Université catholique de Louvain

Institut de Recherche en Mathématique et Physique

Fonds pour la Formation à la Recherche dans l'Industrie et dans l'Agriculture

Fonds de la Recherche Scientifique

Submitted in fulfillment of the requirements for the degree of Doctor of Science

by

Gabriel Kadjo

Semi-abelian aspects of cocommutative Hopf algebras

Supervisors :	Prof.	Marino Gran (UCL)
	Prof.	Joost Vercruysse (ULB, Belgium)
Examination committee :	Prof.	Pierre Bieliavsky (UCL)
	Dr.	Isar Goyvaerts (UNITO, Italy)
	Prof.	Tim Van der Linden (UCL)
	Prof.	Enrico Vitale (UCL)
Chair :	Prof.	Michel Willem (UCL)

Remerciements

Avant tout il y'a l'humain, je remercie donc le ciel d'avoir une épouse exceptionnelle, à l'écoute, et qui m'a toujours encouragée dans la réalisation ma thèse, et d'avoir des parents d'un amour inconditionnel. Je salue et remercie aussi mes nombreux amis, en particulier Christophe, Jalal, Pierre, Didier et Dominique.

C'est un magnifique chemin celui m'a mené de ma découverte des mathématiques à la finalisation de cette thèse. Je souhaite remercier ici ces professeurs rencontrés sur mon chemin: Jacques Vanderstricht qui m'a appris la valeur de l'effort en mathématiques, Jean-Pierre Verbeque qui m'a appris à voir la beauté en mathématiques, Jan Govaerts et Francis Borceux pour leurs cours inoubliables, et Luc Haine qui m'a donné un superbe cours en analyse complexe et a bien voulu m'aider dans mes démarches pour obtenir la bourse du FNRS.

Je suis heureux et honoré d'avoir les membres de mon jury et souhaite les remercier sincèrement pour avoir accepté d'examiner ma thèse.

J'ai eu la "Dream Team" avec mes promoteurs. Au delà du remerciement, j'admire mon promoteur Professeur Marino Gran, jamais je n'oublierai la clarté et la profondeur des cours qu'il m'a enseigné en algèbre multilinéaire et théorie des catégories, et par la suite ses conseils de chercheur toujours porteurs pour ma thèse. C'est naturellement, et pour un grand nombre de raisons, que j'ai rêvé de faire une thèse avec lui. C'est chose faite aujourd'hui, et à mon sens quelle belle période !

J'admire mon promoteur Professeur Joost Vercruysse qui m'a appris à partir du niveau 0, et sans pression, les algèbres de Hopf. Il a toujours été disponible, attentif, jamais précipité. J'ai trouvé nos discussions mathématiques très naturelles, avec un enchainement des résultats parfois inattendu, mais toujours magique !

Abstract

The aim of this thesis is to establish some new interactions between the theory of semi-abelian categories and the theory of Hopf algebras.

We prove that the category $\mathbf{Hopf}_{K,coc}$ of cocommutative Hopf algebras over a field K of characteristic zero is a semi-abelian category, by using the Cartier-Gabriel-Kostant-Milnor-Moore decomposition of any cocommutative Hopf algebra as a semi-direct product of a group Hopf algebra and a primitive Hopf algebra. Then, we show that $\mathbf{Hopf}_{K,coc}$ contains a torsion theory whose torsion-free and torsion parts are given, up to equivalence, by the category of groups and by the category of Lie K-algebras, respectively.

Then we prove that $\mathbf{Hopf}_{K,coc}$ is actually an action representable semi-abelian category, and we describe the split extension classifiers in it in terms of the split extension classifiers in the category of groups and in the category of Lie K-algebras. We explore then the categorical notion of centralizers and centers in $\mathbf{Hopf}_{K,coc}$. We finally show that our categorical and explicit notion of centers in $\mathbf{Hopf}_{K,coc}$ coincides with the one recently introduced and investigated in the context of arbitrary Hopf algebras.

Table of contents

Chapter 0

Introduction

The aim of this thesis is to establish some new interactions between the theory of semi-abelian categories and the theory of Hopf algebras. The starting point is a well-known result due to A. Grothendieck, as outlined in [52], saying that the category of finite-dimensional, commutative and cocommutative Hopf algebras over a field K is abelian. This result was extended by M. Takeuchi [53] to the category $\mathbf{Hopf}_{K,coc,co}$ of cocommutative and commutative Hopf algebras, not necessarily finite-dimensional. In particular, this result implies that the category $\mathbf{Hopf}_{K,coc,co}$ is protomodular [13] in the sense of D. Bourn, and exact [4] in the sense of M. Barr, which are the two properties at the heart of the theory of semi-abelian categories.

In the pointed context, *protomodularity* amounts to the condition that the Split Short Five Lemma holds, i.e. given a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "k"] \arrow[d, "\kappa"'] & A \arrow[r, shift left, "f"] \arrow[d, "\alpha"'] & B \arrow[l, shift left, "s"] \arrow[d, "\beta"] \\
0 \arrow[r] & K' \arrow[r, "k'"'] & A' \arrow[r, shift left, "f'"] & B' \arrow[l, shift left, "s'"]
\end{tikzcd}$$

where $k = ker(f)$, $k' = ker(f')$, $f \circ s = 1_B$, and $f' \circ s' = 1_{B'}$, if both κ and β are isomorphisms, then so is α.

A similar property has been studied by N. Andruskiewitsch and J. Devoto in [3] for the category of arbitrary Hopf algebras, denoted by $\mathbf{Hopf}_K$. These authors prove that α (in the above diagram) is an epimorphism and a monomorphism, thus they link the problem of proving that α is an isomorphism with the problem concerning the surjectivity of epimorphisms and the injectivity of monomorphisms in $\mathbf{Hopf}_K$, which has been later studied by A. Chirvasitu [22] and A.L. Agore [1]. In the cocommutative context, it is known that $\mathbf{Hopf}_{K,coc}$ is protomodular since it can be viewed as the category of internal groups in the category of cocommutative coalgebras (see [51], for instance).

The problem of establishing the *exactness* for the category of arbitrary Hopf algebras is delicate since limits and colimits are difficult to describe. However, an explicit description of binary products and equalizers are given in the cocommutative context [3, 26].

In this thesis, we will first examine exactness properties of $\mathbf{Hopf}_{K,coc}$ and prove that $\mathbf{Hopf}_{K,coc}$ is semi-abelian, by using the Cartier-Gabriel-Kostant-Milnor-Moore decomposition theorem, which decomposes any cocommutative Hopf algebra as a semi-direct product of a group Hopf algebra and a primitive Hopf algebra [52]. Since a category $\mathbf{C}$ is abelian if and only if both $\mathbf{C}$ and its dual $\mathbf{C}^{op}$ are semi-abelian, this observation can be seen as a "non-commutative" version of Takeuchi's theorem mentioned above. Then, we show that $\mathbf{Hopf}_{K,coc}$ contains a torsion theory whose torsion-free and torsion parts are given, up to equivalence, by the category of groups and by the category of Lie K-algebras, respectively. This part of the research is somehow influenced by the workshop "Galois Theory, Hopf Algebras, and Semi-abelian Categories" held in 2002 at the Fields Institute (Toronto, Canada). Many interesting connections have been already obtained between Galois theory and Hopf algebras, or Galois theory and semi-abelian categories, the aim of this thesis is to establish some new interactions between the theory of semi-abelian categories and the theory of cocommutative Hopf algebras. In the semi-abelian context several diagram lemmas of homological algebra hold, such as the Snake lemma, the 3-by-3-lemma, Noether's isomorphism theorems, etc (see [8]). By proving that $\mathbf{Hopf}_{K,coc}$ is semi-abelian, those theorems, and for instance a theory of internal crossed modules and a commutator theory, now become available for Hopf algebraist.

Our theorem proving that $\mathbf{Hopf}_{K,coc}$ is semi-abelian [30] has an important corollary: $\mathbf{Hopf}_{K,coc}$ is actually an *action representable* semi-abelian category as it is the case for the categories of groups and of Lie K-algebras. A category $\mathbf{C}$ is said to be an action representable category if for any object X in $\mathbf{C}$, there is a split extension

$$0 \longrightarrow X \longrightarrow \overline{X} \rightleftarrows [X] \longrightarrow 0 \qquad (0.0.1)$$

with kernel X satisfying the property that any other split extension with kernel X in $\mathbf{C}$

$$0 \longrightarrow X \longrightarrow A \rightleftarrows B \longrightarrow 0 \qquad (0.0.2)$$

can be uniquely obtained by pulling back the split extension (0.0.1) along a suitable morphism $B \to [X]$. In this case, the object $[X]$ in the split extension (0.0.1) is called a *split extension classifier* of X. In the category of groups, the split extension classifier of a group G is given by its automorphism group $Aut(G)$, and the group actions $\rho\colon G' \to Aut(G)$ of groups G' on G correspond bijectively to the isomorphism classes $\mathbf{SplExt}(G', G)$ of split extensions of G' by G. In $\mathbf{Hopf}_{K,coc}$, the categorical notion of the semi-direct product has been introduced by R.K. Molnar and the following theorem is a reformulation of his Theorem 4.1 in [41].

Theorem 0.0.1 *Every split short exact sequence in* $\mathbf{Hopf}_{K,coc}$

$$0 \longrightarrow A \longrightarrow H \rightleftarrows B \longrightarrow 0$$

is canonically isomorphic to a semi-direct product exact sequence

$$0 \longrightarrow A \longrightarrow A \rtimes B \rightleftarrows B \longrightarrow 0$$

It is well known that there are semi-abelian categories that are not action representable: for instance, this is the case for the category of rings. When a semi-abelian category $\mathbf{C}$ is action representable, commutator theory becomes simpler, since the Huq commutator of normal monomorphisms and the Smith commutator of equivalence relations always coincide in $\mathbf{C}$ [19].

We explore then the categorical notion of centralizers and centers in the action representable semi-abelian category $\mathbf{Hopf}_{K,coc}$. We describe the centralizers and the centers in $\mathbf{Hopf}_{K,coc}$ and show that our categorical approach gives an explicit description of the center recently introduced and investigated by A. Chirvasitu and P. Kasprzak in the context of arbitrary Hopf algebras [23].

Our text is organized as follows:

Chapter 1 In the first chapter we give an overview of the semi-abelian context and recall the definition of the category of cocommutative Hopf algebras. We consider some important properties and examples, and consider the connections between them by giving some categorical constructions in $\mathbf{Hopf}_{K,coc}$. Finally, we recall and give a sketch of a proof of the important decomposition due to Cartier-Gabriel-Kostant-Milnor-Moore.

Chapter 2 In the second chapter, we prove that the category $\mathbf{Hopf}_{K,coc}$, over a field K of characteristic zero, is a semi-abelian category, by using the Cartier-Gabriel-Kostant-Milnor-Moore decomposition. Then, we show that $\mathbf{Hopf}_{K,coc}$ contains a torsion theory whose torsion-free and torsion parts are given, up to equivalence, by the category of groups and by the category of Lie K-algebras, respectively. The main results of this chapter have been published in [30].

Chapter 3 In the third chapter, based on results of the second chapter, we describe the split extension classifiers in the category of cocommutative Hopf algebras over an algebraically closed field of characteristic zero, hereby providing a new characterization of semi-direct products (smash products) of cocommutative Hopf algebras. The categorical notions of centralizer and of center in the category of cocommutative Hopf algebras is then explored. In the category of cocommutative Hopf algebras the categorical notion of center coincides with the one that have been recently introduced and investigated in the context of general Hopf algebras.

Chapter 4 In the last chapter, we present some open problems and indicate some directions for future developments. The first one concerns the possibility of extending our second chapter's results to the non-associative context, based on an extended version of the Cartier-Gabriel-Kostant-Milnor-Moore decomposition, proven by V.N. Zhelyabin [55]. The second part of the chapter outlines where the assumption on the characteristic of the field K is important to prove that $\mathbf{Hopf}_{K,coc}$ is semi-abelian.

Chapter 1

The category of cocommutative Hopf algebras

1.1 Introduction to semi-abelian categories

Semi-abelian categories; introduced by G. Janelidze, L. Márki and W. Tholen [33]; are finitely complete, pointed, exact in the sense of M. Barr [4], protomodular in the sense of D. Bourn [13], with finite coproducts. These categories have been introduced to capture some typical algebraic properties valid for non-abelian algebraic structures such as groups, Lie algebras, rings, crossed modules, varieties of Ω-groups in the sense of P. Higgins [31] and compact groups. They also provide a suitable context for the study of non-abelian (co)homology. Every abelian category is in particular semi-abelian, and a category $\mathbf{C}$ is abelian when both $\mathbf{C}$ and its opposite $\mathbf{C}^{op}$ are semi-abelian. A finitely cocomplete Barr-exact category turns out to be semi-abelian precisely when it admits a zero object and when the usual Split Short Five Lemma holds.

1.1.1 Basic categorical notions and notations

In the first section of this chapter, we will recall and fix some elementary categorical notions and notations. For more details, we refer the reader to the introductory books [6, 7, 36].

Definitions 1.1.1 1. Let $\mathbf{C}$ be a category, and $|\mathbf{C}|$ be the class of objects of $\mathbf{C}$. An object $I \in |\mathbf{C}|$ (resp. $F \in |\mathbf{C}|$) is said to be an *initial object* (resp. a *terminal object*) if for any object $X \in |\mathbf{C}|$, there exists a unique morphism from I to X (resp. from X to F). A *zero object*, denoted by 0, is an object which is at the same time an initial and a terminal object. A category is said to be *pointed* when it contains a zero object, in this case a morphism $f\colon A \longrightarrow B$ is called a *zero morphism* if it factors through 0. Each set $\mathrm{Hom}_{\mathbf{C}}(A,B)$ has precisely one zero morphism, which we will denote by $0_{A,B}$.

2. Let A and B be two objects of a category $\mathbf{C}$, the *binary product* of A and B is an object $A \times B$ with two morphisms in $\mathbf{C}$

$$A \xleftarrow{p_A} A \times B \xrightarrow{p_B} B$$

with the following universal property: for any object C and a pair of morphisms $f\colon C \longrightarrow A$ and $g\colon C \longrightarrow B$, there exists a unique morphism $(f,g)\colon C \longrightarrow A \times B$ such that $p_A \circ (f,g) = f$ and $p_B \circ (f,g) = g$, i.e. which makes the following diagram commute

$$\begin{tikzcd}
 & A \times B \arrow[dl, "p_A"'] \arrow[dr, "p_B"] & \\
A & & B \\
 & C \arrow[ul, "f"] \arrow[ur, "g"'] \arrow[uu, "\exists!(f,g)"] &
\end{tikzcd}$$

3. Given a category $\mathbf{C}$, we can define a new category called the *dual (or opposite)* category of $\mathbf{C}$, denoted by $\mathbf{C}^{\mathbf{op}}$, by putting $|\mathbf{C}^{\mathbf{op}}| := |\mathbf{C}|$ and for all $X, Y \in |\mathbf{C}^{\mathbf{op}}|$, $\mathrm{Hom}_{\mathbf{C}^{\mathbf{op}}}(X,Y) := \mathrm{Hom}_{\mathbf{C}}(Y,X)$.

4. The dual notion of the one of binary product is the notion of *binary coproduct*.

Example 1.1.2 (a) In the category $\mathbf{Vect}_K$ of vector spaces over a field K and linear maps, binary products and binary coproducts are given by the direct sum with projections and inclusions, respectively.

(b) In the category $\mathbf{Grp}$ of groups and group homomorphisms, the binary product is given by the direct product of groups with the group law given component by component. The binary coproduct of two groups is usually called the "free product" (see [6], for instance).

(c) In the category $\mathbf{Lie}_K$ of Lie K-algebras over a field K and Lie homomorphisms, the binary product is given by the cartesian product with the bracket law defined component by component.

5. Let $\alpha, \beta \colon A \longrightarrow B$ be two morphisms in a category $\mathbf{C}$. We say that a morphism $u \colon C \longrightarrow A$ is an *equalizer* for α and β if $\alpha \circ u = \beta \circ u$, and the following universal property holds: for any morphism $u' \colon C' \longrightarrow A$ such that $\alpha \circ u' = \beta \circ u'$, there exists a unique morphism $\gamma \colon C' \longrightarrow C$ such that $u' = u \circ \gamma$. We will denote the "object part" of the equalizer of α and β by $Eq(\alpha, \beta)$. Dually, we can define the notion of the *coequalizer*. The coequalizer object will be denoted by $Coeq(\alpha, \beta)$.

   Assume that the category $\mathbf{C}$ has a zero object. If $\alpha \colon A \longrightarrow B$ is a morphism in $\mathbf{C}$, then the equalizer (resp. the coequalizer) of α and $0_{A,B}$ is called the *kernel* (resp. the *cokernel*) of α, and will be denoted by $ker(\alpha)$ (resp. $coker(\alpha)$).

   Example 1.1.3 (a) In $\mathbf{Vect}_K$ and $\mathbf{Lie}_K$, the kernel of a morphism $f \colon A \longrightarrow B$ is given by the usual kernel $f^{-1}(0_B) = \{a \in A$ such that $f(a) = 0_B\}$

   (b) In $\mathbf{Grp}$, the kernel of a morphism $f \colon A \longrightarrow B$ is given by $f^{-1}(1_B) = \{a \in A$ such that $f(a) = 1_B\}$

6. A category $\mathbf{C}$ is said to be *finitely (co)complete* if it has binary (co)products and (co)equalizers.

7. In finitely (co)complete categories, we always have the existence of *pullbacks* (or its dual, i.e. *pushouts*): let $f \colon A \longrightarrow C$ and $g \colon B \longrightarrow C$ be a pair of morphisms with the same codomain. The pullback of f and g over C is an object $A \times_C B$ with two morphisms

$$A \xleftarrow{\pi_A} A \times_C B \xrightarrow{\pi_B} B$$

   satisfying the following universal property: for any pair of morphisms $u \colon E \longrightarrow A$ and $v \colon E \longrightarrow B$, such that $f \circ u = g \circ v$, there exists a unique morphism $\phi \colon E \longrightarrow A \times_C B$ such that $\pi_A \circ \phi = u$ and $\pi_B \circ \phi = v$, i.e. which makes the following diagram commute

$$\begin{tikzcd}
E \arrow[dr, "\exists!\phi"] \arrow[drr, bend left, "v"] \arrow[ddr, bend right, "u"'] & & \\
& A \times_C B \arrow[r, "\pi_B"] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

Pullbacks can be constructed from equalizers and binary products in the following way: we first construct the binary product $(A \times B, p_A, p_B)$ of A and B, then we consider the equalizer of the pair of morphisms

$$f \circ p_A, g \circ p_B \colon A \times B \longrightarrow C$$

denoted by (E, i). It can easily be checked that the pullback of f and g over C is given by $(E, p_A \circ i, p_B \circ i)$.

8. The pullback of a morphism f along itself is called the *kernel pair* of f, and will be denoted by $Eq(f)$.

9. A morphism $f \colon A \longrightarrow B$ in a category is said to be an *epimorphism* (dually, resp. *monomorphism*) if for all morphisms $u, v \colon B \longrightarrow C$ (resp. $u, v \colon C \longrightarrow B$) such that $u \circ f = v \circ f$ (resp. $f \circ u = f \circ v$) one has that $u = v$.

Example 1.1.4 (a) In $\mathbf{Vect}_K$, $\mathbf{Grp}$ and $\mathbf{Lie}_K$, it can be shown that epimorphisms (resp. monomorphisms) are the same as surjective homomorphisms (resp. injective homomorphisms).

(b) In $\mathbf{Ring}$, the category of rings, epimorphisms are not necessarily surjective. A counter-example is the inclusion $\mathbb{Z} \subset \mathbb{Q}$ which is a non-surjective epimorphism.

10. A pair of morphisms $f \colon A \longrightarrow B$ and $g \colon C \longrightarrow B$ is said to be *jointly epimorphic* if for any pair of morphisms $u, v \colon B \longrightarrow X$ such that $u \circ f = v \circ f$ and $u \circ g = v \circ g$, one has that $u = v$. *Jointly monomorphic* pair of morphisms are defined dually.

11. A morphism $f\colon A \longrightarrow B$ in a category is said to be a *regular epimorphism* if f is a coequalizer, i.e. if we can find a pair of morphisms $u, v\colon C \longrightarrow A$ such that $f = Coeq(u, v)$.

Definition 1.1.5 A finitely complete category is said to be *regular* if

- every morphism can be factorized as a regular epimorphism followed by a monomorphism;
- regular epimorphisms are pullback-stable: i.e. for any regular epimorphism f in the pullback

$$\begin{tikzcd} A \times_C B \arrow[r, "\pi_B"] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

then $\pi_B\colon A \times_C B \longrightarrow B$ has to be a regular epimorphism as well.

Example 1.1.6

- The categories $\mathbf{Vect}_K$, $\mathbf{Grp}$ and $\mathbf{Lie}_K$ are regular. More generally, any variety of universal algebras is a regular category.
- The category $\mathbf{Top}$ of topological spaces with continuous maps is not regular (see [6], for instance).

In these categories, the regular epimorphism/monomorphism factorization of a morphism $f\colon A \longrightarrow B$, also called the "*regular image factorization*", is unique (up to isomorphism). This leads to the notion of the *image* of a morphism, denoted by $f(A)$, which is the subobject determined by the monomorphism $m\colon f(A) \longrightarrow B$ in the commutative diagram

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "p"'] & & B \\ & f(A) \arrow[ur, "m"'] & \end{tikzcd}$$

Definition 1.1.7 Two monomorphisms $r : B \longrightarrow A$ and $s : C \longrightarrow A$ having the same codomain are *equivalent* if there is an isomorphism $t : B \longrightarrow C$ such that $s \circ t = r$. An equivalence class of monomorphisms for this equivalence relation is called a *subobject* of A. The subobjects occuring as kernels of morphisms are called *normal subobjects*.

Example 1.1.8 The image $f(A)$ of a morphism $f\colon A \longrightarrow B$ is the subobject of B obtained by the regular epimorphism/monomorphism factorization in regular categories.

The following classes of morphisms will be of importance in our work.

Definition 1.1.9 A morphism $f\colon A \longrightarrow B$ in a category is said to be a *normal epimorphism* if f is a cokernel, i.e. if there exists $u\colon C \longrightarrow A$ such that $coker(u) = f$. Dually, f is said to be a *normal monomorphism* if f is a kernel, i.e. if there exists $u\colon B \longrightarrow C$ such that $ker(u) = f$.

Definition 1.1.10 A morphism $f\colon A \longrightarrow B$ in a category $\mathbf{C}$ is said to be a *split epimorphism* if there exists a morphism $i\colon B \longrightarrow A$ such that $f \circ i = 1_B$.

Note that any epimorphism in $\mathbf{Vect}_K$ is split. This is not the case in the category $\mathbf{Grp}$ of groups.

There is the following natural chain of implications between the described classes of epimorphisms:

Proposition 1.1.11 Let $\mathbf{C}$ be a category with binary products. We have

- split epimorphisms $\subseteq$ regular epimorphisms $\subseteq$ epimorphisms;
- normal epimorphisms $\subseteq$ regular epimorphisms.

Any isomorphism is both a monomorphism and an epimorphism. The converse is not true: in $\mathbf{Ring}$, the canonical inclusion $\mathbb{Z} \subset \mathbb{Q}$ is both a monomorphism and an epimorphism, but not an isomorphism. Any morphism which is both a monomorphism and a regular epimorphism is an isomorphism.

By the *Tierney theorem*, a category is abelian if and only if it is Barr-exact and additive (see [16], for instance). In semi-abelian categories, one replaces additivity with *protomodularity*, a concept introduced by D. Bourn in [13]. In the pointed context, protomodularity amounts to the fact that the following formulation of the

Split Short Five Lemma holds: given a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "k"] \arrow[d, "\kappa"'] & A \arrow[r, shift left, "f"] \arrow[d, "\alpha"'] & B \arrow[l, shift left, "s"] \arrow[d, "\beta"] \\
0 \arrow[r] & K' \arrow[r, "k'"'] & A' \arrow[r, shift left, "f'"] & B' \arrow[l, shift left, "s'"]
\end{tikzcd}$$

where $k = ker(f)$, $k' = ker(f')$, $f \circ s = 1_B$, and $f' \circ s' = 1_{B'}$, if both κ and β are isomorphisms, then α is an isomomorphism.

Example 1.1.12

Among the examples of protomodular categories we have $\mathbf{Grp}$, $\mathbf{Lie}_K$, the category $\mathbf{Ab}$ of abelian groups, the category $\mathbf{Loop}$ of loops (see Definition 4.1.1), and the category $\mathbf{HAlg}$ of Heyting algebras (and of Heyting semilattices, see [34]).

The category of internal groups in a finitely complete category is always protomodular (see [8]).

Definition 1.1.13 An *internal group in a category* $\mathbf{C}$, with binary products and a terminal object F, is an object G in $\mathbf{C}$ with three morphisms, the unit morphism $1\colon F \longrightarrow G$, the inversion morphism $s\colon G \longrightarrow G$ and the multiplication morphism $m\colon G \times G \longrightarrow G$ making the following diagrams commute, where $\delta = (id_G, id_G)\colon G \longrightarrow G \times G$ denotes the diagonal of the binary product $G \times G$.

$$\begin{tikzcd}
G \times G \times G \arrow[r, "1\times m"] \arrow[d, "m\times 1"'] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"'] & G
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[r, "(1{,}id_G)"] \arrow[d, "(id_G{,}1)"'] \arrow[dr, "id_G"'] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"'] & G
\end{tikzcd}$$

$$\begin{tikzcd}
G \arrow[r, "(id_G{,}s)\circ\delta"] \arrow[d, "(s{,}id_G)\circ\delta"'] \arrow[dr, "id_G"'] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"'] & G
\end{tikzcd}$$

Example 1.1.14

- The category $\mathbf{Grp}(\mathbf{Set})$ of internal groups in the category of sets is the usual category of groups.

- The category $\mathbf{Grp}(\mathbf{Top})$ of topological groups is the category of internal groups in the category of topological spaces.
- The category $\mathbf{Grp}(\mathbf{Comp})$ of compact groups is the category of internal groups in the category of compact spaces.
- The category $\mathbf{Hopf}_{K,coc}$ of cocommutative Hopf algebras can be seen as the category $\mathbf{Grp}(\mathbf{CoAlg_{K,coc}})$ of internal groups in the category of cocommutative coalgebras.

Let us recall the following property of protomodular categories.

Proposition 1.1.15 [13] In any protomodular category, a morphism f is a regular epimorphism if and ony if f is a normal epimorphism, with $f = coker\big(ker(f)\big)$.

Any protomodular category $\mathbf{C}$ is a *Mal'tsev* category [21]: this means that every *(internal) reflexive relation* $\mathbf{C}$ is an *(internal) equivalence relation*. Recall that a reflexive relation on an object X is a diagram of the form

$$\begin{tikzcd} R \arrow[r, shift left=2, "p_1"] \arrow[r, shift right=2, "p_2"'] & X \arrow[l, "\delta" description] \end{tikzcd}, \tag{1.1.1}$$

where p_1 and p_2 are jointly monomorphic, and $p_1 \circ \delta = 1_X = p_2 \circ \delta$; such a reflexive relation R is an equivalence relation when, moreover, there exist $\sigma \colon R \longrightarrow R$ and $\tau \colon R \times_X R \longrightarrow R$ as in the following diagram

$$\begin{tikzcd} R \times_X R \arrow[r, "\tau"] & R \arrow[loop above, "\sigma"] \arrow[r, shift left=2, "p_1"] \arrow[r, shift right=2, "p_2"'] & X \arrow[l, "\delta" description] \end{tikzcd}$$

such that $p_1 \circ \sigma = p_2$ and $p_2 \circ \sigma = p_1$ (symmetry), and $p_1 \circ \tau = p_1 \circ \pi_1$ and $p_2 \circ \tau = p_2 \circ \pi_2$ (transitivity), where π_1 and π_2 are the projections in the following pullback:

$$\begin{tikzcd} R \times_X R \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d, "p_1"] \\ R \arrow[r, "p_2"'] & X \end{tikzcd}$$

Definition 1.1.16 A regular category $\mathbf{C}$ is said to be *Barr-exact* if every equivalence relation in $\mathbf{C}$ is *effective*, i.e. every equivalence relation is the kernel pair of a morphism in $\mathbf{C}$.

Example 1.1.17
- The categories $\mathbf{Set}$, $\mathbf{Grp}$, $\mathbf{Lie}_K$, $\mathbf{Vect}_K$, $\mathbf{Grp(Top)}$ are exact;
- All algebraic varieties in the sense of universal algebra are exact.

Definitions 1.1.18
1. A category which is pointed, protomodular and regular is said to be *homological* [8].
2. A category with binary coproducts, pointed, exact and protomodular is said to be *semi-abelian* [33].

A category $\mathbf{C}$ is abelian if and only if both $\mathbf{C}$ and its dual $\mathbf{C}^{op}$ are semi-abelian.

We end this introduction with the following diagram indicating some implications between the different contexts recalled above.

$$\begin{tikzcd}
 & \text{semi-abelian} \arrow[dl] \arrow[dr] & \\
\text{homological} \arrow[d] \arrow[drr] & & \text{Barr-exact} \arrow[d] \\
\text{protomodular} \arrow[d] & & \text{regular} \\
\text{Mal'tsev} & &
\end{tikzcd}$$

1.1.2 Homological lemmas in the semi-abelian context

In the semi-abelian context; several diagram lemmas of homological algebra hold; such as the Snake lemma, the 3-by-3-Lemma, Noether isomorphisms theorems, etc (see [8]).

Definition 1.1.19 In a pointed category, a sequence of morphisms

$$0 \longrightarrow A \xrightarrow{k} B \xrightarrow{q} C \longrightarrow 0$$

is said to be a *short exact sequence* when $k = ker(q)$ and $q = coker(k)$.

In a pointed regular category with finite limits, a sequence of morphisms

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is an *exact sequence* in B when considering the image factorizations of f and g

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "p"'] & & B \arrow[rr, "g"] \arrow[dr, "q"'] & & C \\ & I \arrow[ur, "s"'] & & J \arrow[ur, "t"'] & \end{tikzcd}$$

the following sequence is a short exact sequence

$$0 \longrightarrow I \xrightarrow{s} B \xrightarrow{q} J \longrightarrow 0 \tag{1.1.2}$$

A long sequence of composable morphisms is exact when each pair of consecutive morphisms is exact, and the sequence 1.1.2 is said to be exact if and only if it is a short exact sequence.

Theorem 1.1.20 *Let $A \subseteq B \subseteq C$ (the inclusions are normal subobjects) be objects in a semi-abelian category $\mathbf{C}$. Then the following is an exact sequence*

$$0 \longrightarrow \frac{B}{A} \longrightarrow \frac{C}{A} \longrightarrow \frac{C}{B} \longrightarrow 0$$

Theorem 1.1.21 *[8] (Snake Lemma) Any commutative diagram, in a semi-abelian category, with exact rows as below, such that u, v and w are proper (i.e. their images are kernels)*

$$\begin{tikzcd} & 0 \arrow[d, dotted] & 0 \arrow[d, dotted] & 0 \arrow[d, dotted] & \\ 0 \arrow[r, dotted] & ker(u) \arrow[r, dotted] \arrow[d, dotted] & ker(v) \arrow[r, dotted] \arrow[d, dotted] & ker(w) \arrow[d, dotted] \arrow[dddll, dotted, "\phi"] & \\ & K' \arrow[r] \arrow[d, "u"'] & X' \arrow[r] \arrow[d, "v"'] & Y' \arrow[r] \arrow[d, "w"'] & 0 \\ 0 \arrow[r] & K \arrow[r] \arrow[d, dotted] & X \arrow[r] \arrow[d, dotted] & Y \arrow[d, dotted] & \\ & coker(u) \arrow[r, dotted] \arrow[d, dotted] & coker(v) \arrow[r, dotted] \arrow[d, dotted] & coker(w) \arrow[r, dotted] \arrow[d, dotted] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

can be completed with the dotted morphisms in such a way that all squares commute and the following is an exact sequence

$$ker(u) \longrightarrow ker(v) \longrightarrow ker(w) \xrightarrow{\phi} coker(u) \longrightarrow coker(v) \longrightarrow coker(w)$$

Theorem 1.1.22 *[8] (Five Lemma) In any semi-abelian category, consider the following commutative diagram where the rows are exact sequences and the vertical morphisms b and d are isomorphisms, a is a regular epimorphism, e is a monomorphism.*

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "a"'] & B \arrow[r] \arrow[d, "b"'] & C \arrow[r] \arrow[d, "c"] & D \arrow[r] \arrow[d, "d"] & E \arrow[d, "e"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

Then c is an isomorphism.

Theorem 1.1.23 *[15] (Nine Lemma) In any semi-abelian category, consider the following commutative diagram where the three columns and the middle row are short exact sequences. The first row is a short exact sequence if and only if the last row is a short exact sequence.*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K'' \arrow[r] \arrow[d] & X'' \arrow[r] \arrow[d] & Y'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K' \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

1.1.3 Adjunctions

Definition 1.1.24 Let $\mathbf{C}$ and $\mathbf{D}$ be two categories, $F\colon \mathbf{C} \longrightarrow \mathbf{D}$ and $G\colon \mathbf{D} \longrightarrow \mathbf{C}$ be two functors. We will say that F is a *left adjoint* to G (or G is a *right adjoint* to F) and denote this by

$$\begin{tikzcd}
\mathbf{C} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathbf{D} \arrow[l, shift left, "G"]
\end{tikzcd}$$

if and only if there exists bijections

$$\theta_{C,D}\colon Hom_{\mathbf{C}}\big(C, G(D)\big) \cong Hom_{\mathbf{D}}\big(F(C), D\big)$$

for every objects $C \in |\mathbf{C}|$, $D \in |\mathbf{D}|$ and these bijections are natural both in C and D. By natural, we mean that for every pair of morphisms $\gamma\colon C' \longrightarrow C$ and $\upsilon\colon D \longrightarrow D'$, the following diagram commutes

$$\begin{tikzcd}
Hom_{\mathbf{D}}\big(F(C), D\big) \arrow[r, "\theta_{C,D}"] \arrow[d, "\upsilon\circ - \circ F(\gamma)"'] & Hom_{\mathbf{C}}\big(C, G(D)\big) \arrow[d, "G(\upsilon)\circ - \circ\gamma"] \\
Hom_{\mathbf{D}}\big(F(C'), D'\big) \arrow[r, "\theta_{C',D'}"'] & Hom_{\mathbf{C}}\big(C', G(D')\big)
\end{tikzcd}$$

where $\upsilon \circ - \circ F(\gamma)$ is defined by

$$\big(\upsilon \circ - \circ F(\gamma)\big)(f) = \upsilon \circ f \circ F(\gamma)$$

for any $f \in Hom_{\mathbf{D}}\big(F(C), D\big)$. The definition of $G(\upsilon) \circ - \circ \gamma$ is given similarly.

An equivalent definition is given as follows.

The functor F is a *left adjoint* to the functor G if and only if for every object $C \in |\mathbf{C}|$, there exists a morphism $\eta_C : C \longrightarrow G(F(C))$ which is natural, i.e. for every morphism $\gamma\colon C' \longrightarrow C$, the following diagram commutes

$$\begin{tikzcd}
C \arrow[r, "\eta_C"] & G(F(C)) \\
C' \arrow[u, "\gamma"] \arrow[r, "\eta_{C'}"'] & G(F(C')) \arrow[u, "G\big(F(\gamma)\big)"']
\end{tikzcd}$$

and is universal: for every morphism $f\colon C \longrightarrow G(D)$, there exists a unique morphism $\overline{f}\colon F(C) \longrightarrow D$ such that the following diagram commutes

$$\begin{tikzcd}
C \arrow[r, "\eta_C"] \arrow[dr, "f"'] & G(F(C)) \arrow[d, "G(\overline{f})"] \\
 & G(D)
\end{tikzcd}$$

η is called the *unit of the adjunction*, and η_C is called the C-component of η.

Example 1.1.25
- The forgetful functor $U\colon \mathbf{Ab} \longrightarrow \mathbf{Grp}$ has a left adjoint functor given by the abelianization functor F, i.e. $\forall G \in \mathbf{Grp}$, $F(G) := G/[G, G]$, i.e. $F(G)$ is the quotient of G by its derived subgroup $[G, G]$.

- The underlying set functor $U\colon \mathbf{CURing} \longrightarrow \mathbf{Set}$, from the category of unital and commutative rings to the category of sets, has a left adjoint given by $F\colon \mathbf{Set} \longrightarrow \mathbf{CURing}$ defined for a given set X by $F(X)$ the polynomial ring $\mathbb{Z}[x_1, ..., x_n, ...]$ where the x_i's are the elements of X.

An important property of right adjoints is the following (see [36], for instance).

Proposition 1.1.26 Let $\mathbf{C}$ and $\mathbf{D}$ be finitely complete categories. If the functor $F\colon \mathbf{C} \longrightarrow \mathbf{D}$ has a left adjoint, then F preserves binary products and equalizers.

Definition 1.1.27 Let $\mathbf{C}$ and $\mathbf{D}$ be two categories. A functor $F\colon \mathbf{C} \longrightarrow \mathbf{D}$ is called an *equivalence of categories* if there exists a functor $G\colon \mathbf{D} \longrightarrow \mathbf{C}$ such that $G \circ F \cong 1_{\mathbf{C}}$ and $F \circ G \cong 1_{\mathbf{D}}$, where $1_{\mathbf{C}}$ and $1_{\mathbf{D}}$ are the identity functors on $\mathbf{C}$ and $\mathbf{D}$. If the last isomorphisms are equalities, then $\mathbf{C}$ and $\mathbf{D}$ are said to be *isomorphic categories*.

Definition 1.1.28 A functor $F\colon \mathbf{C} \longrightarrow \mathbf{D}$ induces, for every pair of objects $X, Y \in |\mathbf{C}|$, a morphism

$$F_{X,Y}\colon Hom_{\mathbf{C}}(X,Y) \longrightarrow Hom_{\mathbf{D}}\big(F(X), F(Y)\big)$$

The functor F is said to be *full* (resp. *faithful*) if $F_{X,Y}$ is surjective (resp. injective).

1.1.4 Torsion theories in the semi-abelian context

In the non-abelian context of semi-abelian categories, it is also natural to define and study a general notion of torsion theory, that extends the one introduced by S.E. Dickson in the frame of abelian categories [27]. This study was first initiated in [17], and further developed in [28, 29], also in relationship with semi-abelian homology theory.

Let us recall the definition of a torsion theory in the homological context:

Definition 1.1.29 In a homological category $\mathbf{C}$, a *torsion theory* is given by a pair $(\mathbf{T},\mathbf{F})$ of full and *replete* (i.e. isomorphism closed) subcategories of $\mathbf{C}$ such that:

i. For any object X in $\mathbf{C}$, there exists a short exact sequence, i.e. $t_X = ker(\eta_X), \eta_X = coker(t_X)$:

$$0 \longrightarrow T \xrightarrow{t_X} X \xrightarrow{\eta_X} F \longrightarrow 0$$

where 0 is the zero object in $\mathbf{C}$, $T \in \mathbf{T}$ and $F \in \mathbf{F}$.

ii. The only morphism $f\colon T \longrightarrow F$ from $T \in \mathbf{T}$ to $F \in \mathbf{F}$ is the zero morphism.

When $(\mathbf{T},\mathbf{F})$ is a torsion theory, $\mathbf{T}$ is called the *torsion* subcategory of $\mathbf{C}$, and $\mathbf{F}$ the *torsion-free* subcategory. Among the many examples in the context of homological categories, let us mention the following ones:

Example 1.1.30

1. Every torsion theory in an abelian category $\mathbf{C}$. For instance: the pair $(\mathbf{Ab}_t,\ \mathbf{Ab}_{tf})$ in the category $\mathbf{Ab}$ of abelian groups, where $\mathbf{Ab}_t$ and $\mathbf{Ab}_{tf}$ denote the full and replete subcategories of the category of abelian groups whose objects are torsion and torsion-free abelian groups, respectively.
2. The pair $(\mathbf{NilCRng},\ \mathbf{RedCRng})$ in the category $\mathbf{CRng}$ of non-unital commutative rings, where $\mathbf{NilCRng}$ and $\mathbf{RedCRng}$ denote the full subcategories of nilpotent commutative rings, and of reduced commutative rings (i.e. commutative rings with no nontrivial nilpotent elements), respectively.
3. The pair $(\mathbf{Grp}(\mathbf{Ind}),\ \mathbf{Grp}(\mathbf{Haus}))$ in the category $\mathbf{Grp}(\mathbf{Top})$ of topological groups, where $\mathbf{Grp}(\mathbf{Ind})$ and $\mathbf{Grp}(\mathbf{Haus})$ denote the full subcategories of indiscrete groups and of Hausdorff groups, respectively.

Theorem 1.1.31 *Let $(\mathbf{T},\mathbf{F})$ be a torsion theory in a homological category $\mathbf{C}$. Then the torsion-free subcategoy $\mathbf{F}$ is reflective in $\mathbf{C}$, i.e. the inclusion functor $U\colon \mathbf{F} \longrightarrow \mathbf{C}$ has a left adjoint called the reflector and denoted here by F.*

$$\begin{tikzcd} \mathbf{F} \arrow[r, shift right, "U"'] \arrow[r, phantom, "\perp"] & \mathbf{C} \arrow[l, shift right, "F"'] \end{tikzcd}$$

Any torsion-free subcategory $\mathbf{F}$ of a homological category $\mathbf{C}$ is then a normal epimorphism reflection, i.e. the reflector F is full and for every object $X \in \mathbf{C}$, one has that $\eta_X\colon X \longrightarrow U(F(X))$; the X-component of the unit of the adjunction; is a normal epimorphism. In this case, one has $U\big(F(X)\big) \cong \dfrac{X}{T(X)}$.

1.2 Introduction to cocommutative Hopf algebras

A bialgebra is an algebra on which there exists a dual structure, called a coalgebra structure, such that the two structures satisfy a compatibility relation. A Hopf algebra is a bialgebra with an endomorphism satisfying an invertibility condition which can be expressed using the algebra and the coalgebra structures. The first example of such a structure was observed by H. Hopf in 1941 [32]. Since the end of the 1980s, Hopf algebras have developped a broad range of applications in various fields of mathematics: group theory (the group algebra is a Hopf algebra), Lie theory (the universal enveloping algebra of a Lie algebra is a Hopf algebra), Galois theory [42], combinatorics [37], quantum mechanics [38], etc.

The aim of this section is to recall and fix some notations of the category of cocommutative Hopf algebras. For more details, we refer the reader to the introductory books [26, 52].

1.2.1 Algebras

Definition 1.2.1 Let A be a vector space over a field K. A *unitary and associative K-algebra* on A is a structure $(A, \star)$ where the law $\star$, called multiplication is K-bilinear, internally defined, associative and has a unit denoted by 1_A.

An equivalent definition is given by the following:

A unitary and associative K-algebra over a vector space A is a triple (A, M, u) where $M\colon A \otimes_K A \longrightarrow A$ is the multiplication, and $u\colon K \longrightarrow A$ is the unit. The maps M and u have to make the following diagrams commute:

$$\begin{tikzcd} A \otimes_K A \otimes_K A \arrow[r, "id_A \otimes_K M"] \arrow[d, "M \otimes_K id_A"'] & A \otimes_K A \arrow[d, "M"] \\ A \otimes_K A \arrow[r, "M"'] & A \end{tikzcd} \qquad \begin{tikzcd} & A \otimes_K A \arrow[d, "M"] & \\ K \otimes_K A \arrow[ur, "u \otimes_K id_A"] \arrow[r, "\approx"'] & A & A \otimes_K K \arrow[ul, "id_A \otimes_K u"'] \arrow[l, "\approx"] \end{tikzcd}$$

It is clear that the left diagram in the Definition 1.2.1 expresses associativity of the multiplication, and the right diagram expresses the existence of the unit.

The definition 1.2.1 defines the objects of the category of K-algebras, morphisms are defined by the following

Definition 1.2.2 Let us consider two K-algebras (A, M_A, u_A) and (B, M_B, u_B). A linear map $f\colon A \longrightarrow B$ from A to B is a K*-algebra morphism* if f preserves the multiplication and the unit, i.e. f makes the following diagrams commute

$$\begin{tikzcd} A \otimes_K A \arrow[r, "f \otimes_K f"] \arrow[d, "M_A"'] & B \otimes_K B \arrow[d, "M_B"] \\ A \arrow[r, "f"'] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr, "f"] & & B \\ & K \arrow[ul, "u_A"] \arrow[ur, "u_B"'] & \end{tikzcd}$$

The identity is a K-algebra morphism and the composition of two K-algebra morphisms is still a K-algebra morphism. We will denote by $\mathbf{Alg}_K$ the category of K-algebras.

Remarks 1.2.3
1. We will write $A \otimes A$ to denote $A \otimes_K A$, where the commutative field K is supposed to be fixed.

2. From now, we will say "algebra" to talk about a "unitary associative K-algebra".

3. The definition of an algebra can be given over any commutative ring.

4. We should note that the tensor product $A \otimes B$ of two algebras is still an algebra with the multiplication defined $\forall a_1, a_2 \in A, \forall b_1, b_2 \in B$ by

$$M_{A\otimes B}(a_1 \otimes b_1, a_2 \otimes b_2) := M_A(a_1 \otimes a_2) \otimes M_B(b_1 \otimes b_2)$$

and the unit by $u_{A\otimes B} := u_A \otimes u_B$.

Definition 1.2.4 An algebra (A, M, u) is said to be *commutative* if $\forall x, y \in A$, we have $M(x, y) = M(y, x)$, i.e. M makes the following diagram commute, with $\sigma\colon A \otimes A \longrightarrow A \otimes A$ defined by $\sigma(x \otimes y)\colon = y \otimes x$, $\forall x, y \in A$.

$$\begin{tikzcd} & A \otimes A \arrow[dr, "M"] & \\ A \otimes A \arrow[ur, "\sigma"] \arrow[rr, "M"'] & & A \end{tikzcd}$$

1.2.2 Coalgebras

The dualization of Definitions 1.2.1, 1.2.2 and 1.2.4 leads to the definition of the category of cocommutative coalgebras.

Definition 1.2.5 A *counitary and coassociative coalgebra* over a vector space C is a triple (C, Δ, ϵ) where $\Delta \colon C \longrightarrow C \otimes C$ is the coassociative comultiplication and $\epsilon \colon C \longrightarrow K$ is the counit: the K-linear maps Δ and ϵ have to make the dual diagrams of those of Definition 1.2.1 commute

$$
\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\Delta \otimes id"] \\
C \otimes C \arrow[r, "id \otimes \Delta"'] & C \otimes C \otimes C
\end{tikzcd}
\qquad
\begin{tikzcd}
 & C \otimes C \arrow[dl, "\epsilon \otimes id"'] \arrow[dr, "id \otimes \epsilon"] & \\
K \otimes C & C \arrow[u, "\Delta"'] \arrow[l, "\approx"] \arrow[r, "\approx"'] & C \otimes K
\end{tikzcd}
$$

Definition 1.2.6 Let $(C, \Delta_C, \epsilon_C)$ and $(D, \Delta_D, \epsilon_D)$ be two coalgebras. A linear map $f \colon C \longrightarrow D$ from C to D is a *coalgebra morphisms* if f preserves the comultiplication and the counit, i.e. f makes the dual diagrams of those of the Definition 1.2.2 commute, i.e. makes the following diagrams commute

$$
\begin{tikzcd}
C \arrow[r, "f"] \arrow[d, "\Delta_C"'] & D \arrow[d, "\Delta_D"] \\
C \otimes C \arrow[r, "f \otimes f"'] & D \otimes D
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[rr, "f"] \arrow[dr, "\epsilon_C"'] & & D \arrow[dl, "\epsilon_D"] \\
 & K &
\end{tikzcd}
$$

The category of coalgebras will be denoted by $\mathbf{CoAlg}_K$.

Definition 1.2.7 A coalgebra (C, Δ, ϵ) is said to be *cocommutative* if its comultiplication map Δ makes the dual diagram of the Definition 1.2.4 commute, i.e. makes the following diagram commute, with $\sigma \colon A \otimes A \longrightarrow A \otimes A$ defined by $\sigma(x \otimes y) \colon = y \otimes x$, $\forall x, y \in A$.

$$
\begin{tikzcd}
 & C \otimes C \arrow[dr, "\sigma"] & \\
C \arrow[ur, "\Delta"] \arrow[rr, "\Delta"'] & & C \otimes C
\end{tikzcd}
$$

Remark 1.2.8 1. The following notation for the comultiplication, usually called the "Sweedler notation", proved itself to be very effective. On a coalgebra C:

$\forall c \in C$, we will denote $\Delta(c)$ by

$$\Delta(c) = \sum_i c_{1i} \otimes c_{2i} := \sum c_1 \otimes c_2$$

where we will often omit the summation symbol, and write $\Delta(c) = c_1 \otimes c_2$. By using the notations above, the coassociativity of the comultiplication can be expressed by the equality

$$(id \otimes \Delta) \circ \Delta(c) = (id \otimes \Delta)\big(\sum c_1 \otimes c_2\big) = \sum c_1 \otimes c_{21} \otimes c_{22}$$

should be equal to $\sum c_{11} \otimes c_{12} \otimes c_2$. So we write both of the above quantities as $\sum c_1 \otimes c_2 \otimes c_3$. The counit law can be expressed by the equality

$$\sum \epsilon(c_1)c_2 = c = \sum \epsilon(c_2)c_1$$

2. Remarks 1.2.3 still hold in the context of coalgebras.

Example 1.2.9

1. Let S be a non empty set: $K[S]$ the vector space with basis S. Then $K[S]$ is a coalgebra with the comultiplication Δ and the counit ϵ defined by $\Delta(s) = s \otimes s$ and $\epsilon(s) = 1$, $\forall s \in S$, and linearly extended to all the elements in $K[S]$.

2. The field K has itself a coalgebra structure with the comultiplication $\Delta(k) = k \otimes 1$ and the counit $\epsilon(k) = k$, $\forall k \in K$. We remark that this coalgebra is a particular case of the example 1 for S a set with only one element.

3. Let C be a vector space with basis $\{c_m$ such that $m \in \mathbb{N}\}$. Then C is a coalgebra with the comultiplication Δ and the counit ϵ defined by $\Delta(c_m) = \sum_{i=0}^{m} c_i \otimes c_{m-i}$ and $\epsilon(c_m) = \delta_{0,m}$, $\forall m \in \mathbb{N}$, where δ denotes the Kronecker delta. This coalgebra is called the *divided power coalgebra*.

The notions of subalgebras and quotient of algebras are well known, let us recall the notions of subcoalgebras and coideals which are important to study quotients of coalgebras.

Definition 1.2.10 Let (C, Δ, ϵ) be a coalgebra. A vector subspace D of C is said to be a *subcoalgebra* of C if D is stable under the comultiplication:

$\Delta(D) \subseteq D \otimes D$. The subspace D will have a structure of a coalgebra by the restriction of the comultiplication and the counit from C to D.

Definition 1.2.11 Let (C, Δ, ϵ) be a coalgebra and I a vector subspace of C. We say that I is:

- a *left (resp. right) coideal* if $\Delta(I) \subseteq C \otimes I$ (resp. $\Delta(I) \subseteq I \otimes C$);
- a *coideal* if $\Delta(I) \subseteq C \otimes I + I \otimes C$ and $\epsilon(I) = 0$.

Remark 1.2.12 If I is a coideal then I is not necessarily a left and a right coideal. A left and right coideal of a coalgebra C is a subcoalgebra of C.

Important examples of subcoalgebras and coideals are the following

Proposition 1.2.13 [26] Let $f\colon C \longrightarrow D$ be a coalgebra morphism, then $Im(f)$ is a subcoalgebra of D and $ker(f)$ is a coideal of C.

Proof. Since f is a coalgebra morphism, the following diagram commutes

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\Delta_C"'] & D \arrow[d, "\Delta_D"] \\ C \otimes C \arrow[r, "f\otimes f"'] & D \otimes D \end{tikzcd}$$

Then we have:

$$\begin{aligned} \Delta_D\big(Im(f)\big) &= \Delta_D(f(C)) \\ &= (f \otimes f)\Delta_C(C) \\ &\subseteq (f \otimes f)(C \otimes C) \\ &= f(C) \otimes f(C) \\ &= Im(f) \otimes Im(f) \end{aligned}$$

thus $Im(f)$ is a subcoalgebra of D.

We also have that $(f \otimes f)\Delta_C\big(ker(f)\big) = \Delta_D\Big(f\big(ker(f)\big)\Big) = 0$, so that $\Delta_C\big(ker(f)\big) \subseteq ker(f \otimes f) = ker(f) \otimes C + C \otimes ker(f)$, thus $ker(f)$ is a coideal of C. □

The following result is well known (see [26], for instance).

Theorem 1.2.14 *Let C be a coalgebra, I a coideal of C and $p\colon C \longrightarrow C/I$ the canonical projection of vector spaces. Then there exists a unique coalgebra structure on C/I such that p is a coalgebra morphism.*

Proof. Since $p(I) = 0$, we know that

$$(p \otimes p)\Delta(I) \subseteq (p \otimes p)(I \otimes C + C \otimes I) = 0$$

Hence, by the universal property of quotient vector spaces, one has that there exists a unique linear map $\overline{\Delta}\colon C/I \longrightarrow C/I \otimes C/I$ such that the following diagram commutes

$$\begin{tikzcd} C \arrow[r, "p"] \arrow[d, "\Delta"'] & C/I \arrow[d, "\overline{\Delta}"] \\ C \otimes C \arrow[r, "p\otimes p"'] & C/I \otimes C/I \end{tikzcd}$$

For all $\overline{c} = p(c) \in C/I$, with $c \in C$, this morphism is defined by $\overline{\Delta}(\overline{c}) = \sum \overline{c_1} \otimes \overline{c_2}$.

It is clear that we have

$$(\overline{\Delta} \otimes id_{C/I})\overline{\Delta}(\overline{c}) \quad = \quad (id_{C/I} \otimes \overline{\Delta})\overline{\Delta}(\overline{c}) = \sum \overline{c_1} \otimes \overline{c_2} \otimes \overline{c_3}$$

thus $\overline{\Delta}$ is coassociative.

Since $\epsilon(I) = 0$, we find by the universal property of quotient vector spaces that there exists a unique linear map $\overline{\epsilon}\colon C/I \longrightarrow K$ such that $\overline{\epsilon} \circ p = \epsilon$. We have $\overline{\epsilon}(\overline{c}) = \epsilon(c)$, $\forall c \in C$, hence $\sum \overline{\epsilon}(\overline{c_1})\overline{c_2} = p\Big(\sum \epsilon(c_1)c_2\Big) = p(c) = \overline{c}$.

It follows that $(C/I, \overline{\Delta}, \overline{\epsilon})$ is a coalgebra and p is a coalgebra morphism. The uniqueness of the coalgebra structure on C/I for which one has that p is a coalgebra morphism follows from the uniqueness of $\overline{\Delta}$ and $\overline{\epsilon}$. $\square$

We know that isomorphisms in the category of algebras are the same as bijective algebra morphisms. This is still true in the category of coalgebras.

Proposition 1.2.15 In $\mathbf{CoAlg}_K$, isomorphisms and bijective morphisms coincide.

Proof. Let C, D be two coalgebras and $g\colon D \longrightarrow C$ a coalgebra morphism. Supposing that g is bijective, we have to show that $g^{-1}\colon C \longrightarrow D$ is a coalgebra morphism. Indeed, $\forall c \in C$, we have:

$$\begin{aligned}
\Delta\big(g^{-1}(c)\big) &= \sum g^{-1}(c)_1 \otimes g^{-1}(c)_2 \\
&= \sum g^{-1}\Big(g\big(g^{-1}(c)_1\big)\Big) \otimes g^{-1}\Big(g\big(g^{-1}(c)_2\big)\Big) \\
&= \sum g^{-1}\Big(g\big(g^{-1}(c)\big)_1\Big) \otimes g^{-1}\Big(g\big(g^{-1}(c)\big)_2\Big) \\
&= \sum g^{-1}(c_1) \otimes g^{-1}(c_2)
\end{aligned}$$

So g^{-1} preserves the comultiplication. The map g^{-1} clearly preserves the counit. □

By Proposition 1.2.15, we obtain the first isomorphism theorem for coalgebras.

Theorem 1.2.16 *[26] Let C, D be two coalgebras and $f : C \longrightarrow D$ a coalgebra morphism. Then there exists a canonical coalgebra isomorphism between $C/ker(f)$ and $Im(f)$.*

As it often happens, dual notions can behave quite differently. Coalgebras differ from algebras by a certain finiteness property they have. This can first be seen by the fact that the dual of a coalgebra is always an algebra in a functorial way, but not conversely (see [26], for instance). A second important finiteness property for coalgebras is the following result.

Theorem 1.2.17 *[26] (Fundamental theorem of coalgebras) Let C be a coalgebra over a field K. Every element $c \in C$ is contained in a finite dimensional subcoalgebra of C.*

1.2.3 Bialgebras

Definition 1.2.18 A *bialgebra* on a vector space A is a 5-tuple $(A, M, u, \Delta, \epsilon)$ where (A, M, u) is an algebra and (A, Δ, ϵ) is a coalgebra, such that these two dual structures are compatible to each other in the following sense: the maps M and u are coalgebra morphisms, i.e. M and u have to make the following diagrams commute:

$$\begin{tikzcd}
A \otimes A \arrow[rr, "M"] \arrow[d, "\Delta\otimes\Delta"'] & & A \arrow[d, "\Delta"] \\
A \otimes A \otimes A \otimes A \arrow[r, "\approx", "id\otimes\sigma\otimes id"'] & A \otimes A \otimes A \otimes A \arrow[r, "M\otimes M"'] & A \otimes A
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes A \arrow[r, "\epsilon\otimes\epsilon"] \arrow[d, "M"'] & K \otimes K \arrow[r, "u\otimes u"] \arrow[d, "\approx"] & A \otimes A \\
A \arrow[r, "\epsilon"'] & K \arrow[r, "u"'] & A \arrow[u, "\Delta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
& A \arrow[dr, "\epsilon"] & \\
K \arrow[ur, "u"] \arrow[rr, "id_K"'] & & K
\end{tikzcd}$$

Remark 1.2.19 From diagrams of the Definition 1.2.18, we see that it is equivalent to ask that the comultiplication and the counit have to be algebra morphisms.

Definition 1.2.20 A *bialgebra morphism* is a map that is at the same time an algebra morphism and a coalgebra morphism.

We will denote by $\mathbf{BiAlg}_K$ the category of bialgebras.

1.2.4 Hopf algebras

The inversion map $S\colon G \longrightarrow G$ of a group G satisfies the equality

$$S(g)g = 1 = gS(g), \qquad \forall g \in G$$

i.e. it makes the following diagram commute, where 0 is the zero object in $\mathbf{Grp}$, i.e. 0 is the one-element group.

$$\begin{tikzcd}
G \arrow[r, "\delta"] \arrow[drr, "\epsilon"'] & G \times G \arrow[rr, shift left, "S\otimes id"] \arrow[rr, shift right, "id\otimes S"'] & & G \times G \arrow[r, "M"] & G \\
& & 0 \arrow[urr, "u"'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
G \arrow[r, "\epsilon"] \arrow[d, "\delta"'] & 0 \arrow[r, "u"] & G \\
G \times G \arrow[rr, shift left, "S\times id"] \arrow[rr, shift right, "id\times S"'] & & G \times G \arrow[u, "M"']
\end{tikzcd}$$

In the cocommutative context, bialgebras and Hopf algebras can be viewed, respectively, as monoids and groups in the symmetric monoidal category of cocommutative coalgebras.

The category we study in this section is the category of Hopf algebras over a field K, denoted by $\mathbf{Hopf}_K$.

Definition 1.2.21 The objects in $\mathbf{Hopf}_K$ are *Hopf algebras*, i.e. sextuples $(H, M, u, \Delta, \epsilon, S)$ where $(H, M, u, \Delta, \epsilon)$ is a bialgebra, and S is a linear map, called the *antipode*, making the following diagram commute

$$\begin{tikzcd} H \arrow[r, "\Delta"] \arrow[drr, "\epsilon"'] & H \otimes H \arrow[rr, shift left, "S\otimes id"] \arrow[rr, shift right, "id\otimes S"'] & & H \otimes H \arrow[r, "M"] & H \\ & & K \arrow[urr, "u"'] & & \end{tikzcd}$$

Definition 1.2.22 Morphisms in $\mathbf{Hopf}_K$ are exactly bialgebra morphisms (i.e. morphisms that are both algebra and coalgebra morphisms), since bialgebra morphisms always preserve antipodes (see Proposition 1.2.29).

Definition 1.2.23 A Hopf algebra H is said to be *cocommutative* if its underlying coalgebra is cocommutative.

The category of cocommutative Hopf K-algebras will be denoted by $\mathbf{Hopf}_{K,coc}$.

The first historical examples of Hopf algebras were all commutative or cocommutative. The first example neither commutative nor cocommutative was found more than two decades after the beginning of the research on Hopf algebras. It is the 4-dimensional Sweedler Hopf algebra (see Example 1.2.24 item 4 below).

In the category $\mathbf{Hopf}_{K,coc}$ there are two full subcategories which will be of importance for our work: the category $\mathbf{GrpHopf}_K$ of *group Hopf algebras*, and the category $\mathbf{PrimHopf}_K$ of *primitive Hopf algebras*, whose definitions we now recall.

Example 1.2.24
1. The *group Hopf algebra on a group G*, denoted by $K[G]$, is the free vector space on G over the field K, i.e. $K[G] = \{\sum_{g \in G} \alpha_g g$, where $(\alpha_g)_{g\in G}$ is a family of scalars with only a finite number being non zero$\}$ and $\{g \mid g \in G\}$ is a basis of $K[G]$. The group Hopf algebra $K[G]$ can be equipped

with a structure of cocommutative Hopf algebra: the multiplication is induced by the group law, and the comultiplication $\Delta\colon K[G] \longrightarrow K[G] \otimes K[G]$, the counit $\epsilon\colon K[G] \to K$ and the antipode $S\colon K[G] \to K[G]$ are the linear maps defined on the base elements respectively by $\Delta(g) = g \otimes g$, $\epsilon(g) = 1$ and $S(g) = g^{-1}$, $\forall g \in G$.

This assignment defines a functor $K[-]\colon \mathbf{Grp} \to \mathbf{Hopf}_K$ from the category of groups to the category of Hopf algebras, which has a right adjoint $\mathcal{G}\colon \mathbf{Hopf}_K \to \mathbf{Grp}$, associating to any Hopf algebra H its group of grouplike elements

$$\mathcal{G}(H) = \{x \in H \mid \Delta(x) = x \otimes x, \epsilon(x) = 1\}$$

We also denote $\mathcal{G}(H) := G_H$ for a given Hopf algebra H. We can now consider $\mathbf{GrpHopf}_K$, which is the full subcategory of $\mathbf{Hopf}_K$ whose objects are group Hopf algebras and all Hopf algebras isomorphic to them, that is Hopf algebras generated as vector spaces by grouplike elements. The functor $K[-]\colon \mathbf{Grp} \to \mathbf{GrpHopf}_K$ is an equivalence of categories. Indeed, it is clear that $K[-] \circ \mathcal{G}$ is isomorphic to the identity functor on the category of group Hopf algebras, and let us recall why the functor $\mathcal{G} \circ K[-]$ is the identity functor on the category of groups. If G is a group and $K[G]$ the group Hopf algebra on G, one clearly has that $G \subseteq \mathcal{G}\big(K[G]\big)$ (in fact, this inclusion is the unit of the adjunction described above). Conversely, let $x = \sum\limits_{g\in G} \alpha_g g \in K[G]$ be a group-like element. We have $\Delta(x) = \sum\limits_{g\in G} \alpha_g g \otimes g$ but also $\Delta(x) = x \otimes x = \sum\limits_{g,h\in G} \alpha_g \alpha_h g \otimes h$. Since $\{g \otimes h \mid g, h \in G\}$ forms a basis of $K[G] \otimes K[G]$, $\forall g, h \in G$ we have $\alpha_g^2 = \alpha_g$, and $\alpha_g \alpha_h = 0$ if $g \neq h$. It follows that all α_g's should be zero except one that should be 1_K, thus x is in G.

Group Hopf algebras completely characterize the cocommutative Hopf algebras in the finite dimensional context, by the following result:

Proposition 1.2.25 [26] Let H be a cocommutative finite dimensional Hopf algebra over a field K of characteristic zero. Then there exists a finite group G such that $H \cong K[G]$.

2. The *tensor algebra on a vector space* V, denoted by $T(V)$. Let us denote by $T^0(V) = K$, $T^1(V) = V$, and for $n \geq 2$ by $T^n(V) = V \otimes V \otimes ... \otimes V$, the tensor product of n copies of the vector space V. $T(V)$ is given by

$T(V) = \oplus_{n\geq 0} T^n(V)$. On $T(V)$ one defines the multiplication as follows: for $x = m_1 \otimes ... \otimes m_n \in T^n(V)$, and $y = h_1 \otimes ... \otimes h_r \in T^r(V)$, we define

$$x \cdot y := m_1 \otimes ... \otimes m_n \otimes h_1 \otimes ... \otimes h_r \in T^{n+r}(V)$$

The multiplication of two arbitrary elements of $T(V)$ is obtained by extending the above formula by linearity. In this way $T(V)$ becomes an algebra with identity element $1_K \in T^0(V)$. Defining Δ and ϵ by $\Delta(v) = v \otimes 1 + 1 \otimes v$ and $\epsilon(v) = 0, \forall v \in V$, and $S\colon T(V) \longrightarrow T(V)$ by $S(v) = -v$, $\forall v \in V$, all extend linearly to all $T(V)$.

$T(V)$ is a cocommutative and non-commutative (unless $V = K$) Hopf algebra. It allows to define the *symmetric Hopf algebra* and the *universal enveloping algebra of a Lie algebra*.

3. The *universal enveloping algebra of a Lie algebra L*, denoted by $U(L)$, is defined by the quotient $U(L) = T(L)/I$, where $T(L)$ is the tensor algebra on the vector space underlying L, and I is the two-sided ideal of $T(L)$ generated by the elements of the form $x \otimes x' - x' \otimes x - [x, x']$, $\forall x, x' \in L$. Remark that the elements of L generate $U(L)$ as an algebra. The universal enveloping algebra $U(L)$ can be equipped with a structure of cocommutative (and non commutative) Hopf algebra, by taking the concatenation as multiplication, and comultiplication $\Delta\colon U(L) \longrightarrow U(L) \otimes U(L)$, counit $\epsilon\colon U(L) \to K$ and antipode $S\colon U(L) \to U(L)$ the algebra maps defined on the generators by $\Delta(x) = x \otimes 1 + 1 \otimes x$, $\epsilon(x) = 0$ and $S(x) = -x$, $\forall x \in L$.

   Recall that for any Hopf algebra H an element $x \in H$ is called a *primitive element* if $\Delta(x) = x \otimes 1 + 1 \otimes x$ (and consequently, $\epsilon(x) = 0$). The above constructions lead to a pair of adjoint functors, where the functor $U\colon \mathbf{LieAlg}_K \to \mathbf{Hopf}_K$ is a left adjoint to $\mathcal{P}\colon \mathbf{Hopf}_K \to \mathbf{LieAlg}_K$, with $\mathcal{P}(H) := \{h \in H \text{ such that } \Delta(h) = h \otimes 1 + 1 \otimes h\}$, $\forall H \in \mathbf{Hopf}_K$. We also denote $\mathcal{P}(H) := L_H$ for a given Hopf algebra H. We can now consider the category $\mathbf{PrimHopf}_K$, which is the full subcategory of $\mathbf{Hopf}_K$ whose objects are primitive Hopf algebras, that is Hopf algebras generated as algebras by their primitive elements. In the case where K is a field of characteristic 0, the category $\mathbf{PrimHopf}_K$ is known to be equivalent to the category $\mathbf{LieAlg}_K$ of Lie K-algebras [40, Theorem 5.18].

   As can be seen from the formula of comultiplication, both group Hopf algebras and primitive Hopf algebras are cocommutative. Therefore the categories

$\mathbf{GrpHopf}_K$ and $\mathbf{PrimHopf}_K$ are also full subcategories of $\mathbf{Hopf}_{K,coc}$. The functors $U, \mathcal{P}, K[-], \mathcal{G}$ and their well-known adjunctions are represented in the following diagram:

$$\begin{tikzcd} \mathbf{Grp} \arrow[r, shift left, "K[-]"] \arrow[r, phantom, "\perp"] & \mathbf{Hopf}_{K,coc} \arrow[l, shift left, "\mathcal{G}"] \arrow[r, shift right, "\mathcal{P}"'] \arrow[r, phantom, "\perp"] & \mathbf{LieAlg}_K \arrow[l, shift right, "U"'] \end{tikzcd}$$

In case K is an algebraically closed field of characteristic zero, the functor $K[-]$ is also a right adjoint to $\mathcal{G}$ (see Proposition 1.4.7). The composite functor $\mathcal{P} \circ K[-]$, resp. $\mathcal{G} \circ U$, gives the zero object since there is no non-trivial primitive elements in a group Hopf algebra, resp. there is no non-trivial group-like elements in a universal enveloping algebra.

Examples 1 and 3 are very important in our work since they link $\mathbf{Hopf}_{K,coc}$ with the category of groups and the category of Lie K-algebras, respectively. These latter are classical examples of semi-abelian categories.

4. *Sweedler's 4-dimensional Hopf algebra*[32]. Let H be the algebra given by generators and relations as follows: H is generated as an algebra by c and x satisfying the relations $c^2 = 1, x^2 = 0, xc = -cx$. Then H has dimension 4 as a vector space, with basis $\{1, c, x, cx\}$. The coalgebra structure is induced by $\Delta(c) = c \otimes c$ and $\Delta(x) = c \otimes x + x \otimes 1$, $\epsilon(c) = 1$ and $\epsilon(x) = 0$. In this way, H becomes a bialgebra which also has an antipode S defined by $S(c) = c^{-1}$ and $S(x) = -cx$.

Let us recall now the notion of convolution product of two Hopf algebra morphisms, it will let us clarify Definition 1.2.22 and the fact that $\mathbf{Hopf}_K$ is a full subcategory of $\mathbf{Bialg}_K$.

Definition 1.2.26 Let C be a coalgebra and A an algebra. The *convolution product*, denoted by $*$, on $\mathrm{Hom}_K(C, A)$ is defined by $f * g = M_A \circ (f \otimes g) \circ \Delta_C$. Explicitly, for any $f, g \colon C \longrightarrow A$, by the Sweedler notation, $\forall c \in C$

$$(f * g)(c) = \sum f(c_1) g(c_2)$$

Proposition 1.2.27 [26] Let C be a coalgebra and A an algebra, then $Hom_K(C, A)$ endowed with the convolution product and unit $u_A \circ \epsilon_C$ is an algebra.

Proof. It is clear that $\mathrm{Hom}_K(F, G)$ is a vector space. The convolution product is associative since the comultiplication is coassociative and the multiplication is associative.

The unit of $\mathrm{Hom}_K(F, G)$ is given by the composition of the unit of G and the counit of F: $\forall x \in F$, we have

$$\big(f * (u_G \circ \epsilon_F)\big)(x) = \sum f(x_1)(u_G \circ \epsilon_F)(x_2) = \sum f(x_1)\epsilon(x_2)1_G = f(x)$$

Similarly, we can show that $(u_G \circ \epsilon_F) * f = f$.

□

Remark 1.2.28

1. Let us consider $C = A$ a Hopf algebra in Proposition 1.2.27. In this case, it can be easily shown that $S * id = u \circ \epsilon = id * S$. The antipode S is the inverse of the identity morphism for the convolution product. This proves also the uniqueness of the antipode on a Hopf algebra.

2. Let F and G be two Hopf algebras, the convolution product $u * v$ of two morphisms $u, v \colon F \longrightarrow G$ is not necessarily a Hopf algebra morphism: $u * v$ is an algebra morphism if G is commutative, and a coalgebra morphism if F is cocommutative. The convolution product is the operation giving the additivity of the category of cocommutative and commutative Hopf algebras over a field.

Let us show that the inverse of a Hopf algebra morphism, for the convolution product, is obtained by its composition with the antipode.

Proposition 1.2.29 [26] Let $U \in \mathbf{CoAlg}_K(C, D)$ and $R \in \mathbf{Alg}_K(A, B)$. Defining the linear map

$$\phi\colon \mathrm{Hom}_K(D, A) \longrightarrow \mathrm{Hom}_K(C, B), \phi(f) := R \circ f \circ U$$

Then ϕ is an algebra morphism for the convolution product:

- (i) $\forall f, g \in \mathrm{Hom}_K(D, A)$, we have: $\phi(f * g) = \phi(f) * \phi(g)$;

- (ii) $\phi(u_A \circ \epsilon_D) = u_B \circ \epsilon_C$.

Proof. (i) We have

$$
\begin{aligned}
\phi(f*g) &= R\circ(f*g)\circ U \\
&= R\circ M_A\circ(f\otimes g)\circ\Delta_D\circ U \\
&= M_A\circ(R\otimes R)\circ(f\otimes g)\circ(U\otimes U)\circ\Delta_D \\
&= M_A\circ\big((R\circ f\circ U)\otimes(R\circ g\circ U)\big)\circ\Delta_D \\
&= \phi(f)*\phi(g)
\end{aligned}
$$

(ii) We have

$$
\begin{aligned}
u_B\circ\epsilon_C &= (R\circ u_A)\circ(\epsilon_D\circ U) \\
&= R\circ(u_A\circ\epsilon_D)\circ U \\
&= \phi(u_A\circ\epsilon_D)
\end{aligned}
$$

□

In the particular case where F and G are Hopf algebras in Proposition 1.2.29, and $R\in\mathbf{Hopf}_K(F,G)$, one has that the map ϕ :

$$\phi\colon \mathrm{Hom}_K(F,F)\longrightarrow \mathrm{Hom}_K(F,G)\colon f\longmapsto R\circ f$$

is an algebra morphism. By Remark 1.2.28, we know that

$$R\circ S_F=\phi(S_F)=\phi(id_F^{-1})=\big(\phi(id_F)\big)^{-1}=(R\circ id_F)^{-1}=R^{-1}$$

We have then shown that $R\circ S_F=R^{-1}$. Similarly we can show that $S_G\circ U=U^{-1}$.

If f is a bialgebra morphism, we then have that $S_G\circ f$ is the inverse of f, $f\circ S_F$ is also the inverse of f, but the inverse is unique, which proves that any bialgebra morphism preserves the antipode, i.e. $S_G\circ f=f\circ S_F$. This clarifies the Definition 1.2.22.

Corollary 1.2.30 In the category $\mathbf{Hopf}_K$, isomorphisms are the same as bijective Hopf algebra morphisms.

Proof. Let us consider $f\colon A\longrightarrow B$ a bijective Hopf algebra morphism. We know by Proposition 1.2.15 that there exists $g_c\colon B\longrightarrow A$ a coalgebra morphism such that $g_c \circ f = id_A$ and $f \circ g_c = id_B$. We also know that there exists $g_a \colon B \longrightarrow A$ an algebra morphism such that $g_a \circ f = id_A$ and $f \circ g_a = id_B$. By the associativity of the composition

$$g_a = g_a \circ id_B = g_a \circ (f \circ g_c) = (g_a \circ f) \circ g_c = id_A \circ g_c = g_c$$

Let us denote $g = g_a = g_c$. The morphism g is an algebra and a coalgebra inverse to the morphism f. Then g is a Hopf algebra inverse to the morphism f. □

Let us finally recall the notions of subobjects and quotient objects in $\mathbf{Hopf}_{K,coc}$ and give an important correspondence theorem about them, proved by K. Newman in [45].

Definition 1.2.31 Let A be a Hopf algebra,

- a *Hopf subalgebra* H of A is a subspace of A which is stable under the multiplication, the comultiplication and the antipode;
- a Hopf subalgebra H of A is said to be *normal* if we have that $a_1 h S(a_2) \in H$ and $S(a_1) h a_2 \in H$, $\forall h \in H, \forall a \in A$.

Proposition 1.2.32 [26] Let H be a bialgebra and $I \subset H$ at the same time an ideal and a coideal of H. Then the algebra structure and the coalgebra structure on the quotient H/I define a bialgebra; and the canonical projection $p : H \longrightarrow H/I$ is a bialgebra morphism.

If the bialgebra H is cocommutative (resp. commutative), then the quotient bialgebra H/I is also cocommutative (resp. commutative).

Definition 1.2.33 I is a *Hopf ideal* of a Hopf algebra H if I is an algebra ideal of H, a coalgebra coideal of H, and it is stable under the antipode.

Example 1.2.34 For every Hopf algebra morphism $u \colon F \longrightarrow G$, $ker(u)$ (i.e. the kernel of u in $\mathbf{Vect}_K$) is a Hopf ideal of F.

Proposition 1.2.35 [26] Let H be a Hopf algebra with antipode S and I a Hopf ideal of H, then one can define a Hopf algebra structure on H/I from the one on H. The canonical projection $p \colon H \longrightarrow H/I$ is a Hopf algebras morphism.

Proof. We know that H/I has a bialgebra structure (see Theorem 1.2.14). Since $S(I) \subseteq I$, the morphism $S\colon H \longrightarrow H$ induces a morphism $\overline{S}\colon H/I \longrightarrow H/I$ defined by $\overline{S}(\overline{h}) = \overline{S(h)}$, $\forall h \in H$.

We have then that $\overline{S}$ is an antipode on the vector space H/I since $\forall h \in H$, we have

$$\sum \overline{S}(\overline{h_1})\overline{h_2} = \overline{\sum S(h_1)h_2} = \epsilon(h)\overline{1} = \overline{\epsilon}(\overline{h})\overline{1}$$

Similarly, we can prove that $\sum \overline{h_1}\overline{S}(\overline{h_2}) = \overline{\epsilon}(\overline{h})\overline{1}$. □

K. Newman [45] has shown that for any cocommutative Hopf algebra over a field, there is a bijection between Hopf subalgebras and left bi-ideals (i.e. left ideals which are two-sided coideals).

Theorem 1.2.36 *Letting $H \in \mathbf{Hopf}_{K,coc}$, the correspondence $\tau\colon G \longmapsto HG^+$, from Hopf subalgebras of H to left bi-ideals of H is a bijection, where*

$$G^+ := \{g \in G | \epsilon(g) = 0\}$$

and HG^+ denotes the vector subspace of H whose elements are products of elements of H and elements of G^+. The inverse of τ, which is denoted by μ, is defined as follows: let I be a left bi-ideal in H, then H/I is a coalgebra, and

$$\mu(I) := G = HKer(\pi\colon H \longrightarrow H/I) := \{x \in H | (id \otimes \pi)\Delta(x) = x \otimes 1_A\}$$

The cocommutativity of H implies that G is a subcoalgebra of H and that it is stable under the antipode. Since I is a left ideal, it can be checked that G is closed under the multiplication, thus G is a Hopf subalgebra of H.

1.3 Categorical constructions in $\mathbf{Hopf}_{K,coc}$

The aim of this section, with some personal contributions, is to give explicit constructions of the notions introduced in Section 1.1 applied to cocommutative Hopf algebras defined in Section 1.2.

First of all, it is clear that K with the following structure is the zero object in $\mathbf{Hopf}_{K,coc}$: the multiplication is given by the multiplication on the field K, the comultiplication is given $\forall k \in K$, by $\Delta(k) = k \otimes 1$ (i.e. it is the natural isomorphism $K \cong K \otimes_K K$), the unit, the counit and the antipode on K are given by the identity on K. The zero morphism from a Hopf algebra F to a Hopf algebra G is given by $u_G \circ \epsilon_F$.

1.3.1 $\mathbf{Hopf}_{K,coc}$ is complete and cocomplete

H.E. Porst has proved ([47], Theorem 12) that the category of Hopf algebras on a unitary commutative ring is locally presentable [48], which implies in particular that it admits all limits and colimites.

Explicit constructions of arbitrary (co)products and (co)equalizers in the category of arbitrary Hopf algebras on a commutative field, can be found in the following works of A.L. Agore [1, 2].

Even if the explicit description of binary products for arbitrary Hopf algebras is difficult to deal with in general, let us recall here the construction of binary products in the cocommutative context.

1.3.1.1 Binary products in $\mathbf{Hopf}_{K,coc}$

The following lemma is a characterization of cocommutative coalgebras.

Lemma 1.3.1 The coalgebra (C, Δ, ϵ) is cocommutative if and only if the comultiplication map $\Delta_C \colon C \longrightarrow C \otimes C$ is a coalgebra morphism.

Proof. The comultiplication map $\Delta_C \colon C \longrightarrow C \otimes C$ is a coalgebra morphism if it makes the following diagrams commute

$$\begin{tikzcd} C \arrow[r, "\Delta_C"] \arrow[d, "\Delta_C"'] & C \otimes C \arrow[d, "\Delta_{C\otimes C}"] \\ C \otimes C \arrow[r, "\Delta_C \otimes \Delta_C"'] & C \otimes C \otimes C \otimes C \end{tikzcd} \qquad \begin{tikzcd} C \arrow[rr, "\Delta_C"] \arrow[dr, "\epsilon_C"'] & & C \otimes C \arrow[dl, "\epsilon_{C\otimes C}"] \\ & K & \end{tikzcd}$$

The left-hand side diagram commutes. Indeed, for any $c \in C$, one has:

$$\begin{aligned} \Delta_{C\otimes C}(\Delta_C(c)) &= \Delta_{C\otimes C}(c_1 \otimes c_2) \\ &= (c_1)_1 \otimes (c_2)_1 \otimes (c_1)_2 \otimes (c_2)_2 \\ &= c_1 \otimes c_3 \otimes c_2 \otimes c_4 \end{aligned}$$

and

$$\begin{aligned}(\Delta_C \otimes \Delta_C)\Delta_C(c) &= (\Delta_C \otimes \Delta_C)(c_1 \otimes c_2)\\ &= \Delta_C(c_1) \otimes \Delta_C(c_2)\\ &= (c_1)_1 \otimes (c_1)_2 \otimes (c_2)_1 \otimes (c_2)_2\\ &= c_1 \otimes c_2 \otimes c_3 \otimes c_4\end{aligned}$$

Since C is cocommutative, we have:

$$\begin{aligned}\sum c_1 \otimes c_2 \otimes c_3 \otimes c_4 &= \sum c_1 \otimes (c_2)_1 \otimes (c_2)_2 \otimes c_3\\ &= \sum c_1 \otimes (c_2)_2 \otimes (c_2)_1 \otimes c_3\\ &= \sum c_1 \otimes c_3 \otimes c_2 \otimes c_4\end{aligned}$$

The right-hand side diagram commutes since for any $c \in C$, one has:

$$\epsilon_{C\otimes C}\big(\Delta(c)\big) = \epsilon_{C\otimes C}(c_1 \otimes c_2) = \epsilon_C(c_1)\epsilon_C(c_2) = \epsilon_C\big(c_1\epsilon_C(c_2)\big) = \epsilon_C(c)$$

The converse direction is proved by applying the map $\epsilon \otimes id \otimes id \otimes \epsilon$ to the both sides of the equality

$$(c_1)_1 \otimes (c_2)_1 \otimes (c_1)_2 \otimes (c_2)_2 = (c_1)_1 \otimes (c_1)_2 \otimes (c_2)_1 \otimes (c_2)_2$$

which hold since $\Delta_C \colon C \longrightarrow C \otimes C$ is a coalgebra morphism. $\square$

Proposition 1.3.2 Let A and B be two cocommutative Hopf algebras. Then $(A \otimes B, p_A, p_B)$ is the product of A and B in $\mathbf{Hopf}_{K,coc}$, where $p_A = M_{A\otimes K} \circ \big(id_A \otimes \epsilon_B\big)$ and $p_B = M_{K\otimes B} \circ \big(\epsilon_A \otimes id_B\big)$. The Hopf algebra structure on $A \otimes B$ is given as follows: the multiplication $M_{A\otimes B} = (M_A \otimes M_B) \circ (id_A \otimes \sigma \otimes id_B)$, the unit $u_{A\otimes B} \colon K \cong K \otimes K \longrightarrow A \otimes B$ defined by $u_A \otimes u_B$, the comultiplication $\Delta_{A\otimes B} = (id_A \otimes \sigma \otimes id_B)(\Delta_A \otimes \Delta_B)$, the counit $\epsilon_{A\otimes B} \colon A \otimes B \longrightarrow K \otimes K \cong K$ defined by $\epsilon_A \otimes \epsilon_B$, and the antipode $S_{A\otimes B} = S_A \otimes S_B$.

Proof. It is clear that p_A and p_B are Hopf algebra morphisms. Let us check the universal property of the product: let $f \colon E \longrightarrow A$ and $g \colon E \longrightarrow B$ be two Hopf algebra morphisms. Then we must show that there exists a unique Hopf algebra morphism $\phi\colon E \longrightarrow A \otimes B$ making the following diagram commute

$$\begin{tikzcd}
& A \otimes B \arrow[dl, "p_A"'] \arrow[dr, "p_B"] & \\
A & & B \\
& E \arrow[ul, "f"] \arrow[ur, "g"'] \arrow[uu, "\phi"] &
\end{tikzcd} \tag{1.3.1}$$

We will define ϕ by $\phi := (f \otimes g) \circ \Delta_E$, i.e. $\phi(x) = \sum f(x_1) \otimes g(x_2)$, $\forall x \in E$. The map ϕ is an algebra (resp. coalgebra) morphism as a composite of algebra (resp. coalgebra, see Lemma 1.3.1) morphisms. Then ϕ is Hopf algebra morphism.

The Hopf algebra morphism ϕ makes the Diagram 1.3.1 commute. Indeed, one has that $p_A \circ \phi = f$:

$$\begin{aligned}
\pi_A\big(\phi(x)\big) &= \sum f(x_1)\epsilon_B\big(g(x_2)\big) \\
&= \sum f(x_1)\epsilon_E(x_2) \\
&= f(x)
\end{aligned}$$

One similarly shows that $p_B \circ \phi = g$.

Let us finally prove the uniqueness of ϕ. If $\phi^{'} : E \longrightarrow A \otimes B$ is another Hopf algebra morphism such that $\pi_A \circ \phi^{'} = f$ and $\pi_B \circ \phi^{'} = g$, we have

$$\begin{aligned}
\phi &= (f \otimes g) \circ \Delta_E = \big((\pi_A \circ \phi^{'}) \otimes (\pi_B \circ \phi^{'})\big) \circ \Delta_E \\
&= (\pi_A \otimes \pi_B) \circ (\phi^{'} \otimes \phi^{'}) \circ \Delta_E \\
&= (\pi_A \otimes \pi_B) \circ \Delta_{A \otimes B} \circ \phi^{'} \\
&= (id_A \otimes \epsilon_B \otimes \epsilon_A \otimes id_B) \circ (id_A \otimes \sigma \otimes id_B) \circ (\Delta_A \otimes \Delta_B) \circ \phi^{'} \\
&= (id_A \otimes \epsilon_A \otimes \epsilon_B \otimes id_B) \circ (\Delta_A \otimes \Delta_B) \circ \phi^{'} \\
&= \phi^{'}
\end{aligned}$$

$\square$

1.3.1.2 Equalizers and coequalizers in $\mathbf{Hopf}_{K,coc}$

Explicit descriptions of equalizers and coequalizers in the category of arbitrary Hopf algebra were obtained by N. Andruskiewitsch and J. Devoto in [3]. We are now going to recall them here below.

Definition 1.3.3 Let $f, g\colon A \longrightarrow B$ be two Hopf algebra morphisms. We introduce the following subsets of A:

$$LEqual(f,g) := \{a \in A | (f \otimes id_A)\Delta_A(a) = (g \otimes id_A)\Delta_A(a)\}$$

$$REqual(f,g) := \{a \in A | (id_A \otimes f)\Delta_A(a) = (id_A \otimes g)\Delta_A(a)\}$$

$HEqual(f,g) := \{a \in A$, such that

$$(id_A \otimes f \otimes id_A)(\Delta_A \otimes id)\Delta_A(a) = (id_A \otimes g \otimes id_A)(\Delta_A \otimes id_A)\Delta_A(a)\}$$

Remark 1.3.4
1. If $a \in A$ belongs to one of the three sets of the Definition 1.3.3, then: $f(a) = g(a)$.
2. It is clear that the three subsets in Definition 1.3.3 are subalgebras of A.

Lemma 1.3.5 [3] Let $f, g\colon A \longrightarrow B$ be two Hopf algebra morphisms, then:

1. $S_A\big(LEqual(f,g)\big) = REqual(f,g)$, and $S_A\big(REqual(f,g)\big) = LEqual(f,g)$;
2. $$\Delta\big(LEqual(f,g)\big) \subseteq LEqual(f,g) \otimes A$$
   and
   $$\Delta\big(REqual(f,g)\big) \subseteq A \otimes REqual(f,g)$$
3. $HEqual(f,g)$ is a Hopf subalgebra of A;
4. $\big(HEqual(f,g), i\big)$ where i is the inclusion morphism, is the equalizer of f and g in the category $\mathbf{Hopf}_K$.

Proof. The antipode S and S^{-1} are anticomultiplicative (see [26], for instance), which implies (1). (2) follows from the coassociativity of Δ. (3) is easy to verify and the condition on $HEqual$ in the Definition 1.3.3 is included to guarantee that $HEqual(f,g)$ is a subcoalgebra of A. To prove (4), let $h\colon C \longrightarrow A$ be a Hopf algebra morphism such that $f \circ h = g \circ h$; then the image of h is contained in $HEqual(f, g)$, since h is a Hopf algebra morphism. Hence, there is a unique Hopf algebra morphism $i' \colon C \longrightarrow HEqual(f, g)$ such that $i \circ i' = h$, which proves the universal property of $HEqual(f, g)$ as equalizer. □

Proposition 1.3.6 [3] Let $f, g \colon A \longrightarrow B$ be two of Hopf algebra morphism, then the following conditions are equivalent

1. $LEqual(f, g) = HEqual(f, g)$;
2. $LEqual(f, g)$ is a Hopf subalgebra of A;
3. $LEqual(f, g) = REqual(f, g)$;
4. $REqual(f, g)$ is a Hopf subalgebra of A;
5. $REqual(f, g) = HEqual(f, g)$.

Proof. (1) ⇒ (2) is clear. (2) ⇒ (3) and (3) ⇒ (4) follow from Lemma 1.3.5. (4) ⇒ (5) is a consequence of the universal property of $HEqual(f, g)$. Interchanging right with left, we obtain the proof of (5) ⇒ (1). □

Proposition 1.3.7 Let $f, g \colon A \longrightarrow B$ be two morphisms of cocommutative Hopf algebras, then $LEqual(f, g) = REqual(f, g) = HEqual(f, g)$.

Proof. Let us prove that $LEqual(f, g) \subseteq REqual(f, g)$. Let $a \in LEqual(f, g)$, we have $(f \otimes id_A)(a_1 \otimes a_2) = (g \otimes id_A)(a_1 \otimes a_2)$. Since A is cocommutative, this means that $(f \otimes id_A)(a_2 \otimes a_1) = (g \otimes id_A)(a_2 \otimes a_1)$, thus $a_1 \otimes f(a_2) = a_1 \otimes g(a_2)$ by applying the map σ. This shows that $a \in REqual(f, g)$. We prove similarly that $REqual(f, g) \subseteq LEqual(f, g)$. Then by Proposition 1.3.6 we get that $LEqual(f, g) = REqual(f, g) = HEqual(f, g)$. □

Proposition 1.3.8 Let $f \colon A \longrightarrow B$ be a cocommutative Hopf algebra morphism. By Propositions 1.3.6 and 1.3.7, in the context of cocommutative Hopf algebras, the Hopf kernel of f (kernel of f in the category of cocommutative Hopf algebras) is given by

$$HKer(f) = \{a \in A | (f \otimes id_A)\Delta_A(a) = 1_B \otimes a\}$$

Unlike equalizers, coequalizers are easy to compute in $\mathbf{Hopf}_{K,coc}$.

Proposition 1.3.9 Let $f, g\colon A \longrightarrow B$ be two Hopf algebra morphisms. Note J the subset of B given by the elements of the form $\{f(a) - g(a), a \in A\}$. We can show by direct computations that BJB has a structure of Hopf Ideal in B. Indeed, to see that BJB is a coideal of B, one can use the following equality

$$\Delta\big(f(x) - g(x)\big) = \big(f(x_1) - g(x_1)\big) \otimes f(x_2) + g(x_1) \otimes \big(f(x_2) - g(x_2)\big),$$

then $\big(HCoeq(f, g), \pi\big)$ is the coequalizer of f and g in the category of cocommutative Hopf algebras, where $HCoeq(f, g) = \dfrac{B}{BJB}$ and π is the usual projection of quotient vector spaces. If one wants to consider the cokernel of f, then J becomes $f(A)^+$. Then we have

$$HCoker(f) = \frac{B}{Bf(A)^+B}$$

Remark 1.3.10 We will denote by $HKer$ (resp. $HCoker$) kernels (resp. cokernels) computed in the category $\mathbf{Hopf}_{K,coc}$, and by ker (resp. $coker$) kernels (resp. cokernels) computed in the category $\mathbf{Vect}_K$.

1.3.1.3 Pullbacks in $\mathbf{Hopf}_{K,coc}$

By the constructions of the equalizers and the binary products (Propositions 1.3.2 and 1.3.7) in the category of cocommutative Hopf algebras, the pullback of a morphism f along a morphism g in $\mathbf{Hopf}_{K,coc}$

$$\begin{tikzcd} A \times_C B \arrow[r, "\pi_B"] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

has the following explicit description:

$$A \times_C B = \{a \otimes b \in A \otimes B | f(a_1) \otimes a_2 \otimes b = g(b_1) \otimes a \otimes b_2\}$$

and $\pi_A = M_{A\otimes K} \circ \big(id_A \otimes \epsilon_B\big)$, $\pi_B = M_{K\otimes B} \circ \big(\epsilon_A \otimes id_B\big)$.

1.3.2 Epimorphisms and monomorphisms in $\mathbf{Hopf}_{K,coc}$

Concerning epimorphisms in $\mathbf{Hopf}_{K,coc}$, I. Kaplansky proposed in 1975 a series of ten conjectures [35] concerning the theory of Hopf algebras. Among these conjectures, the first one has a crucial importance for the study of epimorphisms of Hopf algebras, it states that Hopf algebras are free modules over their Hopf subalgebras (under multiplication). It has been proved by W.D. Nichols and M.B. Zoeller in 1989 that this conjecture is true in finite dimension [46]. Counter-examples exist in infinite dimension and it is known today that a non-surjective epimorphism constitutes a counter-example to the first conjecture of I. Kaplansky. Nevertheless, we know that epimorphisms are surjective when the first conjecture of Kaplansky holds, which is true in particular for either finite dimensional or commutative or cocommutative Hopf algebras [22]. Actually, regular epimorphisms are the epimorphisms that "count" in regular categories; a clarification about them will come in Chapter 2.

Concerning monomorphisms in $\mathbf{Hopf}_{K,coc}$, we have the following:

Proposition 1.3.11 If $f\colon A \longrightarrow B$ is a cocommutative Hopf algebra morphism, then $ker(f) = A\big(Hker(f)\big)^+A$. Consequently, monomorphisms are injective in $\mathbf{Hopf}_{K,coc}$.

Proof. We have $A\big(Hker(f)\big)^+A \subseteq A$ since $\big(Hker(f)\big)^+ \subseteq A$: indeed, $\forall x \in \big(Hker(f)\big)^+$ we have $f(x) = \epsilon(x)1_B = 0$. For the converse direction, since $ker(f)$ is a Hopf ideal of A, by Newman's bijection (Theorem 1.2.36) we have

$$ker(f) = A\Big(HKer(A \longrightarrow \frac{A}{ker(f)})\Big)^+$$

We conclude by the fact that $\Big(HKer(A \longrightarrow \frac{A}{ker(f)})\Big)^+ \subseteq \big(Hker(f)\big)^+$ (proved by Theorem 1.2.14).

For f a monomorphism, we know that the kernel in $\mathbf{Hopf}_{K,coc}$ is the zero object K. It follows that $Hker(f)^+ = 0$, thus $ker(f) = 0$ and f is injective (also proved in [44, 54]). □

1.3.3 Exact sequences in $\mathbf{Hopf}_{K,coc}$

Let us recall the notion of exact sequences in the category of cocommutative Hopf algebras in the sense of N. Andruskiewitsch and J. Devoto [3] who gave the following definition in the more general context of arbitrary Hopf algebras.

Definition 1.3.12 Let $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow C$ be two composable cocommutative Hopf algebra morphisms. The following sequence

$$0 \longrightarrow A \xrightarrow{i} C \xrightarrow{\pi} B \longrightarrow 0$$

is said to be *exact* if the following conditions hold:

1. i is injective;
2. π is surjective;
3. $\pi \circ i = u_B \circ \epsilon_A$;
4. $ker(\pi) = CA^+$;
5. $A = \{x \in C$ such that $(\pi \otimes id)\Delta(x) = 1_B \otimes x\} = HKer(\pi)$.

In Chapter 2, we will obtain that these exact sequences become in the cocommutative context the categorical notion of short exact sequences (see Section 2.1.2.1).

1.4 Decomposition theorem in $\mathbf{Hopf}_{K,coc}$

The aim of this section is to recall the Cartier-Gabriel-Kostant-Milnor-Moore decomposition for cocommutative Hopf algebras, and to give a sketch of its proof.

1.4.1 Actions in $\mathbf{Hopf}_{K,coc}$

Let us first recall some definitions and results from [41].

Definition 1.4.1 Let B be a cocommutative Hopf algebra. A *B-module Hopf algebra* is a Hopf algebra A that is at the same time a left B-module, with action denoted by $\rho\colon B \otimes A \to A, \rho(b \otimes a) = b \cdot a$, such that its bialgebra maps (multiplication, unit, comultiplication and counit) are maps of B-modules. Explicitly, the linear map ρ has to satisfy the following axioms for any $b \in B$ and any $a \in A$:

(Axiom 1) $1_B \cdot a = a$;

(Axiom 2) $(bb') \cdot a = b \cdot (b' \cdot a)$;

(Axiom 3) $b \cdot (aa') = (b_1 \cdot a)(b_2 \cdot a')$;

(Axiom 4) $b \cdot 1_A = \epsilon(b)1_A$;

(Axiom 5) $(b \cdot a)_1 \otimes (b \cdot a)_2 = b_1 \cdot a_1 \otimes b_2 \cdot a_2$;

(Axiom 6) $\epsilon(b \cdot a) = \epsilon(b)\epsilon(a)$.

If A is moreover a cocommutative Hopf algebra, we then say that ρ is an *action* of B on A in $\mathbf{Hopf}_{K,coc}$.

Remark that A is a B-module Hopf algebra if and only if A is an internal Hopf algebras in the symmetric monoidal category of B-modules.

Proposition 1.4.2 [41] Let B be a cocommutative Hopf algebra and A a B-module Hopf algebra with action $\rho : B \otimes A \to A$. Then there exists a Hopf algebra $A \rtimes_\rho B$, whose underlying vector space is $A \otimes B$ and whose structure maps are given by

$$\begin{aligned}
u_{A \rtimes B} &= u_A \otimes u_B \\
M_{A \rtimes_\rho B}(a \otimes b \otimes a' \otimes b') &= a(b_1 \cdot a') \otimes b_2 b' \\
\Delta_{A \rtimes B} &= (id_A \otimes \sigma \otimes id_B)(\Delta_A \otimes \Delta_B) \\
\epsilon_{A \rtimes B} &= \epsilon_A \otimes \epsilon_B \\
S_{A \rtimes B}(a \otimes b) &= S_B(b_1) \cdot S_A(a) \otimes S_B(b_2)
\end{aligned}$$

If moreover B is cocommutative, then $A \rtimes_\rho B$ is also cocommutative.

We call $A \rtimes_\rho B$ the *semi-direct product of Hopf algebras* (also known as *smash product*) of B and A.

The following useful lemma about split short exact sequences is a reformulation of Theorem 4.1 in [41].

Lemma 1.4.3 Every split short exact sequence in $\mathbf{Hopf}_{K,coc}$

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "k"] & H \arrow[r, shift left, "p"] & B \arrow[l, shift left, "s"] \arrow[r] & 0 \end{tikzcd}$$

is canonically isomorphic to the semi-direct product exact sequence

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i_1"] & A \rtimes B \arrow[r, shift left, "p_2"] & B \arrow[l, shift left, "i_2"] \arrow[r] & 0 \end{tikzcd}$$

where $i_1 = id_A \otimes u_B$, $i_2 = u_A \otimes id_B$ and $p_2 = \epsilon_A \otimes id_B$.

Proof. The arrow $h\colon A \rtimes B \to H$ in the diagram below is given by $h(a \otimes b) = k(a)s(b)$ for all $a \otimes b \in A \rtimes B$.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i_1"] \arrow[d, equal] & A \rtimes B \arrow[r, shift left, "p_2"] \arrow[d, "h"] & B \arrow[l, shift left, "i_2"] \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r, "k"] & H \arrow[r, shift left, "p"] & B \arrow[l, shift left, "s"] \arrow[r] & 0
\end{tikzcd}$$

This is a morphism of split short exact sequences, and therefore h is an isomorphism by protomodularity of $\mathbf{Hopf}_{K,coc}$. □

Example 1.4.4 1. [41] Let N and M be groups and $\tau\colon M \longrightarrow \mathrm{Aut}(N)$ be a group homomorphism defining an action of M on N. Then the map $M \times N \longrightarrow N$ defined by $(m, n) \mapsto \tau(m)(n)$ induces a coalgebra map $K[M] \otimes K[N] \longrightarrow K[N]$ which makes $K[N]$ into a $K[M]$-module Hopf algebra. Thus $K[N] \rtimes K[M]$ is a Hopf algebra. One actually has that

$$K[N] \rtimes K[M] \cong K[N \rtimes_\tau M]$$

where $N \rtimes_\tau M$ is the usual semi-direct product of groups with the multiplication defined $\forall n_1, n_2 \in N, \forall m_1, m_2 \in M$ by

$$(n_1, m_1)(n_2, m_2) = (n_1 \tau(m_1)(n_2), m_1 m_2).$$

2. [41] Similarly, for Lie algebras L and M and $\nu\colon L \longrightarrow Der(M)$, a Lie algebra homomorphism, the action $L \times M \longrightarrow M$ defined by $(l, m) \mapsto \nu(l)(m)$ (for any $(l, m) \in L \times M$) induces a coalgebra map $U(L) \otimes U(M) \longrightarrow U(M)$ making $U(M)$ into a $U(L)$-module Hopf algebra, and as above the Hopf algebra

$$U(M) \rtimes U(L) \cong U(M \rtimes_\nu L)$$

where $M \rtimes_\nu L$ is the usual semi-direct product of Lie algebra with the Lie bracket defined $\forall m_1, m_2 \in M, \forall l_1, l_2 \in L$ by

$$\big[(m_1, l_1), (m_2, l_2)\big] = \big([m_1, m_2] + \nu(l_1)(m_2) - \nu(l_2)(m_1), [l_1, l_2]\big).$$

3. Let X be a normal Hopf subalgebra of a cocommutative Hopf algebra H. Then X has a structure of H-module Hopf algebra with action defined by

$$h \cdot x = h_1 x S(h_2),$$

for all $x \in X$ and $h \in H$.

4. Let H be any Hopf algebra. Consider $\mathcal{P}(H) = L_H$ the Lie algebra of primitive elements of H and $\mathcal{G}(H) = G_H$ the group of grouplike elements of H. Then $U(L_H)$ and $K[G_H]$ are cocommutative Hopf algebras and

$$g \cdot x = gxS(g),$$

$\forall x \in L_H$ and $\forall g \in G_H$ defines an action of $K[G_H]$ on $U(L_H)$. Furthermore, the map

$$\omega_H : U(L_H) \rtimes K[G_H] \to H, \quad \omega_H(x \otimes g) = xg \tag{1.4.1}$$

is a Hopf algebra morphism between the induced semi-direct product Hopf algebra and the original Hopf algebra H.

We use Lemma 1.4.3 to reformulate the well-known structure theorem for cocommutative Hopf algebras over an algebraically closed field of characteristic zero in terms of split exact sequences (see for instance [52], page 279 in combination with Lemma 8.0.1(c)):

Theorem 1.4.5 (Cartier-Gabriel-Kostant-Milnor-Moore) *For any cocommutative Hopf K-algebra H, over an algebraically closed field K of characteristic 0, the Hopf algebra morphism ω_H (1.4.1) is an isomorphism,*

$$U(L_H) \rtimes K[G_H] \overset{\omega_H}{\cong} H.$$

Consequently, for each H there exists a canonical split exact sequence of cocommutative Hopf algebras of the following form

$$\begin{tikzcd} 0 \arrow[r] & U(L_H) \arrow[r, "i_H"] & H \arrow[r, shift left, "p_H"] & K[G_H] \arrow[l, shift left, "s_H"] \arrow[r] & 0 \end{tikzcd}$$

where $i_H = \omega_H \circ i_1$, $s_H = \omega_H \circ i_2$ and $p_H = p_2 \circ \omega_H^{-1}$ with the notations of Lemma 1.4.3.

Remark 1.4.6 Every morphism $f\colon H_1 \longrightarrow H_2$ of cocommutative Hopf algebras gives rise to a morphism of split exact sequences of the following form

$$\begin{tikzcd} 0 \arrow[r] & U(L_{H_1}) \arrow[r, "i_{H_1}"'] \arrow[d, "f_1:=U(\mathcal{P}(f))"'] & H_1 \arrow[r, shift right, "p_{H_1}"'] \arrow[d, "f"'] & K[G_{H_1}] \arrow[l, bend right, "s_{H_1}"'] \arrow[r] \arrow[d, "f_2:=K[\mathcal{G}(f)]"] & 0 \\ 0 \arrow[r] & U(L_{H_2}) \arrow[r, "i_{H_2}"'] & H_2 \arrow[r, shift right, "p_{H_2}"'] & K[G_{H_2}] \arrow[l, bend right, "s_{H_2}"'] \arrow[r] & 0 \end{tikzcd}$$

Proposition 1.4.7 Over an algebraically closed field K of characteristic 0, the functor

$$K[-]\colon \mathbf{Grp} \to \mathbf{Hopf}_{K,coc}$$

is both a right and a left adjoint to the functor

$$\mathcal{G}\colon \mathbf{Hopf}_{K,coc} \to \mathbf{Grp}.$$

Proof. We already remarked that the functor $K[-]$ is a left adjoint to the functor $\mathcal{G}$ (even in the general case of any commutative base ring K)

$$\begin{tikzcd} \mathbf{Grp} \arrow[r, shift left, "K[-]"] \arrow[r, phantom, "\perp"] & \mathbf{Hopf}_{K,coc}. \arrow[l, shift left, "\mathcal{G}"] \end{tikzcd}$$

Indeed, if $G \in \mathbf{Grp}$, by definition all elements of G are grouplike elements in $K[G]$. Hence there is a canonical inclusion $\eta_G : G \longrightarrow \mathcal{G}\big(K[G]\big)$. One can easily check that η_G is natural and universal. In case that K is a field, it is well-known that $\eta_G = id_G$ (see Examples 1.2.24), and the functor $K[-]$ is then fully faithful.

When K is an algebraic closed field of characteristic zero, as stated, the functor $K[-]$ is actually also a right adjoint to the functor $\mathcal{G}$.

$$\begin{tikzcd} \mathbf{Grp} \arrow[r, shift left, "K[-]"] \arrow[r, phantom, "\top"] & \mathbf{Hopf}_{K,coc} \arrow[l, shift left, "\mathcal{G}"] \end{tikzcd}$$

To see this, let $H \in \mathbf{Hopf}_{K,coc}$. We define the H-component of the unit of the adjunction $\eta_H : H \longrightarrow K[\mathcal{G}(H)]$ as the morphism p_H defined in Theorem 1.4.5.

We clearly have that p_H is natural in H, and p_H is universal since for every morphism $f\colon H \longrightarrow K[G]$ in the following diagram

$$\begin{tikzcd}
0 \arrow[r] & U(L_H) \arrow[r, "i_H"] \arrow[dr] & H \arrow[r, shift left, "p_H"] \arrow[dr, "f"'] & K[G_H] \arrow[l, shift left, "s_H"] \arrow[r] \arrow[d, "\exists!\overline{f}"] & 0 \\
 & & 0 \arrow[r] & K[G] &
\end{tikzcd}$$

we have that $f \circ i_H = 0_{U(L_H),K[G]}$ since primitive elements are preserved by Hopf algebra morphisms and there is no non-trivial primitive elements in a group Hopf algebra. By the universal property of the cokernel p_H, we conclude that there exists a unique morphism $\overline{f}\colon K[G_H] \longrightarrow K[G]$ such that the above diagram commutes. Since the functor $K[-]\colon \mathbf{Grp} \to \mathbf{Hopf}_{K,coc}$ is fully faithful (as explained in the first part of the proof), the required universal property of the unit of the adjunction is satisfied. $\square$

1.4.2 Compatibility condition on semi-direct products

We now give an algebraic condition that will be useful to guarantee the existence of a Hopf algebra morphism between semi-direct products.

First remark that a semi-direct product of cocommutative Hopf algebras $A \rtimes B$, is generated as an algebra by the images of the Hopf algebra morphisms $i_1 : A \to A \rtimes B$ and $i_2 : B \to A \rtimes B$. Indeed, it is enough to observe that

$$(a \otimes b) = (a \otimes 1_B)(1_A \otimes b)$$

in $A \rtimes B$. In fact, this property might be seen as a consequence of [13, Proposition 11], which implies that given any split short exact sequence in a pointed protomodular category

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "k"] & A \arrow[r, shift left, "f"] & B \arrow[l, shift left, "s"] \arrow[r] & 0
\end{tikzcd}$$

the pair of morphisms (k, s) is jointly epimorphic.

Proposition 1.4.8 Consider the solid diagram in the category $\mathbf{Hopf}_{K,coc}$

$$\begin{tikzcd}
0 \arrow[r] & H_1 \arrow[r, "i_1"] \arrow[d, "f"'] & H_1 \rtimes H_2 \arrow[r, shift right] \arrow[d, dotted, "h"'] & H_2 \arrow[l, bend right, "i_2"'] \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & F_2 \arrow[r] & F_1 \rtimes F_2 \arrow[r, shift right] & F_2 \arrow[l, bend right] \arrow[r] & 0
\end{tikzcd}$$

where the sequences are split exact. We write $\cdot_H$ for the action of H_2 on H_1 and $\cdot_F$ for the action of F_2 on F_1. Then there exists a morphism $h\colon H_1 \rtimes H_2 \to F_1 \rtimes F_2$ making the diagram commute if and only if f and g are compatible in the following sense:

$$f(y \cdot_H x) = g(y) \cdot_F f(x) \quad \forall x \in H_1, \forall y \in H_2.$$

If these equivalent conditions hold then $h = f \otimes g$.

Proof. First remark that, by the observation just before this Proposition, if a Hopf algebra morphism h as in the diagram exists, then necessarily

$$h(a \otimes b) = h(a \otimes 1_{H_2})h(1_{H_1} \otimes b) = (f(a) \otimes 1_{F_2})(1_{F_2} \otimes g(b)) = f(a) \otimes g(b),$$

for any $a \otimes b \in H_1 \otimes H_2$. This shows that $h = f \otimes g$.

Furthermore, given any coalgebra morphisms f and g, the map $f \otimes g$ is a coalgebra morphism since the semi-direct product has the tensor product as a coalgebra structure (i.e. the categorical product in the category of cocommutative Hopf algebras).

Finally, $f \otimes g$ is an algebra morphism if and only if for all $x, x' \in H_1$, $y, y' \in H_2$, we have:

$$(f \otimes g)\big((x \otimes y)(x' \otimes y')\big) \quad = \quad \big((f \otimes g)(x \otimes y)\big)\big((f \otimes g)(x' \otimes y')\big).$$

Since f and g are Hopf algebra morphisms, we find that

$$\begin{aligned}
(f \otimes g)\big((x \otimes y)(x' \otimes y')\big) &= (f \otimes g)\big(x(y_1 \cdot_H x') \otimes y_2 y'\big) \\
&= f\big(x(y_1 \cdot_H x')\big) \otimes g(y_2 y') \\
&= f(x) f\big(y_1 \cdot_H x'\big) \otimes g(y_2) g(y')
\end{aligned}$$

and

$$\begin{aligned}\big((f\otimes g)(x\otimes y)\big)\big((f\otimes g)(x'\otimes y')\big) &= \big(f(x)\otimes g(y)\big)\big(f(x')\otimes g(y')\big)\\ &= f(x)\big(g(y)_1\cdot_F f(x')\big)\otimes g(y)_2 g(y')\\ &= f(x)\big(g(y_1)\cdot_F f(x')\big)\otimes g(y_2)g(y').\end{aligned}$$

Therefore $f\otimes g$ is an algebra morphism if and only if

$$f(x)f\big(y_1\cdot_H x'\big)\otimes g(y_2)g(y') = f(x)\big(g(y_1)\cdot_F f(x')\big)\otimes g(y_2)g(y')$$

for all $x, x'\in H_1$ and $y, y'\in H_2$ If the compatibility condition holds, then clearly this condition is satisfied. Conversely, take $x = 1_{H_1}$ and $y' = 1_{H_2}$ and apply $id\otimes\epsilon$ to the above identity to obtain the compatibility condition. □

1.4.3 Sketch of a proof to Theorem 1.4.5

Let us first recall some elementary definitions for coalgebras (for more details and a complete proof, we refer the reader to [52], Chapters 7, 8, 11, 12 and 13).

Definition 1.4.9 Let C be a non-zero coalgebra.

- C is said to be *irreducible* if any two non-zero subcoalgebras have a non-zero intersection;
- C is said to be *simple* if it has no non-zero subcoalgebras;
- C is said to be *pointed* if all simple subcoalgebras of C are 1-dimensional;
- A subcoalgebra D of C is an *irreducible component (IC)* if it is a maximal irreducible subcoalgebra. D is a *pointed irreducible component (PIC)* if it is an irreducible component whose unique simple subcoalgebra is 1-dimensional (i.e. D is pointed).

The following two lemmas will allow us to consider only pointed coalgebras.

Lemma 1.4.10 Let C be a non-zero coalgebra on a field K.

1. Any simple subcoalgebra of C is finite dimensional;

2. Any non-zero subcoalgebra of C contains a simple subcoalgebra;

3. If K is algebraically closed, any cocommutative coalgebra is pointed;

4. C is irreducible $\Leftrightarrow$ C contains a unique simple subcoalgebra R.

Proof. 1. Let $0 \neq d \in D$, with D a simple subcoalgebra. The subcoalgebra E generated by d is finite dimensional and lies in D. But D is simple, so $E = D$.

2. The result clearly holds for finite dimensional subcoalgebras and as in (1) any non-zero subcoalgebra contains a non-zero finite dimensional subcoalgebra.

3. Let R be a simple subcoalgebra of the cocommutative coalgebra C. Then R is cocommutative because C is, and R is finite dimensional by (1). Hence R^* is a commutative finite dimensional algebra. Since the ideals of R^* are in 1-1 correspondence with the subcoalgebras of R, we conclude that R^* is simple, hence a finite field extension of K. Since K is algebraically closed, $dim\ R = dim\ R^* = 1$. Thus C is pointed.

4. $\Rightarrow$ is trivial (consider the intersection of two distinct simple subcoalgebras). To prove the converse implication, let us consider R the unique simple subcoalgebra. By (2) R must lie in the intersection of any two non-zero subcoalgebras, hence C is irreducible.

$\square$

Lemma 1.4.11 Let $C = \sum C_\alpha$, with C_α subcoalgebras of C. Then

1. Any simple subcoalgebra of C lies in one of the C_α.

2. C is irreducible $\Leftrightarrow$ each C_α is irreducible and $\cap_\alpha C_\alpha \neq 0$.

3. C is pointed $\Leftrightarrow$ each C_α is pointed.

By the following theorem, we can decompose any cocommutative coalgebra in its irreducible components.

Theorem 1.4.12 *Let C be any coalgebra. Then we have the followings*

1. Any irreducible subcoalgebra E is contained in an IC.

2. *A sum of (distinct) ICs is direct.*

3. *If C is cocommutative then C is the direct sum of its ICs*

The following two lemmas will let us prove the first part of the decomposition.

Lemma 1.4.13 Let H be a Hopf algebra, $g, h \in \mathcal{G}(H)$ and let us denote by H^g the irreducible component containing g. We have that $H^g H^h \subset H^{gh}$ and H^1 is a Hopf subalgebra of H.

Lemma 1.4.14 Let H be a Hopf algebra. If the underlying coalgebra of H is a pointed cocommutative coalgebra, then $H \cong \bigoplus_{g \in \mathcal{G}(H)} H^g$ as coalgebras.

The following theorem gives a decomposition for each cocommutative Hopf algebra into a semi-direct product of its ICs and a group Hopf algebra over the group of its group-like elements.

Theorem 1.4.15 *Assume H is a cocommutative pointed Hopf algebra and $G = \mathcal{G}(H)$. Then H^1 is a $K[G]$-module algebra via $g \cdot h = ghg^{-1}$, and the multiplication*

$$H^1 \otimes K[G] \xrightarrow{M} H$$

induces a Hopf algebra isomorphism $\phi \colon K[G] \rtimes H^1 \longrightarrow H$ taking $g \otimes h$ to gh.

Proof. H^1 is a $K[G]$-module Hopf algebra. Indeed, it suffices to show that $g\cdot \colon H^1 \longrightarrow H^1 : h \longmapsto ghg^{-1}$ is a Hopf algebra morphism. (Note that $gH^1g^{-1} \subset H^g H^{g^{-1}} = H^1$ so that $g\cdot$ is well defined). Clearly $g\cdot$ is an algebra morphism and so H^1 is a $K[G]$-module algebra and $K[G] \rtimes H^1$ is well defined. We will show that ϕ is actually an algebra map. Let $h, h' \in H^1$, $g, g' \in G$. Then

$$(h \otimes g)(h' \otimes g') = h(g \cdot h') \otimes gg' = h(gh'g^{-1}) \otimes gg'$$

This maps by ϕ to $hgh'g'^{-1}gg' = (hg)(h'g') = \phi(h \otimes g)\phi(h' \otimes g')$. Also $\phi(1 \otimes 1) = 1$, so ϕ is an algebra morphism. It remains to show that ϕ is an isomorphism. This follows from the so-called fundamental theorem of Hopf modules; we omit this part of the proof. For details we refer to [52] (Theorem 4.1.1).

If $H^1 \rtimes K[G]$ is given by the tensor product coalgebra structure and the antipode induced by that of H, then it is easily verified that ϕ is a Hopf algebra morphism.

Note that if H is cocommutative and K is algebraically closed then H is pointed by Lemmas 1.4.13 and 1.4.14 applied together. □

The proof of Theorem 1.4.5 is completed by the Milnor-Moore following theorem which fully characterizes irreducible cocommutative Hopf algebras. Note that the assumption that the characteristic of the field is zero is crucial here (see Section 4.2).

Theorem 1.4.16 *(Milnor-Moore) Let H be an irreducible cocommutative Hopf algebra over a field K of characteristic 0. Then H is isomorphic as a Hopf algebra to $U\big(\mathcal{P}(H)\big)$, the universal enveloping algebra of $\mathcal{P}(H)$, which is a Lie algebra via $[x, y] = xy - yx$. Then $H^1 = U(\mathcal{P}(H))$ for any cocommutative Hopf algebra over a field of characteristic 0.*

Proof. The idea of the proof is to show that H, $U\big(\mathcal{P}(H)\big)$ and $B\big(\mathcal{P}(H)\big)$ have isomorphic coalgebra structures which will induce an isomorphism of Hopf algebras. Recall that $B(V)$, where V is a vector space, is defined by the maximal cocommutative subcoalgebra of $Sh(V)$ where $Sh(V)$ is the *Shuffle Hopf algebra* of V, defined by $Sh(V) = T(V^*)^g =$ the graded dual of $T(V^*)$. For a complete proof, see [52]. □

Chapter 2

A torsion theory in $\mathrm{Hopf}_{K,coc}$

2.1 $\mathbf{Hopf}_{K,coc}$ is semi-abelian

The starting point of this chapter on cocommutative Hopf algebras is a well-known result due to A. Grothendieck, as outlined in [52], saying that the category of finite-dimensional, commutative and cocommutative Hopf K-algebras over a field K is abelian. This result was later extended by M. Takeuchi in the following theorem [53]. An alternative proof was given by K. Newman [45] based on his bijective correspondance between left bi-ideals and normal Hopf subalgebras in the cocommutative context (Theorem 1.2.36).

Theorem 2.1.1 *The category of commutative and cocommutative Hopf K-algebras over a field K is abelian.*

The category $\mathbf{Hopf}_{K,coc}$ of cocommutative Hopf K-algebras is not additive, thus it can not be abelian (see the proof in Section 2.3). In this chapter we investigate some of its fundamental exactness properties, showing that it is a homological category, and that it is Barr-exact, leading to the conclusion that the category $\mathbf{Hopf}_{K,coc}$ is semi-abelian [33] whenever the field K is of characteristic zero (Theorem 2.1.8). This result establishes a new link between the theory of Hopf algebras and the more recent one of semi-abelian categories, and was published [30]. Both these theories can be actually viewed as wide generalizations of group theory. Since a category $\mathbf{C}$ is abelian if and only if both $\mathbf{C}$ and its dual $\mathbf{C}^{op}$ are semi-abelian, this observation can be seen as a "non-commutative" version of

Takeuchi's theorem mentioned above, from which we can actually deduce Takeuchi's theorem as observed by C. Vespa and M. Wambst [54].

In this chapter we also prove the existence of a non-abelian torsion theory $(\mathbf{T}, \mathbf{F})$ in $\mathbf{Hopf}_{K,coc}$ (Theorem 2.2.1), where the torsion subcategory $\mathbf{T}$ is the category of *primitive Hopf K-algebras*, which is equivalent to the category of Lie K-algebras, and the torsion-free subcategory $\mathbf{F}$ is the category of *group Hopf K-algebras*, which is equivalent to the category of groups.

The categories of groups and of Lie K-algebras are two typical examples of semi-abelian categories: this shows again that the theories of cocommutative Hopf algebras and of semi-abelian categories are strongly intertwined. The category $\mathbf{Hopf}_{K,coc}$ inherits some fundamental exactness properties from groups and Lie algebras thanks to the well-known canonical decomposition of Cartier-Gabriel-Kostant-Milnor-Moore (Theorem 1.4.5).

2.1.1 $\mathbf{Hopf}_{K,coc}$ is protomodular

Let us consider the following commutative diagram of short exact sequences in $\mathbf{Hopf}_{K,coc}$:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "id_A"'] & C \arrow[r, "\pi"] \arrow[d, "\theta"] & B \arrow[r] \arrow[d, "id_B"] & 0 \\ 0 \arrow[r] & A \arrow[r, "i'"'] & C' \arrow[r, "\pi'"'] & B \arrow[r] & 0 \end{tikzcd} \tag{2.1.1}$$

Thanks to the explicit descriptions of equalizers and coequalizers recalled in Section 1.3.1.2, one can easily prove that the kernel and the cokernel of θ are the zero object K. Let $c \in LKer(\theta)$. Then we have

$$(\pi \otimes id_C)\Delta(c) = \big((\pi' \circ \theta) \otimes id_C\big)\Delta(c) = 1_B \otimes c$$

thus $c \in A$. Then $(\theta \otimes id_C)\Delta_C(c) = c_1 \otimes c_2 = 1_C \otimes c$, thus $HKer(\theta) = K$.

This proves that θ is a monomorphism and one can check that θ is epimorphism of Hopf K-algebras [3]. Since monomorphisms are injections and epimorphisms are surjections in $\mathbf{Hopf}_{K,coc}$ (see Section 1.3.2), this shows that θ is a bijective Hopf algebra morphism and then an isomorphisms of Hopf algebras (see Corollary 1.2.30). So the category $\mathbf{Hopf}_{K,coc}$ is protomodular.

The category $\mathbf{Hopf}_{K,coc}$ is actually even *strongly protomodular* [14], since it has finite limits and it can be viewed as the category of internal groups in the category of cocommutative coalgebras (see [51], for instance).

In section 3.1.5, we will prove that the category of arbitrary Hopf algebras over a field K, denoted by $\mathbf{Hopf}_{K,coc}$, is protomodular.

2.1.2 $\mathbf{Hopf}_{K,coc}$ is Barr-exact

2.1.2.1 Regular epimorphism/monomorphism factorization in $\mathbf{Hopf}_{K,coc}$

Let $f\colon A \longrightarrow B$ be a cocommutative Hopf algebra morphism. By the protomodularity of $\mathbf{Hopf}_{K,coc}$, it is well known that regular epimorphisms are the same as cokernels, i.e. normal epimorphisms (see Proposition 1.1.15). Accordingly, to construct the regular epimorphism/monomorphism factorization of the morphism f, we consider the kernel $i\colon Hker(f) \to A$ of f and the cokernel $p\colon A \to HCoker(i)$ of i, both computed in the category $\mathbf{Hopf}_{K,coc}$:

$$\begin{tikzcd}
Hker(f) \arrow[r, hook, "i"] & A \arrow[r, two heads, "f"] \arrow[d, two heads, "p"'] & B \\
 & HCoker(i) \arrow[ur, tail, "m"'] &
\end{tikzcd}$$

The existence of this factorization m such that $m \circ p = f$ follows from the universal property of the cokernel p of i. It remains to prove that m is a monomorphism, which is equivalent in $\mathbf{Hopf}_{K,coc}$ to showing that m is an injection. In the category $\mathbf{Hopf}_{K,coc}$ the above factorization is obtained as in the category of vector spaces since $HCoker(i) = \dfrac{A}{A\big(Hker(f)\big)^{+}A}$, and $ker(f) = A\big(Hker(f)\big)^{+}A$ (see Section 1.3.2).

Note that any epimorphism of cocommutative Hopf K-algebras is then a normal epimorphism, and the following classes of epimorphisms coincide in $\mathbf{Hopf}_{K,coc}$:

Corollary 2.1.2 We have the following equalities in $\mathbf{Hopf}_{K,coc}$

normal epimorphisms = regular epimorphisms = epimorphisms = surjective Hopf algebra morphisms

By this corollary, we obtain that exact sequences defined in Section 1.3.3, in the sense of N. Andruskiewitsch and J. Devoto, become in the cocommutative context the categorical notion of short exact sequences (see Section 1.1.2).

We also obtain the first isomorphism theorem in $\mathbf{Hopf}_{K,coc}$:

Theorem 2.1.3 *Any morphism $f\colon A \longrightarrow B$ in $\mathbf{Hopf}_{K,coc}$ induces an isomorphism*

$$\frac{A}{A\big(HKer(f)^{+}\big)A} \cong f(A)$$

2.1.2.2 Pullback stability of regular epis in $\mathbf{Hopf}_{K,coc}$

To prove pullback stability of regular epimorphisms in the category $\mathbf{Hopf}_{K,coc}$, the approach we follow is to apply the pullback stability of regular epimorphisms in the two full subcategories $\mathbf{GrpHopf}_K$ and $\mathbf{PrimHopf}_K$ of $\mathbf{Hopf}_{K,coc}$, which are both semi-abelian, and closed under pullbacks and quotients in $\mathbf{Hopf}_{K,coc}$. From the regularity of these two categories and the decomposition provided by Theorem 1.4.5 we shall deduce the regularity of $\mathbf{Hopf}_{K,coc}$.

Lemma 2.1.4 The two full subcategories $\mathbf{GrpHopf}_K$ and $\mathbf{PrimHopf}_K$, of $\mathbf{Hopf}_{K,coc}$, are semi-abelian categories. Both these categories are closed under quotients and pullbacks in $\mathbf{Hopf}_{K,coc}$.

Proof. As recalled in Section 1.2, the two full subcategories $\mathbf{GrpHopf}_K$ and $\mathbf{PrimHopf}_K$ are equivalent to the category $\mathbf{Grp}$ of groups and to the category $\mathbf{LieAlg}_K$ of Lie K-algebras, respectively. The fact that $\mathbf{Grp}$ and $\mathbf{LieAlg}_K$ are semi-abelian is well known (see [8], for instance).

The categories $\mathbf{GrpHopf}_K$ and $\mathbf{PrimHopf}_K$ are closed under quotients since Hopf K-algebra morphisms preserve both group-like and primitive elements. These categories are closed under products in the category $\mathbf{Hopf}_{K,coc}$, as explained in [40]. To see that $\mathbf{PrimHopf}_K$ is closed under subobjects in $\mathbf{Hopf}_{K,coc}$, let us consider a monomorphism $m\colon A \rightarrowtail U(L)$ in $\mathbf{Hopf}_{K,coc}$ with codomain a primitive Hopf algebra. The morphism m, being a Hopf algebra morphism, preserves group-like elements and the group of group-like elements is trivial in the primitive Hopf algebra $U(L)$. Since m is injective, we see that the group of group-like elements of A is trivial as well. By applying Theorem 1.4.5, we conclude that A has to be a primitive Hopf algebra as well. A similar argument shows that $\mathbf{GrpHopf}_K$ is closed under subobjects in $\mathbf{Hopf}_{K,coc}$. It follows that the subcategories $\mathbf{GrpHopf}_K$ and $\mathbf{PrimHopf}_K$ are closed under pullbacks in $\mathbf{Hopf}_{K,coc}$. □

Remark 2.1.5 In the following we shall assume that K is an algebraically closed field. It can be checked that this is not a restriction: indeed, given a field K and $\phi\colon K \to \overline{K}$ an embedding of K in an algebraic closure $\overline{K}$ of K, one has the adjunction

$$\begin{tikzcd} \mathbf{Hopf}_{\overline{K},coc} \arrow[r, bend right, "R_\phi"'] \arrow[r, phantom, "\perp"] & \mathbf{Hopf}_{K,coc} \arrow[l, bend right, "L_\phi"'] \end{tikzcd}$$

where R_ϕ is the "restriction of scalars functor" and $L_\phi = - \otimes_K \overline{K}$ its left adjoint, the "extension of scalars" functor (this adjunction is a particular case of a more general adjunction between the categories of vector spaces on K and $\overline{K}$, respecitvely. See [50], for instance). Being a left adjoint, L_ϕ preserves regular epimorphisms and moreover L_ϕ reflects regular epimorphisms and preserves finite limits. Accordingly, knowing that $\mathbf{Hopf}_{\overline{K},coc}$ is regular (respectively, exact), one can deduce from this that $\mathbf{Hopf}_{K,coc}$ is regular (resp. exact) as well.

The following result concerning split short exact sequences in $\mathbf{Hopf}_{K,coc}$ will be useful in the proof of the regularity of this category:

Lemma 2.1.6 Given the following commutative diagram of split short exact sequences in $\mathbf{Hopf}_{K,coc}$:

$$\begin{tikzcd} 0 \arrow[r] & A_1 \arrow[r, "i_{H_1}"'] \arrow[d, "h_A"'] & H_1 \arrow[r, "p_{H_1}"'] \arrow[d, "h"'] & B_1 \arrow[r] \arrow[d, "h_B"] \arrow[l, bend right, "s_{H_1}"'] & 0 \\ 0 \arrow[r] & A_2 \arrow[r, "i_{H_2}"'] & H_2 \arrow[r, "p_{H_2}"'] & B_2 \arrow[r] \arrow[l, bend right, "s_{H_2}"'] & 0 \end{tikzcd}$$

We have that h is surjective if and only if both h_A and h_B are surjective.

Proof. We apply Lemma 1.4.3 to the exact sequences in the statement of the Lemma, obtaining the following commutative diagram which is canonically isomorphic to the previous one:

$$\begin{tikzcd}
0 \arrow[r] & A_1 \arrow[r, "i_1"'] \arrow[d, "h_A"'] & A_1 \rtimes B_1 \arrow[l, dotted, bend right, "\xi_1"'] \arrow[r, "p_1"'] \arrow[d, "h_A \otimes h_B"'] & B_1 \arrow[l, bend right, "s_1"'] \arrow[r] \arrow[d, "h_B"] & 0 \\
0 \arrow[r] & A_2 \arrow[r, "i_2"'] & A_2 \rtimes B_2 \arrow[l, dotted, bend right, "\xi_2"'] \arrow[r, "p_2"'] & B_2 \arrow[l, bend right, "s_2"'] \arrow[r] & 0
\end{tikzcd}$$

Hence, the morphism h is surjective if and only if $h_A \otimes h_B \colon A_1 \rtimes B_1 \to A_2 \rtimes B_2$ is surjective. If h_A and h_B are surjective, then $h_A \otimes h_B$ is surjective by considering this morphism on its underlying vector space. For the converse implication, if $h_A \otimes h_B$ is surjective, let us note that for each semi-direct product $A_i \rtimes B_i$, the underlying coalgebra is exactly the categorical product of the coalgebras A_i and B_i; we denote $\xi_i = id_{A_i} \otimes \epsilon_{B_i}$ for the coalgebra-projection of $A_i \rtimes B_i$ onto A_i (which is not a Hopf algebra morphism).

It is clear that $h_A \circ \xi_1 = \xi_2 \circ (h_A \otimes h_B)$, as coalgebra morphisms. Since ξ_2 is a split epimorphism and $h_A \otimes h_B$ is surjective, we conclude that h_A is surjective. It is clear that h_B is surjective whenever h is.

□

Theorem 2.1.7 *Consider the following pullback (P, π_A, π_B) in the category $\mathbf{Hopf}_{K,coc}$:*

$$\begin{tikzcd}
P \arrow[r, "\pi_B"] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\
A \arrow[r, "f"'] & C
\end{tikzcd} \tag{2.1.2}$$

if f is a regular epimorphism then π_B is a regular epimorphism.

Proof. Thanks to Lemma 2.1.4, regular epimorphisms are pullback stable whenever the Hopf algebras A, B and C in diagram (2.1.2) belong to $\mathbf{GrpHopf}_K$, or to $\mathbf{PrimHopf}_K$, respectively.

Let us now consider cocommutative Hopf K-algebras A, B and C over a field K of characteristic zero. By Theorem 1.4.5, we have: $A \cong U(L_A) \rtimes K[G_A]$, $B \cong U(L_B) \rtimes K[G_B]$ and $C \cong U(L_C) \rtimes K[G_C]$.

Thanks to Remark 1.4.6, we can build the following commutative diagram.

$$\begin{tikzcd}
P_1 \arrow[rrrr, "\pi_{B_1}"] \arrow[dr, "i_P"'] \arrow[ddd, "\pi_{A_1}"'] & & & & U(L_B) \arrow[dr, "i_B"] \arrow[ddd, "g_1"] & & \\
& P \arrow[dr, "p_P"'] \arrow[ddd, "\pi_A"'] & & & & B \arrow[llll, "\pi_B"'] \arrow[dr, "p_B"'] \arrow[ddd, "g"] & \\
& & P_2 \arrow[ul, bend right, "s_P"'] \arrow[rrrr, "\pi_{B_2}"] \arrow[ddd, "\pi_{A_2}"] & & & & K[G_B] \arrow[ul, bend right, "s_B"'] \arrow[ddd, "g_2"] \\
U(L_A) \arrow[rrrr, "f_1"] \arrow[dr, "i_A"'] & & & & U(L_C) \arrow[dr, "i_C"] & & \\
& A \arrow[dr, "p_A"'] & & & & C \arrow[llll, "f"'] \arrow[dr, "p_C"'] & \\
& & K[G_A] \arrow[ul, bend right, "s_A"'] \arrow[rrrr, "f_2"'] & & & & K[G_C] \arrow[ul, bend right, "s_C"']
\end{tikzcd}$$

where $(P_1, \pi_{A_1}, \pi_{B_1})$ is the pullback of f_1 and g_1, $(P_2, \pi_{A_2}, \pi_{B_2})$ is the pullback of f_2 and g_2.

When f is surjective, the surjectivity of f_1 and f_2 follows from Lemma 2.1.6 applied to the lower part of the diagram. The front and back squares of the diagram are in $\mathbf{GrpHopf}_K$ and in $\mathbf{PrimHopf}_K$, respectively, thus both π_{B_1} and π_{B_2} are surjective (by Lemma 2.1.4). Applying Lemma 2.1.6 again (in converse direction), we obtain that π_B is also surjective.

□

In order to prove that $\mathbf{Hopf}_{K,coc}$ is semi-abelian, it remains to show that equivalence relations are effective. For this, we shall show that any equivalence relation R in $\mathbf{Hopf}_{K,coc}$ is the kernel pair of its coequalizer $q\colon X \to \overline{X}$. We first apply Theorem 1.4.5 to the equivalence relation R, obtaining the following commutative diagram, where the morphisms q_1, q_2 and q are the coequalizers of p_{11} and p_{21}, p_{12} and p_{22}, p_1 and p_2, respectively, and $(Eq(q)$,π_1,$\pi_2)$ is the kernel pair of q.

$$\begin{tikzcd}
0 \arrow[r] & U(L_R) \arrow[r, "i_R"] \arrow[dd, shift right=2, "p_{11}"'] \arrow[dd, shift left=2, "p_{21}"] \arrow[dr, bend left, "i'_R"] & R \arrow[rr, "p_R"] \arrow[dr, "\theta"] \arrow[dd, shift right=2, "p_1"'] \arrow[dd, shift left=2, "p_2"] & & K[G_R] \arrow[r] \arrow[dd, shift right=2, "p_{12}"'] \arrow[dd, shift left=2, "p_{22}"] & 0 \\
& & & Eq(q) \arrow[ur, bend right, "p'_R"'] \arrow[dl, shift right, "\pi_1"'] \arrow[dl, shift left, "\pi_2"] & & \\
0 \arrow[r] & U(L_X) \arrow[r, "i_X"] \arrow[uu, "\Delta_{X_1}"] \arrow[d, "q_1"] & X \arrow[rr, "p_X"] \arrow[uu, "\Delta_X"] \arrow[d, "q"] & & K[G_X] \arrow[r] \arrow[uu, "\Delta_{X_2}"] \arrow[d, "q_2"] & 0 \\
& \overline{U(L_X)} \arrow[r, "\overline{i_X}"'] & \overline{X} \arrow[rr, "\overline{p_X}"'] & & \overline{K[G_X]} &
\end{tikzcd}$$

Thanks to Lemma 2.1.4, the left column is exact, i.e. $U(L_R) = Eq(q_1)$. On the other hand, let us explain why the right column is also exact. First observe that $(K[G_R], p_{12}, p_{22})$ is a reflexive relation on $K[G_X]$, while the category $\mathbf{GrpHopf}_K$ of group Hopf algebras is exact Mal'tsev and closed under pullbacks and quotients in $\mathbf{Hopf}_{K,coc}$. It follows that $K[G_R]$ is the kernel pair of q_2 in $\mathbf{Hopf}_{K,coc}$.

The universal property of $(Eq(q),\pi_1,\pi_2)$ gives the unique arrow $\theta\colon R \to Eq(q)$ with $\pi_1 \circ \theta = p_1$ and $\pi_2 \circ \theta = p_2$. Since the morphisms i_X and $\overline{i_X}$ are split in the category of cocommutative coalgebras, one can check that the arrow $\overline{i_X}$ is a monomorphism, by using the fact that the coequalizers of equivalence relations in $\mathbf{Hopf}_{K,coc}$ are computed as in $\mathbf{Coalg}_{K,coc}$. Consequently, the lower row in the diagram above is an exact sequence as is then the lower row in the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & U(L_R) \arrow[r, "i_R"] \arrow[d, "id_{U(L_R)}"'] & R \arrow[r, "p_R"] \arrow[d, "\theta"] & K[G_R] \arrow[r] \arrow[d, "id_{K[G_R]}"] & 0 \\
0 \arrow[r] & U(L_R) \arrow[r, "i'_R"'] & Eq(q) \arrow[r, "p'_R"'] & K[G_R] \arrow[r] & 0
\end{tikzcd}$$

By applying the Split Short Five Lemma to the above commutative diagram, it follows that the morphism θ is an isomorphism, and the equivalence relation R is effective. One accordingly has the following:

Theorem 2.1.8 *For any field K of characteristic 0, the category $\mathbf{Hopf}_{K,coc}$ of cocommutative Hopf K-algebras is semi-abelian.*

2.2 A torsion theory in $\mathbf{Hopf}_{K,coc}$

From now on, in the homological category of cocommutative Hopf K-algebras, $\mathbf{T}$ will always denote the (full) subcategory $\mathbf{PrimHopf}_K$ of primitive Hopf algebras, and $\mathbf{F}$ the (full) subcategory $\mathbf{GrpHopf}_K$ of group Hopf algebras.

Theorem 2.2.1 *The pair ($\mathbf{PrimHopf}_K$,$\mathbf{GrpHopf}_K$) is a hereditary torsion theory in $\mathbf{Hopf}_{K,coc}$.*

Proof. Thanks to Theorem 1.4.5, we know that we can associate the following short exact sequence with any cocommutative Hopf K-algebra H:

$$0 \longrightarrow U(L_H) \xrightarrow{i_H} H \xrightarrow{p_H} K[G_H] \longrightarrow 0$$

Any morphism $f\colon U(L) \to K[G]$ from a primitive Hopf algebra $U(L) \in \mathbf{PrimHopf}_K$ to a group Hopf algebra $K[G] \in \mathbf{GrpHopf}_K$ is the zero morphism in $\mathbf{Hopf}_{K,coc}$. Indeed, any primitive Hopf algebra $U(L)$ is generated by its primitive elements, which are preserved by any Hopf algebra morphism. Since a group Hopf algebra $K[G]$ does not contain any non-zero primitive element, it follows that f is the zero morphism. It follows that $(\mathbf{PrimHopf}_K, \mathbf{GrpHopf}_K)$ is a torsion theory, which is actually hereditary thanks to Lemma 2.1.4. $\square$

2.3 $\mathbf{Hopf}_{K,coc}$ is not abelian

The category $\mathbf{Hopf}_{K,coc}$ is not abelian since it contains monomorphisms which are not normal. Indeed, since the functor $K[-]$ is a right and left adjoint to the functor $\mathcal{G}$ (see Proposition 1.4.7), and $\mathcal{G} \circ K[-] = id_{\mathbf{Grp}}$, we have that the inclusion of $K[G']$ into $K[G]$, where G' is a non-normal subgroup of G, is a non-normal monomorphism. Another proof to that $\mathbf{Hopf}_{K,coc}$ is not abelian, is given by using the Tierney theorem, i.e. a category is abelian if and only if it is Barr-exact and additive. This proof will outline where the hypothesis of commutativity is needed to prove that $\mathbf{Hopf}_{K,coc,co}$ is an abelian category. In any additive category, we have that binary products are isomorphic to binary coproducts (see for instance [8]) which we will prove is not the case in $\mathbf{Hopf}_{K,coc}$.

Let X and Y be objects in $\mathbf{Hopf}_{K,coc}$, whose coproduct is supposed to be $X \otimes Y$ with the morphisms i_X and i_Y given by the universal property of products

$$\begin{tikzcd}
 & X \arrow[dl, "id_X"'] \arrow[dr, "u_Y \circ \epsilon_X"] \arrow[d, "\Delta_X"] & \\
 & X \otimes X \arrow[d, "id_X \otimes (u_Y \circ \epsilon_X)"] & \\
X & X \otimes Y \arrow[l, "\pi_X"'] \arrow[r, "\pi_Y"] & Y \\
 & Y \otimes Y \arrow[u, "(u_X \circ \epsilon_Y) \otimes id_X"] & \\
 & Y \arrow[ul, "u_X \circ \epsilon_Y"] \arrow[ur, "id_Y"'] \arrow[u, "\Delta_Y"] &
\end{tikzcd}$$

$$\forall\, x \in X, i_X(x) = \big(id_X \otimes (u_Y \circ \epsilon_X)\big) \circ \Delta_X(x) = x_1 \otimes \epsilon_X(x_1)1_Y = x \otimes 1_Y$$

Similarly, for all $y \in Y$, we show that $i_Y(y) = 1_X \otimes y$. Let us show that the universal property of the binary coproduct is not satisfied: let us consider $f\colon X \longrightarrow Z$ and $g\colon Y \longrightarrow Z$ two Hopf algebra morphisms

$$\begin{tikzcd}
& X \otimes Y \arrow[dd, "\psi"] & \\
X \arrow[ur, "i_X"] \arrow[dr, "f"'] & & Y \arrow[ul, "i_Y"'] \arrow[dl, "g"] \\
& Z &
\end{tikzcd}$$

If there exists a morphism $\psi\colon X \otimes Y \longrightarrow Z$ making the above diagram commute, we should have:

$\forall x \in X\colon\ (\psi \circ i_X)(x) = \psi(x \otimes 1_Y) = f(x)$

$\forall y \in Y\colon\ (\psi \circ i_Y)(y) = \psi(1_X \otimes y) = g(y)$

$\forall x \otimes y \in X \otimes Y\colon$ ψ should satisfy

$$\psi(x \otimes y) = \psi(x \otimes 1_Y)._Z\psi(1_X \otimes y) = f(x)._Z g(y)$$

The morphism ψ necessarily defined in this way is not necessarily an algebra morphism if Z is not commutative.

Chapter 3

Split extension classifiers in $\mathbf{Hopf}_{K,coc}$

In this chapter, we describe the split extension classifiers in the category of cocommutative Hopf algebras over an algebraically closed field of characteristic zero, hereby providing a new characterization of semi-direct products (smash products) of cocommutative Hopf algebras. The categorical notions of centralizer and of center in the category of cocommutative Hopf algebras is then explored. In the category of cocommutative Hopf algebras the categorical notion of center coincides with the one that have been recently introduced and investigated in the context of general Hopf algebras.

3.0.1 Basic notions and notations

Given a group G and its automorphism group $\mathsf{Aut}(G)$, consider the canonical split extension

$$0 \longrightarrow G \xrightarrow{i_1} G \rtimes \mathsf{Aut}(G) \underset{p_2}{\overset{i_2}{\leftrightarrows}} \mathsf{Aut}(G) \longrightarrow 0 \tag{3.0.1}$$

where the action of $\mathsf{Aut}(G)$ on G defining the semidirect product $G \rtimes \mathsf{Aut}(G)$ is simply given by the evaluation, p_2 is the second projection and i_1, i_2 are the canonical injections. As explained in [9, 11, 12] this split extension has a remarkable universal property in the category $\mathbf{Grp}$ of groups: any other split extension with kernel G

$$0 \longrightarrow G \xrightarrow{k} A \underset{f}{\overset{s}{\leftrightarrows}} B \longrightarrow 0 \tag{3.0.2}$$

can be uniquely obtained (up to isomorphism) by pulling back the split extension (3.0.1) along a group homomorphism $\chi\colon B \to \mathsf{Aut}(G)$:

$$\begin{tikzcd}
0 \arrow[r] & G \arrow[r, "k"] \arrow[d, equal] & A \arrow[r, shift right, "f"'] \arrow[d, "\overline{\chi}"] & B \arrow[l, shift right, "s"'] \arrow[r] \arrow[d, "\chi"] & 0 \\
0 \arrow[r] & G \arrow[r, "i_1"] & G \rtimes \mathsf{Aut}(G) \arrow[r, shift right, "p_2"'] & \mathsf{Aut}(G) \arrow[l, shift right, "i_2"'] \arrow[r] & 0
\end{tikzcd}$$

The group action $\chi\colon B \to \mathsf{Aut}(G)$ of B on G corresponds to the semi-direct product $G \rtimes_\chi B$ defining a split extension of B by G isomorphic to (3.0.2).

Given a Lie algebra L in the category $\mathbf{Lie}_K$ of Lie algebras over a field K the same universal property is satisfied by the split extension of the Lie algebra $\mathsf{Der}(L)$ of derivations of L by L

$$0 \longrightarrow L \xrightarrow{i_1} L \rtimes \mathsf{Der}(L) \underset{p_2}{\overset{i_2}{\leftrightarrows}} \mathsf{Der}(L) \longrightarrow 0. \tag{3.0.3}$$

Here the semi-direct product $L \rtimes \mathsf{Der}(L)$ in $\mathbf{Lie}_K$ is defined on the underlying vector space $L \oplus \mathsf{Der}(L)$, and the Lie bracket is given by

$$\big[(x,\phi),(x',\psi)\big] := \Big([x,x'] + \phi(x') - \psi(x), [\phi,\psi]\Big) = \Big([x,x'] + \phi(x') - \psi(x), \phi\circ\psi - \psi\circ\phi\Big),$$

for any $x, x' \in L$, for any $\phi, \psi \in \mathsf{Der}(L)$. In the category $\mathbf{Lie}_K$, the Lie algebra actions $\rho\colon L_1 \to \mathsf{Der}(L)$ of L_1 on L correspond bijectively to the isomorphism classes $\mathbf{SplExt}(L_1, L)$ of split extensions of L_1 by L. Given a split extension

$$0 \longrightarrow L \xrightarrow{k} A \underset{f}{\overset{s}{\leftrightarrows}} L_1 \longrightarrow 0. \tag{3.0.4}$$

the universal property of the split extension (3.0.3) is then expressed by the existence of a unique morphism $\rho\colon L_1 \to Der(L)$ such that the following diagram commutes and the right-hand square is a pullback:

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "k"] \arrow[d, equal] & A \arrow[r, shift right, "f"'] \arrow[d, "\overline{\rho}"] & L_1 \arrow[l, shift right, "s"'] \arrow[r] \arrow[d, "\rho"] & 0. \\ 0 \arrow[r] & L \arrow[r, "i_1"] & L \rtimes \mathsf{Der}(L) \arrow[r, shift right, "p_2"'] & \mathsf{Der}(L) \arrow[l, shift right, "i_2"'] \arrow[r] & 0 \end{tikzcd}$$

In a general pointed protomodular category [13] it is possible to express the same universal property as the one of the automorphism group $\mathsf{Aut}(G)$ of a group G in $\mathbf{Grp}$ and the one of the derivation Lie algebra $\mathsf{Der}(L)$ of a Lie algebra L in $\mathbf{Lie}_K$. One simply requires that, given any object X in $\mathbf{C}$, there is a split extension

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "i_1"] & \overline{X} \arrow[r, shift right, "p_2"'] & {[X]} \arrow[l, shift right, "i_2"'] \arrow[r] & 0 \end{tikzcd} \tag{3.0.5}$$

with kernel X satisfying the property that any other split extension with kernel X in $\mathbf{C}$

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "k"] & A \arrow[r, shift right, "f"'] & B \arrow[l, shift right, "s"'] \arrow[r] & 0 \end{tikzcd} \tag{3.0.6}$$

can be uniquely obtained by pulling back the split extension (3.0.5) along a suitable morphism $B \to [X]$, i.e. there exists a unique $\chi\colon B \to [X]$ such that the following square is a pullback

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\overline{\chi}"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "\chi"] \\ \overline{X} \arrow[r, "p_2"'] & {[X].} \end{tikzcd}$$

Equivalently, one requires the existence of a unique morphism $\chi\colon B \to [X]$ and of a morphism $\overline{\chi}\colon A \longrightarrow \overline{X}$ such that the following diagram of split exact sequences commutes

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "k"] \arrow[d, equal] & A \arrow[r, shift right, "f"'] \arrow[d, "\overline{\chi}"] & B \arrow[l, shift right, "s"'] \arrow[r] \arrow[d, "\chi"] & 0. \\ 0 \arrow[r] & X \arrow[r, "i_1"] & \overline{X} \arrow[r, shift right, "p_2"'] & {[X]} \arrow[l, shift right, "i_2"'] \arrow[r] & 0 \end{tikzcd}$$

If this is the case for any any X in $\mathbf{C}$, one says that the category $\mathbf{C}$ has *representable object actions* or that it is *action representable*, in the sense of [11], and the object $[X]$ in (3.0.5) is called a *split extension classifier*. Besides the categories of groups and of Lie algebras, there are many further examples of categories having this property: crossed modules, boolean rings, commutative von Neumann regular rings, and other ones [11, 12]. It is well known that there are semi-abelian categories that do not have representable object actions: for instance, this is the case for the category of rings. When a semi-abelian category $\mathbf{C}$ has representable object actions, commutator theory becomes simpler, since the Huq commutator of normal monomorphisms and the Smith commutator of equivalence relations always coincide in such a category $\mathbf{C}$ [19].

In Theorem 2.1.8, we showed that the category $\mathbf{Hopf}_{K,coc}$ of cocommutative Hopf algebras over an algebraically closed field of characteristic zero is semi-abelian. In fact, it is widely known that the category of cocommutative Hopf algebras over any commutative base ring K (even not necessarily a field) is isomorphic to the category of internal groups in the category of cocommutative coalgebras over K. Hence, even if in this more general case it is unknown whether $\mathbf{Hopf}_{K,coc}$ is semi-abelian, the category is still protomodular and it makes sense to consider split extension classifiers in $\mathbf{Hopf}_{K,coc}$. Since the category of cocommutative coalgebras is moreover complete and cartesian closed by Theorem 5.3 in [5], it follows by Theorem 2.1 in [10] that $\mathbf{Hopf}_{K,coc}$ is action representable over any commutative ring K.

The aim of the present chapter is to give an explicit description of the split extension classifiers in the category $\mathbf{Hopf}_{K,coc}$ of cocommutative Hopf algebras over an algebraically closed field of characteristic zero. By doing so, we provide a new equivalent characterization of semi-direct products (smash products) of cocommutative Hopf algebras. We also prove that the category $\mathbf{Hopf}_{K}$ of arbitrary Hopf algebras over any field K is protomodular, and observe that the split extension classifier described in the cocommutative case can still be built in the case of a general Hopf algebra H. In the category $\mathbf{Hopf}_{K}$, when K is an algebraically closed field of characteristic 0, this object still has a restricted universal property, with respect to a special type of split exact sequences. We then analyse the abstract notions of center and of centralizer, introduced by D. Bourn and G. Janelidze in [19], in the category $\mathbf{Hopf}_{K,coc}$. This part also uses some nice results concerning centralizers due to A. Cigoli and S. Mantovani in [24]. We finally compare our description of the center in $\mathbf{Hopf}_{K,coc}$ with the defintion given by A. Chirvasitu and P. Kasprzak in [23] for arbitrary (not necessarily cocommutative) Hopf algebras, and observe that they coincide in the cocommutative case. Since our results heavily rely on the Cartier-Gabriel-Konstant-Milnor-Moore decomposition theorem for cocommutative Hopf algebras which is only valid over an algebraically closed field of characteristic zero, our description only holds in this case. It remains an open question how this description can be extended to the general case.

3.1 Split extension classifiers in $\mathbf{Hopf}_{K,coc}$

The aim of this section is to construct the *split extension classifier* $[H]$ for a given cocommutative Hopf algebra H in $\mathbf{Hopf}_{K,coc}$ over an algebraically closed field of characteristic zero K. Thanks to the decomposition theorem for these Hopf algebras, we only have to identify the grouplike and primitive elements of $[H]$. It might be not a big surprise that $\mathcal{G}([H])$ will turn out to be exactly the group $\mathsf{Aut}_{Hopf}(H)$ of Hopf algebra automorphisms of H. The primitive elements of $[H]$ turn out to be what we will call Hopf derivations.

3.1.1 Hopf derivations

Definition 3.1.1 Let H be a Hopf algebra. A *Hopf derivation* on H is a linear endomorphism ψ of H which is at the same time a derivation on H, i.e. ψ satisfies the *Leibniz rule*

$$\psi \circ M = M \circ (\psi \otimes id + id \otimes \psi)$$

and a coderivation on H, i.e. it satifies the *co-Leibniz rule*

$$\Delta \circ \psi = (\psi \otimes id + id \otimes \psi) \circ \Delta$$

The set of all Hopf derivations on H is denoted by $\mathsf{Der}_{\mathsf{Hopf}}(H)$.

Given elements $x, y \in H$, we can express the Leibniz and co-Leibniz rules for $\psi : H \to H$ respectively as

$$\begin{aligned} \psi(xy) &= \psi(x)y + x\psi(y); \\ \psi(x)_1 \otimes \psi(x)_2 &= \psi(x_1) \otimes x_2 + x_1 \otimes \psi(x_2). \end{aligned}$$

It is well-known that the derivations of an algebra are a Lie algebra by means of the commutator bracket. We start this section with the observation that Hopf derivations also form a Lie algebra.

Lemma 3.1.2 Let H be a Hopf algebra. Then $\mathsf{Der}_{Hopf}(H)$ is a Lie algebra for the commutator bracket

$$[\psi_1, \psi_2] = \psi_1 \circ \psi_2 - \psi_2 \circ \psi_1 \quad \forall \psi_1, \psi_2 \in \mathsf{Der}_{\mathsf{Hopf}}(H).$$

Proof. As already mentioned, it is well-known that $[\psi_1, \psi_2]$ is again a derivation on H for all derivations ψ_1 and ψ_2. Let us prove that $[\psi_1, \psi_2]$ is a coderivation for all coderivations ψ_1 and ψ_2:

$$\begin{aligned}
\Delta \circ [\psi_1, \psi_2] &= \Delta \circ (\psi_1 \circ \psi_2 - \psi_2 \circ \psi_1) \\
&= (\Delta \circ \psi_1) \circ \psi_2 - (\Delta \circ \psi_2) \circ \psi_1 \\
&= (\psi_1 \otimes id + id \otimes \psi_1) \circ (\psi_2 \otimes id + id \otimes \psi_2) \circ \Delta \\
&\quad -(\psi_2 \otimes id + id \otimes \psi_2) \circ (\psi_1 \otimes id + id \otimes \psi_1) \circ \Delta \\
&= \big((\psi_1 \circ \psi_2) \otimes id + \psi_1 \otimes \psi_2 + \psi_2 \otimes \psi_1 + id \otimes (\psi_1 \circ \psi_2)\big) \circ \Delta \\
&\quad -\big((\psi_2 \circ \psi_1) \otimes id + \psi_2 \otimes \psi_1 + \psi_1 \otimes \psi_2 + id \otimes (\psi_2 \circ \psi_1)\big) \circ \Delta \\
&= \big((\psi_1 \circ \psi_2) \otimes id + id \otimes (\psi_1 \circ \psi_2)\big) \circ \Delta \\
&\quad -\big((\psi_2 \circ \psi_1) \otimes id + id \otimes (\psi_2 \circ \psi_1)\big) \circ \Delta \\
&= \big((\psi_1 \circ \psi_2 - \psi_2 \circ \psi_1) \otimes id + id \otimes (\psi_1 \circ \psi_2 - \psi_2 \circ \psi_1)\big) \circ \Delta \\
&= \big([\psi_1, \psi_2] \otimes id + id \otimes [\psi_1, \psi_2]\big) \circ \Delta.
\end{aligned}$$

Hence we find that, in particular, Der_{Hopf} is a Lie algebra for the commutator bracket. $\square$

The following known result for derivations and coderivations will be useful.

Lemma 3.1.3 Let H be a Hopf algebra and $\psi \in \mathsf{Der}_{\mathsf{Hopf}}(H)$. Then $\psi \circ u = 0$ and $\epsilon \circ \psi = 0$.

Proof. Let $u(1_K) = 1_H \in H$ be the unit of H. Then we find since ψ is a derivation

$$\psi(1_H) = \psi(1_H 1_H) = \psi(1_H)1_H + 1_H\psi(1_H) = \psi(1_H) + \psi(1_H).$$

Hence $\psi(1_H) = 0$ and thus $\psi \circ u = 0$. Similarly, since ψ is a coderivation, we have for all $z \in H$

$$\begin{aligned}
\epsilon\big(\psi(z)\big) &= \epsilon\Big(\psi(z)_1\epsilon\big(\psi(z)_2\big)\Big) \\
&= \epsilon\Big(\psi(z_1)\epsilon(z_2) + z_1\epsilon\big(\psi(z_2)\big)\Big) \\
&= \epsilon\Big(\psi(z_1)\epsilon(z_2)\Big) + \epsilon\Big(z_1\epsilon\big(\psi(z_2)\big)\Big) \\
&= \epsilon\big(\psi(z)\big) + \epsilon\big(\psi(z)\big)
\end{aligned}$$

Hence $\epsilon \circ \psi(z) = 0$ for all $z \in H$. □

3.1.2 Construction of an action of $K[\mathsf{Aut}_{\mathsf{Hopf}}(H)]$ on $U\Big(\mathsf{Der}_{\mathsf{Hopf}}(H)\Big)$

Given $H \in \mathbf{Hopf}_{K,coc}$, denote by $\mathsf{Aut}_{\mathsf{Hopf}}(H)$ the group of *Hopf algebra automorphisms* of H, i.e. bialgebra automorphisms of H (bijective endomorphisms of H).

Lemma 3.1.4 For any $\phi \in \mathsf{Aut}_{\mathsf{Hopf}}(H)$ and $\psi \in \mathsf{Der}_{\mathsf{Hopf}}(H)$, the linear endomorphism $\phi \cdot \psi := \phi \circ \psi \circ \phi^{-1}$ is a Hopf derivation on H. I.e. there is a well-defined map

$$\rho : \mathsf{Aut}_{\mathsf{Hopf}}(H) \times \mathsf{Der}_{\mathsf{Hopf}}(H) \longrightarrow \mathsf{Der}_{\mathsf{Hopf}}(H), \quad \rho(\phi, \psi) := \phi \circ \psi \circ \phi^{-1}.$$

Proof. Let us first check that $\phi \circ \psi \circ \phi^{-1}$ is a derivation. For any $x, y \in H$ we find

$$\begin{aligned}
(\phi \circ \psi \circ \phi^{-1})(xy) &= (\phi \circ \psi)\big(\phi^{-1}(x)\phi^{-1}(y)\big) \\
&= \phi\Big(\psi\big(\phi^{-1}(x)\big)\phi^{-1}(y) + \phi^{-1}(x)\psi\big(\phi^{-1}(y)\big)\Big) \\
&= (\phi \circ \psi \circ \phi^{-1})(x)y + x(\phi \circ \psi \circ \phi^{-1})(y).
\end{aligned}$$

In the above equalities, we have used the fact that ϕ and ϕ^{-1} are algebra morphisms, and that ψ is a derivation. Let us also prove that $\phi \circ \psi \circ \phi^{-1}$ is a coderivation:

$$\begin{aligned}
\Delta \circ (\phi \cdot \psi) &= \Delta \circ \phi \circ \psi \circ \phi^{-1} \\
&= (\phi \otimes \phi) \circ \Delta \circ \psi \circ \phi^{-1} \\
&= (\phi \otimes \phi) \circ (\psi \otimes id + id \otimes \psi) \circ \Delta \circ \phi^{-1} \\
&= (\phi \otimes \phi) \circ (\psi \otimes id + id \otimes \psi) \circ (\phi^{-1} \otimes \phi^{-1}) \circ \Delta \\
&= \big((\phi \circ \psi \circ \phi^{-1}) \otimes id + id \otimes (\phi \circ \psi \circ \phi^{-1})\big) \circ \Delta \\
&= \big((\phi \cdot \psi) \otimes id + id \otimes (\phi \cdot \psi)\big) \circ \Delta
\end{aligned}$$

In the above equalities, we have used the fact that ϕ and ϕ^{-1} are coalgebra morphisms, and that ψ is a coderivation. □

Lemma 3.1.5 The map ρ from Lemma 3.1.4 induces a linear map

$$\overline{\rho} : K[\mathsf{Aut}_{\mathsf{Hopf}}(H)] \otimes U\big(\mathsf{Der}_{\mathsf{Hopf}}(H)\big) \longrightarrow U\big(\mathsf{Der}_{\mathsf{Hopf}}(H)\big)$$

defined by $\overline{\rho}(\phi \otimes \psi) := \rho(\phi, \psi) = \phi \circ \psi \circ \phi^{-1}$ for $\phi \in \mathsf{Aut}_{\mathsf{Hopf}}(H)$ and $\psi \in \mathsf{Der}_{\mathsf{Hopf}}(H)$.

Proof. Fix any $\phi \in \mathsf{Aut}_{\mathsf{Hopf}}(H)$. Then the induced morphism $\rho_\phi : \mathsf{Der}_{\mathsf{Hopf}}(H) \to \mathsf{Der}_{\mathsf{Hopf}}(H)$ defined by $\rho_\phi(\psi) = \rho(\phi, \psi)$ is a Lie algebra morphism. Indeed:

$$\begin{aligned}
\phi \cdot [\psi, \psi'] &= \phi \cdot (\psi \circ \psi' - \psi' \circ \psi) \\
&= \phi \circ \psi \circ \psi' \circ \phi^{-1} - \phi \circ \psi' \circ \psi \circ \phi^{-1} \\
&= [\phi \circ \psi \circ \phi^{-1}, \phi \circ \psi' \circ \phi^{-1}] \\
&= [\phi \cdot \psi, \phi \cdot \psi'].
\end{aligned}$$

Hence ρ_ϕ induces an algebra map $\overline{\rho}_\phi : U\big(\mathsf{Der}_{\mathsf{Hopf}}(H)\big) \to U\big(\mathsf{Der}_{\mathsf{Hopf}}(H)\big)$. Hence we can define $\overline{\rho}$ by linear extension as stated. □

Proposition 3.1.6 The map $\overline{\rho}$ from Lemma 3.1.5 defines an action of Hopf algebras of $K[\mathsf{Aut}_{\mathsf{Hopf}}(H)]$ on $U\big(\mathsf{Der}_{\mathsf{Hopf}}(H)\big)$. Accordingly, there is a well-defined semi-direct product $U\big(\mathsf{Der}_{\mathsf{Hopf}}(H)\big) \rtimes_{\overline{\rho}} K[\mathsf{Aut}_{\mathsf{Hopf}}(H)]$, that we will denote by $[H]$.

Proof. We have to verify that $\overline{\rho}$ satisfies the axioms of an action of cocommutative Hopf algebras (Definition 1.4.1). Axiom 3 and Axiom 4 follow immediately from the construction of $\overline{\rho}$ in the proof of Lemma 3.1.5. Indeed, by construction for any $\phi \in K[\mathsf{Aut}_{\mathsf{Hopf}}(H)]$, the map $\overline{\rho}_\phi$ is an algebra morphism, which expresses exactly Axiom 3 and Axiom 4. As a consequence, it is enough to verify the remaining axioms on base elements of the vector space $K[\mathsf{Aut}_{\mathsf{Hopf}}(H)]$ (i.e. elements $\phi \in \mathsf{Aut}_{\mathsf{Hopf}}(H)$) and generators of the algebra $U\big(\mathsf{Der}_{\mathsf{Hopf}}(H)\big)$ (i.e. elements of the form $\psi \in \mathsf{Der}_{\mathsf{Hopf}}(H)$).

- (Axiom 1) $id_{K[\mathsf{Aut}_{\mathsf{Hopf}}(H)]} \cdot \psi = \psi$;
- (Axiom 2) $(\phi \circ \phi^{'}) \cdot \psi = \phi \circ \phi^{'} \circ \psi \circ \phi^{'-1} \circ \phi^{-1} = \phi \cdot (\phi^{'} \cdot \psi)$;
- (Axiom 5) Remark first that any $\phi \in \mathsf{Aut}_{\mathsf{Hopf}}(H)$ is a grouplike element of $K[\mathsf{Aut}_{\mathsf{Hopf}}(H)]$ and any $\psi \in \mathsf{Der}_{\mathsf{Hopf}}(H)$ is a primitive element in $U\big(\mathsf{Der}_{\mathsf{Hopf}}(H)\big)$. Since $\phi \cdot \psi \in \mathsf{Der}_{\mathsf{Hopf}}(H)$ this is as well a primitive element in $U\big(\mathsf{Der}_{\mathsf{Hopf}}(H)\big)$. Hence we obtain

$$(\phi \cdot \psi)_1 \otimes (\phi \cdot \psi)_2 = \Delta(\phi \cdot \psi) = (\phi \cdot \psi) \otimes id + id \otimes (\phi \cdot \psi)$$

  and

$$\begin{aligned}(\phi_1 \cdot \psi_1) \otimes (\phi_2 \cdot \psi_2) &= (\phi \cdot \psi) \otimes (\phi \cdot id) + (\phi \cdot id) \otimes (\phi \cdot \psi) \\ &= (\phi \cdot \psi) \otimes id + id \otimes (\phi \cdot \psi),\end{aligned}$$

  from which the axiom follows.
- (Axiom 6) We have $\epsilon(\phi \cdot \psi) = 0 = \epsilon(\phi)\epsilon(\psi)$ since $\phi \cdot \psi$ and ψ are primitive elements.

$\square$

3.1.3 Construction of an action of $U\big(\mathsf{Der}_{\mathsf{Hopf}}(H)\big) \rtimes_{\overline{\rho}} K[\mathsf{Aut}_{\mathsf{Hopf}}(H)]$ on H

Proposition 3.1.7 Let H be a cocommutative Hopf algebra and consider the Hopf algebra $[H] = U\big(\mathsf{Der}_{\mathsf{Hopf}}(H)\big) \rtimes_{\overline{\rho}} K[\mathsf{Aut}_{\mathsf{Hopf}}(H)]$ from Proposition 3.1.6. The map

$$\star : [H] \otimes H \longrightarrow H$$

given by

$$(\psi \otimes \phi) \star h := \psi\big(\phi(h)\big),$$

for any $\psi \in \mathsf{Der}_{\mathsf{Hopf}}(H)$ and $\phi \in \mathsf{Aut}_{\mathsf{Hopf}}(H)$ defines an action of $[H]$ on H.

Proof. Let us check that $\star$ satisfies the axioms of an action of cocommutative Hopf algebras (Definition 1.4.1).

- (Axiom 1) $\big(id_{U\left(\mathsf{Der}_{\mathsf{Hopf}}(H)\right)} \otimes id_{K[\mathsf{Aut}_{\mathsf{Hopf}}(H)]}\big) \star h = h.$

- (Axiom 2) One has the equalities

$$\begin{aligned}
(\psi \otimes \phi)(\psi^{'} \otimes \phi^{'}) \star h &= \big(\psi \overline{\otimes} (\phi \cdot_{\overline{\rho}} \psi^{'}) \otimes (\phi \circ \phi^{'})\big)(h) \\
&= (\psi \circ \phi \circ \psi^{'} \circ \phi^{-1} \circ \phi \circ \phi^{'})(h) \\
&= (\psi \otimes \phi) \star \big((\psi^{'} \otimes \phi^{'}) \star h\big).
\end{aligned}$$

- (Axiom 3) First remark that for any $\phi \in \mathsf{Aut}_{\mathsf{Hopf}}(H)$ and $\psi \in \mathsf{Der}_{\mathsf{Hopf}}(H)$ we have

$$\begin{aligned}
\Delta(\psi \otimes \phi) &= (id \otimes \sigma \otimes id)\big(\Delta(\psi) \otimes \Delta(\phi)\big) \\
&= (id \otimes \sigma \otimes id)((\psi \otimes 1 + 1 \otimes \psi) \otimes (\phi \otimes \phi)) \\
&= \psi \otimes \phi \otimes id \otimes \phi + id \otimes \phi \otimes \psi \otimes \phi
\end{aligned}$$

  where we used that ϕ is grouplike and ψ is primitive. Then we can now verify the axiom.

$$\begin{aligned}
(\psi \otimes \phi) \star hh' &= \psi\big(\phi(hh')\big) \\
&= \psi\big(\phi(h)\phi(h')\big) \\
&= \psi(\phi(h))\phi(h') + \phi(h)\psi(\phi(h')), \\
&= \big((\psi \otimes \phi) \star h\big)\big((id \otimes \phi) \star h'\big) + \big((id \otimes \phi) \star h\big)\big((\psi \otimes \phi) \star h'\big) \\
&= \big((\psi \otimes \phi)_1 \star h\big)\big((\psi \otimes \phi)_2 \star h'\big)
\end{aligned}$$

  where we used that ϕ is an algebra morphism in the second equality and that ψ is a derivation in the third equality.

- (Axiom 4) The right-hand side of the equality is given by

$$(\psi \otimes \phi) \star 1_H = \psi\big(\phi(1_H)\big) = \psi(1_H) = 0,$$

where we applied Lemma 3.1.3. The left-hand side of the equality is given by

$$\epsilon(\psi \otimes \phi)1_H = \epsilon(\psi)\epsilon(\phi)1_H = 0$$

where used $\epsilon(\psi) = 0$ since ψ is a primitive element of $U\big(\mathsf{Der}_{\mathsf{Hopf}}(H)\big)$.

- (Axiom 5) The left-hand side of the identity to be checked is given by

$$\begin{aligned}
\big((\psi \otimes \phi) \star h\big)_1 \otimes \big((\psi \otimes \phi) \star h\big)_2 &= \Delta\big((\psi \otimes \phi) \star h\big) = \Delta \circ \psi \circ \phi(h) \\
&= (\psi \otimes id + id \otimes \psi) \circ \Delta \circ \phi(h) \\
&= (\psi \otimes id + id \otimes \psi) \circ (\phi \otimes \phi) \circ \Delta(h) \\
&= \big((\psi \circ \phi) \otimes \phi + \phi \otimes (\psi \circ \phi)\big) \circ \Delta(h).
\end{aligned}$$

On the other hand, since

$$\Delta(\psi \otimes \phi) = \psi \otimes \phi \otimes id \otimes \phi + id \otimes \phi \otimes \psi \otimes \phi,$$

the right-hand side of the identity is given by

$$\begin{aligned}
\big((\psi \otimes \phi)_1 \star h_1\big) \otimes \big((\psi \otimes \phi)_2 \star h_2\big) &= \big((\psi \otimes \phi) \star h_1\big) \otimes \big((id \otimes \phi) \star h_2\big) \\
&\quad + \big((id \otimes \phi) \star h_1\big) \otimes \big((\psi \otimes \phi) \star h_2\big) \\
&= \psi\big(\phi(h_1)\big) \otimes \phi(h_2) + \phi(h_1) \otimes \psi\big(\phi(h_2)\big) \\
&= \big((\psi \circ \phi) \otimes \phi + \phi \otimes (\psi \circ \phi)\big) \circ \Delta(h)
\end{aligned}$$

- (Axiom 6) The right-hand side of the identity is given by

$$\epsilon(\psi \otimes \phi)\epsilon(h) = \epsilon(\psi)\epsilon(\phi)\epsilon(h) = 0$$

where we used again that $\epsilon(\psi) = 0$ since ψ is a primitive element. The left-hand side of the identity is given by

$$\epsilon\big((\psi \otimes \phi) \star h\big) = \epsilon \circ \psi \circ \phi(h) = 0$$

where we used Lemma 3.1.3.

□

3.1.4 Universal property of the split extension classifier

The aim of this section is to prove that for a given cocommutative Hopf algebra H, the Hopf algebra $[H]$ constructed in Proposition 3.1.6 together with its action on H defined in Proposition 3.1.7 is exactly the split extension classifier of H in $\mathbf{Hopf}_{K,coc}$, i.e. it satisfies the universal property recalled in the introduction.

Consider a split exact sequence in $\mathbf{Hopf}_{K,coc}$

$$\begin{tikzcd} 0 \arrow[r] & H \arrow[r] & B \arrow[r, shift right] & A \arrow[l, shift right] \arrow[r] & 0. \end{tikzcd} \tag{3.1.1}$$

By applying the semi-direct product decomposition of A recalled in Theorem 1.4.5, and by using Lemma 1.4.3, the split exact sequence (3.1.1) is isomorphic to the following split exact sequence in $\mathbf{Hopf}_{K,coc}$.

$$\begin{tikzcd} 0 \arrow[r] & H \arrow[r, "k"] & H \rtimes_A \big(U(L_A) \rtimes K[G_A]\big) \arrow[r, shift right, "f"'] & U(L_A) \rtimes K[G_A] \arrow[l, shift right, "s"'] \arrow[r] & 0. \end{tikzcd}$$

We are now going to construct a morphism

$$\chi \colon U(L_A) \rtimes K[G_A] \longrightarrow U\big(\mathsf{Der}_{\mathsf{Hopf}}(H)\big) \rtimes_{\overline{\rho}} K\big[\mathsf{Aut}_{\mathsf{Hopf}}(H)\big]$$

compatible with the morphism id_H (see Proposition 1.4.8) such that the following diagram commutes, and then prove the uniqueness of such a morphism χ.

$$\begin{tikzcd} 0 \arrow[r] & H \arrow[r, "k"] \arrow[d, equal] & H \rtimes_A \big(U(L_A) \rtimes K[G_A]\big) \arrow[r, shift right, "f"'] \arrow[d, dotted, "id_H \otimes \chi"'] & U(L_A) \rtimes K[G_A] \arrow[l, shift right, "s"'] \arrow[r] \arrow[d, dotted, "\chi"] & 0 \\ 0 \arrow[r] & H \arrow[r, "i_1"] & H \rtimes_\star [H] \arrow[r, shift right, "p_2"'] & {[H]} \arrow[l, shift right, "i_2"'] \arrow[r] & 0 \end{tikzcd} \tag{3.1.2}$$

If $\rho : A \otimes H \to H$ is the action of A on H induced by the split exact sequence (3.1.1), defined by $\rho(a \otimes h) = a \cdot_A h$, one has a group homomorphism defined by $\chi_G : G_A \longrightarrow \mathsf{Aut}_{\mathsf{Hopf}}(H) : \chi_G(g)(h) := g \cdot_A h$, for any $g \in G_A$ and $h \in H$.

Let us first check that $\forall g \in G_A$, the map $\chi_G(g)$ is an endomorphism of Hopf algebras of H:

- $\chi_G(g)$ is an algebra morphism since $\forall h, h' \in H$ we have

$$\begin{aligned} \chi_G(g)(hh') &= g \cdot_A (hh') = (g_1 \cdot_A h)(g_2 \cdot_A h') \\ &= (g \cdot_A h)(g \cdot_A h') = \big(\chi_G(g)(h)\big)\big(\chi_G(g)(h')\big) \\ \chi_G(g)(1_H) &= g \cdot_A 1_H = \epsilon(g)1_H = 1_H \end{aligned}$$

  by axioms 3 and 4 of an action of cocommutative Hopf algebras and by the fact that g is a group-like element of H.

- $\chi_G(g)$ is a coalgebra morphism since we have

$$\begin{aligned} \Delta\big(\chi_G(g)(h)\big) &= \Delta\big(g \cdot_A h) = (g \cdot_A h)_1 \otimes (g \cdot_A h)_2 \\ &= (g_1 \cdot_A h_1) \otimes (g_2 \cdot_A h_2) = (g \cdot_A h_1) \otimes (g \cdot_A h_2) \\ &= \big(\chi_G(g) \otimes \chi_G(g)\big)\Delta(h) \\ \epsilon\big(\chi_G(g)(h)\big) &= \epsilon(g \cdot_A h) = \epsilon(g)\epsilon(h) = \epsilon(h) \end{aligned}$$

  by the axioms 5 and 6 of an action of cocommutative Hopf algebras and by the fact that g is a group-like element of H.

Moreover, the map $\chi_G : G_A \to \mathsf{End}_{\mathsf{Hopf}}(H)$ is a monoid morphism.

- indeed: for all $g, g' \in G$ we have $\chi_G(g'g) = \chi_G(g') \circ \chi_G(g)$.

$$\big(\chi_G(g') \circ \chi_G(g)\big)(h) \quad = \quad g' \cdot_A (g \cdot_A h) = (g'g) \cdot_A h = \chi_G(g'g)(h)$$

  where the second equality follows follows axiom 2. Observe that the map χ_G preserves the neutral element by axiom 1.

In particular, it follows that $\chi_G(g^-1) = \chi_G(g)^{-1}$ and therefore, $\chi_G(g)$ is an automorphism of Hopf algebras for all $g \in G$.

If A acts on H via $\rho : A \otimes H \to H$, with $\rho(a \otimes h) = a \cdot_A h$, then we have a Lie algebra morphism $\chi_L : L_A \longrightarrow \mathsf{Der}_{\mathsf{Hopf}}(H)$ defined by $\chi_L(x)(h) = x \cdot_A h$. Indeed, let us prove that $\chi_L(x)$ is a Hopf derivation in H for all $x \in L_A$.

- $\chi_L(x)$ is a derivation since we have for all $h, h' \in H$

$$\begin{aligned}\chi_L(x)(hh') &= (x \cdot_A h)(1 \cdot_A h') + (1 \cdot_A h)(x \cdot_A h') \\ &= (x \cdot_A h)h' + h(x \cdot_A h') \\ &= \chi_L(x)(h)h' + h\chi_L(x)(h')\end{aligned}$$

  by axiom 1 and 3 of an action of cocommutative Hopf algebras and by the fact that x is a primitive element.

- We also prove that $\chi_L(x)$ is a coderivation since we have for all $h \in H$,

$$\begin{aligned}\big(\Delta \circ \chi_L(x)\big)(h) &= \Delta(x \cdot_A h) \\ &= (x \cdot_A h)_1 \otimes (x \cdot_A h)_2 \\ &= \big(x_1 \cdot_A h_1\big) \otimes \big(x_2 \cdot_A h_2\big) \\ &= \big(x \cdot_A h_1\big) \otimes \big(1 \cdot_A h_2\big) + \big(1 \cdot_A h_1\big) \otimes \big(x \cdot_A h_2\big) \\ &= \big(x \cdot_A h_1\big) \otimes h_2 + h_1 \otimes \big(x \cdot_A h_2\big) \\ &= \chi_L(x)(h_1) \otimes h_2 + h_1 \otimes \chi_L(x)(h_2) \\ &= \big(\chi_L(x) \otimes id + id \otimes \chi_L(x)\big) \circ \Delta(h)\end{aligned}$$

  In the above equalities, we have used the axioms 1 and 5 of an action of cocommutative Hopf algebras, and the fact that x is a primitive element.

Hence $\chi_L : L_A \to \mathsf{Der}_{\mathsf{Hopf}}(H)$ is well-defined. Furthermore, $\forall x, y \in L_A$, $\forall h \in H$, one has:

$$\begin{aligned}\chi_L[x,y](h) &= \chi_L(xy - yx)(h) \\ &= (xy - yx) \cdot_A h \\ &= (xy) \cdot_A h - (yx) \cdot_A h \\ &= x \cdot_A (y \cdot_A h) - y \cdot_A (x \cdot_A h) \\ &= \big(\chi_L(x) \circ \chi_L(y) - \chi_L(y) \circ \chi_L(x)\big)(h) \\ &= \big[\chi_L(x), \chi_L(y)\big](h).\end{aligned}$$

where we used the axiom 2 of action of cocommutative Hopf algebras. Therefore, $\chi_L : L_A \to \mathsf{Der}_{\mathsf{Hopf}}(H)$ is a morphism of Lie algebras.

The morphisms

$$U(\chi_L)\colon U(L_A) \longrightarrow U\big(\mathsf{Der}_{\mathsf{Hopf}}(H)\big)$$

and

$$K[\chi_G]\colon K[G_A] \longrightarrow K\big[\mathsf{Aut}_{\mathsf{Hopf}}(H)\big]$$

satisfy the compatibility condition (Proposition 1.4.8). Indeed, $\forall g \in G_A, \forall x \in L_A$, $\forall h \in H$, we have:

$$\begin{array}{rcl} \chi_L(g \cdot_A x)(h) & = & \big(\chi_G(g) \circ \chi_L(x) \circ \chi_G(g^{-1})\big)(h) \\ \Leftrightarrow (gxg^{-1}) \cdot_A h & = & g \cdot_A \big(x \cdot_A (g^{-1} \cdot_A h)\big) \end{array}$$

which holds by the axiom 2 of action. As a consequence, given a Hopf algebra A acting on H, we obtain the associated morphisms $U(\chi_L)\colon U(L_A) \longrightarrow U\big(\mathsf{Der}_{\mathsf{Hopf}}(H)\big)$ and $K[\chi_G]\colon K[G_A] \longrightarrow K\big[\mathsf{Aut}_{\mathsf{Hopf}}(H)\big]$ which can be recombined to define a Hopf algebra morphism $\chi := U(\chi_L) \otimes K[\chi_G]$ in diagram 3.1.2.

The morphisms id_H and χ satisfy the compatibility condition (Proposition 1.4.8): indeed, for a generator $x \otimes g \in U(L_A) \rtimes_A K[G_A]$, associated to the element $xg \in A$ by the isomorphism of Cartier-Gabriel-Moore-Milnor-Kostant (Theorem 1.4.5), and for all $h \in H$, one has:

$$\begin{array}{rcl} \Big(\big(U(\chi_L) \otimes K[\chi_G]\big)(x \otimes g)\Big) \cdot_\star id_H(h) & = & \Big(U(\chi_L)(x) \otimes K[\chi_G](g)\Big) \cdot_\star id_H(h) \\ & = & \Big(U\big(\chi_L(x)\big) \circ K\big[\chi_G(g)\big]\Big)(h) \\ & = & x \cdot_A (g \cdot_A h) \\ & = & (xg) \cdot_A h \\ & = & id_H\big((xg) \cdot_A h\big) \end{array}$$

by the axiom 2 of an action of cocommutative Hopf algebras.

It remains to prove the uniqueness of the morphism χ. For this, consider two other morphisms ξ and $\overline{\xi}$ making the following diagram commute

$$\begin{tikzcd}
0 \arrow[r] & H \arrow[r, "k"] \arrow[d, equal] & H \rtimes_A \big(U(L_A) \rtimes K[G_A]\big) \arrow[r, shift right, "f"'] \arrow[d, dotted, "\overline{\xi}"'] & U(L_A) \rtimes K[G_A] \arrow[l, shift right, "s"'] \arrow[r] \arrow[d, dotted, "\xi"] & 0 \\
0 \arrow[r] & H \arrow[r, "i_1"] & H \rtimes_{\star} [H] \arrow[r, shift right, "p_2"'] & {[H]} \arrow[l, shift right, "i_2"'] \arrow[r] & 0
\end{tikzcd} \tag{3.1.3}$$

By Proposition 1.4.8 we know that $\overline{\xi} = id_H \otimes \xi$ and id_H and ξ are compatible, hence we obtain that

$$\xi(a) \cdot_{\star} h = a \cdot_A h = \chi(a) \cdot_{\star} h.$$

The morphism ξ induces a group homomorphism $\xi_G \colon G_A \to \mathsf{Aut}_{\mathsf{Hopf}}(H)$ and a Lie algebra homomorphism $\xi_L \colon L_A \to \mathsf{Der}_{\mathsf{Hopf}}(H)$. We see that for all $x \in L_A$ and $g \in G_A$

$$\xi_L(x)(h) = \chi_L(x)(h),$$

and

$$\xi_G(g)(h) = \chi_G(g)(h).$$

One accordingly has the following:

Theorem 3.1.8 *For any algebraically closed field K of characteristic 0, the split extension classifier of any cocommutative Hopf K-algebra H is given by*

$$U\big(\mathsf{Der}_{\mathsf{Hopf}}(H)\big) \rtimes_{\overline{\rho}} K\big[\mathsf{Aut}_{\mathsf{Hopf}}(H)\big]$$

where

$$\overline{\rho} : K[\mathsf{Aut}_{\mathsf{Hopf}}(H)] \otimes U\big(\mathsf{Der}_{\mathsf{Hopf}}(H)\big) \longrightarrow U\big(\mathsf{Der}_{\mathsf{Hopf}}(H)\big)$$

is defined by

$$\overline{\rho}(\phi \otimes \psi) := \phi \cdot \psi = \phi \circ \psi \circ \phi^{-1}$$

on the generators and extended by linearity to all $K[\mathsf{Aut}_{\mathsf{Hopf}}(H)] \otimes U\big(\mathsf{Der}_{\mathsf{Hopf}}(H)\big)$.

3.1.5 Split extension classifiers in $\mathbf{Hopf}_K$

The problem of determining the existence of split extension classifiers can be studied in the context of arbitrary Hopf algebras over a field K, denoted by $\mathbf{Hopf}_K$, which is protomodular by the following theorem:

Theorem 3.1.9 *The category of Hopf algebras over a field K is protomodular.*

Proof. Let us consider the following morphism of split exact sequences in $\mathbf{Hopf}_K$, where we have to prove that θ is an isomorphism of Hopf algebras.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "k"] \arrow[d, "id_A"'] & B \arrow[r, shift left, "f"] \arrow[d, "\theta"] & C \arrow[l, shift left, "s"] \arrow[r] \arrow[d, "id_C"] & 0 \\
0 \arrow[r] & A \arrow[r, "k'"'] & B' \arrow[r, shift left, "f'"] & C \arrow[l, shift left, "s'"] \arrow[r] & 0
\end{tikzcd}$$

By Theorem 4.1 in [41], the above morphism of split exact sequences is isomorphic to the following one

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i_1"] \arrow[d, "id_A"'] & A \rtimes C \arrow[r, shift left, "p_2"] \arrow[d, "\overline{\theta}"] & C \arrow[l, shift left, "i_2"] \arrow[r] \arrow[d, "id_C"] & 0 \\
0 \arrow[r] & A \arrow[r, "i_1'"'] & A \rtimes' C \arrow[r, shift left, "p_2'"] & C \arrow[l, shift left, "i_2'"] \arrow[r] & 0
\end{tikzcd}$$

Hence, to prove that θ is an isomorphism, it is equivalent to prove that $\overline{\theta}$ is an isomorphism. We have that $A \rtimes C$ is generated as an algebra by the images of the Hopf algebra morphisms $i_1 : A \to A \rtimes C$ and $i_2 : C \to A \rtimes C$. Indeed, any element $a \otimes c \in A \rtimes C$ can be written as

$$(a \otimes c) = (a \otimes 1_C)(1_A \otimes c)$$

Then we have

$$\overline{\theta}(a \otimes c) = \overline{\theta}\big((a \otimes 1_C)(1_A \otimes c)\big) = \overline{\theta}(a \otimes 1_C)\overline{\theta}(1_A \otimes c) = (id_A \otimes id_C)(a \otimes c)$$

We obtain that the morphism $\overline{\theta}$ is equal to $id_A \otimes id_C$ and then is an isomorphism.

□

The object $[H]$ given in Theorem 3.1.8 is well-defined in the general context of Hopf algebras, i.e. for H in $\mathbf{Hopf}_K$: indeed, the Hopf algebra $K\big[\mathsf{Aut}_{\mathsf{Hopf}}(H)\big]$ is still cocommutative, and it acts on the cocommutative Hopf algebra $U\big(\mathsf{Der}_{\mathsf{Hopf}}(H)\big)$ via the action

$$\overline{\rho} : K[\mathsf{Aut}_{\mathsf{Hopf}}(H)] \otimes U\big(\mathsf{Der}_{\mathsf{Hopf}}(H)\big) \longrightarrow U\big(\mathsf{Der}_{\mathsf{Hopf}}(H)\big)$$

defined by

$$\overline{\rho}(\phi \otimes \psi) := \phi \cdot \psi = \phi \circ \psi \circ \phi^{-1}$$

on the generators and extended by linearity to all $K[\mathsf{Aut}_{\mathsf{Hopf}}(H)] \otimes U\big(\mathsf{Der}_{\mathsf{Hopf}}(H)\big)$. Then $[H]$ is in $\mathbf{Hopf}_{K,coc}$ and by Theorem 2.13 in [41], the action of $[H]$ on H given by the map

$$\star : [H] \otimes H \longrightarrow H$$

is still well-defined by

$$(\psi \otimes \phi) \star h := \psi\big(\phi(h)\big),$$

for any $\psi \in \mathsf{Der}_{\mathsf{Hopf}}(H)$ and $\phi \in \mathsf{Aut}_{\mathsf{Hopf}}(H)$. Hence, this action $\star$ defines the object $\overline{H} = H \rtimes_\star [H]$ which is in $\mathbf{Hopf}_K$.

Actually, the object $[H]$ can be seen as a kind of "relative" split extension classifier in $\mathbf{Hopf}_K$, where the universal property of $[H]$ can be expressed as follows: the split extension

$$\begin{tikzcd} 0 \arrow[r] & H \arrow[r, "i_1"] & \overline{H} \arrow[r, shift right, "p_2"'] & {[H]} \arrow[l, shift right, "i_2"'] \arrow[r] & 0 \end{tikzcd}$$

is such that, for any other split extension in $\mathbf{Hopf}_K$ with kernel H and $A \in \mathbf{Hopf}_{K,coc}$, depicted as

$$\begin{tikzcd} 0 \arrow[r] & H \arrow[r, "k"] & B \arrow[r, shift right, "f"'] & A \arrow[l, shift right, "s"'] \arrow[r] & 0, \end{tikzcd}$$

there are unique morphisms $\chi\colon A \to [H]$ and $\overline{\chi}\colon B \longrightarrow \overline{H}$ making the following diagram of split exact sequences commute

$$\begin{tikzcd} 0 \arrow[r] & H \arrow[r, "k"] \arrow[d, equal] & B \arrow[r, shift right, "f"'] \arrow[d, "\overline{\chi}"] & A \arrow[l, shift right, "s"'] \arrow[r] \arrow[d, "\chi"] & 0 \\ 0 \arrow[r] & H \arrow[r, "i_1"] & \overline{H} \arrow[r, shift right, "p_2"'] & {[H]} \arrow[l, shift right, "i_2"'] \arrow[r] & 0. \end{tikzcd}$$

3.2 Centers and centralizers in $\mathrm{Hopf}_{K,coc}$

Before investigating the notions of centers and centralizers in $\mathbf{Hopf}_{K,coc}$, we first recall these notions in the categories of groups and of Lie K-algebras.

3.2.1 Centers and centralizers in Grp

Recall that the *center of a group* G, denoted by $Z(G)$, is defined by

$$Z(G) = \{x \in G, xg = gx \ \forall g \in G\}.$$

The *centralizer* $C_G(H)$ *of a subgroup* H of a group G is the set of elements of G which commute with every element of H, i.e.

$$C_G(H) = \{x \in G, xh = hx \ \forall h \in H\}.$$

Proposition 3.2.1 The center of a group G is the kernel of the conjugation map, i.e. the kernel of the morphism $\phi : G \longrightarrow \mathrm{Aut}(G)$ where $\phi(x)$ is defined by $\phi(x)(y) = xyx^{-1}$, $\forall x, y \in G$. The centralizer of a subgroup H of a group G is the kernel of the morphism $\phi : H \longrightarrow \mathrm{Aut}(G)$, where $\phi(h)$ is defined by $\phi(h)(g) = hgh^{-1}$, $\forall h \in H$, $\forall g \in G$.

Definition 3.2.2 A *characteristic subgroup* of a group G is defined as a subgroup H of G which is invariant under all the automorphisms of G.

This means that any automorphism of G restricts to an automorphism of H. Since the automorphism group $\mathrm{Aut}(G)$ of a group G classifies all the group actions on G, a subgroup H of a group G is characteristic if and only if any group action χ on G restricts to an action on H.

Let us also mention the following "transitivity properties" of characteristic subgroups which will allow us to consider the more general case in the category of cocommutative Hopf algebras.

Proposition 3.2.3

1. If K is a characteristic subgroup of H and H is a characteristic subgroup of G, then K is a characteristic subgroup of G.

2. If K is a characteristic subgroup of H and H is a normal subgroup of G, then K is a normal subgroup of G. In particular every characteristic subgroup is a normal subgroup.

3.2.2 Centers and centralizers in $\mathbf{Lie}_K$

Recall that the *center of a Lie algebra* L, denoted by $Z(L)$, is defined by

$$Z(L) = \{x \in L, [x, a] = 0 \ \forall a \in L\}$$

The *centralizer* $C_L(I)$ *of a Lie subalgebra* I *of a Lie algebra* L is the set of elements of L commuting with every element of I, i.e. $C_L(I) = \{x \in L, [x, i] = 0 \ \forall i \in I\}$.

Proposition 3.2.4 The center of a Lie algebra L is the kernel of the adjoint representation, i.e. of the Lie algebra homomorphism defined by $ad : L \longrightarrow End(L) : ad(x) := [x, -]$. The centralizer of a Lie subalgebra I of a Lie algebra L is the kernel of the adjoint representation, i.e. $ad : I \longrightarrow End(L) : ad(l) := [l, -]$.

Definition 3.2.5 A *characteristic ideal* I *of a Lie algebra* L is a subalgebra which is invariant under all the derivations on L.

Similarly to the case of characteristic subgroups in the category of groups, the characteristic ideals also have some useful properties:

Proposition 3.2.6
1. If I is a characteristic Lie subalgebra of L and L is a characteristic Lie subalgebra of M, then I is a characteristic Lie subalgebra of M.
2. If I is a characteristic Lie subalgebra of L and L is a normal Lie subalgebra of M, then I is a normal Lie subalgebra of M.

3.2.3 Centers and centralizers in semi-abelian action representable categories

A. Cigoli and S. Mantovani gave a description of the center and of the centralizer of a normal subobject in the general context of semi-abelian action representable categories [24]. Let $x : X \longrightarrow A$ be a normal subobject of A, the quotient map $p : A \longrightarrow A/X$, and let $\big(Eq(p), p_1, p_2\big)$ be the equivalence relation on A induced by X, i.e. the projections in the following pullback (the kernel pair of the quotient map p)

$$\begin{tikzcd}
Eq(p) \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\lrcorner"] & A \arrow[d, "p"] \\
A \arrow[r, "p"'] & A/X
\end{tikzcd}$$

We write $\Delta : A \longrightarrow Eq(p)$ for the only morphism such that $p_1 \circ \Delta = 1_A = p_2 \circ \Delta$.

Definition 3.2.7 In any semi-abelian action representable category, the *centralizer of a normal subobject* X in A, denoted by $C_A(X)$ is given by the kernel of the unique morphism χ induced by the universal property of the split extension classifier $[X]$ of X:

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "HKer(p_1)"] \arrow[d, equal] & Eq(p) \arrow[r, shift left, "p_1"'] \arrow[d, "\overline{\chi}"'] \arrow[dr, phantom, "\lrcorner"] & A \arrow[l, shift left, "\Delta"'] \arrow[r] \arrow[d, "\chi"] & 0 \\
0 \arrow[r] & X \arrow[r, "i_1"] & X \rtimes_\star [X] \arrow[r, shift right, "p_2"'] & {[X]} \arrow[l, shift right, "i_2"'] \arrow[r] & 0
\end{tikzcd} \tag{3.2.1}$$

Example 3.2.8 In the case of the category of groups (resp. of the category of Lie algebras), the upper split extension sequence in the Diagram 3.2.1 of Definition 3.2.7 gives rise to the conjugation action (resp. to the adjoint representation) of A on X, thus we recover the classical notion of centralizer by Propositions 3.2.1 and 3.2.4.

Definition 3.2.9 [25] Let **C** be a semi-abelian category, F is a *characteristic* subobject of H if, given any normal subobject G of H, then F is a normal subobject of H.

Remark 3.2.10 Definition 3.2.9 is equivalent to Definitions 3.2.2 and 3.2.5 in the category of groups and in the category of Lie K-algebras, respectively.

3.2.4 Centers and centralizers in $\mathbf{Hopf}_{K,coc}$

In the category of cocommutative Hopf algebras, the *centralizer of a normal Hopf subalgebra* H of A, denoted by $C_A(H)$, is given by the kernel of χ in the following morphism of split extensions, where χ is defined as in Section 3.1.4.

$$\begin{tikzcd}
0 \arrow[r] & H \arrow[r, "HKer(p_1)"] \arrow[d, equal] & Eq(p) \arrow[d, "\overline{\chi}"'] \arrow[r, shift right, "p_1"'] & A \cong U(L_A) \rtimes_A K[G_A] \arrow[l, shift right, "\Delta"'] \arrow[r] \arrow[d, "\chi=U(\chi_L)\otimes K[\chi_G]"] & 0 \\
0 \arrow[r] & H \arrow[r, "i_1"] & H \rtimes_\star [H] \arrow[r, shift right, "p_2"'] & {[H] = U\big(\mathsf{Der}_{\mathsf{Hopf}}(H)\big) \rtimes_{\overline{\rho}} K\big[\mathsf{Aut}_{\mathsf{Hopf}}(H)\big]} \arrow[l, shift right, "i_2"'] \arrow[r] & 0
\end{tikzcd}$$

So to describe the centralizer $C_A(H)$, we have to compute the kernel of χ in the category $\mathbf{Hopf}_{K,coc}$, denoted by $HKer(\chi)$. We will show that

$$HKer(\chi) = HKer\big(U(\chi_L) \otimes K[\chi_G]\big) \cong HKer\big(U(\chi_L)\big) \rtimes_A HKer\big(K[\chi_G]\big).$$

This is proved by computing the kernels of $U(\chi_L)$, χ and $K[\chi_G]$ in the following diagram

$$\begin{tikzcd}
0 \arrow[r] & U(L_A) \arrow[r, "i_A"] \arrow[d, "U(\chi_L)"'] & U(L_A) \rtimes_A K[G_A] \arrow[r, shift right, "p_A"'] \arrow[d, "\chi=U(\chi_L)\otimes K[\chi_G]"'] & K[G_A] \arrow[l, shift right, "s_A"'] \arrow[r] \arrow[d, "K[\chi_G]"] & 0 \\
0 \arrow[r] & U\big(\mathsf{Der}_{\mathsf{Hopf}}(H)\big) \arrow[r, "i_{[H]}"] & U\big(\mathsf{Der}_{\mathsf{Hopf}}(H)\big) \rtimes_{\overline{\rho}} K\big[\mathsf{Aut}_{\mathsf{Hopf}}(H)\big] \arrow[r, shift right, "p_{[H]}"'] & K\big[\mathsf{Aut}_{\mathsf{Hopf}}(H)\big] \arrow[l, shift right, "s_{[H]}"'] \arrow[r] & 0
\end{tikzcd}$$

obtaining the following diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & HKer\big(U(\chi_L)\big) \arrow[r, "i"] \arrow[d, "HKer(U(\chi_L))"'] & HKer(\chi) \arrow[r, shift right, "p"'] \arrow[d, "HKer(\chi)"'] & HKer\big(K[\chi_G]\big) \arrow[l, shift right, "s"'] \arrow[r] \arrow[d, "HKer(K[\chi_G])"] & 0 \\
0 \arrow[r] & U(L_A) \arrow[r, "i_A"] \arrow[d, "U(\chi_L)"'] & U(L_A) \rtimes_A K[G_A] \arrow[r, shift right, "p_A"'] \arrow[d, "\chi=U(\chi_L)\otimes K[\chi_G]"'] & K[G_A] \arrow[l, shift right, "s_A"'] \arrow[r] \arrow[d, "K[\chi_G]"] & 0 \\
0 \arrow[r] & U\big(\mathsf{Der}_{\mathsf{Hopf}}(H)\big) \arrow[r, "i_{[H]}"] & U\big(\mathsf{Der}_{\mathsf{Hopf}}(H)\big) \rtimes_{\overline{\rho}} K\big[\mathsf{Aut}_{\mathsf{Hopf}}(H)\big] \arrow[r, shift right, "p_{[H]}"'] & K\big[\mathsf{Aut}_{\mathsf{Hopf}}(H)\big] \arrow[l, shift right, "s_{[H]}"'] \arrow[r] & 0
\end{tikzcd}$$

where the first row is easily seen to be a split short exact sequence. Let us show now that

$$HKer\big(K[\chi_G]\big) = K\big[ker_{Grp}(\chi_G)\big]$$

and

$$HKer\big(U(\chi_L)\big) = U\big(ker_{Lie}(\chi_L)\big)$$

Proposition 3.2.11 Let $f : G \longrightarrow G'$ be a morphism in $\mathbf{Grp}$ and $ker(f)$ its kernel in $\mathbf{Grp}$. Then the Hopf kernel $HKer(\overline{f})$ of $\overline{f} = K[f] : K[G] \longrightarrow K[G']$ in $\mathbf{Hopf}_{K,coc}$ is given by $K\big[ker(f)\big]$.

Proof. It is clear that we have $K\big[ker(f)\big] \subseteq HKer(\overline{f})$: indeed, let us consider an element $\alpha_i g_i$ where $g_i \in ker(f), \forall i \in I$. One has the following equalities

$$(\overline{f} \otimes id_A)\Delta(\alpha_i g_i) = \alpha_i f(g_i) \otimes g_i = 1 \otimes \alpha_i g_i$$

Another way to prove this inclusion is to apply the functor $K[-]$ to f, and to use the universal property of the kernel $HKer(\overline{f})$. To prove that $HKer(\overline{f}) \subseteq K\big[ker(f)\big]$, let us consider an element $\alpha_i g_i \in HKer(\overline{f})$, so we have that $\alpha_i\big(f(g_i) - 1\big) \otimes g_i = 0$, since g_i is a basis of $K[G]$, we obtain $\forall i \in I$ that $f(g_i) = 1$ if $\alpha_i \neq 0$, thus $\alpha_i g_i \in K\big[ker(f)\big]$. $\square$

Remark 3.2.12 An alternative proof for Proposition 3.2.11, valid only over algebraically closed fields K of characteristic zero, comes from the fact that the functor $K[-]$ preserves limits, thus in particular kernels, since it is a right adjoint to the functor $\mathcal{G}$ (see Proposition 1.4.7).

Proposition 3.2.13 Let $f : L \longrightarrow L'$ be a Lie K-algebra morphism and $ker(f)$ its kernel in the category of Lie K-algebras. Then the Hopf kernel $HKer(\overline{f})$ of $\overline{f} = U(f) : U(L) \longrightarrow U(L')$ in the category of cocommutative Hopf algebras is given by $U\big(ker(f)\big)$.

Proof. It is clear that we have $U\big(ker(f)\big) \subseteq HKer\big(U(f)\big)$, since we can apply the functor U to $f : L \longrightarrow L'$, and use the universal property of the kernel $HKer(\overline{f})$. For the converse direction, let $x \in U(L)$ be a generator, i.e. a primitive element, such that $x \in HKer\big(U(f)\big)$, we have

$$\begin{aligned}
(f \otimes id_A)(x \otimes 1 + 1 \otimes x) &= (u \circ \epsilon \otimes id_A)(x \otimes 1 + 1 \otimes x) \\
\text{then } f(x) \otimes 1 + 1 \otimes x &= 1 \otimes x \\
\text{so that } f(x) \otimes 1 &= 0, \text{ and } f(x) = 0.
\end{aligned}$$

Accordingly, x is a generator element of $U\big(ker(f)\big)$. $\square$

Remark 3.2.14 We actually have that the functor U preserves all binary products [40] and equalizers, indeed if $f, g : L \longrightarrow L'$ is a pair of Lie K-algebra morphisms and

$Eq(f,g)$ their equalizer in the category of Lie K-algebras. Then the Hopf equalizer [3] $HEqual(\overline{f},\overline{g})$ of $\overline{f} = U(f) : U(L) \longrightarrow U(L')$ and $\overline{g} = U(g) : U(L) \longrightarrow U(L')$ in the category of cocommutative Hopf algebras is given by $U\big(Eq(f,g)\big)$. Indeed we have

$$\begin{aligned}(f \otimes id_A)(x \otimes 1 + 1 \otimes x) &= (g \otimes id_A)(x \otimes 1 + 1 \otimes x)\\ \text{then } f(x) \otimes 1 + 1 \otimes x &= g(x) \otimes 1 + 1 \otimes x\\ \text{so that } (f-g)(x) \otimes 1 &= 0 \text{ and } (f-g)(x) = 0.\end{aligned}$$

Accordingly, x is a generator of $U\big(Eq(f,g)\big)$.

One can then compute $HKer(\chi)$ by computing $ker_{Lie}(\chi_L)$ and $ker_{Grp}(\chi_G)$ and obtain the following:

Theorem 3.2.15 *The centralizer $C_A(H)$ in the category of cocommutative Hopf algebras is given by*

$$C_A(H) = U\big(ker_{Lie}(\chi_L)\big) \rtimes_A K\big[ker_{Grp}(\chi_G)\big]$$

where

$$ker_{Lie}(\chi_L) = \{x \in L_A \text{ such that } [x,h] = 0, \text{ for all } h \in H\}$$

and

$$ker_{Grp}(\chi_G) = \{g \in G_A \text{ such that } gh = hg, \text{ for all } h \in H\}$$

Let us finally mention some results that follow from the corresponding ones in [25] in the general context of semi-abelian action representable categories.

Proposition 3.2.16 For every normal Hopf subalgebra K of H, the centralizer $Z_H(K)$ of K in H is a normal Hopf subalgebra in H. Then the center $Z(H)$ is a characteristic Hopf subalgebra of H.

Proposition 3.2.17

1. If K is a characteristic Hopf subalgebra of H, and H is a characteristic Hopf subalgebra of G, then K is characteristic Hopf subalgebra of G.
2. If K is a characteristic Hopf subalgebra of H, and H is a normal Hopf subalgebra of G, then K is a normal Hopf subalgebra of G.

3.2.5 Arbitrary Hopf algebra centers

A. Chirvasitu and P. Kasprzak have introduced the following definition of the center for any arbitrary (not necessarily cocommutative) Hopf algebra ([23]).

Definition 3.2.18 Let A be a Hopf algebra. The *Hopf algebra center* $HZ(A)$ of A is the largest Hopf subalgebra of A contained in

$$\overline{Z}(A) = \{x \in A \ \ s.t. \ \ y_1 x S(y_2) = x\epsilon(y) \ \ \forall y \in A\}$$

A. Chirvasitu and P. Kasprzak have added in [23] that in the case of bijective antipodes (which holds for instance in the cocommutative context), the Hopf algebra center of A can be described as:

$$HZ(A) = \{x \in A : \big(id \otimes \Delta\big)\big(\Delta(x)\big) \in A \otimes \overline{Z}(A) \otimes A\}$$

Remark 3.2.19 The center $\overline{Z}(A)$ coincides with the classical notion $Z_{alg}(A)$ of the center of A seen as an algebra, i.e. $Z_{alg}(A) = \{x \in A \ \ s.t. \ \ xy = yx \ \ \forall y \in A\}$. Indeed, to prove that $\overline{Z}(A) \subseteq Z_{alg}(A)$, let us consider $x \in \overline{Z}(A)$, then for all $y \in A$, we have $xy = x\epsilon(y_1)y_2 = y_1 x S(y_2) y_3 = y_1 x \epsilon(y_2) = yx$. To prove that $Z_{alg}(A) \subseteq \overline{Z}(A)$, let us consider $x \in Z_{alg}(A)$, then for all $y \in A$, we have $y_1 x S(y_2) = x y_1 S(y_2) = x\epsilon(y)$.

Proposition 3.2.20 For a cocommutative Hopf algebra A. Let $\chi \colon A \longrightarrow [A]$ denote the unique morphism associated to the action of A on itself by conjugation, and $\chi := U(\chi_L) \otimes K[\chi_G]$. We have

$$Z(A) = U\big(ker_{Lie}(\chi_L)\big) \rtimes_A K\big[ker_{Grp}(\chi_G)\big] = HZ(A)$$

where

$$ker_{Lie}(\chi_L) = \{x \in L_A \text{ such that } [x,a] = 0, \text{ for all } a \in A\}$$

and

$$ker_{Grp}(\chi_G) = \{g \in G_A \text{ such that } ga = ag, \text{ for all } a \in A\}.$$

Proof. Let us first prove that $HKer(\chi) = Z(A) \subseteq HZ(A)$. An element $x \in A$ is in $Z(A)$ if and only if $\chi(x_1) \otimes x_2 = id_{[A]} \otimes x \in [A] \otimes A$. By Remark 3.2.19, it suffices to prove that $Z(A) \subseteq Z_{alg}(A)$, i.e. $xy = yx$, $\forall y \in A$. For this, let us denote the action of $[A]$ on A by

$$\overline{\rho} : [A] \otimes A \to A, \overline{\rho}(u \otimes x) = u.x$$

and define the following map for any $y \in A$

$$r_y : [A] \otimes A \to A, r_y(u \otimes x) = (u.y)x$$

We have $r_y(id_{[A]} \otimes x) = id_A(y)x = yx$, and

$$\begin{aligned} r_y\big(\chi(x_1) \otimes x_2\big) &= \chi(x_1)(y)x_2 \\ &= x_1 y S(x_2) x_3 \\ &= x_1 y \epsilon(x_2) \\ &= xy. \end{aligned}$$

Since $Z(A)$ is a Hopf subalgebra of A, we obtain $Z(A) \subseteq HZ(A)$.

On the other hand one sees that $HZ(A) \subseteq Z(A)$ by applying the universal property of the kernel of $\chi = U(\chi_L) \otimes K[\chi_G]$ which is $Z(A)$, since we have the equalities

$$\chi(a)(h) = a_1 h S(a_2) = h a_1 S(a_2) = h\epsilon(a) = \epsilon(a) id_{[H]}(h), \quad \forall a \in HZ(A), \ \forall h \in [H].$$

□

Chapter 4

Research perspectives

4.1 Non-associative cocommutative Hopf algebras

A crucial point in our proof of Theorem 2.1.8 which states that $\mathbf{Hopf}_{K,coc}$ is a semi-abelian category is the Cartier-Gabriel-Kostant-Milnor-Moore decomposition (Theorem 1.4.5) which cannot be extended to the context of Hopf algebras over a field of non-zero characteristic (see Section 4.2) using the counter-example of divided power Hopf algebras (see Section 4.2). This theorem has been extended by V.N. Zhelyabin [55] to the non-associative context, namely to the category of weakly associative, coassociative and cocommutative Moufang H-bialgebras, over a field of characteristic zero, that we will denote by $\mathbf{MHBialg}_{K,coc,wa}$.

The H-bialgebras, introduced by J.M. Izquierdo in [49], can be tought of as non-associative Hopf algebras: associative H-bialgebras are exactly Hopf algebras. Loop algebras and universal enveloping algebras of Mal'tsev algebras are important examples of H-bialgebras. The Mal'tsev algebras were introduced by A.O. Mal'tsev in [39] as the tangent algebras to locally analytic Moufang loops, they are some generalization of Lie algebras to the non-associative context. The H-bialgebras, satisfying the Moufang identities for H-bialgebras are called Moufang H-bialgebras.

We will investigate in this section some of categorical constructions in $\mathbf{MHBialg}_{K,coc,wa}$.

Definitions 4.1.1
- Let us consider a non-empty set Q equipped with a map $M\colon Q \times Q \longrightarrow Q$ called the multiplication M, and left and right divisions

denoted respectively by $\backslash$ and $/$. The set Q is called a *quasigroup* if $a(a\backslash b) = a\backslash(ab) = b$ and $(b/a)a = (ba)/a = b$, $\forall a, b \in Q$. In case that Q admits an identity element e, the quasigroup Q is said to be a *loop*. Morphisms of loops are maps which preserve the multiplication law (as they always preserve left and right divisions). The category of loops will be denoted by $\mathbf{Loop}$.

- A loop $(Q, M, e, \backslash, /)$ is a *Moufang loop* provided that one of the following equivalent identities holds in Q, $\forall x, y, z \in Q$.

$$x\big(y(xz)\big) = \big((xy)x\big)z \qquad x\big((yz)x\big) = (xy)(zx)$$
$$\big((zx)y\big)x = z\big(x(yx)\big) \qquad \big(x(yz)\big)x = (xy)(zx)$$

  In a Moufang loop Q, each element $x \in Q$ has a two sided inverse $x^{-1} = e/x = x\backslash e$ (for a proof, see for instance [20]). The category of Moufang loops, a full subcategory of $\mathbf{Loop}$, will be denoted by $\mathbf{MLoop}$.

From now, algebras are not supposed to be associative.

Definition 4.1.2 An algebra M with a binary anticommutative bracket operation $[,]$ is said to be a *Mal'tsev algebra* if $[J(x,y,z),x] = J(x,y,[x,z])$ in M, where $J(x,y,z) = [[x,y],z] + [[y,z],x] + [[z,x],y]$ is the Jacobian in x, y and z. Mal'tsev algebra morphisms are maps which preserve the bracket operation. The category of Mal'tsev algebras over a field K will be denoted by $\mathbf{MAlg}_K$.

Remark 4.1.3 It is clear that any Lie algebra is a Mal'tsev algebra since the Jacobian is then equal to zero.

The following theorem, given in [18], describes when a variety (in the sense of the universal algebra) is protomodular.

Theorem 4.1.4 *A variety is pointed protomodular if and only if the corresponding theory contains a unique constant* 0 *and, for some natural number* $n \in \mathbb{N}$*, n binary operations* $\alpha_i(x,y) = 0$*; a (n+1)-ary operations* θ *such that*

$$\theta\big(\alpha_1(x,y), ..., \alpha_n(x,y), y\big) = x$$

Proposition 4.1.5 The categories of Moufang loops and of Mal'tsev algebras are semi-abelian.

Proof. By using Theorem 4.1.4, the category of Moufang loops is protomodular since we can take $n = 1$, $\alpha(x, y) = x/y$ and $\theta(x, y) = xy$. The category of Mal'tsev algebras is protomodular since we can take $n = 1$, $\alpha(x, y) = x - y$ and $\theta(x, y) = x + y$. Then that the categories $\mathbf{MLoop}$ and $\mathbf{MAlg}_K$ are semi-abelian since they are pointed and a variety is semi-abelian if and only if it is pointed and protomodular [13]. □

4.1.1 The category of Moufang H-bialgebras

The category we study in this section is the category of Moufang H-bialgebras over an algebraically closed field K of characteristic zero, denoted by $\mathbf{MHBialg}_K$. The objects in $\mathbf{MHBialg}_K$ are Moufang H-bialgebras, i.e. H-bialgebras satisfying the Moufang condition, defined by J.M. Pérez-Izquierdo in [49], which we are going to recall now.

In this section, a *bialgebra* B over a field K will mean a 5-tuple $(B, M, u, \Delta, \epsilon)$ where (B, M, u) is a non-necessarily associative and unital algebra and (B, Δ, ϵ) is a coassociative and counital coalgebra, such that these two structures are compatible, in the sense that maps M and u are coalgebra morphisms.

Definition 4.1.6

- A bialgebra $(B, M, u, \Delta, \epsilon)$ is an *H-bialgebra* provided that it is equipped with two bilinear mappings $\backslash\colon B \otimes B \to B$ and $/\colon B \otimes B \to B$ called respectively the *left and right divisions* and satisfying $\forall a, b \in B$

  $a_{(1)}\backslash(a_{(2)}b) = \epsilon(a)b = a_{(1)}(a_{(2)}\backslash b)$ and $(ba_{(1)})/a_{(2)} = \epsilon(a)b = (b/a_{(1)})a_{(2)}$

- A H-bialgebra $(B, M, u, \Delta, \epsilon, \backslash, /)$ is a *Moufang H-bialgebra* if the following identities hold $\forall x, y, z \in B$:

  $$\begin{aligned} x_{(1)}\big(y(x_{(2)}z)\big) &= \big((x_{(1)}y)x_{(2)}\big)z & \qquad \big((zx_{(1)})y\big)x_{(2)} &= z\big(x_{(1)}(yx_{(2)})\big) \\ x_{(1)}\big((yz)x_{(2)}\big) &= (x_{(1)}y)(zx_{(2)}) & \qquad \big(x_{(1)}(yz)\big)x_{(2)} &= (x_{(1)}y)(zx_{(2)}) \end{aligned}$$

  These identities are obtained by the linearization of Moufang identities on a Moufang loop, all these identities are also equivalent as shown in [49].

- Morphisms in the category $\mathbf{MHBialg}_K$ are morphisms of bialgebras, preserving left and right divisions.

- Let B be a H-bialgebra. We define by

  $$Q(B) := \{q \in B \mid \Delta(q) = q \otimes q, \epsilon(q) = 1\}$$

the set of *loop-like* elements of B, and by

$$\mathcal{P}(B) := \{b \in B \mid \Delta(b) = b \otimes 1 + 1 \otimes b\}$$

the set of *primitive* elements of B.

- The H-bialgebra B is said to be *weakly associative* provided that $\forall x, y \in \mathcal{P}(B), \forall g, h \in Q(B)$, we have

$$(x, y, g) = (xy)g - x(yg) = 0 = x(gh) - (xg)h = (x, g, h)$$

- The category of cocommutative and weakly associative Moufang H-bialgebras will be denoted by $\mathbf{MHBialg}_{K,coc,wa}$.

Proposition 4.1.7 [49] Let $(B, M, u, \Delta, \epsilon)$ be a bialgebra. There exists maps $\backslash\colon B \otimes B \to B$ and $/\colon B \otimes B \to B$ such that $(B, M, u, \Delta, \epsilon, \backslash, /)$ is an H-bialgebra if and only if the left and right multiplication are invertible in Hom$\big(B, End(B)\big)$. In that case the operations $\backslash$ and $/$ are uniquely determined.

The following proposition explains how H-bialgebras are a good generalization of Hopf algebras to the non-associative context.

Proposition 4.1.8 An associative H-bialgebra B is a Hopf algebra. Moreover, we have: $a\backslash b = S(a)b$ and $a/b = aS(b)$, $\forall a, b \in B$, where S denotes the antipode defined by $S(b) = b\backslash 1 = 1/b$, $\forall b \in B$.

J.M. Pérez-Iquierdo has also proved that the loop-like [49] and primitive elements [43] of H-bialgebras gives also a good generalization to the non-associative context.

Proposition 4.1.9 For any Moufang H-bialgebra $(B, M, u, \Delta, \epsilon, \backslash, /)$ the set $\big(Q(B), M, u, \backslash, /\big)$ is a Moufang loop.

Proposition 4.1.10 For any Moufang H-bialgebra $(B, M, u, \Delta, \epsilon, \backslash, /)$ the set $\big(\mathcal{P}(B), M, u, \backslash, /\big)$ is a Mal'tsev algebra.

Theorem 4.1.11 *(Non-associative Milnor-Moore Theorem) [49] Over a field of characteristic zero, any cocommutative H-bialgebra B generated by $\mathcal{P}(B)$ is isomorphic to the universal enveloping algebra of the Mal'tsev algebra $\mathcal{P}(B)$, i.e. $B \cong U\big(\mathcal{P}(B)\big)$.*

Theorem 4.1.11 and an extended version of the Cartier-Gabriel-Kostant-Milnor-Moore decomposition, proven by V.N. Zhelyabin [55], are very important elements for extending our second chapter's results to the non-associative context.

4.2 Assumption on the characteristic of the field K

Even if the protomodularity and the descriptions of binary products and equalizers in $\mathbf{Hopf}_{K,coc}$ doesn't involve the field characteristic, we would like to outline here that our proof giving that the category $\mathbf{Hopf}_{K,coc}$ is semi-abelian (Theorem 2.1.8) is crucially based on Theorem 1.4.16 which doesn't hold for fields of non-zero characteristic. Moreover, Newman's bijective correspondance between left bi-ideals and Hopf subalgebras in the cocommutative context (Theorem 1.2.36) is also based on Theorem 1.4.16.

Example 4.2.1 A counter-example to Theorem 1.4.16 for Hopf algebras over a non-zero characteristic field is given by the *divided power Hopf algebras* over a field of characteristic p (p prime). Let H be a K-vector space with basis $\{c_i,\, i \in \mathbb{N}\}$ on which we consider the following Hopf algebra structure:

$$\Delta(c_m) = \sum_{i=0}^{m} c_i \otimes c_{m-i}$$

and the counit given by $\epsilon(c_m) = \delta_{0,m}$ for any $m \in \mathbb{N}$. The multiplication given by $c_n c_m = \binom{n+m}{n} c_{n+m}$ for any $n, m \in \mathbb{N}$, extended by linearity to all H.

The bialgebra H has an antipode: since H is cocommutative, it suffices to show that there exists a linear map $S\colon H \longrightarrow H$ such that $\sum S(h_1)h_2 = \epsilon(h)1$ for any h in a basis of H. We define $\forall n \in \mathbb{N}$, $S(c_n) = (-1)^n c_n$. One can verify that H is irreducible as a coalgebra but it is not generated by primitive elements since $\mathcal{P}(H) = K[c_1]$.

Example 4.2.1 makes it difficult to extend Theorem 2.1.8 to the case of cocommutative Hopf algebras over an arbitrary field K.

Bibliography

[1] A. L. Agore. Categorical constructions for Hopf algebras. *Comm. Algebra.*, 39:1476 – 1481, 2011.

[2] A. L. Agore. Limits of coalgebras, bialgebras and Hopf algebras. *Proc. Amer. Math. Soc.*, 139:855 – 863, 2011.

[3] N. Andruskiewitsch and J. Devoto. Extensions of Hopf algebras. *Algebra i Analiz*, 7 (1):22 – 61, 1995.

[4] M. Barr. Exact categories. *1-120*, Exact categories and categories of sheaves, 1971.

[5] M. Barr. Coalgebras over a commutative ring. *Journal of Algebra*, 32:600–610, 1974.

[6] F. Borceux. *Handbook of Categorical Algebra 1: Basic Category Theory*, volume 50 of *Encyclopedia Math. Appl.* Cambridge Univ. Press, 1994.

[7] F. Borceux. *Handbook of Categorical Algebra 2: Categories and Structures*, volume 51 of *Encyclopedia Math. Appl.* Cambridge Univ. Press, 1994.

[8] F. Borceux and D. Bourn. *Mal'cev, Protomodular, Homological and Semi-Abelian Categories*, volume 566 of *Math. Appl.* Kluwer Acad. Publ., 2004.

[9] F. Borceux and D. Bourn. Split extension classifier and centrality. In *Categories in Algebra, Geometry and Math. Physics*, volume 431 of *Contemporary Math.*, pages 85–104, 2007.

[10] F. Borceux, M.M. Clementino, and A. Montoli. On the representability of actions for topological algebras. *Mathematical Texts (University of Coimbra)*, (46):41–66, 2014.

[11] F. Borceux, G. Janelidze, and G. M. Kelly. Internal object actions. *Commentationes Mathematicae Universitatis Carolinae*, 46(2):235 – 255, 2005.

[12] F. Borceux, G. Janelidze, and G.M. Kelly. On the representability of actions in a semi-abelian category. *Theory Appl. Categories*, 14 (11):244 – 286, 2005.

[13] D. Bourn. Normalization equivalence, kernel equivalence and affine categories. *Springer Berlin Heidelberg, Lecture Notes in Mathematics*, 1488:43–62, 1991.

[14] D. Bourn. Normal functors and strong protomodularity. *Theory and Applications of Categories*, 7(9):205 – 218, 2000.

[15] D. Bourn. 3x3 lemma and protomodularity. *Journal of Algebra*, 236(2):778 – 795, 2001.

[16] D. Bourn and M. Gran. Regular, protomodular and abelian categories in *Categorical Foundations: Special Topics in Order, Topology, Algebra, and Sheaf Theory*. *Encyclopedia of Mathematics and its Applications*, 97 :165–201, 2004.

[17] D. Bourn and M. Gran. Torsion theories in homological categories. *Journal of Algebra*, 305(1):18 – 47, 2006.

[18] D. Bourn and G. Janelidze. Characterization of protomodular vartieties of universal algebras. *Theory and applications of categories*, 11(6):143–147, 2003.

[19] D. Bourn and G. Janelidze. Centralizers in action accessible categories. *Cah. Topol. Géom. Différ. Catég.*, 50(3):211–232, 2009.

[20] R. H. Bruck. A survey of binary systems. *Springer Verlag, Berlin-Gottingen-Heidelberg*, 1958.

[21] A. Carboni, J. Lambek, and M. C. Pedicchio. Diagram chasing in Mal'cev categories. *Journal of Pure and Applied Algebra*, 69(3):271 – 284, 1991.

[22] A. Chirvasitu. On epimorphisms and monomorphisms of Hopf algebras. *Journal of Algebra*, 323 (5):1593 – 1606, 2010.

[23] A. Chirvasitu and P. Kasprzak. On the Hopf (co)center of a Hopf algebra. *eprint arXiv:1512.03749*, 12/2015.

[24] A. S. Cigoli and S. Mantovani. Action accessibility via centralizers. *Journal of Pure and Applied Algebra*, 216(8–9):1852 – 1865, 2012. Special Issue devoted to the International Conference in Category Theory 'CT2010'.

[25] A. S. Cigoli and A. Montoli. Characteristic subobjects in semi-abelian categories. *Theory and Applications of Categories*, 30(7):206 – 228, 2015.

[26] S. Dascalescu, C. Nastasescu, and S. Raianu. Hopf algebras, an introduction. *Pure and applied mathematics - A Series of Monographs and Textbooks*, 235, 2001.

[27] S. E. Dickson. A torsion theory for abelian categories. *Trans. Amer. Math. Soc.*, 86:47–62, 1998.

[28] M. Duckerts, T. Everaert, and M. Gran. A description of the fundamental group in terms of commutators and closure operators. *Journal of Pure and Applied Algebra*, 216(8–9):1837 – 1851, 2012.

[29] M. Gran and T. Everaert. Protoadditive functors, derived torsion theories and homology. *Journal of Pure and Applied Algebra*, 219(8):3629 – 3676, 2015.

[30] M. Gran, G. Kadjo, and J. Vercruysse. A torsion theory in the category of cocommutative Hopf algebras. *Applied Categorical Structures*, 24(3):269–282, 2016.

[31] P. J. Higgins. Groups with multiple operators. *Proceedings of the London Mathematical Society*, s3-6(3):366–416, 1956.

[32] H. Hopf. Fundamentalgruppe und zweite Bettische Gruppe. *Comment. Math. Helv.*, 14:257–309, 1942.

[33] G. Janelidze, L. Márki, and W. Tholen. Semi-abelian categories. *Journal of Pure and Applied Algebra*, 168(2–3):367 – 386, 2002.

[34] P.T. Johnstone. A note on the semiabelian variety of heyting semilattices. *Galois Theory Hopf Algebras and Semi-abelian Categories, Fields Inst. Commun.*, (43):317–318, 2004.

[35] I. Kaplansky. Bialgebras. *Department of Mathematics, University of Chicago - Lecture notes in mathematics*, 1975.

[36] S. Mac Lane. *Categories for the Working Mathematician*, volume 5 of *Grad. Texts in Math.* Springer, second edition, 1998.

[37] M. Aguiar S. Mahajan. Monoidal functors, species and Hopf algebras. *American Mathematical Society*, 29 , 2010.

[38] Shahn Majid. Hopf algebras for physics at the planck scale. *Class. Quantum Grav.*, 5:1587–1606, 1988.

[39] A. I. Mal'tsev. Analytic loops. *Mat. Sb.*, 36.3 :569–576, 1955.

[40] J. Milnor and J. Moore. On the structure of Hopf algebras. *Annals of Mathematics, Second Series*, 81 (2):211–264, 1965.

[41] R. K. Molnar. Semi-direct products of Hopf algebras. *J. Algebra*, 47 :29–51, 1977.

[42] S. Montgomery. Hopf Galois theory: a survey. *Geometry & Topology Monographs*, 16:367–400, 2009.

[43] P. J. Morandi, J. M. Pérez-Izquierdo, and S. Pumplün. On the tensor product of composition algebras. *Journal of Algebra*, 243(1):41 – 68, 2001.

[44] C. Nastasescu and B. Torrecillas. Torsion theories for coalgebras. *Journal of Pure and Applied Algebra*, 97 (2):203 – 220, 1994.

[45] K. Newman. A correspondence between bi-ideals and sub-Hopf algebras in cocommutative Hopf algebras. *Journal of Algebra*, 36(1):1 – 15, 1975.

[46] W.D. Nichols and M.B. Zoeller. A Hopf algebra freeness theorem. *American Journal of Mathematics*, 111 :381–385, 1989.

[47] H-E. Porst. Limits and colimits of Hopf algebras. *Journal of Algebra*, 328 (1):254 – 267, 2011.

[48] H-E. Porst. On subcategories of the category of Hopf algebras. *Arabian Journal for Science and Engineering*, 36 (6):1023–1029, 2011.

[49] J. M. Pérez-Izquierdo. Algebras, Hyperalgebras, nonassociative Bialgebras and Loops. *Advances in Mathematics*, 208(2):834 – 876, 2007.

[50] S.Caenepeel, G. Militaru, and S. Zhu. Frobenius and separable functors for generalized module categories and nonlinear equations. *Springer-Verlag Berlin Heidelberg - Lecture Notes in Mathematics*, 1787 , 2002.

[51] R. Street. Quantum groups: A path to current algebra. *Australian Mathematical Society Lecture Series (Book 19), Cambridge University Press*, 2007.

[52] M. E. Sweedler. Hopf algebras. *Benjamin New York*, 1969.

[53] M. Takeuchi. A correspondence between Hopf ideals and sub-Hopf algebras. *Manuscripta Math.*, 7:252–270, 1972.

[54] C. Vespa and M. Wambst. On some properties of the category of cocommutative Hopf algebras. *arXiv:1502.04001*, 2015.

[55] V. N. Zhelyabin. Universal envelopes of Malcev algebras: pointed Moufang bialgebras. *Siberian Mathematical Journal*, 50(6):1011–1026, 2009.