**Noname manuscript No.**
(will be inserted by the editor)

# Exponential Decay of Correlations for the Strongly Coupled Toom Model

**Augustin de Maere · Lise Ponselet**

**Abstract** We prove that, for the two-dimensional probabilistic cellular automaton of Toom in the low-noise regime, there are two classes of initial measures, each of which converges exponentially fast toward one of the two natural invariant measures. We also show that these two invariant measures have exponential decay of correlations in space and in time and are strongly mixing.

**Keywords** Probabilistic Cellular Automata · Phase Transition · Toom Model

## 1 Introduction

Toom's probabilistic cellular automaton, also known as 'North-East-Center model', was introduced in [6] as an example of a two-dimensional probabilistic cellular automaton (PCA) exhibiting a phase transition. This model is a discrete-time random dynamical system with the Markov property. At every site $x$ of $\mathbb{Z}^2$, there is some variable $\omega_x$ in $\{-1,+1\}$, called 'spin', and at every time step, each spin is updated. The simultaneous spin updates follow an asymmetric majority rule, each spin adopting the majority value among three sites: its nearest neighbours to the North, to the East and itself. An error occurs with a probability of order $\varepsilon$, for some $\varepsilon$ in $[0,1]$, the spin then taking the opposite value.

By virtue of the ergodic theorem, the asymptotic behaviour of such a PCA is actually given by its ergodic invariant probability measures. The existence of a phase transition in the Toom model has been rigorously proven in [6]. Below a critical threshold for $\varepsilon$, the system admits two different natural invariant measures, dominated either by positive or by negative spins. In this paper, we will prove that each of them has expo-

A. de Maere · L. Ponselet
UCL, IRMP, Chemin du cyclotron 2, bte L7.01.03, B-1348 Louvain-la-Neuve, Belgium.
Tel.: +32-10-473279
Fax: +32-10-472414
E-mail: lise.ponselet@uclouvain.be

nential decay of correlations in space and in time and is strongly mixing. In particular, they are ergodic and therefore extremal.

## 2 Formalism and Results

Let $S = \{-1,1\}$ and $X = S^{\mathbb{Z}^2}$. For any spin configuration $\underline{\omega} \in X$, $\omega_x$ will denote the value of $\underline{\omega}$ at site $x$, $\underline{\omega}_\Lambda$, the vector $\{\omega_x\}_{x\in\Lambda}$ for any $\Lambda \subseteq \mathbb{Z}^2$ and $\underline{\omega}_{\neq x}$, the vector $\{\omega_y\}_{y\neq x}$. Then $(\underline{\omega}_{\neq x}, a)$ will be the configuration obtained from $\underline{\omega}$ by replacing the spin $\omega_x$ at site $x$ with the value $a \in S$.

Let $\underline{e}_1 = (1,0)$ and $\underline{e}_2 = (0,1)$ denote the canonical basis vectors of $\mathbb{Z}^2$. The North-East-Center majority spin around a site $x$ is given by the function $\phi_x$:

$$\phi_x(\underline{\omega}) = \operatorname{sgn}(\omega_x + \omega_{x+\underline{e}_1} + \omega_{x+\underline{e}_2}).$$

The local transition probabilities $p(\xi_x|\underline{\omega})$ of a Toom model only depend on $\omega_x$, $\omega_{x+\underline{e}_1}$ and $\omega_{x+\underline{e}_2}$. We suppose that they satisfy the following assumptions, for some given values of the parameters $\varepsilon \in [0,1]$ and $\alpha \geq 0$:

(A1) if $\xi_x \neq \phi_x(\underline{\omega})$, then $p(\xi_x|\underline{\omega}) \leq \varepsilon$;
(A2) if $a = \phi_x(\underline{\omega})$, then $|p(\xi_x|\underline{\omega}) - p(\xi_x|\underline{\omega}_{\neq y}, a)| \leq \alpha\, p(\xi_x|\underline{\omega})$ for all $y \in \mathbb{Z}^2$.

This defines a transition probability kernel on $X$, which can be formally written as

$$\mathbb{P}[\underline{\xi}|\underline{\omega}] = \prod_{x\in\mathbb{Z}^2} p(\xi_x|\underline{\omega}). \tag{1}$$

Assume that $S$ is endowed with the discrete topology and $X$ with the product topology. Let $\mathscr{C}(X)$ be the set of continuous functions from $X$ into $\mathbb{R}$ with the norm: $|\varphi|_\infty = \sup_{\underline{\omega}\in X} |\varphi(\underline{\omega})|$. Since $X$ is compact, $\mathscr{C}(X)$ with the norm $|.|_\infty$ is a Banach space, and its dual is $\mathscr{M}(X)$, the space of signed Borel measures on $X$ with the norm:

$$|\mu| = \sup\{\, \mu(\varphi) \mid \varphi \in \mathscr{C}(X), |\varphi|_\infty \leq 1\,\}.$$

For PCA on infinite spaces, it is often useful to consider the following semi-norm on $\mathscr{C}(X)$: if $\delta_x\varphi(\underline{\omega}) = \varphi(\underline{\omega}) - \varphi(\underline{\omega}_{\neq x}, -\omega_x)$, we define:

$$\|\varphi\| = \sum_{x\in\mathbb{Z}^2} |\delta_x\varphi|_\infty\,. \tag{2}$$

Let $T : \mathscr{M}(X) \to \mathscr{M}(X)$ be the transfer operator associated to the transition probability kernel of (1), explicitly defined as:

$$T\mu(\varphi) = \int \mathrm{d}\underline{\omega} \int \mathrm{d}\underline{\xi}\ \varphi(\underline{\xi})\,\mathbb{P}[\underline{\xi}|\underline{\omega}]\,\mu(\underline{\omega}), \tag{3}$$

for any $\varphi$ in $\mathscr{C}(X)$.

In [6], Toom proved the existence of an $\varepsilon_c > 0$ such that for all $\varepsilon \leq \varepsilon_c$, the system defined above admits at least two different invariant measures. Indeed, the Banach-Alaoglu theorem states that the unit ball of $\mathscr{M}(X)$ is compact in the weak-* topology.

Therefore, if $\delta^{(+)}$ and $\delta^{(-)}$ are the Dirac measures concentrated on the configurations $\{+1\}_{x\in\mathbb{Z}^2}$ and $\{-1\}_{x\in\mathbb{Z}^2}$ respectively, we can always extract two weakly-* convergent subsequences of

$$\left(\frac{1}{n}\sum_{k=0}^{n-1} T^k \delta^{(+)}\right)_{n\in\mathbb{N}_0} \quad \text{and} \quad \left(\frac{1}{n}\sum_{k=0}^{n-1} T^k \delta^{(-)}\right)_{n\in\mathbb{N}_0}.$$

The associated limits $\mu_{\text{inv}}^{(+)}$ and $\mu_{\text{inv}}^{(-)}$ are explicitly given by

$$\mu_{\text{inv}}^{(+)}(\varphi) = \lim_{j\to\infty}\left(\frac{1}{n_j}\sum_{k=0}^{n_j-1} T^k \delta^{(+)}(\varphi)\right) \quad \forall \varphi \in \mathscr{C}(X), \tag{4}$$

for a certain subsequence $(n_j)_{j\in\mathbb{N}}$ of increasing positive integers, and by the symmetric expression for $\mu_{\text{inv}}^{(-)}$. The measures $\mu_{\text{inv}}^{(+)}$ and $\mu_{\text{inv}}^{(-)}$ are obviously invariant probability measures and Theorem 1 in [6] implies that $\mu_{\text{inv}}^{(+)} \neq \mu_{\text{inv}}^{(-)}$ as long as $\varepsilon \leq \varepsilon_c$.

Then, if $\varepsilon \leq \varepsilon_c$, there exist an infinite number of invariant probability measures, since any convex combination of $\mu_{\text{inv}}^{(+)}$ and $\mu_{\text{inv}}^{(-)}$ is also an invariant probability measure. But, among these invariant measures, some have interesting properties such as ergodicity and exponential decay of correlations. In Sections 3, 4, 5 and 6, we will use a technique of decoupling in the pure phases previously introduced and developed for Coupled Map Lattices [3,5] to show that any initial probability measure in a suitable basin of attraction of $\mathscr{M}(X)$ converges exponentially fast toward $\mu_{\text{inv}}^{(+)}$ or $\mu_{\text{inv}}^{(-)}$. Let $\mathbb{1}_{-,\Lambda}$ be the indicator function of the subset $\{\underline{\omega} \in X | \omega_x = -1 \ \forall x \in \Lambda\}$, and for any measurable set $Y \subseteq X$, let $\mathbb{1}_Y : \mathscr{M}(X) \to \mathscr{M}(X)$ be the operator defined as:

$$\mathbb{1}_Y \mu(\varphi) = \mu(\varphi \mathbb{1}_Y).$$

We will consider the sets

$$\mathscr{B}^{(+)}(K,\varepsilon') = \left\{ \mu \in \mathscr{M}(X) \;\middle|\; \left|\mathbb{1}_{-,\Lambda}\mu\right| \leq K\varepsilon'^{|\Lambda|} \quad \forall \Lambda \subseteq \mathbb{Z}^2 \right\},$$

with $K \geq 0$ and $\varepsilon' \in [0,1]$, and their symmetric counterparts $\mathscr{B}^{(-)}(K,\varepsilon')$, and prove the following result:

**Theorem 1** *There exist $\alpha_* > 0$ and a positive and strictly decreasing function $\varepsilon_*$ defined on $[0,\alpha_*[$ such that for all $\alpha \in [0,\alpha_*[$ and $\varepsilon \in [0,\varepsilon_*(\alpha)[$, the following assertion is true for any Toom PCA satisfying the corresponding assumptions (A1) and (A2). For any probability measure $\mu$ in $\mathscr{B}^{(+)}(K,\varepsilon')$ with $K \geq 0$ and $\varepsilon' < \varepsilon_*(\alpha)$, there exist some constants $C < \infty$ and $\sigma < 1$ such that, for all $\varphi \in \mathscr{C}(X)$ and $n \in \mathbb{N}$,*

$$\left| T^n \mu(\varphi) - \mu_{\text{inv}}^{(+)}(\varphi) \right| \leq C\sigma^n \|\varphi\|.$$

*The symmetric result for $\mu$ in the class $\mathscr{B}^{(-)}(K,\varepsilon')$ and $\mu_{\text{inv}}^{(-)}$ is also valid.*

Eventually, in Section 7, we will prove that both $\mu_{\text{inv}}^{(+)}$ and $\mu_{\text{inv}}^{(-)}$ have exponential decay of correlations in space and in time and are, consequently, strongly mixing.

**Corollary** *Assume that $\alpha < \alpha_*$ and that $\varepsilon < \varepsilon_*(\alpha)$, with $\alpha_*$ and $\varepsilon_*$ as given by Theorem 1. Then there exists some constant $\sigma < 1$ such that for any $\varphi, \psi$ in $\mathscr{C}(X)$ with $\|\varphi\| < \infty$ and $\|\psi\| < \infty$, and with a positive taxicab distance $d(\varphi, \psi)$ between their supports, we have, for some $C_{\varphi,\psi} < \infty$ independent of $d(\varphi, \psi)$,*

$$\left|\mu_{\text{inv}}^{(+)}(\varphi\psi) - \mu_{\text{inv}}^{(+)}(\varphi)\mu_{\text{inv}}^{(+)}(\psi)\right| \leq C_{\varphi,\psi}\sigma^{d(\varphi,\psi)}.$$

*$\mu_{\text{inv}}^{(-)}$ also satisfies the same result.*

For the exponential decay of correlations in time, we define the operator $T : \mathscr{C}(X) \to \mathscr{C}(X)$:

$$T\varphi(\underline{\omega}) = \int \mathrm{d}\underline{\xi}\ \varphi(\underline{\xi})\ \mathbb{P}[\underline{\xi}|\underline{\omega}],$$

which is simply the dual of the transfer operator acting on signed measures.

**Theorem 2** *Assume that $\alpha < \alpha_*$ and that $\varepsilon < \varepsilon_*(\alpha)$ with $\alpha_*$ and $\varepsilon_*$ as given by Theorem 1. Then, for any continuous functions $\varphi, \psi$ with finite supports, there exist some constants $\sigma < 1$ and $C_{\varphi,\psi} < \infty$ such that, for all $n$,*

$$\left|\mu_{\text{inv}}^{(+)}(\varphi T^n \psi) - \mu_{\text{inv}}^{(+)}(\varphi)\mu_{\text{inv}}^{(+)}(\psi)\right| \leq C_{\varphi,\psi}\,\sigma^n.$$

*The symmetric result for $\mu_{\text{inv}}^{(-)}$ is also true.*

The arguments developed in our proofs of Theorems 1 and 2 also apply to the one-dimensional PCA of Stavskaya [2]. The model of Stavskaya is one of the first PCA for which the existence of a phase transition has been rigorously proven. This phase transition is similar to that of the Toom model and a slight adaptation of our proofs to the Stavskaya model is sufficient to show both the exponential convergence of a large class of initial measures toward the non-trivial natural invariant measure that appears at the phase transition, and the exponential decay of correlations in space and in time for this invariant measure. The modification consists in replacing the Toom graphs which underlie our Peierls argument below with the fences described in [2] for the model of Stavskaya.

## 3 Path Expansion

In this section, we will introduce a path expansion which is essentially equivalent to the Dobrushin criterion [1], using here a formalism which was originally introduced by Keller and Liverani for Coupled Map Lattices [3].

Let $\prec$ be any well-ordering of $\mathbb{Z}^2$, for instance

$$(0,0) \prec (0,1) \prec (-1,0) \prec (1,0) \prec (0,-1) \prec (0,2) \prec (-1,1) \prec (1,1) \prec \ldots$$

The operator $\Pi_x : \mathscr{C}(X) \to \mathscr{C}(X)$ is defined as:

$$\Pi_x\varphi(\underline{\omega}) = \varphi(\underline{\omega}_{\succeq x}, \underline{a}_{\prec x}) - \varphi(\underline{\omega}_{\succ x}, \underline{a}_{\preceq x}),$$

where, from now on, $\underline{a}$ will denote the configuration for which $a_x = a = +1$ for all $x \in \mathbb{Z}^2$. Using telescopic sums, we can check that, for any continuous function $\varphi$,

$$\varphi(\underline{\omega}) = \varphi(\underline{a}) + \sum_{x \in \mathbb{Z}^2} \Pi_x \varphi(\underline{\omega}). \tag{5}$$

With a slight abuse of notation, let us denote by $\Pi_x : \mathscr{M}(X) \to \mathscr{M}(X)$ the dual of the operator $\Pi_x : \mathscr{C}(X) \to \mathscr{C}(X)$. The image of $\mathscr{M}(X)$ under this operator is actually included in the set

$$\mathscr{M}_x = \{\mu \in \mathscr{M}(X) \mid \mu(\varphi) = 0 \text{ if } \varphi(\underline{\omega}) \text{ is independent of } \omega_x\} \tag{6}$$

whose elements verify the following property:

$$\mu \in \mathscr{M}_x \quad \Rightarrow \quad |\mu(\varphi)| \leq |\mu| \, |\delta_x \varphi|_\infty . \tag{7}$$

With equation (5), we can see that any signed measure of zero mass $\mu \in \mathscr{M}(X)$ admits the following decomposition:

$$\mu = \sum_{x \in \mathbb{Z}^2} \Pi_x \mu. \tag{8}$$

While $\Pi_x \mu$ belongs to $\mathscr{M}_x$, it is no longer the case for $T\Pi_x\mu$. Nevertheless, since the interactions are local, we will see that $T\Pi_x\mu$ can be expressed as the sum of three signed measures: a first one in $\mathscr{M}_x$, a second one in $\mathscr{M}_{x-\underline{e}_1}$ and a third one in $\mathscr{M}_{x-\underline{e}_2}$. For this, consider an arbitrary measure $\mu_x \in \mathscr{M}_x$. Using definitions (3) and (6) of $T$ and $\mathscr{M}_x$, it is easy to check that

$$T\mu_x = T^{(x,x)}\mu_x + T^{(x-\underline{e}_1,x)}\mu_x + T^{(x-\underline{e}_2,x)}\mu_x, \tag{9}$$

where three new operators have been defined:

$$T^{(y,x)}\mu(\varphi) = \int \mathrm{d}\underline{\omega} \int \mathrm{d}\underline{\xi} \; \varphi(\underline{\xi}) \, \tau^{(y,x)}(\underline{\xi},\underline{\omega}) \, \mu(\underline{\omega}) \quad \text{for } y \in \{x, x-\underline{e}_1, x-\underline{e}_2\} \tag{10}$$

with the kernels

$$\begin{aligned}
\tau^{(x,x)}(\underline{\xi},\underline{\omega}) &= \left( \prod_{y \neq x, x-\underline{e}_1, x-\underline{e}_2} p(\xi_y|\underline{\omega}) \right) \Big( p(\xi_x|\underline{\omega}) - p(\xi_x|\underline{\omega}_{\neq x}, a) \Big) \cdot \\
&\qquad \cdot p(\xi_{x-\underline{e}_1}|\underline{\omega}) p(\xi_{x-\underline{e}_2}|\underline{\omega}); \\
\tau^{(x-\underline{e}_1,x)}(\underline{\xi},\underline{\omega}) &= \left( \prod_{y \neq x, x-\underline{e}_1, x-\underline{e}_2} p(\xi_y|\underline{\omega}) \right) p(\xi_x|\underline{\omega}_{\neq x}, a) \cdot \\
&\qquad \cdot \Big( p(\xi_{x-\underline{e}_1}|\underline{\omega}) - p(\xi_{x-\underline{e}_1}|\underline{\omega}_{\neq x}, a) \Big) p(\xi_{x-\underline{e}_2}|\underline{\omega}); \\
\tau^{(x-\underline{e}_2,x)}(\underline{\xi},\underline{\omega}) &= \left( \prod_{y \neq x, x-\underline{e}_1, x-\underline{e}_2} p(\xi_y|\underline{\omega}) \right) p(\xi_x|\underline{\omega}_{\neq x}, a) \cdot \\
&\qquad \cdot p(\xi_{x-\underline{e}_1}|\underline{\omega}_{\neq x}, a) \Big( p(\xi_{x-\underline{e}_2}|\underline{\omega}) - p(\xi_{x-\underline{e}_2}|\underline{\omega}_{\neq x}, a) \Big).
\end{aligned} \tag{11}$$

We notice that the images of $\mathscr{M}(X)$ under these three operators are included respectively in $\mathscr{M}_x$, $\mathscr{M}_{x-\underline{e}_1}$ and $\mathscr{M}_{x-\underline{e}_2}$.

Let now $\mu$ be a signed measure of zero mass and consider $T^n\mu$. Using the decomposition (8) and applying (9) iteratively, we find

$$T^n\mu = \sum_{x_0\in\mathbb{Z}^2}\sum_{x_0-x_1\in\{0,\underline{e}_1,\underline{e}_2\}}\cdots\sum_{x_{n-1}-x_n\in\{0,\underline{e}_1,\underline{e}_2\}} T^{(x_n,x_{n-1})}\dots T^{(x_1,x_0)}\Pi_{x_0}\mu.$$

This sum can be rewritten as a sum over paths. Indeed, if we introduce

$$P_x = \left\{\gamma\colon\{0,\dots,n\}\to\mathbb{Z}^2 \mid \gamma_n = x \text{ and } \gamma_{t-1}-\gamma_t\in\{0,\underline{e}_1,\underline{e}_2\}\ \forall t\right\},$$

it is equivalent to the following compact expression:

$$T^n\mu = \sum_{x\in\mathbb{Z}^2}\sum_{\gamma\in P_x}\prod_{t=1}^{n} T^{(\gamma_t,\gamma_{t-1})}\Pi_{\gamma_0}\mu, \tag{12}$$

where the operators $T^{(\gamma_t,\gamma_{t-1})}$ have to be applied in chronological order.

In the case of a weakly interacting system, this sum over paths can be used to prove the existence of a unique invariant probability measure, under the assumptions of the Dobrushin criterion [1]. The system we are considering here is certainly not weakly interacting. However, in order to prove Theorem 1, the idea will be to take advantage of the decoupling in the pure phases conveyed in Assumption (A2). This assumption will provide upper bounds of order $\alpha$ on the couplings $T^{(\gamma_t,\gamma_{t-1})}$ in the positive phase. Indeed, as $a=+1$ in (11), those bounds will be obtained for the instants $t$ such that the considered space-time configuration presents a majority of positive spins at time $t-1$ in the set $\{\gamma_t,\gamma_t+\underline{e}_1,\gamma_t+\underline{e}_2\}$. As for the negative phase, it will be shown to be infrequent enough, for a suitable choice of initial condition.

## 4 Pure Phase Expansion

For fixed $x\in\mathbb{Z}^2$ and $\gamma\in P_x$, if $\gamma$ denotes both the function and its trajectory $\{(t,\gamma_t)\mid t=0,\dots,n\}$, let us partition the trajectory into $\gamma^{(+)}\subseteq\gamma\setminus\{\gamma_0\}$ and $\gamma^{(-)}=(\gamma\setminus\{\gamma_0\})\setminus\gamma^{(+)}$. We define, for $t=0,\dots,n$, the sets $F\left(\gamma^{(+)},t\right)\subseteq X$ with the following indicator functions:

$$\mathbb{1}_{F\left(\gamma^{(+)},t\right)}(\underline{\omega}) = \prod_{x:(t+1,x)\in\gamma^{(+)}}\mathbb{1}_+\left(\phi_x(\underline{\omega})\right)\prod_{x:(t+1,x)\in\gamma^{(-)}}\mathbb{1}_-\left(\phi_x(\underline{\omega})\right).$$

These subsets lead to a partition of the configuration space $X^{\{0,\dots,n\}}$. Inserting this partition in (12),

$$T^n\mu = \sum_{x\in\mathbb{Z}^2}\sum_{\gamma\in P_x}\sum_{\gamma^{(+)}\subseteq\gamma\setminus\{\gamma_0\}}\prod_{t=1}^{n}\left(\mathbb{1}_{F(\gamma^{(+)},t)}T^{(\gamma_t,\gamma_{t-1})}\right)\mathbb{1}_{F(\gamma^{(+)},0)}\Pi_{\gamma_0}\mu,$$

where $\mathbb{1}_{F(\gamma^{(+)},n)}$ is nothing but the identity operator. Using property (7) together with the fact that the image of $\mathscr{M}(X)$ under $T^{(x,\gamma_{n-1})}$ is included in $\mathscr{M}_x$, we have, for all $\varphi \in \mathscr{C}(X)$,

$$|T^n\mu(\varphi)| \leq \sum_{x\in\mathbb{Z}^2}\sum_{\gamma\in P_x}\sum_{\gamma^{(+)}\subseteq\gamma\setminus\{\gamma_0\}}\left|\prod_{t=1}^{n}\left(\mathbb{1}_{F(\gamma^{(+)},t)}T^{(\gamma_t,\gamma_{t-1})}\right)\mathbb{1}_{F(\gamma^{(+)},0)}\Pi_{\gamma_0}\mu\right||\delta_x\varphi|_\infty . \tag{13}$$

Next, for given $x\in\mathbb{Z}^2$, $\gamma\in P_x$ and $\gamma^{(+)}\subseteq\gamma\setminus\{\gamma_0\}$, we define the set

$$\mathscr{E}\left(\gamma^{(+)}\right) = \left\{(\underline{\omega}_t)_{t=0}^n \in X^{\{0,\dots,n\}} \mid \underline{\omega}_t \in F\left(\gamma^{(+)},t\right) \quad \forall t\in\{0,\dots,n\}\right\}.$$

We will now build a graph inspired of the construction in [6] in order to control the extent of the negative phase. We will then rewrite the expansion (13) as a cluster expansion in terms of a combination of paths and graphs.

## 5 Graphs

Toom graphs are at the heart of a Peierls argument in the proof of the existence of a phase transition for a class of PCA, including the Toom model, in [6]. An intuitive review of their complex construction and of the proof in the particular case of the Toom model is given in [4].

In order to bound the probability of finding a negative spin at a given time $t_-$ and at a given site $x_-$ of the lattice, when the initial measure is $\delta^{(+)}$, [6] and [4] associate a graph $G$ on $\{0,\dots,t_-\}\times\mathbb{Z}^2$ to each space-time configuration presenting this negative spin. This graph identifies part of the errors which lead to the negative spin at $(t_-,x_-)$: it traces the origin of this negative spin, via their propagation according to the majority rule, back to some previous errors (when $\phi_x(\underline{\omega}_{t-1}) = +1$ but $\omega_{t,x} = -1$). Such errors, classified in a distinguished subset $\hat{V}_G$ of the set of vertices $V_G$, all carry a probability smaller than $\varepsilon$, because of Assumption (A1). The large number of possible graphs is then an 'entropic' factor which has to be counterbalanced by an 'energetic' factor, i.e. the probability of the graphs, which is bounded by the probability of making all the identified errors, $\varepsilon^{|\hat{V}_G|}$. Now the construction algorithm of the Toom graphs guarantees the following properties:

(P1) The graphs are connected and contain $(t_-,x_-)$. Moreover, they are made of at most 48 different types of edges, taking into account all the admissible displacement vectors, orientations and colors. To any such connected graph corresponds a walk which starts from $(t_-,x_-)$, passes along every edge exactly twice and then comes back to its departure point. We consider the sequence that records, at each step of this walk, the displacement vector, color and orientation of the travelled edge, and call it the 'Eulerian path' associated to the graph. This correspondence is injective. Consequently, the number of different graphs grows only exponentially with their number of edges, $|E_G|$, as the number of the Eulerian paths of length $2|E_G|$ is less than $48^{2|E_G|}$.

(P2) The number $|E_G|$ of edges is in turn related to the number $|\hat{V}_G|$ of identified errors:

$$\frac{1}{4}|E_G| + 1 \leq \left|\hat{V}_G\right|. \tag{14}$$

This is relation (A.5) in [4]. It is due to a current conservation principle which presides over the graph construction so as to take advantage of the erosion property of the Toom model.

References [6] and [4] use these properties to prove that the 'energetic' factor wins as long as $\varepsilon$ is small enough. The probability of finding a negative spin at a given time and a given site can be made as small as desired by taking $\varepsilon$ appropriately small.

Here in our treatment of the space-time configurations with a majority of negative spins at time $t-1$ in $\{\gamma_t, \gamma_t + \underline{e}_1, \gamma_t + \underline{e}_2\}$ for all such $t$ that $(t,\gamma_t) \in \gamma^{(-)}$, we will hardly need to modify Toom's construction, but we will associate to each space-time configuration a collection of Toom graphs instead of only one, as we are interested in the origin of a collection of negative spins, corresponding to the different elements of $\gamma^{(-)}$.

We define a map

$$\begin{aligned} g : \mathscr{E}\left(\gamma^{(+)}\right) &\to \mathscr{G}\left(\gamma^{(+)}\right) = g\left(\mathscr{E}\left(\gamma^{(+)}\right)\right) \\ (\underline{\omega}_t)_{t=0}^n &\mapsto G = g\left((\underline{\omega}_t)_{t=0}^n\right) \end{aligned} \tag{15}$$

where $G$ is a graph made of a disconnected collection of Toom graphs whose construction, for a given $(\underline{\omega}_t)_{t=0}^n \in \mathscr{E}\left(\gamma^{(+)}\right)$, consists in the following steps:

1. We pick $t_1$, the largest $t$ such that $(t_1, \gamma_{t_1}) \in \gamma^{(-)}$. We know that $\phi_{\gamma_{t_1}}(\underline{\omega}_{t_1-1}) = -1$. We consider $(\underline{\tilde{\omega}}_t)_{t=0}^n$ which is obtained from $(\underline{\omega}_t)_{t=0}^n$ by replacing $\omega_{t_1,\gamma_{t_1}}$ with the value $-1$. We construct the Toom graph $G_1$ having the negative spin at $(t_1,\gamma_{t_1})$ as origin and which is associated to $(\underline{\tilde{\omega}}_t)_{t=0}^n$ according to the algorithm in [6] and [4]. The whole algorithm remains valid here, the only difference being that no assumption about the initial condition forbids the presence of negative spins at $t=0$. A rule is then added to the algorithm: when the graph under construction reaches such spins, they are considered equivalent to errors and classed as elements of $\hat{V}_{G_1}$.
   So we end up with a connected graph $G_1$. Its set of vertices $V_{G_1}$ contains $(t_1,\gamma_{t_1})$ and at all points $(t,x) \in V_{G_1} \setminus \{(t_1,\gamma_{t_1})\}$, the space-time configuration $(\underline{\omega}_t)_{t=0}^n$ takes the value $\omega_{t,x} = -1$. It has a distinguished subset of vertices $\hat{V}_{G_1} \subseteq V_{G_1}$ where the space-time configuration presents errors:

   $$\forall (t,x) \in \hat{V}_{G_1} \text{ such that } t \neq 0,\ \phi_x(\underline{\omega}_{t-1}) = +1.$$

   It has a set $E_{G_1}$ of edges connecting its vertices and satisfying (14).
   $G_1$ is the first of the Toom graphs whose union will form the graph $G$.
2. We pick the next maximum $t_2 < t_1$ such that $(t_2,\gamma_{t_2}) \in \gamma^{(-)}$. We perform exactly the same construction as before and associate to $(t_2,\gamma_{t_2})$ the graph $G_2$ with properties analogous to those of $G_1$. If $V_{G_1} \cap V_{G_2} \neq \varnothing$ then we discard $G_2$, as we can't count any error twice. Otherwise the union of $G_1$ and $G_2$ will be part of $G$.

3. We repeat the same process up to the lowest time $t$ such that $(t,\gamma_t) \in \gamma^{(-)}$, discarding any Toom graph $G_i$ which intersects one of the previous retained ones.
4. $G$ is defined as the union of all the retained $G_i$, $i \in \{1,\dots,|\gamma^{(-)}|\}$. $V_G$ is the union of the retained $V_{G_i}$, $\hat{V}_G$, that of the retained $\hat{V}_{G_i}$ and $E_G$, that of the retained $E_{G_i}$.

The map $g$ of (15) is then completely defined. It is usually not injective, as there are many configurations corresponding to one graph. But $\mathscr{E}\left(\gamma^{(+)}\right)$ can always be written as:

$$\mathscr{E}\left(\gamma^{(+)}\right) = \bigcup_{G \in \mathscr{G}\left(\gamma^{(+)}\right)} g^{-1}G. \tag{16}$$

Defining, for all $G \in \mathscr{G}\left(\gamma^{(+)}\right)$ and for $t = 0,\dots,n$, the subsets $E(G,t) \subseteq X$:

$$\mathbb{1}_{E(G,t)}(\underline{\omega}) = \prod_{x:(t,x)\in V_G\setminus\gamma^{(-)}} \mathbb{1}_{-}(\omega_x) \prod_{x:(t+1,x)\in\hat{V}_G} \mathbb{1}_{+}\left(\phi_x(\underline{\omega})\right), \tag{17}$$

we have, by construction of the map $g$,

$$\mathbb{1}_{g^{-1}G}\left((\underline{\omega}_t)_{t=0}^{n}\right) \leq \prod_{t=0}^{n} \mathbb{1}_{E(G,t)}(\underline{\omega}_t). \tag{18}$$

Let us consider any $G \in \mathscr{G}\left(\gamma^{(+)}\right)$. $G$ is the union of $c$ individually connected but pairwise disconnected Toom graphs $(G_{i_1},\dots,G_{i_c})$. From this point on, they will be noted $(G_1,\dots,G_i,\dots,G_c)$. They all possess Properties (P1) and (P2). We show that $G$ itself inherits similar properties.

First we find an upper bound on the number of different graphs with given numbers of edges and of connected parts. We define a map $w$ on $\mathscr{G}\left(\gamma^{(+)}\right)$. For $G \in \mathscr{G}\left(\gamma^{(+)}\right)$, $w(G)$ consists of two elements: first, the list of the origins of $G_1,\dots,G_c$; second, the sequence given by the concatenation of the Eulerian paths associated to $G_1,\dots,G_c$. This map $w : \mathscr{G}\left(\gamma^{(+)}\right) \to w\left(\mathscr{G}\left(\gamma^{(+)}\right)\right)$ is injective.

*Proof* We know by construction of any Toom graph $G_i$ that the Eulerian path will leave from and come back to the origin of $G_i$ exactly three times, because there are always three edges attached to the origin of a Toom graph. Given any element of the set $w\left(\mathscr{G}\left(\gamma^{(+)}\right)\right)$, its unique inverse image can then be deduced from the list of origins and the sequence of steps by the following method. Starting from the first origin recorded in the list, we add the steps of the sequence and redraw the first connected part of the graph, until the origin has been reached three times. Then we jump to the next origin recorded in the list and read on the sequence of steps until we again come back three times to this second departure point, and so on, until we have read the whole sequence. Finally we give back each travelled edge its color and orientation, which are both recorded in the sequence.

For all $c \in \mathbb{N}$ and all $m \in \mathbb{N}$, let $\mathscr{G}\left(\gamma^{(+)},c,m\right)$ be the subset of $\mathscr{G}\left(\gamma^{(+)}\right)$ consisting of the graphs with $c$ connected parts and $m$ edges exactly. Any element of $w\left(\mathscr{G}\left(\gamma^{(+)},c,m\right)\right)$ is made of a list with $c$ origins chosen among the elements of $\gamma^{(-)}$ and of a sequence with exactly $2m$ steps. Therefore Property (P1) is transferred from Toom graphs to graphs $G \in \mathscr{G}\left(\gamma^{(+)}, c, m\right)$:

$$\left|\mathscr{G}\left(\gamma^{(+)}, c, m\right)\right| = \left|w\left(\mathscr{G}\left(\gamma^{(+)}, c, m\right)\right)\right| \le \binom{\left|\gamma^{(-)}\right|}{c}.48^{2m} \le 2^{\left|\gamma^{(-)}\right|}.48^{2m}. \tag{19}$$

On the other hand, every connected part verifies Property (P2), that is to say, satisfies the analog of (14). Summing this inequality over all parts gives a similar property for $G$:

$$\frac{1}{4}\left|E_G\right| + c \le \left|\hat{V}_G\right|. \tag{20}$$

We also have

$$\left|\gamma^{(-)}\right| \le \left|E_G\right| + c. \tag{21}$$

*Proof* It is sufficient to construct an injective map $f : \gamma^{(-)} \to E_G \cup \{1, \ldots, c\}$. Let us consider any $(t, \gamma_t) \in \gamma^{(-)}$. If $(t, \gamma_t)$ belongs to a connected part $G_i$ of $G$ which is contained in $\{0, \ldots, t\} \times \mathbb{Z}^2$, then we know by construction of $G$ that $(t, \gamma_t)$ was the unique origin of $G_i$. We assign to $(t, \gamma_t)$ the image $f((t, \gamma_t)) = i \in \{1, \ldots, c\}$. Otherwise, we know that the Toom graph with origin $(t, \gamma_t)$ has been discarded. Then it intersected at least one of the retained Toom graphs with origin $(s, \gamma_s)$, $s > t$. Now the discarded graph was contained in $\{0, \ldots, t\} \times \mathbb{Z}^2$ and the time component of edges of Toom graphs has maximum absolute value 1. Therefore there exists an edge of the retained graph which arrives onto some point $(t, x)$, $x \in \mathbb{Z}^2$. Such an edge belongs to $E_G$ and we take it to be the image of $(t, \gamma_t)$ under $f$. The map $f$ thus defined is easily seen to be injective.

## 6 Exponential Convergence to Equilibrium

Inserting partition (16) in expansion (13) yields a sum over graphs. It introduces intricate couplings between the configurations at different times, due to the complexity of the Toom graph construction algorithm. Since we do not have the property of factorization over time, we use the upper bound (18) in the equivalent form $\mathbb{1}_{g^{-1}G}\left((\underline{\omega}_t)_{t=0}^n\right) = \mathbb{1}_{g^{-1}G}\left((\underline{\omega}_t)_{t=0}^n\right) \cdot \prod_{t=0}^n \mathbb{1}_{E(G,t)}(\underline{\omega}_t)$, as indicator functions can only take values 0 or 1. Now, for all $(\underline{\omega}_t)_{t=1}^n$, we have

$$\begin{aligned}&\left|\Pi_{\gamma_0}\mu\left[\tau^{(\gamma_1,\gamma_0)}(\underline{\omega}_1,\cdot)\,\mathbb{1}_{F(\gamma^{(+)},0)\cap E(G,0)}(\cdot)\,\mathbb{1}_{g^{-1}G}\left(\cdot\,;(\underline{\omega}_t)_{t=1}^n\right)\right]\right|\\ &\quad\le\left|\Pi_{\gamma_0}\mu\right|\left[\left|\tau^{(\gamma_1,\gamma_0)}(\underline{\omega}_1,\cdot)\right|\mathbb{1}_{F(\gamma^{(+)},0)\cap E(G,0)}(\cdot)\right],\end{aligned}$$

where $\left|\Pi_{\gamma_0}\mu\right|$ is not the norm, but the absolute value of the measure $\Pi_{\gamma_0}\mu$. Consequently, if we define the operator $\tilde{T}^{(y,x)}$ by replacing $\tau^{(y,x)}(\underline{\xi}, \underline{\omega})$ with its absolute value $\left|\tau^{(y,x)}(\underline{\xi},\underline{\omega})\right|$ in (10), the graph expansion gives

$$\left|\prod_{t=1}^{n}\left(\mathbb{1}_{F(\gamma^{(+)},t)}T^{(\gamma_t,\gamma_{t-1})}\right)\mathbb{1}_{F(\gamma^{(+)},0)}\Pi_{\gamma_0}\mu\right| \le \sum_{G\in\mathscr{G}\left(\gamma^{(+)}\right)}\left|\prod_{t=1}^{n}\left(\mathbb{1}_{F(\gamma^{(+)},t)\cap E(G,t)}\tilde{T}^{(\gamma_t,\gamma_{t-1})}\right)\mathbb{1}_{F(\gamma^{(+)},0)\cap E(G,0)}\left|\Pi_{\gamma_0}\mu\right|\right|. \tag{22}$$

This expansion will be the starting point of the proof of Theorem 1. Before, we need the following lemmas:

**Lemma 1** *For all $x\in\mathbb{Z}^2$, $\gamma\in P_x$, $\gamma^{(+)}\subseteq\gamma\setminus\{\gamma_0\}$ and $G\in\mathscr{G}\left(\gamma^{(+)}\right)$, all $\nu\in\mathscr{M}(X)$ and all $t\in\{1,\dots,n\}$,*

$$\left|\mathbb{1}_{F(\gamma^{(+)},t)\cap E(G,t)}\tilde{T}^{(\gamma_t,\gamma_{t-1})}\mathbb{1}_{F(\gamma^{(+)},t-1)\cap E(G,t-1)}\nu\right| \le \left|\mathbb{1}_{F(\gamma^{(+)},t-1)\cap E(G,t-1)}\nu\right|\ \varepsilon^{|\hat{V}_{G,t}|}\ \alpha^{|\gamma_t^{(+)}|}\ 2^{|\gamma_t^{(-)}|},$$

*where we introduced the notation $\gamma_t^{(+)}=\gamma^{(+)}\cap\{(t,x)|x\in\mathbb{Z}^2\}$ and the analogs $\gamma_t^{(-)}$ and $\hat{V}_{G,t}$.*

*Proof* Using definitions (10) and (17) of $T^{(\gamma_t,\gamma_{t-1})}$ and $\mathbb{1}_{E(G,t)}$, we already have

$$\begin{aligned}&\left|\mathbb{1}_{F(\gamma^{(+)},t)\cap E(G,t)}\tilde{T}^{(\gamma_t,\gamma_{t-1})}\mathbb{1}_{F(\gamma^{(+)},t-1)\cap E(G,t-1)}\nu\right|\\ &\quad\le\left|\mathbb{1}_{F(\gamma^{(+)},t-1)\cap E(G,t-1)}\nu\right|\cdot\\ &\qquad\cdot\sup_{\underline{\omega}\in F(\gamma^{(+)},t-1)\cap E(G,t-1)}\int \mathrm{d}\underline{\xi}\prod_{x:(t,x)\in V_G\setminus\gamma^{(-)}}\mathbb{1}_{-}(\xi_x)\left|\tau^{(\gamma_t,\gamma_{t-1})}(\underline{\xi},\underline{\omega})\right|.\end{aligned} \tag{23}$$

Now, keeping definition (11) of $\tau$ in mind, the contributions of the spins $\xi_x$ at different sites $x\in\mathbb{Z}^2$ to the integral in (23) are decoupled and factorize. We first consider the set $\{x\in\mathbb{Z}^2:(t,x)\in\hat{V}_G\setminus\gamma\}$. Because of Assumption (A1), its elements all contribute by a factor bounded by $\varepsilon$, since the supremum in (23) is taken over configurations $\underline{\omega}$ in $E(G,t-1)$. Everywhere else on $\mathbb{Z}^2\setminus\{\gamma_t\}$, the contribution is trivially bounded by 1.

For $x=\gamma_t$, we need upper bounds on $\left|p\left(\xi_{\gamma_t}|\underline{\omega}\right)-p\left(\xi_{\gamma_t}|\underline{\omega}_{\neq\gamma_{t-1}},a\right)\right|$:

- if $\phi_{\gamma_t}(\underline{\omega})=+1$,
  - and if $\xi_{\gamma_t}=-1$, $\left|p\left(\xi_{\gamma_t}|\underline{\omega}\right)-p\left(\xi_{\gamma_t}|\underline{\omega}_{\neq\gamma_{t-1}},a\right)\right|\le\alpha\ p\left(\xi_{\gamma_t}|\underline{\omega}\right)\le\alpha\ \varepsilon$;
  - and if $\xi_{\gamma_t}=+1$, $\left|p\left(\xi_{\gamma_t}|\underline{\omega}\right)-p\left(\xi_{\gamma_t}|\underline{\omega}_{\neq\gamma_{t-1}},a\right)\right|\le\alpha\ p\left(\xi_{\gamma_t}|\underline{\omega}\right)$,

  where we used Assumptions (A1) and (A2);
- if $\phi_{\gamma_t}(\underline{\omega})=-1$, we use the trivial bound $\left|p\left(\xi_{\gamma_t}|\underline{\omega}\right)-p\left(\xi_{\gamma_t}|\underline{\omega}_{\neq\gamma_{t-1}},a\right)\right|\le 1$.

Therefore we obtain the following three types of upper bounds on the contribution of the spin at $\gamma_t$ to the integral in (23): the contribution is bounded by $\alpha\ \varepsilon$ if $(t,\gamma_t) \in \gamma^{(+)} \cap \hat{V}_G$, by $\alpha$ if $(t,\gamma_t) \in \gamma^{(+)} \setminus \hat{V}_G$ or by 2 if $(t,\gamma_t) \in \gamma^{(-)}$. Inserting all these bounds in (23) proves Lemma 1.

**Lemma 2** *There exist $\alpha_* > 0$ and a positive and strictly decreasing function $\varepsilon_*$ defined on $[0,\alpha_*[$ such that for all $\alpha \in [0,\alpha_*[$ and $\varepsilon \in [0,\varepsilon_*(\alpha)[$, the Toom PCA satisfying Assumptions (A1) and (A2) with parameters $\varepsilon$ and $\alpha$ verify the following result. Let $K \geq 0$ and $\varepsilon' \in [0,\varepsilon_*(\alpha)[$. Let $\mu \in \mathscr{M}(X)$ be such that its absolute value $|\mu|$ is in the class $\mathscr{B}^{(+)}(K,\varepsilon')$. Then, there exist some constants $C < \infty$ and $\sigma < 1$ such that, for all $\varphi \in \mathscr{C}(X)$ and $n \in \mathbb{N}$,*

$$\sum_{x\in\mathbb{Z}^2}\sum_{\gamma\in P_x}\sum_{\gamma^{(+)}\subseteq\gamma\setminus\{\gamma_0\}}\left|\prod_{t=1}^{n}\left(\mathbb{1}_{F(\gamma^{(+)},t)}T^{(\gamma_t,\gamma_{t-1})}\right)\mathbb{1}_{F(\gamma^{(+)},0)}\Pi_{\gamma_0}\mu\right||\delta_x\varphi|_\infty \leq C\sigma^n\|\varphi\|.$$

*Proof* Multiple uses of Lemma 1 on the RHS of (22) imply

$$\begin{aligned}\left|\prod_{t=1}^{n}\left(\mathbb{1}_{F(\gamma^{(+)},t)\cap E(G,t)}\tilde{T}^{(\gamma_t,\gamma_{t-1})}\right)\mathbb{1}_{F(\gamma^{(+)},0)\cap E(G,0)}\left|\Pi_{\gamma_0}\mu\right|\right|&\\ \leq \varepsilon^{\sum_{t=1}^{n}|\hat{V}_{G,t}|}\ \alpha^{|\gamma^{(+)}|}\ 2^{|\gamma^{(-)}|}\left|\mathbb{1}_{F(\gamma^{(+)},0)\cap E(G,0)}\left|\Pi_{\gamma_0}\mu\right|\right|&.\end{aligned}$$

The important point is now that we assumed that $|\mu|$ belongs to $\mathscr{B}^{(+)}(K,\varepsilon')$. Indeed, since $\mathbb{1}_{E(G,0)} \leq \mathbb{1}_{-,\hat{V}_{G,0}}$ and $\mathbb{1}_{-,\hat{V}_{G,0}}(\underline{\omega}_{\succeq\gamma_0},\underline{a}_{\prec\gamma_0}) \leq \mathbb{1}_{-,\hat{V}_{G,0}}(\underline{\omega})$, and using the definition of $\Pi_{\gamma_0}$, we obtain

$$\left|\mathbb{1}_{F(\gamma^{(+)},0)\cap E(G,0)}\left|\Pi_{\gamma_0}\mu\right|\right| \leq 2K\varepsilon'^{|\hat{V}_{G,0}|}.$$

Consequently, with $\tilde{\varepsilon} = \max\{\varepsilon,\varepsilon'\}$,

$$\left|\prod_{t=1}^{n}\left(\mathbb{1}_{F(\gamma^{(+)},t)\cap E(G,t)}\tilde{T}^{(\gamma_t,\gamma_{t-1})}\right)\mathbb{1}_{F(\gamma^{(+)},0)\cap E(G,0)}\left|\Pi_{\gamma_0}\mu\right|\right| \leq 2K\ \tilde{\varepsilon}^{|\hat{V}_G|}\ \alpha^{|\gamma^{(+)}|}\ 2^{|\gamma^{(-)}|}. \quad (24)$$

Now for all $G \in \mathscr{G}\left(\gamma^{(+)}\right)$ with $c$ connected parts, (21) implies that the number of edges can be written as $|E_G| = \left|\gamma^{(-)}\right| - c + k$ with a certain $k \in \mathbb{N}$. Therefore $\mathscr{G}\left(\gamma^{(+)}\right) = \bigcup_{\substack{c\in\mathbb{N}\\k\in\mathbb{N}}}\mathscr{G}\left(\gamma^{(+)},c,\left|\gamma^{(-)}\right|-c+k\right)$ and, by virtue of the graph properties established above, we have

$$\begin{aligned}\sum_{G\in\mathscr{G}\left(\gamma^{(+)}\right)}\tilde{\varepsilon}^{|\hat{V}_G|} &\leq \sum_{c=0}^{\infty}\sum_{k=0}^{\infty}\left|\mathscr{G}\left(\gamma^{(+)},c,\left|\gamma^{(-)}\right|-c+k\right)\right|\tilde{\varepsilon}^{\frac{1}{4}(|\gamma^{(-)}|-c+k)+c} && \text{by (20)}\\ &\leq \sum_{c=0}^{\infty}\sum_{k=0}^{\infty}2^{\left|\gamma^{(-)}\right|}.48^{2(|\gamma^{(-)}|-c+k)}\tilde{\varepsilon}^{\frac{1}{4}(|\gamma^{(-)}|+3c+k)} && \text{by (19)}\\ &\leq \frac{1}{(1-48^2\tilde{\varepsilon}^{\frac{1}{4}})(1-48^{-2}\tilde{\varepsilon}^{\frac{3}{4}})}\left(2.48^2\tilde{\varepsilon}^{\frac{1}{4}}\right)^{|\gamma^{(-)}|}, && (25)\end{aligned}$$

provided that $\varepsilon$ and $\varepsilon'$ are such that $48^2\tilde{\varepsilon}^{\frac{1}{4}} < 1$.

Combining (22), (24) and (25), we find

$$\sum_{x\in\mathbb{Z}^2}\sum_{\gamma\in P_x}\sum_{\gamma^{(+)}\subseteq\gamma\setminus\{\gamma_0\}}\left|\prod_{t=1}^{n}\left(\mathbb{1}_{F(\gamma^{(+)},t)}T^{(\gamma_t,\gamma_{t-1})}\right)\mathbb{1}_{F(\gamma^{(+)},0)}\Pi_{\gamma_0}\mu\right|\left|\delta_x\varphi\right|_\infty$$
$$\leq C\sum_{x\in\mathbb{Z}^2}\sum_{\gamma\in P_x}\sum_{\gamma^{(+)}\subseteq\gamma\setminus\{\gamma_0\}}\alpha^{|\gamma^{(+)}|}\left(4.48^2\tilde{\varepsilon}^{\frac{1}{4}}\right)^{|\gamma^{(-)}|}\left|\delta_x\varphi\right|_\infty,$$

where $C=\frac{2K}{(1-48^2\tilde{\varepsilon}^{\frac{1}{4}})(1-48^{-2}\tilde{\varepsilon}^{\frac{3}{4}})}$. Here, we can use Newton's binomial formula: for any finite set $A$ and any $x$ and $y$ in $\mathbb{R}$, $\sum_{B\subseteq A}x^{|B|}y^{|A\setminus B|}=(x+y)^{|A|}$. Finally, since $|P_x|=3^n$ and keeping definition (2) in mind, we get the desired upper bound if we take $\sigma=3(\alpha+4.48^2\tilde{\varepsilon}^{\frac{1}{4}})$. For $\sigma$ to be lower than 1, the parameters must satisfy $\alpha<\alpha_*=1/3$ and $\tilde{\varepsilon}=\max\{\varepsilon,\varepsilon'\}<\varepsilon_*(\alpha)$ where the function $\varepsilon_*$ is strictly decreasing and positive on $[0,\alpha_*[$.

Using Lemmas 1 and 2 and the previous sections, we now prove Theorem 1.

*Proof (Theorem 1)* We take the constant $\alpha_*$ and the function $\varepsilon_*$ obtained in Lemma 2. Let us consider any Toom PCA with parameters $\alpha<\alpha_*$ and $\varepsilon<\varepsilon_*(\alpha)$. We write $T^n\mu(\varphi)-\mu_{\text{inv}}^{(+)}(\varphi)=T^n\bar{\mu}(\varphi)$ where $\bar{\mu}=\mu-\mu_{\text{inv}}^{(+)}\in\mathscr{M}(X)$ is a signed measure of zero mass. Therefore we can apply the above path expansion and pure phase expansion to $T^n\bar{\mu}$ which then satisfies (13). Using definition (4), $\bar{\mu}$ can be rewritten as $\bar{\mu}=\lim^*_{j\to\infty}\frac{1}{n_j}\sum_{k=0}^{n_j-1}\left(\mu-T^k\delta^{(+)}\right)$, so that $|T^n\bar{\mu}(\varphi)|$ is bounded above by

$$\liminf_{j\to\infty}\frac{1}{n_j}\sum_{k=0}^{n_j-1}\sum_{x\in\mathbb{Z}^2}\sum_{\gamma\in P_x}\sum_{\gamma^{(+)}\subseteq\gamma\setminus\{\gamma_0\}}\left[\left|\prod_{t=1}^{n}\left(\mathbb{1}_{F(\gamma^{(+)},t)}T^{(\gamma_t,\gamma_{t-1})}\right)\mathbb{1}_{F(\gamma^{(+)},0)}\Pi_{\gamma_0}\mu\right|\right.$$
$$\left.+\left|\prod_{t=1}^{n}\left(\mathbb{1}_{F(\gamma^{(+)},t)}T^{(\gamma_t,\gamma_{t-1})}\right)\mathbb{1}_{F(\gamma^{(+)},0)}\Pi_{\gamma_0}T^k\delta^{(+)}\right|\right]\left|\delta_x\varphi\right|_\infty. \qquad (26)$$

As the probability measure $\mu$ belongs to $\mathscr{B}^{(+)}(K,\varepsilon')$ with $\varepsilon'<\varepsilon_*(\alpha)$, Lemma 2 applies to the first part of (26):

$$\sum_{x\in\mathbb{Z}^2}\sum_{\gamma\in P_x}\sum_{\gamma^{(+)}\subseteq\gamma\setminus\{\gamma_0\}}\left|\prod_{t=1}^{n}\left(\mathbb{1}_{F(\gamma^{(+)},t)}T^{(\gamma_t,\gamma_{t-1})}\right)\mathbb{1}_{F(\gamma^{(+)},0)}\Pi_{\gamma_0}\mu\right|\left|\delta_x\varphi\right|_\infty\leq C\sigma^n\|\varphi\|, \quad (27)$$

where $C<\infty$ and $\sigma<1$ are given in Lemma 2.

As for the second part of (26), it can be bounded thanks to a slight modification of the same arguments. We don't know whether the Toom model has the property that $T^k\delta^{(+)}$ belongs to some $\mathscr{B}^{(+)}(K,\varepsilon')$. Therefore, we will now extend the Toom graphs up to time $-k$ instead of 0. For fixed $k\in\mathbb{N}$, $x\in\mathbb{Z}^2$, $\gamma\in P_x$ and $\gamma^{(+)}\subseteq\gamma\setminus\{\gamma_0\}$, we define

$$\mathscr{E}_k\left(\gamma^{(+)}\right)=\left\{(\underline{\omega}_t)_{t=-k}^{n}\in X^{\{-k,\dots,n\}}\mid\underline{\omega}_t\in F\left(\gamma^{(+)},t\right)\quad\forall t\in\{0,\dots,n\}\right\}.$$

A map $g_k$ is defined, similarly to the map $g$ above:

$$\begin{aligned} g_k : \mathscr{E}_k\left(\gamma^{(+)}\right) &\to \mathscr{G}_k\left(\gamma^{(+)}\right) = g_k\left(\mathscr{E}_k\left(\gamma^{(+)}\right)\right) \\ (\underline{\omega}_t)_{t=-k}^n &\mapsto G = g_k\left((\underline{\omega}_t)_{t=-k}^n\right). \end{aligned}$$

The only change in the graph construction algorithm takes place whenever a Toom graph $G_i$ under construction reaches a negative spin at time $t=0$. Instead of classing it into the set of identified errors $\hat{V}_{G_i}$, we carry on the construction of this branch of $G_i$, as for positive times, until we meet either an error or a negative spin at time $-k$, which is now considered the initial time, and class it in $\hat{V}_{G_i}$. At the end, we obtain a disconnected union of Toom graphs on $\{-k,,\dots,n\}\times\mathbb{Z}^2$, with origins $(t,\gamma_t)\in\gamma^{(-)}$ and with the properties described above. In particular, all the previous results about $\mathscr{G}\left(\gamma^{(+)}\right)$ still hold for $\mathscr{G}_k\left(\gamma^{(+)}\right)$ and for the associated subsets $\mathscr{G}_k\left(\gamma^{(+)},c,m\right)$ of graphs with $c$ connected parts and $m$ edges.

Extending definition (17) to graphs $G\in\mathscr{G}_k\left(\gamma^{(+)}\right)$ and to negative times, the analogs for $g_k$ of (16) and (18) lead to the following graph expansion for the second part of (26):

$$\begin{aligned} &\left|\prod_{t=1}^n\left(\mathbb{1}_{F(\gamma^{(+)},t)}T^{(\gamma_t,\gamma_{t-1})}\right)\mathbb{1}_{F(\gamma^{(+)},0)}\Pi_{\gamma_0}T^k\delta^{(+)}\right| \\ &\qquad\le\sum_{G\in\mathscr{G}_k\left(\gamma^{(+)}\right)}\left|\prod_{t=1}^n\left(\mathbb{1}_{F(\gamma^{(+)},t)\cap E(G,t)}\tilde{T}^{(\gamma_t,\gamma_{t-1})}\right)\right. \\ &\qquad\qquad\left.\mathbb{1}_{F(\gamma^{(+)},0)\cap E(G,0)}\Theta_{\gamma_0}\prod_{t=-k+1}^{-1}\left(\mathbb{1}_{E(G,t)}T\right)\mathbb{1}_{E(G,-k)}\delta^{(+)}\right|, \end{aligned} \tag{28}$$

where we defined the operator $\Theta_x$:

$$\Theta_x\mu(\varphi)=\int\mathrm{d}\underline{\omega}\int\mathrm{d}\underline{\xi}\ \varphi(\underline{\xi})\left|\left(\Pi_x\mathbb{P}[\,\cdot\,|\underline{\omega}]\right)\left(\underline{\xi}\right)\right|\mu(\underline{\omega}),$$

with $\Pi_x$ acting on the measure $\mathbb{P}[\,\cdot\,|\underline{\omega}]$.

Lemma 1 as well extends immediately to graphs $G\in\mathscr{G}_k\left(\gamma^{(+)}\right)$. Applying it $n$ times to the RHS of (28), we find that it is bounded above by

$$\sum_{G\in\mathscr{G}_k\left(\gamma^{(+)}\right)}\varepsilon^{\sum_{t=1}^n|\hat{V}_{G,t}|}\alpha^{|\gamma^{(+)}|}2^{|\gamma^{(-)}|}\left|\mathbb{1}_{F(\gamma^{(+)},0)\cap E(G,0)}\Theta_{\gamma_0}\prod_{t=-k+1}^{-1}\left(\mathbb{1}_{E(G,t)}T\right)\mathbb{1}_{E(G,-k)}\delta^{(+)}\right|.$$

Then we can use again Assumption (A1) $k$ times. Indeed, we know from definition (17) that $\mathbb{1}_{E(G,t)}\le\mathbb{1}_{-,\hat{V}_{G,t}}$ and from Assumption (A1) that $\left|\mathbb{1}_{-,\hat{V}_{G,t}}T\mathbb{1}_{E(G,t-1)}\nu\right|\le\varepsilon^{|\hat{V}_{G,t}|}\left|\mathbb{1}_{E(G,t-1)}\nu\right|$ for any probability measure $\nu$. The operator $\Theta_{\gamma_0}$ can be handled in the same way as $T$, taking its definition into account, together with the fact that $\mathbb{1}_{-,\hat{V}_{G,0}}(\underline{\omega}_{\succeq x},\underline{a}_{\prec x})\le\mathbb{1}_{-,\hat{V}_{G,0}}(\underline{\omega})$. It will simply introduce an extra factor of 2 due to the operator $\Pi_{\gamma_0}$. Lastly, by definition of $\delta^{(+)}$, $\delta^{(+)}\left(\mathbb{1}_{E(G,-k)}\right)=0$ unless $\hat{V}_{G,-k}$ is empty, in which case $\delta^{(+)}\left(\mathbb{1}_{E(G,-k)}\right)=1$. Consequently, the RHS of (28) is lower than

$$2\sum_{G\in\mathscr{G}_k\left(\gamma^{(+)}\right)}\varepsilon^{|\hat{V}_G|}\,\alpha^{|\gamma^{(+)}|}\,2^{|\gamma^{(-)}|}.$$

And so, performing the same calculations as for the proof of Lemma 2, we obtain, for the second part of (26),

$$\sum_{x\in\mathbb{Z}^2}\sum_{\gamma\in P_x}\sum_{\gamma^{(+)}\subseteq\gamma\setminus\{\gamma_0\}}\left|\prod_{t=1}^{n}\left(\mathbb{1}_{F(\gamma^{(+)},t)}T^{(\gamma_t,\gamma_{t-1})}\right)\mathbb{1}_{F(\gamma^{(+)},0)}\Pi_{\gamma_0}T^k\delta^{(+)}\right||\delta_x\varphi|_\infty \leq C_{\mathrm{inv}}\,\sigma^n\,\|\varphi\|\,, \tag{29}$$

where $C_{\mathrm{inv}}=\frac{2}{(1-48^2\varepsilon^{\frac{1}{4}})(1-48^{-2}\varepsilon^{\frac{3}{4}})}$.

Inserting (27) and (29) in (26), we conclude:

$$\left|T^n\mu(\varphi)-\mu_{\mathrm{inv}}^{(+)}(\varphi)\right|\leq C\,\sigma^n\,\|\varphi\|\,,$$

where we renamed $C+C_{\mathrm{inv}}$ to $C$ for simplicity. Since the Toom model is completely symmetric under the interchange of positive and negative spins, the similar result about $\mu_{\mathrm{inv}}^{(-)}$ and $\mathscr{B}^{(-)}(K,\varepsilon')$ is also true.

## 7 Exponential Decay of Correlations

We will now prove a well-known consequence of Theorem 1: the invariant measures $\mu_{\mathrm{inv}}^{(+)}$ and $\mu_{\mathrm{inv}}^{(-)}$ both present exponential decay of correlations in space.

*Proof (Corollary)* Since $\delta^{(+)}$ belongs to $\mathscr{B}^{(+)}(1,0)$, Theorem 1 implies that for some $\sigma<1$ and some $C<\infty$, we have, for all $k\in\mathbb{N}$,

$$\begin{cases}\left|T^k\delta^{(+)}(\varphi\psi)-\mu_{\mathrm{inv}}^{(+)}(\varphi\psi)\right|\leq C\,\sigma^k\,\|\varphi\psi\|\,;\\ \left|T^k\delta^{(+)}(\varphi)-\mu_{\mathrm{inv}}^{(+)}(\varphi)\right|\leq C\,\sigma^k\,\|\varphi\|\,;\\ \left|T^k\delta^{(+)}(\psi)-\mu_{\mathrm{inv}}^{(+)}(\psi)\right|\leq C\,\sigma^k\,\|\psi\|\,.\end{cases}$$

But $\delta^{(+)}$ is a product measure and the interactions are local, so

$$T^k\delta^{(+)}(\varphi\psi)=T^k\delta^{(+)}(\varphi)T^k\delta^{(+)}(\psi)$$

as long as $k<d(\varphi,\psi)$. Since $d(\varphi,\psi)>0$, we also have

$$\|\varphi\psi\|\leq\|\varphi\|\,|\psi|_\infty+|\varphi|_\infty\,\|\psi\|<\infty.$$

Hence, as long as $k<d(\varphi,\psi)$, there is some constant $C'<\infty$ that depends on $\varphi$ and $\psi$ such that

$$\left|\mu_{\mathrm{inv}}^{(+)}(\varphi\psi)-\mu_{\mathrm{inv}}^{(+)}(\varphi)\mu_{\mathrm{inv}}^{(+)}(\psi)\right|\leq C'\sigma^k.$$

If we take $k=d(\varphi,\psi)-1$ and choose $C_{\varphi,\psi}=\frac{C'}{\sigma}$, we obtain the desired inequality. By symmetry, $\mu_{\mathrm{inv}}^{(-)}$ possesses the same property.

The invariant measures $\mu_{\text{inv}}^{(+)}$ and $\mu_{\text{inv}}^{(-)}$ also exhibit exponential decay of correlations in time.

*Proof (Theorem 2)* Theorem 1 applied to $\delta^{(+)} \in \mathscr{B}^{(+)}(1,0)$ implies that, for all $k \in \mathbb{N}$,

$$\left|\mu_{\text{inv}}^{(+)}(\varphi T^n \psi) - T^k \delta^{(+)}(\varphi T^n \psi)\right| \leq C\, \sigma^k \, \|\varphi T^n \psi\| \,. \tag{30}$$

Now, remembering definition (2) of the semi-norm $\|.\|$, we notice that

$$\|\varphi T^n \psi\| \leq 2\, |\varphi|_\infty \, |\psi|_\infty \, |\text{supp}(\varphi T^n \psi)| \,.$$

But the conical structure of the space-time influence of spins implies

$$|\text{supp}(\varphi T^n \psi)| \leq |\text{supp}(\varphi)| + \big(\text{diam}(\text{supp}(\psi)) + n\big)^2,$$

so there exists a finite constant $C'_{\varphi,\psi}$ such that $\|\varphi T^n \psi\| \leq C'_{\varphi,\psi}\, n^2$ for all $n \in \mathbb{N}_0$.

Theorem 1 also yields

$$\left|\mu_{\text{inv}}^{(+)}(\varphi)\mu_{\text{inv}}^{(+)}(\psi) - \mu_{\text{inv}}^{(+)}(\varphi) T^{k+n}\delta^{(+)}(\psi)\right| \leq C\sigma^{k+n}\, |\varphi|_\infty \, \|\psi\| \,; \tag{31}$$

$$\left|\mu_{\text{inv}}^{(+)}(\varphi) T^{k+n}\delta^{(+)}(\psi) - T^k\delta^{(+)}(\varphi) T^{k+n}\delta^{(+)}(\psi)\right| \leq C\sigma^{k}\, |\psi|_\infty \, \|\varphi\| \,. \tag{32}$$

In order to find an upper bound for $\left|T^k \delta^{(+)}\big(\big(\varphi - T^k\delta^{(+)}(\varphi)\big)\, T^n \psi\big)\right|$, we rewrite it as $\left|T^n \mu_\varphi^{(k)}(\psi)\right|$, where the signed measure $\mu_\varphi^{(k)} \in \mathscr{M}(X)$ is defined by

$$\mu_\varphi^{(k)}(\psi) = T^k \delta^{(+)}\left(\left(\varphi - T^k \delta^{(+)}(\varphi)\right)\psi\right) \quad \forall \psi \in \mathscr{C}(X).$$

$\mu_\varphi^{(k)}$ is a measure of zero mass so it satisfies (13):

$$\left|T^n \mu_\varphi^{(k)}(\psi)\right| \leq \sum_{x\in\mathbb{Z}^2} \sum_{\gamma\in P_x} \sum_{\gamma^{(+)}\subseteq\gamma\setminus\{\gamma_0\}} \left|\prod_{t=1}^{n}\left(\mathbb{1}_{F(\gamma^{(+)},t)} T^{(\gamma_t,\gamma_{t-1})}\right)\mathbb{1}_{F(\gamma^{(+)},0)} \Pi_{\gamma_0}\mu_\varphi^{(k)}\right| |\delta_x \psi|_\infty \,.$$

The last expression is similar to the second term of (26) with $\mu_\varphi^{(k)}$ instead of $T^k\delta^{(+)}$ and we can then perform the same argument as in the proof of Theorem 1 in order to bound $\left|T^n \mu_\varphi^{(k)}(\psi)\right|$, extending again the collections of Toom graphs up to time $-k$. The only change in these calculations is the presence of an extra factor $\varphi - T^k\delta^{(+)}(\varphi)$ whose supremum norm is bounded by $2\,|\varphi|_\infty$. Provided we keep track of this factor, the calculations which lead to (29) still hold here:

$$\left|T^k\delta^{(+)}\left(\left(\varphi - T^k\delta^{(+)}(\varphi)\right) T^n\psi\right)\right| \leq 2\, C_{\text{inv}}\, \sigma^n \, |\varphi|_\infty \, \|\psi\| \,. \tag{33}$$

Combining (30), (31), (32) and (33) and taking the $k \to \infty$ limit ends the proof. The symmetric proof leads to the same result for $\mu_{\text{inv}}^{(-)}$.

Theorem 2 implies that $\mu_{\text{inv}}^{(+)}$ and $\mu_{\text{inv}}^{(-)}$ are not only ergodic but also mixing, that is, for any continuous functions $\varphi, \psi$, we have

$$\lim_{n\to\infty}\left|\mu_{\text{inv}}^{(+)}(\varphi T^n \psi) - \mu_{\text{inv}}^{(+)}(\varphi)\mu_{\text{inv}}^{(+)}(\psi)\right| = 0$$

and the same result holds for $\mu_{\text{inv}}^{(-)}$. Indeed, by definition of $\mathscr{C}(X)$, the set of continuous functions with finite support is dense in $\mathscr{C}(X)$.

**Acknowledgements** The authors would like to thank Jean Bricmont, Carlangelo Liverani and Christian Maes for helpful comments and discussions.
Augustin de Maere was partially supported by the Belgian IAP (Interuniversity Attraction Pole) program P6/02.
Lise Ponselet was supported by a grant from the Belgian F.R.S.-FNRS (Fonds de la Recherche Scientifique) as 'Aspirant FNRS'.

## References

1. Dobrushin, R.: Markov processes with a large number of locally interacting components: existence of a limit process and its ergodicity. Probl. Inf. Transm. **7**(2), 149–164 (1971)
2. Dobrushin, R., Kryukov, V., Toom, A. (eds.): Stochastic cellular systems: ergodicity, memory, morphogenesis. Manchester University Press (1990)
3. Keller, G., Liverani, C.: Uniqueness of the SRB measure for piecewise expanding weakly coupled map lattices in any dimension. Commun. Math. Phys. **262**(1), 33–50 (2006)
4. Lebowitz, J., Maes, C., Speer, E.: Statistical mechanics of probabilistic cellular automata. J. Stat. Phys. **59** (1990)
5. de Maere, A.: Phase transition and correlation decay in coupled map lattices. Commun. Math. Phys. **297**(1), 229–264 (2010). DOI 10.1007/s00220-010-1041-8
6. Toom, A.: Stable and attractive trajectories in multicomponent systems. Adv. Probab. Relat. Top. **6**, 549–575 (1980)