} \boxtimes_{G,\mu} \mathcal{C} \to \mathcal{C}$ satisfying the same unital and associativity axioms as for a strict monoidal category.*

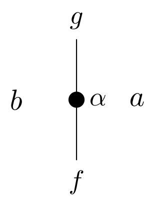
For instance, the (G,μ)-graded tensor product from Definition 2.1 assembles into a G-graded R-linear functor $\otimes_{G,\mu}\colon R\text{-Mod}_G \boxtimes_{G,\mu} R\text{-Mod}_G \to R\text{-Mod}_G$, making $R\text{-Mod}_G$ a (G,μ)-graded-monoidal category.

Any 2-category that is both R-linear and G-graded as a R-linear 2-category is a (G,μ)-graded-2-category, with μ the trivial map. In general though, the converse is not true. In particular, a generic (G,μ)-graded-2-category is *not* a 2-category.

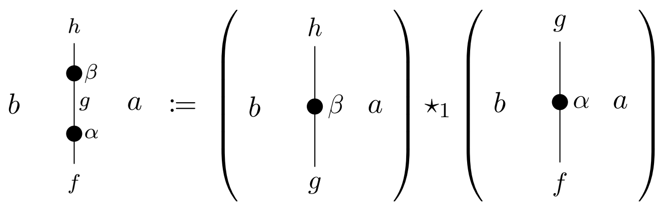
Remark 2.5. Instead, a graded-2-category can be viewed as rather degenerate instance of a *(3,2)-linear Gray category*. Recall that a Gray category is a semi-strict 3-category with particular 3-morphisms, called *interchange generators*, that capture the interchange law on 2-morphisms. It is *(3,2)-linear* if it has a linear structure on both 2-morphisms and 3-morphisms. A graded-2-category is a (3,2)-linear Gray category whose 3-morphisms only consist of interchange generators.

2.1.1 Diagrammatics for graded-2-categories

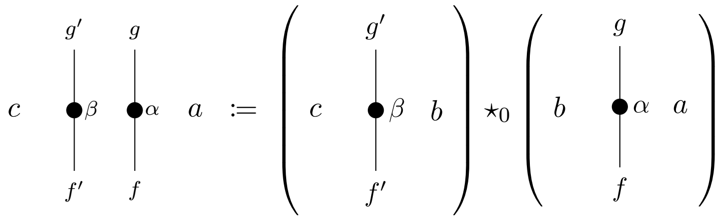
Let $\mathcal{C}$ be a (G,μ)-graded-2-category. Unpacking Definition 2.3, $\mathcal{C}$ consists of objects $\mathcal{C}_0$ together with a G-graded R-linear category $\mathcal{C}(a,b)$ for each pair of objects (a,b). We denote $\mathcal{C}_1(a,b)$ its objects and for $f,g\in\mathcal{C}_1(a,b)$, we denote $\mathcal{C}_2(f,g)$ the Hom-space of morphisms from f to g. Elements of $\mathcal{C}_1(a,b)$ and $\mathcal{C}_2(f,g)$ are respectively called *1-morphisms* and *2-morphisms*. A 2-morphism $\alpha\in\mathcal{C}_2(f,g)$ can be pictured using *string diagrams*, akin to the string diagrammatics for 2-categories:



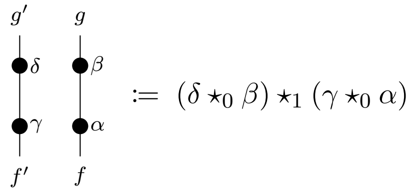
Note that we read from right to left and from bottom to top. 2-morphisms in $\mathcal{C}$ also comes equipped with two compositions, the *horizontal composition* $\star_0$ and the *vertical composition* $\star_1$. The vertical composition denotes all the compositions in the categories G-graded R-linear $\mathcal{C}(a,b)$. It is pictured by stacking the 2-morphisms atop each other:



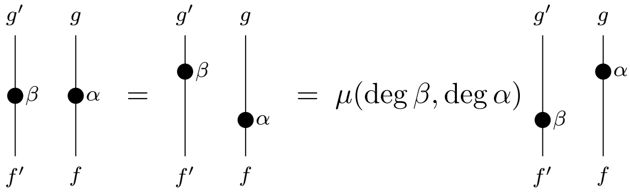
The horizontal composition is the composition for the *underlying category of* $\mathcal{C}$, that is, the category obtained by forgetting the data of the 2-morphisms. It extends to a linear and degree-preserving horizontal composition on 2-morphisms. It is pictured by putting the two 2-morphisms side-by-side:



To avoid clutter, we leave regions unlabelled for now. Ambiguity in the order of compositions is resolved by the rule “compose first horizontally, then vertically”. For instance:



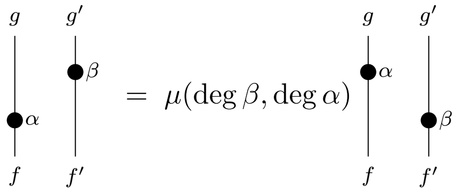
Contrary to 2-categories, it is in general *not* the same as “compose first vertically, then horizontally”. Indeed, in a graded-2-category sliding a 2-morphism past another vertically comes at the price of an invertible scalar. This is the graded interchange law:



In particular, one must be careful with the relative vertical positions of 2-morphisms. To avoid confusion, in this paper we always draw string diagrams such that no two 2-morphisms lie on the same vertical level. One can make an exception for 2-morphisms with trivial grading, as those can slide vertically without adding scalars. Finally, as customary already in the string diagrammatics of 2-categories, we usually do not picture identities.

Definition 2.6. *We say that μ is* symmetric *if $\mu(g,h)\mu(h,g) = 1$ for all $g, h \in G$.*

Note that μ is automatically symmetric in the super case $G = \mathbb{Z}/2\mathbb{Z}$. If μ is symmetric, the scalar appearing in the graded interchange law depends *only* on the relative vertical positions of the 2-morphisms:



In other words, passing α *up* β always adds the scalar $\mu(\deg\alpha, \deg\beta)$, regardless of their respective horizontal position.

2.1.2 More categorical preliminaries

We review the graded additive closure of a (G,μ)-graded-2-category, as well as its Grothendieck group and its Karoubi envelope. This follows the standard definitions for 2-categories.

We refer to [26, chap. 11] for a review on additive and graded closure at the one-categorical level. Fix a (G,μ)-graded-2-category $\mathcal{C}$. We say that $\mathcal{C}$ is $\mathbb{Z}$-*graded* if it is enriched over the category of pre-additive categories. Note that this grading is in principle independent of the G-grading. We say that $\mathcal{C}$ is *with shifts* if it admits a (G,μ)-graded-2-automorphism $\{1\}$ which is the identity on objects. In particular, each G-graded-category $\mathcal{C}(a,b)$ is with shifts. Any $\mathbb{Z}$-graded (G,μ)-graded-2-category $\mathcal{C}$ naturally embeds in a (G,μ)-graded-2-category with shifts $\mathcal{C}^{\mathrm{sh}}$, its *graded closure*, by taking the graded closure of each G-graded-category $\mathcal{C}(a,b)$ and setting

$$(\alpha\colon f\{n\}\to g\{m\})\star_0(\alpha'\colon f'\{n'\}\to g'\{m'\}) = (\alpha\star_0\alpha')\colon (f\star_0 f')\{n+n'\}\to(g\star_0 g')\{m+m'\}$$

The G-grading of $\alpha\colon f\{n\}\to g\{m\}$ is the G-grading of $\alpha\colon f\to g$.

A (G,μ)-graded-2-category $\mathcal{C}$ is *additive* if each G-graded R-linear category $\mathcal{C}(a,b)$ is additive and the horizontal composition is bilinear with respect to the direct sum. Any (G,μ)-graded-2-category $\mathcal{C}$ naturally embeds in an additive (G,μ)-graded-2-category $\mathcal{C}^{\oplus}$, its *additive closure*, by taking the additive closure of each G-graded R-linear category $\mathcal{C}(a,b)$, and extending the horizontal composition bilinearly. If $\mathcal{C}$ is $\mathbb{Z}$-graded, then so is $\mathcal{C}^{\oplus}$. The *graded additive closure of* $\mathcal{C}$ is the additive closure of its graded closure. In the context of an additive (G,μ)-graded-2-category with shifts, we use the shorthand

$$f^{\oplus[n]} = f\{-n+1\}\oplus f\{-n\}\oplus\ldots\oplus f\{n-1\}$$

with f a 1-morphism and $n\in\mathbb{N}$.

If $\mathcal{C}$ is an additive (G,μ)-graded-2-category, its *(split) Grothendieck group* $K_0(\mathcal{C})$ is the pre-additive category with the same objects as $\mathcal{C}$ and with each hom-space $K_0(\mathcal{C})(a,b)$ being the split Grothendieck group $K_0(\mathrm{Hom}_{\mathcal{C}}(a,b))$. If $\mathcal{C}$ is with shifts, then $K_0(\mathcal{C})$ is a $\mathbb{Z}[q,q^{-1}]$-linear category.

If $\mathcal{C}$ is an additive (G,μ)-graded-2-category, its *Karoubi envelope* $\mathrm{Kar}(\mathcal{C})$ is the (G,μ)-graded-2-category with the same objects as $\mathcal{C}$ and with each hom-space $\mathrm{Kar}(\mathcal{C})(a,b)$ being the Karoubi envelope $\mathrm{Kar}(\mathcal{C}(a,b))$. If $\mathcal{C}$ is with shifts, then so is $\mathrm{Kar}(\mathcal{C})$.

2.2 $\mathfrak{gl}_2$-webs

In our context, a $\mathfrak{gl}_2$-web is a certain trivalent graph, smoothly embedded in $\mathbb{R}\times[0,1]$, that we view as a morphism in a certain category defined below. Objects, called *weights*, are elements of the following set:

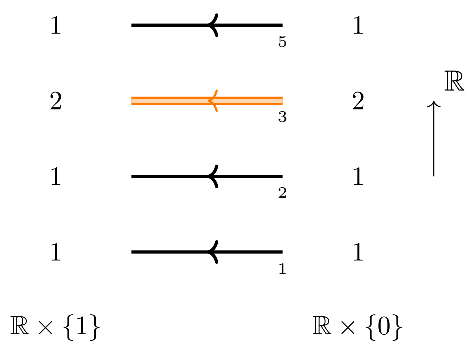
$$\underline{\Lambda}_d \coloneqq \bigsqcup_{k\in\mathbb{N}}\{\lambda\in\{1,2\}^k \mid \lambda_1+\ldots+\lambda_k = d\}.$$

For each $\lambda\in\Lambda_d$ with k coordinates, we can define a labelling on its coordinates

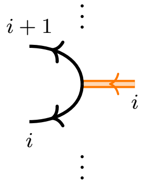
$$l_\lambda\colon\{1,\ldots,k\}\to\{1,\ldots,d\}$$

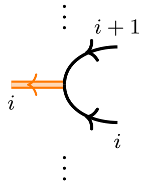
by setting $l_\lambda(i)=\sum_{j<i}\lambda_i+1$. For instance, $l_{(1,1,2,1)}=(1,2,3,5)$. Foreseeing the string diagrammatics, we call this label the *colour* of the coordinate. The identity web of a weight is pictured as a juxtaposition of straight vertical lines in $\mathbb{R}\times[0,1]$, decorated as *single* (black) or *double* (orange) lines:

$\mathrm{id}_{(1,1,2,1)}$ $=$



Note that we read webs from right to left. Note also that each line has a colour, given by the colour of the corresponding coordinate. Effectively, the colour of a line counts the number of preceding lines (plus one), counting twice double lines. A generic $\mathfrak{gl}_2$-*web* is either an identity web or a composition of the following generating webs:

$W_{i,-} =$ 

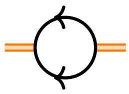
and $W_{i,+} =$ 

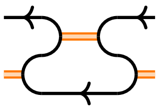
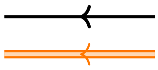
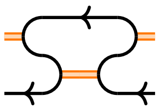
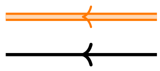
Hereabove the dots denote possibly additional vertical single or double lines, and composition is given by stacking webs atop each other (reading from right to left). Note that a web W has an underlying unoriented flat tangle diagram, denoted $c(W)$, given by forgetting the double lines and the orientations.

Remark 2.7 (Orientation)**.** Our webs are given an orientation "flowing" from $\mathbb{R} \times \{0\}$ to $\mathbb{R} \times \{1\}$, which is more restrictive than some definitions in the literature. Those kind of webs are sometimes called *acyclic* or *left-directed* (e.g. in [49]). As this orientation is canonical, we often omit it.

Definition 2.8. *The category* $\mathbf{Web}_d$ *has objects* $\underline{\Lambda}_d$*, and morphisms are* $\mathbb{Z}[q, q^{-1}]$*-linear combinations of* $\mathfrak{gl}_2$*-webs, up to the following* web relations*:*

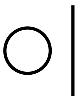
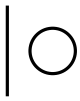
$$W_{i,s_1} W_{j,s_2} = W_{j,s_1} W_{i,s_2}$$
(*for all* $s_1, s_2 \in \{-, +\}$ *and* $|i - j| > 1$) — *exchange relations*

 $= (q + q^{-1})$  — *circle evaluation*

 $=$  *and*  $=$  — *isotopies of flat tangle diagrams*

Recall the following notion:

Definition 2.9. *A* spherical isotopy *of flat tangle diagrams is a usual isotopy with the addition of the following* spherical move*:*

 $\sim$ 

The relations in $\mathbf{Web}_d$ fully capture the spherical isotopy classes of the underlying flat tangle diagrams, in the sense of the following lemma.

Lemma 2.10. *Let* W *and* W' *be two* $\mathfrak{gl}_2$*-webs with the same domain and codomain. Then* W *and* W' *are equal in* $\mathbf{Web}_d$ *if and only if there exists a spherical isotopy between* $c(W)$ *and* $c(W')$*.*

The proof is the analogue of Lemma 3.9 for webs. We leave the details to the reader.

2.3 Graded $\mathfrak{gl}_2$-foams

Foams provide a suitable notion of cobordisms between webs. They are certain singular surfaces locally modeled on the product of the interval $I := [0,1]$ with the letter "Y" (see Fig. 2.1), embedded in $(\mathbb{R} \times I) \times I$. In Fig. 2.1, $\mathbb{R}$ is pictured from front to back, while the last interval is picture from bottom to top. In this context, we refer to the singular curves as *seams*, and call *facets* the components of the complement of the set of seams. Topologically, a facet is simply a two-dimensional disk. Facets have thickness, either single or double, and we refer to them respectively as *1-facets* (or *single facets*, shaded black) and *2-facets* (or *double facets*, shaded orange). We refer to *cross-sections* as cross-sections of the projection $\pi\colon (\mathbb{R} \times I) \times I \to I$ onto the last coordinate (pictured vertically).

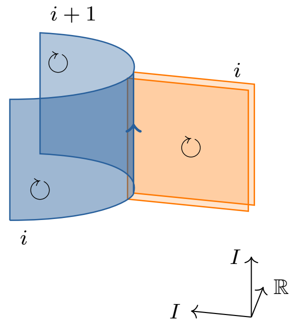
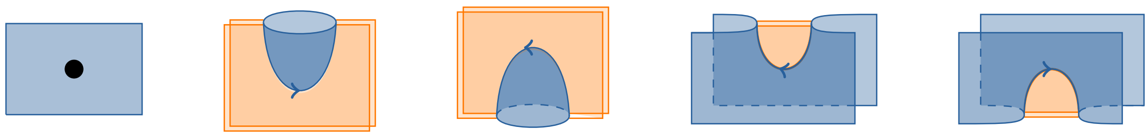


Figure 2.1: Local model for foams.

Definition 2.11. *A* graded $\mathfrak{gl}_2$-foam, *or simply* foam, *is a topological singular surface embedded in $(\mathbb{R} \times I) \times I$, such that generic cross-sections are webs and all non-generic cross-sections respect one the following local behaviours:*

dot   *cup*   *cap*   *zip*   *unzip*



$(-1,-1)$ $(0,1)$ $(1,0)$ $(-1,0)$ $(0,-1)$

where the axes are oriented as in Fig. 2.1. Each such local behaviour F is endowed with a $\mathbb{Z}^2$-degree $\deg_{\mathbb{Z}^2}(F)$*, denoted below the associated picture. Extending additively, this induces a $\mathbb{Z}^2$-grading on graded $\mathfrak{gl}_2$-foams.*

We also call the above local behaviours *generating foams*, or simply *generators*. A foam $F \subset (\mathbb{R} \times I) \times I \to I$ is viewed as a morphism from the web $F \cap \pi^{-1}(\{0\})$ to the web $F \cap \pi^{-1}(\{1\})$, reading from bottom to top. The *dot* in the first picture of Definition 2.11 is a formal decoration that any 1-facet may additionally carry. By removing the 2-facets, a foam F has an underlying surface, denoted $c(F)$. We assume that $c(F)$ is smooth and that the vertical projection π defines a separative Morse function for $c(F)$, where dots are considered as critical points. Note that the local behaviours in Definition 2.11 dictate the local behaviour around critical points of $c(F)$.

The facets of a foam F admit a canonical orientation, induced by the canonical orientation on the web $F \cap \pi^{-1}(\{0\})$. That is, we endow the facets incident to $F \cap \pi^{-1}(\{0\})$ with an orientation compatible with the canonical orientation of $F \cap \pi^{-1}(\{0\})$, and extend globally with the condition that at a given seam, the orientation induced by the 2-facet is opposite to the orientation induced by the two 1-facets (see Fig. 2.1). In particular, the facets incident to $F \cap \pi^{-1}(\{1\})$ induce an orientation on $F \cap \pi^{-1}(\{1\})$ opposite to its canonical orientation. Similarly, facets are endowed with a colour induced by the colour on webs. This also defines an orientation and colour on each seam, induced from the orientation and colour of its incident 2-facet.

As in the non-graded case, we will consider graded $\mathfrak{gl}_2$-foams up to isotopies. In our context, an *isotopy* is a boundary-preserving isotopy which preserves the cross-section condition in Definition 2.13. Sliding a dot along its 1-facet is also considered as an isotopy. Moreover, we assume that the restriction of an isotopy to the underlying surface is a diffeotopy.

We distinguish different kind of isotopies, in analogy with the property of the corresponding diffeotopy for the underlying surface:

- If the isotopy preserves the relative vertical positions of the generating foams, we say that it is a *Morse-preserving isotopy.*
- If the isotopy only interchange the vertical positions of generating foams, we say that it is a *Morse-singular isotopy.*
- If the isotopy is such the underlying diffeotopy is a birth-death diffeotopy,[1] we say that the isotopy is a birth-death isotopy.

We refer to [48] for relevant details on Morse theory.

In the graded case, isotopies lead to additional scalar. We formalize this behaviour with a graded-2-category of graded $\mathfrak{gl}_2$-foams. The graded linear structure is given by the abelian group $\mathbb{Z}^2$, the ring R and bilinear form b defined below.

Definition 2.12. *For $\Bbbk$ a unital commutative ring. Let R and b be respectively the ring*

$$R := \Bbbk[X, Y, Z^{\pm 1}]/(X^2 = Y^2 = 1)$$

and the $\mathbb{Z}$-bilinear map

$$\begin{aligned} \mu\colon \mathbb{Z}^2 \times \mathbb{Z}^2 &\to R^\times, \\ ((a,b),(c,d)) &\mapsto X^{ac}Y^{bd}Z^{ad-bc}. \end{aligned}$$

Note that μ is symmetric in the sense of Definition 2.6.

Definition 2.13. *The $(\mathbb{Z}^2,\mu)$-graded-2-category $\mathbf{GFoam}_d$ has objects $\underline{\Lambda}_d$, $\mathfrak{gl}_2$-webs as 1-morphisms, and R-linear combinations of graded $\mathfrak{gl}_2$-foams as 2-morphisms, regarded up to the following relations:*

(i) If $\varphi\colon F_1 \to F_2$ is a Morse-preserving isotopy, then $F_1 = F_2$ in $\mathbf{GFoam}_d$.

(ii) If $\varphi\colon F_1 \to F_2$ is a Morse-singular isotopy interchanging the vertical positions of exactly two critical points p and q, with p vertically above q in F_1, then $F_1 = \mu(\deg p, \deg q)F_2$ in $\mathbf{GFoam}_d$.

(iii) All the local relations in Fig. 2.2 below.

Remark 2.14. The $\mathbb{Z}^2$-grading induces a $\mathbb{Z}$-grading on graded $\mathfrak{gl}_2$-foams, the *quantum grading* (or *q-grading*). If $q\colon \mathbb{Z}^2 \to \mathbb{Z}$ denotes the map $q(a,b) = a+b$, we set for each foam F:

$$\operatorname{qdeg}(F) := -q(\deg_{\mathbb{Z}^2}(F)).$$

We write $\mathbf{GFoam}_d^{\oplus,\mathrm{cl}}$ the graded additive closure of $\mathbf{GFoam}_d$ (see Subsection 2.1.2) with respect to the quantum grading. We emphasize that although the quantum grading is induced by the $\mathbb{Z}^2$-grading, taking the graded closure does not affect the graded interchange law, as

$$\deg_{\mathbb{Z}^2}(F\colon W_1\{n_1\} \to W_2\{n_2\}) = \deg_{\mathbb{Z}^2}(F).$$

[1] That is, collapsing the two singularities associated to a saddle and a cup or cap by "smoothing out" the surface, or the converse.

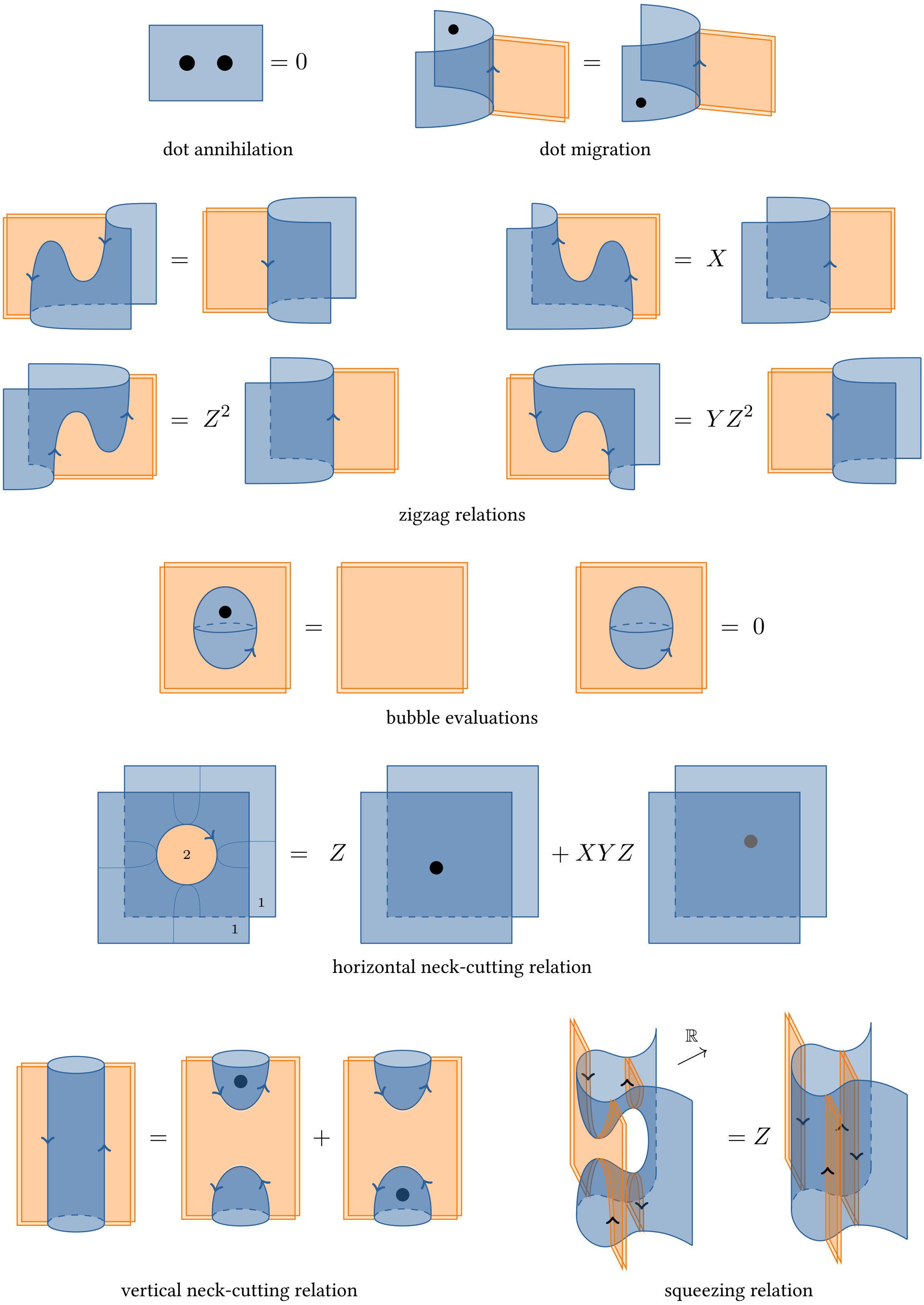


Figure 2.2: Relations in $\mathbf{GFoam}_d$. A gray dot denotes a dot behind a facet. The $\mathbb{R}$ axis is pictured from front to back, except for the squeezing relation for which it is pictured from left to right for better readability.

Similar to webs (Lemma 2.10), the following lemma shows that relations on foams capture the diffeotopy classes of the underlying surfaces; or rather, the underlying dotted surfaces, where sliding a dot along a connected component is considered to be a diffeotopy. Below we write $F \dot{\sim} F'$ whenever there exists an invertible scalar $r \in R^\times$ such that $F = rF'$.

Lemma 2.15. *Let F and F' be two foams in $\mathbf{GFoam}_d$ with the same domain and codomain. If $c(F)$ and $c(F')$ are isotopic, then $F \dot{\sim} F'$ in $\mathbf{GFoam}_d$.*

Proof. By Cerf theory ([11]; see [48, appendix A] for a review), diffeotopic surfaces are related by diffeotopies preserving the relative vertical positions of critical points, Morse-singular diffeotopies interchanging two critical points, and birth-death diffeotopies. These correspond to the relations (i), (ii) and the zigzag relations in $\mathbf{GFoam}_d$, respectively. Finally, if $c(F)$ and $c(F')$ are diffeotopic by sliding a dot along a connected component, then F and F' are related by sliding the dot along 1-facets, possibly crossings seams using dot migration. □

Remark 2.16. One can check that the $\mathbb{Z}$-graded $\Bbbk$-linear 2-category

$$\mathbf{GFoam}_d^{\oplus,\mathrm{cl}}|_{X=Y=Z=1}$$

is precisely the category $n\mathbf{Foam}(N)^\bullet$ from [49, p. 1322] with $n = 2$ and $N = d$. This essentially follows by renormalizing the i-labelled dot, the i-labelled cap and the i-labelled unzip by $(-1)^i$ and comparing the defining relations, with the help of Lemma 2.15.

Remark 2.17. The structure of graded-2-category on $\mathbf{GFoam}_d$ is in some sense universal. Indeed, on one hand the abelian group $\mathbb{Z}^2$ is isomorphic to the abelian group presented by generators $D, \cup^L, \cup^R, \cap^L, \cap^R$ and relations

$$\begin{aligned} \cap^R + \cup^R &= 0 \\ \cap^L + \cup^L &= 0 \\ \cup^R + \cap^L + D &= 0 \\ \cup^L + \cap^R &= D \\ \cup^L + \cup^R + \cap^L + \cap^R &= 0 \end{aligned}$$

obtained from taking the "abelianization" of the defining local relations in Fig. 2.2.[1] Note that this procedure does not depend on the choice of coefficients in the relations. On the other hand, the $\Bbbk$-algebra R is isomorphic to the unital commutative $\Bbbk$-algebra with generators a_{00}, a_{01}, a_{10} and a_{11} subject to the symmetry relations $a_{00}a_{00} = 1$, $a_{11}a_{11} = 1$ and $a_{01}a_{10} = 1$. Hence, the bilinear map μ is the universal symmetric bilinear map with domain $\mathbb{Z}^2 \times \mathbb{Z}^2$.

2.4 String diagrammatics

We now introduce a string diagrammatics for graded $\mathfrak{gl}_2$-foams. This is based on the observation that *a foam is fully described by its seams*;[2] or more precisely, by its domain object, its seams, and their orientation and colour. For identity foams, this holds since webs are generated by $W_{i,\pm}$, and $\mathrm{id}_{W_{i,\pm}}$ is determined by its domain and the seam, together with its orientation and colour. Thus,

[1] Note that in this process, we also identified generators of the same kind but with distinct colouring (see the discussion following Definition 2.11 for the colouring of a generator).

[2] We thank Kris Putyra for pointed that out to us.

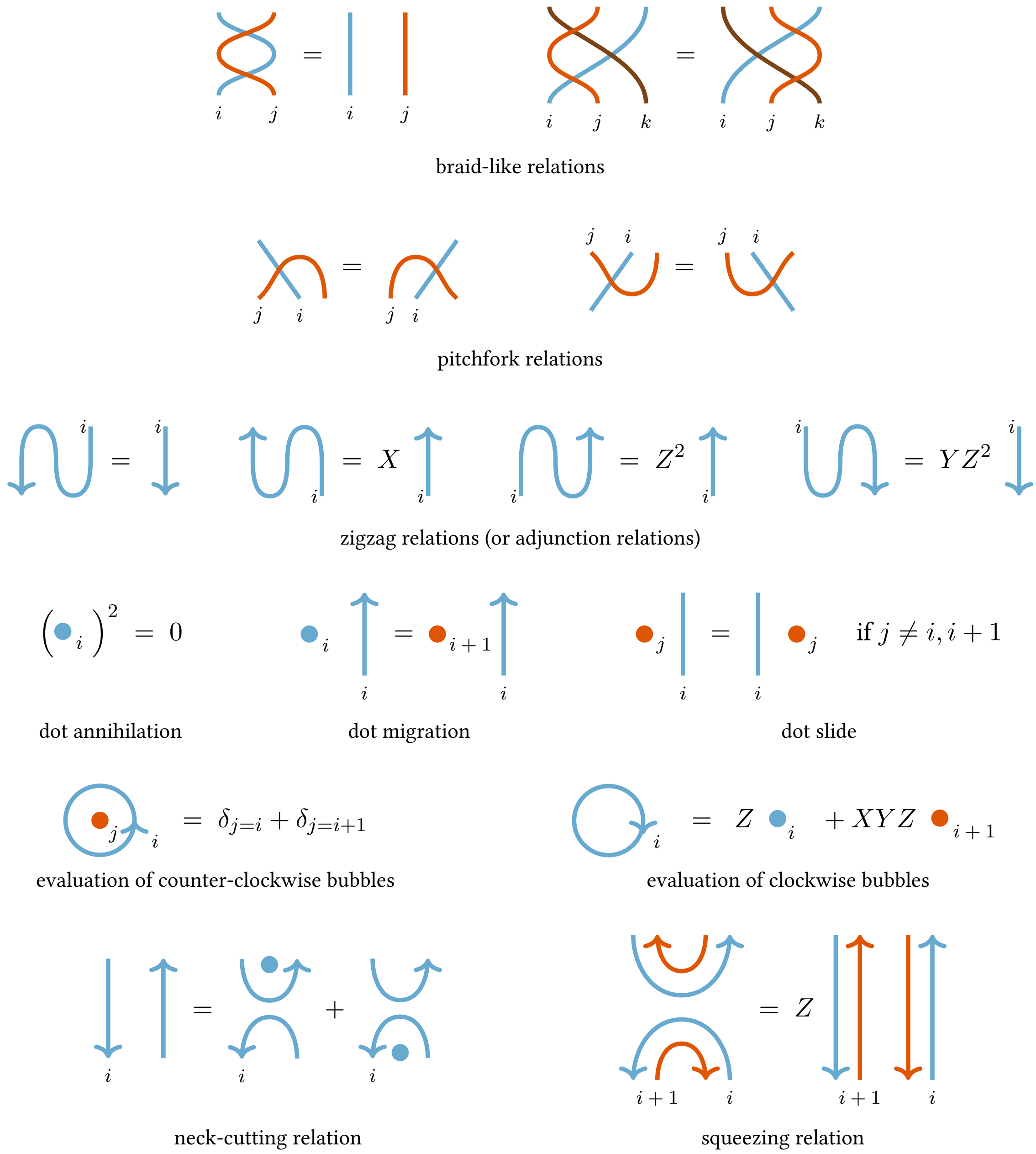
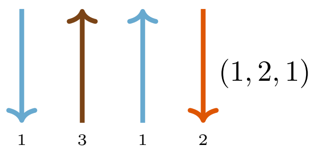
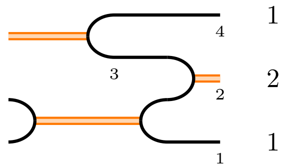


Figure 2.3: Relations in $\mathbf{Diag}_d^\Lambda$. If no orientation is given, the relation holds for all orientations. In the case of the braid-like and pitchfork relations, colours should be so that the crossings exist.

we can represent a web with an *identity foam diagram* (or simply *identity diagram*), an horizontal juxtaposition of oriented vertical strands coloured with elements in $\{1, 2, \dots, d-1\}$. For instance:



a labelled identity diagram



the web it represents

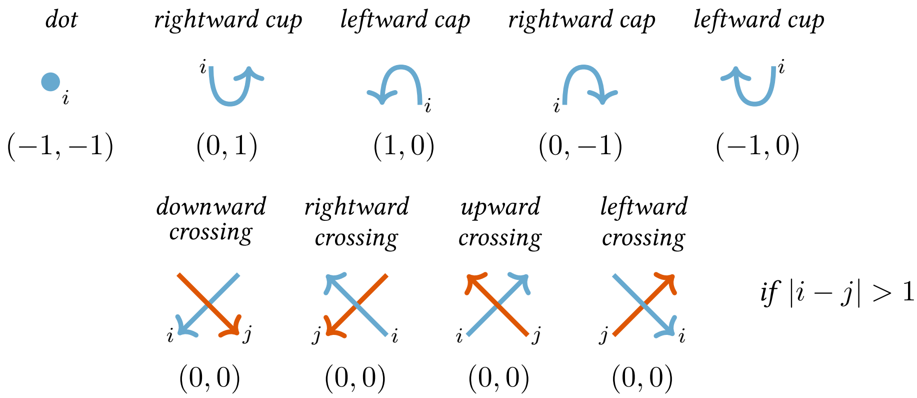
where we remind the reader that webs are read from left to right. In the example above, we have labelled its rightmost region to specify the domain of the web. This induces a labelling on all of its regions thanks the following rule (here the value l_λ (see Subsection 2.2) is given below the corresponding coordinate):

$$(\dots, \underset{i}{1}, \underset{i+1}{1}, \dots) \; \underset{i}{\uparrow} \; (\dots, \underset{i}{2}, \dots) \quad \text{and} \quad (\dots, \underset{i}{2}, \dots) \; \underset{i}{\downarrow} \; (\dots, \underset{i}{1}, \underset{i+1}{1}, \dots) \tag{2}$$

A labelling of the regions of an identity diagram with elements in $\underline{\Lambda}_d$ is said to be *legal* if it locally satisfies (2). In general, if the regions of an identity diagram are labelled with a legal labelling, then it is only necessary to label one region to specify the labelling. Note that the converse does not hold: the data of a label on a region cannot necessary be extended to a legal labelling on the entire diagram. An identity diagram equipped with the data of a legal labelling is called a *labelled identity (foam) diagrams*. By the above discussion, labelled identity diagrams are in one-to-one correspondence with webs.

Given these diagrammatics for webs, it is not difficult to extend it to foams. Indeed, local behaviours in Definition 2.11 are determined by the seam, together with its orientation and colour.

Definition 2.18. *A (generic)* foam diagram*, or simply* diagram*, is a diagram obtained by vertical and horizontal juxtaposition of identity diagrams and generators given below:*



Each generator is equipped with a $\mathbb{Z}^2$-degree, which extends additively to generic diagrams. We further assume that in a generic diagram, generators are in general position with respect to the vertical direction.

In a foam diagram, we refer to strands coloured i as *i-strands*, and dots coloured i as *i-dots*. As for an identity diagram, we can label the regions of a generic diagram with elements of $\underline{\Lambda}_d$. The labelling is said to be *legal* if it satisfies condition (2) as before, and if for each i-dot contained in a region labelled with λ, we have $\lambda_i = 1$. This latter condition corresponds to the fact that dots can only sit on 1-facets. A *labelled (foam) diagram* is a foam diagram equipped with a legal labelling. By the above discussion, labelled diagrams are in one-to-one correspondence with foams.

We now define the string diagrammatics for the graded-2-category $\mathbf{GFoam}_d$.

Definition 2.19. *The $(\mathbb{Z}^2, \mu)$-graded-monoidal category $\mathbf{Diag}_d^\Lambda$ has objects $\underline{\Lambda}_{n,d}$, 1-morphisms labelled identity foam diagrams, and 2-morphisms labelled foam diagrams, regarded up to the following relations:*

(i) If $\varphi\colon D_1 \to D_2$ is a planar isotopy such that φ preserves the relative vertical positions of the generators, then $D_1 = D_2$ in $\mathbf{Diag}_d^\Lambda$.

(ii) If $\varphi\colon D_1 \to D_2$ is a planar isotopy as above except *that it interchanges the vertical positions of two generators p and q, with p vertically above q, then $D_1 = \mu(\deg p, \deg q) D_2$ in $\mathbf{Diag}_d^\Lambda$.*

(iii) All the local relations in Fig. 2.3 above, viewed with a legal labelling.

Proposition 2.20. $\mathbf{GFoam}_d$ *and* $\mathbf{Diag}_d^\Lambda$ *are isomorphic as $(\mathbb{Z}^2, \mu)$-graded-2-categories.*

Proof. The correspondence between foams and diagrams preserve the $\mathbb{Z}^2$-grading. It remains to check that foam relations in Definition 2.13 correspond to diagrammatic relations in Definition 2.19.

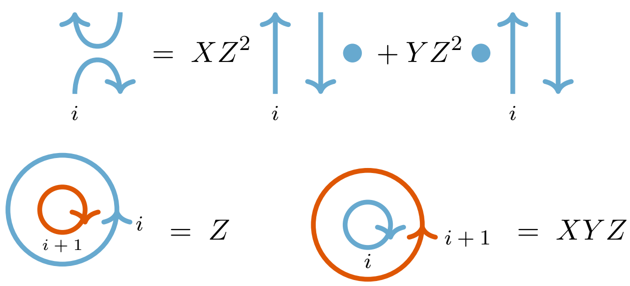
Relations (i) in Definition 2.13 correspond to relations (i) in Definition 2.19, together with graded interchange laws that involve at least one crossing, and braid-like relations, pitchfork relations and dot slide.

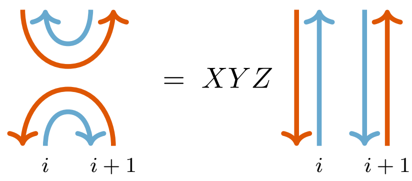
Relations (ii) in Definition 2.13 correspond to graded interchange laws that only involve cups, caps and dots.

Relations in Fig. 2.2 correspond to relations in Fig. 2.3, except braid-like relations, pitchfork relations and dot slide. For instance, the horizontal neck-cutting correspond to the evaluation of clockwise bubbles. ☐

Note that none of the relations in Definition 2.19 depend on the objects, that is, the labelling of the foam diagrams. In the sequel, we mostly leave foam diagrams unlabelled, as the labelling is either irrelevant for the discussion or understood from the context. In particular, this applies when computing secondary relations from the defining relations in $\mathbf{Diag}_d^\Lambda$, as in the following lemma:

Lemma 2.21. *The following local relations hold in $\mathbf{Diag}_d^\Lambda$ for any choice of legal labelling:*





2.5 The categorification theorem

The graded-2-category of foams categories the category of webs:

Theorem 2.22 (Categorification theorem)**.** *Recall that* $\mathbf{GFoam}_d^{\oplus,\mathrm{cl}}$ *denotes the additive graded closure of* $\mathbf{GFoam}_d$ *(see Remark 2.14). Then:*

$$K_0(\mathbf{GFoam}_d^{\oplus,\mathrm{cl}}) \cong \mathbf{Web}_d,$$

where the isomorphism is an isomorphism of $\mathbb{Z}[q,q^{-1}]$*-linear categories.*

This generalizes the analogous statement in the non-graded case, see [2, Theorem 2.11]. The proof follows the same line of thoughts, and as such relies on a non-degeneracy result for graded $\mathfrak{gl}_2$-foams. This results will be shown in future work [53]. To state the non-degeneracy theorem, recall the following non-degenerate [2, Lemma 2.5] pairing on $\mathfrak{gl}_2$-webs:

$$\langle W, W' \rangle := (q+q^{-1})^{\#[c(W) \sqcup_\partial c(W')]}.$$

Recall that for a web W, $c(W)$ is the 1-manifold with boundary obtained by forgetting the double lines in W. Then $c(W) \sqcup_\partial c(W')$ denotes the closed 1-manifold obtained by gluing $c(W)$ and $c(W')$ along their common boundary points. The number $\#\,[c(-W) \sqcup_\partial c(W')]$ counts the number of closed components in $c(W) \sqcup_\partial c(W')$. This pairing follows from the web evaluation formula given in [50]. See also [2, p. 1315].

For $\lambda \in \underline{\Lambda}_d$, let us also define $\#(1:\lambda)$ as the number of ones in λ. Then:

Theorem 2.23 (Non-degeneracy theorem [53])**.** *Let* W *and* W' *be two webs with boundary points* λ, μ *in* $\underline{\Lambda}_d$*. Then the space* $\mathrm{Hom}_{\mathbf{GFoam}_d}(W, W')$ *is a free* $\mathbb{Z}$*-graded* R*-module such that:*

$$\mathrm{rank}_q \mathrm{Hom}_{\mathbf{GFoam}_d}(W, W') = q^{[\#(1:\lambda)+\#(1:\mu)]/2} \langle W, W' \rangle,$$

where rank_q *denotes the graded rank with respect to the* q*-grading.*

Assuming the non-degeneracy theorem, we show the categorification theorem. It closely follows the presentation in [2, Theorem 2.11]. The general strategy is borrowed from [36].

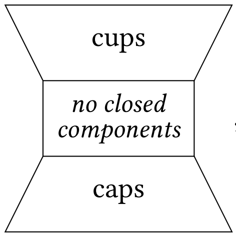
Proof of Theorem 2.22. Let $\gamma\colon \mathbf{Web}_d \to K_0(\mathbf{GFoam}_d)$ sending a web W to its image $[W]$ in $K_0(\mathbf{GFoam}_d)$. The neck-cutting relation, the squeezing relation and the Reidemeister II relation in Fig. 2.3 show that the web relations lift to isomorphisms in $\mathbf{GFoam}_d$, so that γ is well-defined. By Theorem 2.23, γ preserves a non-degenerate pairing, so it must be injective. □

We end the section by describing a basis for $\mathrm{Hom}_{\mathbf{GFoam}_d}(W, W')$, as we shall need in Section 4. It relies on the following lemma:

Lemma 2.24. *Let* W *and* W' *be two webs with boundary points* λ, μ *in* $\underline{\Lambda}_{n,d}$*. Then there exists a foam* $F\colon W \to W'$ *such that* $c(F)$ *is a union of undotted disks.*

This lemma essentially follows from Lemma 2.10, and the "easy" part of the categorification theorem, i.e. the fact that we can lift relations in $\mathbf{Web}_d$ to isomorphisms in $\mathbf{GFoam}_d^{\oplus,\mathrm{cl}}$. The cobordism $c(F)$ has the following cellular form:

$c(F) =$



,

where the cross-section of the middle part consists in a union of closed intervals, without closed components. To each closed interval correspond a disk whose boundary intersect both $c(W)$ and $c(W')$; call this kind of disks *through disks*. The rest of the disks in $c(F)$ are cups or caps, whose boundary only intersect either $c(W')$ or $c(W)$. Note that the number $\#\left[c(W) \sqcup_\partial c(W')\right]$ counts precisely the number of disks in $c(F)$.

For each disk in $c(F)$, one can choose, or not, to add a dot somewhere on the corresponding union of 1-facets in F. If the disk is a cup or a cap, then these two possibilities contribute with $q + q^{-1}$ to the graded rank of $\mathrm{Hom}_{\mathbf{GFoam}_d}(W, W')$. If the disk is a through disk, they contribute with $(q^2 + 1)$ to the graded rank. This justifies our renormalization in front of the pairing in Theorem 2.22: the number $[\#(1 : \lambda) + \#(1 : \mu)]/2$ counts precisely the number of through disks in $c(F)$.

To sum up: If $c(F)$ consists in a union of N disks, a basis of $\mathrm{Hom}_{\mathbf{GFoam}_d}(W, W')$ is given by the 2^N ways of adding dots on F.

3 A graded-categorification of the q-Schur algebra of level two

In this section, we introduce a diagrammatic graded-2-category that categorifies the q-Schur algebra of level two. We then define a graded foamation 2-functor that relates this construction to graded $\mathfrak{gl}_2$-foams. This can be seen as a super analogue of [37] in the $\mathfrak{gl}_2$ case. See also [40] for earlier work.

Diagrammatic categorification of quantum groups was independently introduced by Khovanov, Lauda [36] and Rouquier [51]. A super analogue of this construction was given by Brundan and Ellis [7], building on earlier work of Kang, Kashiwara and Tsuchioka [33]. For odd $\mathfrak{sl}_2$, this was already studied respectively in [28] and in [27]. See also [30, 31, 32] for related work.

In the non-super case, a categorification of the q-Schur algebra appeared in [42]. In [56], the second author defined a supercategorification of the negative half of the q-Schur algebra of level two. A graded version of this construction was given in [44]. In the same paper, Naisse and Putyra also defined a "1-map" from this graded version to their construction. This 1-map is esthetically very similar to our graded foamation 2-functor: we expect the two to coincide once an equivalence between [44] and our category of graded $\mathfrak{gl}_2$-foams is found.

Our presentation of the categorification of the q-Schur algebra of level two is analogous to the presentation of the super Kac–Moody 2-algebras in [7]. However, and contrary to the non-super case, it is not obtained as a cyclotomic quotient of their construction. Indeed, parities of cups and caps do not match. It would be interesting to find a relationship, if there exists any.

Subsection 3.1 review the q-Schur algebra of level two $\dot{S}_{n,d}$. Its graded-categorification, that we call the *graded 2-Schur algebra* $\mathcal{GS}_{n,d}$, is introduced in Subsection 3.2. Subsection 3.3 then defines the *graded foamation 2-functor* from $\mathcal{GS}_{n,d}$ into $\mathbf{GFoam}_d$, our graded-2-category of $\mathfrak{gl}_2$-foams defined in Subsection 2.3. Finally, we show in Subsection 3.4 that $\mathcal{GS}_{n,d}$ categorifies $\dot{S}_{n,d}$.

3.1 The q-Schur algebra of level two

Definition 3.1. *The* (idempotented) q-Schur algebra of level two *is the* $\mathbb{Z}[q, q^{-1}]$*-linear category* $\dot{S}_{n,d}$ *such that:*

- *Objects are weights in the set*

$$\Lambda_{n,d} \coloneqq \{\lambda \in \{0, 1, 2\}^n \mid \lambda_1 + \ldots + \lambda_n = d\}.$$

- *Morphisms are* $\mathbb{Z}[q, q^{-1}]$*-linear combinations of iterated compositions of identity morphisms* $1_\lambda \colon \lambda \to \lambda$ *and generating morphisms*

$$e_i 1_\lambda \colon \lambda \to \lambda + \alpha_i \quad \text{and} \quad f_i 1_\lambda \colon \lambda \to \lambda - \alpha_i \qquad i = 1, \ldots, n-1,$$

where $\alpha_i \coloneqq (0, \ldots, 1, -1, \ldots, 0) \in \mathbb{Z}^n$ *with 1 being on the* i*-th coordinate. Morphisms are subject to the* Schur quotient

$$1_\lambda = 0 \qquad \text{if } \lambda \notin \Lambda_{n,d}$$

and to the following relations:

$$\begin{cases} (e_i f_j - f_j e_i) 1_\lambda = \delta_{ij} [\lambda_i - \lambda_{i+1}]_q, 1_\lambda & \\ (e_i e_j - e_j e_i) 1_\lambda = 0 & \text{for } |i - j| > 1, \\ (f_i f_j - f_j f_i) 1_\lambda = 0 & \text{for } |i - j| > 1. \end{cases} \tag{3}$$

where δ_{ij} *is the Kronecker delta and* $[m]_q = q^{m-1} + q^{m-3} + \ldots + q^{1-m}$ *is the* m*th quantum integer.*

Recall that a $\mathbb{Z}[q, q^{-1}]$-linear category is the same as a $\mathbb{Z}[q, q^{-1}]$-algebra with a distinguished set of idempotents, so that $\dot{S}_{n,d}$ is indeed an algebra. The "level two" stands for the fact that the value of coordinates is at most two. The Schur quotient implies that a morphism that factors through a weight not in $\Lambda_{n,d}$ is set to zero. In the sequel, it is understood that an expression involving a weight that does not belong to $\Lambda_{n,d}$ is set to zero.

Remark 3.2. The q-Schur algebra of level two is an integral form for the fundamental representations of $U_q(\mathfrak{gl}_2)$, analogous to the role of the Temperley–Lieb algebra for $U_q(\mathfrak{sl}_2)$. Let $\bigwedge^k \coloneqq \bigwedge^k(\mathbb{C}(q)^2)$ for $k = 0, 1, 2$ denote the fundamental representations of $U_q q(\mathfrak{gl}_2)$, with $\mathbb{C}(q)^2$ the standard representation. Let $\mathrm{Fund}_{n,d}(U_q(\mathfrak{gl}_2))$ denote the $\mathbb{C}(q)$-linear category consisting of n-fold tensor products $\bigwedge^{k_1} \otimes \ldots \otimes \bigwedge^{k_n}$ with $k_1 + \ldots + k_n = d$ and intertwiners of $U_q(\mathfrak{gl}_2)$-representations. Then:

$$\dot{S}_{n,d} \otimes \mathbb{C}(q) \cong \mathrm{Fund}_{n,d}(U_q(\mathfrak{gl}_2)),$$

where the isomorphism is an isomorphism of $\mathbb{C}(q)$-linear categories (see [10]).

In the sequel, we simply call $\dot{S}_{n,d}$ the *Schur algebra*, and usually drop the "level two".

3.2 The graded 2-Schur algebra

Recall the definitions of R and $\mu\colon \mathbb{Z}^2 \times \mathbb{Z}^2 \to R^\times$ in Definition 2.12 and of $q\colon \mathbb{Z}^2 \to \mathbb{Z}$ in Remark 2.14. We introduce the additional notations $(a,b)^\dagger := (b,a)$ for $(a,b) \in \mathbb{Z}^2$ and

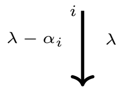
$$p_{ij} := (-(\alpha_j)_{i+1}, (\alpha_j)_i) = \begin{cases} (0,-1) & \text{if } j = i-1, \\ (1,1) & \text{if } j = i, \\ (-1,0) & \text{if } j = i+1, \\ (0,0) & \text{otherwise.} \end{cases}$$

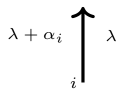
Definition 3.3. *The* graded 2-Schur algebra $\mathcal{GS}_{n,d}$ *is the* $\mathbb{Z}$*-graded* $(\mathbb{Z}^2, \mu)$*-graded-2-category such that:*

- *Objects are elements* λ *for* $\lambda \in \Lambda_{n,d}$*.*
- *1-morphisms are compositions of the generating 1-morphisms*

  $$1_\lambda\colon \lambda \to \lambda, \quad \mathsf{F}_i 1_\lambda\colon \lambda \to \lambda - \alpha_i \quad \textit{and} \quad \mathsf{E}_i 1_\lambda\colon \lambda \to \lambda + \alpha_i,$$

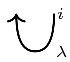
  whenever both λ *and* $\lambda - \alpha_i$ *(resp.* λ *and* $\lambda + \alpha_i$*) are objects of* $\Lambda_{n,d}$*. Using string diagrammatics, the identity* 1_λ *is not pictured, and the non-trivial 1-generators are pictured as follows:*

  $= \mathrm{id}_{\mathsf{F}_i 1_\lambda}$
  

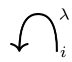
  $= \mathrm{id}_{\mathsf{E}_i 1_\lambda}$

  Note that we read from bottom to top and from right to left.
- *2-morphisms are* R*-linear combination of string diagrams generated by formal vertical and horizontal compositions of the following generating 2-morphisms:*

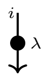
  $: 1_\lambda \to \mathsf{E}_i \mathsf{F}_i 1_\lambda$

  $(-\lambda_{i+1}, \lambda_i) + (0,-1)$

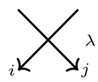
  $: \mathsf{E}_i \mathsf{F}_i 1_\lambda \to 1_\lambda$

  $-(-\lambda_{i+1}, \lambda_i) + (-1,0)$

  $: \mathsf{F}_i 1_\lambda \to \mathsf{F}_i 1_\lambda$

  $(-1,-1)$

  $: \mathsf{F}_i \mathsf{F}_j 1_\lambda \to \mathsf{F}_j \mathsf{F}_i 1_\lambda$

  p_{ij}

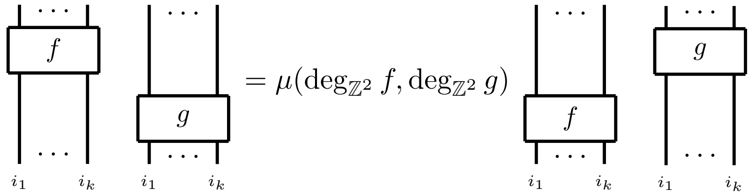
  where the $\mathbb{Z}^2$*-degree* $\deg_{\mathbb{Z}^2}$ *is given below each generator. Such string diagrams are called* Schur diagrams.

2-morphisms are further subject to axioms described below. The q-grading *on* $\mathcal{GS}_{n,d}$ *is the* $\mathbb{Z}$*-grading defined by* $\mathrm{qdeg}(D) := -q(\deg_{\mathbb{Z}^2}(D))$.

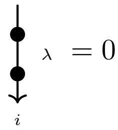
We label only one region with an object in each diagram, as this determines the labelling of all the other regions. If this forces a region to be labelled by a weight λ that does not belong to $\Lambda_{n,d}$, we set this diagram to zero. in that case, we say that the diagram is zero *due to the Schur quotient.*

We assume that the generators in a string diagram are always in generic position, in the sense that the vertical projection defines a separative Morse function. Generating 2-morphisms are subject to the following local relations:

(1) As in any graded-2-category, we have the graded interchange law:

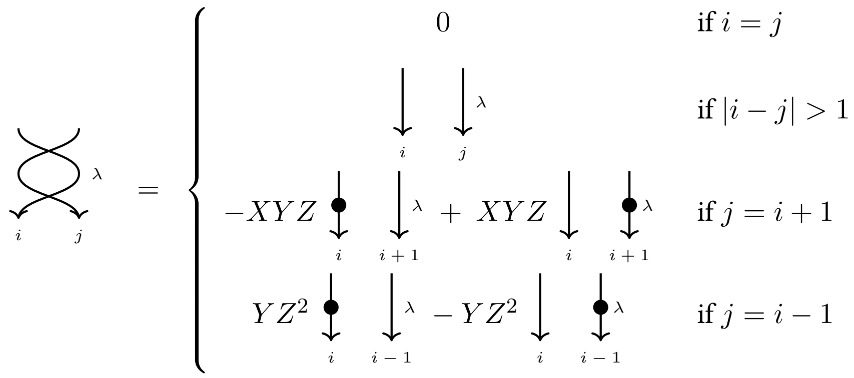


(2) Two dots annihilate:

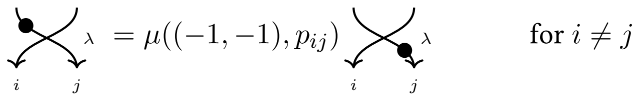


(4)

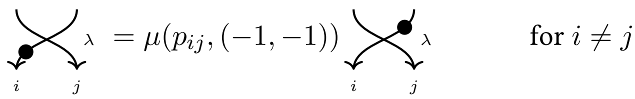
(3) Graded KLR algebra relations for downward crossings:



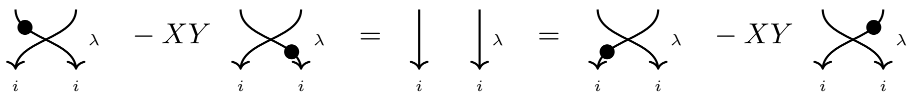
(5)



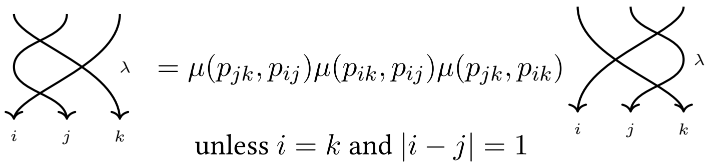
(6)



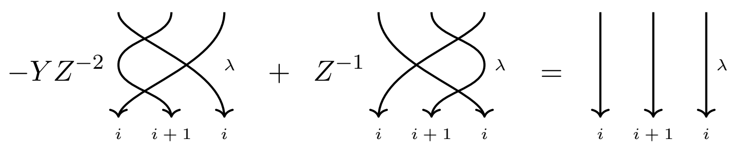
(7)



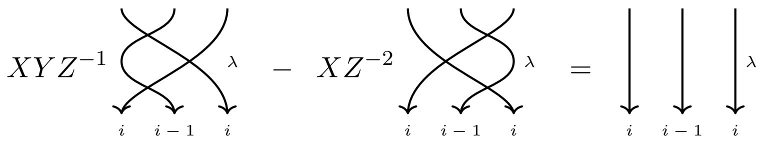
(8)



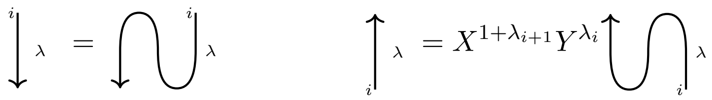
(9)



(10)


(11)

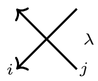
(4) Graded adjunction relations:

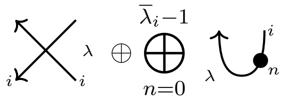

(12)

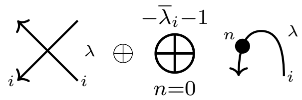
Finally, we require invertibility axioms. To define it, we introduce the following shorthands:

We also write $\overline{\lambda}_i := \lambda_i - \lambda_{i+1}$. Then:

(5) Except if they are zero due to the Schur quotient, the following 2-morphisms are isomorphisms in the graded additive envelope of $\mathcal{GS}_{n,d}$ (see Subsection 2.1.2):


$: \mathsf{F}_i\mathsf{E}_j(\lambda) \to \mathsf{E}_j\mathsf{F}_i(\lambda)$ if $i \neq j$ (13)


$: \mathsf{F}_i\mathsf{E}_i(\lambda) \oplus \lambda^{\oplus[\overline{\lambda}_i]} \to \mathsf{E}_i\mathsf{F}_i(\lambda)$ if $\overline{\lambda}_i \geq 0$ (14)


$: \mathsf{F}_i\mathsf{E}_i(\lambda) \to \mathsf{E}_i\mathsf{F}_i(\lambda) \oplus \lambda^{\oplus[-\overline{\lambda}_i]}$ if $\overline{\lambda}_i \leq 0$ (15)

This ends the definition of the relations on the graded 2-Schur algebra. ◇

Remark 3.4. Let us elaborate on the invertibility axioms above. They are equivalent to the existence of some unnamed generators which are entries of the inverse matrices of (13), (14) and (15), and some unnamed relations that precisely encompass the fact that those generators form inverse matrices. This definition follows Rouquier's approach [51] to 2-Kac–Moody algebras (categorified quantum groups) and Brundan and Ellis' approach [7] to super 2-Kac–Moody algebras. Unravelling the definition would lead to a more explicit (but heavier) definition, similar to Khovanov and Lauda's approach [36] to categorified quantum groups.

In [42, Definition 3.2], a categorification of the q-Schur algebra was constructed, denoted $\mathcal{S}(n,d)$. Let us write $\mathcal{S}(n,d)^\bullet$ for the $\Bbbk$-linear[1] 2-category obtained from $\mathcal{S}(n,d)$ by further imposing relation (4). Then:

Proposition 3.5. *Let $\mathcal{GS}_{n,d}^{\oplus,\mathrm{cl}}$ be the graded additive closure of $\mathcal{GS}_{n,d}$ with respect to the quantum grading (see Subsection 2.1.2 and Remark 2.14). Then:*

$$\mathcal{S}(n,d)^\bullet \cong \mathcal{GS}_{n,d}^{\oplus,\mathrm{cl}}|_{X=Y=Z=1}.$$

[1]The linear 2-category $\mathcal{S}(n,d)$ is defined over $\mathbb{Q}$ in [42], but it can be defined over $\mathbb{Z}$ and hence over any unital commutative ring $\Bbbk$.

Proof. This can be done by defining the unnamed generators implied by the invertibility axioms, and doing a relation chase in order to explicitly exhibit the missing relations. One only need to rescale the (i,i)-downward crossing by (-1). This exactly follows the proof from [4] showing the equivalence between Rouquier's definition of Kac–Moody 2-algebras and Khovanov-Lauda categorified quantum groups. See also [7] for a related statement in the super case. $\square$

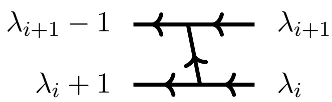
3.3 The graded foamation 2-functor

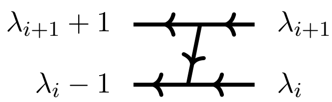
We show that $\mathbf{GFoam}_d$ defines a 2-representation of $\mathcal{GS}_{n,d}$. In other words, for each $n \in \mathbb{N}$ there exists a $(\mathbb{Z}^2,\mu)$-graded-2-functor

$$\mathcal{F}_{n,d} \colon \mathcal{GS}_{n,d} \to \mathbf{GFoam}_d.$$

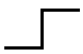
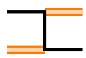
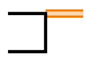
This categorifies the ladder diagrammatics of the Schur algebra, which we now recall.

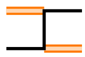
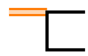
Each generating 1-morphism of $\mathcal{GS}_{n,d}$ can be pictured as a *ladder diagram*:

$\mathsf{E}_i 1_{(\lambda_i,\lambda_{i+1})} \mapsto$ 

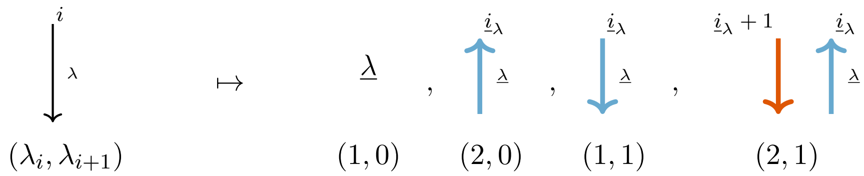
$\mathsf{F}_i 1_{(\lambda_i,\lambda_{i+1})} \mapsto$ 

Removing strands with thickness zero, and replacing strands with thickness one and two by single and double strands, this gives a mapping from generating 1-morphisms in $\mathcal{GS}_{n,d}$ to 1-morphisms in $\mathbf{GFoam}_d$, which locally has the following form:

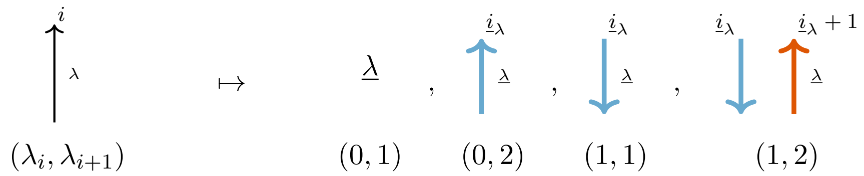
$\mathsf{E}1_{(1,1)} \mapsto$  $\mathsf{E}1_{(0,1)} \mapsto$  $\mathsf{E}1_{(1,2)} \mapsto$  $\mathsf{E}1_{(0,2)} \mapsto$ 

$\mathsf{F}1_{(1,1)} \mapsto$  $\mathsf{F}1_{(1,0)} \mapsto$  $\mathsf{F}1_{(2,1)} \mapsto$  $\mathsf{F}1_{(2,0)} \mapsto$ 

On the level of objects, this maps a weight $\lambda \in \Lambda_{n,d}$ to the weight $\underline{\lambda} \in \underline{\Lambda}_d$, obtained by forgetting all zero entries in λ. Recall the colour of a coordinate defined in Subsection 2.2. For $i \in \{1,\dots,n\}$ such that $\lambda_i \neq 0$, we denote $\underline{i}_\lambda$ the colour of the coordinate of the "image" of i in $\underline{\lambda}$. In the string diagrammatics of foams, the above is given by



(16)



(17)

The local data of $(\lambda_i,\lambda_{i+1})$ is given below each case.

Following [44, p. 59], we shall use the scalar

$$\Gamma_\lambda(i) := (-XY)^{\#\{\lambda_j=1 \mid j\leq i\}}$$

to normalize the graded foamation 2-functor.

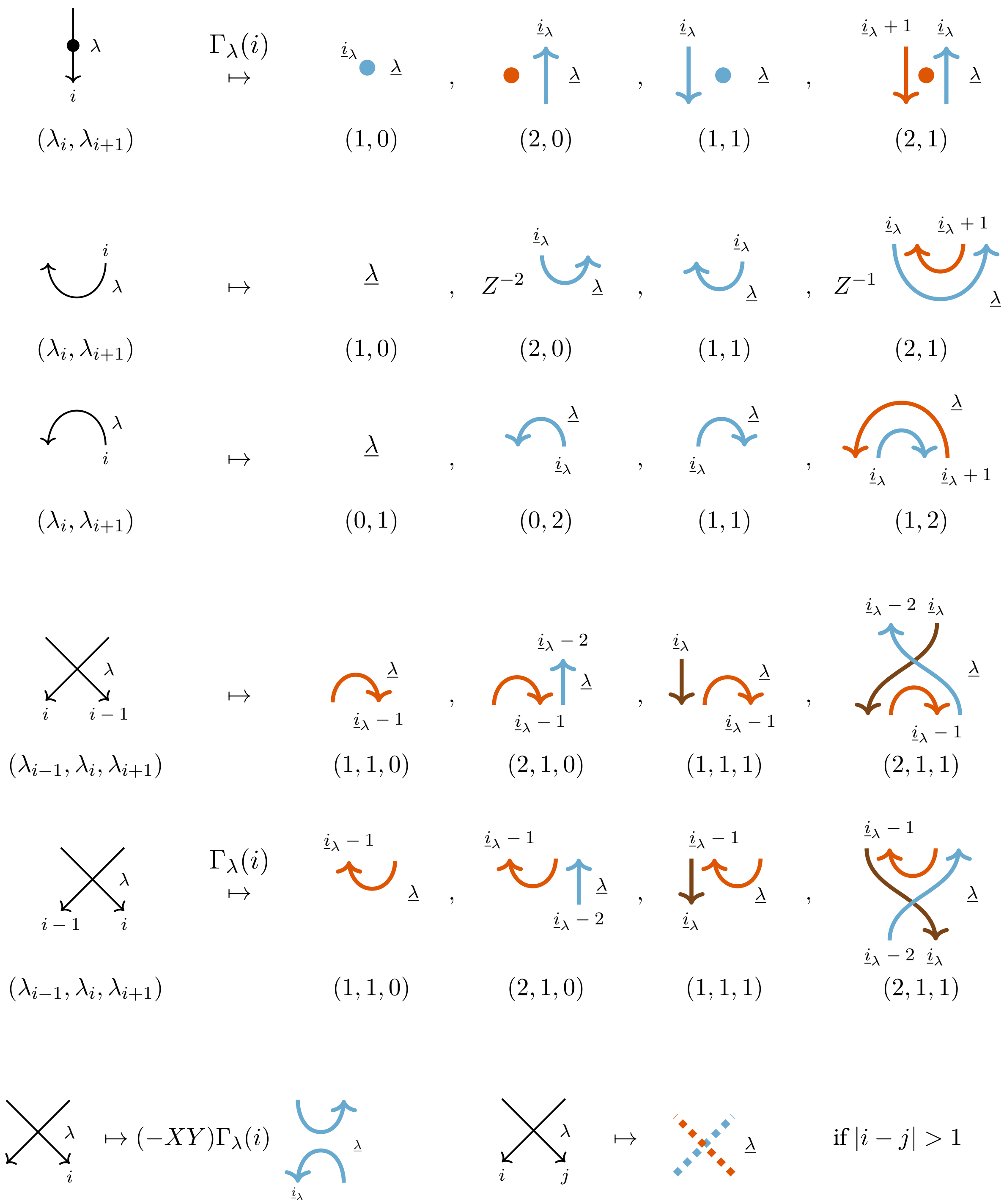


Figure 3.1: Definition of $\mathcal{F}_{n,d}$ on generating 2-morphisms. For dots, rightward cups and caps, and adjacent crossings, it depends on the local value of λ, which is given below each case. A symbol $\Gamma_\lambda(i)$ above a “mapsto arrow” means that the codomain should be multiplied by $\Gamma_\lambda(i)$. For distant crossings, the picture means that one should replace each strand of the Schur crossing by the corresponding strand or pair of strands as prescribed by (16). This defines a foam diagram consisting of one, two or four crossings.

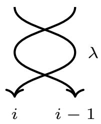
Proposition 3.6. *There exists a $(\mathbb{Z}^2, \mu)$-graded-2-functor*

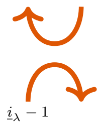
$$\mathcal{F}_{n,d} \colon \mathcal{GS}_{n,d} \to \mathbf{GFoam}_d$$

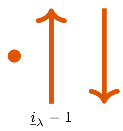
defined on generating 2-morphisms as in Fig. 3.1. We call $\mathcal{F}_{n,d}$ the graded foamation 2-functor.

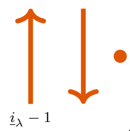
Proof. On checks that $\mathcal{F}_{n,d}$ preserves the $\mathbb{Z}^2$-grading. We need to check that the images through $\mathcal{F}_{n,d}$ of the defining relations of the graded 2-Schur algebra are relations in $\mathbf{GFoam}_d$. This is analogous to the proof of [37, Proposition 3.3]. The main additional work is to check that scalars match. For readability, we leave implicit the labelling of regions for foam diagrams.

The fact that $\mathcal{F}_{n,d}$ respects relation (4) follows from dot annihilation in foams. For relation (5), the case $i = j$ follows from the evaluation of an undotted counterclockwise bubble and the case $|i-j| > 1$ follows from the Reidemeister II braid-like relation. Consider then the case $j = i-1$. For $\lambda_i = 2$, both sides of (5) are zero due to the Schur quotient. If $(\lambda_{i-1}, \lambda_i, \lambda_{i+1}) = (1, 1, 0)$, we have:

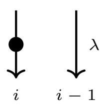
$\mathcal{F}_{n,d}\Bigg($ 

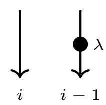
$\Bigg) = \Gamma_\lambda(i)$ 

$= \Gamma_\lambda(i)\Bigg(YZ^2$ 

$+ XZ^2$ 

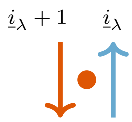
$\Bigg)$

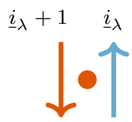
$= \mathcal{F}_{n,d}\Bigg(YZ^2$ 

$- YZ^2$ 

$\Bigg)$

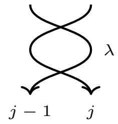
where we used Lemma 2.21 and $\Gamma_{\lambda-\alpha_{i-1}}(i) = \Gamma_\lambda(i) = (-XY)\Gamma_\lambda(i-1)$. Other cases for which $\lambda_i = 1$ are computed similarly. If $\lambda_i = 0$, the left-hand side of (5) is zero due to the Schur quotient. If $(\lambda_{i-1}, \lambda_i, \lambda_{i+1}) = (2, 0, 1)$, the image of the right-hand side is

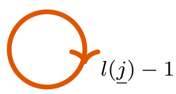
$\Gamma_{\lambda-\alpha_{i-1}}(i)\, YZ^2$ 

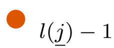
$- \Gamma_\lambda(i-1) YZ^2$ 

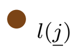
$= 0,$

which follows from $\Gamma_{\lambda-\alpha_{i-1}}(i) = \Gamma_\lambda(i-1)$. Other cases for which $\lambda_i = 0$ are computed similarly. When $j = i+1$, both sides are zero due to the Schur quotient when $\lambda_j = 0$, and only the left-hand side when $\lambda_j = 2$. For the representative case $(\lambda_{j-1}, \lambda_j, \lambda_{j+1}) = (1, 1, 0)$, one gets:

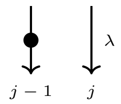
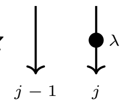
$\mathcal{F}_{n,d}\Bigg($ 

$\Bigg) = \Gamma_\lambda(j)$ 

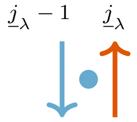
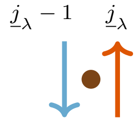
$= \Gamma_\lambda(j)\Big(Z$ 

$+ XYZ$ 

$\Big)$

$$= \mathcal{F}_{n,d}\Bigg(-XYZ$$  $$+ XYZ$$  $$\Bigg)$$

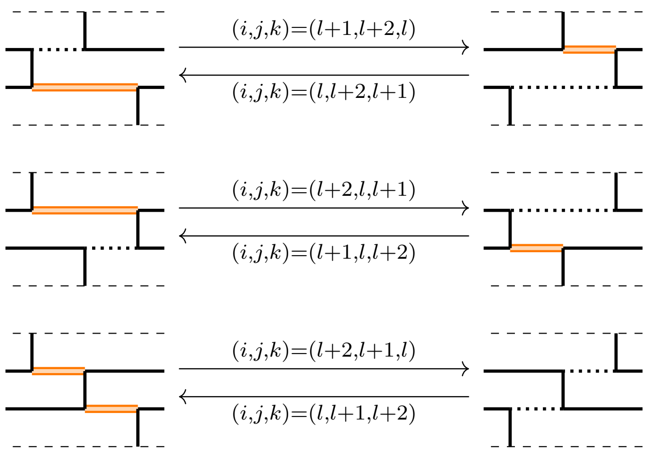
which follows from $\Gamma_{\lambda-\alpha_j}(j-1) = (-XY)\Gamma_\lambda(j)$, and the latter from. For the representative case $(\lambda_{i-1}, \lambda_i, \lambda_{i+1}) = (1, 2, 0)$, the image of the right-hand side of (5) gives:

$$-\Gamma_{\lambda-\alpha_j}(j-1)\; XYZ$$  $$+ \Gamma_\lambda(j) XYZ$$  $$= 0,$$

which follows from $\Gamma_{\lambda-\alpha_j}(j-1) = \Gamma_\lambda(j)$ and dot migration.

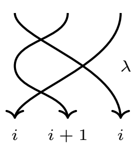
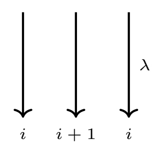
For relations (6) and (7), it follows from the graded interchange law and dot migration. The relation (8) follows direcly from the (vertical) neck-cutting relation.

Consider now relation (9). If all colors are pairwise equal or adjacent (i.e. cases $i = j = k$ and $i = j = k \pm 1$ together with permutations), then either the case is excluded by assumption, or both sides are zero by the Schur quotient. In particular, we can disregard the normalization with the Γ's (this follows from the fact that $\Gamma_{\lambda-\alpha_j}(i) = \Gamma_\lambda(i)$ whenever $i \neq j$). If a color is distant from the two others (i.e. cases $|i-j| > 1$ and $|i-k| > 1$ together with permutations), then the image under $\mathcal{F}_{n,d}$ of relation (9) is a composition of graded interchanges, consisting of moving vertically the image of the only crossing with a (possibly) non-trivial $\mathbb{Z}^2$-grading. The last six cases also consist of composition of graded interchanges, given by the interchange of two saddles (zip or unzip) sharing a common 1-facet. We picture below the domain and codomain of the six cases:

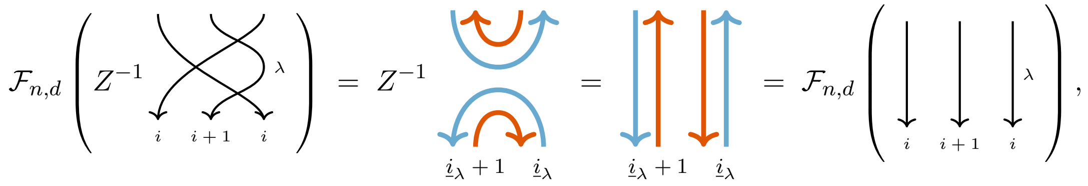


This concludes the proof for the relation (9).

A case-by-case analysis of the possible values for $(\lambda_i, \lambda_{i+1}, \lambda_{i+2})$ shows that except for four possibilities, the three diagrams in (10) are all set to zero by the Schur quotient. In the remaining cases, exactly one summand on the left-hand side is set to zero by the Schur quotient. If $(\lambda_i, \lambda_{i+1}, \lambda_{i+2}) = (2, 0, 0)$, then

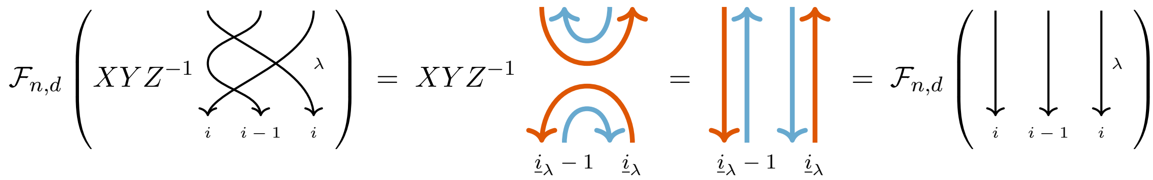
$$\mathcal{F}_{n,d}\Bigg(-YZ^{-2}$$  $$\Bigg) = XZ^{-2}$$  $$=$$  $$= \mathcal{F}_{n,d}\Bigg($$  $$\Bigg),$$

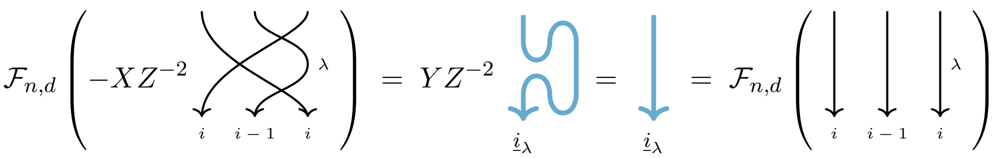
using $\Gamma_{\lambda-\alpha_i}(i+1) = \Gamma_\lambda(i)$. The case $(\lambda_i, \lambda_{i+1}, \lambda_{i+2}) = (2,0,1)$ is similar. If $(\lambda_i, \lambda_{i+1}, \lambda_{i+2}) = (2,1,0)$, then



using $\Gamma_{\lambda-\alpha_{i+1}}(i) = (-XY)\Gamma_\lambda(i+1)$. The case $(\lambda_i, \lambda_{i+1}, \lambda_{i+2}) = (2,1,1)$ is similar.

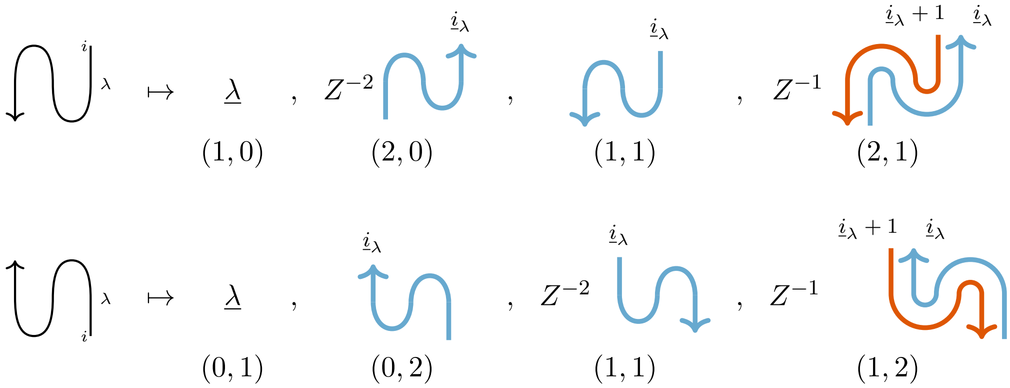
A similar analysis can be done for the relation (11), giving four non-trivial cases, with only two computations needed. We depict the cases $(\lambda_{i-1}, \lambda_i, \lambda_{i+1}) = (1,2,0)$ and $(\lambda_{i-1}, \lambda_i, \lambda_{i+1}) = (1,1,0)$:





using Lemma 2.21, and respectively $\Gamma_{\lambda-\alpha_i}(i) = (-XY)\Gamma_\lambda(i)$ and $\Gamma_{\lambda-\alpha_{i-1}}(i) = \Gamma_\lambda(i)$.

For relation (12), it follows from the zigzag relations for foams:



It remains to check that $\mathcal{F}_{n,d}$ preserves three inverse axioms (13), (14) and (15). Note that this is independent on the coefficients in the defining relations of $\mathbf{GFoam}_d$, as those inverse axioms only impose existence. For relation (13), it follows from graded isotopies and zigzag relations. For $\overline{\lambda}_i = 0$, one checks that the image of the leftward monochromatic crossing is the identity, up to multiplication by an invertible scalar. Otherwise, it is zero by the Schur quotient. For $\overline{\lambda}_i = \pm 1$, The image of the leftward cup (resp. cap) is invertible due to the squeezing relation. Finally, for $\overline{\lambda}_i = \pm 2$ this is precisely the (vertical) neck-cutting relation. $\square$

In [56], the second author defined a thick calculus for the negative half of the super version of the graded 2-Schur $\mathcal{GS}_{n,d}$. One can similarly define a thick calculus for the full graded 2-Schur, defining a graded-2-category $\check{\mathcal{GS}}_{n,d}$. Then, following the line of the proof of the analogous result in the non-graded case as given by Queffelec and Rose [49, Theorem 3.9], one can show that the foamation 2-functor factors through $\check{\mathcal{GS}}_{n,d}$. The inclusion

$$\check{\mathcal{GS}}_{n,d} \hookrightarrow \check{\mathcal{GS}}_{n+1,d}$$

is defined on objects as $(\lambda_1, \ldots, \lambda_n) \mapsto (\lambda_1, \ldots, \lambda_n, 0)$ and similarly for 1-morphisms and 2-morphisms. Finally, we get the graded analogue of [49, Proposition 3.22][1] in the $\mathfrak{gl}_2$-case:

Proposition 3.7. *We have the following equivalence of $(\mathbb{Z}^2, \mu)$-graded-2-categories:*

$$\operatorname{colim}_{n\in\mathbb{N}} \left(\ldots \hookrightarrow \check{\mathcal{GS}}_{n,d} \hookrightarrow \check{\mathcal{GS}}_{n+1,d} \hookrightarrow \ldots\right) \cong \mathbf{GFoam}_d.$$

3.4 The categorification theorem

As proclaimed, the graded 2-Schur algebra categorifies the Schur algebra of level two:

Theorem 3.8. *Let $\mathcal{GS}_{n,d}^{\oplus,\mathrm{cl}}$ denote the additive graded closure of $\mathcal{GS}_{n,d}$, where the graded closure is taken with respect to the q-grading (see Subsection 2.1.2). Then:*

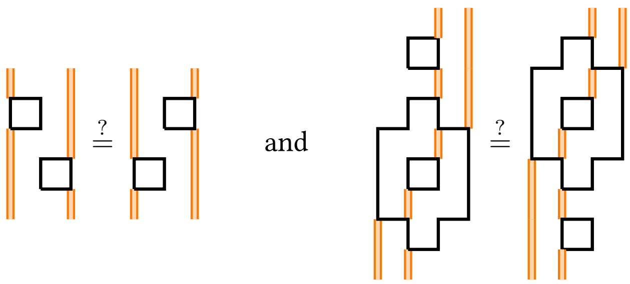
$$K_0(\mathcal{GS}_{n,d}^{\oplus,\mathrm{cl}}) \cong \dot{S}_{n,d},$$

where the isomorphism is an isomorphism of $\mathbb{Z}[q, q^{-1}]$-linear categories.

The proof relies on the graded foamation 2-functor and the analogous result for $\mathbf{GFoam}_d$ (Theorem 2.22), and the following lemma:

Lemma 3.9. *Let $L_1, L_2 \colon \lambda \to \mu$ be two ladder diagrams. If there exists a spherical isotopy from $c(L_1)$ to $c(L_2)$, then L_1 and L_2 are equal in $\dot{S}_{n,d}$.*

Although relations in $\dot{S}_{n,d}$ capture isotopies for the underlying 1-manifolds, it is not clear whether on can lift *all* isotopies to ladders:



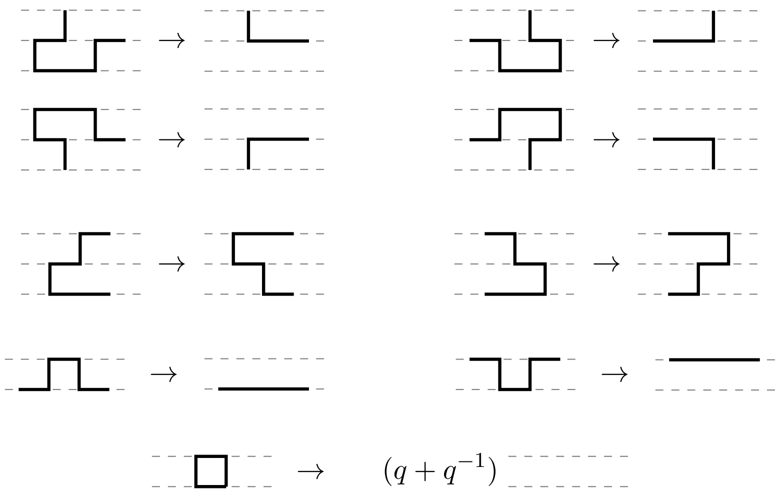
Indeed, in the examples above, one lacks space to perform the relevant isotopies at the level of ladders. That is, we need to embed the diagrams in a diagram with greater width, by adding a double or invisible line in the former case, and by adding a double line in the latter case.[2]

However, in the example on the left for instance, there is another obvious way to show that the two diagrams are equal, without further embeddings: evaluate the circles to $q + q^{-1}$. A similar strategy works in the second case, and in fact we show in the proof below that the ability the evaluate circles is enough to capture all isotopies.

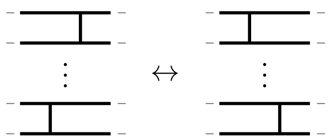
[1] Or rather, the analogue of this proposition when one imposes (4) and dot annihilation respectively.

[2] This distinction corresponds to the fact that the first case become an isotopy in the case of webs, while the second case remains an issue even for webs.

Proof of Lemma 3.9. We define a reduction algorithm on ladder diagrams, using oriented local relations in $\dot{S}_{n,d}$. Below we do not picture the invisible and double lines, and dashed gray lines are pictured to remind the reader of the ladder structure:



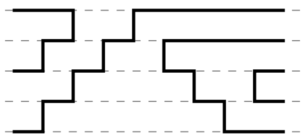
Note that by orienting relations this way, we never require more width than we originally had. By the reduction algorithm, we mean the undeterministic process of applying the above relations in the prescribed direction, modulo sliding distant ladder rungs past one another:[1]



(18)

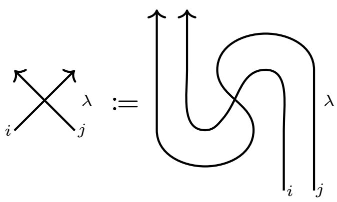
This process terminates. The ladder diagram so obtained, possibly with a leading coefficient, is called a *normal form*. Below is an example of a normal form:

$(q+q^{-1})^3$



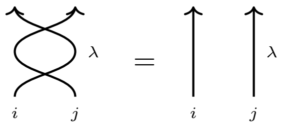
Let L_1 and L_2 be as in the statement of the lemma. If $\hat{L}_1$ and $\hat{L}_2$ are normal forms for L_1 and L_2 respectively, the reader can convince themself that $\hat{L}_1$ and $\hat{L}_2$ are equal modulo (18). In particular, L_1 and L_2 are equal in $\dot{S}_{n,d}$. $\square$

Proof of Theorem 3.8. Relations (3) become 2-isomorphisms in $\mathcal{GS}_{n,d}$. The first relation is categorified by the invertibility axioms (13), (14) and (15). The second relation is categorified by (5) in the case $|i-j|>1$. Finally, the third relation is categorified by the analogue of (5), case $|i-j|>1$, for upward strands. We define the *upward crossing* as:



[1]This could be formalize as a *convergent rewriting system modulo.*

It then follows from adjunction relations (12) and (5), case $|i-j|>1$, that:



if $|i-j|>1$.

This implies that there exists a $\mathbb{Z}[q,q^{-1}]$-linear functor $\dot{S}_{n,d} \to K_0(\mathcal{GS}^{\oplus,\mathrm{cl}}_{n,d})$, full and surjective, fitting into the following commutative diagram:

$$\begin{tikzcd}
\dot{S}_{n,d} \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & K_0(\mathcal{GS}^{\oplus,\mathrm{cl}}_{n,d}) \arrow[d, "K_0(\mathcal{F}_{n,d})"] \\
\mathbf{Web}_d \arrow[r, "\cong"] & K_0(\mathbf{GFoam}^{\oplus,\mathrm{cl}}_d)
\end{tikzcd}$$

The bottom arrow is an isomorphism by Theorem 2.22. By Lemma 3.9, the left arrow is faithful. We conclude that the top arrow is also faithful, and hence it is an isomorphism. □

4 Covering Khovanov homology

Convention: *in this section, we read webs from bottom to top.*

In this section, we define an invariant of oriented tangles and show that it coincides with odd Khovanov homology when restricted to links. More precisely, our invariant coincides with *covering Khovanov homology*, an invariant of links defined by Putyra [48]. Both constructions are defined over the ring $R = \Bbbk[X,Y,Z^{\pm 1}]/(X^2 = Y^2 = 1)$ defined in Definition 2.12, such that setting $X = Y = Z = 1$ recovers (even) Khovanov homology, while setting $X = Z = 1$ and $Y = -1$ recovers odd Khovanov homology.

To distinguish the two constructions, we call Putyra's construction *covering $\mathfrak{sl}_2$-Khovanov homology* and denote it $\mathrm{CKh}_{\mathfrak{sl}_2}(L)$ for an oriented link L, and we call our construction *covering $\mathfrak{gl}_2$-Khovanov homology* and denote it $\mathrm{CKh}_{\mathfrak{gl}_2}(T)$ for an oriented tangle T. The latter coincides with *not even Khovanov homology* as defined by the second author in [56] once we set $X = Z = 1$ and $Y = -1$. See also the introduction for connections with the work of Naisse and Putyra [44].

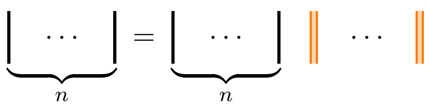
To state our claim precisely, we introduce the following completions:

Definition 4.1. *The set $\underline{\Lambda}$, the $\mathbb{Z}[q,q^1]$-linear category $\mathbf{Web}$ and $(\mathbb{Z}^2,\mu)$-graded-2-category $\mathbf{GFoam}$ are respectively defined as:*

$$\begin{aligned}
\underline{\Lambda} &:= \mathrm{colim}(\ldots \hookrightarrow \underline{\Lambda}_d \hookrightarrow \underline{\Lambda}_{d+2} \hookrightarrow \ldots), \\
\mathbf{Web} &:= \mathrm{colim}(\ldots \hookrightarrow \mathbf{Web}_d \hookrightarrow \mathbf{Web}_{d+2} \hookrightarrow \ldots), \\
\mathbf{GFoam} &:= \mathrm{colim}(\ldots \hookrightarrow \mathbf{GFoam}_d \hookrightarrow \mathbf{GFoam}_{d+2} \hookrightarrow \ldots),
\end{aligned}$$

where the embeddings[1] *denote the addition of a double point, a double line and a double facet on the right.*

For instance, in the category $\mathbf{Web}$ the following identity webs are identified:



in $\mathbf{Web}$.

[1]This follows from Lemma 2.10 and Theorem 2.23.

Here we recall our convention for this section that we read webs from bottom to top. Informally, working in $\underline{\Lambda}$, $\mathbf{Web}$ and $\mathbf{GFoam}$ means that one can always "add a double point, line and facet on the right".

For $\mathcal{C}$ a (G,μ)-graded-2-category, we denote by $\mathrm{Kom}(\mathcal{C})$ the R-linear category of chain complexes in $\mathcal{C}$ and chain morphisms. We say that a chain morphism is *homogeneous* if all its components are homogeneous with the same degree. It is *degree-preserving* if it is homogeneous of degree zero. This does not mean that $\mathrm{Kom}(\mathcal{C})$ is G-graded: a chain morphism may not decompose as a sum of homogeneous chain morphisms.

For each sliced oriented tangle diagram D_T representing an oriented tangle T, we define in Subsection 4.1 a chain complex $\mathrm{Kom}_{\mathfrak{gl}_2}(D_T) \in \mathrm{Kom}(\mathbf{GFoam})$. Then:

Theorem 4.2. *Let $w(D_T)$ be the writhe of D_T. Then the homotopy type of $q^{w(D_T)}\,\mathrm{Kom}_{\mathfrak{gl}_2}(D_T)$ is an invariant of the oriented tangle T. We denote this homotopy type by $\mathrm{CKh}_{\mathfrak{gl}_2}(T)$.*

This construction closely follows the presentation given in [37]. The main difference is the horizontal composition (or "object-adapted" tensor product) of chain complexes, for which we give a minimal introduction in Subsection 4.1.

In Subsection 4.2, we review the definition of covering $\mathfrak{sl}_2$-Khovanov homology. This similarly defines for each oriented link diagram D_L a complex $\mathrm{Kom}_{\mathfrak{sl}_2}(D_L)$ in $\mathrm{Kom}(R\text{-}\mathrm{Mod}_{\mathbb{Z}^2})$, the category of chain complexes in $\mathbb{Z}^2$-graded R-modules. The homotopy type of $q^{w(D_T)}\,\mathrm{Kom}_{\mathfrak{sl}_2}(D_L)$ is an invariant of the oriented link L.

Finally, we show the equivalence between the two constructions when restricted to links in Subsection 4.3. To state it, denote by $\emptyset \in \underline{\Lambda}$ the empty weight and by $\emptyset \coloneqq \mathrm{id}_\emptyset$ its identity, the empty web. Recall that in $\underline{\Lambda}$ (resp. in $\mathbf{Web}$), the empty weight (resp. the empty web) is the same as an arbitrary juxtaposition of double points (resp. double lines). Denote by $\mathbf{GFoam}(\emptyset,\emptyset)$ the $\mathbb{Z}^2$-graded R-linear category obtained by restricting $\mathbf{GFoam}$ to the object $\emptyset$, and

$$\mathcal{A}_{\mathfrak{gl}_2}\colon \mathbf{GFoam}(\emptyset,\emptyset) \to R\text{-}\mathrm{Mod}_{\mathbb{Z}^2}$$

the representable functor $\mathcal{A}_{\mathfrak{gl}_2} \coloneqq \mathrm{Hom}_{\mathbf{GFoam}(\emptyset,\emptyset)}(\emptyset,-)$. We can now state the main result of this section of and of this paper:

Theorem 4.3. *Let D_L be a sliced oriented link diagram presenting an oriented link L. Then we have the following degree-preserving isomorphism of chain complexes of R-modules:*

$$\mathcal{A}_{\mathfrak{gl}_2}(\mathrm{Kom}_{\mathfrak{gl}_2}(D_L)) \cong \mathrm{Kom}_{\mathfrak{sl}_2}(D_L).$$

This theorem is the content of Main theorem B when setting $X = Z = 1$ and $Y = -1$.

4.1 A covering $\mathfrak{gl}_2$-Khovanov homology for oriented tangles

We describe the horizontal composition of two *hypercubic complexes* in a given graded-2-category. Hypercubic complexes are special cases of polyhomogeneous complexes, defined in Definition 5.2. The horizontal composition that we describe is the specialization of the definitions Definition 5.4 and Definition 5.7 (in the context of graded-monoidal categories). This is the minimal description necessary for the construction of covering $\mathfrak{gl}_2$-Khovanov homology.

Notation 4.4. Fix $n \in \mathbb{N}$. We use the shorthand $\mathcal{I} \coloneqq \{1,\dots,n\}$ for the set of indices $1 \le i \le n$. We view $\{0,1\}^n$ as a hypercubic lattice and denote $(\boldsymbol{e}_i)_{i\in\mathcal{I}}$ the canonical basis of $\mathbb{Z}^n$. For $\boldsymbol{r} \in \{0,1\}^n$ and $i \in \mathcal{I}$, we write $\boldsymbol{r} \to \boldsymbol{r}+\boldsymbol{e}_i$ for an edge in the hypercube $\{0,1\}^n$.

We fix $\mathcal{C}$ a (G,μ)-graded-2-category. Whenever we write a composition in $\mathcal{C}$, it is tacitly assumed that the 1-morphisms or 2-morphisms involved are composable.

Definition 4.5. *An* hypercubic complex $\mathbb{A} = (A, \alpha)$ *of dimension* $n \in \mathbb{N}$ *consists of the following data:*

(i) for each vertex $\boldsymbol{r} \in \{0,1\}^n$*, a 1-morphism* $A^{\boldsymbol{r}}$ *in* $\mathcal{C}$*,*

(ii) for each edge $\boldsymbol{r} \to \boldsymbol{r} + \boldsymbol{e}_i$*, a homogeneous 2-morphism* $\alpha_i^{\boldsymbol{r}} \colon A^{\boldsymbol{r}} \to A^{\boldsymbol{r}+\boldsymbol{e}_i}$ *in* $\mathcal{C}$*, such that each square anti-commutes:*

$$\alpha_{i_2}^{\boldsymbol{r}+\boldsymbol{e}_{i_1}} \star_1 \alpha_{i_1}^{\boldsymbol{r}} = -\alpha_{i_1}^{\boldsymbol{r}+\boldsymbol{e}_{i_2}} \star_1 \alpha_{i_2}^{\boldsymbol{r}}$$

for all suitable $\boldsymbol{r} \in \{0,1\}^n$ *and* $i_1, i_2 \in \mathcal{I}$*. Furthermore, we require the grading to be constant in a given direction, in the sense that* $\deg \alpha_{i_1}^{\boldsymbol{r}} = \deg \alpha_{i_2}^{\boldsymbol{r}}$ *for any suitable* $i_1, i_2 \in \mathcal{I}$*.*

Given such an hypercubic complex $\mathbb{A} = (A, \alpha)$, we define the following element of G:

$$|\alpha|\,(\boldsymbol{r}) := \sum_{i\colon \boldsymbol{r}_i = 1} \deg_G(\alpha_i^{\boldsymbol{0}}),$$

where $\boldsymbol{0} := (0, \dots, 0) \in \{0,1\}^n$. Informally, the element $|\alpha|\,(\boldsymbol{r})$ is the sum of the G-degrees along a path from $\boldsymbol{0}$ to $\boldsymbol{r}$. Let $\mathbb{B} = (B, \beta)$ be another hypercubic complex of dimension m. We put $\mathcal{J} := \{1, \dots, m\}$.

Definition 4.6. *The* horizontal composition $\mathbb{A} \star_0 \mathbb{B}$ *of* $\mathbb{A}$ *and* $\mathbb{B}$ *is the hypercubic chain complex of dimension* $n+m$ *defined by the following data:*

(i) on each vertex $(\boldsymbol{r}, \boldsymbol{s}) \in \{0,1\}^{n+m}$*, the 1-morphism* $A^{\boldsymbol{r}} \star_0 B^{\boldsymbol{s}}$*;*

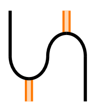
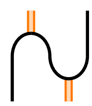
(ii) on each edge $(\boldsymbol{r}, \boldsymbol{s}) \to (\boldsymbol{s}, \boldsymbol{r}) + \boldsymbol{e}_k$*, the homogeneous 2-morphism*

$$(\alpha \star_0 \beta)_k^{(\boldsymbol{r},\boldsymbol{s})} := \begin{cases} \alpha_i^{\boldsymbol{r}} \star_0 \mathrm{id}_{B^{\boldsymbol{s}}} & \text{if } k = i \in \mathcal{I}, \\ (-1)^{|\boldsymbol{r}|} \mu\left(|\alpha|\,(\boldsymbol{r}), \beta_j^{\boldsymbol{s}}\right) \mathrm{id}_{A^{\boldsymbol{r}}} \star_0 \beta_j^{\boldsymbol{s}} & \text{if } k = j \in \mathcal{J}, \end{cases} \tag{19}$$

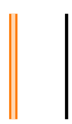
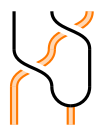
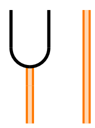
The sign appearing in Eq. (19) is the *graded Koszul rule*. By Theorem 5.10, this horizontal composition is coherent with homotopies (see also Main theorem A).

Note that a length-two chain complex whose differential is homogeneous is exactly a hypercubic complex of dimension one. In particular, if $\mathbb{A}_1, \dots, \mathbb{A}_N$ is a family of horizontally composable length-two chain complexes with homogeneous differential, Definition 4.6 defines their N-fold horizontal composition.

We now define a chain complex $\mathrm{Kom}_{\mathfrak{gl}_2}(D) \in \mathrm{Kom}(\mathbf{GFoam})$ for every sliced oriented tangle diagram D. The reader can follow the procedure on the example given in Fig. 4.1, with D pictured at step ①. We shall need the following definitions of crossings of a single line with a double line:

 :=  and  :=  (20)

These crossings satisfy the following relations in **Web**:

 =  and  =  (21)

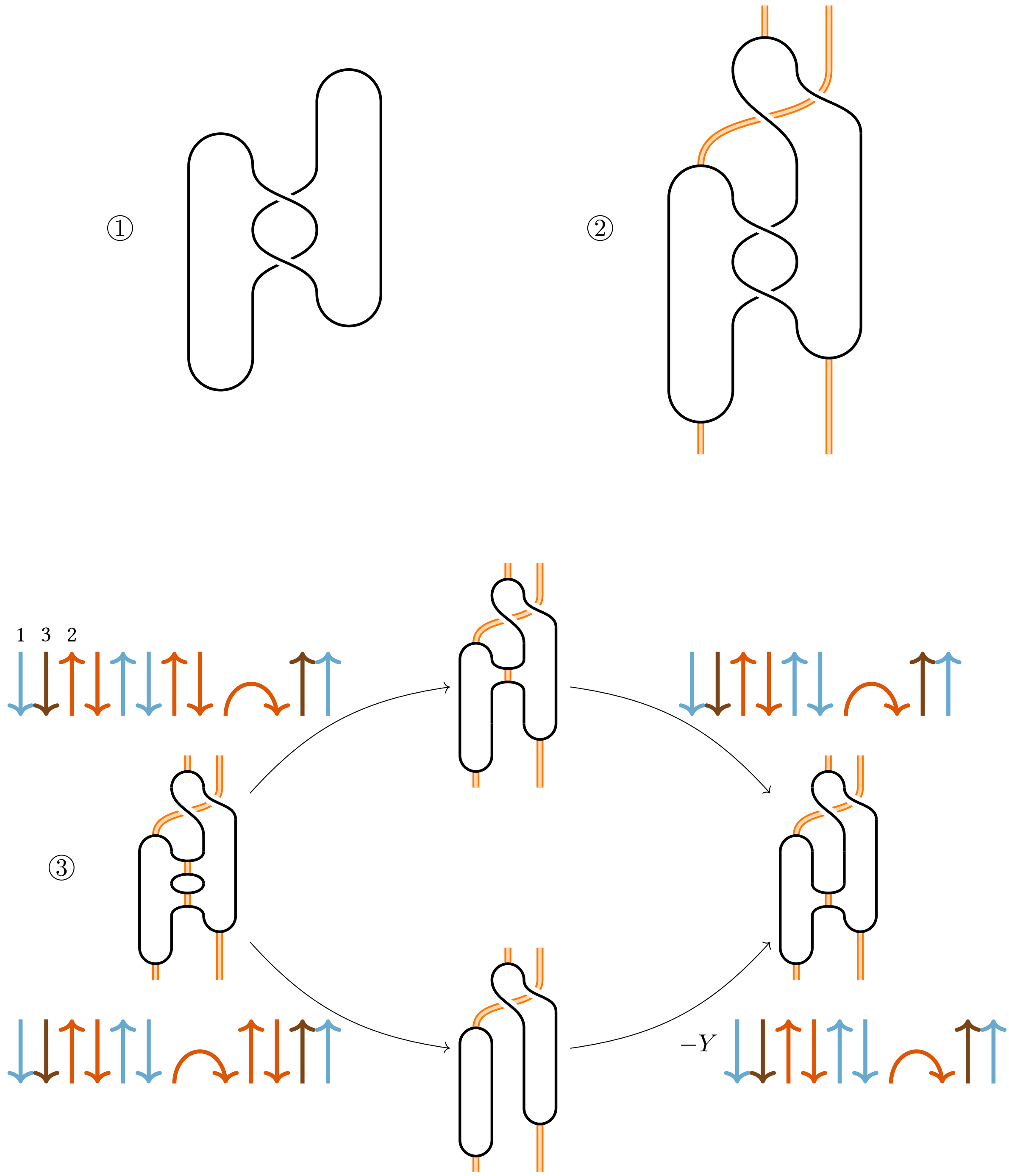
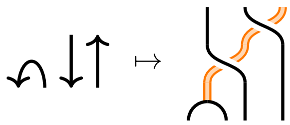


Figure 4.1: Defining procedure for covering $\mathfrak{gl}_2$-Khovanov homology in the case of a sliced tangle diagram presenting the Hopf link. For both differentials given by the categorified Kauffman bracket, the $\mathbb{Z}^2$-degree is $(0, -1)$. One checks that the graded Koszul rule in Definition 4.6 adds the scalar $-Y$ as pictured in the figure.

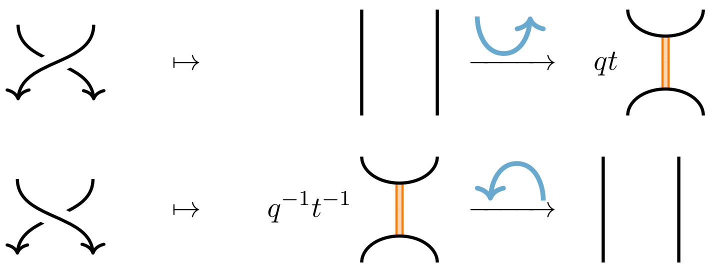
as well as all the relations obtained from the above by reflecting vertically and horizontally.

The procedure starts by telling how to assign webs to elementary oriented flat tangle diagram, that is, caps and cups. This can be done using crossings (20) in an essentially unique way thanks to relations (21). We make a choice by fixing a choice for the endpoints. Say that $\lambda \in \underline{\Lambda}$ is *antidominant* if it is antidominant as a $\mathfrak{gl}_2$-weight, that is if it is weakly increasing. To any set of n points on a line corresponds a unique antidominant weight $\lambda \in \underline{\Lambda}_d$ for $n \leq d$ and $n = d \mod 2$. In turn, those antidominant weights define a unique element in $\underline{\Lambda}$. Given any elementary oriented flat tangle diagram, we pick a web representative whose endpoints are antidominant by "adding a double line to the cup or cap and sliding it to the right". For instance:



Note that fixing a choice for the endpoints ensures that if two elementary flat oriented tangle diagrams are composable, then so are the corresponding webs in **Web**. We may extend this assignment to non-flat oriented tangle diagrams by formally adjoining crossings to our web diagrammatics (see step ② in Fig. 4.1).

The procedure extends to an assignment of a chain complex in $\mathrm{Kom}(\mathbf{GFoam})$. For cups and caps, it is the chain complex concentrated in homological degree 0 corresponding to the previously assigned web. For downward crossings, this is given by the Khovanov-Blanchet bracket, generalized to the graded case:



Here we used the variable t (resp. q) to denote the homological grading (resp. shift in q-grading), so that in each case, the identity web sits in homological degree zero. Note also that it is the only time we explicitly use the orientation data. This extends to a general crossing by choosing a way to rotate it into a downward crossing, possibly with additional cups and caps.

Finally, let D be a sliced oriented tangle diagram. Then $\mathrm{Kom}_{\mathfrak{gl}_2}(D)$ is defined as the horizontal composition (see Definition 4.6) of the chain complexes assigned to each slice of D. This ends the definition of $\mathrm{Kom}_{\mathfrak{gl}_2}(D)$ (see step ③ in Fig. 4.1). ◇

Proof of Theorem 4.3. We need to check invariance under all Turaev moves for sliced oriented tangle diagrams (see for instance [46, fig. 3.6]). Since the horizontal composition of chain complexes is coherent with homotopies, this can be done locally. For planar isotopies, this follows from Lemma 2.10. For Reidemeister moves, this can be shown following the techniques introduced by Bar-Natan [1]. □

4.2 Review of covering $\mathfrak{sl}_2$-Khovanov homology for links

We review the construction of covering $\mathfrak{sl}_2$-Khovanov homology as defined by Putyra [48]. His construction uses a 2-category of *chronological* cobordisms, close in spirit to Bar-Natan's definition of Khovanov homology [1]. For our purpose, we give here a "low-tech" definition of covering $\mathfrak{sl}_2$-Khovanov homology, directly generalizing the original definition of odd Khovanov homology of Ozsváth, Rasmussen and Szabó [47].

We first give some preliminary definitions. For $n \in \mathbb{N}$, let $\wedge_R(a_1, \dots, a_n)$ be the R-algebra generated by variables $a_1, \dots, a_n$ and subject to the following relations:

$$\begin{aligned} a_i a_j &= XY a_j a_i && \text{for } 1 \leq i, j \leq n, \\ a_i^2 &= 0 && \text{for } 1 \leq i \leq n. \end{aligned}$$

Denote by $\wedge_R^r(a_1, \dots, a_n)$ the R-submodule generated by words of length r in the letters $a_1, \dots, a_n$. We endow $\wedge_R(a_1, \dots, a_n)$ with a $\mathbb{Z}$-grading, the *q-grading*, setting $\operatorname{qdeg} p = n - 2r$ whenever $p \in \wedge_R^r(a_1, \dots, a_n)$. Define also the following linear maps:

$$\begin{aligned} m_{a_1, a_2; a} \colon \ \wedge_R\, (a_1, a_2, x_1, \dots, x_n) &\to \wedge_R(a, x_1, \dots, x_n) \\ p &\mapsto p|_{a_1, a_2 \mapsto a} \\ \Delta_{a; a_1, a_2} \colon \ \wedge_R\, (a, x_1, \dots, x_n) &\to \wedge_R(a_1, a_2, x_1, \dots, x_n) \\ p &\mapsto (a_1 + XY a_2) p|_{a \mapsto a_1} = (a_1 + XY a_2) p|_{a \mapsto a_2} \end{aligned}$$

Here $a_1, a_2 \mapsto a$ means that one should replace every instance of a_1 and a_2 by a in p, and similarly for $a \mapsto a_1$ and $a \mapsto a_2$. With respect to the q-grading, these maps are graded maps with q-degree

$$\operatorname{qdeg}(m_{a_1, a_2; a}) = \operatorname{qdeg}(\Delta_{a; a_1, a_2}) = -1.$$

Note that one recovers the algebra $\mathbb{Z}[a_1, \dots, a_n]/(a_1^2 = \dots = a_n^2 = 0)$ with its product and coproduct by setting $X = Y = Z = 1$, and the exterior algebra in variables $a_1, \dots, a_n$ by setting $X = Z = 1$ and $Y = -1$.

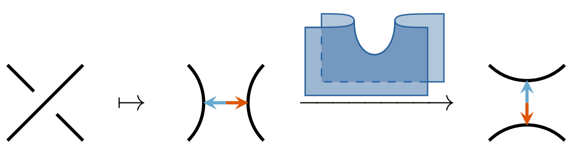
Recall Notation 4.4. For $\boldsymbol{r} \in \{0,1\}^N$ and $k, l \in \{1, \dots, N\}$ where $k < l$, the square

$$\begin{tikzcd} \boldsymbol{r} \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \boldsymbol{r} + \boldsymbol{e}_k \arrow[d] \\ \boldsymbol{r} + \boldsymbol{e}_l \arrow[r] & \boldsymbol{r} + \boldsymbol{e}_k + \boldsymbol{e}_l \end{tikzcd}$$

is given an orientation as depicted, and we denote it by $\square_{k,l}^{\boldsymbol{r}}$.

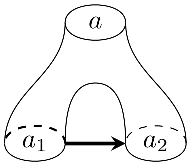
Let then D be an oriented link diagram with N crossings. The complex $\operatorname{Kom}_{\mathfrak{sl}_2}(D)$ is constructed through the following steps:

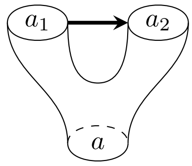
- *Hypercube of resolutions:* fix an arbitrary order on the crossings of D. Each crossing can be *resolved* into two possible planar diagrams, respectively the *0-resolution* (on the left) or the *1-resolution* (on the right):



A *resolution* of D is a choice of resolutions for each crossing. The resolutions of D can be pictured as sitting on the vertices of an hypercube $\{0,1\}^N$, where for $\boldsymbol{r} \in \{0,1\}^N$ the binary $\boldsymbol{r}_i$ encodes the chosen resolution for the i-th crossing. Each edge $\boldsymbol{r} \to \boldsymbol{r} + \boldsymbol{e}_i$ of the hypercube connects two resolutions that only differ at the i-crossing. This edge is decorated with a saddle cobordism, which can either be a merge or a split depending on the global context. Finally, for each crossing one must choose an orientation on the arcs of the two resolutions: the red or the blue orientation. We call it the *arc orientation*. Equivalently, an arc orientation is a choice of arc orientation for the 0-resolution, which induces an arc orientation for the 1-resolution by rotating a quarter of a turn clockwise. We denote $H_{\mathfrak{sl}_2}(D)$ the hypercube $\{0,1\}^N$ decorated as above.

- *Algebrization:* we turn the hypercube of resolutions into a hypercube in the category of $\mathbb{Z}$-graded R-modules. To each vertex $\boldsymbol{r} \in \{0,1\}^N$ we associate the R-module $V_{\boldsymbol{r}} := \wedge_R(a_1, \dots, a_n)$, where n is the number of connected components in the corresponding resolution. One should think of each variable as attached to one connected component. In addition, each edge is replace by a R-linear map between relevant R-modules:

  $\mapsto m_{a_1,a_2;a}$ and 

  $\mapsto \Delta_{a;a_1,a_2}$

  Note the importance of the extra arrows, which gives a preferred choice of ordering between the two circles corresponding to the variables a_1 and a_2. We denote $\mathcal{A}_{\mathfrak{sl}_2}(H_{\mathfrak{sl}_2}(D))$ the hypercube so obtained.

- *Commutativity:* As defined, squares in the hypercube do not necessary commute. In fact, if we consider a generic square

$$\begin{tikzcd} \boldsymbol{r} \arrow[r, "F_{*0}"] \arrow[d, "F_{0*}"'] \arrow[dr, phantom, "\circlearrowright"] & \boldsymbol{r}+\boldsymbol{e}_k \arrow[d, "F_{1*}"] \\ \boldsymbol{r}+\boldsymbol{e}_l \arrow[r, "F_{*1}"'] & \boldsymbol{r}+\boldsymbol{e}_k+\boldsymbol{e}_l \end{tikzcd},$$

  we have:

$$F_{1*} \circ F_{*0} = \psi_{\mathfrak{sl}_2}(\square^{\boldsymbol{r}}_{k,l}) F_{*1} \circ F_{0*}.$$

  Here $\psi_{\mathfrak{sl}_2}$ is the $R^\times$-valued 2-cochain on the hypercube defined by Table 1. As shown in [47, 48], $\psi_{\mathfrak{sl}_2}$ is a cocycle. An $\mathfrak{sl}_2$*-sign assignment* is a choice of a 1-cochain $\epsilon_{\mathfrak{sl}_2}$ such that $\partial\epsilon_{\mathfrak{sl}_2} = \psi_{\mathfrak{sl}_2}$. Such a choice always exists: by contractibility of the hypercube, a 2-cocycle is always a 2-coboundary. Given a choice of $\mathfrak{sl}_2$-sign assignment, we multiply each edge e of the hypercube by $\epsilon(e)$: this makes each square commutes.

- *Grading:* finally, we multiply each edge $\boldsymbol{r} \to \boldsymbol{r} + \boldsymbol{e}_i$ by $(-1)^{\#\{r_j=1|j<i\}}$ (Koszul rule) and define

$$\mathrm{Kom}_{\mathfrak{sl}_2}(D) := \bigoplus_{\boldsymbol{r}\in\{0,1\}^N} q^{|\boldsymbol{r}|-N_-} t^{|\boldsymbol{r}|-N_-} V_{\boldsymbol{r}},$$

  where N_- denotes the number of negative crossings, and as before we used the variable t (resp. q) to denote the homological grading (resp. shift in q-grading). Note that with those shifts, the differential preserves the q-grading. Note also that this is the only time we use the orientation on D.

This ends the definition of $\mathrm{Kom}_{\mathfrak{sl}_2}(D)$. $\diamond$

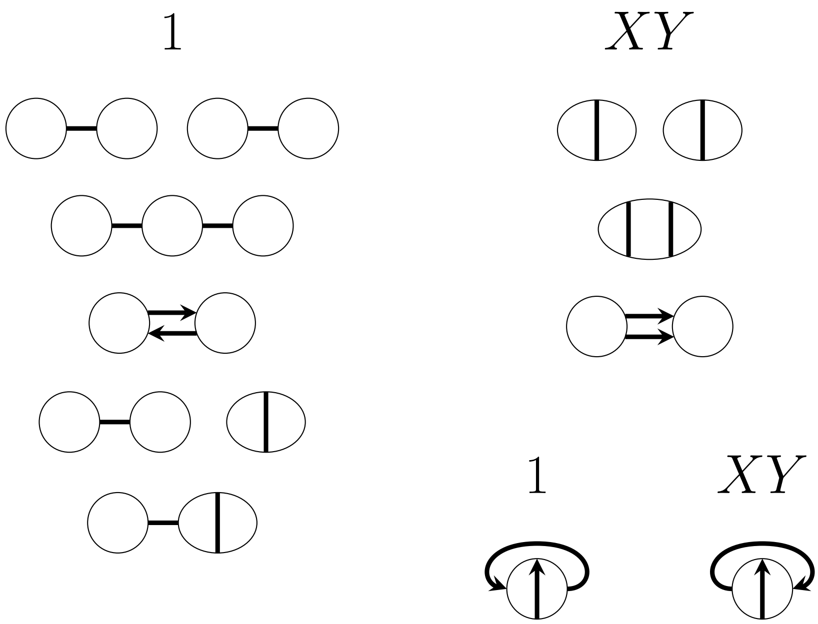


Table 1: Definition of $\psi_{\mathfrak{sl}_2}$ for covering $\mathfrak{sl}_2$-Khovanov homology. Each square is univoquely represented by the (relevant local piece of) resolution at the initial point. If no orientation on the arrows is given, then the value of $\psi_{\mathfrak{sl}_2}$ is independent on the choice of orientations. Note that we do not need to specify the orientation of the squares, as the image of $\psi_{\mathfrak{sl}_2}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

Theorem 4.7 ([47, 48]). *Let L be a link and D a link diagram of L with writhe $w(D)$. Then, the isomorphism class of $\mathrm{Kom}_{\mathfrak{sl}_2}(D)$ is independent on the choice of ordering on crossings, the choice of arc orientations, and the choice of $\mathfrak{sl}_2$-sign assignment. Moreover:*

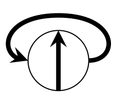
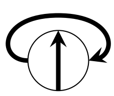
$$\mathrm{CKh}_{\mathfrak{sl}_2}(L) := H^\bullet\left(q^{w(D)}\,\mathrm{Kom}_{\mathfrak{sl}_2}(D)\right)$$

is an invariant of L, called the covering $\mathfrak{sl}_2$-Khovanov homology *of L.*

Remark 4.8. The 2-cocycle $\psi_{\mathfrak{sl}_2}$ is not the only choice that makes the construction above works. Indeed, for the last two cases of Table 1, called the *ladybugs*,[1] we both have

$$F_{1*} \circ F_{*0} = \psi_{\mathfrak{sl}_2}(\Box^r_{k,l})F_{*1} \circ F_{0*} \quad \text{and} \quad F_{1*} \circ F_{*0} = XY\psi_{\mathfrak{sl}_2}(\Box^r_{k,l})F_{*1} \circ F_{0*}$$

Define $\overline{\psi}_{\mathfrak{sl}_2}$ to be the 2-cochain defined as $\psi_{\mathfrak{sl}_2}$ except for the ladybugs, where instead we set

$\overline{\psi}_{\mathfrak{sl}_2}\Bigg($  $\Bigg) = XY$ and $\overline{\psi}_{\mathfrak{sl}_2}\Bigg($  $\Bigg) = 1.$

The results of Theorem 4.7 still holds in this case. Let us write $\mathrm{CKh}_{\mathfrak{sl}_2}(L, \psi_{\mathfrak{sl}_2})$ and $\mathrm{CKh}_{\mathfrak{sl}_2}(L, \overline{\psi}_{\mathfrak{sl}_2})$ to distinguish the two homologies. The homologies $\mathrm{CKh}_{\mathfrak{sl}_2}(L, \psi_{\mathfrak{sl}_2})$ and $\mathrm{CKh}_{\mathfrak{sl}_2}(L, \overline{\psi}_{\mathfrak{sl}_2})$ are respectively called "type X" and "type Y" in [47] (setting $X = Z = 1$ and $Y = -1$; no analogy intended between the scalar X and "type X"). It is shown in Putyra [48] that $\mathrm{CKh}_{\mathfrak{sl}_2}(L, \psi_{\mathfrak{sl}_2})$ and $\mathrm{CKh}_{\mathfrak{sl}_2}(L, \overline{\psi}_{\mathfrak{sl}_2})$ are in fact isomorphic.

[1] This terminology is borrowed from [39].

4.3 Equivalence between covering $\mathfrak{sl}_2$- and $\mathfrak{gl}_2$-Khovanov homology

In this subsection, we give a proof of Theorem 4.3. Here is a quick summary of the proof:

(i) We restate the definition of covering $\mathfrak{gl}_2$-Khovanov homology using a *$\mathfrak{gl}_2$-hypercube of resolutions*. The rest of the proof consists of comparing the two hypercubes. This is done in Subsection 4.3.2.

(ii) To compare the hypercubes, we need to compare the R-modules at each vertex. This requires a choice of basis for each R-module $\mathrm{Hom}_{\mathbf{GFoam}}(\emptyset, W)$, called *cup foams*, that we describe in Subsection 4.3.1.

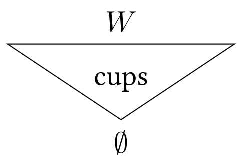
(iii) In Subsection 4.3.3, we use the above basis to define a family of isomorphisms on the level of vertices. This defines a proper morphism of hypercubes only *up to invertible scalar*; we call it a *projective morphism*. We state a certain 2-cocycle condition such that, if satisfied, the aforementioned family of isomorphisms can be rescaled into a genuine isomorphism of hypercube.

(iv) The proof of Theorem 4.3 then reduces to the analysis of this 2-cocycle condition. Subsection 4.3.4 shows that this can be done locally, looking only at the cases pictured in Table 1. In most cases, general considerations show that the 2-cocycle condition is necessary verified.

(v) However, this does not work for the ladybugs (see Remark 4.8). To deal with these two cases, we require finer results on the independence on all choices involved of the above family of isomorphisms. This is done in Subsection 4.3.5, which concludes the proof.

4.3.1 Cup foams

Call a web W *closed* if $c(W)$ is a closed 1-manifold. Recall the basis described in Subsection 2.5. In the special case where the domain is the empty web and the codomain is a closed web W (recall that we work in **GFoam**), the foam $F\colon \emptyset \to W$ given by Lemma 2.24 is such that

$c(F) =$ 

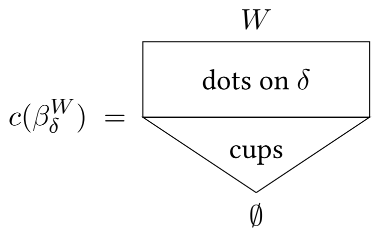
We call such an F an *undotted cup foam on W*. If we wish to allow F to (possibly) carry dots, we simply say that F is a *cup foam on W*. We write $|c(W)|$ for the set of closed components of $c(W)$. In this context, Theorem 2.23 is restated as follows:

Proposition 4.9. *Let W be a closed web. Let B be a set containing precisely one cup foam*

$$\beta_\delta\colon \emptyset \to W$$

for each subset $\delta \subset |c(W)|$, so that for each $c \in |c(W)|$, the corresponding disk in $c(\beta_\delta)$ is dotted if and only if $c \in \delta$. Then B is basis for the R-module $\mathrm{Hom}_{\mathbf{GFoam}}(\emptyset, W)$.

Fix an undotted cup foam β^W for W and pick a total order on $|c(W)|$. For each subset $\delta \subset c(W)$, denote by $\mathrm{id}_W^\delta\colon W \to W$ the foam identical to id_W except for an additional dot on the connected component c for each $c \in \delta$, ordering the dots increasingly with respect to the total order on $|c(W)|$, reading from bottom to top. This defines id_W^δ uniquely. We denote $\beta_\delta^W := \mathrm{id}_W^\delta \circ \beta^W$. Schematically:



It follows from Proposition 4.9 that

Corollary 4.10. *for any choice of undotted cup foam β^W for W and total order on $|c(W)|$, the family $\{\beta_\delta^W\}_{\delta \subset c(W)}$ defines a basis for the R-module $\mathrm{Hom}_{\mathbf{GFoam}}(\emptyset, W)$.*

Let V and W be two closed webs. If $F\colon V \to W$ is a zip or an unzip, then $c(F)$ is either a merge or a split. Below we sometimes speak of the closed components in the domain and codomain of $c(F)$ to refer only to the closed components making up the boundary of the closed component in $c(F)$ containing the saddle. The distinction should be clear by the context.

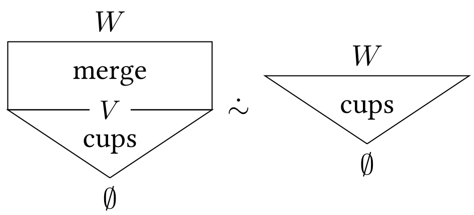
Recall the symbol $\dot{\sim}$ to mean "equal up to multiplying by an invertible scalar", defined in the paragraph before Lemma 2.15.

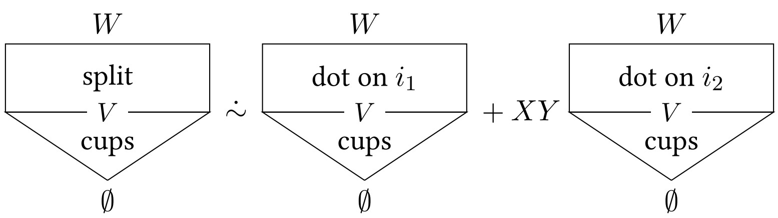
Proposition 4.11. *Let V and W be closed webs. Let β^V (resp. β^W) be a choice of undotted cup foam for V (resp. W). Let $F\colon V \to W$ be a zip or an unzip. Then:*

(i) If $c(F)$ is a merge, then
$$F \circ \beta^V \mathrel{\dot{\sim}} \beta^W.$$

(ii) If $c(F)$ is split, then
$$F \circ \beta^V \mathrel{\dot{\sim}} \beta_{i_1}^W + XY\beta_{i_2}^W,$$
where i_1 and i_2 are the connected components of W corresponding to the codomain of the split $c(F)$.

Here is the schematic for Proposition 4.11:





Proof. Fix a total order on $|c(V)|$ and $|c(W)|$. Using the neck-cutting successively on the identity of W, one can decompose it as

$$\mathrm{id}_W = \sum_{\delta\subset|c(W)|} \beta^W_\delta \circ \beta^c_{\delta^c},$$

where $\delta^c := |c(W)| \setminus \delta$, and each $\beta^c_\delta \colon W \to \emptyset$ is a *cap foam* in the sense that $c(\beta^c_\delta)$ is an union of disks, each disk being dotted depending on δ as in Proposition 4.9. This allows us to write:

$$F \circ \beta^V = \sum_{\delta\subset|c(W)|} \beta^W_\delta \circ \left(\beta^c_{\delta^c} \circ F \circ \beta^V\right),$$

where each $\beta^c_{\delta^c} \circ F \circ \beta^V$ is a *closed foam*, that is a foam with domain and codomain the empty web. Then, we apply the following result on the evaluation of closed foams, which is a consequence of Theorem 2.23:

Lemma 4.12. *Let $U \colon \emptyset \to \emptyset$ be a closed foam. Then:*

$$U \mathrel{\dot\sim} \begin{cases} \mathrm{id}_\emptyset & \text{if each closed component of } c(U) \text{ is a sphere with a single dot,} \\ 0 & \text{otherwise.} \end{cases}$$

By the above lemma, there exist invertible scalars τ, τ_1 and τ_2 such that:

$$F \circ \beta^V = \tau\beta^W \quad \text{or} \quad F \circ \beta^V = \tau_1\beta^W_{i_1} + \tau_2\beta^W_{i_2},$$

depending on whether $c(F)$ is a merge or a split. It remains to show that $\tau_1/\tau_2 = XY$ in the latter case. For that, we use Lemma 4.13 below. Assume $i_1 < i_2$ for the purpose of the computation, so that $\mathrm{id}_{i_1,i_2} = \mathrm{id}_{i_2} \circ \mathrm{id}_{i_1}$:

$$\begin{aligned} \tau_1\beta^W_{i_1,i_2} &= \mathrm{id}^W_{i_2} \circ (\tau_1\beta^W_{i_1} + \tau_2\beta^W_{i_2}) = \mathrm{id}^W_{i_2} \circ F \circ \beta^V \\ &\overset{4.13}{=} \mathrm{id}^W_{i_1} \circ F \circ \beta^V = \mathrm{id}^W_{i_1} \circ (\tau_1\beta^W_{i_1} + \tau_2\beta^W_{i_2}) = XY\tau_2\beta^W_{i_1,i_2}. \end{aligned}$$

Since $\beta^W_{i_1,i_2} \neq 0$ belongs to a free family by Proposition 4.9, we must have $(\tau_1 - XY\tau_2) = 0$, which concludes. ☐

Lemma 4.13. *Let $F \colon W \to W'$ and $F' \colon W' \to W''$ be two foams and let D_1 and D_2 be two foams identical to $\mathrm{id}_{W'}$ except for a dot sitting on a 1-facet. If the two dots belong to the same closed component of $c(F' \circ F)$, then*

$$F' \circ D_1 \circ F = F' \circ D_2 \circ F$$

in **GFoam**.

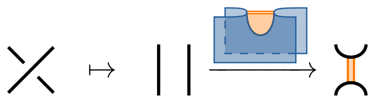
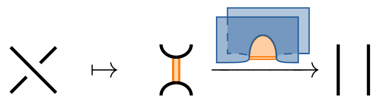
Proof. This is a consequence of the fact that a dot can slide across 2-facets at no cost of scalar, and along 1-facets past generators depending on μ. Thanks to Definition 2.6, the latter scalar only depends on the relative vertical position of the dot. ☐

4.3.2 The $\mathfrak{gl}_2$-hypercube of resolutions

We reformulate the definition of covering $\mathfrak{gl}_2$-Khovanov homology to emphasize the similarities with covering $\mathfrak{sl}_2$-Khovanov homology.

Let D be a sliced oriented link diagram with N crossings. As described in Subsection 4.1, we can associate to D a knotted web W_D. Then, starting with W_D, the complex $\mathcal{A}_{\mathfrak{gl}_2}(\mathrm{Kom}_{\mathfrak{gl}_2}(D))$ can be defined as follows:

(i) *Hypercube of resolutions:* each crossing can be resolved into a *web 0-resolution* or a *web 1-resolution*:

 and 

A *web resolution* is a choice of web resolutions for each crossing. Fixing an ordering on the crossings, they can be pictured as sitting on the vertices of the hypercube $\{0,1\}^N$, whose edges are decorated with a zip or an unzip, depending on whether the associated crossing is positive or negative. Denote $H_{\mathfrak{gl}_2}(D)$ the decorated hypercube so obtained.

(ii) *Algebrization:* we apply the functor $\mathcal{A}_{\mathfrak{gl}_2}$ to obtain the hypercube $\mathcal{A}_{\mathfrak{gl}_2}(H_{\mathfrak{gl}_2}(D))$.

(iii) *Commutativity:* a $\mathfrak{gl}_2$*-sign assignment*[1] is an $R^\times$-valued 1-cochain $\epsilon_{\mathfrak{gl}_2}$ on the hypercube $\{0,1\}^N$, such that $\partial\epsilon_{\mathfrak{gl}_2} = \psi_{\mathfrak{gl}_2}$ where $\psi_{\mathfrak{gl}_2}$ is a 2-cocycle defined as

$$\psi_{\mathfrak{gl}_2}(\square^{\boldsymbol{r}}_{k,l}) := \mu(\deg F_{0*}, \deg F_{*0})^{-1} = \mu(\deg F_{*1}, \deg F_{1*}).$$

We multiply each edge e by $\epsilon_{\mathfrak{gl}_2}(e)$. This makes each square commutes. This defines an hypercube $H_{\mathfrak{gl}_2}(D;\epsilon_{\mathfrak{gl}_2})$.

(iv) *Grading:* we apply the Koszul rule and shift the grading, in exactly the same way as for the covering $\mathfrak{sl}_2$-Khovanov homology.

As a consequence of Lemma 5.8, the isomorphism class of the complex obtained does not depend on the choice of $\mathfrak{gl}_2$-sign assignment. It is easily checked that the construction above coincide with the definition of covering $\mathcal{A}_{\mathfrak{gl}_2}(H_{\mathfrak{gl}_2}(D))$ given in Subsection 4.1. In particular, it is shown in Definition 5.7 that the graded Koszul rule is a $\mathfrak{gl}_2$-sign assignment.[2]

4.3.3 A projective isomorphism of hypercubes

Let D be a sliced oriented link diagram. To show Theorem 4.3, it suffices to exhibit an isomorphism between the two hypercubes $H_{\mathfrak{sl}_2}(D;\epsilon_{\mathfrak{sl}_2})$ and $H_{\mathfrak{gl}_2}(D;\epsilon_{\mathfrak{gl}_2})$, where an *isomorphism of hypercubes* is a family of isomorphisms at each vertex, such that all squares involved commute. Indeed, as step (iv) is identical for both constructions, we may disregard grading. Note that in particular, the orientation of D is irrelevant for rest of the discussion.

Let then D be a sliced link diagram with N crossings, with a fixed choice of ordering on crossings and arc orientations. We often do not distinguish between $H_{\mathfrak{sl}_2}(D)$ (resp. $H_{\mathfrak{gl}_2}(D)$) and $\mathcal{A}_{\mathfrak{sl}_2}(H_{\mathfrak{sl}_2}(D))$ (resp. $\mathcal{A}_{\mathfrak{gl}_2}(H_{\mathfrak{gl}_2}(D))$). The relevant hypercube should be clear by the context. Also, we use $\mathfrak{sl}_2$ and $\mathfrak{gl}_2$ as subscripts to distinguish features of the two constructions. For instance, we write $\boldsymbol{r}_{\mathfrak{gl}_2}$ to denote the decoration on the vertex $\boldsymbol{r} \in \{0,1\}^N$ of the hypercube $H_{\mathfrak{gl}_2}(D)$ (that is, a web W, or the graded R-module $\mathrm{Hom}_{\mathbf{GFoam}}(\emptyset, W)$ depending on the context).

Looking at step (i) in the construction of the hypercubes, it is clear that $c(H_{\mathfrak{gl}_2}(D)) = H_{\mathfrak{sl}_2}(D)$: that is, $c(\boldsymbol{r}_{\mathfrak{gl}_2}) = \boldsymbol{r}_{\mathfrak{sl}_2}$ for each $\boldsymbol{r} \in \{0,1\}^N$, and similarly for edges. For each vertex $W = \boldsymbol{r}_{\mathfrak{gl}_2}$ of $H_{\mathfrak{gl}_2}(D)$, fix a choice of undotted cup foam β^W and a choice of total ordering on the set of connected components $|c(W)|$. By Corollary 4.10, $\boldsymbol{r}_{\mathfrak{gl}_2}$ has basis given by the set $\{\beta^W_\delta\}_{\delta\subset|c(W)|}$.

[1]The notion of sign assignment here is slightly different from Section 5, as the latter already includes the Kozsul rule.

[2]With the caveat of the previous footnote.

On the other hand, $\boldsymbol{r}_{\mathfrak{sl}_2}$ has basis given by the set $\{a_\delta\}_{\delta\subset|c(W)|}$, where $a_\delta = a_{i_k}\dots a_{i_1}$ with $\delta = \{i_k,\dots,i_1\}$ and $i_k > \dots > i_1$. Hence, the map

$$\iota_W\colon \beta^W_\delta \mapsto \mu(|\delta|\,(-1,-1),\beta^W)a_\delta$$

defines an isomorphism of graded R-modules.[1] Note that ι_W depends on the choice of β^W, but not on the choice of ordering on $|c(W)|$. Moreover, the family of those isomorphisms at each vertex defines a *projective* isomorphism of hypercubes, in the sense that for each edge $e\colon V\to W$ in $H_{\mathfrak{gl}_2}(D)$, the square

$$\square_e := \quad \begin{tikzcd} V \arrow[r, "e"] \arrow[d, "\iota_V"'] \arrow[dr, phantom, "\circlearrowright"] & W \arrow[d, "\iota_W"] & H_{\mathfrak{gl}_2}(D) \\ c(V) \arrow[r, "c(e)"'] & c(W) & H_{\mathfrak{sl}_2}(D) \end{tikzcd} \tag{22}$$

commutes up to invertible scalar. This scalar is computed in the following lemma:

Lemma 4.14. *Denote by $\tau\in R^\times$ the invertible scalar given by Proposition 4.11. Then either*

$$e\circ\beta^V = \tau\beta^W \quad \text{or} \quad e\circ\beta^V = \tau\left(\beta^W_{i_1} + XY\beta^W_{i_2}\right),$$

depending on whether $c(e)$ is a merge or a split. We differentiate between i_1 and i_2 using the arc orientation: it goes from i_1 to i_2. Define

$$\psi_\beta(e) := \begin{cases} \tau & \text{if } c(e) \text{ is a merge,} \\ \tau\mu((-1,-1),\beta^W) & \text{if } c(e) \text{ is a split.} \end{cases}$$

Then $\iota_W\circ e = \psi_\beta(e)(c(e)\circ\iota_V)$.

Note that $\psi_\beta(e)$ depends in general on the choices of arc orientation on e and undotted cup foams β^V and β^W, but not on the choice of ordering on closed components. The proof of Lemma 4.14 is postponed to after this discussion.

Ultimately, we are interested in comparing the hypercubes $H_{\mathfrak{sl}_2}(D;\epsilon_{\mathfrak{sl}_2})$ and $H_{\mathfrak{gl}_2}(D;\epsilon_{\mathfrak{gl}_2})$. The above suggests the strategy of finding a 0-cochain φ on $\{0,1\}^N$ such that the square

$$\begin{tikzcd} c(V) \arrow[r, "\epsilon_{\mathfrak{gl}_2}(e)e"] \arrow[d, "\varphi(V)\iota_V"'] \arrow[dr, phantom, "\circlearrowright"] & W \arrow[d, "\varphi(W)\iota_W"] & H_{\mathfrak{gl}_2}(D;\epsilon_{\mathfrak{gl}_2}) \\ c(V) \arrow[r, "\epsilon_{\mathfrak{sl}_2}(e)c(e)"'] & c(W) & H_{\mathfrak{sl}_2}(D;\epsilon_{\mathfrak{sl}_2}) \end{tikzcd}$$

commutes. That would define an isomorphism of hypercubes, and prove Theorem 4.3.

We can rephrase the problem as follows. Denote by $H_\iota(D)$ the $(N+1)$-dimensional hypercube decorated as $H_{\mathfrak{gl}_2}(D)$ on $\{0,1\}^N\times\{0\}$, as $H_{\mathfrak{gl}_2}(D)$ on $\{0,1\}^N\times\{1\}$, and decorated with ι on edges $\boldsymbol{r}\times\{0\}\to\boldsymbol{r}\times\{1\}$. Define also the following 2-cochain ψ on $H_\iota(D)$:

$$\psi := \begin{cases} \psi_{\mathfrak{gl}_2} & \text{on } \{0,1\}^N\times\{0\}, \\ \psi_{\mathfrak{sl}_2} & \text{on } \{0,1\}^N\times\{1\}, \\ \psi_\beta(e)^{-1} & \text{on } \square_e, \end{cases}$$

[1] One can think of a_δ as corresponding to $\beta^W\circ\mathrm{id}^W_\delta$: this makes no sense when it comes to composition, but this is coherent with the interchange relation. Thinking this way makes some of the identities below clearer.

$$
\boxbox_S := \quad
\begin{tikzcd}
W_{00} \arrow[rr, "e_{0*}"] \arrow[dr, "e_{*0}"] \arrow[ddd, "\iota_{W_{00}}"'] & & W_{01} \arrow[dr, "e_{*1}"] \arrow[ddd, "\iota_{W_{01}}"] & & S \\
& W_{10} \arrow[rr, "e_{1*}"] \arrow[ddd, "\iota_{W_{10}}"] & & W_{11} \arrow[ddd, "\iota_{W_{11}}"] & \\
& & & & \\
c(W_{00}) \arrow[rr, "c(e_{0*})"'] \arrow[dr, "c(e_{*0})"'] & & c(W_{01}) \arrow[dr, "c(e_{*1})"'] & & \\
& c(W_{10}) \arrow[rr, "c(e_{1*})"'] & & c(W_{11}) & c(S)
\end{tikzcd}
$$

Figure 4.2: The 3-dimensional cube $\boxbox_S$, where S is a square of $H_{\mathfrak{gl}_2}(D)$, pictured on the top.

where we recall $\square_e$ from (22).

Assume that ψ is a 2-cocycle. Then by contractibility of the hypercube, it is a coboundary, and there exists some 1-cochain ϵ such that $\partial\epsilon = \psi$. Denote by $H_\iota(D;\epsilon)$ the hypercube $H_\iota(D)$ obtained by multiplying each edge by its value on ϵ. By definition, $\epsilon_{\mathfrak{gl}_2} := \epsilon|_{\{0,1\}^N\times\{0\}}$ (resp. $\epsilon_{\mathfrak{sl}_2} := \epsilon|_{\{0,1\}^N\times\{1\}}$) is a $\mathfrak{gl}_2$-sign assignment (resp. a $\mathfrak{sl}_2$-sign assignment). In other words, the hypercube $\{0,1\}^N \times \{0\}$ (resp. $\{0,1\}^N \times \{1\}$) in $H_\iota(D;\epsilon)$ coincides with $H_{\mathfrak{gl}_2}(D;\epsilon_{\mathfrak{gl}_2})$ (resp $H_{\mathfrak{sl}_2}(D;\epsilon_{\mathfrak{sl}_2})$). Moreover, by Lemma 4.14 all squares of the kind $\square_e$ commute. In other words, the $(N+1)$-direction in the hypercube $H_\iota(D;\epsilon)$ defines an isomorphism of hypercubes between $H_{\mathfrak{gl}_2}(D;\epsilon_{\mathfrak{gl}_2})$ and $H_{\mathfrak{sl}_2}(D;\epsilon_{\mathfrak{sl}_2})$.

When is ψ a 2-cocycle? Note that it suffices that ψ is a 2-cocycle on every 3-dimensional cube of $H_\iota(D)$. As $\psi_{\mathfrak{gl}_2}$ and $\psi_{\mathfrak{sl}_2}$ are already 2-cocycles, it is only necessary that ψ is a 2-cocycle on every 3-dimensional cube $\boxbox_S := S \times \{0,1\}$ for S a square of $H_{\mathfrak{gl}_2}(D)$ (see Fig. 4.2). We give an orientation on $\boxbox_S$ such that it agrees with the orientation of e_{0*}.

To sum up, we have shown that:

Proposition 4.15. *Let D be a sliced link diagram with N crossings. Assume given an ordering on crossings, a choice of arc orientations, and a choice of undotted cup foams on the webs decorating $H_{\mathfrak{gl}_2}(D)$. If for every square S of $H_{\mathfrak{gl}_2}(D)$, the identity*

$$\partial\psi(\boxbox_S) = 1$$

holds, then there exist sign assignments $\epsilon_{\mathfrak{gl}_2}$ and $\epsilon_{\mathfrak{sl}_2}$ such that $H_{\mathfrak{gl}_2}(D;\epsilon_{\mathfrak{gl}_2})$ and $H_{\mathfrak{sl}_2}(D;\epsilon_{\mathfrak{sl}_2})$ are isomorphic. $\square$

We end this section with the proof of Lemma 4.14.

Proof of Lemma 4.14. We have two cases: either $c(e)$ is a merge, or it is a split. In both cases, we compare where the basis element β^W_δ is mapped through the two paths defining the square. The first case is depicted below, where the two end results are separated by a dashed line:

$$\begin{tikzcd}
\beta^V_\delta \arrow[r] \arrow[dd] & \tau\mu(\deg e, |\delta|\,(-1,-1))\;\beta^W_\delta \arrow[d] \\
& \tau\mu(\deg e, |\delta|\,(-1,-1))\mu(|\delta|\,(-1,-1), \deg \beta^W)\; a^W_\delta \\
\mu(|\delta|\,(-1,-1), \deg \beta^V)\; a^V_\delta \arrow[r] & \mu(|\delta|\,(-1,-1), \deg \beta^V)\; a^W_\delta
\end{tikzcd}$$

We use the identity $\deg \beta^W = \deg \beta^V + \deg e$ to conclude. Similarly, the computation for the second case gives:

$$\begin{tikzcd}
\beta^V_\delta \arrow[r] \arrow[dd] & \tau\mu(\deg e, |\delta|\,(-1,-1))\;\mathrm{id}^\delta_W \circ \left(\beta^W_{i_1} + XY\beta^W_{i_2}\right) \arrow[d] \\
& \tau\mu(\deg e, |\delta|\,(-1,-1))\mu((|\delta|+1)(-1,-1), \deg \beta^W)\; a^W_\delta\; \left(a^W_{i_1} + XY a^W_{i_2}\right) \\
\mu(|\delta|\,(-1,-1), \deg \beta^V)\; a^V_\delta \arrow[r] & \mu(|\delta|\,(-1,-1), \deg \beta^V)\; \left(a^W_{i_1} + XY a^W_{i_2}\right) a^W_\delta
\end{tikzcd}$$

and the identity $\deg \beta^W + (-1,-1) = \deg \beta^V + \deg e$ concludes. $\square$

4.3.4 Local analysis

For a sliced link diagram D together with choices as in Proposition 4.15, we need to verify

$$\partial(\square_S) = 1$$

for every square S in $H_{\mathfrak{gl}_2}(D)$. Note that this certainly does not depend on the choice of ordering on crossings. We implicitly assume such a choice in the sequel.

Note also that given a square S in $H_{\mathfrak{gl}_2}(D)$, the value of $\partial(\square_S)$ only depends on *local* choices: a choice of arc orientations for the two crossings involved and choices of undotted cup foams for the four webs involved. (This is essentially because ι only depends on local choices.) In other words, we can restrict our analysis to the set of squares that can appear in a $\mathfrak{gl}_2$-hypercube of resolutions, that is, to diagrams with exactly two crossings:

Lemma 4.16. *Assume that for every sliced link diagram S with exactly two crossings, and for all choices of arc orientations and undotted cup foams, the identity $\partial(\square_S) = 1$ holds. Then Theorem 4.3 holds.* $\square$

Recall the pictures of Table 1: they describe each possible isotopy class, together with the data of arc orientations, associated to such a sliced link diagram S. They are obtained by recording only the 0-resolution for both crossings together with their arc orientation. More precisely, pictures in

Table 1 only picture the non-trivial local part, obtained by removing the simple closed loops that do not contain the boundary of an arc. We call them *local arc presentations.*

Recall the ladybug local arc presentation from Remark 4.8: this was the only case were the value of the $\mathfrak{sl}_2$-2-cocycle $\psi_{\mathfrak{sl}_2}$ could be set differently. As we could expect, this extra freedom in the definition of covering $\mathfrak{sl}_2$-Khovanov homology forces some extra work to deal with the ladybug in our proof of Theorem 4.3. For the other cases, a generic argument is sufficient:

Proposition 4.17. *Let S be a sliced link diagram with exactly two crossings, together with choices of arc orientations and undotted cup foams. Then:*

(i) if the local arc presentation of S is not a ladybug, then $\partial(\text{⧉}_S) = 1$,

(ii) if the local arc presentation of S is a ladybug, then $\partial(\text{⧉}_S) = 1$ or $\partial(\text{⧉}_S) = XY$.

Proof. Recall the notations of Fig. 4.2. As ψ is the 2-cochain controlling the commutativity in ⧉_S, by definition we have that $p = \partial\psi(\text{⧉}_S)p$ for $p = c(e_{*1}) \circ c(e_{0*})$. In particular:

$$(1 - \partial\psi(\text{⧉}_S))p(1) = 0. \tag{23}$$

the element $p(1)$ admits a unique decomposition into basis elements: $p(1) = \sum_{i=1}^{n} \lambda_i a_{\delta_i}$ for some scalars $\lambda_i \in R$ and some subset $\delta_i \subset c(W_{11})$. The above relation and the unicity of the decomposition implies that $(1 - \partial\psi(\text{⧉}_S))\lambda_i = 0$ for all $i = 1, \dots, n$. Hence if any of the λ_i's is invertible, we automatically get that $\partial\psi(\text{⧉}_S) = 1$.

The only case where we get non-invertible coefficients is when $c(e_{0*})$ is a split and $c(e_{*1})$ is a merge, that is, in the ladybug cases. In that case we have $p(1) = (1+XY)a_j$ for some $j \in c(W_{11})$, which forces either $\partial\psi(\text{⧉}_S) = 1$ or $\partial\psi(\text{⧉}_S) = XY$. $\square$

If $\partial(\text{⧉}_S) = 1$ holds for all choices with a ladybug local arc presentation, then Theorem 4.3 holds. If on the contrary $\partial(\text{⧉}_S) = XY$ holds for all choices with a ladybug local arc presentation, then Theorem 4.3 still holds, as we can apply the same reasoning using the $\mathfrak{sl}_2$-2-cocycle $\overline{\psi}_{\mathfrak{sl}_2}$ defined in Remark 4.8 instead, effectively comparing our $\mathfrak{gl}_2$-construction with the $\mathfrak{sl}_2$-construction of type Y. In either case, we construct an isomorphism between the $\mathfrak{gl}_2$-hypercube and the $\mathfrak{sl}_2$-hypercubes of type X or of type Y, the latter two being isomorphic.

In other words, what matters is that, whatever the value of $\partial(\text{⧉}_S)$, it remains the same for all the choices involved. This is given by the following proposition:

Proposition 4.18. *Let S be a sliced link diagram with exactly two crossings, together with choices of arc orientations and undotted cup foams. Then the value of $\partial(\text{⧉}_S)$ only depends on the local arc presentation of S.*

The proof of Proposition 4.18 is given in Subsection 4.3.5.

Remark 4.19. A direct computation shows that in fact, we do have $\partial(\text{⧉}_S) = 1$ even in the case of a ladybug local arc presentation. It is interesting to note that, if we change the defining zigzag relations as follows:

leaving the rest of the definition identical, we get $\partial(\text{⧉}_S) = XY$ in the ladybug case. It is not clear to the authors whether this other graded-2-category of $\mathfrak{gl}_2$-foams is isomorphic to $\mathbf{GFoam}_d$.

4.3.5 Independence on choices

We prove Proposition 4.18. With the notation of Fig. 4.2, recall that S is oriented as follows:

$$S = \begin{tikzcd} W_{00} \arrow[r, "e_{*0}"] \arrow[d, "e_{0*}"'] \arrow[dr, phantom, "\circlearrowright"] & W_{10} \arrow[d, "e_{1*}"] \\ W_{01} \arrow[r, "e_{*1}"'] & W_{11} \end{tikzcd}$$

and similarly for $c(S)$. We compute that:

$$\partial\psi(\text{⧉}_S) = \psi_{\mathfrak{gl}_2}(S)\psi_{\mathfrak{sl}_2}(c(S))^{-1}\partial\psi_\beta(S).$$

Lemma 4.20. *Let S be a sliced link diagram with exactly two crossings, together with choices of arc orientations and undotted cup foams. Then the value of $\partial(\text{⧉}_S)$ does not depend on the choices of arc orientations and undotted cup foams.*

Proof. Assume we swap the arc orientation of the kth crossing. Then $\partial\psi_\beta(S)$ will contribute an additional factor XY for every split in direction k. Looking case by case at Table 1, one checks that this change is exactly compensated by the contribution of $\psi_{\mathfrak{sl}_2}(c(S))$. The value of $\psi_{\mathfrak{gl}_2}(S)$ does not change. Hence, changing the arc orientations does not change the value of $\partial(\text{⧉}_S)$.

Assume we change the choice of undotted cup foams instead. Let β and $\overline{\beta}$ be two choice of undotted cup foams, identical everywhere except at some vertex W. Denote by τ the invertible scalar such that $\overline{\beta}^W = \tau\beta^W$.[1] Then:

$$\psi_{\overline{\beta}}(\to W) = \psi_\beta(\to W)/\tau \quad \text{and} \quad \psi_{\overline{\beta}}(W \to) = \psi_\beta(W \to)\cdot\tau,$$

where $\to W$ (resp $W \to$) denotes an incoming (resp. outgoing) edge in $H_{\mathfrak{gl}_2}(S)$. In particular, $\partial\psi_\beta(S) = \partial\psi_{\overline{\beta}}(S)$. The values of $\psi_{\mathfrak{gl}_2}(S)$ and $\psi_{\mathfrak{sl}_2}(c(S))$ do not change. Hence, changing the choice of undotted cup foams does not change the value of $\partial(\text{⧉}_S)$. □

Next we check Proposition 4.18 for planar isotopies. Actually, we show a bit more, namely the the value of $\partial(\text{⧉}_S)$ is independent on the choice of spherical representative for S. A *spherical sliced representative* of a link diagram S is a sliced diagram that is a representative of S up to spherical isotopies (see Definition 2.9).

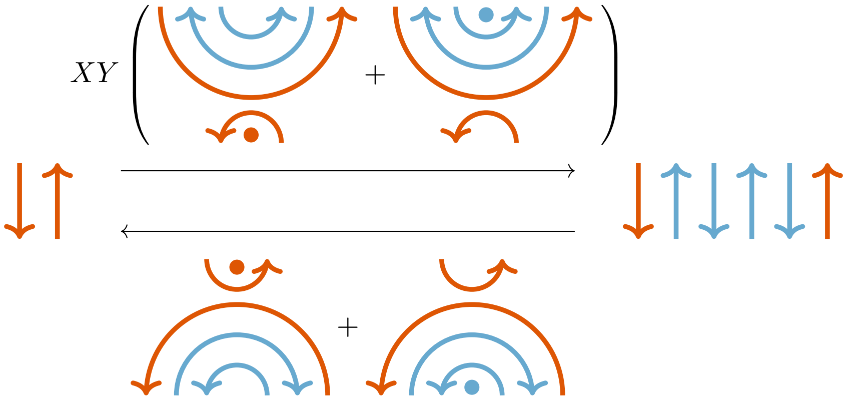
Lemma 4.21. *Let S be a sliced link diagram with exactly two crossings, together with choices of arc orientations and undotted cup foams. Then the value of $\partial(\text{⧉}_S)$ does not depend on the spherical sliced representative.*

Proof. Let S and $\overline{S}$ be two sliced link diagrams in the same spherical isotopy class. Throughout the proof we use the notation $\overline{(-)}$ to distinguish features related to $\overline{S}$. By Lemma 4.20, we may freely choose arc orientations and undotted cup foams.

Pick a choice of arc orientations on S. Then there are arc orientations on $\overline{S}$ naturally associated to the one on S: for instance, one can use an orientation on S to record the arc orientations, and orientation of link diagrams is preserved by spherical isotopies. With this choice, we have $\overline{\psi}_{\mathfrak{sl}_2} = \psi_{\mathfrak{sl}_2}$.

[1]The existence of such a scalar can be deduced from Theorem 2.23

Recall that if S and $\overline{S}$ are planar isotopic, then the proof of Theorem 4.2 showed that there exist sign assignments ϵ and $\overline{\epsilon}$ such that $H_{\mathfrak{gl}_2}(S;\epsilon)$ and $H_{\mathfrak{gl}_2}(\overline{S};\overline{\epsilon})$ are isomorphic. This extends to the spherical case, thanks to the following pair of isomorphisms in **GFoam**:



Denote $\varphi\colon H_{\mathfrak{gl}_2}(S;\epsilon)\to H_{\mathfrak{gl}_2}(\overline{S};\overline{\epsilon})$ such an isomorphism. If β is a choice of undotted cup foams for S, then $\varphi\circ\beta$ is a choice of undotted cup foams for $\overline{S}$. Then, for each edge $e\colon \boldsymbol{r}\to\boldsymbol{s}$ in $\{0,1\}^2$, denoting F_e and $\overline{F}_e$ the corresponding foams in $H_{\mathfrak{gl}_2}(S)$ and $H_{\mathfrak{gl}_2}(\overline{S})$ respectively, we have that:

$$\overline{\epsilon}(e)\left(\overline{F}_e\circ\overline{\beta}^{\boldsymbol{r}}\right)=\epsilon(e)\left(\varphi_{\boldsymbol{s}}\circ F_e\circ\beta^{\boldsymbol{r}}\right)=\epsilon(e)\psi_\beta(e)\begin{cases}\beta^{\boldsymbol{s}} & \text{if } F_e, \overline{F}_e \text{ are merges,}\\ (\beta^{\boldsymbol{s}}_{i_1}+XY\beta^{\boldsymbol{s}}_{i_2}) & \text{if } F_e, \overline{F}_e \text{ are splits.}\end{cases}$$

Hence we have $\psi_{\overline{\beta}}\overline{\epsilon}=\psi_\beta\epsilon$. That implies $\partial(\psi_{\overline{\beta}})\overline{\psi}_{\mathfrak{gl}_2}=\partial(\psi_\beta)\psi_{\mathfrak{gl}_2}$. This concludes the proof. □

Finally, we need to check that $\partial(\text{⧉}_S)$ only depends on the local arc presentation. Up to spherical isotopies, we can slide away all closed simple loops. The next lemma concludes the proof of Proposition 4.18.

Lemma 4.22. *Let D_0 and D_1 be two sliced link diagrams, such that D_1 has two crossings and D_0 none. Then $\partial\psi(\text{⧉}_{D_1})=\partial\psi(\text{⧉}_{D_0\star_0 D_1})$.*

Proof. If W_0 is the web corresponding to D_0 and β^{W_0} is a choice of undotted cup foam for W_0, then $(\mathrm{id}_{W_0}\star_0\beta^W)\star_1\beta^{W_0}$ is an undotted cup foam for every undotted cup foam β^W. This defines a choice of undotted cup foams on $D_0\star_0 D_1$, given one on D_1. It is also clear how to define arc orientations on $D_0\star_0 D_1$ given such a choice on D_1. With this choices, $\psi_{\mathfrak{gl}_2}$, $\psi_{\mathfrak{sl}_2}$ and ψ_β remain identical. This concludes. □

5 Chain complexes in graded-monoidal categories

In this section, we define the tensor product of two chain complexes in a given graded-monoidal category; more precisely, we restrict our study to *homogeneous polycomplexes.* Crucially, we then show that this tensor product leaves homotopy classes invariant. Although this section is set in the context of graded-monoidal categories for simplify, all definitions and results extend in a straightforward way to graded-2-categories.

Although not difficult, this section is more technical than the others. We refer the reader to the beginning of Subsection 4.1 for a minimal version of this section, sufficient for the purpose of Section 4.

Notation 5.1. Fix $n \in \mathbb{N}$. Recall Notation 4.4. For $\boldsymbol{r} \in \mathbb{Z}^n$, we write $|\boldsymbol{r}| := r_1 + \ldots + r_n$. Fix R, G and μ as in Subsection 2.1, and fix a (G, μ)-graded-monoidal category $\mathcal{C}$. In order to reduce clutter, we often abuse notation and write f instead of $\deg f$, where f is a morphism of $\mathcal{C}$. The distinction should be clear by the context. We also write $*$ instead of μ. For instance, for f, g and h morphisms, we may write $(f+g)*h$ for $\mu(\deg f + \deg g, \deg h)$. We sometimes use the subscripted equal signs $=_R$ and $=_G$ to emphasize where an equality holds.

5.1 A graded Koszul rule

Definition 5.2. *A* homogeneous n-polycomplex $\mathbb{A} = (A, \alpha, \psi_{\mathbb{A}})$ *is the data of:*

(i) a family $A := (A^{\boldsymbol{r}})_{\boldsymbol{r}\in\mathbb{Z}^n}$ of objects $A^{\boldsymbol{r}} \in \mathcal{C}$;

(ii) a family $\alpha := (\alpha_i^{\boldsymbol{r}})_{\boldsymbol{r}\in\mathbb{Z}^n, i\in\mathcal{I}}$ of homogeneous *morphisms $\alpha_i^{\boldsymbol{r}} \colon A^{\boldsymbol{r}} \to A^{\boldsymbol{r}+\boldsymbol{e}_i}$, such that each square anti-commutes:*

$$\alpha_j^{\boldsymbol{r}+\boldsymbol{e}_i} \circ \alpha_i^{\boldsymbol{r}} = -\alpha_i^{\boldsymbol{r}+\boldsymbol{e}_j} \circ \alpha_j^{\boldsymbol{r}}$$

(in particular, $\alpha_i^{\boldsymbol{r}+\boldsymbol{e}_i} \circ \alpha_i^{\boldsymbol{r}} = 0$ for all i);

(iii) a G-valued 1-cocycle $\psi_{\mathbb{A}}$ on $\mathbb{Z}^n$ such that $\psi_{\mathbb{A}}(\boldsymbol{r} \to \boldsymbol{r}+\boldsymbol{e}_i) = \deg \alpha_i^{\boldsymbol{r}}$ whenever $\alpha_i^{\boldsymbol{r}} \neq 0$.

Thanks to condition (ii), we associate to $\mathbb{A}$ a chain complex $(\mathrm{Tot}(\mathbb{A}), \mathrm{Tot}(\alpha))$, its *total complex*, given by

$$\mathrm{Tot}(\mathbb{A})_t := \bigoplus_{\boldsymbol{r}\in\mathbb{Z}^n, |\boldsymbol{r}|=t} A^{\boldsymbol{r}} \quad \text{and} \quad \mathrm{Tot}(\alpha)|_{A^{\boldsymbol{r}}} := \sum_{1\leq i\leq n} \alpha_i^{\boldsymbol{r}}.$$

In what follows, we will not distinguish between a homogeneous polycomplex and its associated total complex. In particular, chain maps and chain homotopies are defined in the usual sense.

Condition (iii) is a technical condition that is not restrictive for our purpose, but allows us to mainly use combinatorial arguments throughout the section. Note that if a square is non-zero, that is if

$$\alpha_j^{\boldsymbol{r}+\boldsymbol{e}_i} \circ \alpha_i^{\boldsymbol{r}} = -\alpha_i^{\boldsymbol{r}+\boldsymbol{e}_j} \circ \alpha_j^{\boldsymbol{r}} \neq 0,$$

then on the edges of this square, the definition of $\psi_{\mathbb{A}}$ is both forced and automatically compatible with the cocycle condition. On the other hand, if the square is zero, then there is a priori no condition imposed on the degrees of the edges. Condition (iii) is a way of forcing the cocycle condition on zero squares.

For each $\boldsymbol{r} \in \mathbb{Z}^n$, let p be a path in $\mathbb{Z}^n$ from $\boldsymbol{0}$ to $\boldsymbol{r}$. Since $\psi_{\mathbb{A}}$ is a cocycle, the value $|\alpha|(\boldsymbol{r}) := \psi_{\mathbb{A}}(p)$ does not depend on the choice of path. Most importantly, it verifies the following:

$$|\alpha|(\boldsymbol{r}+e_i) - |\alpha|(\boldsymbol{r}) = \psi_{\mathbb{A}}(\boldsymbol{r} \to \boldsymbol{r}+\boldsymbol{e}_i) = \deg \alpha_i^{\boldsymbol{r}} \qquad \text{if } \alpha_i^{\boldsymbol{r}} \neq 0. \tag{24}$$

We now introduce the notion of tensor product of homogeneous polycomplexes, after setting some further notations.

Notation 5.3. Recall our notations from Notation 5.1 and Definition 5.2. We introduce analogous notations for a homogeneous m-polycomplex $\mathbb{B} = (B, \beta, \psi_{\mathbb{B}})$: we let $\mathcal{J} = \{1, \ldots, m\}$ denotes the set of indices and use generically the letter j for an index $j \in \mathcal{J}$ and $\boldsymbol{s} = (s_1, \ldots, s_m)$ for a vertex in $\mathbb{Z}^m$. Finally, we generically use the letter k for an index $k \in \mathcal{I} \sqcup \mathcal{J}$ and the notation $(\boldsymbol{r}, \boldsymbol{s})$ for a vertex in $\mathbb{Z}^{n+m}$.

Definition 5.4. *Let ϵ be a $R^\times$-valued 1-cochain on $\mathbb{Z}^{n+m}$. The* ϵ-tensor product $(\mathbb{A} \otimes \mathbb{B})(\epsilon)$ *of $\mathbb{A}$ and $\mathbb{B}$ is the following data:*

(i) the family $A \otimes B = ((A \otimes B)^{(\boldsymbol{r},\boldsymbol{s})})$ *of objects* $(A \otimes B)^{(\boldsymbol{r},\boldsymbol{s})} := A^{\boldsymbol{r}} \otimes B^{\boldsymbol{s}}$*;*

(ii) the family $\alpha \otimes \beta = ((\alpha \otimes \beta)_k^{(\boldsymbol{r},\boldsymbol{s})})$ *of homogeneous morphisms*

$$(\alpha \otimes \beta)_k^{(\boldsymbol{r},\boldsymbol{s})} := \epsilon(e) \begin{cases} \alpha_i^{\boldsymbol{r}} \otimes \mathrm{id}_{B^{\boldsymbol{s}}} & k = i \in \mathcal{I}, \\ \mathrm{id}_{A^{\boldsymbol{r}}} \otimes \beta_j^{\boldsymbol{s}} & k = j \in \mathcal{J}, \end{cases}$$

where e *denotes the edge* $(\boldsymbol{r}, \boldsymbol{s}) \to (\boldsymbol{r}, \boldsymbol{s}) + \boldsymbol{e}_k$*;*

(iii) the 1-cocycle $\psi_{\mathbb{A}} \otimes \psi_{\mathbb{B}}$ *on* $\mathbb{Z}^{n+m}$ *given on* $e = (\boldsymbol{r}, \boldsymbol{s}) \to (\boldsymbol{r}, \boldsymbol{s}) + \boldsymbol{e}_k$ *by*

$$(\psi_{\mathbb{A}} \otimes \psi_{\mathbb{B}})(e) := \begin{cases} \psi_{\mathbb{A}}(\boldsymbol{r} \to \boldsymbol{r} + \boldsymbol{e}_i) & k = i \in \mathcal{I}; \\ \psi_{\mathbb{B}}(\boldsymbol{s} \to \boldsymbol{s} + \boldsymbol{e}_j) & k = j \in \mathcal{J}. \end{cases}$$

For $(\mathbb{A} \otimes \mathbb{B})(\epsilon)$ to define a homogeneous $(n+m)$-polycomplex, ϵ needs to be such that squares anti-commute. This is encapsulated in the following lemma, where $\square_{k,l}^{(\boldsymbol{r},\boldsymbol{s})}$ with $k < l$ denotes the following oriented square:

$$\begin{tikzcd}
(\boldsymbol{r}, \boldsymbol{s}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & (\boldsymbol{r}, \boldsymbol{s}) + \boldsymbol{e}_k \arrow[d] \\
(\boldsymbol{r}, \boldsymbol{s}) + \boldsymbol{e}_l \arrow[r] & (\boldsymbol{r}, \boldsymbol{s}) + \boldsymbol{e}_k + \boldsymbol{e}_l
\end{tikzcd}$$

Lemma 5.5. *Say that a 1-cochain* ϵ *on* $\mathbb{Z}^{n+m}$ *is* compatible *if*

$$\partial\epsilon(\square_{k,l}^{(\boldsymbol{r},\boldsymbol{s})}) = \begin{cases} 0 & \text{if } k, l \in \mathcal{I} \text{ or } k, l \in \mathcal{J}, \\ -\psi_{\mathbb{A}}(\boldsymbol{r} \to \boldsymbol{r} + \boldsymbol{e}_i) * \psi_{\mathbb{B}}(\boldsymbol{s} \to \boldsymbol{s} + \boldsymbol{e}_j) & \text{if } k = i \in \mathcal{I} \text{ and } l = j \in \mathcal{J}. \end{cases}$$

If ϵ *is compatible, then* $(\mathbb{A} \otimes \mathbb{B})(\epsilon)$ *defines a homogeneous* $(n+m)$*-polycomplex.*

Proof. The case $k, l \in \mathcal{I}$ or $k, l \in \mathcal{J}$ is clear. In the case $k = i \in \mathcal{I}$ and $l = j \in \mathcal{J}$, the square $\square_{k,l}^{(\boldsymbol{r},\boldsymbol{s})}$ has the following form:

$$\begin{tikzcd}
(\boldsymbol{r}, \boldsymbol{s}) \arrow[r, "\alpha_i^{\boldsymbol{r}} \otimes \mathrm{id}_{B^{\boldsymbol{s}}}"] \arrow[d, "\mathrm{id}_{A^{\boldsymbol{r}}} \otimes \beta_j^{\boldsymbol{s}}"'] \arrow[dr, phantom, "\circlearrowright"] & (\boldsymbol{r}, \boldsymbol{s}) + \boldsymbol{e}_k \arrow[d, "\mathrm{id}_{A^{\boldsymbol{r}+\boldsymbol{e}_k}} \otimes \beta_j^{\boldsymbol{s}}"] \\
(\boldsymbol{r}, \boldsymbol{s}) + \boldsymbol{e}_l \arrow[r, "\alpha_i^{\boldsymbol{r}} \otimes \mathrm{id}_{B^{\boldsymbol{s}+\boldsymbol{e}_l}}"'] & (\boldsymbol{r}, \boldsymbol{s}) + \boldsymbol{e}_k + \boldsymbol{e}_l
\end{tikzcd}$$

The morphism corresponding to the path $(\boldsymbol{r}, \boldsymbol{s}) \to (\boldsymbol{r}, \boldsymbol{s}) + \boldsymbol{e}_k \to (\boldsymbol{r}, \boldsymbol{s}) + \boldsymbol{e}_k + \boldsymbol{e}_l$ is $(\beta_j^{\boldsymbol{s}} * \alpha_i^{\boldsymbol{r}}) \alpha_i^{\boldsymbol{r}} \otimes \beta_j^{\boldsymbol{s}}$, while the morphism corresponding to the other path is $\alpha_i^{\boldsymbol{r}} \otimes \beta_j^{\boldsymbol{s}}$. If the square is zero, it automatically anti-commutes. Otherwise, it is sufficient to have

$$\partial\epsilon(\square_{k,l}^{(\boldsymbol{r},\boldsymbol{s})}) =_R -(\beta_j^{\boldsymbol{s}} * \alpha_i^{\boldsymbol{r}})^{-1} =_R -\psi_{\mathbb{A}}(\boldsymbol{r} \to \boldsymbol{r} + \boldsymbol{e}_i) * \psi_{\mathbb{B}}(\boldsymbol{s} \to \boldsymbol{s} + \boldsymbol{e}_j). \qquad \square$$

Remark 5.6. Note that the compatibility condition is a sufficient condition to ensure that all squares anti-commute, but a priori not a necessary one. Indeed, there might be some liberty on squares that are zero. In particular, we could have $\alpha_i^{\boldsymbol{r}} \otimes \beta_j^{\boldsymbol{s}} = 0$ even if $\alpha_i^{\boldsymbol{r}} \neq 0$ and $\beta_j^{\boldsymbol{s}} \neq 0$.

Definition 5.4 defines a tensor product of homogeneous polycomplexes, *provided* that a compatible 1-cochain exists. Definition 5.7 below gives the existence.

Definition 5.7 (graded Koszul rule)**.** *The* standard 1-cochain $\epsilon_{\mathbb{A}\otimes\mathbb{B}}$ *is the compatible 1-cochain on* H^{n+m} *given on* $e = (\boldsymbol{r}, \boldsymbol{s}) \to (\boldsymbol{r}, \boldsymbol{s}) + \boldsymbol{e}_k$ *by*

$$\epsilon_{\mathbb{A}\otimes\mathbb{B}}(e) = \begin{cases} 0 & k = i \in \mathcal{I}; \\ (-1)^{|\boldsymbol{r}|} \, |\alpha| \, (\boldsymbol{r}) * \psi_{\mathbb{B}}(\boldsymbol{s} \to \boldsymbol{s} + \boldsymbol{e}_j) & k = j \in \mathcal{J}. \end{cases}$$

Note that if $\psi_{\mathbb{A}} = \psi_{\mathbb{B}} = 1$, then $|\alpha| \, (\boldsymbol{r}) * \psi_{\mathbb{B}}(\boldsymbol{s} \to \boldsymbol{s} + \boldsymbol{e}_j) = 1$ and we recover the usual Koszul rule. Checking that $\epsilon_{\mathbb{A}\otimes\mathbb{B}}$ is compatible is straightforward:

Proof. Consider the square $\square_{k,l}^{(\boldsymbol{r},\boldsymbol{s})}$ as above. If $k, l \in \mathcal{I}$ or $k, l \in \mathcal{J}$, $\epsilon_{\mathbb{A}\otimes\mathbb{B}}$ is clearly compatible. Otherwise if $k \in \mathcal{I}$ and $l \in \mathcal{J}$, we get (using property (24)):

$$\begin{aligned} \partial\epsilon(\square_{k,l}^{(\boldsymbol{r},\boldsymbol{s})}) &= \Big((-1)^{|\boldsymbol{r}|+1} \, |\alpha| \, (\boldsymbol{r} + \boldsymbol{e}_k) * \psi_{\mathbb{B}}(\boldsymbol{s} \to \boldsymbol{s} + \boldsymbol{e}_j)\Big)(-1)^{|\boldsymbol{r}|} \, |\alpha| \, (\boldsymbol{r}) * \psi_{\mathbb{B}}(\boldsymbol{s} \to \boldsymbol{s} + \boldsymbol{e}_j)^{-1} \\ &= -\big(|\alpha| \, (\boldsymbol{r} + \boldsymbol{e}_k) - |\alpha| \, (\boldsymbol{r})\big) * \psi_{\mathbb{B}}(\boldsymbol{s} \to \boldsymbol{s} + \boldsymbol{e}_j) \\ &= -\psi_{\mathbb{A}}(\boldsymbol{r} \to \boldsymbol{r} + \boldsymbol{e}_i) * \psi_{\mathbb{B}}(\boldsymbol{s} \to \boldsymbol{s} + \boldsymbol{e}_j). \end{aligned}$$

□

We next check that this choice of compatible 1-cochain is essentially unique, *among compatible cochains* (see Remark 5.6).[1]

Lemma 5.8. *Let* ϵ *and* ϵ' *be two compatible 1-cochains. Then* $(\mathbb{A} \otimes \mathbb{B})(\epsilon)$ *and* $(\mathbb{A} \otimes \mathbb{B})(\epsilon')$ *are isomorphic as chain complexes.*

Proof. If ϵ and ϵ' are both compatible, then $\partial(\epsilon(\epsilon')^{-1})$ is zero on all squares. As squares generate all cycles in $\mathbb{Z}^n$, $\epsilon(\epsilon')^{-1}$ is a 1-cocycle. In particular, $\epsilon(\epsilon')^{-1}$ is a 1-coboundary and there exists a 0-cochain φ such that $\partial\varphi = \epsilon(\epsilon')^{-1}$. Consider the map $\Phi\colon (\mathbb{A} \otimes \mathbb{B})(\epsilon) \to (\mathbb{A} \otimes \mathbb{B})(\epsilon')$ that corresponds to multiplication by $\varphi(\boldsymbol{r}, \boldsymbol{s})$ when restricted to $(\boldsymbol{r}, \boldsymbol{s})$. It is straightforward to check that this map defines an isomorphism for the associated total complexes. □

From now on, we simply call $(\mathbb{A} \otimes \mathbb{B})(\epsilon_{\mathbb{A}\otimes\mathbb{B}})$ the *tensor product* of $\mathbb{A}$ and $\mathbb{B}$ and denote it $\mathbb{A} \otimes \mathbb{B} = (A \otimes B, \alpha \otimes \beta, \psi_{\mathbb{A}} \otimes \psi_{\mathbb{B}})$.

Example 5.9. A chain complex $\mathbb{C} = (C_\bullet, \partial_\bullet)$ whose differentials ∂_t are all homogeneous is a homogeneous 1-polycomplex. An n-fold tensor product of homogeneous 1-polycomplexes is a homogeneous n-polycomplex. This is the specific construction used in Subsection 4.1 to define covering $\mathfrak{gl}_2$-Khovanov homology.

We now come to the main result of this section:

Theorem 5.10. *Let* $\mathbb{A}_1$, $\mathbb{A}_2$, $\mathbb{B}_1$ *and* $\mathbb{B}_2$ *be homogeneous polycomplexes. Then:*

$$\mathbb{A}_1 \simeq \mathbb{B}_1 \quad \textit{and} \quad \mathbb{A}_2 \simeq \mathbb{B}_2 \qquad \textit{implies} \qquad \mathbb{A}_1 \otimes \mathbb{A}_2 \simeq \mathbb{B}_1 \otimes \mathbb{B}_2,$$

where $\simeq$ *denotes homotopy equivalence (in the usual sense).*

The idea of the proof of Theorem 5.10 is straightforward: define induced morphisms and induced homotopies on tensor product. Precisely, Theorem 5.10 holds if the following holds:

[1] The proof given below is nearly identical to the proof of Lemma 2.2 in [47].

(1) Given homogeneous polycomplexes $\mathbb{A}_k, \mathbb{B}_k$ and chain maps $F_k \colon \mathbb{A}_k \to \mathbb{B}_k$ ($k = 1, 2$), there exists a chain complex $F_1 \otimes F_2 \colon \mathbb{A}_1 \otimes \mathbb{A}_2 \to \mathbb{B}_1 \otimes \mathbb{B}_2$. This definition is such that $F_1 \otimes F_2 = \mathrm{Id}_{\mathbb{A}_1 \otimes \mathbb{A}_2}$ if $F_1 = \mathrm{Id}_{\mathbb{A}_1}$ and $F_2 = \mathrm{Id}_{\mathbb{A}_2}$.

(2) Given homogeneous polycomplexes $\mathbb{A}_k, \mathbb{B}_k$, chain maps $F_k, G_k \colon \mathbb{A}_k \to \mathbb{B}_k$ and homotopies $H_k \colon F_k \to G_k$ ($k = 1, 2$), there exists a homotopy $H_1 \otimes H_2 \colon F_1 \otimes F_2 \to G_1 \otimes G_2$.

This is the content of Subsection 5.2: part (1) is shown by Proposition 5.14, while part (2) is shown by Proposition 5.16.

5.1.1 Further directions

As it is unnecessary for our purpose, we did not investigate the categorical properties of the tensor product. For example, it is not clear whether it is functorial, or functorial up to homotopy. Other possible questions include:

- Can we extend the definition of the tensor product to higher structures? That is, can we define induced n-fold homotopies on the tensor product?
- Are the definitions of induced morphisms and induced homotopies unique up to higher structures, similarly to Lemma 5.8?
- What is the most general case where on can define a (sensible) tensor product on chain complexes in a monoidal category? In particular, how far can we weaken condition (iii) in Definition 5.2?

5.2 Induced morphisms and homotopies on the tensor product

5.2.1 Preliminary remarks

Recall the conventions of Notation 5.1 and Notation 5.3. We extend them using subscripts to $\mathbb{A} = (A_k, \alpha_k, \psi_{\mathbb{A}_k})$ and $\mathbb{B} = (B_k, \beta_k, \psi_{\mathbb{B}_k})$ for $k = 1, 2$. From now on, we denote $\mathbb{A} = \mathbb{A}_1 \otimes \mathbb{A}_2$ and $\mathbb{B} = \mathbb{B}_1 \otimes \mathbb{B}_2$. We also use different shortcuts, such as $\boldsymbol{r} = (\boldsymbol{r}_1, \boldsymbol{r}_2)$ and $\boldsymbol{s} = (\boldsymbol{s}_1, \boldsymbol{s}_2)$, that should be clear from the context.

Note that in Definition 5.4, the scalar $\epsilon^{\boldsymbol{r},\boldsymbol{s},j}_{\mathbb{A}\otimes\mathbb{B}}$ always appears in front of an expression involving $\beta^{\boldsymbol{s}}_j$. Thus, when doing computation with $\epsilon^{\boldsymbol{r},\boldsymbol{s},j}_{\mathbb{A}\otimes\mathbb{B}}$ in this context, we can assume $\beta^{\boldsymbol{s}}_j \neq 0$ and in particular write $\deg \beta^{\boldsymbol{s}}_j$ instead of $\psi_{\mathbb{B}}(\boldsymbol{s} \to \boldsymbol{s} + \boldsymbol{e}_j)$. In the proofs below, we will always encounter similar situations. In order to simplify the exposition, we shall act as if $\deg f$ is always well-defined, and avoid the use of ψ altogether.

Before entering the main proofs, we define a generic $R^\times$-valued 0-cochain, satisfying generic properties. In fact, Definition 5.13 and Definition 5.15 solely depend on this generic definition, and all "degree-wise" computations follow from those generic properties.

Definition 5.11. *For $\lambda_k = \left(\lambda_k^{\boldsymbol{r}_k,\boldsymbol{s}_k}\right)_{(\boldsymbol{r}_k,\boldsymbol{s}_k)\in A_k\otimes B_k}$ a pair of G-valued 0-cochains ($k = 1, 2$), let $\epsilon^{\boldsymbol{r},\boldsymbol{s}}_{\lambda_1,\lambda_2}$ be the following $R^\times$-valued 0-cochain:*

$$\epsilon^{\boldsymbol{r},\boldsymbol{s}}_{\lambda_1,\lambda_2} := \Big(\left[\lambda_1^{\boldsymbol{r}_1,\boldsymbol{s}_1} + |\alpha_1|(\boldsymbol{r}_1) - |\beta_1|(\boldsymbol{s}_1)\right] * |\beta_2|(\boldsymbol{s}_2)\Big)^{-1} \Big(|\alpha_1|(\boldsymbol{r}_1) * \lambda_2^{\boldsymbol{r}_2,\boldsymbol{s}_2}\Big).$$

Lemma 5.12. *The generic $R^\times$-valued 0-cochain defined above satisfies the following:*

(i) Assume $\mu_1 \coloneqq_G \lambda_1^{\boldsymbol{r}_1+\boldsymbol{e}_{i_1},\boldsymbol{s}_1} + \alpha_{i_1}^{\boldsymbol{r}_1} =_G \lambda_1^{\boldsymbol{r}_1,\boldsymbol{s}_1-\boldsymbol{e}_{j_1}} + \beta_{j_1}^{\boldsymbol{s}_1-\boldsymbol{e}_{j_1}}$ *holds for all* $i_1 \in \mathcal{I}_1$ *and* $j_1 \in \mathcal{J}_1$*; in particular, the* G*-valued 0-cochain* μ_1 *is defined independently of the choice of* i_1 *and* j_1*. Then the following identities hold, for all* $i_1 \in \mathcal{I}_1$ *and* $j_1 \in \mathcal{J}_1$*:*

$$\epsilon_{\mu_1,\lambda_2}^{\boldsymbol{r},\boldsymbol{s}} =_R \epsilon_{\lambda_1,\lambda_2}^{\boldsymbol{r}+e_{i_1},\boldsymbol{s}} \left(\lambda_2^{\boldsymbol{r}_2,\boldsymbol{s}_2} * \alpha_{i_1}^{\boldsymbol{r}_1}\right) =_R \epsilon_{\lambda_1,\lambda_2}^{\boldsymbol{r},\boldsymbol{s}-\boldsymbol{e}_{j_1}}.$$

(ii) Assume $\mu_2 \coloneqq_G \lambda_2^{\boldsymbol{r}_2+\boldsymbol{e}_{i_2},\boldsymbol{s}_2} + \alpha_{i_2}^{\boldsymbol{r}_2} =_G \lambda_2^{\boldsymbol{r}_2,\boldsymbol{s}_2-\boldsymbol{e}_{j_2}} + \beta_{j_2}^{\boldsymbol{s}_2-\boldsymbol{e}_{j_2}}$ *holds for all* $i_2 \in \mathcal{I}_2$ *and* $j_2 \in \mathcal{J}_2$*; in particular, the* G*-valued 0-cochain* μ_2 *is defined independently of the choice of* i_2 *and* j_2*. Then the following identities hold, for all* $i_2 \in \mathcal{I}_2$ *and* $j_2 \in \mathcal{J}_2$*:*

$$\epsilon_{\lambda_1,\mu_2}^{\boldsymbol{r},\boldsymbol{s}} =_R (-1)^{|\boldsymbol{r}_1|}\epsilon_{A_1\otimes A_2}^{\boldsymbol{r}_1,\boldsymbol{r}_2,j_2}\epsilon_{\lambda_1,\lambda_2}^{\boldsymbol{r}+\boldsymbol{e}_{i_2},\boldsymbol{s}} =_R (-1)^{|\boldsymbol{s}_1|}\epsilon_{B_1\otimes B_2}^{\boldsymbol{s}_1,\boldsymbol{s}_2-\boldsymbol{e}_{j_2},j_2}\epsilon_{\lambda_1,\lambda_2}^{\boldsymbol{r},\boldsymbol{s}-\boldsymbol{e}_{j_2}} \left(\beta_{j_2}^{\boldsymbol{s}_2-\boldsymbol{e}_{j_2}} * \lambda_1^{\boldsymbol{r}_1,\boldsymbol{s}_1}\right).$$

Proof. It follows from direct computation (recall relation (24)):

$$\begin{aligned}
\epsilon_{\lambda_1,\lambda_2}^{\boldsymbol{r}+e_{i_1},\boldsymbol{s}} &=_R \left(\left[\lambda_1^{\boldsymbol{r}_1+\boldsymbol{e}_{i_1},\boldsymbol{s}_1} + \alpha_{i_1}^{\boldsymbol{r}_1} + |\alpha_1|(\boldsymbol{r}_1) - |\beta_1|(\boldsymbol{s}_1)\right] * |\beta_2|(\boldsymbol{s}_2)\right)^{-1}\left(\left[|\alpha_1|(\boldsymbol{r}_1) + \alpha_{i_1}^{\boldsymbol{r}_1}\right] * \lambda_2^{\boldsymbol{r}_2,\boldsymbol{s}_2}\right)\\
&=_R \epsilon_{\mu_1,\lambda_2}^{\boldsymbol{r},\boldsymbol{s}}\left(\alpha_{i_1}^{\boldsymbol{r}_1} * \lambda_2^{\boldsymbol{r}_2,\boldsymbol{s}_2}\right)
\end{aligned}$$

$$\begin{aligned}
\epsilon_{\lambda_1,\lambda_2}^{\boldsymbol{r},\boldsymbol{s}-e_{j_1}} &=_R \left(\left[\lambda_1^{\boldsymbol{r}_1,\boldsymbol{s}_1-\boldsymbol{e}_{j_1}} + \beta_{j_1}^{\boldsymbol{s}_1-\boldsymbol{e}_{j_1}} + |\alpha_1|(\boldsymbol{r}_1) - |\beta_1|(\boldsymbol{s}_1)\right] * |\beta_2|(\boldsymbol{s}_2)\right)^{-1}\left(|\alpha_1|(\boldsymbol{r}_1) * \lambda_2^{\boldsymbol{r}_2,\boldsymbol{s}_2}\right)\\
&=_R \epsilon_{\mu_1,\lambda_2}^{\boldsymbol{r},\boldsymbol{s}}
\end{aligned}$$

$$\begin{aligned}
&(-1)^{|\boldsymbol{r}_1|}\epsilon_{A_1\otimes A_2}^{\boldsymbol{r}_1,\boldsymbol{r}_2,j_2}\epsilon_{\lambda_1,\lambda_2}^{\boldsymbol{r}+\boldsymbol{e}_{i_2},\boldsymbol{s}}\\
&\quad=_R \left(|\alpha_1|(\boldsymbol{r}_1) * \alpha_{i_2}^{\boldsymbol{r}_2,\boldsymbol{s}_2}\right)\left(\left[\lambda_1^{\boldsymbol{r}_1,\boldsymbol{s}_1} + |\alpha_1|(\boldsymbol{r}_1) - |\beta_1|(\boldsymbol{s}_1)\right] * |\beta_2|(\boldsymbol{s}_2)\right)^{-1}\left(|\alpha_1|(\boldsymbol{r}_1) * \lambda_2^{\boldsymbol{r}_2+\boldsymbol{e}_{i_2},\boldsymbol{s}_2}\right)\\
&\quad=_R \left(\left[\lambda_1^{\boldsymbol{r}_1,\boldsymbol{s}_1} + |\alpha_1|(\boldsymbol{r}_1) - |\beta_1|(\boldsymbol{s}_1)\right] * |\beta_2|(\boldsymbol{s}_2)\right)^{-1}\left(|\alpha_1|(\boldsymbol{r}_1) * \left[\lambda_2^{\boldsymbol{r}_2+\boldsymbol{e}_{i_2},\boldsymbol{s}_2} + \alpha_{i_2}^{\boldsymbol{r}_2,\boldsymbol{s}_2}\right]\right)\\
&\quad=_R \epsilon_{\lambda_1,\mu_2}^{\boldsymbol{r},\boldsymbol{s}}
\end{aligned}$$

$$\begin{aligned}
&(-1)^{|\boldsymbol{s}_1|}\epsilon_{B_1\otimes B_2}^{\boldsymbol{s}_1,\boldsymbol{s}_2-\boldsymbol{e}_{j_2},j_2}\epsilon_{\lambda_1,\lambda_2}^{\boldsymbol{r},\boldsymbol{s}-\boldsymbol{e}_{j_2}}\left(\lambda_1^{\boldsymbol{r}_1,\boldsymbol{s}_1} * \beta_{j_2}^{\boldsymbol{s}_2-\boldsymbol{e}_{j_2}}\right)^{-1}\\
&\quad=_R \left(|\beta_1|(\boldsymbol{s}_1) * \beta_{j_2}^{\boldsymbol{r}_2,\boldsymbol{s}_2-\boldsymbol{e}_{j_2}}\right)\left(\left[\lambda_1^{\boldsymbol{r}_1,\boldsymbol{s}_1} + |\alpha_1|(\boldsymbol{r}_1) - |\beta_1|(\boldsymbol{s}_1)\right] * |\beta_2|(\boldsymbol{s}_2 - \boldsymbol{e}_{j_2})\right)^{-1}\\
&\qquad\qquad\cdot\left(|\alpha_1|(\boldsymbol{r}_1) * \lambda_2^{\boldsymbol{r}_2,\boldsymbol{s}_2-\boldsymbol{e}_{j_2}}\right)\left(\lambda_1^{\boldsymbol{r}_1,\boldsymbol{s}_1} * \beta_{j_2}^{\boldsymbol{s}_2-\boldsymbol{e}_{j_2}}\right)^{-1}\\
&\quad=_R \left(\left[\lambda_1^{\boldsymbol{r}_1,\boldsymbol{s}_1} + |\alpha_1|(\boldsymbol{r}_1) - |\beta_1|(\boldsymbol{s}_1)\right] * |\beta_2|(\boldsymbol{s}_2)\right)^{-1}\left(|\alpha_1|(\boldsymbol{r}_1) * \left[\lambda_2^{\boldsymbol{r}_2,\boldsymbol{s}_2-\boldsymbol{e}_{j_2}} + \beta_{j_2}^{\boldsymbol{s}_2-\boldsymbol{e}_{j_2}}\right]\right)\\
&\quad=_R \epsilon_{\lambda_1,\mu_2}^{\boldsymbol{r},\boldsymbol{s}}.
\end{aligned}$$

□

5.2.2 Induced morphism on tensor product

A chain map $F\colon \mathbb{A} \to \mathbb{B}$ is a set of morphisms $F^{\boldsymbol{r},\boldsymbol{s}}\colon A^{\boldsymbol{r}} \to B^{\boldsymbol{s}}$ for all $|\boldsymbol{r}| = |\boldsymbol{s}|$ such that $\beta \circ F = F \circ \alpha$, that is:

$$\sum_{j\in\mathcal{J}} \beta_j^{\boldsymbol{s}-\boldsymbol{e}_j} \circ F^{\boldsymbol{r},\boldsymbol{s}-\boldsymbol{e}_j} = \sum_{i\in\mathcal{I}} F^{\boldsymbol{r}+\boldsymbol{e}_i,\boldsymbol{s}} \circ \alpha_i^{\boldsymbol{r}}.$$

Definition 5.13. *Let* $F_k\colon \mathbb{A}_k \to \mathbb{B}_k$ *be chain maps for each* $k = 1, 2$*. Their* induced morphism $F = F_1 \otimes F_2$ *is the chain map* $F\colon \mathbb{A}_1 \otimes \mathbb{A}_2 \to \mathbb{B}_1 \otimes \mathbb{B}_2$ *given by the data:*

$$F^{\boldsymbol{r},\boldsymbol{s}} = \epsilon_{F_1,F_2}^{\boldsymbol{r},\boldsymbol{s}} F_1^{\boldsymbol{r}_1,\boldsymbol{s}_1} \otimes F_2^{\boldsymbol{r}_2,\boldsymbol{s}_2},$$

where $\epsilon_{F_1,F_2}^{\boldsymbol{r},\boldsymbol{s}}$ *is defined as in Definition 5.11 (here we used the abuse of notation* $F_i = \deg F_i$*).*

Proposition 5.14. *Definition 5.13 gives a well-defined chain map. Moreover, if F_1 and F_2 are identities then F is the identity.*

Proof. Note that if both F_1 and F_2 are identities, then

$$\epsilon^{\boldsymbol{r},\boldsymbol{s}}_{F_1,F_2} = \Big(\big[0 + |\alpha_1|\,(\boldsymbol{r}_1) + |\alpha_1|\,(\boldsymbol{r}_1)\big] * |\alpha_2|\,(\boldsymbol{s}_2)\Big)^{-1}\big(|\alpha_1|\,(\boldsymbol{r}_1) * 0\big) = 1,$$

so that two identities induce the identity on the tensor product. To show the first part of the statement, we must check that $\beta \circ F = F \circ \alpha$. First, we unfold both sides of the equation:

$$\begin{aligned}
&\sum_{j\in\mathcal{J}} \beta_j^{\boldsymbol{s}-\boldsymbol{e}_j} \circ F^{\boldsymbol{r},\boldsymbol{s}-\boldsymbol{e}_j} \\
&\quad= \sum_{j_1\in\mathcal{J}_1} \Big[\beta_{j_1}^{\boldsymbol{s}_1-\boldsymbol{e}_{j_1}} \otimes \mathrm{Id}\Big] \circ \Big[\epsilon^{\boldsymbol{r},\boldsymbol{s}-\boldsymbol{e}_{j_1}}_{F_1,F_2}\ \ F_1^{\boldsymbol{r}_1,\boldsymbol{s}_1-\boldsymbol{e}_{j_1}} \otimes F_2^{\boldsymbol{r}_2,\boldsymbol{s}_2}\Big] \\
&\qquad+ \sum_{j_2\in\mathcal{J}_2} \Big[\epsilon^{\boldsymbol{s}_1,\boldsymbol{s}_2-\boldsymbol{e}_{j_2},j_2}_{B_1\otimes B_2}\ \ \mathrm{Id} \otimes \beta_{j_2}^{\boldsymbol{s}_2-\boldsymbol{e}_{j_2}}\Big] \circ \Big[\epsilon^{\boldsymbol{r},\boldsymbol{s}-\boldsymbol{e}_{j_2}}_{F_1,F_2}\ \ F_1^{\boldsymbol{r}_1,\boldsymbol{s}_1} \otimes F_2^{\boldsymbol{r}_2,\boldsymbol{s}_2-\boldsymbol{e}_{j_2}}\Big] \\
&\quad= \left[\sum_{j_1\in\mathcal{J}_1} \epsilon^{\boldsymbol{r},\boldsymbol{s}-\boldsymbol{e}_{j_1}}_{F_1,F_2}\ \ \beta_{j_1}^{\boldsymbol{s}_1-\boldsymbol{e}_{j_1}} \circ F_1^{\boldsymbol{r}_1,\boldsymbol{s}_1-\boldsymbol{e}_{j_1}}\right] \otimes F_2^{\boldsymbol{r}_2,\boldsymbol{s}_2} && ① \\
&\qquad+ F_1^{\boldsymbol{r}_1,\boldsymbol{s}_1} \otimes \left[\sum_{j_2\in\mathcal{J}_2} \Big(\beta_{j_2}^{\boldsymbol{s}_2-\boldsymbol{e}_{j_2}} * F_1^{\boldsymbol{r}_1,\boldsymbol{s}_1}\Big)\, \epsilon^{\boldsymbol{s}_1,\boldsymbol{s}_2-\boldsymbol{e}_{j_2},j_2}_{B_1\otimes B_2}\, \epsilon^{\boldsymbol{r},\boldsymbol{s}-\boldsymbol{e}_{j_2}}_{F_1,F_2}\ \ \beta_{j_2}^{\boldsymbol{s}_2-\boldsymbol{e}_{j_2}} \circ F_2^{\boldsymbol{r}_2,\boldsymbol{s}_2-\boldsymbol{e}_{j_2}}\right] && ②
\end{aligned}$$

$$\begin{aligned}
&\sum_{i\in\mathcal{I}} F^{\boldsymbol{r}+\boldsymbol{e}_i,\boldsymbol{s}} \circ \alpha_i^{\boldsymbol{r}} \\
&\quad= \sum_{i_1\in\mathcal{I}_1} \Big[\epsilon^{\boldsymbol{r}+\boldsymbol{e}_{i_1},\boldsymbol{s}}_{F_1,F_2}\ \ F_1^{\boldsymbol{r}_1+\boldsymbol{e}_{i_1},\boldsymbol{s}_1} \otimes F_2^{\boldsymbol{r}_2,\boldsymbol{s}_2}\Big] \circ \Big[\alpha_{i_1}^{\boldsymbol{r}_1} \otimes \mathrm{Id}\Big] \\
&\qquad+ \sum_{i_2\in\mathcal{I}_2} \Big[\epsilon^{\boldsymbol{r}+\boldsymbol{e}_{i_2},\boldsymbol{s}}_{F_1,F_2}\ \ F_1^{\boldsymbol{r}_1,\boldsymbol{s}_1} \otimes F_2^{\boldsymbol{r}_2+\boldsymbol{e}_{i_2},\boldsymbol{s}_2}\Big] \circ \Big[\epsilon^{\boldsymbol{r}_1,\boldsymbol{r}_2,i_2}_{A_1\otimes A_2}\ \ \mathrm{Id} \otimes \alpha_{i_2}^{\boldsymbol{r}_2}\Big] \\
&\quad= \left[\sum_{i_1\in\mathcal{I}_1} \big(F_2^{\boldsymbol{r}_2,\boldsymbol{s}_2} * \alpha_{i_1}^{\boldsymbol{r}_1}\big)\, \epsilon^{\boldsymbol{r}+\boldsymbol{e}_{i_1},\boldsymbol{s}}_{F_1,F_2}\ \ F_1^{\boldsymbol{r}_1+\boldsymbol{e}_{i_1},\boldsymbol{s}_1} \circ \alpha_{i_1}^{\boldsymbol{r}_1}\right] \otimes F_2^{\boldsymbol{r}_2,\boldsymbol{s}_2} && ① \\
&\qquad+ F_1^{\boldsymbol{r}_1,\boldsymbol{s}_1} \otimes \left[\sum_{i_2\in\mathcal{I}_2} \epsilon^{\boldsymbol{r}+\boldsymbol{e}_{i_2},\boldsymbol{s}}_{F_1,F_2}\, \epsilon^{\boldsymbol{r}_1,\boldsymbol{r}_2,i_2}_{A_1\otimes A_2}\ \ F_2^{\boldsymbol{r}_2+\boldsymbol{e}_{i_2},\boldsymbol{s}_2} \circ \alpha_{i_2}^{\boldsymbol{r}_2}\right] && ②
\end{aligned}$$

We want to show that the two terms labelled ① (resp. ②) are equal, using the chain map relation of F_1 (resp. F_2). To do so, we only need to check that the scalars are the same. In each case, we can assume that:

① $\beta_{j_1}^{s_{j_1}-1} + F_1^{\boldsymbol{r}_1,\boldsymbol{s}_1-\boldsymbol{e}_{j_1}} =_G F_1^{\boldsymbol{r}_1+\boldsymbol{e}_{i_1},\boldsymbol{s}_1} + \alpha_{i_1}^{r_{i_1}}$ for all $i_1 \in \mathcal{I}_1$ and $j_1 \in \mathcal{J}_1$;

② $\beta_{j_2}^{s_{j_2}-1} + F_2^{\boldsymbol{r}_2,\boldsymbol{s}_2-\boldsymbol{e}_{j_2}} =_G F_2^{\boldsymbol{r}_2+\boldsymbol{e}_{i_2},\boldsymbol{s}_2} + \alpha_{i_2}^{r_{i_2}}$ for all $i_2 \in \mathcal{I}_2$ and $j_2 \in \mathcal{J}_2$, and $|\boldsymbol{r}_1| = |\boldsymbol{s}_1|$.

These are exactly the assumptions needed to apply Lemma 5.12. □

5.2.3 Induced homotopies on tensor product

A homotopy H between morphisms $F\colon \mathbb{A}\to\mathbb{B}$ and $G\colon \mathbb{A}\to\mathbb{B}$ is a set of morphisms $H^{\boldsymbol{r},\boldsymbol{s}}\colon A^{\boldsymbol{r}}\to B^{\boldsymbol{s}}$ for all $|\boldsymbol{r}|=|\boldsymbol{s}|+1$ such that

$$F^{\boldsymbol{r},\boldsymbol{s}}-G^{\boldsymbol{r},\boldsymbol{s}}=\sum_{i\in\mathcal{I}}H^{\boldsymbol{r}+\boldsymbol{e}_i,\boldsymbol{s}}\circ\alpha_i^{\boldsymbol{r}}+\sum_{j\in\mathcal{J}}\beta_j^{\boldsymbol{s}-\boldsymbol{e}_j}\circ H^{\boldsymbol{r},\boldsymbol{s}-\boldsymbol{e}_j}.$$

Definition 5.15. *Let $F_k\colon A_k\to B_k$ (resp. G_k) be chain maps and $H_k\colon F_k\to G_k$ homotopies for each $k=1,2$. Denote F (resp. G) the chain map induced by F_1 and F_2 (resp. G_1 and G_2). Their* induced homotopy *is the homotopy $H\colon F\to G$ given by the data:*

$$H^{\boldsymbol{r},\boldsymbol{s}}=\epsilon^{\boldsymbol{r},\boldsymbol{s}}_{H_1,F_2}H_1^{\boldsymbol{r}_1,\boldsymbol{s}_1}\otimes F_2^{\boldsymbol{r}_2,\boldsymbol{s}_2}+(-1)^{|\boldsymbol{r}_1|}\epsilon^{\boldsymbol{r},\boldsymbol{s}}_{G_1,H_2}G_1^{\boldsymbol{r}_1,\boldsymbol{s}_1}\otimes H_2^{\boldsymbol{r}_2,\boldsymbol{s}_2}$$

where the ϵ's are defined as in Definition 5.11.

Proposition 5.16. *Definition 5.15 gives a well-defined homotopy.*

Proof. We must check that $F-G=\beta\circ H+H\circ\alpha$. We unfold the two terms on the right-hand side (RHS), with first the computation of $\beta\circ H$ restricted to the paths from $A^{\boldsymbol{r}}$ to $B^{\boldsymbol{s}}$:

$$\begin{aligned}
(\beta\circ H)|_{A^{\boldsymbol{r}}}^{B^{\boldsymbol{s}}}&=\sum_{j\in\mathcal{J}}\beta_j^{\boldsymbol{s}-\boldsymbol{e}_j}\circ H^{\boldsymbol{r},\boldsymbol{s}-\boldsymbol{e}_j}\\
&=\sum_{j_1\in\mathcal{J}_1}\left[\beta_{j_1}^{\boldsymbol{s}_1-\boldsymbol{e}_{j_1}}\otimes\mathrm{Id}\right]\\
&\qquad\circ\left[\epsilon^{\boldsymbol{r},\boldsymbol{s}-\boldsymbol{e}_{j_1}}_{H_1,F_2}H_1^{\boldsymbol{r}_1,\boldsymbol{s}_1-\boldsymbol{e}_{j_1}}\otimes F_2^{\boldsymbol{r}_2,\boldsymbol{s}_2}+(-1)^{|\boldsymbol{r}_1|}\epsilon^{\boldsymbol{r},\boldsymbol{s}-\boldsymbol{e}_{j_1}}_{G_1,H_2}G_1^{\boldsymbol{r}_1,\boldsymbol{s}_1-\boldsymbol{e}_{j_1}}\otimes H_2^{\boldsymbol{r}_2,\boldsymbol{s}_2}\right]\\
&\quad+\sum_{j_2\in\mathcal{J}_2}\left[\epsilon^{\boldsymbol{s}_1,\boldsymbol{s}_2-\boldsymbol{e}_{j_2},j_2}_{B_1\otimes B_2}\mathrm{Id}\otimes\beta_{j_2}^{\boldsymbol{s}_2-\boldsymbol{e}_{j_2}}\right]\\
&\qquad\circ\left[\epsilon^{\boldsymbol{r},\boldsymbol{s}-\boldsymbol{e}_{j_2}}_{H_1,F_2}H_1^{\boldsymbol{r}_1,\boldsymbol{s}_1}\otimes F_2^{\boldsymbol{r}_2,\boldsymbol{s}_2-\boldsymbol{e}_{j_2}}+(-1)^{|\boldsymbol{r}_1|}\epsilon^{\boldsymbol{r},\boldsymbol{s}-\boldsymbol{e}_{j_2}}_{G_1,H_2}G_1^{\boldsymbol{r}_1,\boldsymbol{s}_1}\otimes H_2^{\boldsymbol{r}_2,\boldsymbol{s}_2-\boldsymbol{e}_{j_2}}\right]\\
&=\sum_{j_1\in\mathcal{J}_1}\epsilon^{\boldsymbol{r},\boldsymbol{s}-\boldsymbol{e}_{j_1}}_{H_1,F_2}\left[\beta_{j_1}^{\boldsymbol{s}_1-\boldsymbol{e}_{j_1}}\circ H_1^{\boldsymbol{r}_1,\boldsymbol{s}_1-\boldsymbol{e}_{j_1}}\right]\otimes F_2^{\boldsymbol{r}_2,\boldsymbol{s}_2} && (1)\\
&\qquad+(-1)^{|\boldsymbol{r}_1|}\epsilon^{\boldsymbol{r},\boldsymbol{s}-\boldsymbol{e}_{j_1}}_{G_1,H_2}\left[\beta_{j_1}^{\boldsymbol{s}_1-\boldsymbol{e}_{j_1}}\circ G_1^{\boldsymbol{r}_1,\boldsymbol{s}_1-\boldsymbol{e}_{j_1}}\right]\otimes H_2^{\boldsymbol{r}_2,\boldsymbol{s}_2} && (2)\\
&\quad+\sum_{j_2\in\mathcal{J}_2}\left(\beta_{j_2}^{\boldsymbol{s}_2-\boldsymbol{e}_{j_2}}*H_1^{\boldsymbol{r}_1,\boldsymbol{s}_1}\right)\epsilon^{\boldsymbol{s}_1,\boldsymbol{s}_2-\boldsymbol{e}_{j_2},j_2}_{B_1\otimes B_2}\epsilon^{\boldsymbol{r},\boldsymbol{s}-\boldsymbol{e}_{j_2}}_{H_1,F_2}\;H_1^{\boldsymbol{r}_1,\boldsymbol{s}_1}\otimes\left[\beta_{j_2}^{\boldsymbol{s}_2-\boldsymbol{e}_{j_2}}\circ F_2^{\boldsymbol{r}_2,\boldsymbol{s}_2-\boldsymbol{e}_{j_2}}\right] && (3)\\
&\qquad+\left(\beta_{j_2}^{\boldsymbol{s}_2-\boldsymbol{e}_{j_2}}*G_1^{\boldsymbol{r}_1,\boldsymbol{s}_1}\right)(-1)^{|\boldsymbol{r}_1|}\epsilon^{\boldsymbol{s}_1,\boldsymbol{s}_2-\boldsymbol{e}_{j_2},j_2}_{B_1\otimes B_2}\epsilon^{\boldsymbol{r},\boldsymbol{s}-\boldsymbol{e}_{j_2}}_{G_1,H_2}\;G_1^{\boldsymbol{r}_1,\boldsymbol{s}_1}\otimes\left[\beta_{j_2}^{\boldsymbol{s}_2-\boldsymbol{e}_{j_2}}\circ H_2^{\boldsymbol{r}_2,\boldsymbol{s}_2-\boldsymbol{e}_{j_2}}\right] && (4)
\end{aligned}$$

and then the computation of $H\circ\alpha$ restricted to the paths from $A^{\boldsymbol{r}}$ to $B^{\boldsymbol{s}}$:

$$\begin{aligned}
(H\circ\alpha)|_{A^{\boldsymbol{r}}}^{B^{\boldsymbol{s}}}&=\sum_{i\in\mathcal{I}}H^{\boldsymbol{r}+\boldsymbol{e}_i,\boldsymbol{s}}\circ\alpha_i^{\boldsymbol{r}}\\
&=\sum_{i_1\in\mathcal{I}_1}\left[\epsilon^{\boldsymbol{r}+\boldsymbol{e}_{i_1},\boldsymbol{s}}_{H_1,F_2}H_1^{\boldsymbol{r}_1+\boldsymbol{e}_{i_1},\boldsymbol{s}_1}\otimes F_2^{\boldsymbol{r}_2,\boldsymbol{s}_2}+(-1)^{|\boldsymbol{r}_1|+1}\epsilon^{\boldsymbol{r}+\boldsymbol{e}_{i_1},\boldsymbol{s}}_{G_1,H_2}G_1^{\boldsymbol{r}_1+\boldsymbol{e}_{i_1},\boldsymbol{s}_1}\otimes H_2^{\boldsymbol{r}_2,\boldsymbol{s}_2}\right]\\
&\qquad\circ\left[\alpha_{i_1}^{\boldsymbol{r}_1}\otimes\mathrm{Id}\right]
\end{aligned}$$

$$+\sum_{i_2\in\mathcal{I}_2}\left[\epsilon_{H_1,F_2}^{\boldsymbol{r}+e_{i_2},\boldsymbol{s}}H_1^{\boldsymbol{r}_1,\boldsymbol{s}_1}\otimes F_2^{\boldsymbol{r}_2+e_{i_2},\boldsymbol{s}_2}+(-1)^{|\boldsymbol{r}_1|}\epsilon_{G_1,H_2}^{\boldsymbol{r}+e_{i_2},\boldsymbol{s}}G_1^{\boldsymbol{r}_1,\boldsymbol{s}_1}\otimes H_2^{\boldsymbol{r}_2+e_{i_2},\boldsymbol{s}_2}\right]$$
$$\circ\left[\epsilon_{A_1\otimes A_2}^{\boldsymbol{r}_1,\boldsymbol{r}_2,i_2}\mathrm{Id}\otimes\alpha_{i_2}^{\boldsymbol{r}_2}\right]$$
$$=\sum_{i_1\in\mathcal{I}_1}\left(F_2^{\boldsymbol{r}_2,\boldsymbol{s}_2}*\alpha_{i_1}^{\boldsymbol{r}_1}\right)\epsilon_{H_1,F_2}^{\boldsymbol{r}+e_{i_1},\boldsymbol{s}}\left[H_1^{\boldsymbol{r}_1+e_{i_1},\boldsymbol{s}_1}\circ\alpha_{i_1}^{\boldsymbol{r}_1}\right]\otimes F_2^{\boldsymbol{r}_2,\boldsymbol{s}_2} \qquad (1)$$
$$+\left(H_2^{\boldsymbol{r}_2,\boldsymbol{s}_2}*\alpha_{i_1}^{\boldsymbol{r}_1}\right)(-1)^{|\boldsymbol{r}_1|+1}\epsilon_{G_1,H_2}^{\boldsymbol{r}+e_{i_1},\boldsymbol{s}}\left[G_1^{\boldsymbol{r}_1+e_{i_1},\boldsymbol{s}_1}\circ\alpha_{i_1}^{\boldsymbol{r}_1}\right]\otimes H_2^{\boldsymbol{r}_2,\boldsymbol{s}_2} \qquad (2)$$
$$+\sum_{i_2\in\mathcal{I}_2}\epsilon_{A_1\otimes A_2}^{\boldsymbol{r}_1,\boldsymbol{r}_2,i_2}\epsilon_{H_1,F_2}^{\boldsymbol{r}+e_{i_2},\boldsymbol{s}}\ H_1^{\boldsymbol{r}_1,\boldsymbol{s}_1}\otimes\left[F_2^{\boldsymbol{r}_2+e_{i_2},\boldsymbol{s}_2}\circ\alpha_{i_2}^{\boldsymbol{r}_2}\right] \qquad (3)$$
$$+(-1)^{|\boldsymbol{r}_1|}\epsilon_{A_1\otimes A_2}^{\boldsymbol{r}_1,\boldsymbol{r}_2,i_2}\epsilon_{G_1,H_2}^{\boldsymbol{r}+e_{i_2},\boldsymbol{s}}\ G_1^{\boldsymbol{r}_1,\boldsymbol{s}_1}\otimes\left[H_2^{\boldsymbol{r}_2+e_{i_2},\boldsymbol{s}_2}\circ\alpha_{i_2}^{\boldsymbol{r}_2}\right] \qquad (4)$$

We find four different pairs of terms, labelled (1) to (4), that we simplify pairwise using chain map or chain homotopy relations:

(1) We can assume that for all $i_1\in\mathcal{I}_1$ and all $j_1\in\mathcal{J}_1$:

$$\beta_{j_1}^{\boldsymbol{s}_1-e_{j_1}}+H_1^{\boldsymbol{r}_1,\boldsymbol{s}_1-e_{j_1}}=_G H_1^{\boldsymbol{r}_1+e_{i_1},\boldsymbol{s}_1}+\alpha_{i_1}^{\boldsymbol{r}_1}=_G F_1^{\boldsymbol{r}_1,\boldsymbol{s}_1}=_G G_1^{\boldsymbol{r}_1,\boldsymbol{s}_1}.$$

Thanks to Lemma 5.12 (i), the sum of the two (1)-terms is:

$$\left[\epsilon_{F_1,F_2}^{\boldsymbol{r}_1,\boldsymbol{s}_1}F_1^{\boldsymbol{r}_1,\boldsymbol{s}_1}-\epsilon_{G_1,F_2}^{\boldsymbol{r}_1,\boldsymbol{s}_1}G_1^{\boldsymbol{r}_1,\boldsymbol{s}_1}\right]\otimes F_2^{\boldsymbol{r}_2,\boldsymbol{s}_2}.$$

The computation is similar for the case (4).

(3) We can assume $|\boldsymbol{r}_1|+1=|\boldsymbol{s}_1|$ and that:

$$\beta_{j_2}^{\boldsymbol{s}_2-e_{j_2}}+F_2^{\boldsymbol{r}_2,\boldsymbol{s}_2-e_{j_2}}=_G F_2^{\boldsymbol{r}_2+e_{i_2},\boldsymbol{s}_2}+\alpha_{i_2}^{\boldsymbol{r}_2}.$$

Then, Lemma 5.12 (ii) shows that:

$$\left(H_1^{\boldsymbol{r}_1,\boldsymbol{s}_1}*\beta_{j_2}^{\boldsymbol{s}_2-e_{j_2}}\right)\epsilon_{B_1\otimes B_2}^{\boldsymbol{s}_1,\boldsymbol{s}_2-e_{j_2},j_2}\epsilon_{H_1,F_2}^{\boldsymbol{r},\boldsymbol{s}-e_{j_2}}=-\epsilon_{A_1\otimes A_2}^{\boldsymbol{r}_1,\boldsymbol{r}_2,i_2}\epsilon_{H_1,F_2}^{\boldsymbol{r}+e_{i_2},\boldsymbol{s}}$$

independently of j_2 and i_2. We conclude that the sum of the pair (3) is zero. The computation is similar for the case (2).

We conclude:

$$\begin{aligned}\mathrm{RHS}&=\left[\epsilon_{F_1,F_2}^{\boldsymbol{r}_1,\boldsymbol{s}_1}F_1^{\boldsymbol{r}_1,\boldsymbol{s}_1}-\epsilon_{G_1,F_2}^{\boldsymbol{r}_1,\boldsymbol{s}_1}G_1^{\boldsymbol{r}_1,\boldsymbol{s}_1}\right]\otimes F_2^{\boldsymbol{r}_2,\boldsymbol{s}_2}+G_1^{\boldsymbol{r}_1,\boldsymbol{s}_1}\otimes\left[\epsilon_{G_1,F_2}^{\boldsymbol{r}_1,\boldsymbol{s}_1}F_2^{\boldsymbol{r}_2,\boldsymbol{s}_2}-\epsilon_{G_1,G_2}^{\boldsymbol{r}_1,\boldsymbol{s}_1}G_2^{\boldsymbol{r}_2,\boldsymbol{s}_2}\right]\\&=\epsilon_{F_1,F_2}^{\boldsymbol{r}_1,\boldsymbol{s}_1}F_1^{\boldsymbol{r}_1,\boldsymbol{s}_1}\otimes F_2^{\boldsymbol{r}_2,\boldsymbol{s}_2}-\epsilon_{G_1,G_2}^{\boldsymbol{r}_1,\boldsymbol{s}_1}G_1^{\boldsymbol{r}_1,\boldsymbol{s}_1}\otimes G_2^{\boldsymbol{r}_2,\boldsymbol{s}_2}\\&=(F-G)|_{A^{\boldsymbol{r}}}^{B^{\boldsymbol{s}}}.\end{aligned}$$

□

References

[1] D. Bar-Natan, Khovanov's homology for tangles and cobordisms, *Geom. Topol.* Vol. 9, no. 3 (2005), pp. 1443–1499.

[2] A. Beliakova, M. Hogancamp, K. K. Putyra, and S. M. Wehrli, On the functoriality of $\mathfrak{sl}_2$ tangle homology, *Algebr. Geom. Topol.* Vol. 23, no. 3 (2023), pp. 1303–1361.

[3] C. Blanchet, An oriented model for Khovanov homology, *J. Knot Theory Ramifications*, vol. 19, no. 2 (2010), pp. 291–312.

[4] J. Brundan, On the definition of Kac–Moody 2-category, *Math. Ann.* Vol. 364, no. 1-2 (2016), pp. 353–372.

[5] J. Brundan and N. Davidson, Categorical actions and crystals, in: *Categorification and Higher Representation Theory*, vol. 683, Contemp. Math. Amer. Math. Soc., Providence, RI, 2017, pp. 105–147.

[6] J. Brundan and A. P. Ellis, Monoidal supercategories, *Comm. Math. Phys.* Vol. 351, no. 3 (2017), pp. 1045–1089.

[7] J. Brundan and A. P. Ellis, Super Kac–Moody 2-categories, *Proc. London Math. Soc.* Vol. 115, no. 5 (2017), pp. 925–973.

[8] J. Brundan and A. Kleshchev, Odd Grassmannian bimodules and derived equivalences for spin symmetric groups, preprint 2022, arXiv: 2203.14149.

[9] C. L. Caprau, $sl(2)$ tangle homology with a parameter and singular cobordisms, *Algebr. Geom. Topol.* Vol. 8, no. 2 (2008), pp. 729–756.

[10] S. Cautis, J. Kamnitzer, and S. Morrison, Webs and quantum skew Howe duality, *Math. Ann.* Vol. 360, no. 1-2 (2014), pp. 351–390.

[11] J. Cerf, Sur les difféomorphismes de la sphère de dimension trois ($\Gamma_4 = 0$), vol. No. 53, Lecture Notes in Mathematics, Springer-Verlag, Berlin-New York, 1968, xii+133.

[12] C. Chung, T. Sale, and W. Wang, Quantum supergroups VI: roots of 1, *Lett Math Phys*, vol. 109, no. 12 (2019), pp. 2753–2777.

[13] D. Clark, S. Morrison, and K. Walker, Fixing the functoriality of Khovanov homology, *Geom. Topol.* Vol. 13, no. 3 (2009), pp. 1499–1582.

[14] S. Clark, Odd knot invariants from quantum covering groups, *Algebr. Geom. Topol.* Vol. 17, no. 5 (2017), pp. 2961–3005.

[15] S. Clark, Quantum supergroups IV: the modified form, *Math. Z.* Vol. 278, no. 1-2 (2014), pp. 493–528.

[16] S. Clark, Z. Fan, Y. Li, and W. Wang, Quantum supergroups III. Twistors, *Comm. Math. Phys.* Vol. 332, no. 1 (2014), pp. 415–436.

[17] S. Clark and D. Hill, Quantum supergroups V. Braid group action, *Comm. Math. Phys.* Vol. 344, no. 1 (2016), pp. 25–65.

[18] S. Clark, D. Hill, and W. Wang, Quantum supergroups I. Foundations, *Transformation Groups*, vol. 18, no. 4 (2013), pp. 1019–1053.

[19] S. Clark, D. Hill, and W. Wang, Quantum supergroups II. Canonical basis, *Represent. Theory*, vol. 18, no. 9 (2014), pp. 278–309.

[20] J. N. Eberhardt, G. Naisse, and A. Wilbert, Real Springer fibers and odd arc algebras, *J. London Math. Soc.* Vol. 103, no. 4 (2021), pp. 1415–1452.

[21] M. Ebert, A new presentation of the osp(1|2)-polynomial link invariant and categorification, preprint 2022, arXiv: 2210.09583.

[22] M. Ebert, A. D. Lauda, and L. Vera, Derived superequivalences for spin symmetric groups and odd sl(2)-categorifications, preprint 2022, arXiv: 2203.14153.

[23] M. Ehrig, C. Stroppel, and D. Tubbenhauer, Generic $\mathfrak{gl}_2$-foams, web and arc algebras, preprint 2020, arXiv: 1601.08010.

[24] M. Ehrig, C. Stroppel, and D. Tubbenhauer, The Blanchet-Khovanov algebras, in: *Categorification and Higher Representation Theory*, vol. 683, Contemp. Math. Amer. Math. Soc., Providence, RI, 2017, pp. 183–226.

[25] M. Ehrig, D. Tubbenhauer, and P. Wedrich, Functoriality of colored link homologies, *Proc. London Math. Soc.* Vol. 117, no. 5 (2018), pp. 996–1040.

[26] B. Elias, S. Makisumi, U. Thiel, and G. Williamson, Introduction to Soergel bimodules, RSME Springer Series volume 5, Cham, Switzerland: Springer, 2020, 588 pp.

[27] A. P. Ellis, M. Khovanov, and A. D. Lauda, The odd nilHecke algebra and its diagrammatics, *Int. Math. Res. Not. IMRN*, no. 4 (2014), pp. 991–1062.

[28] A. P. Ellis and A. D. Lauda, An odd categorification of $U_q(\mathfrak{sl}_2)$, *Quantum Topol.* Vol. 7, no. 2 (2016), pp. 329–433.

[29] A. P. Ellis and Y. Qi, The differential graded odd nilHecke algebra, *Comm. Math. Phys.* Vol. 344, no. 1 (2016), pp. 275–331.

[30] D. Hill and W. Wang, Categorification of quantum Kac-Moody superalgebras, *Trans. Amer. Math. Soc.* Vol. 367, no. 2 (2015), pp. 1183–1216.

[31] S.-J. Kang, M. Kashiwara, and S.-j. Oh, Supercategorification of quantum Kac-Moody algebras, *Adv. Math.* Vol. 242 (2013), pp. 116–162.

[32] S.-J. Kang, M. Kashiwara, and S.-j. Oh, Supercategorification of quantum Kac–Moody algebras II, *Adv. Math.* Vol. 265 (2014), pp. 169–240.

[33] S.-J. Kang, M. Kashiwara, and S. Tsuchioka, Quiver Hecke superalgebras, *J. Reine Angew. Math.* Vol. 711 (2016), pp. 1–54.

[34] M. Khovanov, A categorification of the Jones polynomial, *Duke Math. J.* Vol. 101, no. 3 (2000), pp. 359–426.

[35] M. Khovanov, sl(3) link homology, *Algebr. Geom. Topol.* Vol. 4, no. 2 (2004), pp. 1045–1081.

[36] M. Khovanov and A. Lauda, A diagrammatic approach to categorification of quantum groups I, *Represent. Theory*, vol. 13, no. 14 (2009), pp. 309–347.

[37] A. D. Lauda, H. Queffelec, and D. E. V. Rose, Khovanov homology is a skew Howe 2–representation of categorified quantum $\mathfrak{sl}_m$, *Algebr. Geom. Topol.* Vol. 15, no. 5 (2015), pp. 2517–2608.

[38] A. D. Lauda and H. M. Russell, Oddification of the cohomology of type A Springer varieties, *Int. Math. Res. Not. IMRN*, no. 17 (2014), pp. 4822–4854.

[39] R. Lipshitz and S. Sarkar, A Khovanov stable homotopy type, *J. Amer. Math. Soc.* Vol. 27, no. 4 (2014), pp. 983–1042.

[40] M. Mackaay, sl(3)-Foams and the Khovanov-Lauda categorification of quantum sl(k), preprint 2009, arXiv: 0905.2059.

[41] M. Mackaay, M. Stošić, and P. Vaz, $\mathfrak{sl}(N)$–link homology ($N \geq 4$) using foams and the Kapustin–Li formula, *Geom. Topol.* Vol. 13, no. 2 (2009), pp. 1075–1128.

[42] M. Mackaay, M. Stošić, and P. Vaz, A diagrammatic categorification of the q-Schur algebra, *Quantum Topol.* Vol. 4, no. 1 (2013), pp. 1–75.

[43] V. Mikhaylov and E. Witten, Branes and Supergroups, *Commun. Math. Phys.* Vol. 340, no. 2 (2015), pp. 699–832.

[44] G. Naisse and K. Putyra, Odd Khovanov homology for tangles, preprint 2020, arXiv: 2003.14290.

[45] G. Naisse and P. Vaz, Odd Khovanov's arc algebra, *Fund. Math.* Vol. 241, no. 2 (2018), pp. 143–178.

[46] T. Ohtsuki, Quantum invariants: a study of knots, 3-manifolds, and their sets, K & E Series on Knots and Everything v. 29, Singapore ; River Edge, NJ: World Scientific, 2002, 489 pp.

[47] P. S. Ozsváth, J. Rasmussen, and Z. Szabó, Odd Khovanov homology, *Algebr. Geom. Topol.* Vol. 13, no. 3 (2013), pp. 1465–1488.

[48] K. K. Putyra, A 2-category of chronological cobordisms and odd Khovanov homology, in: *Knots in Poland III. Part III*, vol. 103, Banach Center Publ. Polish Acad. Sci. Inst. Math., Warsaw, 2014, pp. 291–355.

[49] H. Queffelec and D. E. V. Rose, The $\mathfrak{sl}_n$ foam 2-category: A combinatorial formulation of Khovanov–Rozansky homology via categorical skew Howe duality, *Adv. Math.* Vol. 302 (2016), pp. 1251–1339.

[50] L.-H. Robert, A new way to evaluate MOY graphs, preprint 2015, arXiv: 1512.02370.

[51] R. Rouquier, 2-Kac–Moody algebras, preprint 2008, arXiv: 0812.5023.

[52] T. Sano, Fixing the functoriality of Khovanov homology: a simple approach, *J. Knot Theory Ramifications*, vol. 30, no. 11 (2021), Paper No. 2150074, 12.

[53] L. Schelstraete, Higher Rewriting Theory and Non-Degeneracy of the Graded-2-Category of Foams, in preparation.

[54] L. Schelstraete, Supercategorification and Khovanov-like tangle invariants, master's thesis, UCLouvain, 2020.

[55] A. N. Shumakovitch, Patterns in odd Khovanov homology, *J. Knot Theory Ramifications*, vol. 20, no. 1 (2011), pp. 203–222.

[56] P. Vaz, Not even Khovanov homology, *Pacific J. Math.* Vol. 308, no. 1 (2020), pp. 223–256.

[57] P. Vogel, Functoriality of Khovanov homology, *J. Knot Theory Ramifications*, vol. 29, no. 04 (2020), p. 2050020.

[58] B. Webster, Knot invariants and higher representation theory, *Mem. Amer. Math. Soc.* Vol. 250, no. 1191 (2017), pp. v+141.