On the representability of actions of non-associative algebras

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Actions / Split extensions of groups

- Let B, X be groups. An action of B on X is a map

$$\xi\colon B \times X \to X\colon (b,x) \mapsto b \cdot x$$

  such that $(bb')\cdot x = b\cdot(b'\cdot x)$, $1_B \cdot x = x$ and $b\cdot(xx') = (b\cdot x)(b\cdot x')$.
- One can construct the *semi-direct product* $B \ltimes X$ with respect to ξ, that is $B \times X$ with multiplication

$$(b,x)(b',x') = (bb', x(b\cdot x')),$$

  where $1_{B\ltimes X} = (1_B, 1_X)$ and $(b,x)^{-1} = (b^{-1}, b^{-1}\cdot x^{-1})$.
- We can associate with ξ the split extension

$$\begin{tikzcd} 1 \arrow[r] & X \arrow[r, "i_2"] & B \ltimes X \arrow[r, shift left, "\pi_1"] & B \arrow[l, shift left, "i_1"] \arrow[r] & 1. \end{tikzcd}$$

- Conversely, given

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "k"] & A \arrow[r, shift left, "\alpha"] & B \arrow[l, shift left, "\beta"] \arrow[r] & 0 \end{tikzcd}$$

  then we can define an action

$$B \times X \to X \colon (b, x) \mapsto \beta(b)k(x)\beta(b)^{-1}$$

  and we have a bijection between $\mathrm{Act}(B, X)$ and $\mathrm{SplExt}(B, X)$.
- Split extensions of groups are in bijection with the homomorphisms $B \to \mathrm{Aut}(X)$: given a split extension of groups as above, one can define

$$\varphi \colon B \to \mathrm{Aut}(X) \colon b \mapsto \varphi_b,$$

  where $\varphi_b(x) = \beta(b)k(x)\beta(b)^{-1}$.
- Conversely, any homomorphism $\varphi \colon B \to \mathrm{Aut}(X)$ defines a semi-direct product $B \ltimes X$ with multiplication

$$(b, x)(b', x') = (bb', x\varphi_b(x'))$$

  and thus a split extension of B by X.

The semi-abelian context

- Semi-abelian categories were introduced in order to provide a categorical setting which would capture algebraic properties of groups, rings and algebras.
- Let $\mathcal{C}$ be a semi-abelian category (i.e. $\mathcal{C}$ is pointed, Barr-exact, Bourn-protomodular with finite coproducts).
- Let B, X be objects of $\mathcal{C}$. A *split extension* of B by X is a diagram

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "k"] & A \arrow[r, shift left, "\alpha"] & B \arrow[l, shift left, "\beta"] \arrow[r] & 0 \end{tikzcd}$$

in $\mathcal{C}$ such that $\alpha \circ \beta = 1_B$ and (X, k) is a kernel of α.

- For any object X in $\mathcal{C}$, we consider the functor

$$\mathrm{SplExt}(-, X) \colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$$

where $\mathrm{SplExt}(B, X)$ is the set of isomorphism classes of split extensions of B by X.
- We have a natural isomorphism

$$\mathrm{SplExt}(-, X) \cong \mathrm{Act}(-, X).$$

Action representable categories

Definition (F. Borceux, G. Janelidze and G. M. Kelly, 2005)

A semi-abelian category $\mathcal{C}$ is *action representable* if for every object X in $\mathcal{C}$, the functor $\mathrm{SplExt}(-, X)$ is representable. This means that there exists an object $[X]$ of $\mathcal{C}$, called the *actor* of X, and a natural isomorphism

$$\mathrm{SplExt}(-, X) \cong \mathrm{Hom}_{\mathcal{C}}(-, [X]).$$

- Prototype examples: **Grp** and **Lie**.
- In **Grp**, the actor of X is $\mathrm{Aut}(X)$.
- In **Lie**, the actor of X is $\mathrm{Der}(X)$.

- The notion of action representable category has proven to be quite restrictive.

Theorem (X. García Martínez, M. Tsishyn, T. Van der Linden and C. Vienne, 2021)

Let $\mathbb{F}$ be an infinite field with $\mathrm{char}(\mathbb{F}) \neq 2$ and let $\mathcal{V}$ be a variety of non-associative algebras over $\mathbb{F}$. If $\mathcal{V}$ is action representable, then $\mathcal{V} = \mathbf{AbAlg}$ or $\mathcal{V} = \mathbf{Lie}$.

- More recently G. Janelidze introduced the notion of *weakly action representable category*, which includes a wider class of semi-abelian categories.

Weakly action representable categories

Definition (G. Janelidze, 2022)

A semi-abelian category $\mathcal{C}$ is *weakly action representable* if for every object X in $\mathcal{C}$, the functor $\mathsf{SplExt}(-, X)$ admits a weak representation. This means that there exist an object T of $\mathcal{C}$ and a monomorphism of functors

$$\tau \colon \mathsf{SplExt}(-, X) \rightarrowtail \mathrm{Hom}_{\mathcal{C}}(-, T).$$

An object T as above is called *weak actor* of X, the pair (T, τ) is called *weak representation* of $\mathsf{SplExt}(-, X)$ and $(\varphi \colon B \to T) \in \mathrm{Im}(\tau_B)$ is called *acting morphism*.

Example

Every action representable category $\mathcal{C}$ is weakly action representable. In this case $T = [X]$ is the actor of X and τ is a natural isomorphism.

Associative algebras and Leibniz algebras

- **G. Janelidze, 2022.** The category **Assoc** is weakly action representable. A weak actor of X is the associative algebra of *bimultipliers* of X.

$$\mathrm{Bim}(X) = \{(f{*}-, -{*}f) \in \mathrm{End}(X) \times \mathrm{End}(X)^{\mathrm{op}} \mid \cdots$$

$$\cdots \mid f{*}(xy) = (f{*}x)y, (xy){*}f = x(y{*}f), x(f{*}y) = (x{*}f)y, \ \forall x, y \in X\}.$$

- **A. S. Cigoli, M. M., G. Metere, 2023.** The category **Leib** is weakly action representable. A weak actor of X is the Leibniz algebra of *biderivations* of X.

$$\mathrm{Bider}(X) = \{(d, D) \in \mathrm{End}(X)^2 \mid d(xy) = d(x)y + xd(y), \cdots$$

$$\cdots \ D(xy) = D(x)y - D(y)x, \ xd(y) = xD(y), \ \forall x, y \in X\}$$

A counterexample: Jordan algebras

Definition

A *Jordan algebra* over a field $\mathbb{F}$ is a commutative algebra $(X, \cdot)$ over $\mathbb{F}$ which satisfies the *Jordan identity*

$$(xy)(xx) = x(y(xx)), \ \ \forall x, y \in X.$$

Theorem (G. Janelidze, 2022)

Every weakly action representable category is action accessible.

Remark (A. S. Cigoli and S. Mantovani, 2012)

The category **Jord** of Jordan algebras over a field $\mathbb{F}$ is not action accessible.

Conclusion: Jord is not weakly action representable.

W.R.A. of non-associative algebras

- Let $\mathcal{V}$ be a variety of non-associative algebras over a field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \neq 2$;
- We suppose that $\mathcal{V}$ is *operadic*, i.e. all the identities of $\mathcal{V}$ are *multilinear*. This happens, for instance, when $\mathrm{char}(\mathbb{F}) = 0$.
- We also suppose that $\mathcal{V}$ is *action accessible*, or equivalently that $\mathcal{V}$ is an *Orzech category of interest*;
- This is equivalent to saying that the λ/μ *rules* hold, i.e. there exist $\lambda_1, \dots \lambda_8, \mu_1, \dots \mu_8 \in \mathbb{F}$ such that

$$x(yz) = \lambda_1(xy)z + \lambda_2(yx)z + \lambda_3 z(xy) + \lambda_4 z(yx) +$$

$$+\lambda_5(xz)y + \lambda_6(zx)y + \lambda_7 y(xz) + \lambda_8 y(zx),$$

$$(yz)x = \mu_1(xy)z + \mu_2(yx)z + \mu_3 z(xy) + \mu_4 z(yx) +$$

$$+\mu_5(xz)y + \mu_6(zx)y + \mu_7 y(xz) + \mu_8 y(zx)$$

are identities in $\mathcal{V}$.

- We fix a set of multilinear identities

$$\Phi_{k,i}(x_1, \dots, x_k) = 0, \quad k = \deg(\Phi_{k,i}),\ i = 1, \dots, n,$$

which determine $\mathcal{V}$.

- Our goal is to find a vector space $\mathcal{E}(X)$ such that

$$\mathrm{Inn}(X) = \{(L_x, R_x) \mid x \in X\} \leq \mathcal{E}(X) \leq \mathrm{End}(X)^2$$

and we want to endow it with a bilinear partial operation

$$\langle -, - \rangle \colon \Omega \subseteq X \times X \to X,$$

such that we can associate in a natural way a morphism of partial algebras $B \to \mathcal{E}(X)$, with every split extension of B by X in $\mathcal{V}$.

Definition

A *partial algebra* is a vector space X endowed with a *bilinear partial operation* $\Omega \subseteq X \times X \to X$. A homomorphism of partial algebras is a linear map $f \colon X \to Y$ such that $f(xy) = f(x)f(y)$, whenever xy and $f(x)f(y)$ are defined.

Split extensions

- Let

$$0 \longrightarrow X \xrightarrow{k} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B \longrightarrow 0$$

be a split extension in $\mathcal{V}$.

- We have that

$$A \cong B \ltimes X = (B \oplus X, \cdot)$$

with

$$(b, x) \cdot (b', x') = (bb', xx' + b * x' + x * b').$$

- The action of B on X is defined by the pair of bilinear maps

$$l \colon B \times X \to X, \quad r \colon X \times B \to X,$$

with $b * x' = l(b, x') = \beta(b) \cdot_A k(x')$, $x * b' = r(x, b') = k(x) \cdot_A \beta(b')$.

Conversely, if we define the product $(\cdot)$ on $B \oplus X$ as above:

Proposition

Let X and B be two algebras in $\mathcal{V}$. Given a pair of bilinear maps

$$l\colon B \times X \to X, \quad r\colon X \times B \to X,$$

we construct $(B \oplus X, \cdot)$ as before. Then $(B \oplus X, \cdot)$ is an object of $\mathcal{V}$ if and only if

$$\Phi_{k,i}(\alpha_1, \ldots, \alpha_k) = 0, \quad \forall i = 1, \ldots, n,$$

where at least one of the α_1, ..., α_k is an element of of the form $(0, x)$, with $x \in X$, and the others are of the form $(b, 0)$, with $b \in B$. The resulting algebra is the semi-direct product of B and X, denoted by $B \ltimes X$.

- We denote $b * x \coloneqq (b, 0) \cdot (0, x)$ and $x * b \coloneqq (0, x) \cdot (b, 0)$.

The partial algebra $\mathcal{E}(X)$

$\mathcal{E}(X)$ is the subspace of all pairs $(f * -, - * f) \in \mathrm{End}(X)^2$ satisfying

$$\Phi_{k,i}(\alpha_1, \dots, \alpha_k) = 0, \quad \forall i = 1, \dots, n,$$

for each choice of $\alpha_j = f$ and $\alpha_t \in X$, where $t \neq j \in \{1, \dots, k\}$ and $fx := f * x$, $xf := x * f$.
We define $\langle -, - \rangle \colon \mathcal{E}(X) \times \mathcal{E}(X) \to \mathrm{End}(X)^2$

$$\langle (f * -, - * f), (g * -, - * g) \rangle = (h * -, - * h),$$

where

$$\begin{aligned} x * h = & \lambda_1 (x * f) * g + \lambda_2 (f * x) * g + \lambda_3 g * (x * f) + \lambda_4 g * (f * x) \\ & + \lambda_5 (x * g) * f + \lambda_6 (g * x) * f + \lambda_7 f * (x * g) + \lambda_8 f * (g * x) \end{aligned}$$

and

$$\begin{aligned} h * x = & \mu_1 (x * f) * g + \mu_2 (f * x) * g + \mu_3 g * (x * f) + \mu_4 g * (f * x) \\ & + \mu_5 (x * g) * f + \mu_6 (g * x) * f + \mu_7 f * (x * g) + \mu_8 f * (g * x). \end{aligned}$$

Example

If $\mathcal{V} = \textbf{Assoc}$, then $\mathcal{E}(X) = \mathrm{Bim}(X)$. If $\mathcal{V} = \textbf{Leib}$, then $\mathcal{E}(X) \cong \mathrm{Bider}(X)$.

Remark

A split extension of B by X in $\mathcal{V}$ is the same as a linear map $B \to \mathcal{E}(X)$, defined by $b \mapsto (b * -, - * b)$, such that

$$((bb') * -, - * (bb')) = \langle (b * -, - * b), (b' * -, - * b') \rangle$$

and $\Phi_{k,i}(\alpha_1, \ldots, \alpha_k) = 0$, for $i = 1, \ldots, n$, where α_1, ..., α_k are as before.

Remark

The bilinear map $\langle -, - \rangle \colon \mathcal{E}(X) \times \mathcal{E}(X) \to \mathrm{End}(X)^2$ defines a partial operation $\langle -, - \rangle \colon \Omega \to \mathcal{E}(X)$, where Ω is the preimage $\langle -, - \rangle^{-1}(\mathcal{E}(X))$ of the inclusion $\mathcal{E}(X) \hookrightarrow \mathrm{End}(X)^2$.

Theorem (J. Brox, X. García Martínez, M. M., T. Van der Linden and C. Vienne, 2024)

*Let $\mathcal{V}$ be an action accessible, operadic variety of non-associative algebras over a field $\mathbb{F}$ and let $U\colon \mathcal{V} \to$ **PAlg** denote the forgetful functor.*

- *Let X be an object of $\mathcal{V}$. There exists a monomorphism of functors*

$$\tau\colon\ \mathsf{SplExt}(-,X) \rightarrowtail \mathsf{Hom}_{\mathbf{PAlg}}(U(-),\mathcal{E}(X))$$

  and, for every object B of $\mathcal{V}$, τ_B is the injection which sends an element of $\mathsf{SplExt}(B,X)$ to the corresponding partial algebra homomorphism

$$B \to \mathcal{E}(X)\colon\ b \mapsto (b * -, - * b).$$

- *Let B, X be objects of $\mathcal{V}$. The homomorphism of partial algebras $B \to \mathcal{E}(X)\colon b \mapsto (b * -, - * b)$ belongs to $\mathsf{Im}(\tau_B)$ if and only if $\Phi_{k,i}(\alpha_1, \ldots, \alpha_k) = 0$, as before.*

Theorem

- *If* $(\mathcal{E}(X), \langle -,- \rangle)$ *is an object of* $\mathcal{V}$*, then* $(\mathcal{E}(X), \tau)$ *is a weak representation of* $\mathsf{SplExt}(-,X)$*.*
- *If* $\mathcal{V}$ *is a variety of commutative or anti-commutative algebras, then* $\mathcal{E}(X)$ *is isomorphic to the partial algebra*

  $$\{f \in \mathrm{End}(X) \mid \Phi_{k,i}(f, x_2, \ldots, x_k) = 0,\ \forall x_2, \ldots, x_k \in X,\ \forall i = 1, \ldots, n\}$$

  endowed with the bilinear partial operation $\langle f, g \rangle = \alpha(f \circ g) + \beta(g \circ f)$*, where* α*,* $\beta \in \mathbb{F}$ *are given by the* λ/μ *rules.*

Definition

The partial algebra $\mathcal{E}(X)$ is called *external weak actor* of X. The pair $(\mathcal{E}(X), \tau)$ is called *external weak representation* of $\mathsf{SplExt}(-,X)$. When τ is a natural isomorphism, we say that $\mathcal{E}(X)$ is an *external actor* of X.

Example

We may check that, with the *obvious* choices of the λ/μ rules:

1. if $\mathcal{V} = \mathbf{AbAlg}$, then $\mathcal{E}(X) = 0$ is the actor of X;
2. if $\mathcal{V} = \mathbf{CAssoc}$, then $\mathcal{E}(X) \cong \mathrm{M}(X)$ is an external actor of X;
3. if $\mathcal{V} = \mathbf{JJord}$, then the external actor is isomorphic to the partial algebra $\mathrm{ADer}(X)$ of anti-derivations of X;
4. if $\mathcal{V} = \mathbf{Lie}$, then $\mathcal{E}(X) \cong \mathrm{Der}(X)$ is the actor of X;
5. if $\mathcal{V} = \mathbf{Nil}_2(\mathbf{CAlg})$, then $\mathcal{E}(X)$ is a weak actor of X and it is an abelian algebra;
6. if $\mathcal{V} = \mathbf{Alt}$ over a field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \neq 2$ and X is unitary, then $\mathcal{E}(X) \cong X$ is an alternative algebra and

$$\mathrm{SplExt}(-, X) \cong \mathrm{Hom}_{\mathbf{Alt}}(-, X)$$

i.e. X is the actor of itself. In particular, the algebra of octonions $\mathbb{O}$ has representable actions.

What happens if $(\mathcal{E}(X), \langle -, - \rangle)$ is not an object of $\mathcal{V}$?

Example

If $\mathcal{V} = \mathbf{Leib}$, then $\mathcal{E}(X) \cong \mathrm{Bider}(X)$ as vector spaces. Choosing the λ/μ rules as

$$x(yz) = (xy)z - (xz)y, \qquad (yz)x = (yx)z - y(xz),$$

we get the standard multiplication of $\mathrm{Bider}(X)$ that defines a weak actor in **Leib**. If we choose the λ/μ rules as

$$x(yz) = (xy)z - (xz)y, \qquad (yz)x = (yx)z + y(zx),$$

we get a non-associative multiplication which, in general, does not define a Leibniz algebra structure on $\mathrm{Bider}(X)$.

Commutative associative algebras

Remark

Let X be a commutative associative algebra. Then

$$\mathsf{SplExt}(-, X) \cong \mathsf{Hom}_{\mathbf{Assoc}}(U(-), \mathsf{M}(X)),$$

where $\mathsf{M}(X)$ is the associative algebra of *multipliers* of X and $U\colon \mathbf{CAssoc} \to \mathbf{Assoc}$ denotes the forgetful functor.

Theorem (F. Borceux, G. Janelidze, G. M. Kelly, 2005)

Let X be a commutative associative algebra. Then $\mathsf{M}(X)$ is a commutative algebra if and only if the functor $\mathsf{SplExt}(-, X)$ is representable.

Corollary (F. Borceux, G. Janelidze, G. M. Kelly, 2005)

*The variety **CAssoc** is not action representable.*

Amalgamation Property (AP)

Definition

For a diagram in a category $\mathcal{C}$, an *amalgam* is a monic cocone, i.e. a cocone which is a monomorphic natural transformation.

Definition

A category with pushouts $\mathcal{C}$ has the *amalgamation property* (AP) when each span of monomorphisms admits an amalgam, i.e. for each pushout square

$$\begin{tikzcd} I \arrow[r, "t"] \arrow[d, "s"'] & T \arrow[d, "\iota_T"] \\ S \arrow[r, "\iota_S"'] & S +_I T \end{tikzcd}$$

if s and t are monomorphisms then so are ι_S and ι_T.

- **BooRg**, **Grp** and **Lie** satisfy the condition (AP).
- **CAssoc** and **Assoc** don't satisfy the condition (AP).

Representability of actions and (AP)

Proposition (F. Borceux, G. Janelidze, G. M. Kelly, 2005)

Let X be an object of a semi-abelian category $\mathcal{C}$. If $\mathrm{SplExt}(-, X)$ *is representable, then the (AP) holds in $\mathcal{C}$ for all protosplit monomorphisms with domain X, i.e. monomorphisms which form the kernel part of a split extension in $\mathcal{C}$ with kernel X.*

Remark (F. Borceux, G. Janelidze, G. M. Kelly, 2005)

The converse is not true in general: if $\mathcal{C} = \mathbf{Alg}$, then the (AP) holds for protosplit monomorphisms but the actions of an object X are representable if and only if X is an abelian algebra.

Remark (F. Borceux, G. Janelidze, G. M. Kelly, 2005)

If $\mathcal{C}$ is an Orzech category of interest, then the representability of actions is equivalent to the (AP) for protosplit monomorphisms.

W.R.A. and (AP)

Remark

The associative algebra $\mathrm{M}(X)$ is an amalgam in **Assoc** of the diagram consisting of the commutative subalgebras M_ξ of $\mathrm{M}(X)$.

Proposition (J. Brox, X. García Martínez, M. M., T. Van der Linden and C. Vienne, 2024)

For a commutative associative algebra X, a weak representation (T, τ) of $\mathrm{SplExt}(-, X)$ *exists if and only if an amalgam in* **CAssoc** *exists for the diagram of commutative subalgebras of* $\mathrm{M}(X)$.

Theorem (J. Brox, X. García Martínez, M. M., T. Van der Linden and C. Vienne, 2024)

If $S \leftarrow I \rightarrow T$ is a span of commutative associative algebras for which an amalgam exists in **Assoc**, *then it has an amalgam in* **CAssoc**.

Corollary

An amalgam in **CAssoc** *exists for any diagram of commutative associative algebras for which an amalgam exists in* **Assoc**.

Theorem (J. Brox, X. García Martínez, M. M., T. Van der Linden and C. Vienne, 2024)

The category **CAssoc** *of commutative associative algebras is weakly action representable.*

Theorem (J. Brox, X. García Martínez, M. M., T. Van der Linden and C. Vienne, 2024)

Let X be an object of a variety of non-associative algebras $\mathcal{V}$. If the diagram in $\mathcal{V}$ given by those subalgebras of $\mathcal{E}(X)$ which occur as a codomain of a morphism $(B \to \mathcal{E}(X)) \in \mathrm{Im}(\tau_B)$ admits an amalgam in $\mathcal{V}$, then the colimit of that diagram determines a weak representation of $\mathrm{SplExt}(-, X)$.

Open problems and future directions

- Does the converse of the implication

  weakly action representable category $\Rightarrow$ *action accessible category*

  hold in the context of varieties of algebras over a field?
- Is it possible to generalize the construction of the external weak actor to any operadic and action accessible variety of algebras with two not necessarily associative bilinear operations? We already have positive answers for the categories **Pois** and **CPois** of (commutative) Poisson algebras.
- We can use the construction of the external weak actor to study the representability of actions of varieties of unitary algebras, which are *ideally exact categories* in the sense of G. Janelidze. For instance, we have that the category of unitary associative algebras and that of unitary alternative algebras are action representable.

Thank you!

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