I N S T I T U T D E
S T A T I S T I Q U E

UNIVERSITÉ CATHOLIQUE DE LOUVAIN

D I S C U S S I O N
P A P E R

0419

NONPARAMETRIC EFFICIENCY ANALYSIS: A MULTIVARIATE CONDITIONAL QUANTILE APPROACH

DAOUIA, A. and L. SIMAR

http://www.stat.ucl.ac.be

Nonparametric Efficiency Analysis: A Multivariate Conditional Quantile Approach

Abdelaati Daouia*
GREMAQ, Université de Toulouse I
and LSP, Université de Toulouse III, France
Abdelaati.Daouia@math.ups-tlse.fr

Léopold Simar†
Institut de Statistique
Université Catholique de Louvain, Belgium
simar@stat.ucl.ac.be

July 05, 2004

Abstract

This paper proposes a unifying probabilistic framework for efficiency and productivity analysis in a complete multivariate setup (multiple inputs and multiple outputs). Properties of the Farrell's efficiency scores are derived in terms of the characteristics of the probability distribution of the data generating process. This allows to introduce a notion of α-quantile efficiency scores related to a non-standard conditional α-quantile frontier. Nonparametric estimators are then naturally introduced providing estimators of the production frontier more robust to outliers and/or extreme values than the traditional envelopment estimators (FDH/DEA). The paper defines a new concept of efficiency measurement, analyzes its properties and proposes a nonparametric estimator with all its asymptotic properties. Numerical examples (simulated data and mutual funds data) illustrate its practical use. This extends and generalizes previous works of Cazals, Florens and Simar (2002), Aragon, Daouia and Thomas-Agnan (2002) and Daraio and Simar (2003).

Key words : Efficiency; Frontier estimation; Robust nonparametric estimators; Conditional quantiles.

JEL Classification: C13, C14, D20.

---

*This work was supported in part by "Actions Thématiques de l'Université Paul Sabatier", Toulouse III.
†Research support from "Projet d'Actions de Recherche Concertées" (No. 98/03–217) and from the "Interuniversity Attraction Pole", Phase V (No. P5/24) from the Belgian Government are also acknowledged.

1 Introduction and Basic Concepts

Foundations of the economic theory on productivity and efficiency analysis date back to the works of Koopmans (1951) and Debreu (1951) on activity analysis. Shephard (1970) proposes a modern formulation of the problem. Following these lines, we consider a production technology where the activity of the production units is characterized by a set of inputs $x \in \mathbb{R}_+^p$ used to produce a set of outputs $y \in \mathbb{R}_+^q$. In this framework the production set is the set of technically feasible combinations of (x, y). It is defined as

$$\Psi = \{(x, y) \in \mathbb{R}_+^{p+q} \mid x \text{ can produce } y\}. \tag{1.1}$$

Assumptions are usually done on this set, such as free disposability of inputs and outputs, meaning that if $(x, y) \in \Psi$, then $(x', y') \in \Psi$, as soon as[1] $x' \geq x$ and $y' \leq y$. Often convexity of Ψ is also assumed, and no free lunches (if $y \geq 0$ with $y \neq 0$, $(0, y) \notin \Psi$, see Shephard, 1970, for more details). The production set can be described in terms of its sections:

$$\begin{aligned} &\text{Input requirement sets} \;\;: \quad \forall y \in \mathbb{R}_+^q, \quad X(y) = \{x \in \mathbb{R}_+^p \mid (x, y) \in \Psi\} \\ &\text{Output requirement sets}: \quad \forall x \in \mathbb{R}_+^p, \quad Y(x) = \{y \in \mathbb{R}_+^q \mid (x, y) \in \Psi\} \end{aligned}$$

As far as efficiency is of concern, the boundaries of Ψ are of interest. The efficient boundary (frontier) of Ψ is the locus of optimal production scenarios (minimal achievable input level for a given output or maximal achievable output given the input). The Farrell-Debreu efficient frontier is defined in a "radial sense". It can be described either in the input space:

$$\begin{aligned} \forall y \in \mathbb{R}_+^q, X^\partial(y) &= \{(x^\partial(y), y) \mid x^\partial(y) \in X(y) : \theta x^\partial(y) \notin X(y), \ \forall \theta < 1\} \qquad (1.2) \\ &= \{(x^\partial(y), y) \mid x^\partial(y) \in X(y) : (\theta x^\partial(y), y) \notin \Psi, \ \forall \theta < 1\}, \end{aligned}$$

where the points $x^\partial(y)$ represent the minimal inputs necessary to produce the output y, or in the output space:

$$\begin{aligned} \forall x \in \mathbb{R}_+^p, Y^\partial(x) &= \{(x, y^\partial(x)) \mid y^\partial(x) \in Y(x) : \lambda y^\partial(x) \notin Y(x), \ \forall \lambda > 1\} \qquad (1.3) \\ &= \{(x, y^\partial(x)) \mid y^\partial(x) \in Y(x) : (x, \lambda y^\partial(x)) \notin \Psi, \ \forall \lambda > 1\}, \end{aligned}$$

where the points $y^\partial(x)$ are the maximal outputs a unit operating at the level x can produce. Finally the Farrell-Debreu efficiency scores for a given production scenario $(x, y) \in \Psi$, are defined as:

$$\text{Input oriented} \;\;: \quad \theta(x, y) = \inf\{\theta \mid (\theta x, y) \in \Psi\} = \inf\{\theta \mid \theta x \in X(y)\} \tag{1.4}$$

$$\text{Output oriented}: \quad \lambda(x, y) = \sup\{\lambda \mid (x, \lambda y) \in \Psi\} = \sup\{\lambda \mid \lambda y \in Y(x)\} \tag{1.5}$$

[1]From here and below inequalities between vectors $a, b \in \mathbb{R}^k$ have to be understood element by element. Writing $a \leq b$ means $a_i \leq b_i$, for $i = 1, \ldots, k$.

If (x,y) is inside Ψ, $\theta(x,y) \leq 1$ is the proportionate reduction of inputs a unit working at the level (x,y) should perform to achieve efficiency. If $\theta(x,y) = 1$, the unit is on the efficient frontier of Ψ. This frontier is then described as the set $\cup_{y\in\mathbb{R}^q_+} X^\partial(y)$, where $X^\partial(y) = \{(\theta(x,y)\,x, y)\,|\, x \text{ is such that } (x,y) \in \Psi\}$. Each optimal achievable level of input $x^\partial(y) = \theta(x,y)\,x$ represents the radial projection of (x,y) on the frontier, in the input direction (orthogonal to the vector y).

In the output direction, we have $\lambda(x,y) \geq 1$ represents the proportionate increase of outputs the unit operating at level (x,y) should attain to be considered as being efficient. The efficient frontier of Ψ can thus also be described as the set $\cup_{x\in\mathbb{R}^p_+} Y^\partial(x)$, where $Y^\partial(x) = \{(x, \lambda(x,y)\,y)\,|\, y \text{ is such that } (x,y) \in \Psi\}$. Here, $y^\partial(x) = \lambda(x,y)\,y$ is the radial projection of (x,y) on the frontier, in the output direction (orthogonal to the vector x).

In practice Ψ is unknown and so has to be estimated from a random sample of production units $\{(X_i, Y_i)\,|\, i = 1, \ldots, n\}$, where we assume that $\text{Prob}((X_i, Y_i) \in \Psi) = 1$ (refered in the literature as deterministic frontier models). So the problem is related to the problem of estimating the support of the random variable (X, Y) where, for mathematical convenience, we will assume that Ψ is compact. The most popular nonparametric estimators are based on the envelopment ideas: we search for estimators of Ψ which envelops at best the observed data points. The statistical properties of these estimators are now well established (see *e.g.* Simar and Wilson, 2000, for a recent survey).

The most flexible nonparametric estimator, initiated by Deprins, Simar and Tulkens (1984), is the Free Disposal Hull (FDH) estimator. It is provided by the free disposal hull of the sample points:

$$\widehat{\Psi}_{FDH} = \left\{(x,y) \in \mathbb{R}^{p+q}_+ \,|\, y \leq Y_i,\ x \geq X_i, \quad i = 1, \ldots, n\right\}. \tag{1.6}$$

The FDH efficiency scores are obtained by plugging $\widehat{\Psi}_{FDH}$ in equations (1.4) and (1.5) in place of the unknown Ψ. The asymptotic properties of the resulting estimators are provided by Park, Simar and Weiner (2000). In summary, the error of estimation converges at a rate $n^{1/(p+q)}$ to a limiting Weibull distribution.

If we assume that Ψ is convex, the convex hull of $\widehat{\Psi}_{FDH}$ provides the Data Envelopment Analysis (DEA) estimator of Ψ, initiated by Farrell (1957) and popularized as linear programming estimator by Charnes, Cooper and Rhodes (1978). It is defined as

$$\begin{aligned}\widehat{\Psi}_{DEA} \quad = \quad & \{(x,y) \in \mathbb{R}^{p+q}_+ \,|\, y \leq \sum_{i=1}^n \gamma_i Y_i\,;\, x \geq \sum_{i=1}^n \gamma_i X_i \text{ for } (\gamma_1, \ldots, \gamma_n) \\ & \qquad \text{such that } \sum_{i=1}^n \gamma_i = 1\,;\, \gamma_i \geq 0, i = 1, \ldots, n\}. \end{aligned} \tag{1.7}$$

It is the smallest free disposal convex set covering all the data points. When using $\widehat{\Psi}_{DEA}$ as the estimated attainable set, the asymptotic properties of resulting DEA efficiency scores have been investigated in Kneip, Park and Simar (1998) and Kneip, Simar and Wilson (2003). In summary, the error of estimation converges at a rate $n^{2/(p+q+1)}$ to a limiting nondegenerate distribution (when $p = q = 1$, Gijbels, Mammen, Park and Simar (1999) provide an explicit expression for this limiting distribution).

The FDH/DEA estimators envelop all the data points and so are very sensitive to outliers and/or to extreme values. Cazals, Florens and Simar (2002) have introduced the concept of partial frontiers (order-m frontiers) with a nonparametric estimator which does not envelop all the data points. For instance, in the input oriented case, in place of looking for the lower boundary of $X(y)$, as it is the case for defining $\theta(x, y)$, the order-m efficiency score $\theta_m(x, y)$ can roughly be viewed as the expectation of the minimal input efficiency score of the unit (x, y), when compared to m units randomly drawn from the population of units producing more outputs than the level y. So, the preformance of a unit operating at the level (x, y) is compared with the average performance of m peers randomly drawn from the population of units producing at least the quantity y of output. The value of m may be considered as a trimming parameters and as $m \to \infty$ the partial order-m frontier converges to the full-frontier.

A nonparametric estimator of order-m efficiency scores is proposed by Cazals, Florens and Simar (2002). It is shown that by selecting the value of m as an appropriate function of n, the estimator of the order-m efficiency scores provides a robust estimator of the corresponding efficiency scores sharing the same asymptotic properties as the FDH estimators but being less sensitive to outliers and/or extreme values. These properties have been investigated from the robustness perspective by Daouia and Ruiz-Gazen (2003).

Recently Aragon, Daouia and Thomas-Agnan (2002) have proposed an alternative to order-m partial frontiers by introducing quantile based partial frontiers. The idea is to replace this concept of "discrete" order-m partial frontier by a "continuous" order-α partial frontier where $\alpha \in [0, 1]$ corresponds to the level of an appropriate non-standard conditional quantile frontier. Roughly said, in the input oriented case, the unit operating at the level (x, y) is benchmarked against the input level not exceeded by $100(1-\alpha)\%$ of firms producing more outputs than the level y. Here when $\alpha \to 1$, the order-α partial frontier converges to the full-frontier. A nonparametric estimator of the frontier is proposed which shares similar properties than the order-m estimators. As pointed out in Aragon, Daouia and Thomas-Agnan (2002) and in Daouia and Ruiz-Gazen (2003), partial frontiers based on α-quantile estimators have better robustness properties than the ones based on the order-m estimators.

Unlike the order-m partial frontiers, due to the absence of natural ordering of Euclidean spaces for dimension greater than one, the α-quantile approach is limited to one-dimensional input for the input oriented frontier and to one-dimensional output for the output oriented frontier. In this paper, we overcome this difficulty and we propose an extention to the full multivariate case, introducing the concept of α-quantile efficiency scores and the corresponding α-quantile frontier set.

The paper is organized as follows: in the next section, we reformulate the concept of frontier and efficiency in a probabilistic framework, in the lines of Daraio and Simar (2003). We characterize the monotonicity properties of the resulting efficiency scores and show that in the case of free disposability of Ψ, these scores coincide with the Farrell-Debreu efficiency scores defined above. Due to this unifying presentation it is then easy in Section 3 to define a concept of α-quantile efficiency scores in a full multivariate framework and to investigate its properties. In Section 4 we analyze the asymptotic properties of the corresponding nonparametric estimators. Then Section 5 shows how environemental variables can be introduced to evaluate efficiencies. Section 6 illustrates with some numerical examples and Section 7 concludes.

2 Probabilistic Formulation and Nonparametric Estimation

Daraio and Simar (2003), extending previous works of Cazals, Florens and Simar (2002), propose a probabilistic formulation of efficiency concepts. The Data Generating Process (DGP) of (X, Y) is completely characterized by the knowledge of the probability function $H_{XY}(\cdot, \cdot)$ defined as

$$H_{XY}(x, y) = \text{Prob}(X \leq x, Y \geq y). \tag{2.1}$$

The support of $H_{XY}(\cdot, \cdot)$ is Ψ and $H_{XY}(x, y)$ can be interpreted as the probability for a unit operating at the level (x, y) to be dominated. Note that this function is a non-standard distribution function, having a cumulative distribution form for X and a survival form for Y. So $H_{XY}(\cdot, \cdot)$ is monotone nondecreasing with x and monotone nonincreasing[2] with y.

This joint probability can be decomposed as follows:

$$\begin{aligned} H_{XY}(x, y) &= \text{Prob}(X \leq x \,|\, Y \geq y) \, \text{Prob}(Y \geq y) = F_{X|Y}(x|y) \, S_Y(y) & (2.2) \\ &= \text{Prob}(Y \geq y \,|\, X \leq x) \, \text{Prob}(X \leq x) = S_{Y|X}(y|x) \, F_X(x), & (2.3) \end{aligned}$$

where we suppose the conditional probabilities exits (*i.e.*, when needed, $F_X(x) > 0$ or $S_Y(y) > 0$). Note that the conditional distribution $F_{X|Y}$ and the conditional survival $S_{Y|X}$

[2] A function f from $\mathbb{R}^k$ to $\mathbb{R}$ is monotone nonincreasing if $a_1 \leq a_2$ implies $f(a_1) \geq f(a_2)$. We will say that f is monotone decreasing, if $a_1 \leq a_2$ and $a_1 \neq a_2$ implies $f(a_1) > f(a_2)$.

are non-standard due to the event describing the condition. We can now define, as in Daraio and Simar (2003) efficiency scores in terms of the support of theses probabilties. An input oriented efficiency score $\tilde{\theta}(x, y)$ for $(x, y) \in \Psi$ is defined for all y with $S_Y(y) > 0$ as

$$\widetilde{\theta}(x, y) = \inf\{\theta \mid F_{X|Y}(\theta x|y) > 0\} = \inf\{\theta \mid H_{XY}(\theta x, y) > 0\}. \tag{2.4}$$

For the output oriented case, for all x such that $F_X(x) > 0$, we define the output efficiency score as

$$\widetilde{\lambda}(x, y) = \sup\{\lambda \mid S_{Y|X}(\lambda y|x) > 0\} = \sup\{\lambda \mid H_{XY}(x, \lambda y) > 0\}. \tag{2.5}$$

This input (resp. output) efficiency score can be interpreted as the proportionate reduction (resp. increase) of inputs (resp. outputs) a unit working at the level (x, y) should perform to be dominated with probability zero. These efficiency scores share the following properties.

Proposition 2.1. *Whenever defined, $\widetilde{\theta}(x, y)$ is monotone nonincreasing with x and monotone nondecreasing with y. Whenever defined, $\widetilde{\lambda}(x, y)$ is monotone nondecreasing with x and monotone nonincreasing with y.*

Proof: We prove the proposition for the input case, the same argument can be followed, *mutatis mutandis*, for the output case. The fact that $\widetilde{\theta}(x, y)$ is monotone nonincreasing with x is obvious. Consider now a fixed x, we have for $S_Y(y) > 0$

$$\widetilde{\theta}(x, y) = \inf\{\theta \mid F_{X|Y}(\theta x|y) > 0\} = \sup\{\theta \mid F_{X|Y}(\theta x|y) = 0\}.$$

Consider the set

$$D_y = \{\theta > 0 | F_{X|Y}(\theta x|y) = 0\} = \{\theta > 0 | H_{XY}(\theta x, y) = 0\}.$$

Clearly, if $y_1 \leq y_2$, $D_{y_1} \subseteq D_{y_2}$. Therefore, $\sup\{\theta \mid \theta \in D_{y_2}\} \geq \sup\{\theta \mid \theta \in D_{y_1}\}$ and this completes the proof. $\square$

If we now define $\widetilde{x}^\partial(y) = \widetilde{\theta}(x, y)\, x$, as the radial projection of (x, y) on the efficient frontier, in the input direction (orthogonal to the vector y), we have that for a fixed x, $\widetilde{x}^\partial(y)$ is monotone nondecreasing with y. Similarly, if $\widetilde{y}^\partial(x) = \widetilde{\lambda}(x, y)\, y$, for fixed y, this is monotone nondecreasing with x. This result can be seen as a multivariate extension of Theorem 2.1 of Cazals, Florens and Simar (2002).

The efficient frontier, according this probabilistic definition of efficiency, can be described either for all y such that $S_Y(y) > 0$ by the set $\{(\widetilde{\theta}(x, y)\, x, y) \mid x \text{ s.t. } (x, y) \in \Psi\}$ or for all x such that $F_X(x) > 0$ by the set $\{(x, \widetilde{\lambda}(x, y)\, y) \mid y \text{ s.t. } (x, y) \in \Psi\}$. By construction and by

Proposition 2.1, the set bounded by this frontier is the free disposal hull of Ψ. If Ψ is free disposal, the two sets coincide and $\widetilde{\theta}(x,y) \equiv \theta(x,y)$ and $\widetilde{\lambda}(x,y) \equiv \lambda(x,y)$.

From now on, we will assume that Ψ is free disposal. This implies that for all $y_1 \le y_2$, $X(y_2) \subseteq X(y_1)$ and that for all $x_1 \le x_2$, $Y(x_1) \subseteq Y(x_2)$. Note also that in this case,

$$X(y) = \{x \mid \theta(x,y) \le 1\} \tag{2.6}$$
$$Y(x) = \{y \mid \lambda(x,y) \ge 1\}. \tag{2.7}$$

Natural nonparametric estimators of $\theta(x,y)$ and of $\lambda(x,y)$ are obtained by plugging the empirical distribution $\widehat{H}_{XY,n}$ in place of H_{XY} in the definition of the efficiency scores. Defining

$$\widehat{H}_{XY,n}(x,y) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{I}(X_i \le x, Y_i \ge y), \tag{2.8}$$

the most natural nonparametric estimators of the efficiency scores are given for the input orientation[3], by

$$\widehat{\theta}_n(x,y) = \inf\{\theta \mid \widehat{F}_{X|Y,n}(\theta x|y) > 0\} = \min_{i|Y_i \ge y} \max_{j=1,\dots,p} X_i^j/x^j, \tag{2.9}$$

where

$$\widehat{F}_{X|Y,n}(x|y) = \frac{\widehat{H}_{XY,n}(x,y)}{\widehat{H}_{XY,n}(\infty,y)}.$$

For the output oriented case we have:

$$\widehat{\lambda}_n(x,y) = \sup\{\lambda \mid \widehat{S}_{Y|X,n}(\lambda y|x) > 0\} = \max_{i|X_i \le x} \min_{j=1,\dots,q} Y_i^j/y^j, \tag{2.10}$$

where

$$\widehat{S}_{Y|X,n}(y|x) = \frac{\widehat{H}_{XY,n}(x,y)}{\widehat{H}_{XY,n}(x,0)}.$$

As pointed out in Daraio and Simar (2003), these estimators are the FDH estimators of the Farrell-Debreu efficiency scores. Note also that the FDH efficiency scores share the properties of Proposition 2.1.

[3]For a vector $a \in \mathbb{R}^k$, we denote by a^j its j-th component.

3 Conditional Quantile Based Efficiency Scores

Aragon, Daouia and Thomas-Agnan (2002) have introduced the conditional quantile frontier function for a production (output) function when the output is unidimensional and for a cost (input) function when the input is one dimensional. We extend the ideas to a full multivariate setup. Since a natural ordering of Euclidean spaces of dimension greater than one does not exist, we overcome the difficulty by defining α-quantile efficiency scores as follows.

Definition 3.1. *For all y such that $S_Y(y) > 0$ and for $\alpha \in]0,1]$, the α-quantile input efficiency score for the unit $(x,y) \in \Psi$ is defined as*

$$\theta_\alpha(x,y) = \inf\{\theta \mid F_{X|Y}(\theta x|y) > 1-\alpha\}. \tag{3.1}$$

For all x such that $F_X(x) > 0$ and for $\alpha \in]0,1]$, the α-quantile output efficiency score for the unit $(x,y) \in \Psi$ is defined as

$$\lambda_\alpha(x,y) = \sup\{\lambda \mid S_{Y|X}(\lambda y|x) > 1-\alpha\} \tag{3.2}$$

These efficiency scores can be interpreted as follows. For the input case, suppose for instance that $\theta_\alpha(x,y) = 1$, we will say that the unit (x,y) is input efficient at the level $\alpha \times 100\%$ in the sense that this unit is dominated by firms producing more than the output level y with probability $1-\alpha$. More generally, $\theta_\alpha(x,y)$ is the proportionate reduction (if < 1) or increase (if > 1) of inputs, a unit working at the level (x,y) should perform to be input efficient at the level $\alpha \times 100\%$. Clearly when $\alpha = 1$, this is, under free disposability of Ψ, the Farrell-Debreu input efficiency score. In a certain sense, we can say that $\theta_\alpha(x,y)$ is the input efficiency score of (x,y) at the level $\alpha \times 100\%$.

For the output case, $\lambda_\alpha(x,y)$ is the proportionate reduction (if < 1) or increase (if > 1) of outputs, a unit working at the level (x,y) should perform to be dominated by firms using less input than the level x with probability $1-\alpha$. Again when $\alpha = 1$, this is, under free disposability of Ψ, the Farrell-Debreu output efficiency score and roughly said, $\lambda_\alpha(x,y)$ is the output efficiency score of (x,y) at the level $\alpha \times 100\%$.

We can now for all y such that $S_Y(y) > 0$ define the α-quantile efficient frontier in the input direction (or the input efficient frontier at the level $\alpha \times 100\%$) as the set

$$X^\partial_\alpha(y) = \{(\theta_\alpha(x,y)\, x, y) \mid x \text{ s.t. } (x,y) \in \Psi\}. \tag{3.3}$$

Note that the points $x^\partial_\alpha(y) = \theta_\alpha(x,y)\, x$ can be seen as the efficient inputs for the output y at the level $\alpha \times 100\%$ and that $(x^\partial_\alpha(y), y)$ has a probability $H_{XY}(x^\partial_\alpha(y), y) = (1-\alpha) S_Y(y) \le 1-\alpha$ of being dominated. Of course the first equality holds if the distribution function $H_{XY}(\cdot,\cdot)$ is continuous on its support Ψ.

Similarly, in the output direction we have the α-quantile efficient frontier defined for all x such that $F_X(x) > 0$, as the set

$$Y_\alpha^\partial(x) = \{(x, \lambda_\alpha(x,y)\, y) \mid y \text{ s.t. } (x,y) \in \Psi\}. \tag{3.4}$$

Here again, the points $y_\alpha^\partial(x) = \lambda_\alpha(x,y)\, y$ represent the efficient outputs for the input x at the level $\alpha \times 100\%$ and the pairs $(x, y_\alpha^\partial(x))$ have a probability $H_{XY}(x, y_\alpha^\partial(x)) = (1-\alpha)F_X(x) \leq 1-\alpha$ of being dominated if $H_{XY}(\cdot,\cdot)$ is continuous on Ψ.

Note that in the particular case of $p = 1$, for any y such that $S_Y(y) > 0$, the input efficient frontier at the level $\alpha \times 100\%$ may be described as the set $X_\alpha^\partial(y) = \{(\phi_\alpha(y), y)\}$ where $\phi_\alpha(y) = \theta_\alpha(1,y)$ is the conditional quantile cost (input) function of order α introduced by Aragon *et al.* (2002). In the same way, in the particular case of $q = 1$, for any x such that $F_X(x) > 0$, the output efficient frontier at the level $\alpha \times 100\%$ may be described as the set $Y_\alpha^\partial(x) = \{(x, \varphi(x))\}$ where $\varphi_\alpha(x) = \lambda_\alpha(x,1)$ is the conditional quantile production function of order α of Aragon *et al.* (2002).

As shown below, the α-quantile efficiency scores share most of the properties of their univariate correspondant. We remind here that Ψ is assumed to be free disposal. We define Ψ^* as being the interior of Ψ:

$$\Psi^* = \{(x,y) \in \Psi \mid 0 < H_{XY}(x,y) < S_Y(y) \wedge F_X(x)\}. \tag{3.5}$$

Proposition 3.1. *Assume that $F_{X|Y}$ is continuous and monotone increasing in x and that $S_{Y|X}$ is continuous and monotone decreasing in y. Then, for all $(x,y) \in \Psi^*$, there exist α and β in $]0,1]$ such that*

$$\theta_\alpha(x,y) = 1, \quad \textit{where } \alpha = 1 - F_{X|Y}(x|y) \tag{3.6}$$

$$\lambda_\beta(x,y) = 1, \quad \textit{where } \beta = 1 - S_{Y|X}(y|x). \tag{3.7}$$

Proof: Since $F_{X|Y}(\cdot|y)$ is continuous, we have

$$\theta_\alpha(x,y) = \inf\{\theta \mid F_{X|Y}(\theta x|y) > 1-\alpha\} = \sup\{\theta \mid F_{X|Y}(\theta x|y) = 1-\alpha\}.$$

Set $\alpha = 1 - F_{X|Y}(x|y)$, then we have $\theta_\alpha(x,y) = \sup\{\theta \mid F_{X|Y}(\theta x|y) = F_{X|Y}(x|y)\}$. Since $F_{X|Y}(\theta x|y)$ is continuous and monotone increasing in θ, we must have $\theta_\alpha(x,y) = 1$. The same argument can be followed for the second part of the proposition. $\square$

Proposition 3.1 shows that any point (x,y) in the interior of Ψ, belongs to an appropriate α-quantile efficient frontier in both directions (input and output). For instance in the output orientation, it can be described as the set $Y_\alpha^\partial(x)$ where $\alpha = 1 - S_{Y|X}(y|x)$. Since for any (x,y)

belonging to the efficient frontier of Ψ, $\theta(x, y) = \theta_1(x, y) = 1$ and $\lambda(x, y) = \lambda_1(x, y) = 1$, we can use the value of $\alpha = \alpha(x, y)$ and of $\beta = \beta(x, y)$ of the proposition to define a new concept of input and output efficiency score. This is in the same spirit as in Aragon, Daouia and Thomas-Agnan (2003), this will not be pursued here.

Proposition 3.2. *For all y such that $S_Y(y) > 0$, we have $\lim_{\alpha\to 1} \searrow \theta_\alpha(x, y) = \theta(x, y)$ and for all x such that $F_X(x) > 0$, $\lim_{\alpha\to 1} \nearrow \lambda_\alpha(x, y) = \lambda(x, y)$.*

Proof: For all $(x, y) \in \Psi$ such that $S_Y(y) > 0$, and for any $\alpha \in]0, 1]$, $\theta_\alpha(x, y) \geq \theta_1(x, y) = \theta(x, y)$. The family $\{\theta_\alpha(x, y)\}_{0<\alpha\leq 1}$ is monotone nonincreasing and bounded by $\theta_1(x, y)$. Therefore $\theta_\alpha(x, y)$ converges pointwize to $\theta_1(x, y)$ when $\alpha \to 1$. The same argument can be followed for the output case. $\square$

The α-quantile input efficiency score $\theta_\alpha(x, y)$ is clearly monotone nonincreasing with x but it is in general not monotone in y, unless we add an assumption on $F_{X|Y}$.

Proposition 3.3. *Assume that $F_{X|Y}(\cdot|y)$ is continuous for any y. Then, for points (x, y) such that $F_{X|Y}(x|y) < 1$, the two following properties are equivalent.*

$$F_{X|Y}(x|y) \text{ is monotone nonincreasing with } y \tag{3.8}$$

$$\theta_\alpha(x, y) \text{ is monotone nondecreasing with } y \text{ for all } \alpha. \tag{3.9}$$

Proof: We first prove that (3.8) implies (3.9). Define the set $A_y = \{\theta > 0 \,|\, F_{X|Y}(\theta x|y) > 1 - \alpha\}$. Consider $y_1 \leq y_2$, if $F_{X|Y}(x|y)$ is monotone nonincreasing with y, clearly $A_{y_2} \subseteq A_{y_1}$ and so, $\theta_\alpha(x, y_1) \leq \theta_\alpha(x, y_2)$.

Conversely, we have seen that $\theta_\alpha(x, y) = \sup\{\theta|F_{X|Y}(\theta x|y) = 1 - \alpha\}$. Consider $y_1 \leq y_2$ and let $\alpha = 1 - F_{X|Y}(x|y_1)$. Since $\theta_\alpha(x, y_1) = \sup\{\theta \,|\, F_{X|Y}(\theta x|y_1) = F_{X|Y}(x|y_1)\} \geq 1$, we obtain $1 \leq \theta_\alpha(x, y_2)$ and so, $F_{X|Y}(x|y_2) \leq F_{X|Y}(\theta_\alpha(x, y_2)x|y_2) = 1 - \alpha = F_{X|Y}(x|y_1)$. $\square$

Note that both conditions of the proposition are quite reasonable in production analysis. The first relation (3.8) says that there is less probability to observe a level of input lower than a fixed value x for firms producing more than a level y_2, than for firms producing more than a level $y_1 \leq y_2$. It is more difficult to reduce inputs when producing higher level of outputs. Whereas (3.9) states that everything else being kept constant, the α-quantile input efficiency score cannot decrease when the output increases.

Of course, *mutatis mutandis*, we have the same property in the output direction. $\lambda_\alpha(x, y)$ is monotone nonincreasing with y, but for the monotonicity with respect to x, we have:

Proposition 3.4. *Assume that $S_{Y|X}(\cdot|x)$ is continuous for any x. Then, for points (x, y) such that $S_{Y|X}(y|x) < 1$, the two following properties are equivalent.*

$$S_{Y|X}(y|x) \text{ is monotone nondecreasing with } x \tag{3.10}$$

$$\lambda_\alpha(x, y) \text{ is monotone nondecreasing with } x \text{ for all } \alpha. \tag{3.11}$$

Proof: We first prove that (3.10) implies (3.11). If $x_1 \le x_2$ then $\{\lambda \,|\, S_{Y|X}(\lambda y|x_1) > 1-\alpha\} \subseteq \{\lambda \,|\, S_{Y|X}(\lambda y|x_2) > 1-\alpha\}$ and so, $\lambda_\alpha(x_1, y) \le \lambda_\alpha(x_2, y)$.

Conversely, consider $x_1 \le x_2$ and let $1-\alpha = S_{Y|X}(y|x_2)$. We have $\lambda_\alpha(x_1, y) \le \lambda_\alpha(x_2, y) \le 1$ since $\lambda_\alpha(x_2, y) = \sup\{\lambda \,|\, S_{Y|X}(\lambda y|x_2) > S_{Y|X}(y|x_2)\} \le 1$. Then $S_{Y|X}(\lambda_\alpha(x_1, y)y|x_1) \ge S_{Y|X}(y|x_1)$. We also have from the definition of $\lambda_\alpha(x_1, y)$ that $S_{Y|X}(\lambda_\alpha(x_1, y)y|x_1) \le 1-\alpha = S_{Y|X}(y|x_2)$, and thus $S_{Y|X}(y|x_1) \le S_{Y|X}(y|x_2)$. □

4 Nonparametric Estimator

A natural nonparametric estimator of the α-quantile efficiency scores is obtained by plugging the empirical $\widehat{H}_{XY,n}(x, y)$ in the above fromulas so we have:

$$\widehat{\theta}_{\alpha,n}(x, y) = \inf\{\theta \,|\, \widehat{F}_{X|Y,n}(\theta x|y) > 1-\alpha\}, \tag{4.1}$$

$$\widehat{\lambda}_{\alpha,n}(x, y) = \sup\{\lambda \,|\, \widehat{S}_{Y|X,n}(\lambda y|x) > 1-\alpha\}, \tag{4.2}$$

where $\widehat{F}_{X|Y,n}$ and $\widehat{S}_{Y|X,n}$ were defined in Section 2.

These nonparametric estimators can be computed very easily. Indeed, define

$$\mathcal{Y}_i = \min_{k=1,\cdots,q} \frac{Y_i^k}{y^k}, \quad i = 1, \cdots, n,$$

and let $N_x = n\widehat{H}_{XY,n}(x, 0)$ be non nul. For $j = 1, \cdots, N_x$, denote by $\mathcal{Y}^x_{(j)}$ the j-th order statistic of the observations $\mathcal{Y}_i$ such that $X_i \le x$: $\mathcal{Y}^x_{(1)} \le \mathcal{Y}^x_{(2)} \le \ldots \le \mathcal{Y}^x_{(N_x)}$. We have,

$$\begin{aligned}
\widehat{S}_{Y|X,n}(\lambda y|x) &= \frac{\sum_{i|X_i\le x} \mathbb{1}(Y_i \ge \lambda y)}{N_x} = \frac{\sum_{i|X_i\le x} \mathbb{1}(\lambda \le \mathcal{Y}_i)}{N_x} = \frac{\sum_{j=1}^{N_x} \mathbb{1}(\lambda \le \mathcal{Y}^x_{(j)})}{N_x} \\
&= \begin{cases} 1 \quad \text{if} \quad \lambda \le \mathcal{Y}^x_{(1)} \\ \frac{N_x - j}{N_x} \quad \text{if} \quad \mathcal{Y}^x_{(j)} < \lambda \le \mathcal{Y}^x_{(j+1)}, \; j = 1, \cdots, N_x - 1 \\ 0 \quad \text{if} \quad \lambda > \mathcal{Y}^x_{(N_x)}. \end{cases}
\end{aligned}$$

It follows,

$$\widehat{\lambda}_{\alpha,n}(x, y) = \begin{cases} \mathcal{Y}^x_{(\alpha N_x)} & \text{if} \quad \alpha N_x \in \mathbb{N}^* \\ \mathcal{Y}^x_{([\alpha N_x]+1)} & \text{otherwise}, \end{cases} \tag{4.3}$$

where $\mathbb{N}^*$ denotes the set of positive integers and $[\alpha N_x]$ denotes the integral part of αN_x.

Likewise, let $M_y = n\widehat{H}_{XY,n}(\infty, y) > 0$, and define

$$\mathcal{X}_i = \max_{k=1,\cdots,p} \frac{X_i^k}{x^k}, \quad i = 1, \cdots, n.$$

For $j = 1, \cdots, M_y$, denote by $\mathcal{X}^y_{(j)}$ the j-th order statistic of the observations $\mathcal{X}_i$ such that $Y_i \geq y$: $\mathcal{X}^y_{(1)} \leq \mathcal{X}^y_{(2)} \leq \ldots \leq \mathcal{X}^y_{(M_y)}$. We have,

$$\begin{aligned}
\widehat{F}_{X|Y,n}(\theta x|y) &= \frac{\sum_{i|Y_i\geq y} \mathbb{1}(\mathcal{X}_i \leq \theta)}{M_y} = \frac{\sum_{j=1}^{M_y} \mathbb{1}(\mathcal{X}^y_{(j)} \leq \theta)}{M_y} \\
&= \begin{cases} 0 \quad \text{if} \quad \theta < \mathcal{X}^y_{(1)} \\ \frac{j}{M_y} \quad \text{if} \quad \mathcal{X}^y_{(j)} \leq \theta < \mathcal{X}^y_{(j+1)}, \ j = 1, \cdots, M_y - 1 \\ 1 \quad \text{if} \quad \theta \geq \mathcal{X}^y_{(M_y)}. \end{cases}
\end{aligned}$$

It follows,

$$\widehat{\theta}_{\alpha,n}(x, y) = \begin{cases} \mathcal{X}^y_{((1-\alpha)M_y)} & \text{if} \quad (1-\alpha)M_y \in \mathbb{N} \\ \mathcal{X}^y_{([(1-\alpha)M_y]+1)} & \text{otherwise,} \end{cases} \tag{4.4}$$

where $\mathbb{N}$ denotes the set of all nonnegative integers.

The nonparametric α-quantile efficiency scores $\widehat{\theta}_{\alpha,n}(x, y)$ and $\widehat{\lambda}_{\alpha,n}(x, y)$ share the following properties.

Proposition 4.1. *For all y such that $\widehat{H}_{XY,n}(\infty, y) > 0$, we have $lim_{\alpha\to1} \searrow \widehat{\theta}_{\alpha,n}(x, y) = \widehat{\theta}_n(x, y)$ and for all x such that $\widehat{H}_{XY,n}(x, 0) > 0$, $lim_{\alpha\to1} \nearrow \widehat{\lambda}_{\alpha,n}(x, y) = \widehat{\lambda}_n(x, y)$.*

Proof: We can use the same argument as in the proof of Proposition 3.2. □

Now we investigate some of the asymptotic properties of our estimators. In what follows, we limit the presentation for the nonparametric estimator in the output oriented case. The same properties hold for the input oriented case and are summarized in the Appendix.

Theorem 4.1. *Let $(x, y) \in \Psi$ be such that $F_X(x) > 0$ and let $0 < \alpha < 1$. Assume that $\lambda \mapsto S_{Y|X}(\lambda y|x)$ is decreasing in a neigborhood of $\lambda_\alpha(x, y)$. Then, for every $\varepsilon > 0$,*

$$Prob(|\widehat{\lambda}_{\alpha,n}(x, y) - \lambda_\alpha(x, y)| > \varepsilon) \leq 2e^{-2n\delta^2_{\alpha,\varepsilon,x,y}}, \quad \textit{for all } n \geq 1,$$

where

$$\delta_{\alpha,\varepsilon,x,y} = \frac{F_X(x)}{2-\alpha} \min \left\{(1-\alpha) - S_{Y|X}((\lambda_\alpha(x, y) + \varepsilon)y|x); S_{Y|X}((\lambda_\alpha(x, y) - \varepsilon)y|x) - (1-\alpha)\right\}.$$

Proof: Let $\varepsilon > 0$. We have

$$\begin{aligned} Prob(|\widehat{\lambda}_{\alpha,n}(x,y) - \lambda_\alpha(x,y)| > \varepsilon) &= Prob(\widehat{\lambda}_{\alpha,n}(x,y) > \lambda_\alpha(x,y) + \varepsilon) \\ &\quad + Prob(\widehat{\lambda}_{\alpha,n}(x,y) < \lambda_\alpha(x,y) - \varepsilon). \end{aligned}$$

By applying the fact that $\widehat{\lambda}_{\alpha,n}(x,y) > \lambda$ implies $\widehat{S}_{Y|X,n}(\lambda y|x) > 1-\alpha$, we obtain

$$\begin{aligned} Prob(\widehat{\lambda}_{\alpha,n}(x,y) > \lambda_\alpha(x,y) + \varepsilon) &\leq Prob[\widehat{S}_{Y|X,n}((\lambda_\alpha(x,y)+\varepsilon)y|x) > 1-\alpha] \\ &= Prob[\widehat{H}_{XY,n}(x,(\lambda_\alpha(x,y)+\varepsilon)y) > (1-\alpha)\widehat{H}_{XY,n}(x,0)] \\ &= Prob(\sum_{i=1}^{n} V_i - \sum_{i=1}^{n} E(V_i) > n\delta_1), \end{aligned}$$

where

$$\begin{aligned} V_i &= 1\!\!I(X_i \leq x, Y_i \geq (\lambda_\alpha(x,y)+\varepsilon)y) - (1-\alpha)1\!\!I(X_i \leq x), \\ \delta_1 &= -E(V_1) = F_X(x)[(1-\alpha) - S_{Y|X}((\lambda_\alpha(x,y)+\varepsilon)y|x)] > 0. \end{aligned}$$

Since $Prob(\alpha - 1 \leq V_i \leq 1) = 1$, for each i, we obtain by applying Hoeffding's Inequality (Hoeffding, 1963),

$$Prob(\widehat{\lambda}_{\alpha,n}(x,y) > \lambda_\alpha(x,y) + \varepsilon) \leq e^{-2n\delta_1^2/(2-\alpha)^2}.$$

Likewise, by applying the fact that $\widehat{\lambda}_{\alpha,n}(x,y) < \lambda$ implies $\widehat{S}_{Y|X,n}(\lambda y|x) \leq 1-\alpha$, we get

$$\begin{aligned} Prob(\widehat{\lambda}_{\alpha,n}(x,y) < \lambda_\alpha(x,y) - \varepsilon) &\leq Prob[\widehat{S}_{Y|X,n}((\lambda_\alpha(x,y)-\varepsilon)y|x) \leq 1-\alpha] \\ &= Prob(\sum_{i=1}^{n} W_i - \sum_{i=1}^{n} E(W_i) \geq n\delta_2), \end{aligned}$$

where

$$\begin{aligned} W_i &= (1-\alpha)1\!\!I(X_i \leq x) - 1\!\!I(X_i \leq x, Y_i \geq (\lambda_\alpha(x,y)-\varepsilon)y), \\ \delta_2 &= -E(W_1) = F_X(x)[S_{Y|X}((\lambda_\alpha(x,y)-\varepsilon)y|x) - (1-\alpha)] > 0. \end{aligned}$$

Therefore, by using Hoeffding's Inequality, we obtain

$$Prob(\widehat{\lambda}_{\alpha,n}(x,y) < \lambda_\alpha(x,y) - \varepsilon) \leq e^{-2n\delta_2^2/(2-\alpha)^2}.$$

Putting $\delta_{\alpha,\varepsilon,x,y} = \min\{\delta_1;\delta_2\}/(2-\alpha)$, the proof is complete. $\square$

Thus $Prob(|\widehat{\lambda}_{\alpha,n}(x,y) - \lambda_\alpha(x,y)| > \varepsilon) \to 0$ exponentially fast, which implies that $\widehat{\lambda}_{\alpha,n}(x,y)$ converges completely to $\lambda_\alpha(x,y)$. This generalizes the exponential probability inequality obtained in Daouia (2003, see Theorem 2.3) for the nonparametric α-quantile frontier $\widehat{y}^{\partial}_{\alpha,n}(x) = \widehat{\lambda}_{\alpha,n}(x,y)y$ in the univariate case where $p \geq 1$ and $q = 1$.

We also obtain the following asymptotic normality result which extends the one established in Aragon, Daouia and Thomas-Agnan (2002, see Theorem 4.1) to the more general case where $p, q \geq 1$.

Theorem 4.2. *Let $0 < \alpha < 1$ be a fixed order and let $(x, y) \in \Psi$ be a fixed unit such that $F_X(x) > 0$. Assume that $G(\lambda) = S_{Y|X}(\lambda y|x)$ is differentiable at $\lambda_\alpha(x, y)$ with negative derivative $G'(\lambda_\alpha(x, y))$. Then,*

$$\sqrt{n}\left(\widehat{\lambda}_{\alpha,n}(x,y) - \lambda_\alpha(x,y)\right) \xrightarrow{\mathcal{L}} N\left(0, \sigma^2_\alpha(x,y)\right) \quad as \quad n \to \infty,$$

where

$$\sigma^2_\alpha(x,y) = \frac{\alpha(1-\alpha)}{[G'(\lambda_\alpha(x,y))]^2 F_X(x)}.$$

Note that this theorem requires a slightly stronger hypothesis on the function $G(\lambda) = S_{Y|X}(\lambda y|x)$ than in the preceding theorem. This assumption is standard in quantile theory for the generalized inverse of the cdf $S_{Y|X}(\lambda y|x)$ to coincide with the reciprocal.

Proof: Consider the following three random variables

$$\begin{aligned}
V_n &= \sqrt{n}\left(\widehat{\lambda}_{\alpha,n}(x,y) - \lambda_\alpha(x,y)\right), \\
W_n &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \frac{(1-\alpha)\mathbb{1}(X_i \leq x) - \mathbb{1}(X_i \leq x, Y_i \geq \lambda_\alpha(x,y)y)}{G'(\lambda_\alpha(x,y))F_X(x)}, \\
R_n &= V_n - W_n.
\end{aligned}$$

By using the fact that $H_{XY}(x, \lambda_\alpha(x, y)y) = (1 - \alpha)F_X(x)$, we obtain in view of the central limit theorem that $W_n \xrightarrow{\mathcal{L}} N\left(0, \sigma^2_\alpha(x, y)\right)$. To show that V_n has the same asymptotic normal distribution, it suffices to prove that

$$R_n \xrightarrow{p} 0. \tag{4.5}$$

Using $\widehat{\lambda}_{\alpha,n}(x, y) \geq \lambda \Leftrightarrow \widehat{S}_{Y|X,n}(\lambda y|x) > 1-\alpha$ (this can be easily proved from the definition of $\widehat{\lambda}_{\alpha,n}$ and by using the left-continuity of $\lambda \mapsto \widehat{S}_{Y|X,n}(\lambda y|x)$), we get for any real t,

$$\{V_n \geq t\} = \left\{\widehat{S}_{Y|X,n}\left((\lambda_\alpha(x,y) + \frac{t}{\sqrt{n}})y|x\right) > 1-\alpha\right\} = \{Z_{t,n} > T_n\}, \tag{4.6}$$

where

$$\begin{aligned}
Z_{t,n} &= \frac{\sqrt{n}\widehat{H}_{XY,n}(x,0)}{G'(\lambda_\alpha(x,y))F_X(x)}\left[G(\lambda_\alpha(x,y) + \frac{t}{\sqrt{n}}) - \widehat{S}_{Y|X,n}((\lambda_\alpha(x,y) + \frac{t}{\sqrt{n}})y|x)\right] \\
T_n &= \frac{\sqrt{n}\widehat{H}_{XY,n}(x,0)}{G'(\lambda_\alpha(x,y))F_X(x)}\left[G(\lambda_\alpha(x,y) + \frac{t}{\sqrt{n}}) - (1-\alpha)\right].
\end{aligned}$$

Since $G(\lambda_\alpha(x,y)+\frac{t}{\sqrt{n}})-(1-\alpha)=\frac{t}{\sqrt{n}}G'(\lambda_\alpha(x,y))+\frac{t}{\sqrt{n}}o(1)$ and $\widehat{H}_{XY,n}(x,0)\xrightarrow{p}F_X(x)$, as $n\to\infty$, we obtain

$$T_n \xrightarrow{p} t \quad \text{as} \quad n\to\infty. \tag{4.7}$$

We also have

$$\begin{aligned}Z_{t,n}-W_n=\frac{\sqrt{n}\widehat{H}_{XY,n}(x,0)}{G'(\lambda_\alpha(x,y))F_X(x)}\bigg[\bigg(G(\lambda_\alpha(x,y)+\frac{t}{\sqrt{n}})-\widehat{S}_{Y|X,n}((\lambda_\alpha(x,y)+\frac{t}{\sqrt{n}})y|x)\bigg)\\ -\Big((1-\alpha)-\widehat{S}_{Y|X,n}((\lambda_\alpha(x,y)y|x)\Big)\bigg].\end{aligned}$$

An easy computation shows that,

$$\begin{aligned}E[(Z_{t,n}-W_n)^2]\left\{G'(\lambda_\alpha(x,y))F_X(x)\right\}^2 &= (1-\alpha)F_X(x)-(1-\alpha)^2F_X(x)\\ &- F_X(x)G^2(\lambda_\alpha(x,y)+\frac{t}{\sqrt{n}})+(1+2(1-\alpha))F_X(x)G(\lambda_\alpha(x,y)+\frac{t}{\sqrt{n}})\\ &- 2F_X(x)G\left(\lambda_\alpha(x,y)+(\frac{t}{\sqrt{n}}\vee 0)\right).\end{aligned}$$

Using the continuity of $G(\cdot)$ in $\lambda_\alpha(x,y)$, we then obtain $E[(Z_{t,n}-W_n)^2]\to 0$, and so

$$Z_{t,n}-W_n \xrightarrow{p} 0 \quad \text{as} \quad n\to\infty. \tag{4.8}$$

Now, using the results (4.6)-(4.8), we will show that V_n and W_n satisfy the following two conditions:
i) For all $\delta>0$, there exists a λ (depending on δ) such that $Prob(|W_n|>\lambda)<\delta$
ii) For all k and all $\varepsilon>0$

$$\lim_{n\to\infty} Prob(V_n\geq k+\varepsilon, W_n\leq k) = 0 \tag{4.9}$$
$$\lim_{n\to\infty} Prob(V_n< k, W_n\geq k+\varepsilon) = 0. \tag{4.10}$$

By using the fact that $E[W_n^2]=\sigma_\alpha^2(x,y)$ and by choosing λ such that $\lambda^2>\sigma_\alpha^2(x,y)/\delta$, the first condition follows from a trivial application of the Markov inequality. On the other hand, for any k and any $\varepsilon>0$, putting $t=k+\varepsilon$, we obtain in view of (4.6),

$$\begin{aligned}Prob(V_n\geq k+\varepsilon, W_n\leq k) &= Prob(Z_{t,n}>T_n, W_n\leq t-\varepsilon)\\ &\leq Prob\left(|(Z_{t,n}-W_n)-(T_n-t)|\geq\varepsilon\right).\end{aligned}$$

The result (4.9) follows then immediately from (4.7) and (4.8). Similarly, by applying (4.6) to $t=k$, we obtain

$$Prob(V_n<k, W_n\geq k+\varepsilon)\leq Prob\left(|(W_n-Z_{t,n})+(T_n-t)|\geq\varepsilon\right)\longrightarrow 0 \quad \text{as} \quad n\to\infty,$$

and so the result (4.10) is also satisfied. Thus, by making use of a lemma of Ghosh (1971, Lemma 1, p. 1958), we conclude that $R_n \xrightarrow{p} 0$. $\square$

Note that this convergence result can be extended to the analysis of the asymptotic properties of a vector $\left(\sqrt{n}\left(\widehat{\lambda}_{\alpha,n}(x^1,y^1)-\lambda_\alpha(x^1,y^1)\right),\cdots,\sqrt{n}\left(\widehat{\lambda}_{\alpha,n}(x^r,y^r)-\lambda_\alpha(x^r,y^r)\right)\right)$. We still have the asymptotic r-variate normal distribution with asymptotic covariances given by $\Sigma_{k,l}=E\left[\Gamma(x^k,y^k,X,Y)\Gamma(x^l,y^l,X,Y)\right]$, where

$$\Gamma(x,y,X,Y)=\frac{(1-\alpha)\mathbb{1}(X\le x)-\mathbb{1}(X\le x,Y\ge\lambda_\alpha(x,y)y)}{G'(\lambda_\alpha(x,y))F_X(x)}. \tag{4.11}$$

The expression of the variance factors can be used to derive asymptotic confedence intervals for the order-α efficiency scores. For instance, consistent estimators for the factors $\sigma_\alpha^2(x,y)$ and $\Sigma_{k,l}$ can be obtained by plugging nonparametric estimators for $G'(\lambda)=<\bigtriangledown S_{Y|X}(\lambda y|x),y>$ and $F_X(x)$ and taking the empirical mean for the expectation.

A more robust estimator of the Farrell-Debreu efficiency scores $\lambda(x,y)$ than the standard FDH estimator $\widehat{\lambda}_n(x,y)$, which however shares similar asymptotic properties with this later one, can be derived as follows.

Lemma 4.1. *Assume that the support of Y is bounded. Then, for any $(x,y)\in\Psi$,*

$$n^{1/(p+q)}\left(\widehat{\lambda}_n(x,y)-\widehat{\lambda}_{\alpha(n),n}(x,y)\right)\xrightarrow{a.s.}0 \quad as \quad n\to\infty,$$

where the order $\alpha(n)>0$ is such that

$$n^{(p+q+1)/(p+q)}\left(1-\alpha(n)\right)\longrightarrow 0 \quad as \quad n\to\infty.$$

Proof: We have from (4.3),

$$\begin{aligned}\widehat{\lambda}_n(x,y)-\widehat{\lambda}_{\alpha(n),n}(x,y) &= \left(\mathcal{Y}^x_{(N_x)}-\mathcal{Y}^x_{(\alpha(n)N_x)}\right)\mathbb{1}(\alpha(n)N_x\in\mathbb{N}^*)\\ &\quad+\left(\mathcal{Y}^x_{(N_x)}-\mathcal{Y}^x_{([\alpha(n)N_x]+1)}\right)\mathbb{1}(\alpha(n)N_x\notin\mathbb{N}^*).\end{aligned}$$

Let $C_{x,y,k}(n)=\frac{\mathcal{Y}^x_{(N_x)}-\mathcal{Y}^x_{(k)}}{1-\frac{k}{N_x}}$ for $k\in\{1,\cdots,N_x-1\}$, and let $C_{x,y}(n)=\max_{1\le k\le N_x-1}C_{x,y,k}(n)$. Then, we obtain in the same way as Aragon, Daouia and Thomas-Agnan (2002, see Equations (18) and (19)),

$$\begin{aligned}&\left(\mathcal{Y}^x_{(N_x)}-\mathcal{Y}^x_{(\alpha(n)N_x)}\right)\mathbb{1}(\alpha(n)N_x\in\mathbb{N}^*)\le C_{x,y}(n)(1-\alpha(n))\mathbb{1}(\alpha(n)N_x\in\mathbb{N}^*),\\ &\left(\mathcal{Y}^x_{(N_x)}-\mathcal{Y}^x_{([\alpha(n)N_x]+1)}\right)\mathbb{1}(\alpha(n)N_x\notin\mathbb{N}^*)\le C_{x,y}(n)(1-\alpha(n))\mathbb{1}(\alpha(n)N_x\notin\mathbb{N}^*).\end{aligned}$$

Which gives,

$$n^{1/(p+q)}\left(\widehat{\lambda}_n(x,y)-\widehat{\lambda}_{\alpha(n),n}(x,y)\right)\le n^{1/(p+q)}C_{x,y}(n)(1-\alpha(n)).$$

Since the support of Y is bounded, there exists a constant $M_y > 0$ (depending on y) such that $\mathcal{Y}_i \leq M_y$ almost surely, for any $i = 1, \cdots, n$. Hence $\mathcal{Y}^x_{(N_x)} - \mathcal{Y}^x_{(k)} \leq \mathcal{Y}^x_{(N_x)} \leq M_y$ almost surely, for any $k = 1, \cdots, N_x - 1$. Using the fact that $\frac{1}{1-\frac{k}{N_x}} \leq N_x$, we therefore obtain $C_{x,y}(n) \leq M_y N_x$ almost surely, and so

$$\begin{aligned} n^{1/(p+q)}\left(\widehat{\lambda}_n(x,y) - \widehat{\lambda}_{\alpha(n),n}(x,y)\right) &\leq n^{1/(p+q)} M_y N_x (1-\alpha(n)) \\ &= M_y \widehat{H}_{XY,n}(x,0) n^{(p+q+1)/(p+q)}(1-\alpha(n)) \end{aligned}$$

almost surely. The conclusion follows by applying the strong law of large numbers. □

Making use of this lemma and the following decomposition

$$n^{1/(p+q)}(\lambda(x,y) - \widehat{\lambda}_{\alpha(n),n}(x,y)) = n^{1/(p+q)}(\lambda(x,y) - \widehat{\lambda}_n(x,y)) + n^{1/(p+q)}(\widehat{\lambda}_n(x,y) - \widehat{\lambda}_{\alpha(n),n}(x,y))$$

we get immediately from Corollary 3.2 of Park, Simar and Weiner (2000) the following result.

Theorem 4.3. *Under Assumptions AI-AIII of Park, Simar and Weiner (2000), we have for any (x, y) interior to Ψ,*

$$n^{1/(p+q)}\left(\lambda(x,y) - \widehat{\lambda}_{\alpha(n),n}(x,y)\right) \xrightarrow{\mathcal{L}} Weibull(\mu^{p+q}_{NW,0}, p+q) \quad as \quad n \to \infty,$$

where $\mu_{NW,0}$ is a constant.

An explicit expression of the Weibull parameter $\mu_{NW,0}$ is given in Park et al. (2000, see Definition A.2 of the appendix). A consistent estimator of this unknown parameter is also provided (see Park et al., 2000, Theorem 3.4).

The nonparametric conditional quantile efficiency scores $\widehat{\lambda}_{\alpha(n),n}(x,y)$ lead to an estimator of the full frontier $Y^\partial(x)$. For all x such that $\widehat{H}_{XY,n}(x,0) > 0$, we have:

$$\widehat{Y}^\partial_{\alpha(n),n}(x) = \left\{(x, \widehat{\lambda}_{\alpha(n),n}(x,y)\, y) \mid y \text{ s.t. } (x,y) \in \widehat{\Psi}_{FDH}\right\}. \quad (4.12)$$

This estimator does not envelop all the observed data points and so, is more resistant to extreme points and/or to outliers than the usual nonparametric envelopment estimators (FDH, DEA).

It is important to note that the order-α efficiency scores $\widehat{\lambda}_{\alpha,n}(x,y)$ are representable as a functional T^α_{xy} of the empirical version of the probability function H_{XY} which characterizes completely the DGP:

$$\begin{aligned} \lambda_\alpha(x,y) &= \sup\left\{\lambda \,|\, \frac{H_{XY}(x,\lambda y)}{H_{XY}(x,0)} > 1-\alpha\right\} = T^\alpha_{xy}(H_{XY}), \\ \widehat{\lambda}_{\alpha,n}(x,y) &= \sup\left\{\lambda \,|\, \frac{\widehat{H}_{XY,n}(x,\lambda y)}{\widehat{H}_{XY,n}(x,0)} > 1-\alpha\right\} = T^\alpha_{xy}(\widehat{H}_{XY,n}). \end{aligned}$$

Therefore, The reliability of $T^{\alpha}_{xy}(\widehat{H}_{XY,n})$ in estimating $T^{\alpha}_{xy}(H_{XY})$ can be analyzed from a robustness theory point of view. The richest robustness information is provided by the influence function $(X_i, Y_i) \mapsto IF((X_i, Y_i); T^{\alpha}_{xy}, H_{XY})$ of T^{α}_{xy} at H_{XY} (Hampel, 1974). It is defined as the first Gâteaux derivative of T^{α}_{xy} at H_{XY} in the direction of $\Delta_{X_iY_i}(\cdot,\cdot) = \mathbb{1}(X_i \leq \cdot, Y_i \geq \cdot)$. Formally, $IF\left((X_i, Y_i); T^{\alpha}_{xy}, H_{XY}\right) = \frac{\partial}{\partial\delta} T^{\alpha}_{xy}(H_{XY} + \delta(\Delta_{X_iY_i} - H_{XY}))|_{\delta=0+}$.

The importance of the IF lies in the fact that it allows to assess the relative influence of individual observations towards the value of the estimate. If it is unbounded, even a single outlier may cause trouble. Its maximum absolute value $\gamma^*(T^{\alpha}_{xy}, H_{XY}) = \sup_{u\in\mathbb{R}^{p+q}} |IF(u; T^{\alpha}_{xy}, H_{XY})|$ defines the gross-error sensitivity of T^{α}_{xy} at H_{XY}. It measures the effect of contamination of the data by gross errors, whereby some of the observations (X_i, Y_i) may have a distribution grossly different from H_{XY}. Specifically, γ^* is interpreted as the worst possible influence which a fixed amount of contamination can have upon the estimator.

Proposition 4.2. *Under the same conditions of Theorem 4.2,*
1. the gross-error sensitivity:

$$\gamma^*(T^{\alpha}_{xy}, H_{XY}) = \frac{\min(-\alpha, \alpha - 1)}{G'(\lambda_\alpha(x, y))F_X(x)}$$

2. almost surely:

$$IF\left((X_i, Y_i); T^{\alpha}_{xy}, H_{XY}\right) = \Gamma(x, y, X_i, Y_i)$$

where Γ *is described in* (4.11).

Proof: We can use the same techniques of proof as in the input-oriented direction (see the appendix). □

As a consequence, we get in view of (4.5) the following asymptotic bias representation:

$$\begin{aligned}
\widehat{\lambda}_{\alpha,n}(x, y) - \lambda_\alpha(x, y) &= \frac{1}{n}\sum_{i=1}^{n} IF\left((X_i, Y_i); T^{\alpha}_{xy}, H_{XY}\right) + \frac{1}{\sqrt{n}}R_n, \\
&= \frac{\widehat{H}_{XY,n}(x, 0)}{G'(\lambda_\alpha(x, y))F_X(x)}\left[(1 - \alpha) - \widehat{S}_{Y|X,n}(\lambda_\alpha(x, y)y|x)\right] + R^{\alpha,xy}_n,
\end{aligned}$$

where $\sqrt{n}R^{\alpha,xy}_n = o_p(1)$ as $n \to \infty$. This confirms the fact that the IF describes approximately the effect of an infinitesimal contamination at the observations (X_i, Y_i) on the estimate. In particular, the sequence of nonparametric estimators $\{\widehat{\lambda}_{\alpha,n}(x, y)\}$ is bias-robust (Rousseeuw, 1981) since its gross-error sensitivity is finite.

5 Introducing Environmental Variables

The analysis of the preceding section can easily be extended to the case where additional information is provided by other variables $Z \in \mathbb{R}^r$, exogenous to the production process itself, but which may explain a part of it. The basic idea for introducing this additional information in the model is to condition the production process to a given value of $Z = z$. Inspired from Cazals et al. (2002), Daraio and Simar (2003) introduce the concepts of conditional efficiency measure and of partial conditional efficiency measure of order-m, where m is a positive integer. Similarly, we propose below the idea for conditional quantile efficiency measure of continuous order $\alpha \in [0, 1]$. The presentation here is limited to the output oriented case to save place.

If the joint distribution of (X, Y) conditional on $Z = z$ defines the production process, the efficiency measure $\lambda(x, y)$ defined in (1.5) and (2.5) has to be adapted to the condition $Z = z$ as follows:

$$\lambda(x, y|z) = \sup\{\lambda \mid S_{Y|X,Z}(\lambda y|x, z) > 0\}, \tag{5.1}$$

where $S_{Y|X,Z}(y|x, z) = Prob(Y \geq y|X \leq x, Z = z)$.

A nonparametric estimator of the conditional full-frontier efficiency $\lambda(x, y|z)$ is given by plugging in its formula a nonparametric estimator of $S_{Y|X,Z}(y|x, z)$. We can use the following smoothed estimator:

$$\widehat{S}_{Y|X,Z,n}(y|x, z) = \frac{\sum_{i=1}^n 1\!\!I(X_i \leq x, Y_i \geq y) K\left((z - Z_i)/h_n\right)}{\sum_{i=1}^n 1\!\!I(X_i \leq x) K\left((z - Z_i)/h_n\right)} \tag{5.2}$$

where K is the kernel and h_n is the bandwidth of approriate size. Practical bandwidth selection issues, based on a k-Nearest Neighbor method, are addressed in Section 4 of Daraio and Simar (2003) in the input oriented framework. As also pointed there, the estimate of the conditional full-frontier efficiency for kernels with unbounded support is unable to detect any influence of the environmental factors. Therefore, kernels with compact support have to be used.

By following the idea of Definition (3.2), we can define the conditional order-α output efficiency measure as follows:

Definition 5.1. *For any $y \in \mathbb{R}^q_+$, the conditional order-α output efficiency measure given that $Z = z$, denoted by $\lambda_\alpha(x, y|z)$ is defined for all x in the interior of the support of X as:*

$$\lambda_\alpha(x, y|z) = \sup\{\lambda \mid S_{Y|X,Z}(\lambda y|x, z) > 1 - \alpha\}. \tag{5.3}$$

Therefore, for any $y \in \mathbb{R}^q_+$, the conditional order-α quantile frontier given that $Z = z$, is defined as the set of points $y^\partial_\alpha(x|z) = \lambda_\alpha(x, y|z)y$, $y \in \mathbb{R}^q_+$. As above we have immediately the following result.

Proposition 5.1. *For any $y \in \mathbb{R}_+^q$ and for all x in the interior of the support of X,*

$$\lim_{\alpha \to 1} \nearrow \lambda_\alpha(x, y|z) = \lambda(x, y|z).$$

A nonparametric estimator of $\lambda_\alpha(x, y|z)$ is provided by plugging in its formula the nonparametric estimator of $S_{Y|X,Z}(y|x, z)$. Formally, it is defined as:

$$\widehat{\lambda}_{\alpha,n}(x, y|z) = \sup\{\lambda \mid \widehat{S}_{Y|X,Z,n}(\lambda y|x, z) > 1 - \alpha\}.$$

Here also we have

$$\lim_{\alpha \to 1} \nearrow \widehat{\lambda}_{\alpha,n}(x, y|z) = \widehat{\lambda}_n(x, y|z) := \sup\{\lambda \mid \widehat{S}_{Y|X,Z,n}(\lambda y|x, z) > 0\}.$$

Note that the asymptotic properties of $\widehat{\lambda}_n(x, y|z)$ have not yet been derived in the literature. We would return to this topic in another paper. The asymptotic properties of $\widehat{\lambda}_{\alpha,n}(x, y|z)$ would be investigating, too.

These conditional nonparametric estimators are very easy to implement and very fast to compute in practice. Indeed, for $j = 1, \cdots, N_x$, denote by $Z^x_{[j]}$ the observation Z_i corresponding to the order statistic $\mathcal{Y}^x_{(j)}$, and let $R_{x,z} = \sum_{i=1}^n 1\!\!I(X_i \le x) K(\frac{z - Z_i}{h_n}) > 0$. Then,

$$\begin{aligned}
\widehat{S}_{Y|X,Z,n}(\lambda y|x, z) &= \frac{1}{R_{x,z}} \sum_{j=1}^{N_x} 1\!\!I(\lambda \le \mathcal{Y}^x_{(j)}) K\left((z - Z^x_{[j]})/h_n\right) \\
&= \begin{cases} 1 \quad \text{if} \quad \lambda \le \mathcal{Y}^x_{(1)} \\ L_{k+1} \quad \text{if} \quad \mathcal{Y}^x_{(k)} < \lambda \le \mathcal{Y}^x_{(k+1)}, \ k = 1, \cdots, N_x - 1 \\ 0 \quad \text{if} \quad \lambda > \mathcal{Y}^x_{(N_x)}, \end{cases}
\end{aligned}$$

where $L_{k+1} = (1/R_{x,z}) \sum_{j=k+1}^{N_x} K\left((z - Z^x_{[j]})/h_n\right)$. It follows,

$$\widehat{\lambda}_{\alpha,n}(x, y|z) = \begin{cases} \mathcal{Y}^x_{(k)} \quad \text{if} \quad L_{k+1} \le 1 - \alpha < L_k, \ k = 1, \cdots, N_x - 1 \\ \mathcal{Y}^x_{(N_x)} \quad \text{if} \quad 0 \le 1 - \alpha < L_{N_x}. \end{cases} \tag{5.4}$$

Stressing the influence of Z on the production process

The comparison of $\widehat{\lambda}_n(x, y|z)$ with $\widehat{\lambda}_n(x, y)$ is certainly of interest for analyzing the global influence of Z on the production process. When Z is univariate, Daraio and Simar (2003) suggest that the use of a scatter plot of the ratios $\widehat{\lambda}_n(x, y|z)/\widehat{\lambda}_n(x, y)$ against Z and its smoothed nonparametric regression line would be helpful to describe the influence of Z on efficiency. An increasing regression corresponds to favorable environmental factor and a decreasing regression indicates an unfavorable factor. The correspondent α-quantile efficiency scores provide a more robust analysis, robust to extremes or outliers. Of course, we do not propose any inference here, but only an easy and useful descriptive diagnostic tool.

In the Appendix, the α-quantile based efficiency scores in the input oriented case is described in details. Here, the interpretation of the shape of the regression line of the ratios $\widehat{\theta}_n(x, y|z)/\widehat{\theta}_n(x, y)$ against Z (and of their correspondent α-quantile based measures), is in the opposite direction.

6 Numerical Illustrations

We illustrate the α-quantile efficiency scores and their estimation by using some of the data set used in Daraio and Simar (2003), in a multivariate setup.

6.1 A multivariate simulated example

Here a multi-input ($p = 2$) and multi-output ($q = 2$) data set is simulated. In this set-up, the function describing the efficient frontier is given by:

$$y^{(2)} = 1.0845(x^{(1)})^{0.3}(x^{(2)})^{0.4} - y^{(1)}$$

where $y^{(j)}$, ($x^{(j)}$), denotes the jth component of y, (of x), for $j = 1, 2$. We draw $X_i^{(j)}$ independent uniforms on $(1, 2)$ and $\tilde{Y}_i^{(j)}$ independent uniforms on $(0.2, 5)$. Then the generated random rays in the output space are characterized by the slopes $S_i = \tilde{Y}_i^{(2)}/\tilde{Y}_i^{(1)}$. Finally, the generated random points on the frontier are defined by:

$$\begin{aligned} Y_{i,eff}^{(1)} &= \frac{1.0845(X_i^{(1)})^{0.3}(X_i^{(2)})^{0.4}}{S_i + 1} \\ Y_{i,eff}^{(2)} &= 1.0845(X_i^{(1)})^{0.3}(X_i^{(2)})^{0.4} - Y_{i,eff}^{(1)}. \end{aligned}$$

The efficiencies are generated by $\exp(-U_i)$ where U_i are drawn from an exponential with mean $\mu = 1/3$. Finally, in a standard setup (without environmental factors), we define $Y_i = Y_{i,eff} * \exp(-U_i)$.

Now we introduce the dependency on Z in the latter expression as follows: Z is uniform on $(1, 4)$ and (see Case 1 of Daraio and Simar ,2003), Z is favorable to output production but differently for $Y^{(1)}$ than for $Y^{(2)}$. We define $V = Z$ and set

$$\begin{aligned} Y_i^{(1)} &= V^2 * Y_{i,eff}^{(1)} * \exp(-U_i) \\ Y_i^{(2)} &= V^{1/2} * Y_{i,eff}^{(2)} * \exp(-U_i). \end{aligned}$$

The results are displayed in Figure 1. We see that all the ratios allow to detect the favorable effect of Z on the production process. The α-quantile measures being less sensitive to extreme values, give a better picture. We have chosen a triangle kernel for the smoothing: the results are very stable with respect to other choice of the kernel with compact support.

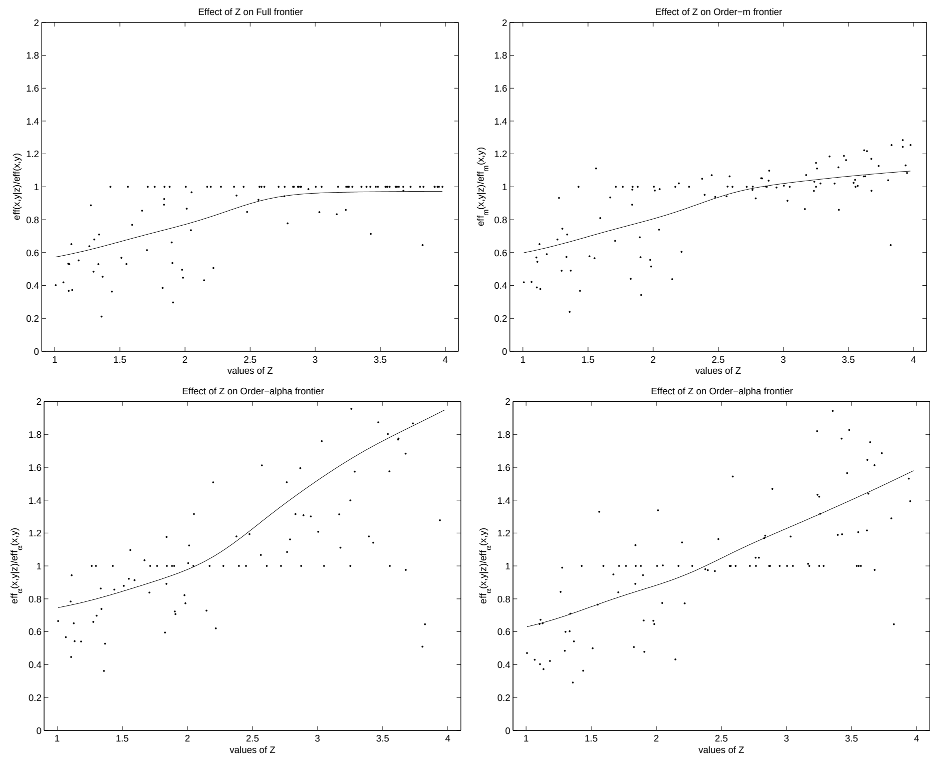

Figure 1: *Simulated example, $n = 100$: "positive" effect of Z on production efficiency (output oriented framework). Scatterplot and smoothed regression of the ratios $\hat{\lambda}_n(x, y \mid z)/\hat{\lambda}_n(x, y)$ on Z (top left), of $\hat{\lambda}_{m,n}(x, y \mid z)/\hat{\lambda}_{m,n}(x, y)$ on Z (top right, with $m = 25$) and of $\hat{\lambda}_{\alpha,n}(x, y \mid z)/\hat{\lambda}_{\alpha,n}(x, y)$on Z (bottom panel, left $\alpha = 0.80$ and right $\alpha = 0.90$). Here k-NN=17.*

In order to appreciate the robustness to outliers, and compare the performance of the order-m and of the α-quantile measures, we introduce in the same data set 5 outliers. They are introduced at the following values of X: (1.25,1.25), (1.25, 1.75), (1.5,1.5), (1.75, 1.25) and (1.75, 1.75), the corresponding values for the slopes in the Y space are (0.25, 0.5, 1, 1.5, 2). Then the random effect of the Z variables were introduced as above and finally, the obtained points in the Y coordinates were projected to be outliers by a radial expansion by a factor 1/0.6. The results of this data set with $n = 105$ points are shown in Figure 2. It is clear that the full frontier approach is unable to detect the favorable effect of Z, at least for values larger than the mean of Z (2.5), the order-m does better but again fails for large values of Z. On the contrary, the order-α quantile frontier are much more robust to the 5 outliers and we obtain similar results as in Figure 1, where no outliers where introduced.

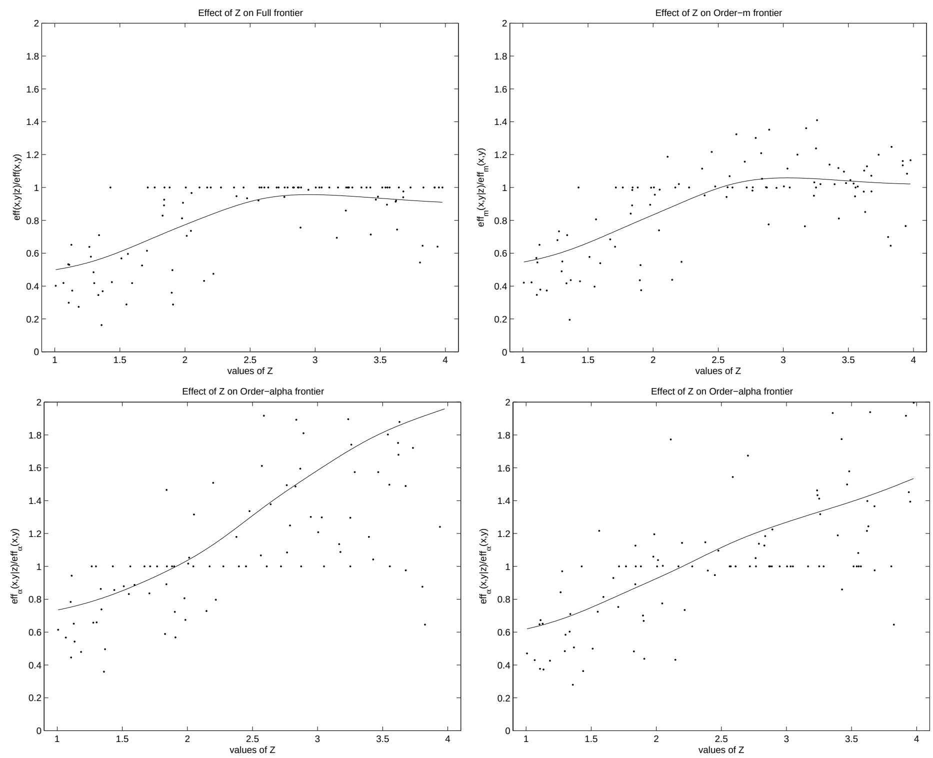

Figure 2: *Simulated example, $n = 105$ including 5 outliers: "positive" effect of Z on production efficiency (output oriented framework). Scatterplot and smoothed regression of the ratios $\hat{\lambda}_n(x, y \mid z)/\hat{\lambda}_n(x, y)$ on Z (top left), of $\hat{\lambda}_{m,n}(x, y \mid z)/\hat{\lambda}_{m,n}(x, y)$ on Z (top right, with $m = 25$) and of $\hat{\lambda}_{\alpha,n}(x, y \mid z)/\hat{\lambda}_{\alpha,n}(x, y)$ on Z (bottom panel, left $\alpha = 0.80$ and right $\alpha = 0.90$). Here k-NN=20.*

6.2 Mutual funds data

We also illustrate our methodology analyzing US Mutual Funds data. We use a cross-section data set, collected by the reputed Morningstar, which consists of the US Mutual Funds universe updated at 05-31-2002. Among this universe we select the Aggressive-Growth (AG) category of Mutual Funds. These are funds that seek rapid growth of capital and that may invest in emerging market growth companies. For details about the data, the variables and references to this literature, see Daraio and Simar (2004).

We have a sample of 129 mutual funds and we apply an input oriented framework. The traditional output in this framework is the total return of funds (the annual return at the 05-31-2002, expressed in percentage terms). Most returns were negative in this period, hence we shift them to get all positive returns by adding 100. This does not change our input oriented analysis. The inputs are risk (standard deviation, or volatility of the return), expense ratio (the percentage of fund assets paid for operating expenses, management fees, administrative fees, and all other asset-based costs) and turnover ratio (a measure of the fund's trading activity). In our illustration we use the market risks of mutual funds (the percentage of fund's movements that can be explained by movements in its benchmark index) as environmental variable, to investigate its effect on our data, *i.e.* if it is detrimental or favorable to the performance of mutual funds in the period under consideration.

We compare FDH, order-m (with $m = 25$) and α-quantile efficiency scores, unconditional and condtional to Z. The results are displayed in Figure 3. They confirm the global positive effect of the risk market on the performance of the funds, as expected from the literature (Sengupta, 2000 used this variable as additional input in this framework), but here this interpretation is a result of our analysis. The effect appears more clearly with the order-α measures, because they are less sensitive to extreme values. Table 1 gives the complete set of results for the 129 funds, with $\alpha = 0.80$ and $m = 25$. The table allows to compare the different efficiency scores: clearly the 80% efficiency scores are easier to interpret than the order-25 efficiency scores. Both being robust versions of the FDH efficiency scores.

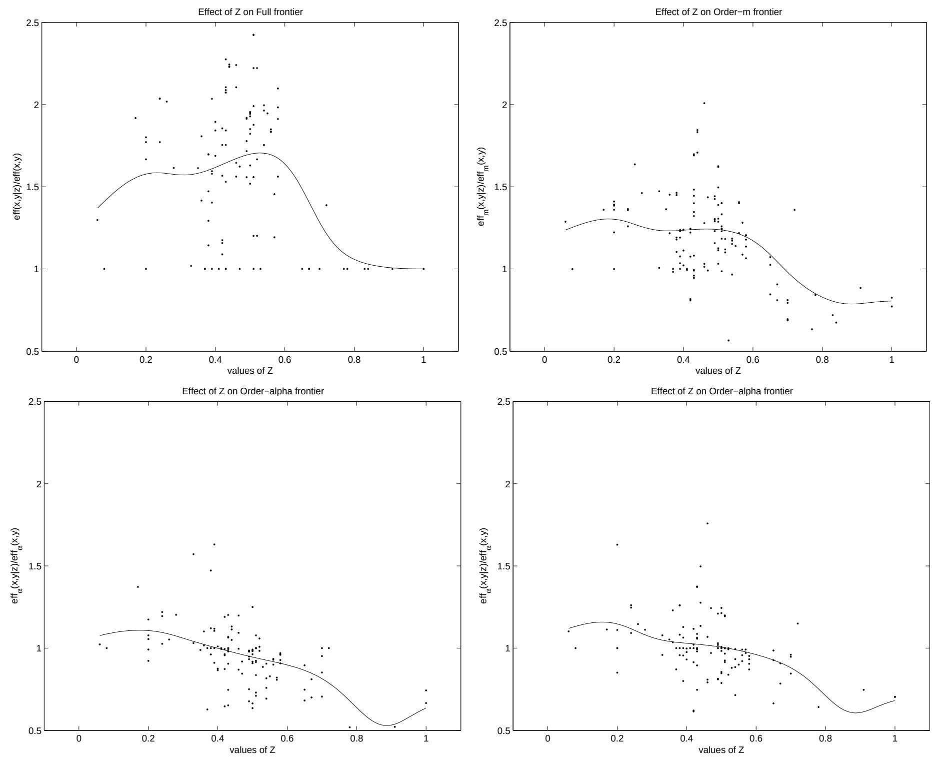

Figure 3: Aggressive-Growth US Mutual Funds. Scatterplot and smoothed regression of the ratios $\hat{\theta}_n(x, y \mid z)/\hat{\theta}_n(x, y)$ on Z (top left), of $\hat{\theta}_{m,n}(x, y \mid z)/\hat{\theta}_{m,n}(x, y)$ on Z (top right, with $m = 25$) and of $\hat{\theta}_{\alpha,n}(x, y \mid z)/\hat{\theta}_{\alpha,n}(x, y)$on Z (bottom panel, left $\alpha = 0.80$ and right $\alpha = 0.90$). Here k-NN=21.

7 Conclusions

In this paper we develop a new concept of efficiency measure, the α-quantile efficient scores, related to a non-standard conditional α-quantile frontier in a full multivariate set-up. The approach can be viewed as an alternative to the order-m efficiency scores and order-m efficient frontier developed by Cazals *et al.* (2002) and Daraio and Simar (2003).

Both approaches provide nonparametric estimators of the efficient frontier which are more robust than the usual envelopment estimators (like FDH/DEA estimators). The α-quantile approach is more easy to interpret since the parameter α is just the selected level of the quantile. The choice of m in order-m efficient frontier is more delicate although it can be interpreted as the number of potential firms against which the benchmark is done to determine the efficiency score of a particular firm. The choice of m can also be indirectly piloted by the percentage of observed firms staying above the frontier for a given m, but the α-quantile approach seems to be more direct.

The asymptotic theory of our estimator is provided and we show how it can be easily calculated. As for the order-m frontiers, the α-quantile frontier can also be used to detect outliers in the spirit of Simar (2003). In this framework, it is also easy to introduce environmental factors and we propose useful tools for detecting their influence on efficiency.

Finally, since for every attainable point (x_i, y_i), there exists a α such that $\widehat{\theta}_{\alpha,n} = 1$ (or $\widehat{\lambda}_{\alpha,n} = 1$), this α could serve as an alternative measure of input (or output) efficiency. In other words, one may set the performance measure for the unit (x_i, y_i) to be the order α of the quantile frontier which passes through this unit. For instance, in the output orientation, it can be easily seen that $\alpha(x_i, y_i) = 1 - \widehat{S}_{Y|X,n}(y_i|x_i) + (1/n\widehat{H}_{XY,n}(x_i, 0))$.

Appendix: Input-oriented quantile efficiency scores

A Some Robust and Asymptotic Theory

The order-α quantile efficiency score in the input-direction and its empirical estimator are representable, respectively, as a functional T^{α}_{xy} of the nonstandard distribution H_{XY} and its empirical version $\widehat{H}_{XY,n}$:

$$\begin{cases} \theta_{\alpha}(x,y) = \inf\{\theta | \frac{H_{XY}(\theta x,y)}{H_{XY}(\infty,y)} > 1-\alpha\} = T^{\alpha}_{xy}(H_{XY}) \\ \widehat{\theta}_{\alpha,n}(x,y) = \inf\{\theta | \frac{\widehat{H}_{XY,n}(\theta x,y)}{\widehat{H}_{XY,n}(\infty,y)} > 1-\alpha\} = T^{\alpha}_{xy}(\widehat{H}_{XY,n}). \end{cases}$$

According the robust estimation literature, the contribution or influence of the observations (X_i, Y_i) toward the estimation error $T^{\alpha}_{xy}(\widehat{H}_{XY,n}) - T^{\alpha}_{xy}(H_{XY})$ can be measured approximately by the following influence function (see, e.g., Hampel et al. (1985), Definition 1, p. 84):

$$IF\left((X_i, Y_i); T^{\alpha}_{xy}, H_{XY}\right) = \frac{\partial}{\partial \delta} T^{\alpha}_{xy}(H_{XY} + \delta(\Delta_{X_iY_i} - H_{XY}))|_{\delta=0+}$$

where $\Delta_{X_iY_i}(x_0, y_0) = 1\!\!1(X_i \leq x_0, Y_i \geq y_0)$ for any $(x_0, y_0) \in \mathbb{R}^{p+q}$.

Proposition A.1. *Let $(x,y) \in \Psi$ be such that $S_Y(y) > 0$ and let $0 < \alpha < 1$. Assume that $G(\theta) = F_{X|Y}(\theta x|y)$ is differentiable at $\theta_{\alpha}(x,y)$ with positive derivative $G'(\theta_{\alpha}(x,y))$. Then*

$$IF\left((X_i, Y_i); T^{\alpha}_{xy}, H_{XY}\right) = \frac{(1-\alpha)\,1\!\!1(Y_i \geq y) - 1\!\!1(X_i \leq \theta_{\alpha}(x,y)x, Y_i \geq y)}{G'(\theta_{\alpha}(x,y))S_Y(y)}$$

with probability 1, *and the gross-error sensitivity is given by*

$$\gamma^*(T^{\alpha}_{xy}, H_{XY}) = \frac{\max(\alpha, 1-\alpha)}{G'(\theta_{\alpha}(x,y))S_Y(y)}.$$

Thus the sequence of nonparametric α-quantile efficiency scores $\{\widehat{\theta}_{\alpha,n}(x,y)\}$ is bias-robust (Rousseeuw, 1981) since it possesses a finite gross-error sensitivity. The approximate influence of the observations (X_i, Y_i) toward the error of estimation in estimating $T^{\alpha}_{xy}(H_{XY})$ by $T^{\alpha}_{xy}(\widehat{H}_{XY,n})$ is showed by the following asymptotic bias representation:

Theorem A.1. *Under the same conditions of the above Proposition, we have*

$$\widehat{\theta}_{\alpha,n}(x,y) - \theta_{\alpha}(x,y) = \frac{1}{n}\sum_{i=1}^{n} IF\left((X_i, Y_i); T^{\alpha}_{xy}, H_{XY}\right) + R^{\alpha,xy}_n,$$

where $\sqrt{n}R^{\alpha,xy}_n = o_p(1)$ as $n \to \infty$.

As an immediate consequence, we obtain the following asymptotic Bahadur-type representation:

$$\sqrt{n}\left(\widehat{\theta}_{\alpha,n}(x,y)-\theta_\alpha(x,y)\right)=\frac{\sqrt{n}\widehat{H}_{XY,n}(\infty,y)}{G'(\theta_\alpha(x,y))S_Y(y)}\left[(1-\alpha)-\widehat{F}_{X|Y,n}(\theta_\alpha(x,y)x|y)\right]+o_p(1).$$

Which gives the asymptotic normality of $\widehat{\theta}_{\alpha,n}(x,y)$ in view of the central limit theorem.

Corollary A.1. *Under the same conditions of the above Theorem, we have*

$$\sqrt{n}\left(\widehat{\theta}_{\alpha,n}(x,y)-\theta_\alpha(x,y)\right)\xrightarrow{\mathcal{L}}N\left(0,\sigma_\alpha^2(x,y)\right)\quad as\quad n\to\infty,$$

where $\sigma_\alpha^2(x,y)=\alpha(1-\alpha)/[G'(\theta_\alpha(x,y))]^2S_Y(y)$.

By applying the multivariate central limit theorem, we also obtain the asymptotic r-variate normal distribution for a vector $(\widehat{\theta}_{\alpha,n}(x^1,y^1),\cdots,\widehat{\theta}_{\alpha,n}(x^r,y^r))$. Moreover, the estimator $\widehat{\theta}_{\alpha,n}(x,y)$ converges completely to $\theta_\alpha(x,y)$. Even more strongly, we have the following exponential probability inequality:

Theorem A.2. *Let $(x,y)\in\Psi$ be such that $S_Y(y)>0$ and let $0<\alpha<1$. Assume that $\theta\mapsto F_{X|Y}(\theta x|y)$ is continuous and increasing in a neigborhood of $\theta_\alpha(x,y)$. Then, for every $\varepsilon>0$,*

$$Prob(|\widehat{\theta}_{\alpha,n}(x,y)-\theta_\alpha(x,y)|>\varepsilon)\le 2e^{-2n\delta^2_{\alpha,\varepsilon,x,y}},\quad for\ all\ \ n\ge 1,$$

where

$$\delta_{\alpha,\varepsilon,x,y}=\frac{S_Y(y)}{2-\alpha}\min\left\{(1-\alpha)-F_{X|Y}((\theta_\alpha(x,y)-\varepsilon)x|y);F_{X|Y}((\theta_\alpha(x,y)+\varepsilon)x|y)-(1-\alpha)\right\}.$$

The asymptotic distribution of $\widehat{\theta}_{\alpha,n}(x,y)$, as an estimator of the efficiency score $\theta(x,y)$ itself, can easily be adapted from Theorem 4.3.

B Introducing Environmental Variables

Here, following Daraio and Simar (2004), the conditional input efficiency score is defined as:

$$\theta(x,y\mid z)=\inf\{\theta\mid F_{X|Y,Z}(\theta x\mid y,z)>0\},\tag{B.1}$$

where $F_{X|Y,Z}(x\mid y,z)=\text{Prob}(X\le x\mid Y\ge y,Z=z)$. A nonparametric estimator of the conditional full-frontier efficiency $\theta(x,y|z)$ is given by plugging a nonparametric estimator of $F_{X|Y,Z}(x|\ y,z)$, defined as:

$$\widehat{F}_{X|Y,Z,n}(x\mid y,z)=\frac{\sum_{i=1}^n \mathbb{1}(x_i\le x,y_i\ge y)K((z-z_i)/h_n)}{\sum_{i=1}^n \mathbb{1}(y_i\ge y)K((z-z_i)/h_n)},\tag{B.2}$$

where $K(\cdot)$ is the kernel and h_n is the bandwidth of appropriate size.

The conditional quantile of order-α is defined as follows:

Definition B.1. *For any $x \in \mathbb{R}^p_+$, the conditional order-α input efficiency measure given that $Z = z$, denoted by $\theta_\alpha(x, y|z)$ is defined for all y in the interior of the support of Y as:*

$$\theta_\alpha(x, y|z) = \inf\{\theta \mid F_{X|Y,Z}(\theta x|y, z) > 1 - \alpha\}. \tag{B.3}$$

A nonparametric estimator is given by plugging a nonparametric estimator of $F_{X|Y,Z}(\cdot)$ in (B.3). We have

$$\begin{aligned}
\widehat{F}_{X|Y,Z,n}(\theta x|y, z) &= \frac{1}{Q_{y,z}} \sum_{j=1}^{M_y} 1\!\!I(\mathcal{X}^y_{(j)} \le \theta) K\left((z - Z^y_{[j]})/h_n\right) \\
&= \begin{cases} 0 & \text{if} \quad \theta < \mathcal{X}^y_{(1)} \\ \ell_k & \text{if} \quad \mathcal{X}^y_{(k)} \le \theta < \mathcal{X}^y_{(k+1)}, \ k = 1, \cdots, M_y - 1 \\ 1 & \text{if} \quad \theta \ge \mathcal{X}^y_{(M_y)}, \end{cases}
\end{aligned}$$

where, for $j = 1, \cdots, M_y$, $Z^y_{[j]}$ denotes the observation Z_i corresponding to the order statistic $\mathcal{X}^y_{(j)}$, and

$$\begin{aligned}
Q_{y,z} &= \sum_{i=1}^{n} 1\!\!I(Y_i \ge y) K(\frac{z - Z_i}{h_n}), \\
\ell_k &= (1/Q_{y,z}) \sum_{j=1}^{k} K\left((z - Z^y_{[j]})/h_n\right).
\end{aligned}$$

Thus

$$\widehat{\theta}_{\alpha,n}(x, y|z) = \begin{cases} \mathcal{X}^y_{(1)} & \text{if} \quad 0 \le 1 - \alpha < \ell_1 \\ \mathcal{X}^y_{(k+1)} & \text{if} \quad \ell_k \le 1 - \alpha < \ell_{k+1}, \quad k = 1, \cdots, M_y - 1. \end{cases} \tag{B.4}$$

C Proofs

Proof of Proposition A.1: We have

$$\begin{aligned}
&T^\alpha_{xy}\left(H_{XY} + \delta[\Delta_{X_iY_i} - H_{XY}]\right) \\
&\quad = \inf\left\{\theta | \frac{H_{XY}(\theta x, y) + \delta[\Delta_{X_iY_i}(\theta x, y) - H_{XY}(\theta x, y)]}{S_Y(y) + \delta[1\!\!I(Y_i \ge y) - S_Y(y)]} > 1 - \alpha\right\} \\
&\quad = \inf\left\{\theta | \ \frac{(1 - \delta)H_{XY}(\theta x, y) + \delta\Delta_{X_iY_i}(\theta x, y)}{(1 - \delta)S_Y(y) + \delta 1\!\!I(Y_i \ge y)} > 1 - \alpha\right\} \\
&\quad = \inf\left\{\theta | \ F_{X|Y}(\theta x|y) > (1 - \alpha) + \frac{\delta(1 - \alpha)1\!\!I(Y_i \ge y) - \delta\Delta_{X_iY_i}(\theta x, y)}{(1 - \delta)S_Y(y)}\right\}
\end{aligned}$$

$$= \inf \left\{ \theta \geq \mathcal{X}_i |\ F_{X|Y}(\theta x|y) > (1-\alpha) + \frac{\delta(1-\alpha)\, 1\!\!1(Y_i \geq y) - \delta \Delta_{X_iY_i}(\theta x, y)}{(1-\delta) S_Y(y)} \right\}$$
$$\wedge \inf \left\{ \theta < \mathcal{X}_i |\ F_{X|Y}(\theta x|y) > (1-\alpha) + \frac{\delta(1-\alpha)\, 1\!\!1(Y_i \geq y) - \delta \Delta_{X_iY_i}(\theta x, y)}{(1-\delta) S_Y(y)} \right\}$$
$$= \inf \left\{ \theta \geq \mathcal{X}_i |\ G(\theta) > (1-\alpha) - \frac{\alpha\delta\, 1\!\!1(Y_i \geq y)}{(1-\delta) S_Y(y)} \right\}$$
$$\wedge \inf \left\{ \theta < \mathcal{X}_i |\ G(\theta) > (1-\alpha) + \frac{\delta(1-\alpha)\, 1\!\!1(Y_i \geq y)}{(1-\delta) S_Y(y)} \right\}.$$

The last equality is obtained by making use of the fact that $\theta \geq \mathcal{X}_i \Leftrightarrow X_i \leq \theta x$. If $\theta_\alpha(x,y) \geq \mathcal{X}_i$ then the second infimum on the right-hand side of this equality is equal to infinity and, if $\theta_\alpha(x,y) < \mathcal{X}_i$, then the first infimum is $\mathcal{X}_i$. Therefore, for δ sufficiently small, we get

$$\begin{aligned}
& T^\alpha_{xy}\left(H_{XY} + \delta[\Delta_{X_iY_i} - H_{XY}]\right) \\
&\quad = \inf \left\{ \theta \geq \mathcal{X}_i |\ G(\theta) > (1-\alpha) - \frac{\alpha\delta\, 1\!\!1(Y_i \geq y)}{(1-\delta) S_Y(y)} \right\} 1\!\!1(\theta_\alpha(x,y) \geq \mathcal{X}_i) \\
&\qquad + G^{-1}\left((1-\alpha) + \frac{\delta(1-\alpha)\, 1\!\!1(Y_i \geq y)}{(1-\delta) S_Y(y)} \right) 1\!\!1(\theta_\alpha(x,y) < \mathcal{X}_i) \\
&\quad = G^{-1}\left((1-\alpha) - \frac{\alpha\delta\, 1\!\!1(Y_i \geq y)}{(1-\delta) S_Y(y)} \right) 1\!\!1(\theta_\alpha(x,y) > \mathcal{X}_i) \\
&\qquad + \theta_\alpha(x,y)\, 1\!\!1(\theta_\alpha(x,y) = \mathcal{X}_i) \\
&\qquad + G^{-1}\left((1-\alpha) + \frac{\delta(1-\alpha)\, 1\!\!1(Y_i \geq y)}{(1-\delta) S_Y(y)} \right) 1\!\!1(\theta_\alpha(x,y) < \mathcal{X}_i).
\end{aligned}$$

Hence

$$IF\left((X_i,Y_i); T^\alpha_{xy}, H_{XY}\right) = \frac{-\alpha\, 1\!\!1(Y_i \geq y)\, 1\!\!1(\theta_\alpha(x,y) > \mathcal{X}_i) + (1-\alpha)\, 1\!\!1(Y_i \geq y)\, 1\!\!1(\theta_\alpha(x,y) < \mathcal{X}_i)}{G'(\theta_\alpha(x,y)) S_Y(y)}.$$

More generally, we have for any $(x_0, y_0) \in \mathbb{R}^{p+q}$

$$\begin{aligned}
& IF\left((x_0,y_0); T^\alpha_{xy}, H_{XY}\right) = \\
&\quad \frac{1\!\!1(y_0 \geq y)}{G'(\theta_\alpha(x,y)) S_Y(y)} \left\{ -\alpha\, 1\!\!1\left(\theta_\alpha(x,y) > \max_{1\leq k\leq p} \frac{x_0^k}{x^k} \right) + (1-\alpha)\, 1\!\!1\left(\theta_\alpha(x,y) < \max_{1\leq k\leq p} \frac{x_0^k}{x^k} \right) \right\}.
\end{aligned}$$

Thus

$$\begin{aligned}
\gamma^*(T^\alpha_{xy}, H_{XY}) &= \sup_{(x_0,y_0)\in\mathbb{R}^{p+q}} \left| IF\left((x_0,y_0); T^\alpha_{xy}, H_{XY}\right) \right| \\
&= \frac{\max(\alpha, 1-\alpha)}{G'(\theta_\alpha(x,y)) S_Y(y)}.
\end{aligned}$$

We also have almost surely

$$\begin{aligned}
&IF\left((X_i,Y_i);T^{\alpha}_{xy},H_{XY}\right)\\
&= \frac{-\alpha 1\!\!I(Y_i\geq y)1\!\!I(\theta_\alpha(x,y)\geq \mathcal{X}_i)+(1-\alpha)1\!\!I(Y_i\geq y)[1-1\!\!I(\theta_\alpha(x,y)\geq \mathcal{X}_i)]}{G'(\theta_\alpha(x,y))S_Y(y)}\\
&= \frac{(1-\alpha)1\!\!I(Y_i\geq y)-1\!\!I(X_i\leq \theta_\alpha(x,y)x, Y_i\geq y)}{G'(\theta_\alpha(x,y))S_Y(y)}.
\end{aligned}$$

This ends the proof. $\square$

Proof of Theorem A.1: Set

$$V_n=\sqrt{n}\left(\widehat{\theta}_{\alpha,n}(x,y)-\theta_\alpha(x,y)\right),\quad W_n=\frac{1}{\sqrt{n}}\sum_{i=1}^{n} IF\left((X_i,Y_i);T^{\alpha}_{xy},H_{XY}\right).$$

Using the fact that $\widehat{\theta}_{\alpha,n}(x,y)\leq\theta \Leftrightarrow \widehat{F}_{X|Y,n}(\theta x|y)>1-\alpha$, we obtain for any real t,

$$\begin{aligned}
\{V_n\leq t\} &= \left\{\widehat{\theta}_{\alpha,n}(x,y)\leq \theta_\alpha(x,y)+\frac{t}{\sqrt{n}}\right\}\\
&= \left\{\widehat{F}_{X|Y,n}\left(\left(\theta_\alpha(x,y)+\frac{t}{\sqrt{n}}\right)x|y\right)>1-\alpha\right\}=\{Z_{t,n}<T_n\}, \qquad \text{(C.1)}
\end{aligned}$$

where

$$\begin{aligned}
Z_{t,n} &= \frac{\sqrt{n}\widehat{H}_{XY,n}(\infty,y)}{G'(\theta_\alpha(x,y))S_Y(y)}\left[G(\theta_\alpha(x,y)+\frac{t}{\sqrt{n}})-\widehat{F}_{X|Y,n}((\theta_\alpha(x,y)+\frac{t}{\sqrt{n}})x|y)\right]\\
T_n &= \frac{\sqrt{n}\widehat{H}_{XY,n}(\infty,y)}{G'(\theta_\alpha(x,y))S_Y(y)}\left[G(\theta_\alpha(x,y)+\frac{t}{\sqrt{n}})-(1-\alpha)\right].
\end{aligned}$$

Since $G(\theta_\alpha(x,y)+\frac{t}{\sqrt{n}})-(1-\alpha)=\frac{t}{\sqrt{n}}G'(\theta_\alpha(x,y))+\frac{t}{\sqrt{n}}o(1)$ and $\widehat{H}_{XY,n}(\infty,y)\xrightarrow{p}S_Y(y)$, as $n\to\infty$, we obtain

$$T_n\xrightarrow{p} t \quad \text{as} \quad n\to\infty. \qquad \text{(C.2)}$$

Since

$$\begin{aligned}
Z_{t,n}-W_n=\frac{\sqrt{n}\widehat{H}_{XY,n}(\infty,y)}{G'(\theta_\alpha(x,y))S_Y(y)}&\left[\left(G(\theta_\alpha(x,y)+\frac{t}{\sqrt{n}})-\widehat{F}_{X|Y,n}((\theta_\alpha(x,y)+\frac{t}{\sqrt{n}})x|y)\right)\right.\\
&\left.-\left((1-\alpha)-\widehat{F}_{X|Y,n}(\theta_\alpha(x,y)x|y)\right)\right],
\end{aligned}$$

we obtain by a straightforward computation

$$\begin{aligned}
E[(Z_{t,n}-W_n)^2]\left\{G'(\theta_\alpha(x,y))S_Y(y)\right\}^2 &= (1-\alpha)S_Y(y)-(1-\alpha)^2S_Y(y)\\
&- S_Y(y)G^2(\theta_\alpha(x,y)+\frac{t}{\sqrt{n}})+[1+2(1-\alpha)]S_Y(y)G(\theta_\alpha(x,y)+\frac{t}{\sqrt{n}})\\
&- 2S_Y(y)G\left(\theta_\alpha(x,y)+(\frac{t}{\sqrt{n}}\wedge 0)\right).
\end{aligned}$$

It follows that $E[(Z_{t,n}-W_n)^2]\to 0$, and so

$$Z_{t,n}-W_n \xrightarrow{p} 0 \quad \text{as} \quad n\to\infty. \tag{C.3}$$

To prove that $\sqrt{n}R_n^{\alpha,xy}=V_n-W_n=o_p(1)$, it suffices to show that $\{V_n\}$ and $\{W_n\}$ satisfy the two conditions of Ghosh (1971, Lemma 1, p. 1958).

Since W_n converges in law in view of the central limit theorem, it is uniformly tight and thus the first Ghosh's condition is satisfied. On the other hand, for all k and all $\varepsilon>0$, setting $t=k$, we obtain from (C.1),

$$\{V_n\le k, W_n\ge k+\varepsilon\}=\{Z_{t,n}<T_n, W_n\ge t+\varepsilon\}\subset\{(W_n-Z_{t,n})\ge\varepsilon-(T_n-t)\}.$$

Which gives via (C.2) and (C.3)

$$Prob(V_n\le k, W_n\ge k+\varepsilon)\le Prob\left(|(W_n-Z_{t,n})+(T_n-t)|\ge\varepsilon\right)\to 0 \quad \text{as} \quad n\to\infty.$$

Likewise, applying (C.1) to $t=k+\varepsilon$ and then combining (C.2) and (C.3), we get

$$Prob(V_n> k+\varepsilon, W_n\le k)\le Prob\left(|(Z_{t,n}-W_n)-(T_n-t)|\ge\varepsilon\right)\to 0 \quad \text{as} \quad n\to\infty.$$

Thus the second Ghosh's condition is also satisfied, which completes the proof. $\square$

Proof of Theorem A.2: Let $\varepsilon>0$. Using the fact that $\widehat{\theta}_{\alpha,n}(x,y)\le\theta\Leftrightarrow\widehat{F}_{X|Y,n}(\theta x|y)>1-\alpha$, we obtain

$$\begin{aligned}
Prob(\widehat{\theta}_{\alpha,n}(x,y)>\theta_\alpha(x,y)+\varepsilon) &= Prob[\widehat{H}_{XY,n}((\theta_\alpha(x,y)+\varepsilon)x,y)\le(1-\alpha)\widehat{H}_{XY,n}(\infty,y)] \\
&= Prob(\sum_{i=1}^n V_i-\sum_{i=1}^n E(V_i)\ge n\delta_1),
\end{aligned}$$

where

$$\begin{aligned}
V_i &= (1-\alpha)\,1\!\!1(Y_i\ge y)-1\!\!1(X_i\le(\theta_\alpha(x,y)+\varepsilon)x, Y_i\ge y), \\
\delta_1 &= -E(V_1)=S_Y(y)[F_{X|Y}((\theta_\alpha(x,y)+\varepsilon)x|y)-(1-\alpha)]>0.
\end{aligned}$$

Therefore, Hoeffding's Inequality (Hoeffding, 1963) yields

$$Prob(\widehat{\theta}_{\alpha,n}(x,y)>\theta_\alpha(x,y)+\varepsilon)\le e^{-2n\delta_1^2/(2-\alpha)^2}.$$

Likewise, we have

$$\begin{aligned}
Prob(\widehat{\theta}_{\alpha,n}(x,y)<\theta_\alpha(x,y)-\varepsilon) &\le Prob[\widehat{F}_{X|Y,n}((\theta_\alpha(x,y)-\varepsilon)x|y)>1-\alpha] \\
&= Prob(\sum_{i=1}^n W_i-\sum_{i=1}^n E(W_i)\ge n\delta_2),
\end{aligned}$$

where

$$
\begin{aligned}
W_i &= 1\!\!1(X_i \le (\theta_\alpha(x,y)-\varepsilon)x, Y_i \ge y) - (1-\alpha)1\!\!1(Y_i \ge y),\\
\delta_2 &= -E(W_1) = S_Y(y)[(1-\alpha) - F_{X|Y}((\theta_\alpha(x,y)-\varepsilon)x|y)] > 0.
\end{aligned}
$$

Which gives in view of Hoeffding's Inequality

$$
Prob(\widehat{\theta}_{\alpha,n}(x,y) < \theta_\alpha(x,y) - \varepsilon) \le e^{-2n\delta_2^2/(2-\alpha)^2}.
$$

This completes the proof by putting $\delta_{\alpha,\varepsilon,x,y} = \min\{\delta_1;\delta_2\}/(2-\alpha)$. $\square$

References

[1] Aragon, Y. and Daouia, A. and Thomas-Agnan, C. (2002), Nonparametric Frontier Estimation: A Conditional Quantile-based Approach, Discussion paper, GREMAQ et LSP, Université de Toulouse (http://www.univ-tlse1.fr/GREMAQ/Statistique/adt1202.pdf). Forthcoming in *Econometric Theory*

[2] Aragon, Y. and Daouia, A. and Thomas-Agnan, C. (2003), Efficiency Measurement: A Nonparametric Approach, Discussion paper, GREMAQ et LSP, Université de Toulouse (http://www.univ-tlse1.fr/GREMAQ/Statistique/mesure.pdf).

[3] Cazals, C., J.P. Florens and L. Simar (2002), Nonparametric frontier estimation: a robust approach, *Journal of Econometrics*, 106, 1-25.

[4] Charnes, A., Cooper, W.W. and E. Rhodes (1978), Measuring the inefficiency of decision making units. *European Journal of Operational Research*, 2, 429–444.

[5] Daouia, A. (2003), Asymptotic Representation Theory for Nonstandard Conditional Quantiles, Discussion paper, GREMAQ et LSP, Université de Toulouse (http://www.univ-tlse1.fr/GREMAQ/Statistique/these3.pdf). Forthcoming in *Journal of Nonparametric Statistics.*

[6] Daouia, A. and Ruiz-Gazen, A. (2003), Robust Nonparametric Frontier Estimators: Influence Function and Qualitative Robustness, Discussion paper (http://www.univ-tlse1.fr/GREMAQ/Statistique/ar0903.pdf).

[7] Daraio, C. and L. Simar (2003), Introducing environmental variables in nonparametric frontier models: a probabilistic approach, Discussion paper 0313, Institut de Statistique, Université Catholique de Louvain, Belgium, to appear in *Journal of Productivity Analysis.*

[8] Daraio, C. and L. Simar (2004), A robust nonparametric approach to evaluate and explain the performance of mutual funds, Discussion paper 0412, Institut de Statistique, Université Catholique de Louvain, Belgium.

[9] Debreu, G. (1951), The coefficient of ressource utilization,*Econometrica*, 19:3, 273-292.

[10] Deprins, D., Simar, L. and H. Tulkens (1984), Measuring labor inefficiency in post offices. In *The Performance of Public Enterprises: Concepts and measurements*. M. Marchand, P. Pestieau and H. Tulkens (eds.), Amsterdam, North-Holland, 243–267.

[11] Farrell, M.J. (1957), The measurement of productive efficiency. *Journal of the Royal Statistical Society, Series A*, 120, 253–281.

[12] Florens, J.P. and L. Simar, (2004), Parametric Approximations of Nonparametric Frontier, Discussion paper 0222, Institut de Statistique, Université Catholique de Louvain, Belgium, to appear in *Journal of Econometrics*.

[13] Ghosh, J.K. (1971), A new proof of the Bahadur representation of quantiles and an application, *The Annals of Mathematical Statistics*, 42, 1957–1961.

[14] Gijbels, I., E. Mammen, B.U. Park and L. Simar (1999), On Estimation of Monotone and Concave Frontier Functions, *Journal of the American Statistical Association*, vol 94, 445, 220–228.

[15] Hampel, F.R. (1974), The influence curve and its role in robust estimation, *Journal of the American Statistical Association*, vol 69, 383–393.

[16] Hampel, F.R., E.M. Ronchetti, P.J. Rousseeuw, and A.S. Werner (1985), *Robust Statistics: The Approach Based on Influence Functions*, New York: Wiley.

[17] Hoeffding, W. (1963), Probability inequalities for sums of bounded random variables, *Journal of the American Statistical Association*, vol 58, 13–30.

[18] Kneip, A., B.U. Park, and L. Simar (1998), A note on the convergence of nonparametric DEA estimators for production efficiency scores, *Econometric Theory*, 14, 783–793.

[19] Kneip, A, L. Simar and P.W. Wilson (2003), Asymptotics for DEA Estimators in Nonparametric Frontier Models, Discussion paper #0317, Institut de Statistique, UCL, Belgium (http://www.stat.ucl.ac.be).

[20] Koopmans, T.C.(1951), An analysis of production as an efficient combination of activities, in Koopmans, T.C. (ed) *Activity Analysis of Production and Allocation*, Cowles Commision for Research in Economics, Monograph 13, John-Wiley, New-York.

[21] Park, B. Simar, L. and Ch. Weiner (2000), The FDH Estimator for Productivity Efficiency Scores : Asymptotic Properties, *Econometric Theory*, Vol 16, 855–877.

[22] Rousseeuw, P.J. (1981), A new infinitesimal approach to robust estimation, *Z. Wahrsch. verw. Geb*, Vol 56, 127–132.

[23] Sengupta, J. K. (2000), *Dynamic and Stochastic Efficiency Analysis, Economics of Data Envelopment Analysis*, World Scientific, Singapore.

[24] Simar, L. (2003), Detecting Outliers in Frontiers Models: a Simple Approach, *Journal of Productivity Analysis*, 20, 391–424.

[25] Simar, L., and P.W. Wilson (2000), Statistical inference in nonparametric frontier models: The state of the art, *Journal of Productivity Analysis* 13, 49–78.

Individual Efficiency Scores for the 129 funds from the Aggressive-Growth US Mutual Funds

Units	N	$\hat{\theta}_n(x,y)$	$\hat{\theta}_{\alpha,n}(x,y)$	h	N_z	$\hat{\theta}_n(x,y\|z)$	$\hat{\theta}_{\alpha,n}(x,y\|z)$	$\hat{\theta}_{m,n}(x,y)$	$\hat{\theta}_{m,n}(x,y\|z)$
1	20	0.4200	0.9653	0.2593	3	0.8056	1.3245	0.6923	0.9413
2	7	0.4956	1.0075	0.1426	0	1.0000	1.0600	0.6113	1.0004
3	0	1.0000	1.0087	0.2204	0	1.0000	1.0000	1.0005	1.0000
4	4	0.6138	0.8519	0.2204	4	0.6138	1.0000	0.6314	0.8787
5	3	0.6138	0.8519	0.2204	3	0.6138	1.0000	0.6266	0.8520
6	3	0.5138	1.2881	0.0519	0	1.0000	1.0673	0.8777	1.0003
7	4	0.5407	1.1696	0.0648	0	1.0000	1.0524	0.8224	1.0018
8	6	0.5446	1.0777	0.0648	0	1.0000	1.0071	0.7112	1.0004
9	6	0.5456	1.0777	0.0648	0	1.0000	1.0018	0.7145	1.0001
10	5	0.5147	1.0894	0.0130	0	1.0000	1.0000	0.6686	1.0000
11	4	0.5129	1.1009	0.0130	0	1.0000	1.0000	0.7198	1.0000
12	7	0.5185	1.0183	0.0130	1	1.0000	1.0000	0.6170	1.0000
13	6	0.5185	1.0183	0.0130	0	1.0000	1.0000	0.6155	1.0000
14	7	0.4750	1.0896	0.0130	0	1.0000	1.0896	0.7152	1.0013
15	9	0.4396	1.0052	0.0130	0	1.0000	1.0052	0.6925	1.0007
16	17	0.4560	0.9280	0.2204	6	0.8080	1.0000	0.6231	0.8789
17	2	0.6000	1.3625	0.2204	0	1.0000	1.4375	0.8299	1.0145
18	6	0.5552	1.0900	0.2204	1	1.0000	1.0057	0.7223	1.0004
19	18	0.5095	1.0263	0.0259	5	0.7938	1.0107	0.6737	0.8288
20	34	0.3497	0.7853	0.0519	11	0.6318	0.8650	0.5334	0.6494
21	5	0.5278	1.2407	0.0259	1	1.0000	1.0741	0.8996	1.0007
22	29	0.3735	0.9900	0.0259	4	0.6304	1.0000	0.6700	0.6847
23	2	0.6333	1.4889	0.0389	0	1.0000	1.6455	1.0038	1.0388
24	6	0.7931	0.9699	0.0778	1	0.8080	1.0000	0.8088	0.8142
25	13	0.4317	1.0476	0.0130	3	0.8102	0.9583	0.6622	0.8141
26	3	0.5093	1.2456	0.0389	0	1.0000	1.0175	0.8681	1.0001
27	7	0.5107	1.1047	0.0389	2	0.8952	1.0000	0.7876	0.9330
28	5	0.4648	1.4437	0.0389	1	0.9275	1.0000	0.8013	0.9397
29	0	1.0000	1.2340	0.0130	0	1.0000	1.1170	1.0427	1.0006
30	0	1.0000	1.3400	0.0130	0	1.0000	1.0000	1.0101	1.0000
31	0	1.0000	1.3400	0.0130	0	1.0000	1.0000	1.0056	1.0000
32	2	0.6537	1.0105	0.0130	1	1.0000	1.0000	0.7565	1.0000
33	0	1.0000	1.5333	0.0130	0	1.0000	1.0000	1.0576	1.0000
34	4	0.4500	1.6613	0.0259	0	1.0000	1.7581	0.8576	1.0138
35	6	0.4855	1.0833	0.0130	1	0.9667	1.0000	0.7255	0.9671
36	7	0.4860	1.0395	0.0130	1	0.8855	1.0010	0.7144	0.9331
37	15	0.4910	0.9656	0.0130	4	0.7457	0.9572	0.6179	0.7951
38	15	0.4904	0.9724	0.0130	4	0.7447	0.9560	0.6218	0.8003
39	8	0.4896	1.0298	0.0259	1	0.8406	1.0074	0.7161	0.9240
40	3	0.8323	1.0000	0.0130	0	1.0000	1.0000	0.8449	1.0000
41	3	0.6418	1.0004	0.0130	0	1.0000	1.0000	0.7141	1.0000
42	4	0.6416	1.0000	0.0130	1	1.0000	1.0000	0.7139	1.0000
43	1	1.0000	1.4084	0.0130	0	1.0000	1.0000	1.0140	1.0000
44	6	0.4790	1.0245	0.0130	0	1.0000	1.0038	0.6746	1.0003
45	13	0.4825	0.9463	0.0130	0	1.0000	1.0073	0.5918	1.0006
46	13	0.4823	0.9460	0.0130	0	1.0000	1.0107	0.5894	1.0007
47	3	0.5700	1.1891	0.0130	0	1.0000	1.1891	0.7466	1.0052
48	5	0.5429	1.1429	0.0259	0	1.0000	1.0000	0.8070	1.0000
49	19	0.4042	0.8944	0.0389	3	0.6444	1.0000	0.5872	0.7221
50	20	0.4041	0.8944	0.0389	4	0.6444	1.0000	0.5907	0.7310
51	2	0.7125	1.4375	0.0389	0	1.0000	1.3094	0.9431	1.0149
52	5	0.6196	1.3898	0.0908	1	0.9674	1.2609	0.9513	1.0129
53	0	1.0000	1.4286	0.1426	0	1.0000	1.0000	1.1030	1.0000
54	0	1.0000	1.1837	0.0389	0	1.0000	1.0000	1.0095	1.0000
55	0	1.0000	2.8246	0.2593	0	1.0000	1.0000	1.5839	1.0030
56	0	1.0000	2.5841	0.3242	0	1.0000	1.0000	1.3897	1.0000
57	0	1.0000	2.7556	0.3371	0	1.0000	1.0000	1.4835	1.0000
58	0	1.0000	1.0000	0.0389	0	1.0000	1.0000	1.0000	1.0000
59	0	1.0000	1.0000	0.0389	0	1.0000	1.0000	1.0000	1.0000
60	1	1.0000	1.0000	0.0389	1	1.0000	1.0000	1.0000	1.0000
61	5	0.5212	1.1333	0.0259	0	1.0000	1.0571	0.7692	1.0033
62	10	0.5226	1.0571	0.0259	1	1.0000	1.0027	0.6932	1.0000
63	7	0.5226	1.0571	0.0259	0	1.0000	1.0381	0.7019	1.0005
64	1	0.5700	1.8727	0.0389	0	1.0000	1.4200	1.0591	1.0234
65	21	0.3913	1.0145	0.0778	0	1.0000	1.5942	0.6888	1.0142

Units	N	$\hat{\theta}_n(x,y)$	$\hat{\theta}_{\alpha,n}(x,y)$	h	N_z	$\hat{\theta}_n(x,y\|z)$	$\hat{\theta}_{\alpha,n}(x,y\|z)$	$\hat{\theta}_{m,n}(x,y)$	$\hat{\theta}_{m,n}(x,y\|z)$
66	1	0.8507	1.9403	0.0130	0	1.0000	1.2532	1.2603	1.0195
67	1	0.8636	1.8283	0.0130	0	1.0000	1.8182	1.2598	1.0293
68	1	0.6404	1.4533	0.0389	0	1.0000	1.2629	0.9775	1.0083
69	5	0.5098	1.1028	0.0648	1	0.8227	1.0905	0.7110	0.9696
70	5	0.6000	1.2632	0.0259	0	1.0000	1.2737	0.9246	1.0179
71	1	0.8385	1.2385	0.0778	0	1.0000	1.0000	0.9198	1.0000
72	2	0.6872	1.2188	0.0778	0	1.0000	1.0000	0.7803	1.0000
73	2	0.8327	1.6648	0.2723	2	0.8327	0.8632	0.9902	0.8337
74	8	0.4831	1.1780	0.1815	2	0.4831	1.0035	0.8772	0.6035
75	23	0.4458	1.0000	0.1815	9	0.4458	1.0000	0.6961	0.5644
76	22	0.4458	1.0076	0.1815	8	0.4458	0.9591	0.6948	0.5517
77	2	0.6129	1.4301	0.1815	1	0.6129	1.0084	0.9795	0.6808
78	3	0.6163	1.0882	0.0389	0	1.0000	1.0000	0.6963	1.0000
79	6	0.6193	1.0543	0.1297	0	1.0000	1.2679	0.6857	1.0023
80	0	1.0000	1.0000	0.3760	0	1.0000	1.0000	1.0009	1.0000
81	24	0.4191	0.9891	0.0389	7	0.6896	0.9853	0.6900	0.6993
82	54	0.3301	0.7630	0.0389	13	0.6948	0.8341	0.5468	0.6993
83	12	0.6793	0.8923	0.0389	4	0.7769	1.0000	0.7154	0.7896
84	6	0.6793	1.0000	0.0389	1	0.8783	1.0000	0.7502	0.8846
85	19	0.3335	0.6793	0.0389	3	0.5659	1.0000	0.4509	0.6596
86	19	0.3333	0.6793	0.0389	3	0.5659	1.0000	0.4484	0.6500
87	1	0.6793	1.1667	0.0389	0	1.0000	1.1222	0.8402	1.0010
88	0	1.0000	1.0000	0.0259	0	1.0000	1.0000	1.0061	1.0000
89	0	1.0000	1.0052	0.0259	0	1.0000	1.0000	1.0001	1.0000
90	22	0.4711	0.9917	0.0130	6	0.8264	0.9551	0.6990	0.8699
91	61	0.3777	0.7908	0.0130	9	0.5918	0.9411	0.5585	0.6933
92	61	0.3777	0.7908	0.0130	9	0.5918	0.9411	0.5585	0.6933
93	6	0.5279	1.1458	0.0130	1	0.9792	1.0000	0.8069	0.9856
94	1	0.6404	1.7079	0.1297	1	0.6404	1.5294	1.0990	0.9295
95	8	0.4172	1.3399	0.1297	2	0.4172	1.0014	0.8174	0.8765
96	6	0.4166	1.4748	0.1297	2	0.4166	1.0052	0.8385	0.8594
97	4	0.5182	1.2182	0.0908	1	0.9909	1.1810	0.8847	1.0050
98	15	0.3884	1.0292	0.0908	7	0.7701	0.9905	0.6541	0.7890
99	14	0.3885	1.0326	0.0908	6	0.7703	0.9905	0.6526	0.7858
100	12	0.3873	1.0571	0.0908	6	0.8125	0.9810	0.6994	0.8244
101	6	0.6991	1.0892	0.4020	2	0.9075	1.1139	0.7352	0.9467
102	2	0.7207	1.0000	0.1945	0	1.0000	1.0000	0.7356	1.0000
103	5	0.6706	1.2833	0.5446	3	0.6706	0.8546	0.8301	0.6847
104	3	0.6674	1.3459	0.5446	2	0.6674	1.0000	0.9093	0.7018
105	2	0.6136	1.9091	0.0130	0	1.0000	2.3864	1.0064	1.0386
106	2	0.5403	1.5909	0.0130	0	1.0000	1.0565	0.8998	1.0026
107	2	0.5115	1.5909	0.0130	0	1.0000	1.0102	0.8889	1.0008
108	1	0.5625	1.4771	0.0259	0	1.0000	1.0000	0.8641	1.0000
109	2	0.5625	1.3333	0.0259	0	1.0000	1.0000	0.7694	1.0000
110	5	0.5649	0.9837	0.0519	2	0.8000	1.0000	0.6556	0.9518
111	1	0.9182	1.0455	0.0130	0	1.0000	1.0000	0.9300	1.0000
112	5	0.7976	1.0168	0.0259	1	0.9587	1.0000	0.8571	0.9590
113	14	0.4464	0.8346	0.0389	0	1.0000	1.0000	0.4979	1.0000
114	1	0.8189	1.0000	0.0389	1	0.8189	1.0000	0.8228	0.8229
115	10	0.4457	0.9524	0.0259	0	1.0000	1.0000	0.5857	1.0002
116	16	0.4486	0.9038	0.0259	0	1.0000	1.0065	0.5459	1.0003
117	16	0.4482	0.8888	0.0259	0	1.0000	1.0056	0.5421	1.0002
118	0	1.0000	4.6667	0.0389	0	1.0000	4.1333	2.3707	1.3393
119	8	0.4128	1.3714	0.0130	1	1.0000	1.0008	0.7950	1.0011
120	6	0.4125	1.4714	0.0130	0	1.0000	1.5857	0.8183	1.0167
121	10	0.4142	1.2000	0.0130	2	0.9203	1.0033	0.7648	0.9532
122	0	1.0000	1.9200	0.4279	0	1.0000	1.0000	1.1304	1.0000
123	0	1.0000	1.5941	0.0389	0	1.0000	1.0000	1.0217	1.0040
124	19	0.4790	1.0084	0.1686	4	0.8487	1.0347	0.7182	0.9047
125	65	0.3461	0.7750	0.1686	13	0.7050	0.9450	0.5564	0.7593
126	64	0.3463	0.7750	0.1686	12	0.7050	0.9259	0.5557	0.7547
127	47	0.3098	0.8321	0.0130	5	0.5707	1.0000	0.5472	0.5916
128	7	0.5848	1.2105	0.1426	2	0.5848	0.9812	0.8348	0.6765
129	5	0.4453	1.3214	0.0389	2	0.9062	2.1548	0.7846	0.9345
mean	9.2	0.6083	1.2120	0.0821	1.7	0.8873	1.0990	0.8149	0.9245

Table 1: *Results from the 129 funds from the Aggressive-Growth US Mutual Funds. N is the number of observations dominating (x, y) and N_z the number of dominating points given $Z = z$. h is the selected local bandwidth (k-NN=21). Here, $m = 25$ and $\alpha = 0.80$. Last row is the average over all the 129 observations*