FROM RISK SHARING TO RISK TRANSFER: THE ANALYTICS OF COLLABORATIVE INSURANCE

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Abstract

Denuit (2019, 2020b) demonstrated that conditional mean risk sharing introduced by Denuit and Dhaene (2012) is the appropriate theoretical tool to share losses in collaborative P2P insurance schemes. Denuit and Robert (2020a,b,c) studied this risk sharing mechanism and established several attractive properties including linear approximations when total losses or the number of participants get large. It is also shown there that the conditional expectation defining the conditional mean risk sharing is asymptotically increasing in the total loss (under mild technical assumptions). This ensures that the risk exchange is Pareto-optimal and that all participants have an interest to keep total losses as small as possible. In this paper, we design a flexible system where entry prices can be made attractive compared to the premium of a regular, commercial insurance contract. Participants can also be grouped according to some meaningful criterion, resulting in a hierarchical decomposition of the community. The particular case where realized losses are allocated in proportion of the pure premiums is studied. This applies to losses obeying infinitely divisible distributions, such as compound Poisson or compound Negative Binomial ones as long as severities are identically distributed. Also, participants can just opt for such a proportional mean risk sharing, for simplicity.

Keywords: Peer-to-Peer (P2P) insurance, conditional mean risk sharing, size-biased transform, comonotonicity.

1 Introduction and motivation

In collaborative, Peer-to-Peer (P2P) insurance, participants agree to pool the risks they face. P2P insurance thus builds on the fundamental principle of mutuality at the origin of insurance without the equity buffer provided by insurer's capital. In case large losses may occur, pooling can be restricted to the first layer to avoid that the scheme runs out of money. Higher losses are then transferred to a third party, typically an insurance or reinsurance company. In this paper, we consider that the pool retains the total losses up to a given threshold and the part above the threshold is covered by some (re-)insurer with the help of a stop-loss protection. Unclaimed money can be paid back, given to a charity or used to fund a common project to which the community adheres. We refer to Clemente and Marano (2020) and the references therein for an overview of P2P insurance.

Collaborative, P2P insurance schemes may well become an additional sales channel, complementing classical ones (brokers, direct, bank, as well as blank products provided in B2B). This new channel also addresses one of the key challenges faced by the insurance industry in the 21st century: offering an appropriate coverage to every customer, meeting his or her own personal expectations. In that respect, the parallel with the blend of consumers' profiles existing in electricity markets seems to be instructive: these profiles

- range from consumers staying with the ex-monopolistic provider, buying electricity produced from coal or nuclear plant, corresponding to policyholders buying traditional contracts, sold by brokers or through large bank networks (or by direct insurers in case price is an important aspect);

- to those buying shares in windmill projects near to their place, corresponding to individuals joining a mutual insurer, owned by its members;

- passing through consumers buying green electricity, corresponding to policyholders opting for a product supporting sustainable development, e.g. through ethic investment, still sold by classical insurance actors;

- and prosumers installing solar panels on their roof and equipping their buildings with batteries, staying connected to the public electricity network to face the extreme case where their own system does not suffice, corresponding to individuals prone to self-insurance, wishing to access high deductibles reducing premium to a large extent;

- not to forget local energy communities providing inhabitants with electricity produced nearby by their members, corresponding to P2P insurance communities.

There seems to be a strong potential for developing insurance offers along these lines from an economic point of view, since yearly electricity bill for a typical household lies is of the same order than motor insurance premium, or even larger in case of electric warming systems and vehicles. Also, younger generations question existing models, including those dominating the insurance industry, and it appears to be of crucial importance to address their concerns.

Setting a collaborative insurance scheme raises the following question: how to fairly share, in an understandable and transparent way, that part of the premium that has not been used to cover the claims? As everybody is well aware that insurance risks are heterogeneous, equally sharing the total losses among participants may well appear to be unfair. To be successful, P2P insurance schemes require an appropriate risk sharing mechanism recognizing the different distributions of the risks brought to the pool. Intuitively speaking, participants pooling "smaller" or "less variable" risks should contribute less to the realized total loss. Denuit (2019) proposed a simple and mathematically correct solution based on the conditional mean risk sharing rule introduced by Denuit and Dhaene (2012). Being based on the concept of mean value, this sharing rule turns out to be quite intuitive as most participants have at least a vague idea about averaging and its risk-reducing effect. Since participants can be informed when they enter the pool about the amount they will have to contribute in function of the total realized loss, this approach ensures full transparency.

Let us explain how such a system operates. Consider n participants to an insurance pool, numbered $i = 1, 2, \ldots, n$. Each of them faces a risk X_i. By risk, we mean a non-negative random variable representing a monetary loss caused by some insurable peril over one period (a calendar year, say). In the remainder of this paper, we assume that $X_1, X_2, \ldots, X_n$ are independent with $0 < \mathrm{E}[X_i] < \infty$ for all i. This assumption is not restrictive for applications to P2P insurance. Henceforth, we use the notation

$$S = \sum_{i=1}^{n} X_i$$

for the total risk of the pool.

In the design of a risk pooling scheme, it is important that the sharing rule is both intuitively acceptable and transparent. In that respect, the conditional mean risk sharing (or allocation) proposed by Denuit and Dhaene (2012) seems to be particularly attractive. In that risk pooling scheme, each participant contributes ex-post the amount $\mathrm{E}[X_i|S = s]$ where $s = \sum_{i=1}^{n} x_i$ is the sum of the realizations $x_1, x_2, \ldots, x_n$ of $X_1, X_2, \ldots, X_n$. Clearly, the conditional mean risk sharing allocates the full risk S as we obviously have

$$\sum_{i=1}^{n} \mathrm{E}[X_i|S] = S.$$

In words, participant i must contribute the expected value of the risk X_i he or she brings to the pool, given the total loss S experienced by the entire community. Stated differently, the contribution of each participant is the average part of the total loss S that can be attributed to the risk he or she brought to the pool. Explained in this way, the conditional mean risk sharing rule appears to be very intuitive and easily understandable by the members of the P2P insurance community. We will also discuss simplifications, sharing losses in proportion to premiums paid by participants making the system even more transparent while still based on firm methodological grounds.

The key point to explain to the members of a P2P insurance community is that the actual realization of X_i is mostly due to chance, and only partly reveals the underlying magnitude of the risk brought to the community. Over one period, the conditional mean risk sharing splits the risk X_i into a random part $X_i - \mathrm{E}[X_i|S]$ shared among participants by virtue of the mutuality (or random solidarity) principle and a structural part $\mathrm{E}[X_i|S]$ to be supported by participant i, individually. Formally, the risk X_i brought by participant i is decomposed into

$$X_i = \underbrace{\mathrm{E}[X_i|S]}_{=\text{structural part}} + \underbrace{X_i - \mathrm{E}[X_i|S]}_{=\text{random departure } \mathcal{E}_i}. \tag{1.1}$$

Both terms entering this split are uncorrelated and random departures $\mathcal{E}_i$ always sum to 0. The latter property ensures that the community bears its own risk, without transferring (part of) it to another economic agent.

In commercial insurance, the variations in S are absorbed by insurer's capital and individual i must contribute a fixed amount to compensate the cost of insured events, equal to the pure premium $\mathrm{E}[X_i]$ supplemented with shareholder's profit for providing the risk-bearing capital. With P2P insurance, each participant must contribute ex-post the amount $\mathrm{E}[X_i|S]$ to the pool whereas the random departures $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n$ entering the decomposition (1.1) are re-allocated among them according to the mutuality principle. The expected contribution is equal to the expected loss, that is,

$$\mathrm{E}\big[\mathrm{E}[X_i|S]\big] = \mathrm{E}[X_i], \quad i = 1, 2, \ldots, n.$$

The conditional mean risk allocation is thus considered to be fair: on average, individual contributions match the corresponding losses so that no participant gains in the operation. Stated otherwise, there are no transfers among participants to the pool, on average, so that each of them bears the consequence of his or her own risk.

The conditional mean risk sharing rule enjoys the following attractive properties:

(i) In the expected utility setting, every risk-averse economic agent prefers $\mathrm{E}[X_i|S]$ over the initial risk X_i so that the conditional mean risk sharing rule appears to be beneficial to all participants (as an application of Jensen's inequality).

(ii) Denuit and Dhaene (2012) established that the conditional mean risk sharing rule is Pareto-optimal for all risk-averse expected-utility maximizers, as long as every function $s \mapsto \mathrm{E}[X_i|S = s]$ is non-decreasing (see assumption (2.4) below). It turns out that this condition is asymptotically valid under mild technical conditions, as established by Denuit and Robert (2020c). Denuit (2020b) demonstrated that explicit formulas can be derived to describe the risk sharing mechanism in this setting.

(iii) The conditional mean risk sharing rule does not require the prior elicitation of individual preferences beyond risk aversion. This is useful for application to P2P insurance where some standardization is required.

(iv) Denuit and Robert (2020b) established that, under mild technical conditions, $\mathrm{E}[X_i|S]$ tends to $\mathrm{E}[X_i]$ as the number n of participants increases, with probability 1. The pure premium is thus recovered as a limiting case.

However, several issues need to be addressed before a possible implementation of the conditional mean risk sharing in collaborative insurance. First and foremost, the entry price must be attractive and an upper limit should be placed on personal contributions to convince participants to join the community. This is especially important when the new scheme is launched. Unfortunately, this is not guaranteed with the system designed by Denuit (2020b) where individual contributions derive from the total amount of retention selected by the community. The present paper innovates in that we start from the amounts paid ex-ante by participants (replacing premiums charged by commercial insurers) and design the system in such a way that no additional contribution is needed ex-post (thanks to the stop-loss treaty protecting the community).

Also, participants are often grouped in "teams" based on existing networks (family or relatives, for instance), common economic interests or geographic location. We adapt the system to account for these two levels: the intermediate group level in addition to the individual one. We also discuss possible simplifications to the conditional mean risk sharing rule. Precisely, we show that this sharing may reduce to a simple proportional allocation of losses, according to pure premiums. Notice that participants can also opt for this allocation rule, just for simplicity. In the two-level case, the theoretically correct conditional mean risk sharing rule can be applied at the upper level (to ensure fairness between groups, avoiding any undue transfer) whereas within the groups, the simpler proportional mean risk allocation may be applied (even if it is not formally supported by theoretical arguments).

The remainder of this paper is organized as follows. In Section 2, we recall the definition of the size-bias transform and its close link to the conditional mean risk sharing. Then, we carefully discuss the monotonicity and linearity of the conditional mean risk sharing rule. Section 3 is devoted to the design of the proposed collaborative P2P insurance scheme. In Section 4, we adapt this scheme to the case where participants are partitioned into subgroups as it is often the case in practice. The final Section 5 briefly discusses the result obtained in this paper.

2 Conditional mean risk sharing rule and size-biased transform

2.1 Size-biased transform

The size-biased transform appears to be useful to study the conditional mean risk sharing rule, as pointed out in Denuit (2019). Given a non-negative random variable X with distribution function F_X and strictly positive expected value $\mathrm{E}[X]$, define $\widetilde{X}$ with distribution function

$$\mathrm{P}[\widetilde{X} \leq t] = \frac{\mathrm{E}\big[X\mathrm{I}[X \leq t]\big]}{\mathrm{E}[X]},$$

where $\mathrm{I}[\cdot]$ denotes the indicator function (equal to 1 if the event appearing within the brackets is realized, and to 0 otherwise). Then, $\widetilde{X}$ is said to be a size-biased version of X, and the operator mapping the distribution function F_X of X to the distribution function $F_{\widetilde{X}}$ of $\widetilde{X}$ is called the size-biased transform. Henceforth, we assume that X and $\widetilde{X}$ are mutually independent.

The size-biased transform can be traced back to the late 1960s in the statistical literature. It has proven to be useful in the study of risk measures after the pioneering work by Furman and Landsman (2005, 2008) and Furman and Zitikis (2008a,b). The size-biased transform is an example of weighted distribution. Initially developed in order to unify various sampling distributions when the chance of being recorded by an observer varies, weighted distributions are closely related to weighted risk measures and weighted capital allocation rules. See Furman and Zitikis (2009) for an overview. Among these weighted distributions, the size-biased, or length-biased one corresponds to the identity weight function. We refer the reader to Aaratia et al. (2019) for a general presentation of the size-biased transform and to Denuit (2020a) for applications to risk measurement.

Compound sums with absolutely continuous severities have a probability mass at 0 and possess a probability density function over $(0,\infty)$. Such a distribution is said to be zero-augmented and it appears to be relevant to examine the effect of size-biasing in this case. This is because zero-augmented distributions are appropriate for modeling losses X_i to be shared within P2P insurance communities.

Example 2.1 (Size-biased transform of zero-augmented distributions)**.** *Assume that X is a zero-augmented risk, i.e. it is equal to 0 with probability $\mathrm{P}[X=0]>0$ or strictly positive with probability $\mathrm{P}[X>0]$ and possesses the probability density function $f_{X|X>0}$ over $(0,\infty)$. Then, $\widetilde{X}$ is a strictly positive random variable with probability density function*

$$f_{\widetilde{X}}(x)=\frac{xf_{X|X>0}(x)}{\mathrm{E}[X|X>0]}. \tag{2.1}$$

See e.g. Property 2.1 in Denuit (2019).

2.2 Conditional mean risk sharing in terms of size-biasing

If the random variables X_1, X_2, ..., are assumed to be independent and identically distributed, it is well known that

$$\mathrm{E}[X_i|S=s]=\frac{s}{n}. \tag{2.2}$$

Whatever the common distribution of the random variables X_1, X_2, ..., we thus see that the conditional expectation is an increasing function of S in this case. The homogeneity assumption is however very restrictive for applications and the conditional mean risk sharing extends (2.2) to heterogeneous losses X_i so that each participant supports the right share of the total loss S.

Let us now give the reason why size-biasing appears to be useful in relation with the conditional mean risk sharing. Consider independent, zero-augmented risks $X_1,X_2,\ldots,X_n$ with positive expectations. Let $\widetilde{X}_1,\widetilde{X}_2,\ldots,\widetilde{X}_n$ be their corresponding size-biased versions, assumed to be independent and independent of $X_1,X_2,\ldots,X_n$. It is proved in Denuit (2019, Proposition 2.2, item iii) that, for any $s>0$,

$$\mathrm{E}[X_i|S=s]=\frac{\mathrm{E}[X_i]f_{S-X_i+\widetilde{X}_i}(s)}{\sum_{j=1}^n\mathrm{E}[X_j]f_{S-X_j+\widetilde{X}_j}(s)}s. \tag{2.3}$$

If the random variables $X_1,X_2,\ldots,X_n$ are identically distributed then the ratio appearing in (2.3) is equal to $1/n$ and we recover (2.2).

2.3 Comonotonic structural losses

As recalled in the introduction, when $s \mapsto \mathrm{E}[X_i|S = s]$ is non-decreasing for each $i = 1, 2, \ldots, n$, the conditional mean risk sharing is comonotonic and hence Pareto-optimal (Denuit and Dhaene, 2012, Proposition 4.2). Henceforth, we make the following assumption:

$$\text{Assumption: } s \mapsto \mathrm{E}[X_i|S = s] \text{ non-decreasing for every } i. \tag{2.4}$$

Condition (2.4) is a natural requirement in P2P insurance since it ensures that all participants have an interest to keep the total loss S as small as possible because enlarging S always increases the contribution $\mathrm{E}[X_i|S]$ they have to pay. It is important to stress that (2.4) is not always valid. Counterexamples are easy to build with discrete losses X_i assuming just a few values (because the value of s may bring some information about the realization of individual losses in this simplistic setting). In a couple of papers, Zabell (1980, 1993) studied the behavior of random variables $\mathrm{E}[X_i|S]$ when the contribution of X_i to the sum S is asymptotically negligible. Intuitively speaking, the impact of X_i on S should vanish when the number n of participants gets large, as long as X_i does not dominate the remaining terms and the phenomenon occurring in the discrete case with a limited number of participants should disappear. Several sufficient conditions for (2.4) to hold have been derived in the literature, often involving log-concavity (see below).

All the calculations are performed with $\mathrm{E}[X_i|S]$ instead of X_i, exploiting the fact that

$$S = \sum_{i=1}^{n} X_i = \sum_{i=1}^{n} \mathrm{E}[X_i|S].$$

Thus, we substitute the structural part $\mathrm{E}[X_i|S]$ of the loss to the actual one X_i, following (1.1). As explained above, this is because the actual loss X_i is not very informative about the underlying riskiness for participant i, being subject to a lot of random variability: it is equal to 0 with a high probability, even for riskier participants (in other words, the actual losses are just 0 for most participants, regardless of their respective degrees of riskiness) and when actual losses are positive they appear to be very volatile. On the contrary, $\mathrm{E}[X_i|S]$ appropriately quantifies the relative riskiness of X_i within the pool S.

Assumption (2.4) means that the conditional expectations $\mathrm{E}[X_i|S]$ are comonotonic random variables (that is, they all move in the same direction with the total loss S, increasing when S gets larger and decreasing when S gets smaller). Properties of such random variables have been thoroughly reviewed by Dhaene et al. (2002a,b). Denuit (2020b) demonstrated that comonotonicity allows us to derive the allocation of S to the n participants. From a theoretical point of view, requiring that each $\mathrm{E}[X_i|S]$ is a non-decreasing function of S ensures that the conditional mean risk sharing is Pareto-optimal for expected-utility maximizers. In practice, this means that all participants have an interest to keep S as low as possible because their contributions increase with total losses.

Remark 2.2. *Conditional expectations of the form* $\mathrm{E}[X|S]$ *are encountered in many applications. This is for instance the case when we cannot observe directly a variable* X *of interest but we only have at our disposal a signal* S *for* X*. See for instance Denuit (2010), Ganuza and Penalva (2010) and Shaked et al. (2012). Often, the analyst is interested in the* *monotonicity of* $\mathrm{E}[X|S]$ *in the conditioning variable* S*. This may be imposed as a condition or needs to be established to allow for further developments. This happens for instance in the case where the signal is subject to measurement error and the analyst wants to check whether this affects the monotonicity of the regression function. The increasingness of* $\mathrm{E}[X|S]$ *in the conditioning variable* S *is referred to as (positive) regression dependence in the literature. Positive regression dependence thus appears to play a central role in the development of collaborative insurance.*

As explained before, the increasingness of $\mathrm{E}[X_i|S]$ in the total S is crucial in the applications, to ensure that the conditional mean risk sharing principle produces a Pareto-optimal allocation. Considering (2.2), we see that the conditional expectation increases in S when the losses X_i are independent and identically distributed. Turning to heterogeneous losses X_i, general conditions on the distributions of the random variables X_i ensuring that $\mathrm{E}[X_i|S]$ is non-decreasing in S are difficult to establish. A sufficient condition is that the independent random variables X_i possess log-concave densities. This is the case studied by Efron (1965) that is recalled next.

2.4 Efron's asymptotic monotonicity property

Consider independent losses X and Y with respective probability density functions f_X and f_Y such that f_Y is continuously differentiable and denote as f_Y' its first derivative. Saumard and Wellner (2014) established in the proof of their Proposition 6.7 that

$$\frac{\partial}{\partial s}\mathrm{E}[X|X+Y=s] = \mathrm{Cov}\left[X, \frac{f_Y'(Y)}{f_Y(Y)}\middle|X+Y=s\right]. \tag{2.5}$$

Considering condition (2.4), formula (2.5) appears to be intimately connected to the problem investigated in the present paper by taking $X = X_i$ and $Y = S - X_i = \sum_{k\neq i} X_k$.

Assuming that Y possesses a log-concave density, that is, $\ln f_Y$ is a concave function, is equivalent to assume that f_Y'/f_Y decreases. Now, given $X+Y=s$, it is reasonable to believe that X and Y are negatively related so that the covariance appearing in formula (2.5) is positive. This has been formally established by Joag-Dev and Proschan (1983) when X and Y both possess log-concave probability density functions. Hence, formula (2.5) shows that $\frac{\partial}{\partial s}\mathrm{E}[X|X+Y=s] \geq 0$ directly follows in that case.

Denuit and Robert (2020c) proposed an alternative approach, considering that a Central-Limit Theorem applies applies to S so that it asymptotically obeys the Normal distribution. The latter being log-concave, it seems reasonable to expect from Efron (1965) that $\mathrm{E}[X_i|S]$ becomes non-decreasing in S when the number of terms is large enough. Precisely, assume that $\mathrm{Var}[X_i] = \sigma_i^2 < \infty$ holds for all i and denote as $s_n^2 = \sum_{i=1}^n \sigma_i^2$. Denuit and Robert (2020c) established that, under mild technical conditions,

$$\lim_{n\to\infty} s_n \frac{\partial}{\partial\alpha}\mathrm{E}\left[X_i - \mu_i \middle| s_n^{-1}\left(\sum_{k=1}^n (X_k - \mu_k)\right) = \Phi^{-1}(\alpha)\right] = \frac{\sigma_i^2}{\phi\left(\Phi^{-1}(\alpha)\right)} > 0$$

for any probability level $\alpha \in (0,1)$, where Φ denotes the distribution function of the standard Normal distribution and $\phi = \Phi'$ denotes the associated probability density function. Thus, assumption (2.4) appears to be reasonable when the number n of participants is large enough.

2.5 Proportional mean risk sharing

Considering (2.2), we see that the conditional expectation increases linearly in S when individual losses X_i are independent and identically distributed. This can be rewritten as $\mathrm{E}[X_i|S=s]=\frac{\mathrm{E}[X_i]}{\mathrm{E}[S]}s$. When conditional expectations fulfill this relationship, the analytics of P2P collaborative insurance becomes particularly simple and elegant, as shown in the next sections. This is why it is interesting to discuss the conditions under which the latter representation is valid for heterogeneous losses X_i.

Furman et al. (2018) characterized the situation where the identity $\mathrm{E}[X_i|S=s]=\frac{\mathrm{E}[X_i]}{\mathrm{E}[S]}s$ holds true (see Theorem 3.2 and Theorem 4.1 in that paper). Their fundamental result is recalled next.

Property 2.3. *Given two non-negative random variables X and Y with finite, positive means, we have*

$$\mathrm{E}[X|X+Y]=\frac{\mathrm{E}[X]}{\mathrm{E}[X+Y]}(X+Y)$$

if, and only if, one of the two following equivalent statements holds true:

(i) the function $t\mapsto \ln L_X(t)/\ln L_Y(t)$ is constant over $[0,\infty)$, where $L_X(t)=\mathrm{E}[\exp(-tX)]$ and $L_Y(t)=\mathrm{E}[\exp(-tY)]$ are the respective Laplace transforms of X and Y. The constant must be $\mathrm{E}[X]/\mathrm{E}[Y]$.

(ii) the distributional equality $\widetilde{X}+Y=_{\mathrm{d}}X+\widetilde{Y}$ holds true.

Considering Property 3.1 in Denuit (2019), we known that size-biasing a sum of independent random variables $Z_1,\ldots,Z_n$ is equivalent to selecting at random (according to the discrete distribution assigning probability $\mathrm{E}[Z_i]/(\mathrm{E}[Z_1]+\ldots+\mathrm{E}[Z_n])$ to index i) one element appearing in the sum and replacing it with its size-biased version. Thus, the risks Z_i with larger means are more likely to be selected. Item (ii) in Property 2.3 thus implies that

$$\widetilde{X+Y}=_{\mathrm{d}}\widetilde{X}+Y=_{\mathrm{d}}X+\widetilde{Y}.$$

These distributional equalities are obviously true when X and Y are identically distributed but extend beyond this particular case. Here, we identify a large class of risks satisfying this condition. These risks appear to be particularly appealing for applications to insurance.

Assume that X is infinitely divisible. This means that for any positive integer n, X can be represented as the sum of n independent and identically distributed random variables. Infinitely divisible distributions on $[0,\infty)$ correspond either to compound Poisson sums if there is a positive probability mass at the origin, or to weak limits of compound Poisson sums. We refer the reader to Theorems 3.2-3.3 in Steutel and van Harn (2003, Chapter 3). The Poisson distribution, the Negative Binomial distribution, the Gamma distribution and the degenerate distribution are examples of infinitely divisible distributions. The uniform distribution and the Binomial distribution are not infinitely divisible, nor are any other distributions with bounded (finite) support.

Infinite divisibility appears to be intimately connected with size-biasing. Pakes et al. (1996, Theorem 2.1) established that the size-biased transform of any infinitely divisible distributed random variable can be written as the sum of the initial random variable plus a non-negative, independent one. Precisely, the following distributional equality

$$\widetilde{X} =_{\mathrm{d}} X + \Delta \text{ for some random variable } \Delta \geq 0, \text{ independent of X.} \tag{2.6}$$

is valid if, and only if, the distribution of X is infinitely divisible. This result can easily be established by rewriting (2.6) in terms of Laplace transforms. The Laplace transform of $\widetilde{X}$ is given by

$$L_{\widetilde{X}}(t) = \frac{\mathrm{E}[X \exp(-tX)]}{\mathrm{E}[X]} = -\frac{L'_X(t)}{\mathrm{E}[X]}$$

so that (2.6) can be equivalently re-stated as

$$L_{\widetilde{X}}(t) = -\frac{L'_X(t)}{\mathrm{E}[X]} = L_X(t) L_\Delta(t) \Leftrightarrow L_\Delta(t) = -\frac{\frac{\mathrm{d}}{\mathrm{d}t} \ln L_X(t)}{\mathrm{E}[X]}$$

which is a Laplace transform if, and only if, the distribution of X is infinitely divisible, because infinite divisibility is equivalent to $-\frac{\mathrm{d}}{\mathrm{d}t} \ln L_X(t)$ being completely monotone (Theorem 4.1 in Steutel and van Harn, 2003, Chapter 3). Recall that any completely monotone function can be represented as a combination of negative exponential functions according to Bernstein theorem. We refer the reader to Arratia and Goldstein (2010) for a detailed analysis of various situations where (2.6) is valid.

Example 2.4. *We know from Theorem 4.2 in Chapter 3 of Steutel and van Harn (2003) that the Laplace transform of any random variable X possessing and infinitely divisible distribution is of the form*

$$L_X(t) = \exp\left(-\int_0^t \rho(s)\mathrm{d}s\right)$$

where ρ is a completely monotone function on $(0, \infty)$. Let $X_1, X_2, \ldots, X_n$ be independent positive random variable with respective Laplace transforms

$$L_{X_i}(t) = \exp\left(-\int_0^t \lambda_i \rho(s)\mathrm{d}s\right)$$

where ρ is a completely monotone function on $(0, \infty)$ and λ_i is a positive constant. Then, item (i) in Property 2.3 applies and we get

$$\mathrm{E}[X_i | S = s] = \frac{\lambda_i}{\lambda_\bullet} s$$

where $\lambda_\bullet = \lambda_1 + \ldots + \lambda_n$.

Example 2.5 (Semi-homogeneous risk model)**.** *Example 2.4 applies for instance to the case where $X_1, X_2, \ldots, X_n$ obey compound Poisson distributions with identically distributed severities, so that $L_{X_i}(t) = \exp(\lambda_i(L_C(t) - 1))$ where $L_C(\cdot)$ is the Laplace transform of the severities. Precisely, assume that the heterogeneity is confined to claim frequencies, in the sense that*

$$X_i = \sum_{k=1}^{N_i} C_{ik}, \ \ i = 1, 2, \ldots, n,$$

where N_i is Poisson(λ_i) and $C_{i1}, C_{i2}, \ldots$ are positive and identically distributed, all these random variables being independent. If the severities C_{ik} are identically distributed for all i and k then Example 2.4 ensures that

$$\mathrm{E}[X_i|S=s] = \frac{\lambda_i}{\lambda_\bullet}s.$$

See also Denuit (2019, Section 3.4). This result can also be obtained directly from Property 2.3(i) as

$$L_{X_i}(t) = \exp\big(\lambda_i(L_C(t)-1)\big) \Rightarrow \frac{\ln L_{X_i}(t)}{\ln L_{\sum_{j\neq i} X_j}(t)} = \frac{\lambda_i}{\sum_{j\neq i}\lambda_j}$$

is constant. This extends the well-known property of the Poisson distribution (obtained when $C_{ik}=1$) according to which X_i given $S=s$ obeys the Binomial distribution with mean $\frac{\lambda_i}{\lambda_\bullet}s$.

To end with, let us mention that infinite divisibility is only a sufficient condition for the conditional mean risk sharing rule to be linear. For instance, the conditional expectation defining the conditional mean risk sharing is also linear when X_i obeys Binomial distributions with the same success parameter but different exponents whereas the Binomial distribution is not infinitely divisible.

3 Design of the P2P insurance scheme

3.1 Entry price

To ensure that the system does not run out of money, participants must pay a provision, ex-ante so that the sum of the provisions is enough to cover the entirety of the losses, in all cases. This means that the upper layer of the total losses is ceded to a (re-)insurer and that collaborative insurance is restricted to the lower layer.

We assume that premiums and provisions are computed according to the mean-value principle, that is, they are equal to the pure premium (or expected loss) increased by a proportional loading. This is in line with the way premiums are generally computed in commercial lines. The amount π_i paid by participant i who brings a loss X_i is thus supposed to be of the form

$$\pi_i = (1+\theta_{\text{P2P}})\mathrm{E}[X_i] \tag{3.1}$$

where θ_{P2P} is the loading applied within the P2P community. Whatever the actual losses S, the system is designed in such a way that participants will never have to pay more than π_i given in (3.1) and they may even be rewarded by a cash-back mechanism in case of favorable experience.

Denote as θ_{comm} the loading of a regular, commercial insurance product covering the same peril and as θ_{SL} the loading applied by the (re-)insurer covering the higher layer, exceeding the community's risk-bearing capacity. It is reasonable to have $\theta_{\text{SL}} < \theta_{\text{comm}}$ because of the reduced expenses faced when covering only the higher layer. In the remainder of the paper, we assume that the P2P community selects a loading θ_{P2P} comprised between θ_{SL} and θ_{comm}, that is,

$$\theta_{\text{SL}} < \theta_{\text{P2P}} \leq \theta_{\text{comm}}.$$

It is thus possible to charge a provision π_i smaller compared to the premium $(1+\theta_{\text{comm}})\mathrm{E}[X_i]$ of the corresponding regular, commercial insurance contract, making the system attractive. If $\theta_{\text{P2P}} = \theta_{\text{comm}}$ then participants are not granted premium discounts ex-ante but may benefit from the cash-back mechanism, ex-post. This shows that the system operates like a participating insurance policy that could be sold by a commercial insurer, as discussed in Denuit (2020b).

3.2 Assumptions

Throughout this section, we assume that

(i) the losses X_i are valued in $[0,\infty)$ and obey zero-augmented absolutely continuous distributions, that is, $\mathrm{P}[X_i = 0] > 0$ and X_i possesses a probability density function $f_{X_i|X_i>0}$ over $(0,\infty)$.

(ii) the losses $X_1,\ldots,X_n$ are mutually independent, with finite means $\mathrm{E}[X_i] = \mu_i > 0$.

(iii) the conditional expectations $\mathrm{E}[X_i|S]$ are continuously increasing in S for all i so that the functions $s \mapsto \mathrm{E}[X_i|S]$ are one-to-one.

Notice that X_i corresponds to the risk brought by participant i to the pool, after deduction of some individual retention. For instance, this participant could specify per-claim or aggregate deductibles so that X_i only corresponds to the excess over the individual retention level. Also, the pool could (re-)insure part of X_i in case large losses may occur, with the help of any individual treaty (such as excess-of-loss, for instance). In this case, the price of the treaty must simply be added to π_i.

3.3 Determination of the retention capacity of the P2P community

The amount w retained by the pool is the solution to the equation

$$\sum_{i=1}^{n} \pi_i - (1+\theta_{\text{SL}})\mathrm{E}[(S-w)_+] = w. \tag{3.2}$$

Stated in words, (3.2) ensures that the total income $\sum_{i=1}^{n} \pi_i$ allows the organizer to pay for losses up to w and to transfer the upper layer (w,∞) to a (re-)insurer in exchange of the premium

$$\pi_{\text{SL}} = (1+\theta_{\text{SL}})\mathrm{E}[(S-w)_+]$$

where w fulfills equation (3.2). The cash-back mechanism then restores fairness ex-post, when $S < w$.

The next result establishes the existence and unicity of the solution w to equation (3.2), under the assumed inequality $\theta_{\text{SL}} < \theta_{\text{P2P}}$.

Property 3.1. *The solution w to equation* (3.2) *exists and is unique. Moreover, it fulfills* $F_S(w) > \frac{\theta_{SL}}{1+\theta_{SL}}$.

Proof. Consider the continuous function φ defined as

$$\varphi(\xi)=\xi+(1+\theta_{\mathrm{SL}})\mathrm{E}[(S-\xi)_+].$$

Equation (3.2) then requires to find where φ crosses the horizontal line at height $\sum_{i=1}^n \pi_i$. Notice that

$$\sum_{i=1}^n \pi_i=\sum_{i=1}^n(1+\theta_{\mathrm{P2P}})\mathrm{E}[X_i]>\sum_{i=1}^n(1+\theta_{\mathrm{SL}})\mathrm{E}[X_i]=\varphi(0)$$

so that the solution cannot be located at the origin. Since

$$\varphi'(\xi)=1-(1+\theta_{\mathrm{SL}})\overline{F}_S(\xi),$$

the function φ is increasing if

$$\varphi'(\xi)>0\Leftrightarrow\overline{F}_S(\xi)<\frac{1}{1+\theta_{\mathrm{SL}}}\Leftrightarrow\xi>F_S^{-1}\left(\frac{\theta_{\mathrm{SL}}}{1+\theta_{\mathrm{SL}}}\right)=\xi_0.$$

Thus, the function φ starts below $\sum_{i=1}^n \pi_i$ at the origin, then it decreases until ξ_0 where it reaches its minimum before starting to increase at ξ_0. Since $\varphi(\xi)\to\infty$ as $\xi\to\infty$, there must be a unique solution to equation (3.2), as announced. □

Notice that $F_S(w)$ is the probability that there is some relief to be distributed among participants. We thus see from Property 3.1 that the probability that the cash-back mechanism operates is at least equal to $\frac{\theta_{\mathrm{SL}}}{1+\theta_{\mathrm{SL}}}$. Thus, this probability is at least equal to 1/2 if $\theta_{\mathrm{SL}}=1$, for instance.

Remark 3.2. *In the limiting case $\theta_{P2P}=\theta_{SL}$, we see that $w=0$ solves equation* (3.2) *but that there is also another positive root $w>0$ to* (3.2)*. This means that the community could just cede the entirety of the losses to the (re-)insurer, without any cash-back mechanism (total risk transfer), or to set up a P2P insurance scheme covering the first layer* $(0,w)$*, with $w>0$, to benefit from a possible cash-back mechanism, with probability $F_S(w)$. The latter option is clearly preferable since it comprises a possible gain, ex-post, thanks to the cash-back mechanism.*

3.4 Individual, ex-ante contributions

Proceeding as in Denuit (2020b), each participant is liable for an amount

$$w_i=F^{-1}_{\mathrm{E}[X_i|S]}\big(F_S(w)\big)\text{ with }\sum_{i=1}^n w_i=w.\tag{3.3}$$

The amounts $w_1,\ldots,w_n$ defined in this way are such that

$$(S-w)_+=\sum_{i=1}^n\big(\mathrm{E}[X_i|S]-w_i\big)_+\text{ and }(w-S)_+=\sum_{i=1}^n\big(w_i-\mathrm{E}[X_i|S]\big)_+.\tag{3.4}$$

The key to understand decompositions appearing in (3.4) is as follows: the total loss S is not larger than w with probability $F_S(w)$. Since each $\mathrm{E}[X_i|S]$ is a continuously increasing function of S, we have

$$S \leq w \Leftrightarrow \mathrm{E}[X_i|S] \leq w_i \text{ for } i = 1, 2, \ldots, n,$$

where w_i is defined according to (3.3). This shows that the identities (3.4) are indeed valid.

Considering (3.4), the initial contribution π_i in (3.1) can then be split into

$$\pi_i = \pi_i^{\mathrm{P2P}} + \pi_i^{\mathrm{SL}} \text{ where } \pi_i^{\mathrm{SL}} = (1+\theta_{\mathrm{SL}})\mathrm{E}\big[\big(\mathrm{E}[X_i|S] - w_i\big)_+\big].$$

The total loss S is well covered: the upper layer (w, ∞) is transferred to the (re-)insurer for the payment π_i^{SL} and

$$\sum_{i=1}^n \pi_i - \sum_{i=1}^n \pi_i^{\mathrm{SL}} = w \text{ with } \pi_{\mathrm{SL}} = \sum_{i=1}^n \pi_i^{\mathrm{SL}},$$

by virtue of (3.2). Thus, the lower layer $(0, w)$ is covered by the contributions

$$\begin{aligned} \pi_i^{\mathrm{P2P}} &= (1+\theta_{\mathrm{P2P}})\mathrm{E}\big[X_i\big] - (1+\theta_{\mathrm{SL}})\mathrm{E}\big[\big(\mathrm{E}[X_i|S] - w_i\big)_+\big] \\ &= (\theta_{\mathrm{P2P}} - \theta_{\mathrm{SL}})\mathrm{E}\big[X_i\big] + (1+\theta_{\mathrm{SL}})\Big(\mathrm{E}\big[X_i\big] - \mathrm{E}\big[\big(\mathrm{E}[X_i|S] - w_i\big)_+\big]\Big) \\ &= (\theta_{\mathrm{P2P}} - \theta_{\mathrm{SL}})\mathrm{E}\big[X_i\big] + (1+\theta_{\mathrm{SL}})\mathrm{E}\big[\min\big\{\mathrm{E}[X_i|S], w_i\big\}\big] \end{aligned}$$

paid ex-ante by participants. In the limiting case $\theta_{\mathrm{P2P}} = \theta_{\mathrm{SL}}$, we see that

$$\pi_i^{\mathrm{P2P}} = (1+\theta_{\mathrm{SL}})\mathrm{E}\big[\min\big\{\mathrm{E}[X_i|S], w_i\big\}\big].$$

However, the provisions paid ex-ante by the participants may individually depart from the liability levels w_i. This mismatch will be solved in Section 3.6. Before, we compare the proposed mechanism to insurance.

3.5 Comparison with insurance

Let us carefully compare traditional insurance with its collaborative counterpart. In both cases, the loss X_i caused by the peril under consideration is reimbursed to individual i so that we concentrate on premiums or provisions paid ex-ante and possible cash-backs accompanying collaborative insurance.

3.5.1 Classical insurance solution

In a traditional insurance scheme involving one insurance company covering n policyholders for a given peril, individual premiums are of the form $\pi_i^{\mathrm{comm}} = (1+\theta_{\mathrm{comm}})\mathrm{E}[X_i]$. The sum of individual premiums $\sum_{i=1}^n \pi_i^{\mathrm{comm}}$ is used to pay for claims filed by the policyholders amounting to S. A solvency capital is used to face situations where S exceeds the total premium collected and π_i^{comm} comprises the dividends to be paid to stock-holders bringing risk-bearing capacities.

3.5.2 Collaborative insurance and cash-back mechanism

In the P2P insurance scheme proposed in this paper, involving one community of n participants and a (re-)insurance company protecting the upper risk layer, individual provisions of the form $\pi_i = (1+\theta_{\text{P2P}})\mathrm{E}[X_i]$ are paid ex-ante, see (3.1), where $\theta_{\text{P2P}} \leq \theta_{\text{comm}}$ so that the offer is competitive with respect to the traditional insurance scheme. The key point here is that π_i does not depend on the number of participants, nor of the losses X_k, $k \neq i$, they bring to the community, but only on the expected loss for participant i. Exactly as in traditional insurance, the inclusion of a new participant leaves π_i unchanged. But recruiting new members modifies the distribution of S which materializes in the value of the retention level w solving equation (3.2) and on the cash-back mechanism.

Notice also that the individual retention level w_i defined according to (3.3) can be rewritten as

$$w_i = \mathrm{E}\big[X_i \big| S = F_S^{-1}\big(F_S(w)\big)\big] = \mathrm{E}[X_i|S=w]$$

that obviously sum to w. This is because $s \mapsto \mathrm{E}[X_i|S=s]$ is continuously increasing in s; we refer the interested reader to Denuit (2020b) for more general formulas. This shows how recruiting new members impacts individual i: the provision π_i paid ex-ante is left unchanged but the cash-back mechanism generating the end-of-period cash-flow $\big(w_i - \mathrm{E}[X_i|S]\big)\mathrm{I}[S \leq w]$ is modified. The latter may be more or less favorable to the participant, depending on the community he or she joins. But with the pricing structure adopted here, collaborative insurance cannot be more expensive compared to traditional insurance cover.

3.6 Proportional mean risk allocation

If the conditional expectations defining the conditional mean risk sharing principle is linear in the total loss S then we obtain intuitive formulas that comply with intuition. Precisely, assume that

$$\mathrm{E}[X_i|S] = \frac{\mathrm{E}[X_i]}{\mathrm{E}[S]} S \text{ for } i = 1, 2, \ldots, n. \tag{3.5}$$

This is the case under the conditions stated in Property 2.3. With entry price π_i given in (3.1), the latter identity can be rewritten as

$$\mathrm{E}[X_i|S] = \frac{\pi_i}{\sum_{k=1}^n \pi_k} S \text{ for } i = 1, 2, \ldots, n,$$

so that losses are shared in proportion of the initial contributions paid by participants. This makes the system particularly simple to explain to the members of the community.

The individual levels w_i defined according to (3.3) can then be rewritten as

$$w_i = F^{-1}_{\mathrm{E}[X_i|S]}\big(F_S(w)\big) = \frac{\mathrm{E}[X_i]}{\mathrm{E}[S]} F_S^{-1}\big(F_S(w)\big) = \frac{\mathrm{E}[X_i]}{\mathrm{E}[S]} w. \tag{3.6}$$

Considering (3.2), we obtain

$$\begin{aligned}
w_i &= \frac{\mathrm{E}[X_i]}{\mathrm{E}[S]}\Big((1+\theta_{\text{P2P}})\mathrm{E}[S] - (1+\theta_{\text{SL}})\mathrm{E}[(S-w)_+]\Big) \\
&= \pi_i - (1+\theta_{\text{SL}})\frac{\mathrm{E}[X_i]}{\mathrm{E}[S]}\mathrm{E}[(S-w)_+] \\
&= \pi_i - (1+\theta_{\text{SL}})\mathrm{E}\big[\big(\mathrm{E}[X_i|S]-w_i\big)_+\big] \text{ by (3.5) and (3.6)} \\
&= \pi_i - \pi_i^{\text{SL}} \\
&= \pi_i^{\text{P2P}}
\end{aligned}$$

so that the identity $\pi_i^{\text{P2P}} = w_i$ holds true for every participant. Hence, the provisions paid ex-ante by the participants match their individual liability level.

The proposed system is fair. Indeed, the result of the operation for participant i is

$$\text{benefit - initial contribution + cash-back} = X_i - \pi_i + (w_i - \mathrm{E}[X_i|S])_+.$$

In the linear case, this can be rewritten as

$$X_i - w_i - (1+\theta_{\text{SL}})\mathrm{E}\big[\big(\mathrm{E}[X_i|S]-w_i\big)_+\big] + (w_i - \mathrm{E}[X_i|S])_+.$$

On average, this is equal to $-\theta_{\text{SL}}\mathrm{E}\big[\big(\mathrm{E}[X_i|S]-w_i\big)_+\big]$, that is, to the loading charged by the (re-)insurer protecting the higher layer. Hence, the operation is fair among participants since it induces no transfer within the P2P community, on average. Recall that the system designed by Denuit (2020b) requires $\pi_i = w_i$. This may exclude some risk profiles because the amount w_i may be high, but this is needed to restore fairness. In the linear case, we see that we can match the ex-ante provision w_i with π_i^{P2P} and keep a fair system.

Example 3.3. *These results apply to the case where all individual losses X_i obey compound Poisson distribution with severities identically distributed among participants (who only differ in their respective expected loss frequencies λ_i) as considered in Example 2.5. Specifically, we know that the identity*

$$w_i = \mathrm{E}[X_i|S=w] = \frac{\lambda_i}{\lambda_\bullet}w$$

holds true in that case. As pointed out by Denuit (2020b, Section 4.4), this equality holds true if we select w_i of the form

$$w_i = (1+\theta)\lambda_i m \textit{ where } m \textit{ is the expected severity,}$$

whatever the retention rate θ provided it applies to all participants. The retention of the group then amounts to a multiple of the mean loss:

$$w = (1+\rho)\sum_{i=1}^{n}\lambda_i m = (1+\theta)\mathrm{E}[S].$$

This is in line with the specification (3.1) selected for the entry price π_i.

4 Two-level P2P insurance community

4.1 Hierarchical risk sharing

Often, participants are partitioned into smaller communities corresponding to social groups (friends or relatives), or to individuals with common economic interest or located in the same geographic area, for instance. The system then proceeds in two steps: the risk is first shared between the different groups and the part allocated to each group is then shared among its members. Precisely, we consider here that the n participants to the P2P insurance pool are divided into p groups of respective sizes n_j for $j = 1, ..., p$, with $n = \sum_{j=1}^{p} n_j$. We denote as $X_{j,i}$ the risk brought by participant i, belonging to group j. By risk, we mean a non-negative random variable representing a monetary loss, with positive expected value.

Throughout this section, we assume that

(i) the losses $X_{j,i}$ are valued in $[0,\infty)$ and obey zero-augmented absolutely continuous distributions, that is, $\mathrm{P}[X_{j,i} = 0] > 0$ and $X_{j,i}$ possesses a probability density function $f_{X_{j,i}|X_{j,i}>0}$ over $(0,\infty)$.

(ii) the losses are mutually independent, with finite means $\mathrm{E}[X_{j,i}] = \mu_{j,i} > 0$.

(iii) the conditional expectations $\mathrm{E}[X_{j,i}|S]$ are continuously increasing in S for all i and j.

As before, $X_{j,i}$ corresponds here to the risk pooled by participant i within group j, after deduction of some individual retention (per-claim or aggregate deductibles, for instance).

In the remainder of this section, we denote the total loss for group j as

$$S_j = \sum_{i=1}^{n_j} X_{j,i}.$$

Clearly, the grand total for the whole community can be obtained from $S = \sum_{j=1}^{p} S_j$. Notice that by assumption (iii), the conditional expectations $\mathrm{E}[S_j|S]$ are also continuously increasing in S for all j. Since the conditional expectation $\mathrm{E}[S_j|S] = \sum_{i=1}^{n_j} \mathrm{E}[X_{j,i}|S]$ is one-to-one in S, we have

$$\mathrm{E}\big[X_{i,j}\big|\mathrm{E}[S_j|S]\big] = \mathrm{E}\big[X_{i,j}\big|S\big] \tag{4.1}$$

and we recover the conditional mean risk sharing rule. We refer to Lemma 2.2 in Shaked et al. (2012) for a formal proof of (4.1). Identity (4.1) shows that we can proceed in two steps: first, we allocate $\mathrm{E}[S_j|S]$ to group j and then we allocate the conditional expectation of $X_{j,i}$ given the structural loss $\mathrm{E}[S_j|S]$ of group j, individually to participant i in that group. This is equivalent to allocating $\mathrm{E}[X_{j,i}|S]$ to this participant.

As before, participants must pay a provision, ex-ante so that the sum of the provisions is enough to cover the entirety of the losses, in all cases. In line with the specification (3.1), the amount $\pi_{j,i}$ paid by participant i in group j is equal to

$$\pi_{j,i} = (1 + \theta_{\mathrm{P2P}})\mathrm{E}[X_{j,i}]$$

where θ_{P2P} is the loading applied by the organizer, with $\theta_{\mathrm{SL}} < \theta_{\mathrm{P2P}} \leq \theta_{\mathrm{comm}}$.

4.2 Determination of the retention capacity of the P2P community

At the level of the entire community, we determine a retention capacity as explained previously. Precisely, the amount retained by the pool is the solution w to the equation

$$\sum_{j=1}^{p}\sum_{i=1}^{n_j}\pi_{j,i} - (1+\theta_{\mathrm{SL}})\mathrm{E}[(S-w)_+] = w. \tag{4.2}$$

Proceeding as in the proof of Property 3.1, we can show that the solution w to equation (4.2) exists and is unique. Moreover, it fulfills $F_S(w) > \frac{\theta_{\mathrm{SL}}}{1+\theta_{\mathrm{SL}}}$.

4.3 Determination of retention levels for each group

The total loss S is first allocated to each group j according to the risk sharing rule

$$S = \sum_{j=1}^{p}\mathrm{E}[S_j|S]$$

where each $\mathrm{E}[S_j|S]$ is continuously increasing in S. Each group thus bears its structural part $\mathrm{E}[S_j|S]$ in S and is liable for an amount w_j given by

$$w_j = F^{-1}_{\mathrm{E}[S_j|S]}\big(F_S(w)\big) \text{ such that } \sum_{j=1}^{p} w_j = w.$$

The following useful identities are valid:

$$(S-w)_+ = \sum_{j=1}^{p}(\mathrm{E}[S_j|S]-w_j)_+ \text{ and } (w-S)_+ = \sum_{j=1}^{p}(w_j-\mathrm{E}[S_j|S])_+.$$

These formulas describe how to distribute the total loss S between the groups. They are easily obtained as follows:

$$\begin{aligned}
\sum_{j=1}^{p}(w_j-\mathrm{E}[S_j|S])_+ &= \sum_{j=1}^{p}(w_j-\mathrm{E}[S_j|S])\mathrm{I}[\mathrm{E}[S_j|S]\leq w_j] \\
&= \sum_{j=1}^{p}(w_j-\mathrm{E}[S_j|S])\mathrm{I}[S\leq w] \\
&= \mathrm{I}[S\leq w]\sum_{j=1}^{p}(w_j-\mathrm{E}[S_j|S]) \\
&= (w-S)_+.
\end{aligned}$$

The levels w_j corresponding to quantiles of $\mathrm{E}[S_j|S]$ at probability level $F_S(w)$ are those for which this decomposition holds.

These levels also give the respective contributions of each group in the price of the stop-loss protection as

$$\mathrm{E}[(S-w)_+] = \sum_{j=1}^{p} \mathrm{E}\big[\big(\mathrm{E}[S_j|S] - w_j\big)_+\big].$$

Formula (68) in Dhaene et al. (2002a) shows that the inequality

$$\sum_{j=1}^{p} \mathrm{E}\big[\big(\mathrm{E}[S_j|S] - v_j\big)_+\big] \geq \sum_{j=1}^{p} \mathrm{E}\big[\big(\mathrm{E}[S_j|S] - w_j\big)_+\big] = \mathrm{E}\left[(S-w)_+\right].$$

holds true for any other set of group-specific retentions $v_1, \ldots, v_p$ such that $\sum_{j=1}^{p} v_j = w$. Hence, the stop-loss protection written on S is equivalent to write p group-specific covers on $\mathrm{E}[S_j|S]$ with retention w_j, which appears to be cheaper compared to any other group-specific stop-loss protection with retention v_j summing to w. The contribution to the upper layer for group j is

$$\pi_j^{\mathrm{SL}} = (1+\theta_{\mathrm{SL}})\mathrm{E}\big[\big(\mathrm{E}[S_j|S] - w_j\big)_+\big].$$

4.4 Cash-back within groups

We can then split the amounts of cash-back among participants within each group using formulas

$$(\mathrm{E}[S_j|S] - w_j)_+ = \sum_{i=1}^{n_j} \big(\mathrm{E}\big[X_{j,i}\big|\mathrm{E}[S_j|S]\big] - w_{j,i}\big)_+ = \sum_{i=1}^{n_j} \big(\mathrm{E}[X_{j,i}|S] - w_{j,i}\big)_+$$

and

$$(w_j - \mathrm{E}[S_j|S])_+ = \sum_{i=1}^{n_j} \big(w_{j,i} - \mathrm{E}\big[X_{j,i}\big|\mathrm{E}[S_j|S]\big]\big)_+ = \sum_{i=1}^{n_j} \big(w_{j,i} - \mathrm{E}[X_{j,i}|S]\big)_+$$

$$\text{with } w_{j,i} = F^{-1}_{\mathrm{E}[X_{j,i}|S]}\big(F_{\mathrm{E}[S_j|S]}(w_j)\big) \text{ such that } \sum_{i=1}^{n_j} w_{j,i} = w_j.$$

We will see in the next section that these formulas become particularly simple to apply when the conditional mean risk sharing is linear.

4.5 Proportional mean risk allocation

If the conditional expectation defining the conditional mean risk sharing is linear in the total loss S for every participant as stated in (3.5), that is, if the conditions of Property 2.3 are fulfilled, then we also have

$$\mathrm{E}[S_j|S] = \sum_{i=1}^{n_j} \mathrm{E}[X_{j,i}|S] = \sum_{i=1}^{n_j} \frac{\mathrm{E}[X_{j,i}]}{\mathrm{E}[S]} S = \frac{\mathrm{E}[S_j]}{\mathrm{E}[S]} S.$$

With entry price given in (3.1), the latter identity can be rewritten as

$$\mathrm{E}[S_j|S] = \frac{\sum_{i=1}^{n_j} \pi_{j,i}}{\sum_{k=1}^{p} \sum_{i=1}^{n_k} \pi_{k,i}} S \text{ for } j = 1, 2, \ldots, p,$$

so that losses are shared between groups in proportion of the total contributions paid by their respective participants. This makes the system particularly simple and transparent.

Hence, the liability for group j

$$\begin{aligned} w_j &= F^{-1}_{\mathrm{E}[S_j|S]}\big(F_S(w)\big) \\ &= \frac{\mathrm{E}[S_j]}{\mathrm{E}[S]}w \\ &= \frac{\mathrm{E}[S_j]}{\mathrm{E}[S]}\Big((1+\theta_{\mathrm{P2P}})\mathrm{E}[S]-(1+\theta_{\mathrm{SL}})\mathrm{E}[(S-w)_+]\Big) \\ &= (1+\theta_{\mathrm{P2P}})\mathrm{E}[S_j]-(1+\theta_{\mathrm{SL}})\frac{\mathrm{E}[S_j]}{\mathrm{E}[S]}\mathrm{E}[(S-w)_+] \\ &= (1+\theta_{\mathrm{P2P}})\mathrm{E}[S_j]-(1+\theta_{\mathrm{SL}})\mathrm{E}\big[\big(\mathrm{E}[S_j|S]-w_j\big)_+\big] \end{aligned}$$

matches the total provision $\sum_{i=1}^{n_j}\pi_{j,i}$ minus the contribution to the stop-loss cover.

The proposed system is fair between groups. Indeed, the result of the operation for group j is

$$\text{benefit - initial contribution + cash-back} = S_j - \sum_{i=1}^{n_j}\pi_{j,i} + (w_j - \mathrm{E}[S_j|S])_+.$$

This can be rewritten as

$$S_j - w_j - (1+\theta_{\mathrm{SL}})\mathrm{E}\big[\big(\mathrm{E}[S_j|S]-w_j\big)_+\big] + (w_j - \mathrm{E}[S_j|S])_+.$$

On average, this is equal to $-\theta_{\mathrm{SL}}\mathrm{E}\big[\big(\mathrm{E}[S_j|S]-w_j\big)_+\big]$, that is, to the loading charged by the (re-)insurer. Hence, the operation is fair among groups of participants since it induces no transfer between groups within the P2P community, on average. It can also be shown that the proposed system is fair within groups. Fairness in the hierarchical case can also be seen as a direct consequence of the fairness for each participant, so that this property also holds when participants are grouped, creating an intermediate level at which fairness also automatically holds true.

Remark 4.1. *An additional degree of flexibility is obtained by allowing for different risk sharing mechanisms between the groups and among the members of each specific group. For instance, we might allocate the total losses between groups with the help of the conditional mean risk sharing rule, in order to avoid any undue transfer, but participants within groups might be happy to resort to the simpler proportional mean risk allocation, assigning the amount* $\frac{\pi_{j,i}}{\sum_{k=1}^{n_j}\pi_{j,k}}\mathrm{E}[S_j|S]$ *to participant i in group j even if the conditions of Property 2.3 are not valid.*

5 Conclusion

In this paper, we have designed a collaborative insurance scheme limiting the contributions paid by participants to an initial, ex-ante provision calculated by adding a proportional loading to the pure premium (or expect loss). A cash-back mechanism operates ex-post, to restore fairness. Because the realization of actual losses X_i is mostly due to chance, revealing only partly the underlying magnitude of the risk brought to the community, the system assigns structural losses $\mathrm{E}[X_i|S]$ to participants whereas the departures $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n$ are re-allocated among them according to the mutuality principle. See decomposition (1.1).

Risk sharing is limited to the lower layer which acts as a pooled deductible. Within P2P insurance communities, the risk borne by each participant can be determined in proportion of the initial contributions (proportional mean risk sharing). The total risk-bearing capacity for the lower layer is then determined accordingly. Whereas pure P2P insurance does not include any guarantee, the collaborative insurance scheme designed here supplements risk sharing with risk transfer of a higher layer, operated by a (re-)insurer.

To ensure full transparency, it is important to inform participants by issuing regular reports that disclose meaningful indicators. A yearly summary can be issued to policyholders to describe loss experience over the past period and make explicit the amount of risk sharing and risk transfer, listing for participant i actual costs X_i together with the part $\mathrm{E}[X_i|S]$ that can be attributed to participant i, individually and the departure $\mathcal{E}_i$ mutualized inside the community. All calculations are based on incurred amounts. This is particularly important as the yearly account must assess the overall costs. All open and IBNR cases at the end of the period are "sold" to the (re-)insurer which settles them for a given price.

Insurance activities require risk-bearing capital provided by stock-holders. The amount of capital required by regulatory authorities is then paid by each policyholder, in a cost-of-capital approach. This burden is avoided for the lower layer but still present for the cover protecting the upper layer. The corresponding premium can be considered as part of running expenses and benefits can be allocated to each policy. In case the community also buys excess-of-loss covers, the policy-specific reinsurance premium can be included in the yearly account whereas the stop-loss premium can be allocated to each participant according to the respective "use" of the upper layer.

The collaborative insurance offer can also broaden the range of risks that can be covered. Traditionally, risks with low severity have been excluded from insurance because administrative and settlement expenses were considered to be too expensive. With the automation of the customer relationship and of the claim settlement process, this point of view becomes somewhat outdated and this offers a huge potential new market for insurance providers.

In this way, insurers could broaden the range of risks covered to a very large extent, diversifying the offered services and investing in the coverage of small risks that have been so far excluded from insurance policies. Insurance has traditionally been restricted to low frequency-high severity risks, because of expensive claim-handling costs. But nowadays, consumers are ready to pay for serenity/peace of mind. Consequently, they also cover small pieces of equipment (like smartphones), pay "premiums" for extended warranty or buy comprehensive maintenance and assistance contracts (e.g. from energy providers). There is thus a trend towards covering high frequency-low severity risks to get rid of their time-consuming management or to smooth household budget by paying fixed-amount monthly fees (for instance medical out-patient insurance cover). These risks are even easier to model and manage compared to traditional insurance risks, but their coverage requires automated handling and adapted procedures (such as direct repair though a network of carefully selected, continuously assessed and rewarded sub-contractors) to be economically feasible. Thus, the approach proposed in this paper turns out to be effective to expand coverages to smaller risks that are easy to manage through P2P insurance.

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