CORE DISCUSSION PAPER
2004/57

DYNAMIC OPTIMAL PORTFOLIO SELECTION IN A VaR FRAMEWORK *

E.W. Rengifo † **J.V.K.Rombouts** ‡

July 29, 2004

Abstract

We propose a dynamic portfolio selection model that maximizes expected returns subject to a Value-at-Risk constraint. The model allows for time varying skewness and kurtosis of portfolio distributions estimating the model parameters by weighted maximum likelihood in a increasing window setup. We determine the best daily investment recommendations in terms of percentage to borrow or lend and the optimal weights of the assets in the risky portfolio. Two empirical applications illustrate in an out-of-sample context which models are preferred from a statistical and economic point of view. We find that the APARCH(1,1) model outperforms the GARCH(1,1) model. A sensitivity analysis with respect to the distributional innovation hypothesis shows that in general the skewed-t is preferred to the normal and Student-t.

Keywords: Portfolio Selection; Value-at-Risk; Skewed-t distribution; Weighted Maximum Likelihood.

JEL Classification: C32, C35, G10

---

*The authors would like to thank Luc Bauwens, Sébastien Laurent, Bruce Lehmann and Samuel Mongrut for helpful discussions and suggestions. The usual disclaimers apply.

This text presents research results of the Belgian Program on Interuniversity Poles of Attraction initiated by the Belgian State, Prime Minister's Office, Science Policy Programming. The scientific responsibility is assumed by the authors.

†Corresponding author. Center of Operations Research and Econometrics, Catholic University of Louvain, 34 Voie du Roman Pays, 1348 Louvain-la-Neuve, Belgium, Telephone (32) 10 474358 e-mail:rengifo@core.ucl.ac.be.

‡Institut d'Economie Appliquée, HEC Montréal, Canada; CORE, Université catholique de Louvain, Louvain-la-Neuve, Belgium

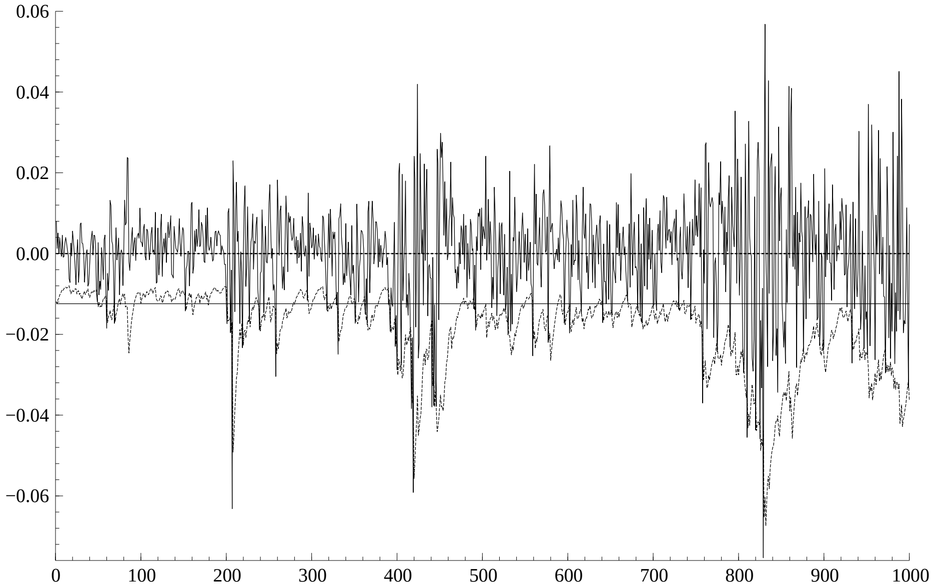
1 Introduction

One important venue of portfolio allocation research started with Markowitz (1952). According to the mean-variance model, investors maximize the expected return for a given risk level, where risk is measured by the variance. In this framework Fleming, Kirby, and Ostdiek (2001) study the economic value of volatility timing and de Roon, Nijman, and Werker (2003) show its usefulness in currency hedging for international stock portfolios. Recently, models have been proposed where the variance is replaced by another risk measure, the Value-at-Risk (VaR) being one of them. The VaR is defined as the maximum expected loss on an investment over a specified horizon given some confidence level, see Jorion (1997) for more information. Campbell, Huisman, and Koedijk (2001) propose a model which allocates financial assets by maximizing the expected return subject to the constraint that the expected maximum loss should meet the VaR limits. Their model is applied in a static context to find optimal weights between stocks and bonds for a past period. In this context the VaR is estimated by computing the quantiles from parametric distributions or by non parametric procedures such as empirical quantiles or smoothing techniques. See Gourieroux, Laurent, and Scaillet (2000) for an example of the latter techniques.

Contrary to many papers that evaluate statistically the accuracy of the VaR estimation for individual assets (see for example Mittnik and Paolella (2000), Giot and Laurent (2003) and Giot and Laurent (2004)), this paper proposes to generalize the work of Campbell, Huisman, and Koedijk (2001), CHK hereafter, to a flexible forward looking dynamic portfolio selection framework. A dynamic portfolio selection model which combines assets in order to maximize the portfolio expected return subject to a VaR risk constraint, allowing to give future investment recommendations. We determine, from both a statistical and economic point of view, the best daily investment recommendations in terms of percentage to borrow or lend and the optimal weights of the assets in the risky portfolio. For the estimation of the VaR we use ARCH-type models and we investigate the importance of several parametric innovation distributions.

Figure 1 shows the importance of estimating the 95% level-VaR dynamically in an out-of-sample, or forward looking, context using Russell2000 index return data (see Section 4.3 for more details), a GARCH(1,1) model and a skewed-t innovation distribution. In the dynamic case, the failure rate is 6.7% and in the constant case is 13.5%, i.e. in the last case the realized confidence level is more than twice the desired one. Thus, from a risk management point of view it could pay off to shift from the static to the dynamic framework.

Guermat and Harris (2002) working with three equity return series find more accurate VaR forecasts using a model that allows for time variation not only in the variance but also in the kurtosis of the return distribution. Jondeau and Rockinger (2003), investigating the time-series behavior of five stock indices and of six foreign exchange rates, find time dependence of the



Returns, constant and dynamic VaR of Russell2000, during the out-of-sample period. The dynamic VaRs are estimated using the GARCH(1,1) model with the skewed-t innovation distribution. The confidence level is 95%. The constant VaR is equal to −0.012. Out-of-sample period from 02/01/1997 until 20/12/2000 (1000 days).

Figure 1: **Returns, constant and dynamic VaR**

asymmetry parameter and generally a constant degrees of freedom parameter. Patton (2004) in the context of asset allocation studies the skewness in the distribution of individual stocks and the asymmetry in the dependence between stocks. Our approach, apart from time variation in the variance, also allows for an evolution of the skewness and kurtosis of the portfolio distributions. This is done by estimating the model parameters by Weighted Maximum Likelihood (WML) in a increasing window setup.

For two datasets, one consisting of indices and another of stocks, we perform out-of-sample forecasts applying our dynamic portfolio selection model to determine the daily optimal portfolio allocations. We work with two stock indices and two individual stocks and not with bonds indices in order to capture the asymmetric dependence documented only for stock returns, see Patton (2004). The dynamic model we propose outperforms the CHK model in terms of failure rates, defined as the number of times the desired confidence level used for the estimation of the VaR is violated. Based on this statistical criterion, the APARCH model gives as good results as the GARCH model. However, if we consider not only the failure rate but also an economic criterion like the achieved wealth, we find that for similar levels of risk, the APARCH model outperforms the GARCH model. A sensitivity analysis with respect to the distribution innovation shows that the skewed-t is preferred to the normal and Student-t.

The plan of the paper is as follows: In Section 2 we present the optimal portfolio selection in a VaR framework. In Section 3, we describe different model specifications for the estimation of the VaR. Section 4 presents two empirical applications using out-of-sample forecasts to determine the optimal investment strategies. We use portfolios formed by either two US indices (SP500-RUSELL2000) or by two stocks (Colgate-IBM). We compare the performance of the different models using the failure rates and the wealth achieved as instruments to determine the best model. Section 5 evaluates several related aspects of the models and Section 6 concludes and provides an outlook for future research.

2 Optimal portfolio selection

This section follows Campbell, Huisman, and Koedijk (2001). The portfolio model allocates financial assets by maximizing the expected return subject to a risk constraint, where risk is measured by a Value-at-Risk (VaR). The optimal portfolio is such that the maximum expected loss should not exceed the VaR for a chosen investment horizon at a given confidence level α. We consider the possibility of borrowing and lending at the market interest rate, considered as given.

Define W_t as the investor's wealth at time t, b_t the amount of money that can be borrowed ($b_t > 0$) or lent ($b_t < 0$) at the risk free rate r_f. Consider n financial assets with prices at time t given by $p_{i,t}$, with $i = 1, \ldots, n$. Define $X_t \equiv [x_t \in R^n : \sum_{i=1}^{n} x_{i,t} = 1]$ as the set of portfolios weights at time t, with well-defined expected rates of return, such that $w_{i,t} = x_{i,t}(W_t + b_t)/p_{i,t}$ is the number of shares of asset i at time t. The budget constraint of the investor is given by:

$$W_t + b_t = \sum_{i=1}^{n} w_{i,t} p_{i,t} = w_t' p_t. \tag{1}$$

The value of the portfolio at $t+1$ is:

$$W_{t+1}(w_t) = (W_t + b_t)(1 + R_{t+1}(w_t)) - b_t(1 + r_f), \tag{2}$$

where $R_{t+1}(w_t)$ is the portfolio return at maturity. The VaR of the portfolio is defined as the maximum expected loss over a given investment horizon and for a given confidence level α:

$$P_t[W_{t+1}(w_t) \leq W_t - VaR^*] \leq 1 - \alpha, \tag{3}$$

where P_t is the probability conditioned on the available information at time t and VaR^* is the cutoff return or the investor's desired VaR level. Note that $(1-\alpha)$ is the probability of occurrence. Equation (3) represents the second constraint that the investor has to take into account. The portfolio optimization problem can be expressed in terms of the maximization of the expected returns $E_t W_{t+1}(w_t)$, subject to the budget restriction and the VaR-constraint:

$$w_t^* \equiv \arg\max_{w_t} \, (W_t + b_t)(1 + E_t R_{t+1}(w_t)) - b_t(1 + r_f), \tag{4}$$

s.t. (1) and (3). $E_t R_{t+1}(w_t)$ represents the expected return of the portfolio given the information at time t. The optimization problem may be rewritten in an unconstrained way. To do so, replacing (1) in (2) and taking expectations yields:

$$E_t W_{t+1}(w_t) = w_t' p_t (E_t R_{t+1}(w_t) - r_f) + W_t(1 + r_f). \tag{5}$$

Equation (5) shows that a risk-averse investor wants to invest a fraction of his wealth in risky assets if the expected return of the portfolio is bigger than the risk free rate. Substituting (5) in (3) gives:

$$P_t[w_t' p_t (R_{t+1}(w_t) - r_f) + W_t(1 + r_f) \leq W_t - VaR^*] \leq 1 - \alpha, \tag{6}$$

so that,

$$P_t \left[R_{t+1}(w_t) \leq r_f - \frac{VaR^* + W_t r_f}{w_t' p_t} \right] \leq 1 - \alpha, \tag{7}$$

defines the quantile $q(w_t, \alpha)$ of the distribution of the return of the portfolio at a given confidence level α or probability of occurrence of $(1 - \alpha)$. Then, the portfolio can be expressed as:

$$w_t' p_t = \frac{VaR^* + W_t r_f}{r_f - q(w_t, \alpha)}. \tag{8}$$

Finally, substituting (8) in (5) and dividing by the initial wealth W_t we obtain:

$$\frac{E_t(W_{t+1}(w_t))}{W_t} = \frac{VaR^* + W_t r_f}{W_t r_f - W_t q(w_t, \alpha)} (E_t R_{t+1}(w_t) - r_f) + W_t(1 + r_f), \tag{9}$$

and therefore,

$$w_t^* \equiv \arg\max_{w_t} \, \frac{E_t R_{t+1}(w_t) - r_f}{W_t r_f - W_t q(w_t, \alpha)}. \tag{10}$$

The two fund separation theorem applies, i.e. the investor's initial wealth and desired $VaR = W_t q(w_t, \alpha)$ do not affect the maximization procedure. As in traditional portfolio theory, investors first allocate the risky assets and second the amount of borrowing and lending. The latter reflects by how much the VaR of the portfolio differs according to the investors' degree of risk aversion measured by the selected VaR level. The amount of money that the investor wants to borrow or lend is found by replacing (1) in (8):

$$b_t = \frac{VaR^* + W_t q(w_t^*, \alpha)}{r_f - q(w_t^*, \alpha)}. \tag{11}$$

In order to solve the optimization problem (10) over a large investment horizon T, we partition this in one-period optimizations, i.e. if T equals 30 days, we optimize 30 times one-day periods to achieve the desired final horizon.

We illustrate the framework by a simple hypothetical example with $n = 2$, an initial investor's wealth of US$ 10000 and an annual risk free rate equal to 1.24%. We also assume any non-normal innovation distribution. The hypothetical values were selected to show a fact noted by Campbell, Huisman, and Koedijk (2001): the portfolio VaR in absolute value increases when the confidence level increases. However, the portfolio weights are non-monotonic functions of the confidence level, unless the normal distribution is used. Table 1 presents these hypothetical results.

Table 1: **Optimal portfolio selection under VaR.**

α(%)	Asset1(%)	Asset2(%)	$Portfolio\ VaR$($)
90	30	70	-5.0
94	35	65	-5.6
95	40	60	-6.5
97	30	70	-7.5
99	25	75	-8.5

Next, we determine the amount of money to borrow or lend. First, assume that the desired VaR^* is equal to 6.5 (that corresponds to the 95% confidence level) and that we have two kinds of investors. One that is less risk averse (Investor 1) and chooses a confidence level of 90% and the other that is more risk averse (Investor 2) and chooses a confidence level of 99%. Table 2 presents the decisions based on their particular types.

Table 2: **Investment decision of different type of investors.**

Type of Investor	b(%)	Asset1(%)	Asset2(%)	Tot-portfolio
Less risk averse	28.08	38.42	89.66	128.08
More risk averse	-22.62	19.35	58.03	77.38

We observe that Investor 1 borrows ($b > 0$) at the risk-free rate an amount equivalent to 28.08% of his initial wealth investing everything (128.08%) in the portfolio made of the two assets. Investor 2 prefers to lend ($b < 0$) 22.62% of his wealth at the risk-free rate, investing the difference (77.38%) in the risky portfolio.

3 Methodology

We observe the following steps in the estimation of the optimal portfolio allocation and its evaluation:

1. Estimation of portfolio returns:

   A typical model of the portfolio return R_t may be written as follows:

   $$R_t = \mu_t + \epsilon_t, \tag{12}$$

   where μ_t is the conditional mean and ϵ_t an error term. As mentioned for example by Merton (1980) and Fleming, Kirby, and Ostdiek (2001), predicting returns is more difficult than predicting of variances and covariances.

   In the empirical application we predict the expected return by the unconditional mean using observations until day $t-1$. We also modelled the expected return by autoregressive processes, but the results were not satisfactory, either in terms of failure rates or in terms of wealth evolution.

2. Estimation of the conditional variance:

   The error term ϵ_t in equation (12) can be decomposed as $\sigma_t z_t$ where z_t is an IID innovation with mean zero and variance 1. We distinguish three different specifications for the conditional variance σ_t^2:

   - The CHK model, similar to the model presented in Section 2, where σ_t^2 is estimated as the empirical variance using data until $t-1$. In fact, this can be interpreted as a straightforward dynamic extension of the application presented in Campbell, Huisman, and Koedijk (2001).
   - The GARCH(1,1) model of Bollerslev (1986), where

     $$\sigma_t^2 = \omega + \alpha\epsilon_{t-1}^2 + \beta\sigma_{t-1}^2.$$

   - The APARCH(1,1) model of Ding, Granger, and Engle (1993), where

     $$\sigma_t^\delta = \omega + \alpha_1(|\epsilon_{t-1}| - \alpha_n\epsilon_{t-1})^\delta + \beta_1\sigma_{t-1}^\delta.$$

     with ω, α_1, α_n, β_1 and δ parameters to be estimated. The parameter δ ($\delta > 0$) is the Box-Cox transformation of σ_t. The parameter α_n ($-1 < \alpha_n < 1$), reflects the leverage effect such that a positive (negative) value means that the past negative (positive) shocks have a deeper impact on current conditional volatility than the past positive shocks of the same magnitude. Note that if $\delta = 2$ and $\alpha_n = 0$ we get the GARCH(1,1) model.

With respect to the innovation distribution, several parametric alternatives are available in the literature. In the empirical application, see Section 4, we consider the normal, Student-t and skewed-t distributions. The skewed-t distribution was proposed by Hansen (1994) and reparameterized in terms of the mean and the variance by Lambert and Laurent (2001) in such a way that the innovation process has zero mean and unit variance. The skewed-t distribution depends on two parameters, one for the thickness of tails (degrees of freedom) and the other for to the skewness.

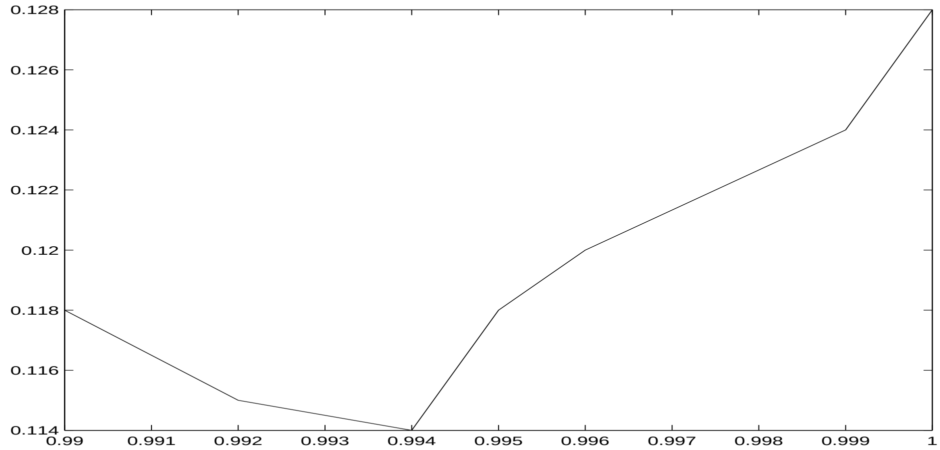
Following Mittnik and Paolella (2000) the parameters of the models are estimated by Weighted Maximum Likelihood (WML). We use weights which multiply the log-likelihood contributions of the returns in period t, $t = 1, \ldots, T$. This allows to give more weight to recent data in order to obtain parameter estimates that reflect the "current" value of the "true" parameter. The weights are defined by:

$$\omega_t = \rho^{T-t}. \tag{13}$$

If $\rho < 1$ more weight is given to recent observations than those far in the past. The case $\rho = 1$ corresponds to usual maximum likelihood estimation. The decay factor ρ is obtained by minimizing the failure rate (defined later in this section) for a given confidence level. Figure 2 illustrates the failure rate-ρ relationship for portfolios made of Russell2000 and SP500 indices for an investor using VaR at 90% level. The model used is the GARCH(1,1) with normal innovation distribution. The optimal ρ that minimizes the failure rate is equal to 0.994. We find similar results for other cases. Moreover, the value of the optimal ρ is robust to different innovation distributions. We use WML in an increasing window setup, i.e. the number of observations of the sample increases through time in order to consider the new information available. The improvement, in terms of better approximation to the desired confidence levels, using WML in an increasing window setup instead of ML is of the order of 10%. See also Section 5 for more details.

By using WML in a increasing window setup, $q_{1-\alpha}$ in (14) takes into account the time evolution of the degrees of freedom and asymmetry parameters when we use the skewed-t distribution. We do not specify an autoregressive structure for the degrees of freedom and for the asymmetry parameter like Jondeau and Rockinger (2003). They find that this approach is subject to numerical instabilities.

3. Estimation of the VaR:



Failure rates (vertical axis) obtained with different ρ values (horizontal axis) using the geometric weighting scheme for a 1000 out-of-sample period. Portfolios made of Russell2000 and SP500 indices for an investor with VaR-90. The model used is the GARCH with normal innovation distribution Out-of-sample period from 02/01/1997 until 20/12/2000 (1000 days).

Figure 2: **Failure rates-ρ relationship**

The VaR is a quantile of the distribution of the return of a portfolio, see Equations (3) and (7). In an unconditional setup the VaR of a portfolio may be estimated by the quantile of the empirical distribution at a given confidence level α. In parametric models, such as the ones we are using, the quantiles are functions of the variance of the portfolio return R_t. The $VaR_{t,\alpha}$ (VaR for time t at the confidence level α) is calculated as:

$$VaR_{t,\alpha} = \hat{\mu}_t + \hat{\sigma}_t q_{1-\alpha}, \tag{14}$$

where $\hat{\mu}_t$ and $\hat{\sigma}_t$ are the forecasted conditional mean and variance using data until $t-1$ and, $q_{1-\alpha}$ is the $(1-\alpha)$-th quantile of the innovation distribution.

4. Determine the optimal risky portfolio allocation:

   Once we have determined the VaR for each of the risky portfolios, we use equation (10) to find the optimal portfolio weights. These weights correspond to the portfolio that maximizes the expected returns subject to the VaR constraint.

5. Determine the optimal amount to borrow or lend:

   As shown in section 2, the two fund separation theorem applies. Then, in order to determine the amount of money to borrow or lend we simply use equation (11).

6. Evaluate the models:

   A criterion to evaluate the models is the failure rate:

$$f = \frac{1}{n} \sum_{t=T-n+1}^{T} \mathbf{1}[R_t < -VaR_{t-1,\alpha}], \tag{15}$$

where, n is the number of out-of-sample days, T is the total number of observations, R_t is the observed return at day t and $VaR_{t-1,\alpha}$ is the threshold value determined at time $t-1$. A model is correctly specified if, the observed return is bigger than the threshold values in 100α percent of the predictions.

We also evaluate the models by analyzing the wealth evolution generated by the application of the portfolio recommendations of the different models. With this economic criterion, the best model will be the one that reports the highest wealth for similar risk levels.

4 Empirical application

We develop two applications of the model presented in the previous sections. We construct 1000 daily out-of-sample portfolio allocations based on conditional variance predictions of GARCH and APARCH models and compare the results with the ones obtained with the CHK model. The parameters are estimated using WML in a rolling window setup. Moreover, we use the normal, Stutent-t and skewed-t distributions to investigate the importance of the choice of several innovation densities for different confidence levels. Each of the three models can be combined with the three innovation distributions resulting in nine different specifications. In the applications we consider an agent's problem of allocating his wealth (set to 1000 US dollars) among two different American indices and two stocks, Russell2000-SP500 and Colgate-IBM respectively. For the riskfree rate we use the one-year Treasury bill rate in January 1998 (approximately 4.47% annual). We have considered only the trading days in which both indices or stocks where traded. We define the daily returns as log price differences from the adjusted closing price series .

With the information until time t, the models forecast one-day ahead the percentage of the cumulated wealth that should be borrowed ($b_t > 0$) or lent ($b_t < 0$) according to the agent's risk aversion expressed by his confidence level α, and the percentage that should be invested in the portfolio made of the two indices or the two stocks. The models give the optimal weights of each of the indices or stocks in the optimal risky portfolio. Then, with the investment recommendations of the previous day, we use the real returns and determine the agent's wealth evolution according to each model suggestions. Since the parameters of the GARCH and APARCH models change slowly from one day to another, these parameters are re-estimated every 10 days to take into account the expanding information and to keep the computation time low. We also re-estimate the parameters daily, every 5, 15 and 20 days (results not shown). We find similar results in terms of the parameter estimates. However, in the case of daily and 5-day re-estimation, the computational time was about 10-times bigger.

For the estimation of the programs we use a Pentium Xeon 2.6 Ghz. The time required for the GARCH and APARCH models is 90 and 120 minutes on average, respectively. Estimating the models with a fixed window requires 60 and 90 minutes on average to run the GARCH and APARCH models respectively.

In the next section we present the statistical characteristics of the data. Then, we present generally how the models work only for two specific examples due to space limitations. Finally, we present the key results for all the models in terms of failure rates and total achieved wealth and stress their differences.

4.1 Description of the data

4.1.1 SP500 - Russell2000

We use daily data of the SP500 composite index (large stocks) and the Russell2000 index (small stocks). The sample period goes from 02/01/1990 to 20/12/2000 (2770 observations). Descriptive statistics are given in the left panel of Table 3. We see that for all indices skewness and excess kurtosis is present and that the means and standard deviations are similar. Figure 3 presents the daily returns during the out-of-sample period for both indices.

Note that our one-day ahead forecast horizon is four years (more or less 1000 days). During this period we observe mainly a bull market, except for the last days, when the indices start a sharp fall. The lower panel of Table 3 presents the descriptive statistics corresponding to the out-of-sample period. Note that the volatility in this period is higher than the previous one.

4.1.2 Colgate - IBM

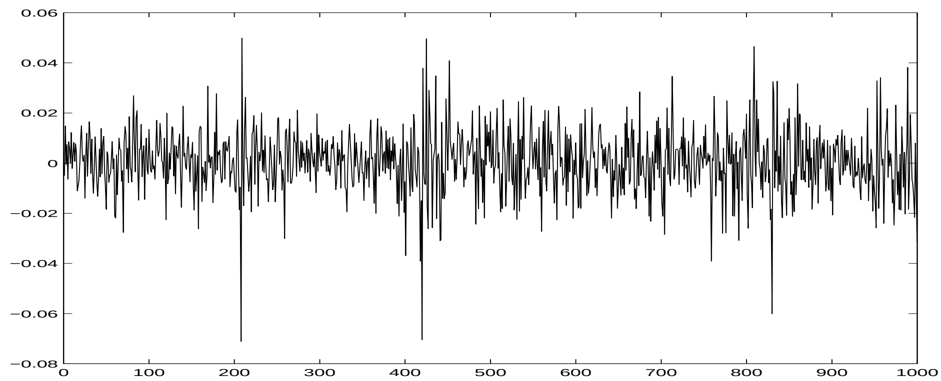
The daily sample period for these two stocks goes from 10/01/1990 to 31/12/2000 (2,870 observations). Descriptive statistics are given in the right panel of Table 3. Both series present skewness and excess kurtosis. However, Colgate is positively skewed meanwhile IBM is negatively skewed. The excess of kurtosis is higher than in the indices case due to the presence of more extreme returns (either positive or negative), which is a common finding when stocks are used instead of indices. The mean of the Colgate returns is higher than the mean of the IBM returns and interestingly, the standard deviation of Colgate is also smaller. In Figure 4 we present the daily returns during the out-of-sample period for both assets.

As observed in the case of the indices, during the forecast period we observe mainly a bull market, except for the last days, where the stock prices start to fall. The right panel of Table 3 also presents the descriptive statistics of the out-of-sample period. As noted in the previous case, the volatility in this out-of-sample period is higher than the previous period.

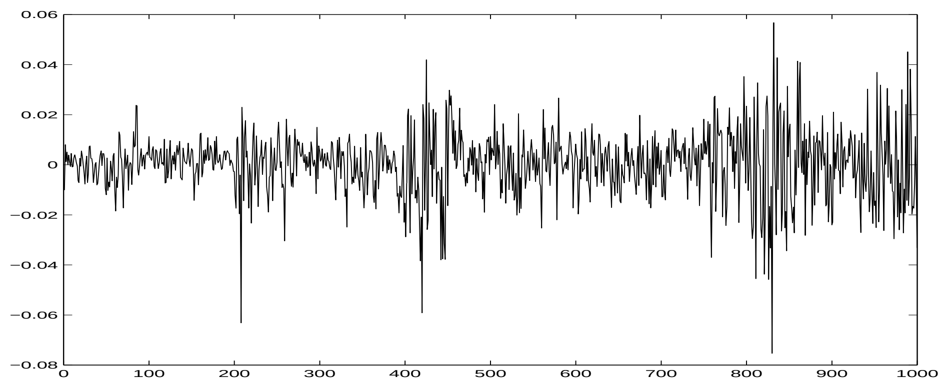
Table 3: Descriptive statistics

	02/01/1990 - 20/12/2000 N=2770		10/01/1990 - 31/12/2000 N=2870	
	SP500	**Russell2000**	**Colgate**	**IBM**
Mean	0.045	0.035	0.073	0.045
Standard deviation	0.946	0.937	1.730	2.012
Skewness	-0.293	-0.642	0.012	-0.101
Kurtosis	7.741	9.084	13.108	10.203
Minimum	-7.114	-7.533	-17.329	-16.889
Maximum	4.990	5.678	18.499	12.364
	02/01/1997 - 20/12/2000 N=1000		10/01/1997 - 31/12/2000 N=1000	
	SP500	**Russell2000**	**Colgate**	**IBM**
Mean	0.053	0.020	0.090	0.085
Standard deviation	1.247	1.279	2.311	2.481
Skewness	-0.306	-0.454	0.035	-0.317
Kurtosis	6.059	6.308	10.915	8.648
Minimum	-7.114	-7.533	-17.329	-16.889
Maximum	4.990	5.678	18.499	12.364

Descriptive statistics for the daily returns of the corresponding indices (left panel) and stocks (right panel). The mean, the standard deviation, the minimum and maximum values are expressed in %.

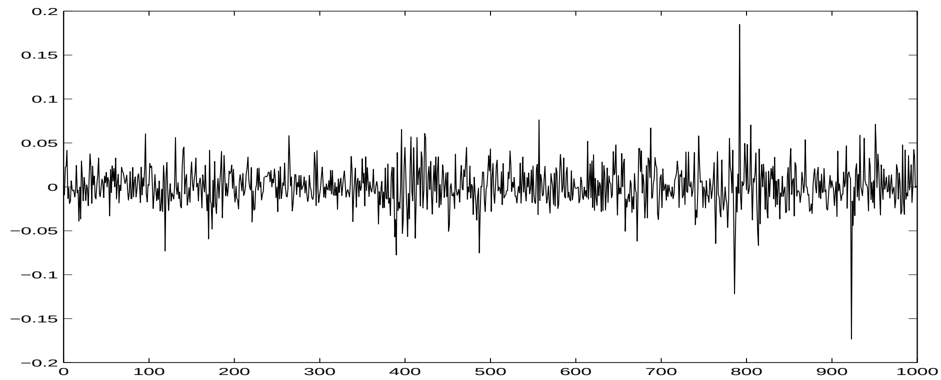


(a) SP500 daily returns. Out-of-sample period from 02/01/1997 until 20/12/2000 (1000 days)

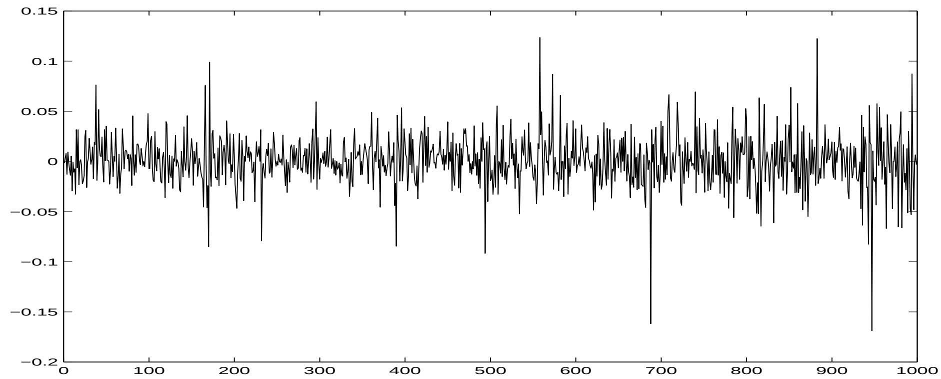


(b) Russell2000 daily returns. Out-of-sample period from 02/01/1997 until 20/12/2000 (1000 days)

Figure 3: **SP500 and Russell2000 out-of-sample returns**



(a) Colgate daily returns. Out-of-sample period from 10/01/1997 until 31/12/2000 (1000 days)



(b) IBM daily returns. Out-of-sample period from 10/01/1997 until 31/12/2000 (1000 days)

Figure 4: **Colgate and IBM out-of-sample returns**

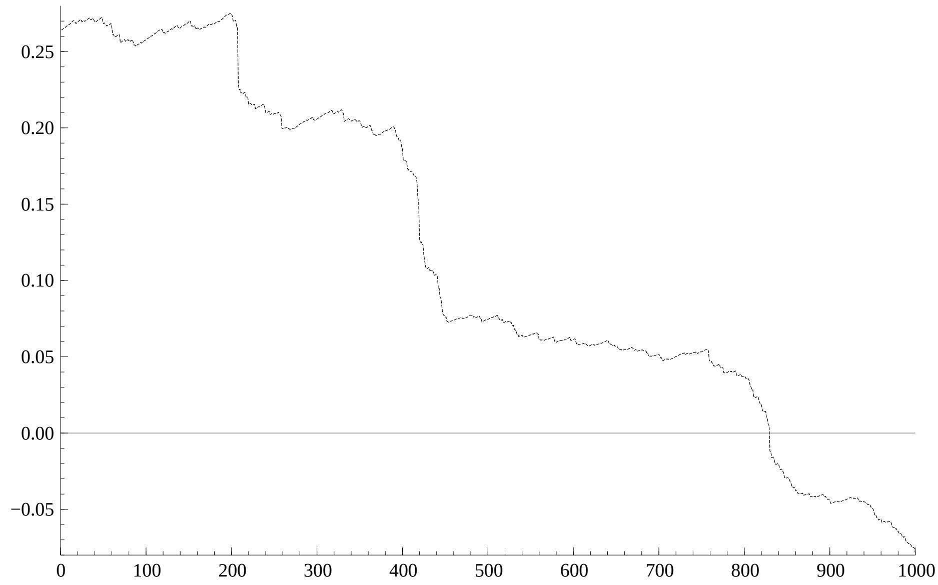
4.2 A general view of the daily recommendations

We present two examples of model configurations to illustrate the main results. For all the cases the investor's desired VaR_t^* is set to 1% of his cumulated wealth at time $t-1$. First, we explain the investment decisions based on the CHK model using the normal distribution for portfolios made of Russell2000-SP500. The agent desired VaR confidence level is $\alpha = 90\%$, i.e. a less risk-averse investor. Figure 5 shows the evolution of the percentage of the total wealth to be borrowed ($b_t > 0$) or lent ($b_t < 0$). In this case the model suggests until day 829 to borrow at the risk-free rate and to invest everything in the risky portfolio. However, after that day the model recommendation is to change from borrowing to lending. This is a natural response to the negative change in the trend of the indices and to the higher volatility observed in the stock market during the last days of the out-of-sample period (Figure 3). Figure 6 presents the evolution of the share of the risky portfolio to be invested in the Russell2000 index. The model suggests for 807 days to invest 70% of the wealth (on average) in Russell2000 index and the difference in SP500 index. After that day, the model recommendations change drastically favoring the investment in SP500, which increases its portfolio weights to 66%, i.e. going from 30% to almost 50% at the end of the out-of-sample period. Again, this responds to the higher volatility of the Russell2000 compared with the SP500 during the last days. Thus, the model recommend to shift from the more risky index to the less risky one and from the risky portfolio to the risk free investment.

Figure 7 compares the wealth evolution obtained by applying the CHK model suggestions with investments made in either one or the other index. The wealth evolution is higher than the one that could be obtained by investing only in Russell2000 but lower if investing only in SP500 during the out-of-sample forecast period. We also include the wealth evolution that an agent can realize when investing everything at the risk-free rate (assumed constant during the whole forecasted period). More details can be found in Section 4.3.

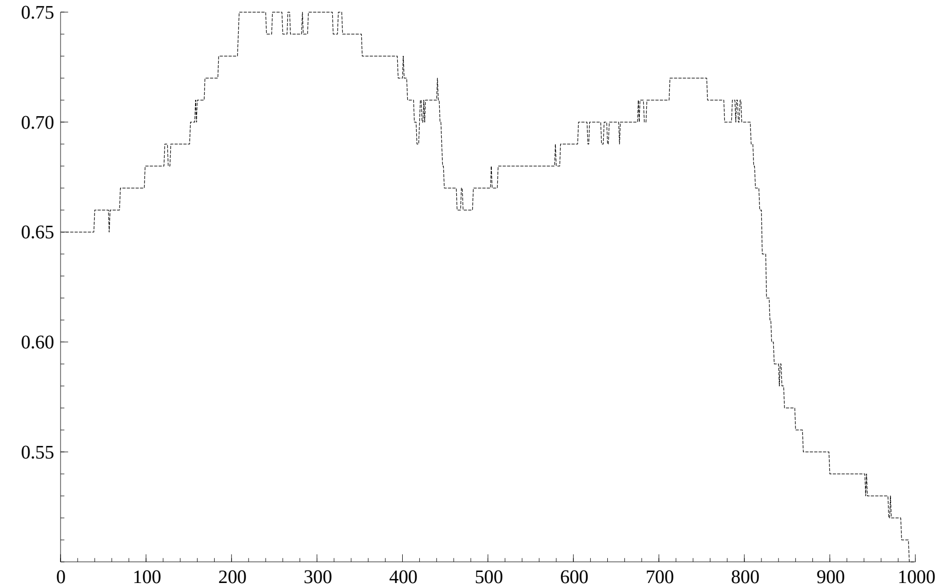
As a second example we present the results of applying our dynamic optimal portfolio selection model to the Colgate-IBM data for which the conditional variance is estimated using the APARCH model. The agent's desired VaR confidence level is $\alpha = 99\%$, i.e. a risk-averse investor and the distribution is the skewed-t distribution. In Figure 8 we observe how the model accommodates its recommendations to higher risk aversion. The model suggests during the whole forecasted period to lend a big proportion of the wealth at the risk free rate (70% on average) which comes as no surprise given his desired confidence level. Figure 9 shows the model recommendations with respect to the weight invested in Colgate. It varies considerably, showing how the model adjusts its suggestions in order to maximize the expected return subject to the VaR constraint.

Figure 10 presents the wealth evolution obtained by applying the model suggestions and compares it with investments in either one or the other stock. An agent that desires a 99% VaR



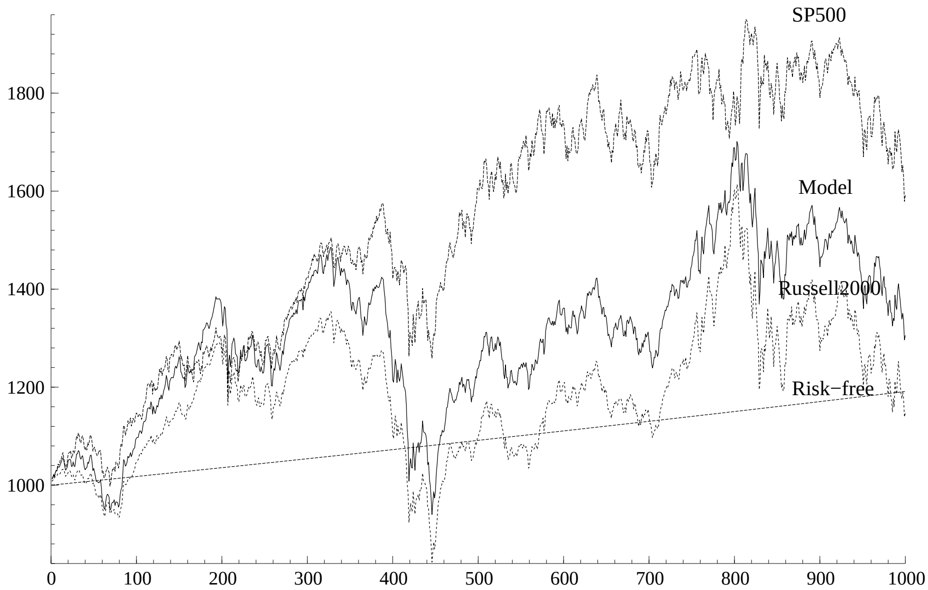
Riskfree weights for portfolios made of Russell2000 and SP500 indices for an investor with VaR-90, based on the CHK model using the normal distribution. Out-of-sample period from 02/01/1997 until 20/12/2000 (1000 days).

Figure 5: **Riskfree weights using CHK model with normal distribution**



Risky weights of Russell2000 for an investor with VaR-90, based on the CHK model using the normal distribution. Out-of-sample period from 02/01/1997 until 20/12/2000 (1000 days).

Figure 6: **Risky weights on Russell2000 using CHK model with normal distribution**



Portfolios made of Russell2000 and SP500 indices for an investor with VaR-90. Wealth evolution for 1000 out-of-sample forecast using the model recommendations (Model) compared with the wealth evolution obtained by investments made on Russell2000 or SP500 alone and with investments done at the risk-free rate. Out-of-sample period from 02/01/1997 until 20/12/2000.

Figure 7: **Wealth evolution using CHK model**

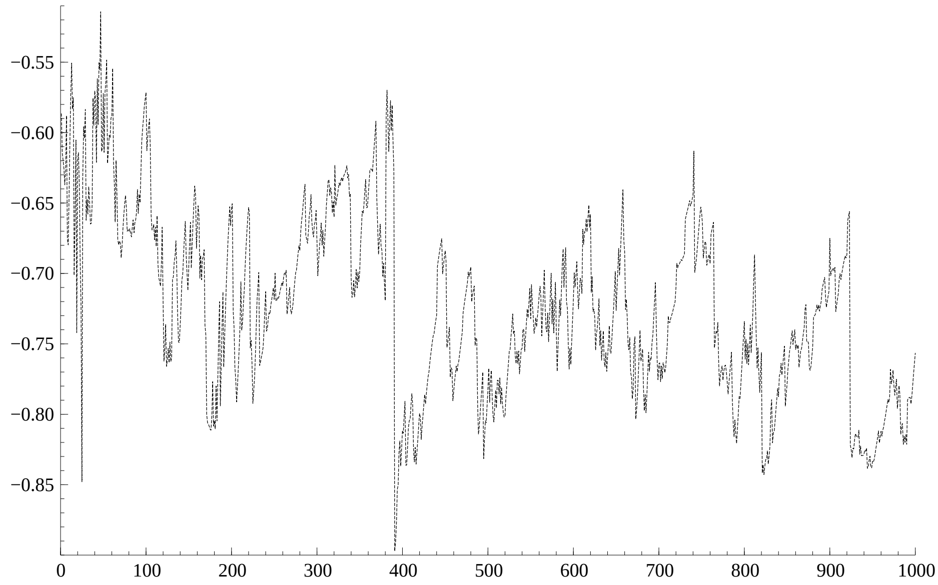
confidence level is a highly risk-averse investor. As a result, the investment decisions are very conservative, since his risk constraint is tight. Even though the returns are lower than the ones obtained by investing in either one of the two stocks, it is higher (during the whole period) than the investment at the risk-free rate.[1]

The two previous illustrations show how the model recommendations change according to new information coming to the market, allowing the agent to maximize expected return subject to budget and risk constraints in a dynamic way. The next section presents more synthetically the comparison of all models for different distributional assumptions and different confidence levels.

4.3 Results

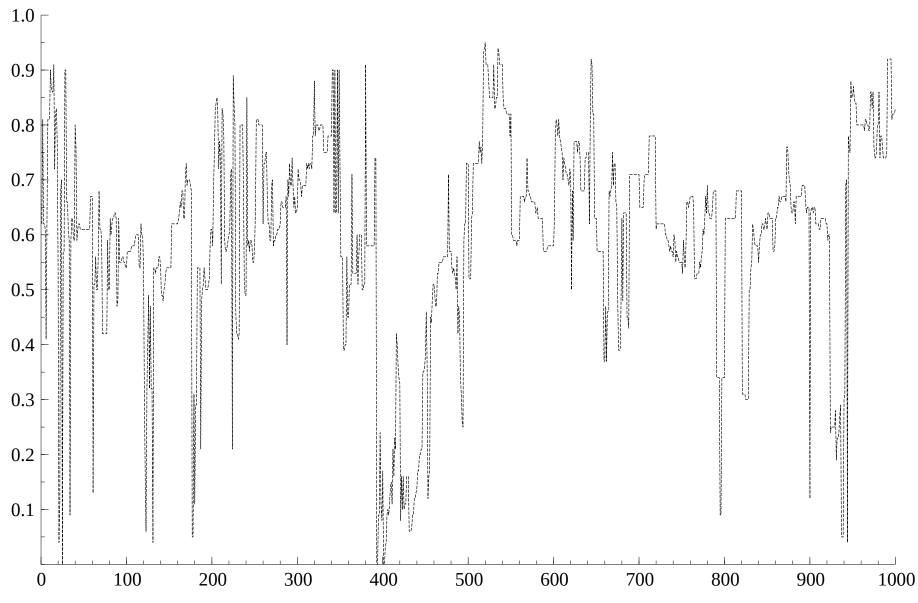
This section presents concisely the results of all the model configurations we used. We compare the three different models explained in Section 3: the CHK model in which the variance is estimated simply from the observed past returns and the parametric dynamic model in which the conditional variance is estimated using either the GARCH or the APARCH model. Moreover, we

[1]The same graph for a more risky investor, i.e. with a desired VaR confidence level of 90% for example, shows that the wealth evolution is always higher than the one resulting of investing only in Colgate and sometimes higher than only investing in IBM. Moreover, the final wealth attained with the model recommendations is higher than the final wealth achieved by investing only in IBM.



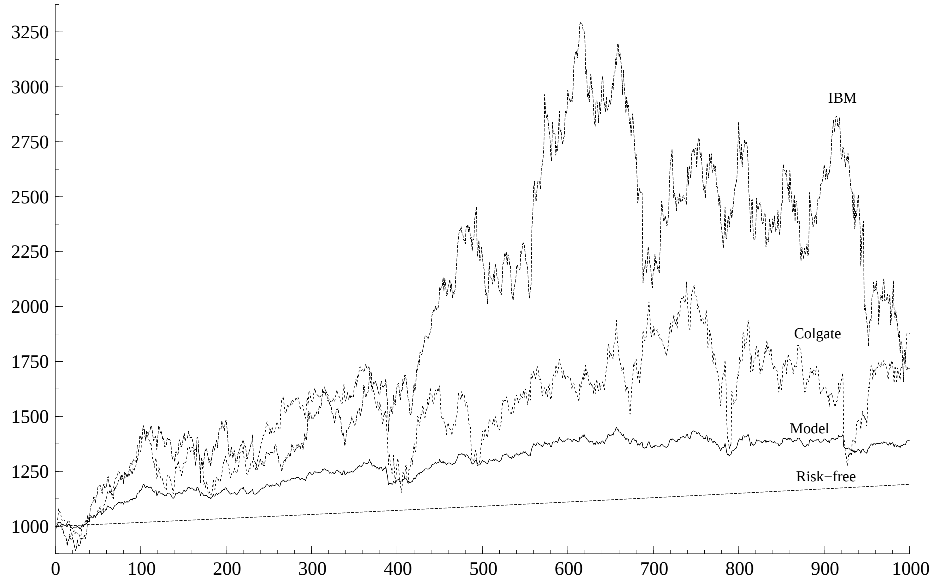
Riskfree weights for portfolios made of Colgate and IBM for an investor with VaR-99, based on the APARCH model using the skewed-t distribution. Out-of-sample period from 10/01/1997 until 31/12/2000 (1000 days).

Figure 8: **Riskfree weights using APARCH model with skewed-t distribution**



Risky weights on Colgate for an investor with VaR-99, based on the APARCH model using the skewed-t distribution. Out-of-sample period from 10/01/1997 until 31/12/2000 (1000 days).

Figure 9: **Risky weights on Colgate using APARCH model with skewed-t distribution**



Portfolios made of Colgate and IBM for an investor with VaR-99. Wealth evolution for 1000 out-of-sample forecast using the model recommendations (Model) compared with the wealth evolution obtained by investments made on Colgate or IBM alone and with investments made at the risk-free rate. Out-of-sample period from 10/01/1997 until 31/12/2000.

Figure 10: **Wealth evolution using APARCH model**

investigate three different distributional assumptions: the normal, the Student-t and the skewed-t. We consider three VaR confidence levels: 90%, 95% and 99%, corresponding to increasing risk aversion and show how these levels affect the results. The parameters are estimated using WML in a rolling window setup.

From the optimization procedure presented in Section 2, see Equation (10), we determine the weights of the risky portfolio and, considering the agent's desired risk expressed by the desired VaR (VaR^*), the amount to borrow or lend, see Equation (11). With this information at time t the investment strategy for day $t+1$ is set: percentage of wealth to borrow or lend and percentage to be invested in the risky portfolio. In order to evaluate the models we consider the wealth evolution of the initial invested amount and the failure rate of the returns obtained by applying the strategies with respect to the desired VaR level. A model is good when the wealth is high and when the failure rate is respected.

We expect that the forecasted VaR's by the different models be less or equal than the threshold values. To test this we perform a likelihood ratio test comparing the failure rate with the desired VaR level, as proposed by Kupiec (1995). We present the Kupiec-LR test for the portfolios made of Russell2000-SP500 (Table 4) and of Colgate-IBM (Table 6), for the probabilities of occurrence of $1-\alpha = 10\%$ (upper panel), 5% (middle panel) and 1% (lower panel). Several failure rates are significantly different from their nominal levels when we do out-of-sample forecasts. For in-sample forecast (results not presented) we found p-values as high as those presented by Giot and Laurent (2004) for example. This is understandable since the information set, on which we condition, contains only past observations so that the failure rates tend to be significantly different from their nominal levels. However, these failures rates are not completely out of scope of the desired confidence level, see for example Mittnik and Paolella (2000) for similar results.

Table 4 presents the failure rates and p-values for the Kupiec LR ratio test for portfolios made of Russell2000 and SP500. In general we observe that the dynamic model performs considerably better than its CHK counterpart for any VaR confidence level α. In terms of the distributional assumption we see that in the case of the probability of occurrence of $1 - \alpha = 10\%$ the normal distribution performs better than the Student-t even for low degrees of freedom (7 on average). This happens because the two densities cross each other at more or less that confidence level. See Guermat and Harris (2002) for similar results. Looking at lower probabilities of occurrence (higher confidence levels), one remarks that the skewed-t distribution performs better than the other two distributions. This is due to the fact that the skewed-t distribution allows not only for fatter tails but it can also capture the asymmetry present in the long and short sides of the market. This result is consistent with the findings of Mittnik and Paolella (2000), Giot and Laurent (2003) and Giot and Laurent (2004) who used single indices, stocks, exchange rates or a portfolio with unique weights.

With respect to the conditional variance models, we observe that for all the confidence levels, the APARCH model performs almost as good as the GARCH model, but from inspection of Table 4 we cannot conclude which model is better. However, considering that an agent wants to maximize his expected return subject to a risk constraint, we look after good results for the portfolio optimization (in terms of the wealth achieved), respecting the desired VaR confidence level (measured by the failure rate). To have a complete picture of the model performances, Table 5 presents the final wealth attained with portfolios made of Russell2000-SP500. From Table 5 we can appreciate the following facts: first, it happens that the final wealth obtained by the static model not only is lower than the wealth attained by the dynamic models but also, as pointed out before, has a higher risk. Second, even though we cannot select a best model between the APARCH and GARCH models in terms of failure rates, we can see that for almost the same level of risk the APARCH model investment recommendations allow the agent to get the highest final wealth. Therefore, we infer that the APARCH model outperforms the GARCH model. Thus, if an investor is a less risk averse ($1 - \alpha = 10\%$) he could have earned an annual rate return of 9.5%, two times bigger than simple investing at the risk-free rate.

Tables 6 and 7 present the results for the Colgate-IBM dataset. Like for the previous dataset, the dynamic models outperform the CHK model in terms of the failure rate. The normal distribution behaves better than the Student-t when the VaR confidence level is set to 90% ($1-\alpha = 10\%$).

Table 4: Failure rates for portfolios made of Russell2000 - SP500

$1-\alpha$	Model	Normal	p	Student-t	p	Skewed-t	p
0,10	**CHK**	0,177	0,000	0,200	0,000	0,188	0,000
	GARCH	0,114	0,148	0,130	0,002	0,117	0,080
	APARCH	0,126	0,008	0,129	0,003	0,118	0,064
0,05	**CHK**	0,127	0,000	0,135	0,000	0,120	0,000
	GARCH	0,071	0,004	0,074	0,001	0,060	0,159
	APARCH	0,083	0,000	0,081	0,000	0,062	0,093
0,01	**CHK**	0,068	0,000	0,048	0,000	0,032	0,000
	GARCH	0,029	0,000	0,021	0,002	0,011	0,754
	APARCH	0,030	0,000	0,027	0,000	0,012	0,538

Empirical tail probabilities for the out-of-sample forecast for portfolios made of linear combinations of Russell2000 and SP500 indices. The Kupiec-LR test is used to determine the specification of the models. The null hypothesis is that the model is correctly specified, i.e. that the failure rate equal to the probability of occurrence $1-\alpha$. Results obtained using WML with $\rho = 0.994$.

Table 5: Final wealth achieved by investing in portfolios made of Russell2000-SP500

$1-\alpha$	Model	Normal	r	Student-t	r	Skewed-t	r
0,10	**CHK**	1306	6,9	1303	6,8	1303	6,8
	GARCH	1355	7,9	1351	7,8	1346	7,7
	APARCH	1586	12,2	1630	13,0	1439	9,5
0,05	**CHK**	1297	6,7	1300	6,8	1296	6,7
	GARCH	1324	7,3	1328	7,3	1317	7,1
	APARCH	1497	10,6	1517	11,0	1368	8,2
0,01	**CHK**	1277	6,3	1270	6,2	1263	6,0
	GARCH	1290	6,6	1296	6,7	1281	6,4
	APARCH	1409	8,9	1388	8,5	1310	7,0

Final wealth achieved by investing in portfolios made of Russell2000-SP500. r is the annual rate of return in (%). The risk-free interest rate is 4.47% annual.

In general, we see that the skewed-t distribution outperforms the other distributions. In terms of the failure rate, the APARCH is slightly better than the GARCH but this difference is not striking enough to conclude that the APARCH model outperforms the GARCH model. If we also consider the wealth achieved by the application of the model recommendations (Table 7) we see that the APARCH outperforms the GARCH.

Table 6: Failure rates for portfolios made of Colgate - IBM

$1-\alpha$	**Model**	**Normal**	**p**	**Student-t**	**p**	**Skewed-t**	**p**
0,10	**CHK**	0,145	0,000	0,175	0,000	0,166	0,000
	GARCH	0,100	1,000	0,112	0,214	0,122	0,024
	APARCH	0,097	0,751	0,115	0,122	0,114	0,148
0,05	**CHK**	0,092	0,000	0,102	0,000	0,085	0,000
	GARCH	0,060	0,159	0,065	0,037	0,066	0,027
	APARCH	0,058	0,257	0,063	0,069	0,064	0,051
0,01	**CHK**	0,037	0,000	0,028	0,000	0,020	0,005
	GARCH	0,024	0,000	0,022	0,001	0,016	0,079
	APARCH	0,025	0,000	0,018	0,022	0,015	0,139

Empirical tail probabilities for the out-of-sample forecast for portfolios made of linear combinations of Colgate and IBM. The Kupiec-LR test is used to determine the specification of the models. The null hypothesis is that the model is correctly specified, i.e. that the failure rate equal to the desired probability of occurrence $1-\alpha$.

Finally, in order to be sure that the final wealth is not just caused by an outlier, we present as examples, the wealth evolution of the portfolios made of Russell2000 - SP500 (Figure 11) and Colgate - IBM (Figure 12). The distributional assumption used was the skewed-t. The VaR confidence level used in the first case was 90% and in the second case 99%. Figures 11 and 12 show that the final wealth achieved by the recommendations of the APARCH model is consistently larger than the wealth achieved by the GARCH model suggestions.

5 Evaluation

5.1 Risk-free interest rate sensitivity

We have used as a risk-free interest rate the one-year Treasury bill rate in January 1998 (approximately 4.47% annual) as an approximation for the average risk-free rate during the whole out-of-sample period (January 1997 to December 2000). In order to analyze the sensitivity of our

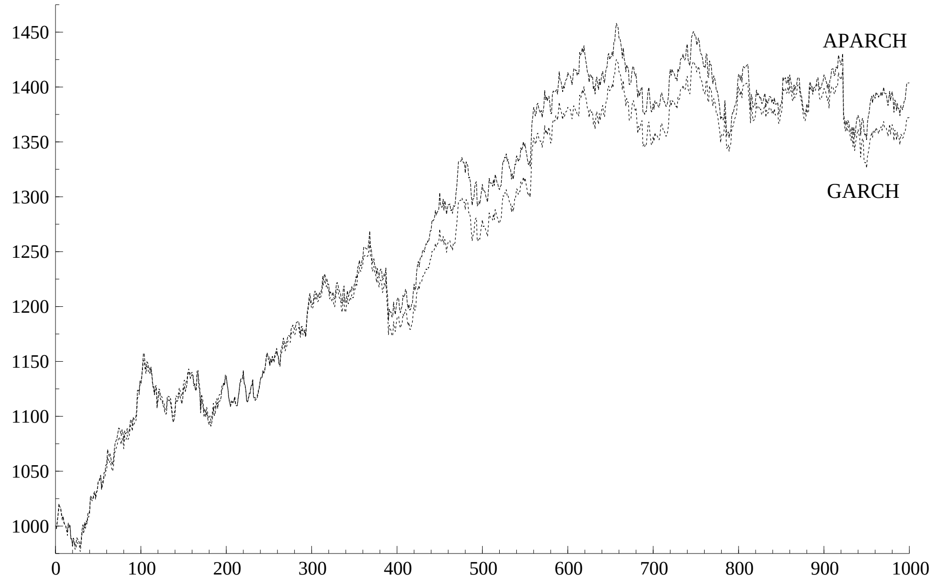
Table 7: Final wealth achieved by investing in portfolios made of Colgate-IBM

$1-\alpha$	Model	Normal	r	Student-t	r	Skewed-t	r
0,10	**CHK**	1758	15,2	1830	16,3	1799	15,8
	GARCH	1559	11,7	1602	12,5	1624	12,9
	APARCH	1658	13,5	1641	13,2	1691	14,0
0,05	**CHK**	1638	13,1	1661	13,5	1622	12,8
	GARCH	1491	10,5	1476	10,2	1491	10,5
	APARCH	1577	12,1	1521	11,1	1506	10,8
0,01	**CHK**	1506	10,8	1470	10,1	1432	9,4
	GARCH	1415	9,1	1353	7,8	1392	8,6
	APARCH	1496	10,6	1446	9,7	1400	8,8

Final wealth achieved by investing in portfolios made of Colgate-IBM. r is the annual rate of return in (%). The risk-free interest rate is 4.47% annual.

Wealth evolution of portfolios made of Russell2000 - SP500. The distribution used is the skewed-t and the confidence level for the VaR is 90%. Out-of-sample period goes from 02/01/1997 until 20/12/2000.

Figure 11: **Compared Wealth evolution using GARCH and APARCH models**



Wealth evolution of portfolios made of Colgate - IBM. The distribution used is the skewed-t and the confidence level for the VaR is 99%. Out-of-sample period goes from 10/01/1997 until 31/12/2000.

Figure 12: **Compared Wealth evolution using GARCH and APARCH models**

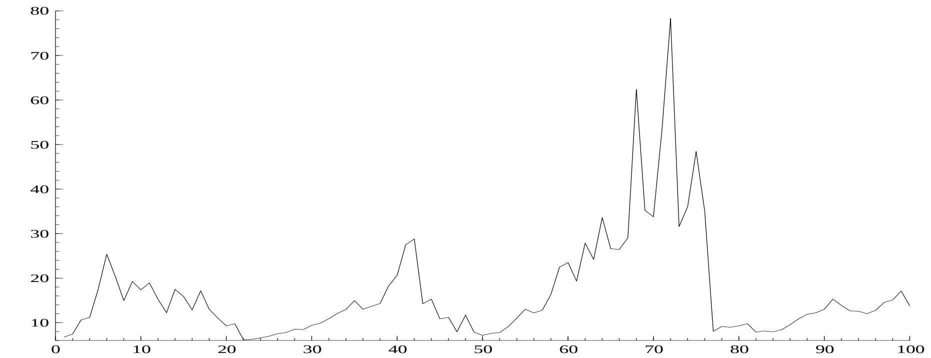
our results to changes of the risk-free rate, we develop four scenarios based on increments (+1% and +4%) or decrements (-1% and -4%) with respect to the benchmark.

The results show that neither the borrowing/lending nor the risky portfolios weights are strongly affected by either of these scenarios. This is due to the fact that we are working with daily optimizations, and that those interest rates at a daily frequency are low. For example 4.47% annual equals 0.01749% daily (based on 250 days).

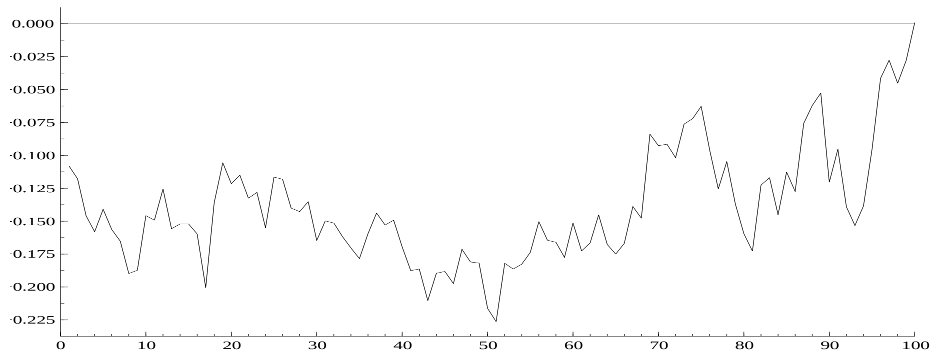
5.2 Time varying kurtosis and asymmetry

As in Guermat and Harris (2002), our framework allows for time varying degrees of freedom parameters, related to the kurtosis, when working with either the Student-t or the skewed-t distributions. Moreover, when the skewed-t distribution is used we allow for time varying asymmetry parameters. Figure 13 presents the pattern of the degrees of freedom and asymmetry parameter of the skewed-t distribution estimated using the APARCH model. Similarly to Jondeau and Rockinger (2003), we find time dependence of the asymmetry parameter but we also remark that the degrees of freedom parameter is time varying.

We also test the significance of the asymmetry parameter and of the asymmetry and degrees of freedom parameters, for the Student-t and skewed-t respectively. We find that they are highly significant. As an example, Table 8 presents the results for the first out-of-sample day for portfolios



(a) Degrees of freedom



(b) Asymmetry parameter

Time varying degrees of freedom and Asymmetry for the skewed-t innovation distribution estimated using the APARCH model. The parameters are estimated every 10 days during the out-of-sample forecast. The figure corresponds to a portfolio made only of RUSSELL2000.

Figure 13: **Time varying degrees of freedom and asymmetry parameters**

made of linear combinations of Russell2000 and SP500 using the WML procedure with $\rho = 0.994$. The skewed-t distribution was estimated using the APARCH model. Similar results are observed in the other procedures.

Table 8: **Significance of the asymmetry and degrees of freedom parameters.**
This table presents the parameter estimates and the statistical significance of the asymmetry and degrees of freedom parameters of the skewed-t distribution estimated using the APARCH model. The results correspond to the first day of the out-of-sample forecast for portfolios made of linear combinations of Russell2000 and SP500 using the WML procedure with $\rho = 0.994$.

Parameter	Estimates	Std-errors	T-value	p-value
asymmetry	-0.064	0,025	-2,582	0.000
degrees of freedom	6,918	0,897	7,712	0.000

5.3 Weighted Maximum Likelihood vs Maximum Likelihood

We study the effect of using Weighted Maximum Likelihood (WML) instead of Maximum Likelihood (ML). Note that when $\rho = 1$ WML is equal to ML. Table 9 presents a comparison of failure rates for portfolios made of Russell2000-SP500. Both dynamic models improve their failure rates by using WML in a rolling window setup instead of ML. In terms of the p-values (not presented) it turns out that when ML is used almost none of the failure rates were significant at any level. Thus, using WML helps to satisfy the investor's desired level of risk.

5.4 Rolling window of fixed size

We analyze the effect of using a rolling window of fixed size. The idea behind this procedure is that we assume that information until n days in the past convey some useful information for the prices, meanwhile the rest does not. We use a rolling window of fixed size of $n = 1000$ days for performing the out-of-sample forecasts. The results presented in Tables 10 and 11 show that the gains in better model specification are nil: the failure rates are worse and the final wealth achieved are almost the same. The computational time decreases (about 30% less).

5.5 VaR subadditivity problem

According to Artzner, Delbaen, Eber, and Heath (1999), Frey and McNeil (2002) and Szegö (2002), a coherent risk measure satisfies the following axioms: translation invariance, subadditivity, posi-

Table 9: Comparison of failure rates

		Normal		Student-t		Skewed-t	
α	**Model**	**ML**	**WML**	**ML**	**WML**	**ML**	**WML**
0,90	**CHK**	0,177	0,177	0,200	0,200	0,188	0,188
	GARCH	0,128	0,114	0,153	0,130	0,139	0,117
	APARCH	0,132	0,126	0,149	0,129	0,126	0,118
0,95	**CHK**	0,127	0,127	0,135	0,135	0,120	0,120
	GARCH	0,085	0,071	0,094	0,074	0,069	0,060
	APARCH	0,085	0,083	0,086	0,081	0,068	0,062
0,99	**CHK**	0,068	0,068	0,048	0,048	0,032	0,032
	GARCH	0,037	0,029	0,026	0,021	0,011	0,011
	APARCH	0,040	0,030	0,030	0,027	0,014	0,012

Comparison of empirical tail probabilities for the out-of-sample forecast for portfolios made of linear combinations of Russell2000 and SP500 using the ML procedure ($\rho = 1$) with WML with $\rho = 0.994$.

Table 10: Failure rates for portfolios made of Russell2000-SP500, using ML with rolling window of fixed size

$1-\alpha$	**Model**	**Normal**	**p**	**Student-t**	**p**	**Skewed-t**	**p**
0,10	**CHK**	0,177	0,000	0,200	0,000	0,188	0,000
	GARCH	0,133	0,001	0,148	0,000	0,129	0,003
	APARCH	0,141	0,000	0,145	0,000	0,130	0,002
0,05	**CHK**	0,127	0,000	0,135	0,000	0,120	0,000
	GARCH	0,081	0,000	0,088	0,000	0,069	0,009
	APARCH	0,085	0,000	0,087	0,000	0,067	0,019
0,01	**CHK**	0,068	0,000	0,048	0,000	0,032	0,000
	GARCH	0,039	0,000	0,028	0,000	0,012	0,538
	APARCH	0,045	0,000	0,031	0,000	0,016	0,079

Empirical tail probabilities for the out-of-sample forecast for portfolios made of linear combinations of Russell2000 and SP500 using a rolling window of fixed size of 1000 days. The Kupiec-LR test is used to determine the specification of the models. The null hypothesis is that the model is correctly specified, i.e. that the failure rate equal to the desired probability of occurrence $1 - \alpha$.

Table 11: Final wealth achieved by investing in portfolios made of Russell2000-SP500, using ML with rolling window of fixed size

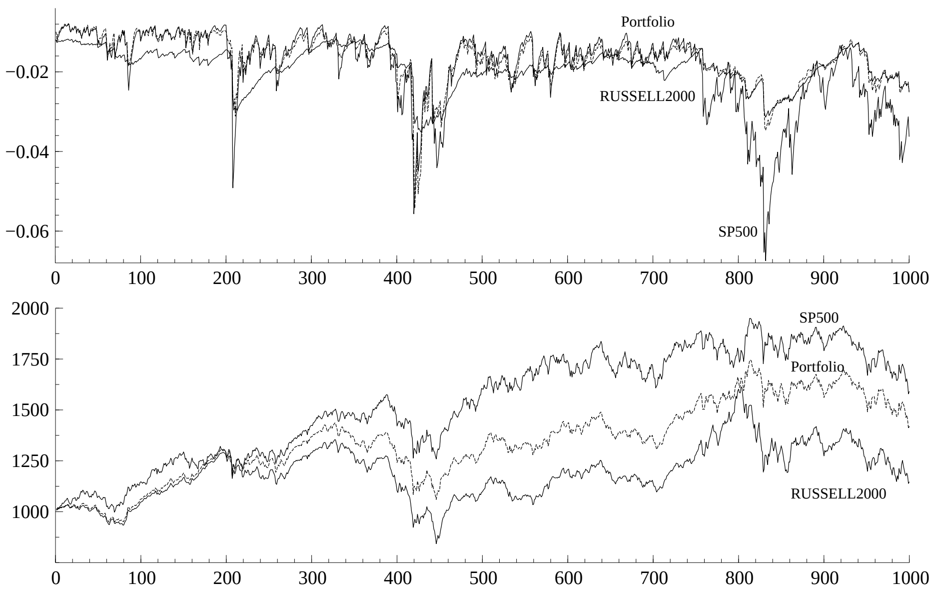
$1-\alpha$	**Model**	**Normal**	**r**	**Student-t**	**r**	**Skewed-t**	**r**
0,10	Static	1306	6,9	1303	6,8	1303	6,8
	GARCH	1311	7,0	1283	6,4	1300	6,8
	APARCH	1663	13,6	1704	14,3	1461	9,9
0,05	Static	1297	6,7	1300	6,8	1296	6,7
	GARCH	1292	6,6	1284	6,5	1284	6,5
	APARCH	1579	12,1	1564	11,8	1390	8,6
0,01	Static	1277	6,3	1270	6,2	1263	6,0
	GARCH	1271	6,2	1258	5,9	1248	5,7
	APARCH	1465	10,0	1436	9,5	1336	7,5

Final wealth achieved by investing in portfolios made of Russell2000-SP500. r is the annual rate of return in (%). The risk-free interest rate is 4.47% annual.

tive homogeneity and monotonicity. They show that VaR satisfies all but one of the requirements to be considered as a coherent risk measure: the subadditivity property. Subadditivity means that "a merger does not create extra risk", i.e. that diversification must reduce risk. Moreover, the VaR is not convex with respect to portfolio rebalancing no matter what is the assumption made on the return distribution. Following Consigli (2002), we do not discuss the limits of the VaR and instead we try to generate more accurate VaR estimates considering the asymmetry and kurtosis of the financial data.

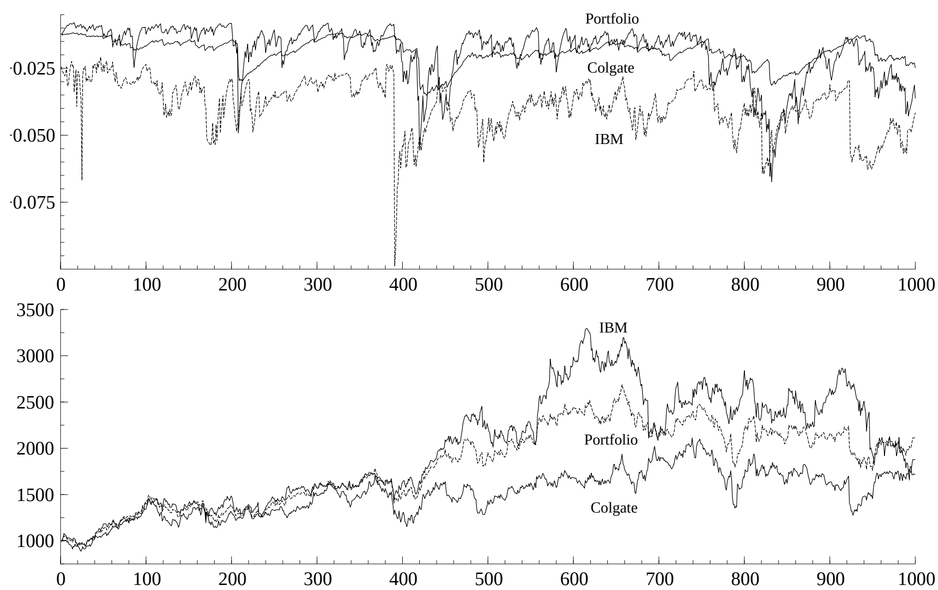
Figure 14 presents the VaR and wealth evolution for an investor whose desired confidence level is 5%, the model used is GARCH and the innovation distribution is the skewed-t. The optimal portfolio VaR's are consistently smaller that the VaR's of the individual series. This is the case for all the models in our empirical application implying that by combining the two indices or stocks optimally we are reducing the risk. Moreover, the portfolio model not only allows to decrease risk but also to obtain portfolio returns between the returns of the individual indices.

Figure 15 presents the same graph for portfolios made of Colgate-IBM. The VaR of the optimal portfolios are always smaller than the individual VaRs. Moreover, we can appreciate the advantages of diversification by looking at the wealth evolution at the end of the out-of-sample period (lower panel). The wealth evolution by only investing on IBM reduces rapidly while the portfolio wealth does not. At the end of the out-of-sample period the final wealth is almost the same.



VaR-95 evolution (above) and Wealth evolution (below) for SP500, Russell2000 and for the optimal portfolios using GARCH model with skewed-t innovation distribution. Out-of-sample period goes from 02/01/1997 until 20/12/2000.

Figure 14: **Compared VaR and Wealth evolution: Russell2000-Sp500**



VaR-95 evolution (above) and Wealth evolution (below) for Colgate, IBM and for the optimal portfolios using APARCH model with skewed-t innovation distribution. Out-of-sample period goes from 02/01/1997 until 20/12/2000.

Figure 15: **Compared VaR and Wealth evolution: Colgate-IBM**

6 Conclusions and future work

The dynamic portfolio selection model we propose performs well out-of-sample statiscally in terms of failure rates, defined as the number of times the desired confidence level used for the estimation of the VaR is violated. Based on this criterion, the APARCH model gives as good results as the GARCH model. However, if we consider not only the failure rate but also the wealth achieved, we find that for similar levels of risk, the APARCH model outperforms the GARCH model. A sensitivity analysis with respect to the distributional innovation hypothesis shows that in general the skewed-t is preferred to the normal and Student-t. Estimating the model parameters by Weighted Maximum Likelihood in an increasing window setup allows us to account for a changing time pattern of the degrees of freedom and asymmetry parameters of the innovation distribution and to improve the forecasting results in the statistical and economical sense: smaller failure rates and larger final wealth.

There are a number of directions for further research along the lines presented here. A potential extension could use the dynamic model to study the optimal time of portfolio rebalancing, as day-to-day portfolio rebalancing may be neither practicable nor economically viable. A more ambitious extension is to work in a multivariate setting, where a group of different financial instruments are used to maximize the expected return subject to a risk constraint. Another interesting extension of the model is to investigate its intra-daily properties. This extension could be of special interest for traders who face the market second by second during the trading hours in the financial markets.

References

ARTZNER, P., F. DELBAEN, J. EBER, AND D. HEATH (1999): "Coherent Measures of Risk," *Mathematical Finance*, 9, 203–228.

BOLLERSLEV, T. (1986): "Generalized Autoregressive Conditional Heteroskedasticity," *Journal of Econometrics*, 52(4), 5–59.

CAMPBELL, R., R. HUISMAN, AND K. KOEDIJK (2001): "Optimal Portfolio Selection in a Value-at-Risk Framework," *Journal of Banking and Finance*, 25, 1789–1804.

CONSIGLI, G. (2002): "Tail Estimation and mean-VaR Portfolio Selection in Markets Subject to Financial Inestability," *Journal of Banking and Finance*, 26, 1355–1382.

DE ROON, F., T. NIJMAN, AND B. WERKER (2003): "Currency Hedging for International Stock Portfolios: the Usefulness of Mean-Variance Analysis," *Journal of Banking and Finance*, 27, 327–349.

DING, Z., C. GRANGER, AND R. ENGLE (1993): "A Long Memory Property of Stock Market Returns and a New Model," *Journal of Empirical Finance*, 1, 83–106.

FLEMING, J., C. KIRBY, AND B. OSTDIEK (2001): "The Economic Value of Volatility Timing," *Journal of Finance*, (1), 329–352.

FREY, R., AND A. MCNEIL (2002): "VaR and Expected Shorfall in Portfolios of Dependent Credit Risks: Conceptual and Practical Insights," *Journal of Banking and Finance*, 26, 1317–1334.

GIOT, P., AND S. LAURENT (2003): "Value-at-Risk for Long and Short Trading Positions," *Journal of Applied Econometrics*, 18, 641–663.

——— (2004): "Modelling daily Value-at-Risk using realized volatility and ARCH type models," *Journal of Empirical Finance*, 11, 379–398.

GOURIEROUX, C., J. LAURENT, AND O. SCAILLET (2000): "Sensitivity Analysis of Values at Risk," *Journal of Empirical Finance*, 7, 225–245.

GUERMAT, C., AND D. HARRIS (2002): "Forecasting Value at Risk Allowing for Time Variation in the Variance and Kurtosis of Portfolio Returns," *International Journal of Forecasting*, 18, 409–419.

HANSEN, B. (1994): "Autoregressive Conditional Density Estimation," *International Economic Review*, 35, 705–730.

Jondeau, E., and M. Rockinger (2003): "Conditional Volatility, Skewness, and Kurtosis: Existence, Persistence, and Comovements," *Journal of Economic Dynamics and Control*, 27, 1699–1737.

Jorion, P. (1997): *Value-at-Risk: the New Benchmark for Controlling Market Risk*. McGraw-Hill, New York.

Kupiec, P. (1995): "Techniques for Verifying the Accuracy of Risk Measurement Models," *The Journal of Derivatives*.

Lambert, P., and S. Laurent (2001): "Modelling Financial Time Series using GARCH-type Models with a Skewed Student Distribution for the Innovations," Discussion Paper 0125, Institut de Statistique, Université Catholique de Louvain, Louvain-la-Neuve, Belgium.

Markowitz, H. M. (1952): "Portfolio Selection," *Journal of Finance*, 7(1), 77–91.

Merton, R. (1980): "On Estimating the Expected Return on the Market: An Exploratory Investigation," *Journal of Financial Economics*, (8), 323–361.

Mittnik, S., and M. Paolella (2000): "Conditional Density and Value-at-Risk Prediction of Asian Currency Exchange Rates," *Journal of Forecasting*, 19, 313–333.

Patton, A. (2004): "On the Out-of-sample Importance of Skewness and Asymmetric Dependence for Asset Allocation," *Journal of Financial Econometrics*, 2(1), 130–168.

Szegö, G. (2002): "Measures of Risk," *Journal of Banking and Finance*, 26, 1253–1272.