Strategic complementarities and endogenous heterogeneity in oligopolistic markets

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Acknowledgments:

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CHAPTER 1

INTRODUCTION

1.1 Problem: heterogeneity

Encompassing strategic relations among economic agents demands a meaningful typology of the underlying interactions. Strategic complementarity is one of the most important to be distinguished, both with respect to the scope of applications, and for methodological reasons. In this thesis strategic complementarity plays a central role in all four essays. We use this approach to modelling micro- and macroeconomic characteristics of markets, and with special attention to competition of oligopolistic type. The common problematic for all the essays is the issue of heterogeneity. In the two first papers we deal with endogenous heterogeneity, while the third and fourth paper concentrate on some market characteristics in models with product differentiation, hence the heterogeneity is in this case rather exogenous.

Understanding the origins and/or evolution of diversity across economic agents or disparities in economic performance across regions is increasingly perceived as a central goal of economic and social research in a number of different areas, see e.g. Geroski et al. (2003) and Ghemawat (1986). Macroeconomists attempt to explain the causes of booms and recessions. Development economists wish to understand the forces behind poor and strong economic performances. Labor economists attempt to get a handle on discriminatory treatment of some groups of workers. Business strategists and industrial economists devote a lot of attention to the sources and sustainability of inter-firm heterogeneity within and across industries. Overall, much effort has been expanded with a view to explain "the diversity across space, time and groups" (Matsuyama, 2002).

In industrial economics there are a number of models dealing in some way with strategic endogenous heterogeneity as connected with product differentiation. These are basically two-stage games where each firm chooses a quality level or a horizontal characteristic in the first stage, and then a price for its product in the second stage. Endogenous heterogeneity naturally emerges out of the firms' perception that identical choices in the first stage will lead to zero profits in the second stage Bertrand competition due to the resulting homogeneity of their products. Related literature is listed in Chapter 2. We aim there to provide an approach which generalizes this kind of models.

The next strand of literature that deals with symmetry-breaking or endogenous heterogeneity, is recent and has applications in various fields of economic inquiry including macroeconomics (Matsuyama, 2002), industrial organization (Amir and Wooders, 2000), and business strategy (Nelson, 1991). The key question in this literature is whether strategic interaction could turn ex ante identical players into necessarily heterogeneous actions at equilibrium. In different words, can symmetric games possess exclusively asymmetric equilibria, or in case both asymmetric and symmetric equilibria coexist, can one invoke some plausible arguments to select out the latter? We provide a contribution to this literature in Chapter 2 and 3.

Chapters 4 and 5 concentrate on one-stage models with differentiated products. Last essay deals with sufficient conditions for the Cournot competition model to display complementarities. This problem was solved for homogenous product case (Amir, 1996), and here we generalize that approach. Bertrand-type competition is considered in Chapter 4, where imperfect market transparency is assumed. We investigate whether an increase in the transparency level is always positive for consumers and negative for producers.

In all the announced problems we use the same methodology, based on games with strategic complementarity.

1.2 Methodology: complementarity

Consider a $n-$player game. The strategies of the agents display complementarity when an increase in the actions of some agents implies a larger marginal payoff to the others when they increase their own actions. In different words, we can say that the best replies are increasing. This kind of interactions can be described by supermodular payoff functions. There is also a dual notion of strategic substitutes. In this case, the best replies are decreasing since an increase in some agents' actions leads to a decrease in the marginal payoff of the others. Complementarity and substitutability can arise in various familiar economic settings: consumer theory, monopoly and oligopoly competition, but also in assortative matching and growth theory (cf. Amir, 2005).

While the importance of this class of games has been known to economists for a long time, see e.g. Bulow, Geanakoplos and Klemperer (1985) and Fudenberg and Tirole (1984), a complete understanding of their theoretical foundations as well as of their general properties has only recently been achieved with the advent of the lattice-theoretic approach, proposed by Topkis (1978) and further developed by several authors, which are listed for instance in Chapter 3.

A key feature of the games of strategic complementarity, which is invariably critical for economic applications, is that they always admit pure-strategy Nash equilibrium points, even if the best responses are not continuous in any way (or the payoff functions satisfy no quasi-concavity assumptions whatsoever). In addition, these games possess several other general properties that make them especially attractive in a variety of settings in economics, including, under quite general conditions, dominance solvability and convergence of a broad class of natural learning algorithms. Moreover, monotonic best replies enable to conduct comparative statics relying essentially on critical assumptions and dispensing with smoothness, interiority and concavity assumptions.

Increasing reaction correspondences characterize games with strategic complementarities, while games of strategic substitutes have downward sloping best replies. The former kind of games always possesses pure strategy Nash equilibria. The latter kind of games, in the case of $2-$player game, can be converted into the game of strategic complementarities by reordering of one player's action set (Milgrom and Roberts, 1990). Unfortunately, this trick does not work for $n-$player games, if $n > 2$. Hence, the existence of pure strategy Nash equilibria cannot be guaranteed.

1.3 Results

Chapter 2 of this thesis takes advantage of the complementarity methodology to address endogenous heterogeneity issues. Chapter 3 studies the properties of pure strategy Nash equilibria of symmetric supermodular games in a lattice theoretical context. Chapter 4 investigates an imperfectly transparent market in the framework of Bertrand model and reaction of prices and profits to the increase in transparency level. Chapter 5 provides general conditions for a Cournot oligopoly to have monotonic best replies, which is a direct generalization of Amir (1996) for the case of differentiated product. In the following paragraphs we describe in more details the main results of the respective chapters.

Chapter 2 considers a broad class of $2-$player symmetric games, which display a fundamental non-concavity when actions of both players are about to be the same. This implies that no symmetric equilibrium is possible. We distinguish different properties of the payoff functions, like strategic substitutes, complements and quasi-concavity, which are not necessarily imposed globally on the joint action space. For each of these cases we provide conditions to secure the existence of exclusively asymmetric equilibria. Moreover we consider the case of convex payoff functions. A number of applications from industrial organization and applied microeconomics literature are provided. This is joint work with R. Amir and F. Garcia.

In Chapter 3 we generalize to the extent possible the known results for the case of games with one-dimensional action sets to the general case of games with action spaces that are complete lattices. One key issue addressed is the extent to which all equilibria tend to be symmetric for the general case of multi-dimensional (i.e. only partially ordered) strategy spaces. We find that the scope for asymmetric equilibrium behavior is definitely broader than in the one-dimensional case, though still quite limited. Specifically, while asymmetric equilibria are possible even for strictly supermodular games, they cannot involve one player using a larger strategy (in the product order) than another player. For instance, in case each player's action is a Euclidean vector, there can be no pure strategy Nash equilibrium where the coordinates of one player's action are all larger than those of another player. Another key question investigated here is whether asymmetric pure strategy Nash equilibria are always Pareto dominated by symmetric pure strategy Nash equilibria. While this need not hold in general for games with strategic complementarities, we identify different sufficient conditions that guarantee that such dominance holds. This is joint work with R. Amir and M. Jakubczyk.

In Chapter 4 we deal with the effects of market transparency on prices in the Bertrand duopoly model. The analysis is intuitive and simple when we consider two types of strategic interaction between firms in an industry - strategic complementarities and substitutabilities. We present also traditional comparative statics analysis, demanding additionally some other regularity conditions, to cover those problems, when neither of these situations is the case. In the first case, the results are close to conventional wisdom, especially, when in the same time products are substitutes. Namely, equilibrium prices and profits are always decreasing in transparency level, while the consumer's surplus is increasing. For a special case when supermodularity holds, but products are not substitutes, the result on profits is not valid anymore. Considering price competition with strategic substitutes, an ambiguity in the direction of change of prices may appear. This leads to ambiguity concerning equilibrium profits and surplus changes caused by increasing transparency. Assuming in addition complementarity of the products, which promotes submodularity of the profits, enables us to conclude that consumer surplus increase, but firms profits remain ambiguous.

In Chapter 5 we provide general conditions for Cournot oligopoly with products differentiation to have monotonic reaction correspondences. We give a proof for the conditions stated by Vives (1999). Moreover we elaborate more general requirements. They allow for identifying increasing best responses even in case when inverse demand is submodular, and similarly, decreasing best responses in case of supermodular inverse demand. Examples illustrating the scope of applicability of these results are provided.

This approach gives a significant value added to the problem of equilibrium existence and comparative statics. The standard approach demands profit function to be quasi-concave in own quantities. This is quite restrictive, particularly because non-concavities in costs are not uncommon and very convex demand functions cannot be ruled out (Vives, 1999). The lack of quasi-concavity of payoffs causes discontinuities in the best response correspondences of firms and makes possible the nonexistence of equilibrium. Our approach covers games with monotonic reaction correspondences and does not rely on the regularity condition.

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CHAPTER 2

ENDOGENOUS HETEROGENEITY IN STRATEGIC MODELS: SYMMETRY BREAKING VIA STRATEGIC SUBSTITUTES AND NON-CONCAVITIES

This is joint work with Rabah Amir and Filomena Garcia.

2.1 Introduction

One of the most pervasive presumptions in modern economic analysis is the symmetric nature of interacting agents. While often intended solely as a simplifying assumption on *a priori* grounds, this presumption has also permeated economic thinking for a variety of other reasons. When considering noncooperative games, analysts often restrict attention to symmetric equilibrium points even when other asymmetric outcomes exist and may reflect more rationalizable or more pertinent behavior. In mechanism design or policy games, the social planner typically assumes identical treatment of identical agents, although global optimality might dictate otherwise. The design of various forms of joint ventures is also subject to a similar observation.

In most cases, the only justification beyond simplicity is what Schelling (1960) convincingly termed the focal nature of symmetric equilibrium outcomes. Indeed, it is widely recognized that inter-agent heterogeneity is often a critical dimension of several economic and social phenomena. From a positive perspective, heterogeneity is simply a necessary postulate to account for the simple fact that in the real world, one seldom observes identical agents, be it individuals, firms, industries or countries. In a similar vein, even from a normative standpoint, differences across interacting agents often constitute a necessary condition for many important economic activities such as trade or risk-sharing.

Understanding the origins and/or evolution of diversity across economic agents or disparities in economic performance across regions is increasingly perceived as a central goal of economic and social research in a number of different areas, see e.g. Geroski et al. (2003) and Ghemawat (1986). Macroeconomists attempt to explain the causes of booms and recessions. Development economists wish to understand the forces behind poor and strong economic performances. Labor economists attempt to get a handle on discriminatory treatment of some groups of workers. Business strategists and industrial economists devote a lot of attention to the sources and sustainability of inter-firm heterogeneity within and across industries. Overall, much effort has been expanded with a view to explain "the diversity across space, time and groups" (Matsuyama, 2002).

In view of the diversity of economic research areas involved in this effort, it is not surprising that various conceptual and methodological approaches have been developed in connection with this complex task. While often tailored to a specific area, each of these approaches is broad in explanatory scope and has wide potential applicability. We now briefly review three of these general paradigms that share some relationship to the present paper.

The dominant approach, based on coordination failures, postulates a game with strategic complementarities and multiple Pareto-ranked pure-strategy Nash equilibrium points. Diversity is then synonymous with making different equilibrium selections, with the high-performing entity picking the Pareto-dominant equilibrium and the low-performing entity failing to do so. This argument is thus generally predicated on the presence of two identical and non-interacting economies, each operating under a different equilibrium out of the same equilibrium set. It may also be invoked to explain diversity across time within the same economy, with booms and recessions corresponding to operation under the Pareto dominant and inferior equilibria respectively. This literature includes as key studies Cooper and John (1988), Murphy, Schleifer and Vishny (1989), and is surveyed by Cooper (1999).

The coordination failures approach has been criticized for failing to offer any compelling argument for the diversity in equilibrium selections in the two-economy model or for the regime switch in the one-economy model. Matsuyama (2002) proposes a modification of the former model by creating an interactive link between the two sub-economies and allowing the two players to take two decisions, one in each sub-economy. Under some conditions on the larger game with *a priori* identical players, namely that the players' actions are pair-wise strategic complements and each player's actions are substitutes due to a fixed total resource constraint, multiple equilibria arise with the property that the symmetric equilibrium is unstable while the two asymmetric equilibria are stable, both in the sense of Cournot dynamics. Endogenous heterogeneity in this approach is then predicated on the central postulate that only Cournot-stable equilibria are observable outcomes of this complex game, any of which would involve each agent taking different actions in the two *ex ante* identical sub-economies. Matsuyama (2000, 2002, 2004) coined the term "symmetry-breaking" to refer to this heterogeneity-generating process.

The third approach originates in the business strategy literature, and is often presented as part of a general critique of economic theory. With their traditional emphasis on investigating the workings of firms as complex organizations, strategy scholars have been particularly concerned with understanding the sources and dynamics of inter-firm heterogeneity along various functions and characteristics. In the dominant view, as articulated by Nelson and Winter (1982), firms operate in such highly complex and ever-changing environments that they entertain no hope of ever accumulating enough knowledge about their world to view it as a strategic game or formulate a precise game-theoretic strategy to guide their overall behavior. Rather, firms grope for economic performance via a heuristic learning process of trial and error and the continual updating of routines and rules of thumb eschewing optimization. In this evolutionary vision, heterogeneity is simply an inevitable outcome of this groping behavior, with firms ending up with different heuristic strategies and core capabilities to implement them. These "discretionary" differences can then be sustained over extended periods of time due to the presence of barriers to successful imitation generated by the differences in core competencies, and also by forces of path dependence in the evolution of firms' choices. This literature often criticizes economic theory for not adequately accounting for inter-firm differences, other than postulating them either as reflecting variations in initial conditions, or as exogenous consequences of the luck of a draw in stochastic models. This failure is attributed to the fact that economic theorists persist, as part of their excessive reliance on complete rationality, in "taking a firm's choice sets as obvious to it and the best choice similarly clear and obvious" (Nelson, 1991).[1]

The present paper constitutes an attempt to contribute to this rich debate along standard lines of argument in applied game theory and industrial organization. Consider a $2-$player symmetric

[1]An interesting development over the last two decades is a strand of literature straddling the traditional boundaries between industrial organization and business strategy and addressing issues of interest to both fields, making them increasingly related: See Shapiro (1989), Rumelt, Schendel and Teece (1991), Roller and Sinclair-Desgagne (1996).

normal-form game characterized by two key properties: (a) actions form strategic substitutes, and (b) each player's payoff, though continuous, admits a key non-concavity along the diagonal in action space, which results in a jump of the reaction correspondence across the 45^o line. Such a game always admits pure-strategy Nash equilibrium points due simply to the property of strategic substitutes. Furthermore, due to property (b), no such equilibrium could ever be symmetric. At any of the possibly multiple equilibria, which obviously occur in pairs due to the symmetry of the game, otherwise identical agents will necessarily take different equilibrium actions. While this description exactly fits the main result of the paper, we consider two other related classes of games that always possess asymmetric, but never symmetric, pure-strategy equilibria although they are, strictly speaking, not of strategic substitutes. This suggests that the latter property is not as critical as the diagonal non-concavity property in generating exclusively asymmetric outcomes.

Since payoffs are continuous in actions in all three classes of games under consideration, these games will typically admit a symmetric mixed-strategy Nash equilibrium (Dasgupta and Maskin, 1986). As this would be the only focal equilibrium in the sense of Schelling (1960), it may reasonably be advanced as a plausible outcome of such a game. Nevertheless, in the actual realization of the equilibrium randomizations, the players will still end up playing different actions with high, if not full, probability. Hence, given a focus on explaining observed heterogeneity, this approach need not rule out mixed strategies *a priori*.

Towards the goal of generating endogenous heterogeneity, this approach is obviously closest in spirit to Matsuyama's symmetry-breaking explanation. By allowing for suitable discontinuities in the players' reaction curves, it dispenses with the need to interconnect two separate games in the somewhat complex (and subtle) manner proposed by Matsuyama. More importantly, it also provides a framework that is independent of the somewhat controversial argument of outright rejecting Cournot-unstable equilibria. Indeed, even when one ignores the focal nature derived from their symmetry, it is worthwhile to observe that these equilibria cannot be ruled out on account of any of the standard Nash equilibrium refinements, such as normal-form perfection or strategic stability (Kohlberg and Mertens, 1986)[2]. Furthermore, in an experimental setting involving a symmetric $2-$player game with one unstable symmetric equilibrium and a pair of asymmetric equilibria, Cox and Walker (1998) found little support in the data for *any of the three equilibria.* This provocative finding suggests that while a Cournot-unstable equilibrium of a given game may be justifiably regarded as unobservable, it does not thereby follow that some Cournot-stable equilibrium of the same game will necessarily prevail and thus be observable. Rather, the presence of both Cournot-stable and unstable equilibria

[2] Indeed, they are typically strict Nash equilibria (in the sense that a unilateral deviation will lead to a strict loss for the deviator), and thus would survive any of the well-known refinements.

may engender a high level of indeterminacy, which may critically reduce the predictive power of the game.

Our findings may also be advanced as a rebuttal to the aforementioned criticism from the business strategy literature. Indeed, while sharing their motivation for understanding intra-industry heterogeneity, this approach underlies a general methodology for generating inter-firm differences out of a strategic game with fully rational and completely informed players. The contrast with the evolutionary explanation is rather striking. Instead of discretionary differences that inevitably arise out of the idiosyncratic heuristic response that each firm develops in isolation from other firms in response to its multi-faceted operation in an extremely complex environment, we uncover strategic differences that arise out of a fully-fledged game-theoretic interaction amongst firms in a simple and completely known environment. We will return to this contrast in the specific context of an R&D game in a subsequent section.

The present paper may also be motivated along more narrow lines, relating to various broad strands of literature in industrial economics dealing in some way with strategic endogenous heterogeneity along lines similar to ours here. The first literature that comes to mind is concerned with product differentiation. In a myriad of two-stage games where each firm chooses a quality level or a horizontal characteristic in the first stage, and then a price for its product in the second stage, endogenous heterogeneity naturally emerges out of the firms' perception that identical choices in the first stage will lead to zero profits in the second stage Bertrand competition due to the resulting homogeneity of their products.[3]

The second, extensive literature deals with infinite-horizon industry dynamics allowing for entry and exit. One class of models, exemplified by Jovanovic (1982), postulates perfectly competitive firms for which differences emerge due to *exogenous* idiosyncratic technology shocks. Another class is formed by studies that do generate endogenous heterogeneity in long run dynamics by considering firms that invest in capacity expansion (e.g. Besanko and Doraszelski, 2002) or R&D (e.g. Doraszelski and Satterthwaite, 2004). Simpler two-stage models with similar flavor but without entry and exit also generate endogenous differences amongst competing firms: Reynolds and Wilson (2000) and Maggi (1996) for capacity expansion and Amir and Wooders (1999, 2000) for R&D.

There are several other studies in various areas of industrial organization where endogenous heterogeneity emerges in a strategic setting. Hermalin (1994) deals with a two-stage game where firms' choices of managerial structures take place before market competition. Mills (1996) and Amir (2000) deal with R&D games giving rise to equilibrium outcomes with maximal heterogeneity only, i.e. full R&D by one firm and no R&D by the rival.[4] In public eco-

[3] See in particular Gabszewics and Thisse (1979) and Shaked and Sutton (1982). The present paper will directly generalize Motta (1993) and Aoki and Prusa (1996).

[4] Another strand of literature, not directly related to our setting, deals with endogenous heterogeneity arising out of hybrid models of joint ventures where firms make a cooperative decision in the first stage followed by product market competition in the second stage: See e.g. Salant and Shaffer (1998,1999) and Long and Soubeyran (2001).

nomics Mintz and Tulkens (1986) exhibit asymmetric tax rates for identical member states.

As a second motivation, the present paper is an attempt to develop a unifying approach to understanding symmetry-breaking mechanisms in general classes of $2-$player games, encompassing many of the cited studies. These two-stage models share two key features that are critical for the symmetry-breaking arguments they present. The first is a fundamental non-concavity in the payoffs, which may be confined to the diagonal in action space or hold globally, and the second is some form of strategic substitutes in first-period actions, possibly of an abstract sort (more on this in Section 2.4). While there is quite some variation in the precise manner versions of these two features are present and interact across all the models, we will be able to capture most of them in three separate general results, which though quite distinct at first sight, nonetheless bear some definite relationship at an abstract level.

The paper is organized as follows. Section 2.2 contains the overall set-up. Section 2.3 provides the results on the exclusive existence of asymmetric equilibria for submodular payoff functions. Section 2.4 deals with nonsubmodular payoff functions and Section 2.5 presents the results for games with convex payoffs. Each section provides a summary of the relevant applications the results pertain to. Section 2.6 discusses an extension to ordinal complementarity and substitutability conditions. The appendix provides a brief overview of the supermodularity notions and results.

2.2 Setup

This section lays out the general notation for use throughout the paper. The non-cooperative game described below may be a simple one-shot game or it may represent the payoffs of a two-stage game as a function of the first period actions, where the unique second stage pure-strategy equilibrium has been substituted in. In the latter case, which actually covers most of the applications of this paper, we obviously restrict consideration to subgame-perfect equilibria and analyze the resulting one-shot game.

Consider a $2-$player normal form game Γ given by the tuple (X, Y, F, G). Let X and Y be the action sets of player 1 and 2 respectively, such that $X = Y = [0, c] \subset R$. The maps F and G $: X \times Y \to R$ are the payoff functions of players 1 and 2 respectively. F can be expressed as:

$$F(x, y) = \begin{cases} U(x, y), x \geq y \\ L(x, y), x < y \end{cases} \tag{2.1}$$

By symmetry of the game Γ, G can be expressed as:

$$G(x, y) = \begin{cases} L(y, x), x \geq y \\ U(y, x), x < y \end{cases} \tag{2.2}$$

Observe that, somewhat contrary to standard practice, the first argument of F is the action of player 1 while the first argument of G is the action of player 2. It is useful to define the following sets:

$$\Delta_U = \{(x, y) \in R^2 : x \geq y\} \text{ and } \Delta_L = \{(x, y) \in R^2 : x \leq y\}.$$

It will be assumed throughout the paper that U, L, F and G are jointly continuous functions of the two actions. Define the best response correspondences (reaction curves) for players 1 and 2 respectively as $r_1(y) = \arg\max\{F(x,y) : x \in [0,c]\}$ and $r_2(x) = \arg\max\{G(y,x) : y \in [0,c]\}$.

As usual, a pure strategy Nash equilibrium (or PSNE for short) $(x^*, y^*) \in [0,c]^2$ is said to be symmetric if $x^* = y^*$, and asymmetric otherwise. It follows from the symmetry of the game that if (x^*, y^*) is a PSNE, (y^*, x^*) is also a PSNE.

Each of the next three sections investigates a separate class of normal-form symmetric games that always possess asymmetric Nash equilibria and no symmetric Nash equilibria. For each of the three classes, we provide a general result establishing both the existence and the inexistence conclusions and an illustration based on previous studies where a special case of the result was derived in a specific setting.

The definitions and main results from the theory of supermodular games used in this paper are reviewed in the appendix in a very simple way, which is sufficient for the purposes of this paper.

2.3 Endogenous heterogeneity with strategic substitutes

In this section, we consider a $2-$player symmetric normal-form game characterized by two key properties. The first is that actions form strategic substitutes. This means that a more aggressive strategy by one player lowers the other player's marginal returns to increasing his own strategy. As a result, players respond optimally to an increase of the opponent's choice with a decrease of their own variable. In different words, the best reply correspondences are downward-sloping and a pure-strategy Nash equilibrium exists (see, Vives, 1990 and Milgrom and Roberts, 1990).

The second key property is that each player's payoff, though jointly continuous in the two actions, admits a fundamental non-concavity along the 45^o line, giving rise to a canyon shape along the diagonal. A key consequence of this feature is that a player would never optimally respond to an action of the rival by playing that same action.

Taken together, these two properties imply that each best reply is a decreasing correspondence[5] with a (downward) jump over the 45^o line. Hence, no PSNE could ever be symmetric. At any of the possibly multiple equilibria, which obviously occur in pairs due to the symmetry of the game, *ex ante* identical agents will necessarily take different equilibrium actions.

While all three results presented in this paper share this same flavor, the main result in terms of the generality of the assumptions and thus of the scope of applicability is this section's.

[5] In this paper, we will say that a function $f : R \rightarrow R$ is increasing (strictly increasing) if $x' > x$ implies $f(x') \geq (>) f(x)$. A correspondence is increasing if its maximal and minimal selections are increasing functions (as in the conclusion of Topkis's Monotonicity Theorem).

2.3.1 The results

Different subsets of the following assumptions will be needed for our conclusions below. The notation is as laid out in Section 2.2.[6] A full discussion of the assumptions and results is presented at the end of the section. Most of the proofs can be found in the appendix.

A1 U, L are submodular

A2 $U_1(x,x) > L_1(x,x), \forall x \in (0,c)$

A3 $U_1(0,0) > 0, L_1(c,c) < 0$

A1 says that on either side of the diagonal, but not necessarily globally, each player's marginal returns to increasing his action decrease with the rival's action. A2 holds that each player's payoff, though globally continuous in the two actions, has a kink along the diagonal in the shape of a "valley". The role of A3 is simply to rule out PSNEs at $(0,0)$ or (c,c).

These assumptions form a sufficiently general framework to encompass many of the studies mentioned in Section 2.1 as illustrated below. Furthermore, all three assumptions are easy to check in a particular model.

[6]In addition, throughout the paper, partial derivatives are denoted by a subindex corresponding to the relevant variable, i.e. $U_1(x,y) = \frac{\partial U(x,y)}{\partial x}$ and $U_2(x,y) = \frac{\partial U(x,y)}{\partial y}$.

Theorem 2.1 *Assume that $A1 - A3$ hold. Then the game Γ is of strategic substitutes, has at least one pair of asymmetric PSNEs and no symmetric PSNEs.*

The idea of the proof is that overall submodularity of the payoff function is inherited from the submodularity of its components U and L in the presence of assumption $A2$. We know from Topkis's Monotonicity Theorem that global submodularity of the payoff function implies globally decreasing best replies. Assumptions $A2 - A3$ imply that the best replies have a downward jump that crosses over the diagonal. This situation is depicted in figure 2.1.[7] (We caution the reader that the continuity of the reaction curves in each triangle over and below the diagonal is only there for the sake of a clearer figure. It needs not hold under assumptions A1- A3). The first result gives us existence of equilibrium via the strategic substitutes property, and the second precludes symmetric equilibria. The complete proof can be found in the appendix.

Theorem 2.1 does not rule out the existence of multiple pairs of PSNEs. Indeed, the two reaction curves may intersect several times above and below the diagonal. In case of multiple pairs of PSNEs, there will typically be co-existence of pairs of Cournot-stable and pairs of Cournot-unstable PSNEs. Nevertheless, Theorem 2.1 does imply that all of these PSNEs are asymmetric. Hence symmetry-breaking in this context does not rely on the rejection of

[7]Notice that unusually x is the variable in the vertical axis. This corresponds to analyzing the game from the point of view of player 1 that chooses x as a response to y. We maintain this convention throughout.

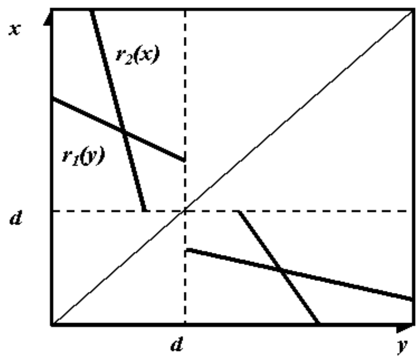

Figure 2.1: Decreasing reaction curves have a jump along the diagonal, and there is no symmetric equilibrium.

Cournot-unstable symmetric PSNEs. In the same vein, this type of symmetry-breaking is not at odds with Schelling's (1960) notion of focalness of PSNEs.

The next result adds further restrictions of a general nature on the payoff components of our game that lead to a unique pair of PSNEs, which are then necessarily Cournot-stable.[8] In this case, symmetry-breaking is coupled with more predictive power of the game, although the selection of one PSNE from the pair still remains indeterminate, as is standard in symmetric settings.[9]

[8] It is worthwhile to point out here that our results indicate an even total number of PSNEs, in apparent conflict with the well-known odd number results. The explanation is that the latter results are based on degree theory and require continuity of the best-response form. Given our systematic and robust jump across the diagonal, our findings are actually consistent with the odd number result in a generic sense.

[9] As mentioned in the Section 2.1, assumptions $A1-A3$ imply that our game admits a *symmetric* mixed-strategy Nash equilibrium. However, in the actual realization of such an equilibrium, the two players will be heterogeneous with high, if not full, probability.

Theorem 2.2 *Assume that U and L are twice continuously differentiable and that the following holds:*

$$U_{11}(r(z), z) - U_{12}(r(z), z) \geq 0 \tag{2.3}$$

$$L_{11}(r(z), z) - L_{12}(r(z), z) \geq 0 \tag{2.4}$$

then there is exactly one pair of PSNEs.

The next result is devoted to comparing the two asymmetric PSNEs from any given pair from the point of view of the players' welfare. Given any pair of asymmetric PSNEs, it is often of interest to determine circumstances where a given equilibrium secures better payoffs for a player. In different words, under what conditions would each player prefer the PSNE where he is the high or the low-activity player? To this end, we need to impose a condition of monotonicity on the payoff function of the player in question along his opponent's best reply as stated in the following result[10], which lays out conditions for player 1 (say) to prefer the PSNE where he is the low-activity player.

Theorem 2.3 *Let $x^* > y^*$, so that (x^*, y^*) and (y^*, x^*) are equilibria in Δ_U and Δ_L, respectively. If $A1 - A3$ hold and moreover $U(r_1(y), y)$ and $L(r_1(y), y)$ are increasing in $y \in [0, c]$ then $F(x^*, y^*)$ $\leq F(y^*, x^*)$.*

There is a dual statement giving conditions under which player 1 would prefer the high-activity equilibrium, given any pair of

[10] This monotonicity assumption is clearly more general than assuming that each player's payoff is increasing in the rival's action. For an example illustrating this point, see von Stengel (2003).

PSNEs. Being obvious from Theorem 2.3, it is omitted for the sake of brevity.

2.3.2 Applications

In this section we present examples of economic models that constitute special cases of the general framework developed above. While the assumptions validating Theorem 2.1 might at first appear somewhat special, they are satisfied in several *a priori* unrelated studies that have established endogenous heterogeneity in strategic settings. There are also some studies where asymmetric equilibria are produced via a mechanism similar to our Theorem 2.1, without being a special case in a formal sense. Going over some of these examples illustrates the unifying character of our results and allows us to provide some contextual interpretations of endogenous heterogeneity, or our version of symmetry-breaking.

2.3.2.1 R&D investment

The first example is based on the model by Amir and Wooders (2000). Two *a priori* identical firms with initial unit cost c are engaged in a two stage game of R&D investment and production. In the first stage, autonomous cost reductions x and y for firms 1 and 2, respectively, are chosen. The novel feature of this study is that spillovers are postulated to flow only from the more R&D active firm to the rival, but not vice versa. The effective (post-spillover) cost reductions X and Y when $x \geq y$ are given by:

$$X = x \text{ and } Y = \begin{cases} x \text{ with probability } \beta \\ y \text{ with probability } 1-\beta \end{cases} \tag{2.5}$$

Second stage product market competition, be it Cournot or Bertrand, is assumed to have a unique PSNE with equilibrium payoffs given by $\Pi : [0, c]^2 \to R$.[11] $\Pi(x, y)$ is the payoff of the firm whose unit cost is the first argument. $f : [0, c] \to R$ is a known R&D cost schedule. Amir and Wooders (2000) assume the following:

C1 Π and f are twice continuously differentiable

C2 Π is strictly submodular and $\Pi_1(x, y) < 0$ and $\Pi_2(x, y) > 0$

C3 $\Pi(x, x) < \Pi(y, y)$ if $x > y$

C4 $|\Pi_1(x, x)| > |\Pi_2(x, x)|, \forall x \in [0, c]$

C5 $f'(x) \geq 0$ and $f(0) \geq 0$

C6 $f\prime(0) < -\beta\Pi_2(c, c) - \Pi 1(c, c)$ and $f\prime(c) > -(1-\beta)\Pi_1(0, 0)$

The overall payoff of firm 1, $F(x, y)$, defined as in (2.1), is given by the difference between its second stage profit and first stage R&D cost. The payoff of firm 2 is $G(y, x)$ by symmetry.

$$U(x, y) = \beta\Pi(c-x, c-x) + (1-\beta)\Pi(c-x, c-y) - f(x) \tag{2.6}$$

$$L(x, y) = \beta\Pi(c-y, c-y) + (1-\beta)\Pi(c-x, c-y) - f(x) \tag{2.7}$$

[11] This is a standard assumption in the literature.

We can easily check that assumptions $A1, A2$ and $A3$ indeed hold in order to apply Theorem 2.1. U and L are continuous and differentiable because they result from the sum of continuous and differentiable functions. $U(x,x) = L(x,x), \forall x \in [0,c]$, so F and G are continuous. $A1$ can be checked by using the cross-partial test and the fact that $\Pi(x,y)$ is submodular (Assumption $C1$). Using $C2$ we can conclude that $U(x,y) < L(x,y)$ when $x < y$, which checks $A2$. Also,

$$U_1(x,x) = -[\Pi_1(c-x,c-x) + \beta\Pi_2(c-x,c-x)] - f'(x) \tag{2.8}$$

$$L_1(x,x) = -(1-\beta)\Pi_1(c-x,c-x) - f'(x) \tag{2.9}$$

So $U_1(x,x) \neq L_1(x,x)$ given $C4$, and therefore $A3$ is verified.

From Theorem 2.1, we can conclude that payoff functions for both players are submodular and thus reaction curves are downward sloping and there exists at least one PSNE. Moreover the reaction curves do not intersect the diagonal and by $C6$, $U_1(0,0) > 0$ and $L_1(c,c) < 0$, so there is no symmetric equilibrium in $x \in [0,c]$. The uniqueness of a pair of asymmetric PSNEs is shown by imposing conditions that secure that the reaction curves are contractions, so that Theorem 2.2 can be applied. Similarly, extra conditions are needed to apply Theorem 2.3 and Amir and Wooders conclude that player 1 prefers the equilibrium in Δ_U (for details, see Amir and Wooders, 2000).

This model provides a good opportunity for a typical economic interpretation of strategic endogenous heterogeneity in a context that is of particular interest to business strategy scholars. Indeed that field typically attaches a great deal of importance to the innovation process and to its central role in dynamic competition. The key driving force behind asymmetric equilibrium outcomes here is the one-way nature of the spillover process. A firm will always react by performing either less R&D than its rival knowing that it may free ride on the difference in R&D levels, or, in case the rival's R&D is simply too low, by overtaking it. In this vision, firms will endogenously settle into R&D innovator and imitator roles simply as a reflection of the nature of the R&D spillover process. This critical difference arises as a consequence of strategic thinking in a fully interactive setting: though facing equal and known opportunities in all respects, firms emerge as fundamentally different in all possible equilibria of a robust and general model. In strategic settings, there may simply be no single "best choice" (to paraphrase Nelson, 1991) for all *ex ante* identical firms facing the same available choices, simply because one firm's choice has a direct influence on what becomes best for its rivals.

This difference in one key component of firms' overall strategies will then be a causal factor, through natural complementarity-reinforcing developments, for heterogeneity in other aspects of firms' strategies, including in particular firm size and organization (see Amir and Wooders, 1999 for details). This perspective stands in sharp contrast to the explanation for inter-firm differences characterized by idiosyncratic groping behavior on the part of firms and weak interaction amongst them in a world of high uncertainty and complexity, as often envisioned in the strategy literature.

2.3.2.2 Provision of information

The second example deals with the provision of information in Bertrand oligopoly, see Irland (1993). Two *a priori* symmetric firms produce a homogeneous product and play a two stage game. In the first stage, each firm sets the level of its product information and in the second stage they compete in prices. Information regards only the existence of the product. Consumers may obtain costless information about prices of products that they know to exist. The number of consumers is normalized to 1. x and y are the proportions of consumers who know about product 1 and 2 respectively. Each consumer is not willing to pay a unit price higher than 1. Firm's 1 sales are given by:

$$Q_1 = \begin{cases} x \text{ if } p_1 < p_2 \\ x - \frac{xy}{2} \text{ if } p_1 = p_2 \\ x - xy \text{ if } p_1 > p_2 \end{cases} \tag{2.10}$$

For $x = y = 1$ the Bertrand oligopoly has a pure strategy Nash equilibrium at $p_1 = p_2 = 0$. If information is not full (x or y or both are less than 1), no pure strategy Nash equilibrium exists. There exists a mixed strategy Nash equilibrium given by the distribution function $G_i\,(p_i)$ that has the following form respectively for firm 1 and 2 .

$$G_1(p) = \begin{cases} \frac{1-(1-x)/p}{x} \text{ if } 1-x \leq p \leq 1 \\ 0 \text{ if } 0 \leq p \leq 1-x \end{cases} \quad (2.11)$$

$$G_2(p) = \begin{cases} \frac{1-(1-x)/p}{y} \text{ if } 1-x \leq p \leq 1 \\ 0 \text{ if } 0 \leq p \leq 1-x \end{cases}$$

The overall payoff for (say) firm 1 in the game, upon substituting the second stage equilibrium payoffs, is given by $F(x, y) = E_{all\ p}(p_1 Q_1)$ or

$$F(x, y) = \begin{cases} U(x, y) = x(1-y) \text{ if } x \geq y \\ L(x, y) = x(1-x), \text{ if } x \leq y \end{cases} \quad (2.12)$$

It is trivial to show that assumptions $A1, A2$ and $A3$ are verified in this example. There exists an equilibrium and this equilibrium cannot be symmetric for $p \in (0, 1)$. Moreover $U_1(0, 0) > 0$ and $L_1(1, 1) < 0$. So we can conclude from Theorem 2.1 that no symmetric equilibria exist for $p \in [0, 1]$.

The number of equilibria may be obtained by finding the explicit form of the reaction curves.

$$r_1(y) = \begin{cases} 1 \text{ if } y \leq 1/2 \\ 1/2 \text{ if } y > 1/2 \end{cases} \qquad r_2(x) = \begin{cases} 1 \text{ if } x \leq 1/2 \\ 1/2 \text{ if } x > 1/2 \end{cases} \quad (2.13)$$

Figure 2.2 illustrates that there exists exactly one pair of pure strategy Nash equilibria, namely $(1, \frac{1}{2})$ and $(\frac{1}{2}, 1)$.

We can compare equilibria from the point of view of player 1, using the dual to the Theorem 2.3. The payoff function of player 1

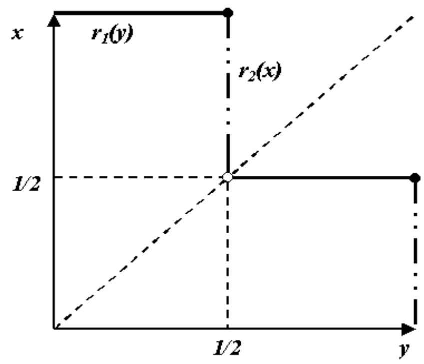

Figure 2.2: Reaction curves are constant except for a jump down, which precludes symmetric equilibrium.

when $x \leq y$, is constant along his best reply, i.e. $L_2(r_1(y), y) = 0$. Therefore we conclude that player 1 prefers the equilibrium where he is more active.

2.4 Endogenous heterogeneity without monotonic best replies

In the previous section we discussed symmetry breaking via strategic substitutes and non-concavity of the payoff function. In this section we extend the analysis from the previous section to encompass other forms of strategic interaction. The main difference is that agents' strategies form (partially) strategic complements. This occurs when a more aggressive strategy from one agent rises the other player's marginal returns to increasing his own strategy. Consequently, an increase of the opponent's choice is responded to by an increase of own choice variable. This property implies that best-replies are increasing in own action. As stated above, strategic complementarities are partial in the sense that they are not observed overall. Reaction curves are piecewise increasing, however due to a symmetry breaking non-concavity in the payoff function, they possess a jump down across the diagonal. The common aspect to the whole analysis is the canyon shape of the agents' payoff functions along the 45^o line. Once again, a player would never optimally respond to an action of the rival by playing that same action. Whereas in the previous section the submodularity of the payoff function (or alternatively the strategic substitutability) was a global feature, in this section we cannot state global supermodularity (or strategic complementarity). The main consequence is that we cannot guarantee without further assumptions the existence of a PSNE. Nevertheless, when it exists, it will never be symmetric.

When strategies are strategic complements, there is no need to assume the quasi-concavity of the payoff function in order to guarantee the existence of a PSNE. The reason is that existence might be based on the fact that reaction curves are increasing and continuity plays no role. Likewise, when payoff functions are quasi-concave supermodularity is not crucial for arguing the existence of a PSNE. When player's payoff function is partially quasi-concave and possesses a non-concavity along the diagonal, we obtain the same type of symmetry-breaking already discussed. In this case, reaction curves are continuous (not necessarily increasing), except for the jump across the diagonal. Existence of PSNE can be assured (as before) via added conditions, however it is clear that no PSNE can be symmetric.

We provide now the main results of this section, first for the case in which strategic complementarities are present and then for the case of partially quasi-concave payoff functions.

2.4.1 The results

In this section we analyze the case in which the components of the payoff function, U and L, are not submodular. We first consider the case in which U and L are supermodular and then the case where U and L do not have this property, which is replaced by quasi-concavity.

Consider the following assumptions:

B1 U and L are supermodular.

B2 U and L are differentiable and $U_1(x,x) > L_1(x,x), \forall x \in (0,c)$

Define $\overline{r}_1(y) = \arg\max\{U(x,y) : (x,y) \in \Delta_U\}$ and $\underline{r}_1(y) = \arg\max\{L(x,y) : (x,y) \in \Delta_L\}$

B3 $U_2(\overline{r}_1(y), y) < L_2(\underline{r}_1(y), y), \forall y \in Y$

B4 $U_1(0,0) > 0, L_1(c,c) < 0$

B5 $U_1(x,0) > 0$, $\forall x < d$ and $L_1(x,c) < 0$, $\forall x > d$, for d : $r_1(d-\varepsilon) > d > r_1(d+\varepsilon)$

$B1$ says that on either side of the diagonal, but not necessarily globally, each player's marginal returns to increasing his action increase with the rival's action. $B2$ holds that each player's payoff, though globally continuous in the two actions, has a valley-like shape along the diagonal. $B3$ is responsible for the uniqueness of the jump in the reaction curves. $B4$ excludes the existence of equilibria in $(0,0)$ and (c,c). Finally, $B5$ restricts the reaction curves to a compact subset of the action space which enables us to prove the existence of an asymmetric PSNE.

These assumptions form a sufficiently general framework to encompass many of the studies mentioned in Section 2.1 as illustrated below. Furthermore, all assumptions are possible to check in a particular model.

Lemma 2.4 *If $B1-B4$ hold, then there exists exactly one $d \in [0,c]$ such that $r_1(d-\varepsilon) > d > r_1(d+\varepsilon)$, $\forall \varepsilon > 0$.*

This lemma guarantees that the reaction curves possess a jump and that this jump is unique. The flavor of the proof is the following: given submodularity of U and L, we know that reaction curves are increasing in Δ_U and in Δ_L. Assumption $B2$ implies that there is no interior symmetric equilibrium due to the presence of a canyon along the diagonal. Furthermore, assumption $B4$ rules out symmetric equilibrium on the boundary. This is equivalent to say that the reaction curves do not intersect the diagonal on the whole strategy space. Hence, there must be a jump across the diagonal. Finally assumption $B3$, that entails an idea of monotonicity

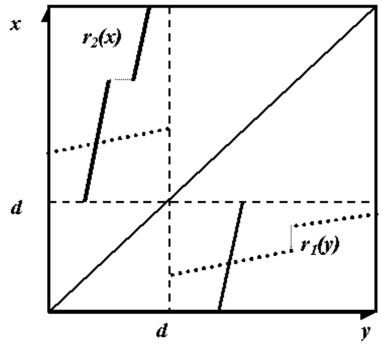

Figure 2.3: Reaction curves are partially increasing, but posses an unique jump down, which precludes any symmetric equilibrium.

of maxima along y, guarantees that in case a jump occurs, it is unique. From this lemma we can conclude that the reaction curves possess a jump across the diagonal in a point d and that once it occurs, the reaction curves never jump back again. The point d is useful to define subsets of the strategy space, which are necessary in the next theorem.

The following theorem implies that no symmetric PSNE exists for a game where assumptions $B1 - B5$ hold and that a PSNE exists. From these premises we can conclude that there are only asymmetric PSNEs.

Theorem 2.5 *Assume that $B1 - B5$ hold, then there exist at least one pair of asymmetric PSNEs and no symmetric one.*

The intuition behind this result is the following: from Lemma 2.4 we have that the reaction curves are partially increasing (as they increase in each side of the diagonal) and possess a unique downward jump at point d. Since reaction curves are not overall increasing we cannot guarantee without further assumptions the existence of a PSNE. Introducing assumption $B5$ guarantees that the reaction functions are well defined in $R = \{(x, y) : d \leq x \leq c, 0 \leq y \leq d\}, R \subset \Delta_U$ and $R' = \{(x, y) : 0 \leq x \leq d, d \leq y \leq c\}, R' \subset \Delta_L$, in the sense that they are completely contained in these compact subsets of the strategy space. We obtain, hence, increasing maps in compact sets and Tarski's Fixed Point Theorem can be applied to show that within these sets a PSNE exists.

Note that we can reorder one player's action set in a nonstandard way to obtain a submodular game. Consider action space of (say) player 1 as $(d, 0] \cup [c, d)$, ordered from left to right. The other player's action space order remains without change. It can be verified, that the game satisfying $B1 - B5$ then becomes a game of strategic substitutes.

Now consider another property of U and L.

B1' U and L are quasi-concave.

The assumption $B1$ can be replaced by the assumption $B1'$ which guarantees that player's best replies are continuous (not overall) and still the result holds. In different words, the supermodularity of U and L is not necessary (even though often observed in applications) for the existence of only asymmetric PSNE in this framework. In particular, assuming that U and L are quasi-concave implies that the reaction curves are partially continuous (even though not monotone) and thus, as long as the unique jump across the diagonal exists and the reaction curves are completely contained in compact subsets of the domain, we can show the existence of asymmetric PSNE.

Lemma 2.6 *If $B1'$ and $B2-B4$ hold, then there exists exactly one $d \in [0,c]$ such that $\forall \varepsilon > 0$: $r_1(d-\varepsilon) > d > r_1(d+\varepsilon)$,*

Theorem 2.7 *Assume that $B1'$ and $B2-B5$ hold then there exist at least one pair of asymmetric PSNEs and no symmetric one.*

The proofs of these results follow the same reasoning as the proofs of the precedent ones, however, existence of PSNE is now guaranteed through the application of Brouwer's Fixed Point Theorem (which is suitable given that payoff functions are quasi-concave).

Uniqueness of a pair of equilibria can be shown if reaction functions are contractions by using Banach's Fixed Point Theorem as in the previous section.

A comparison of equilibria when payoff functions are partially supermodular (or quasi-concave) can be done in the same spirit of Theorem 2.3. Instead of assuming $A1-A3$ we assume $B1-B5$ ($B1'-B5$).

Adopting the results of Echenique (2004), wherein the order on the action spaces is not exogenously given would enlarge the scope of supermodular games. In particular, since our game here has at least two PSNEs, one can always find a partial order such that it becomes a supermodular game (Echenique, 2004, Theorem 5).

2.4.2 Applications: quality investment

We illustrate the results of this section with two papers dealing with quality investment problems. The first paper we analyze is Aoki and Prusa (1996). In this paper, two identical firms produce products differentiated by quality, in a two-stage setting. In the first stage firms 1 and 2 decide the level of quality investment $x \in [0, c]$ and $y \in [0, c]$ respectively, and in the second stage they simultaneously announce prices.[12]

Consumers are diversified in their willingness to pay for quality. Production cost is assumed to be 0 and firm 1 (firm 2) incurs a cost of quality investment $f(x) = kx^2$ $[f(y) = ky^2]$, $k > 0$. A detailed study of the second stage equilibrium of this game can be found in Gabszewicz and Thisse (1979) and Shaked and Sutton (1982). The subgame perfect equilibrium of the whole game can be obtained by backward induction and first stage overall payoffs for firm 1 are given as follows:

$$F\left(x,y\right) = \begin{cases} U(x,y) = \frac{4x^2(x-y)}{(4x-y)^2} - kx^2 \text{ if } x \geq y \\ L(x,y) = \frac{yx(y-x)}{(4y-x)^2} - kx^2 \text{ if } x < y \end{cases} \tag{2.14}$$

Now we can check whether assumptions of Theorem 2.5 are fulfilled. Given that U and L are differentiable we can check supermodularity (Assumption $B1$) by recurring to Topkis's Characterization Theorem. Consider that $U_1(x,x) = \frac{4}{9} - 2x$ and $L_1\left(x,x\right) = -\frac{1}{9} - 2x$, so $U_1\left(x,x\right) > L_1\left(x,x\right)$ which verifies $B2$. To check $B3$

[12] Aoki and Prusa (1996) consider unlimited quality investments. We impose the upper limit c, arbitrarily big, such that the strategy spaces are compact.

we compute:

$$U_2\left(\overline{r}_1\left(y\right),y\right) = -4\overline{r}_1^2\left(y\right)\frac{2\overline{r}_1\left(y\right)+y}{\left(4\overline{r}_1\left(y\right)-y\right)^3} \tag{2.15}$$

$$L_2\left(\underline{r}_1\left(y\right),y\right) = -\underline{r}_1^2\left(y\right)\frac{2y+\underline{r}_1\left(y\right)}{\left(-4y+\underline{r}_1\left(y\right)\right)^3}$$

Since $\overline{r}_1\left(y\right) > y > \underline{r}_1\left(y\right)$ the condition $B3$ holds. We know, hence, that the reaction curves are upward sloping except for a downward jump at a point d. Since $U(0,0)$ is not defined we cannot check the first part of $B4$ directly, we must compute $\lim_{x\to o} U_1\left(x,x\right)$. We obtain $\lim_{x\to o} U_1\left(x,x\right) = \frac{4}{9} > 0$ which means that increasing quality investment at point $(0,0)$ is profitable for both firms. Point $(0,0)$ is thus ruled out as a pure strategy Nash equilibrium of the game. As for the second part of $B4$, it is easily verified by the fact that $L_1\left(c,c\right) = -\left(\frac{1}{9}+2kc\right) < 0$. From Lemma 2.4 we have that if there is an equilibrium, it cannot be symmetric in $\left[0,c\right]^2$.

To apply Theorem 2.5 we must also check, whether $B5$ is verified. To this end, we must find the point d where the reaction curve has a jump. If the reaction curve is given implicitly, there is no algorithm to find this point d, however it is possible to find it through numerical methods. In the case of the model by Aoki and Prusa (1996), $d = \frac{1}{12k}$. It is possible to compute that $U_1\left(x,0\right) = \frac{1}{4} - 2kx \geq 0$ for $x > d$ and that for sufficiently big c $L_1\left(x,c\right) = \lim_{y\to\infty} L_1(x,y) = -2kx + \frac{1}{16} < 0$ for $x > d$. So the assumptions of Theorem 2.5 are satisfied and we can conclude that there exist only asymmetric equilibria in the game. Furthermore, reaction curves in this model are contractions and so a unique pair of equilibria exists.

The second paper that can be analyzed under our framework is Motta (1993). He develops two versions of a vertical product differentiation model, one with fixed and the other with variable cost of investment in quality. In each of them, he compares price versus quality competition. The model with fixed cost and price competition can illustrate the results of this section. Considering the same notation as in Aoki and Prusa (1996), the payoff function is defined as follows:

$$F(x,y)=\begin{cases}\frac{4v^2x^2(x-y)}{(4x-y)^2}-\frac{x^2}{2} \text{ if } x\geq y\\ \frac{xyv^2(y-x)}{(4y-x)^2}-\frac{x^2}{2} \text{ if } x\leq y\end{cases} \tag{2.16}$$

Where v is the upper limit of the set of consumer's taste parameters. This model is analogous to Aoki and Prusa (1996) if we let $v=1$ and $k=\frac{1}{2}$. All the results follow directly from the above discussion.

2.5 Convex Payoffs

Certain features of the production technologies or consumer preferences may lead to a situation where payoff functions are convex. In particular, the presence of highly convex demand functions or strongly concave costs translating intensely decreasing elasticity of demand or decreasing marginal costs might have this effect. This property of the payoff functions implies that agents prefer corner solutions. Moreover certain additional conditions on the payoffs might generate asymmetric equilibria and even rule out the symmetric ones. In this section we analyze a class of games in which players have convex payoff functions and only asymmetric PSNEs arise.

Let $S = [0, c]$ be the strategy space of a player in a $2-$player game, Γ. Consider $F : S \times S \rightarrow R$ as the payoff function of player 1. $G(y, x) = F(x, y)$ is the payoff of player 2 because the game is *a priori* symmetric. Then we can apply the following theorem to conclude about the properties of the equilibria of Γ.

Theorem 2.8 *Assume that the following assumptions hold:*

1. *F is strictly quasi-convex in own strategy*
2. *$F(c, 0) > F(0, 0)$ and $F(0, c) > F(c, c)$*

Then, the game has no symmetric equilibrium and it has exactly one pair of asymmetric equilibria given by $(0, c)$ and $(c, 0)$.

Proof. From the definition of strict quasi-convexity, we know that:

$$F(\lambda z_1 + (1 - \lambda) z_2) < \max\{F(z_1), F(z_2)\}, \forall z_1, z_2 \in S \times S$$

Let $z_1 = (0, y)$ and $z_2 = (c, y)$, then we know that any $x \in (0, c)$ yields a lower payoff than $x = 0$ or $x = c$, $\forall y \in [0, c]$. This means that $\forall y \in [0, c]$, $r_1(y) = 0$ or $r_1(y) = c$. Analogously, for player 2 we have the same result, due to symmetry.

Finally, from Assumption 2 we have that $(c, 0)$ and $(0, c)$ are the only PSNEs of the game. ■

Figure 2.4 illustrates the results of this section. Notice that we depicted reaction curves which are not continuous, specifically, they

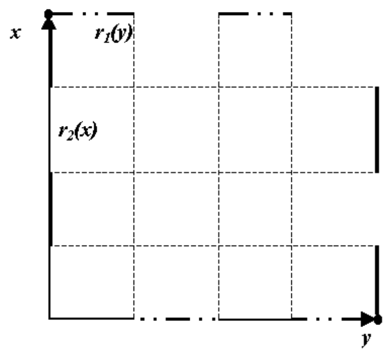

Figure 2.4: Quasi-convexity of the payoffs implies that players prefer corner solutions.

possess an odd number of jumps. This is a direct consequence of the Assumptions 1 and 2 that imply either that the reaction curves are continuous, or that, if they jump, they must jump an odd number of times.

2.5.1 Applications

This theorem generalizes the results of Amir (2000) and Mills and Smith (1996), whose models are usually presented within the literature about endogenous heterogeneity of firms.[13] In these papers it is considered a two-stage duopoly game. In the first stage, firms make long term investment decisions that affect the production costs. In the second stage, firms compete *à la* Cournot. Both

[13] Another example where the game can be reduced to a two person normal form game is presented in Boyer and Moreaux (1997). Conditions for the non-existence of symmetric equilibria are the same, even if the payoff function is not convex.

firms face a linear demand function and for some parameterization of the cost functions, profits are convex in own quantity. In both papers it is then concluded that if some conditions on the parameters hold, asymmetric equilibria might arise. It is easily shown that the conditions presented in the papers can be deduced from the assumptions of Theorem 2.8.

2.6 Extensions

Most of the results of the previous sections depend on some form of monotonicity of the reaction curves. We used the cardinal notions of complementarity and substitutability to obtain this property mainly due to its convenient characterization through the cross partial derivatives of the payoff functions. Supermodularity and submodularity are cardinal notions that are not preserved by monotone transformations of the objective function. The usefulness of ordinal properties under which comparative statics results are invariant is clear. Milgrom and Shannon (1994) proved that the result of Topkis holds when the single crossing property substitutes supermodularity of the objective function. In this section we show that our results can be generalized to the ordinal definitions of complementarity.

Let F be defined as in (2.1) and consider the following assumptions:

$A1'$ U and L have the dual single crossing property

$A4$ $U_2(\overline{r}_1(y), y) < L_2(\underline{r}_1(y), y), \forall y \in [0, c]$

The Assumption $A1'$ is alternative to Assumption $A1$ defined in Section 2.3 and provides the ordinal condition for the reaction curves to be decreasing. Assumption $A4$ expresses monotonicity of U and L with respect to the opponent's action along the best replies.

We now show the main result of this section in the following steps: first we conclude about decreasing best-replies; then we observe that there is a downward jump in the reaction function at point d and that this jump is unique. Finally we use Topkis Fixed Point Theorem to conclude of the existence of equilibria in the subsets of the strategy space defined using point d.

Finally we treat the setting of Section 2.4, extending its results to the ordinal definitions of complementarity.

Theorem 2.9 *If* $A1', A2, A3$ *and* $A4$ *hold, then the game* Γ *has at least one pair of asymmetric PSNEs and no symmetric one.*

The idea of the proof is that within Δ_U and Δ_L the dual single crossing property of the payoff function F holds from assumption $A1'$. As Milgrom and Shannon (1994) showed, the dual single crossing property allows us to draw the same conclusions as the submodularity of the payoff function in terms of monotonicity of the reaction curves. Furthermore the dual single crossing property has the advantage of being more general and preserved by monotonic transformations. Then assumption $A2$ implies that there exists a jump down in the reaction curves at a certain point d and $A4$ implies that this jump is unique. With point d we define compact subsets of the strategy space where reaction curves are decreasing and thus we can guarantee the existence of a PSNE by Tarski's Fixed Point Theorem.

Some results from Section 2.4 can also be extended into an ordinal version.

Let F be defined as in (2.1) and consider the following assumption:

$B1''$ U and L have the single crossing property

Theorem 2.10 *If the assumptions $B1''$ and $B2 - B5$ hold, then there exist at least one pair of asymmetric PSNEs and no symmetric one.*

From Milgrom and Shannon's Theorem (see Appendix) Assumption $B1''$ implies that reaction curves are partially increasing in Δ_U and Δ_L. From this fact and as long as they are well defined in a compact subset of the strategy space, we can obtain existence of PSNEs. The valley-shape of the profit precludes symmetric equilibria in the same spirit as Theorem 2.5.

2.7 Conclusion

Our theorems assert that, under specific conditions, heterogeneity in agent's behavior might arise even when they are *a priori* identical. This paper constitutes, hence, a contribution to the discussion about the sources of diversity across economic agents and disparities in economic performances. While previous literature stands on arguments related to multiplicity of equilibria and strategic complementarities (Cooper, 1999) or on strategic substitutability and stability of equilibria (Matsuyama, 2002), our approach stands on the existence of a fundamental non-concavity of the payoff function and on some form of strategic substitutability. It is, thus, similar in spirit to Matsuyama's work. However, we show that endogenous heterogeneity does not rely on the idea that only stable equilibria are observable as in Matsuyama. With respect to Cooper's approach, where agents can still choose symmetrically, our results guarantee that symmetric equilibria can never arise in a $2-$player setting. Even though we have, in our model some form of strategic substitutability (notice that when talking about $2-$player games strategic substitutability can be converted into complementarity through a simple inversion of one agent's strategy space), the critical assumption for the inexistence of symmetric equilibria is the non-concavity of the payoffs. In fact, we show that strategic substitutability can be replaced by partial quasi-concavity and still the results follow.

An alternative explanation for endogenous heterogeneity can be found in business strategy literature. Our results can also be considered as a response to its critique as asymmetries arise here in a completely deterministic and rational setup. We should thus expect that heterogeneity generated in such a framework can be the origin of long-term diversity.

One may ask about possibility to generalize our results to $n-$player case. Results of Sections 2.4 and 2.5 can be extended to include larger number of players. Unfortunately, generalization is limited for the main result concerning the case with submodular payoff function (Section 2.3). Namely, expanding the number of players would lead to a game, where existence of PSNE is no more guaranteed. Well known property of submodular games that they can be transformed to supermodular games by reordering of the strategy spaces of one of the players, works only for $2-$player games.

2.8 Appendix

2.8.1 Summary of supermodular/submodular games

We give an overview of the main definitions and results in the theory of supermodular games that are used in the paper, in a simplified setting that is sufficient for our purposes. Details may be found in Topkis (1978).[14]

Let I_1 and I_2 be compact real intervals and $F : I_1 \times I_2 \to R$. F is (strictly) supermodular if $\forall x_1, x_2 \in I_1, x_2 > x_1$ and $\forall y_1, y_2 \in I_2, y_2 > y_1$ we have $F(x_2, y_2) - F(x_2, y_1)(>) \geq F(x_1, y_2) - F(x_1, y_1)$. F is (strictly) submodular if $-F$ is (strictly) supermodular.

Theorem 2.11 (Topkis's Characterization Theorem) *Let F*

[14] Other aspects of the theory may be found in Topkis (1979), Milgrom and Roberts (1990) and Vives (1990).

be twice continuously differentiable. Then

(i) $F_{12} = \frac{\partial^2 F}{\partial x \partial y} \geq 0$ *[* ≤ 0*] for all* $x, y \Leftrightarrow F$ *is supermodular [submodular].*

(ii) $F_{12} = \frac{\partial^2 F}{\partial x \partial y} > 0$ *[*< 0 *] for all* $x, y \Rightarrow F$ *is strictly supermodular [submodular].*

The supermodularity property is not preserved by monotonic transformations of the function F. An alternative notion (ordinal) is the single crossing property defined as follows: F has the [dual] single crossing property in (x, y) if $\forall x_1, x_2 \in I_1, x_2 > x_1$ and $\forall y_1, y_2 \in I_2, y_2 > y_1$ we have

$$F(x_1, y_2) - F(x_1, y_1) \geq 0 \, [\leq 0] \Rightarrow F(x_2, y_2) - F(x_2, y_1) \geq 0 \, [\leq 0] \, .$$

The single crossing property does not have a correspondent differential characterization and thus it is often more difficult to check. Now we present the main monotonicity theorems.

Theorem 2.12 (Topkis's Monotonicity Theorem) *If* F *is continuous in* y *and (strictly) supermodular [submodular] in* (x, y)*, then* $\arg\max_{y \in I_2} F(x, y)$ *has (all of its) maximal and minimal selections increasing [decreasing] in* $x \in I_1$.

Theorem 2.13 (Milgrom and Shannon) *The conclusion of Topkis's Theorem continues to hold when supermodularity [submodularity] is replaced by the [dual] single crossing property.*

We can introduce now the notion of supermodular game and of its properties. A $2-$player game is supermodular (submodular) if both payoff functions are continuous, supermodular (submodular) and both action spaces are compact real intervals.[15] The fixed point theorems associated with this framework are due to Tarski (1955).

Theorem 2.14 (Tarski's Fixed Point Theorem) *Let $f : I_1 \times I_2 \to I_1 \times I_2$ be an increasing function, then f has a fixed point.*

Theorem 2.15 *A $2-$player supermodular (submodular) game has a pure strategy Nash equilibrium.*

In general, this theory dispenses with assumptions of concavity or differentiability of payoff functions, making it an extremely general framework to study the properties of equilibria.

2.8.2 Proofs of Section 2.3

The proof of Theorem 2.1 is organized as follows: we begin with proving four preliminary lemmas, and then present the main proof in two steps: first we show existence of PSNE and afterwards that all PSNEs must be asymmetric.

The first lemma states that for a small enough square of points on the diagonal we have submodularity.

Lemma 2.16 *Consider the following points as depicted in figure 2.5. If $A1 - A2$ hold, then for small enough $\alpha > 0 : U(x, x-\alpha) - U(x-\alpha, x-\alpha) \geq L(x, x) - L(x-\alpha, x), \forall x \in [0, c]$.*

[15] Compactness is not necessary, it is required in order to use a simplified version of Tarski's Fixed Point Theorem, without referring to lattices.

Proof. Take any point (x,x) on the diagonal, belonging to the domain of F. For $\alpha > 0$ small enough

$$U_1(x,x) \simeq U(x+\alpha,x) - U(x,x) \text{ and } L_1(x,x) \simeq (x,x) - L(x-\alpha,x). \tag{2.17}$$

Hence, from $A2$:

$$U(x+\alpha,x) - U(x,x) > L(x,x) - L(x-\alpha,x). \tag{2.18}$$

From $A1$ we know that

$$U(x+\alpha,x-\alpha) - U(x,x-\alpha) \geq U(x+\alpha,x) - U(x,x). \tag{2.19}$$

Take $\widehat{\varepsilon} > 0$ such that

$$U(x+\alpha,x) - U(x,x) \geq L(x,x) - L(x-\alpha,x) + \widehat{\varepsilon}. \tag{2.20}$$

From continuity of $U(x,y)$ in x (for fixed $y = x-\alpha$) it follows that

$$\forall 0 < \varepsilon \leq \frac{\widehat{\varepsilon}}{2}\ \exists \delta > 0 \text{ such that}$$

$$|x-(x-\alpha)| = \alpha \leq \delta \text{and } |U(x,x-\alpha) - U(x-\alpha,x-\alpha)| \leq \varepsilon \leq \frac{\widehat{\varepsilon}}{2}$$

and, moreover

$$\forall 0 < \varepsilon \leq \frac{\widehat{\varepsilon}}{2}\ \exists \delta > 0 \text{ such that}$$

$$|x+\alpha-x| = \alpha \leq \delta \text{ and } |U(x+\alpha,x-\alpha) - U(x,x-\alpha)| \leq \varepsilon \leq \frac{\widehat{\varepsilon}}{2}.$$

This allows us to establish that

$$(U(x+\alpha,x-\alpha) - U(x,x-\alpha)) -$$
$$(U(x,x-\alpha) - U(x-\alpha,x-\alpha)) \leq$$
$$\leq 2\varepsilon \leq \widehat{\varepsilon}$$

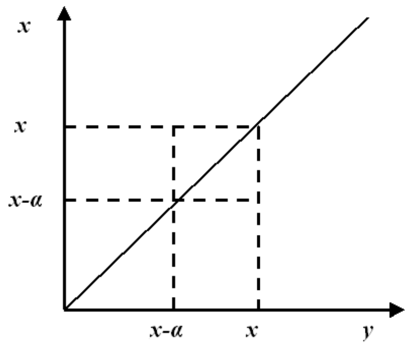

Figure 2.5: Square of the length α.

and then

$$U(x+\alpha, x-\alpha) - U(x, x-\alpha) \leq U(x, x-\alpha) - U(x-\alpha, x-\alpha) + \widehat{\varepsilon} \quad (2.21)$$

Summarizing, we have

$$L(x,x) - L(x-\alpha, x) + \widehat{\varepsilon} \leq$$
$$\leq U(x+\alpha, x) - U(x,x) \leq$$
$$\leq U(x+\alpha, x-\alpha) - U(x, x-\alpha) \leq$$
$$\leq U(x, x-\alpha) - U(x-\alpha, x-\alpha) + \widehat{\varepsilon}$$

where, the first inequality comes from (2.20), the second one from (2.19) and the last one from (2.21). So we obtain,

$$U(x, x-\alpha) - U(x-\alpha, x-\alpha) + \widehat{\varepsilon} \geq (x,x) - L(x-\alpha, x) + \widehat{\varepsilon}. \quad (2.22)$$

Subtracting $\widehat{\varepsilon}$ from both sides finishes the proof. ■

The next lemma extends the property of submodularity of F form the small square of length α to any square with two vertices on the diagonal.

Lemma 2.17 *If Lemma 2.16 holds, then for any square with points in the diagonal, such as depicted in figure 2.6, we have,*

$$F(z,x) - F(z,z) \geq F(x,x) - F(x,z)$$

Proof. Consider the square formed by the four points defined in the lemma. Divide this square into rectangles such that their height is equal to the original height of the square and its length is not bigger than α, as defined in Lemma 2.16. We will now show that for points in the vertices of such rectangles the thesis holds and *a fortiori* it is possible to obtain the conclusion for the whole square.

Let $x - z = k\alpha$, where $k \in R$, and $\alpha > 0$ is small enough. Now consider the rectangle defined by the following points $(x,x), (x, x - \alpha)$, (z,x) and $(x, x-\alpha)$. From Lemma 2.16 we know that

$$F(x, x-\alpha) - F(x-\alpha, x-\alpha) \geq F(x,x) - F(x-\alpha, x)$$

Also, from $A1$ we know that

$$F(x-\alpha, x-\alpha) - F(z, x-\alpha) \geq F(x-\alpha, x) - F(z,x)$$

Adding these two inequalities we obtain that

$$F(x, x-\alpha) - F(z, x-\alpha) \geq F(x,x) - F(z,x) \tag{2.23}$$

Repeating the procedure, consider the rectangle defined by: $(x, x-\alpha)$, $(x, x-2\alpha)$, $(z, x-\alpha)$, $(z, x-2\alpha)$. From $A1$ we know that:

$$F(x, x-2\alpha) - F(x-\alpha, x-2\alpha) \geq F(x, x-\alpha) - F(x-\alpha, x-\alpha)$$

and also

$$F(x-2\alpha, x-2\alpha) - F(z, x-2\alpha) \geq F(x-2\alpha, x-\alpha) - F(z, x-\alpha).$$

Using Lemma 2.16 we know that

$$F(x-\alpha, x-2\alpha) - F(x-2\alpha, x-2\alpha) \geq F(x-\alpha, x-\alpha) - F(x-2\alpha, x-\alpha)$$

Adding the three inequalities we obtain:

$$F(x, x-2\alpha) - F(z, x-2\alpha) \geq F(x, x-\alpha) - F(z, x-\alpha)$$

From (2.23) we obtain

$$F(x, x-2\alpha) - F(z, x-2\alpha) \geq F(x, x) - F(z, x)$$

We can repeat this argument until getting a rectangle whose length is not bigger than α. Once again we apply assumption $A1$ and Lemma to show that submodularity holds for this rectangle as well and we can conclude that,

$$F(x, z) - F(z, z) \geq F(x, x) - F(x, z).$$

Hence submodularity is satisfied for any square with vertices coinciding with the diagonal. ■

The following Lemma establishes that the analysis of submodularity of any rectangle formed by four points of the domain $[0, c]^2$ can be reduced to the analysis of submodularity for points placed in such a way that they form a square with vertices coinciding with the diagonal.

Lemma 2.18 *If $A1$ and $A2$ hold, then F and G are submodular on* $[0, c]^2$.

Proof. Due to the kink along the diagonal, one cannot invoke Topkis's simple cross-partial test (Topkis's Characterization Theorem in the Appendix) to verify submodularity of F and G. Instead, we use definition of submodularity for any configuration of four points in the domain, constituting a rectangle. If the rectangle is completely contained in either Δ_U or Δ_L, the submodularity condition follows from $A1$. Every other situation can be reduced by adding subrectangles, each of which lying fully in either Δ_U or Δ_L, to the situation depicted in figure 2.5 as we now show, say for F. Consider the case of figure 2.6 with the four points (x, z), (z, z), (x, y), (z, y) as shown. With $z < x < y$, we know from $A1$ that, since $F = U$ on Δ_U,

$$F(x, y) - F(x, x) \geq F(z, y) - F(z, x).$$

From Lemma 2.17 submodularity holds for the configuration of the square (x, x), (z, x), (z, z), (z, x), hence we have

$$F(x, x) - F(x, z) \geq F(z, x) - F(z, z).$$

Adding the two inequalities yields

$$F(x, y) - F(x, z) \geq F(z, y) - F(z, z),$$

which is just the definition of submodularity for the original points (x, z), (z, z), (x, y) and (z, y).

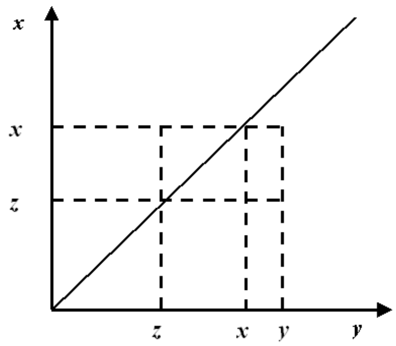

Figure 2.6: If F satisfies submodularity on the square on the diagonal, this implies it satisfies submodularity on the rectangle.

It can be shown via analogous steps that the submodularity of F for any other configuration of points can be reduced to showing submodularity for squares with two vertices on the diagonal. The details are left out. ■

The next result allows us to conclude that the two reaction curves always admit a discontinuity that skips over the diagonal, a key step for our endogenous heterogeneity result.

Lemma 2.19 *Given $A1 - A3$, there exists exactly one point $d \in (0, c)$, such that $r_i(d - \varepsilon) > d > r_i(d + \varepsilon), i = 1, 2$.*

Proof. (of Lemma 2.19) From Topkis's Monotonicity Theorem and Lemma 2.18 , all the selections from the best reply correspondences are downward sloping. Hence, both $r_i(d - \varepsilon)$ and $r_i(d + \varepsilon)$ exist for any selection of r_i and are independent of the selection.

From assumption $A3$, we know that $(0, 0) \notin$ *Graph* r_i and

$(c, c) \notin Graph\ r_i$ (i.e. r_i does not go through $(0,0)$ or (c,c)). These two properties imply that r_i cannot be identically 0 or c.

We next show that the reaction correspondence r_1 (say) cannot ever cross the 45^o line at an interior point, i.e. in $(0,c)$. The generalized first order condition for a maximum of F (say) to occur at a point (x,x) with $x \in (0,c)$, which applies even in the absence of differentiability, is that $U_1(x,x) \leq_1 (x,x)$. Assumption $A2$ rules out this case . Hence no $x \in (0,c)$ can ever be a best reply to itself, meaning that the reaction curves do not cross the 45^o line at any interior point.

Since r_1 starts strictly above 0 (for $y = 0$) and ends strictly below c (for $y = c$), the above properties of r_1 imply that there exists exactly one $d \in (0,c)$ such that $r_1(d-\varepsilon) > d > r_1(d+\varepsilon)$. In words, there must exist a jump in the best reply function past the diagonal as in figure 2.1. ■

Using the Lemmas 2.18 and 2.19 we can now prove Theorem 2.1.

Proof. (of Theorem 2.1) From Lemma 2.18 we have overall submodularity of the payoff function. This guarantees that a PSNE exists.

Consider now the behavior of the reaction curves in the area Δ_U. The same conclusion follows for Δ_L by symmetry. Define the following restricted reaction curves: $r_1|_{\Delta_U}(y) : [0,d] \to [d,c]$ and $r_2|_{\Delta_U}(x) : [d,c] \to [0,d]$ both decreasing as implied by Lemma 2.18. Define the mapping $B : [d,c] \to [d,c]$, $B(x) = r_1|_{\Delta_U} \circ r_2|_{\Delta_U}(x)$, which is increasing given that each of $r_1|_{\Delta_U}$ and $r_2|_{\Delta_U}$ is decreasing. From Tarski's Fixed Point Theorem, we know that there exists $\bar{x}$ such that $B(\bar{x}) = r_1|_{\Delta_U} \circ r_2|_{\Delta_U}(\bar{x})$, therefore $(\bar{x}, r_2(\bar{x})|_{\Delta_U})$ is a PSNE. From Lemma 2.19, there is no symmetric PSNE in $[0, c]$. Hence, there must exist at least one pair of asymmetric PSNEs. ■

Theorem 2.2 rules out the existence of multiple pairs of asymmetric equilibria.

Proof. (of Theorem 2.2) Once again we concentrate on the area Δ_U. Conclusions follow for the area Δ_L by symmetry. Whenever r_1 $[r_2]$ is interior, first order condition $U_1(r_1(y), y) = 0$ $[L_1(r_2(x), x) = 0]$, the implicit function theorem and the assumptions (2.3) and (2.4), yield that r_1 $[r_2]$ is differentiable in Δ_U and that $r_1'(y) = -\frac{U_{12}(r_1(y),y)}{U_{11}(r_1(y),y)} \geq -1$, also $r_2'(x) = -\frac{L_{12}(r_2(x),x)}{L_{11}(r_2(x),x)} \geq -1$. Hence, $r_1(y)|_{\Delta_U}$ and $r_2(x)|_{\Delta_U}$ are contractions. Using Banach's Fixed Point Theorem we can conclude that there exists exactly one pure strategy Nash equilibrium in Δ_U.[16] In the same way there exists exactly one pure strategy Nash equilibrium in Δ_L. Concluding, we have exactly one pair of pure strategy Nash equilibria. ■

Finally we provide a proof of Theorem 2.3.

Proof. (of Theorem 2.3) Since $x^* > d > y^*$ we have $F(x^*, y^*) = U(x^*, y^*)$ and $F(y^*, x^*) = L(y^*, x^*)$. Also $U(r_1(d), d) = L(r_1(d), d)$ if d denotes the unique point of jump of reaction curve between Δ_U

[16](Banach's Fixed Point Theorem): Let $S \subset \mathcal{R}^n$ be closed and $f : S \to S$ be a contraction mapping, then there exists $x \in S : f(x) = x$.

and Δ_L, as defined in Lemma 2.19. Then

$$\begin{aligned}F(x^*, y^*) &= U(x^*, y^*) = \\ &= U(r_1(y^*), y^*) \leq U(r_1(d), d) = \\ &= L(r_1(d), d) \leq L(r_1(x^*), x^*) = L(y^*, x^*) = F(y^*, x^*)\end{aligned}$$

where both inequalities follow from the monotonicity of $U(r_1(y), y)$ and $L(r_1(y), y)$ in y. ■

2.8.3 Proofs of Section 2.4

First we prove Lemma 2.4 and Lemma 2.6 since these proofs are similar. We then move to proving Theorems 2.5 and 2.7.

Proof. (of Lemma 2.4) We consider the area Δ_U. From $B1$ and Topkis's Monotonicity Theorem, we know that the reaction curves are increasing. The generalized first order condition for a maximum to occur in the point (x, x) in the absence of differentiability of F (and G, by symmetry) is that $U(x, x) \leq L(x, x)$. Assumption $B2$ rules out this possibility so we have that no y (nor x), belonging to $(0, c)$, can be best reply to itself, meaning that the reaction curves do not cross the 45^o line. Assumption $B4$ excludes that 0 can be a best reply to 0 and that c can be a best reply to c. Hence, there must exist a $d \in [0, c]$, such that $r_1(d - \varepsilon) > d > r_1(d + \varepsilon)$.

To exclude the possibility of another jump we use assumption $B3$. Consider $\underline{r}_1(y)$ and $\overline{r}_1(y)$ defined as in Section 2.4. Denote $W(y) = L(\underline{r}_1(y), y)$ and $V(y) = U(\overline{r}_1(y), y)$. From the Envelope Theorem, $\frac{\partial W(y)}{y} = L_2(\underline{r}_1(y), y)$, and $\frac{\partial V(y)}{y} = U_2(\overline{r}_1(y), y)$. Hence, when $B3$ holds, we know that W increases in y quicker then V does. It means that when the overall reaction curve jumps down along the diagonal, it never jumps up again. ■

Proof. (of Lemma 2.6) If $B1'$ holds reaction curves are continuous in Δ_U and in Δ_L. $B2$ and $B4$ rules out the possibility that F (and G, by symmetry) has a maximum in a point $[x, x]$, that secures that the reaction curve must have a jump down in a point $d \in [0, c]$. As in the proof of Lemma 2.4 $B3$ rules out other possible upward jumps between Δ_L and Δ_U. ■

We may now show that only asymmetric pure strategy Nash equilibria exist.

Proof. (of Theorem 2.5) Consider $R \subset \Delta_U$ as defined in Section 2.4. Define as before the restricted reaction curves as $r_1(y)|_{\Delta_U}$ and $r_2(x)|_{\Delta_U}$. From assumption $B5$, the best reply of player 1 to $y = 0$ cannot be less than d as U is decreasing in x for $y = 0$ when $x < d$. For $y > 0$, and given assumption $B1$ (supermodularity), Topkis's Monotonicity Theorem allows us to conclude that the best reply of 1 is increasing. Hence $r_1(y)|_{\Delta_U} \in R$. Seemingly the best reply of player 2 for $x = c$ cannot exceed d as L is decreasing in y when $x = c$. Also by Topkis's, the reaction curve is an increasing map. Therefore $r_2(x)|_{\Delta_U} \in R$. Consider the mapping, $B : R \to R$ such that $B(x, y) = (r_1(y), r_2(x))$. $B(x, y)$ is an increasing correspondence, given that both its components are increasing. R is a compact set and hence we may use Tarski's Fixed Point Theorem to conclude that there exists a pair $(\bar{x}, \bar{y})$ such that

$B(\bar{x}, \bar{y}) = (r_1(\bar{y}), r_2(\bar{x}))$. $(\bar{x}, \bar{y})$ is a pure strategy Nash equilibrium.

Finally, by Lemma 2.4 we know that no equilibrium can be symmetric. ■

Proof. (of Theorem 2.7) Define R, $r_1(y)|_{\Delta_U}$ and $r_2(x)|_{\Delta_U}$ as before. From assumption $B5$, $r_1(y)|_{\Delta_U} \in R$ and $r_2(x)|_{\Delta_U} \in R$ and from assumption $B1'$ they are continuous. Consider the mapping $B : R \to R$ such that $B(x, y) = (r_1(y), r_2(x))$. B is a continuous correspondence, given that both its components are continuous, R is a compact set and hence we may use Brouwer's Fixed Point Theorem to conclude that there exists a pair $(\bar{x}, \bar{y})$ such that $B(\bar{x}, \bar{y}) = (r_1(\bar{y}), r_2(\bar{x}))$. $(\bar{x}, \bar{y})$ is a pure strategy Nash equilibrium.

By Lemma 2.6 we know that no equilibrium can be symmetric.

■

2.8.4 Proofs of Section 2.6

To prove the theorem we first formulate a useful lemma.

Lemma 2.20 *If $A1\prime, A2$ and $A4$ hold, then there exist exactly one point $d \in [0, c]$ such that $r_1(d - \varepsilon) > d > r_1(d + \varepsilon), \varepsilon > 0$.*

Proof. Consider Δ_U. From $A1'$ and Topkis's Monotonicity Theorem we know that reaction curve is decreasing in this area. The generalized first order condition for a maximum to occur in (x, x) is (in the absence of differentiability in this point) $U_1(x, x) \leq L_1(x, x)$. Assumption $A2$ rules out this possibility, thus no $x \in (0, c)$ can be a best response to itself. Moreover, neither $(0, 0)$ nor (c, c) can be an equilibrium, since from $A3$ follows, that for player 1 it is always profitable to deviate from any of these points. Hence, the reaction curve does not cross the 45^o line, and there must exist a point, call it $d \in [0, c]$ such that $r_1(d - \varepsilon) > d > r_1(d + \varepsilon)$.

Now, we prove uniqueness of this point. Consider $\underline{r}_1(y)$ and $\overline{r}_1(y)$ defined as in Section 2.4. Denote $W(y) = L(\underline{r}_1(y), y)$ and $V(y) = U(\overline{r}_1(y), y)$. From the Envelope Theorem it follows that $\frac{\partial W(y)}{y} = L_2(\underline{r}_1(y), y)$, and $\frac{\partial V(y)}{y} = U_2(\overline{r}_1(y), y)$. Hence, if $A3$ holds, we know that W increases in y quicker then V does. It means that when the overall reaction curve jumps down along the diagonal, it never jumps up again. ■

Proof. (of Theorem 2.9) Consider restricted reaction curves $r_1|_{\Delta_U}(y)$ and $r_2|_{\Delta_U}(x)$, both decreasing as implied by $A1'$ and Topkis's Monotonicity Theorem. From Lemma 2.20 and the monotonicity it follows that $c \geq r_1|_{\Delta_U}(0) \geq r_1|_{\Delta_U}(d) > d$ and $0 < r_2|_{\Delta_U}(0) \leq r_2|_{\Delta_U}(d) \leq d$, therefore $r_1|_{\Delta_U}(y) : [0, d] \to [d, c]$ and $r_2|_{\Delta_U}(x) : [d, c] \to [0, d]$ are well defined. Define the mapping $B : [d, c] \to [d, c]$, $B(x) = r_1|_{\Delta_U} \circ r_2|_{\Delta_U}(x)$, which is increasing given that each of $r_1|_{\Delta_U}$ and $r_2|_{\Delta_U}$ is decreasing. From Tarski's Fixed Point Theorem, we know that there exists $\bar{x}$ such that $B(\bar{x}) = r_1|_{\Delta_U} \circ r_2|_{\Delta_U}(\bar{x})$, therefore $(\bar{x}, r_2(\bar{x})|_{\Delta_U})$ is a PSNE.

From Lemma 2.20, there is no symmetric PSNE in $[0, c]$. Hence, there must exist at least one pair of PSNEs. ■

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CHAPTER 3

SYMMETRIC VERSUS ASYMMETRIC EQUILIBRIA IN SYMMETRIC $N-$PLAYER SUPERMODULAR GAMES

This is joint work with Rabah Amir and Michal Jakubczyk

3.1 Introduction

The class of games with strategic complementarities encompasses a substantial part of all strategic games that arise naturally in various branches of economic analysis. This class of games is characterized by the properties that higher actions by a subset of the players increases the marginal returns to the remaining players of increasing their actions, and that higher components of the action vector of any player likewise increase the marginal returns to the same player of increasing the other components of his action. While the importance of this class of games has been known to economists for a long time[1], a complete understanding of their theoretical foundations as well as of their general properties has only recently been achieved with the advent of the lattice-theoretic ap-

[1] See e.g. Bulow, Geanakoplos and Klemperer (1985) and Fudenberg and Tirole (1984).

proach, proposed by Topkis (1978) and further developed by several authors.[2] A key feature of these games, which is invariably critical for economic applications, is that they always admit pure-strategy Nash equilibrium points, even if the best responses are not continuous in any way (or the payoff functions satisfy no quasi-concavity assumptions whatsoever). In addition, these games possess several other general properties that make them especially attractive in a variety of settings in economics, including, under quite general conditions, nice comparative statics properties, dominance solvability and convergence of a broad class of natural learning algorithms. In particular, another property that will be of quite some interest for the present paper is that, with the additional assumption of positive externalities, i.e. that each player payoff is monotonically increasing in other players' strategies, the set of pure-strategy Nash equilibrium points is Pareto-ranked, with the greatest equilibrium being the Pareto-dominant equilibrium. This property has been invoked in many applications in industrial organization and most critically in the macroeconomic literature dealing with Keynesian-type coordination failure (Cooper, 1999).

Owing to simplicity motivations, most applications in industrial organization (Vives, 1999), macroeconomics (Cooper, 1999) and other fields that utilize the framework of games with strategic complementarities postulate identical players, and thus symmetric

[2] A partial list of studies, which have some relationship to the present paper, are Vives (1990), Milgrom and Roberts (1990), Milgrom and Shannon (1994), Amir (1996), Amir and Lambson (2000), and Echenique (2003). Book or survey treatments include Topkis (1998), Vives (1999) and Amir (2005).

games, as well as one-dimensional action sets. Under these assumptions, though possibly multiple, the pure-strategy Nash equilibria are essentially all symmetric, i.e. they involve all players using the same equilibrium action (for a proof, see Amir, 1996, p. 145).

In particular, one line of literature relying on this framework, with the additional assumptions of positive externalities and sufficiently strong strategic complementarities, deals with game-theoretic models of Keynesian coordination failure. These studies typically develop arguments and results that rely critically on the presence of multiple, symmetric, and Pareto-ranked pure-strategy Nash equilibria (PSNEs).

The main purpose of the present paper is to investigate thoroughly the general properties of the class of symmetric games with strategic complementarities. We generalize to the extent possible the known results for the case of games with one-dimensional action sets to the general case of games with action spaces that are complete lattices. One key issue addressed is the extent to which all equilibria tend to be symmetric for the general case of multi-dimensional (i.e. only partially ordered) strategy spaces. We find that the scope for asymmetric equilibrium behavior is definitely broader than in the one-dimensional case, though still quite limited. Specifically, while asymmetric equilibria are possible even for strictly supermodular games, they cannot involve one player using a larger strategy (in the product order) than another player.[3] An-

[3]For instance, in case each player's action is a Euclidean vector, there can be no PSNE where the coordinates of one player's action are all larger than

other key question investigated here is whether asymmetric PSNEs are always Pareto dominated by symmetric PSNEs. While this need not hold in general for games with strategic complementarities, we identify different sufficient conditions that guarantee that such dominance holds.

While these results may be useful to the applied researcher in a number of different ways, there are two areas of application that often consider issues for which the results of the present paper may present a particular interest. One of these is the aforementioned macroeconomics literature. Given that the models typically considered require the assumption of positive externality for the game under consideration, our results on the Pareto dominance of symmetric PSNE relying on assumptions other than positive externality may well allow for a broader scope of applicability of the underlying theories. Furthermore, the multi-dimensional nature of our analysis may constitute another feature of interest in this case in view of the macroeconomic nature of these studies.

The second literature is recent and deals with symmetry-breaking or endogenous heterogeneity, and has strands in various fields of economic inquiry including macroeconomics (Matsuyama, 2002), industrial organization (Amir and Wooders, 2000), and business strategy (Nelson, 1991). The key question in this literature is whether strategic interaction could turn ex ante identical players into necessarily heterogeneous players at equilibrium. In different

those of another player.

words, can symmetric games possess exclusively asymmetric equilibria, or in case both asymmetric and symmetric equilibria coexist, can one invoke some plausible arguments to select out the latter?[4] The results of the present paper may well turn out to be relevant to this literature as well. For instance, one possible conclusion from this paper might say that in a supermodular game with two-dimensional action sets, at an asymmetric equilibrium, if a player's first action is larger than his rival's first action, the second actions must be ranked in the opposite manner. Hence, with strategic complementarity, the scope for endogenous heterogeneity is limited in this specific sense.

The remainder of this paper is organized as follows. The next section presents all mathematical preliminaries that are useful for our results. Section 3.3 contains all our results, along with some examples illustrating the fact that the underlying assumptions cannot be weakened. The last section contains some remarks on the scope of these results.

3.2 Lattice-theoretic preliminaries

A set S together with a binary relation $\leq$ that is reflexive, antisymmetric and transitive is a *partially ordered set* (henceforth *poset*). Denote the induced asymmetric and transitive relation by $<$.

[4] An example of such argument is Cournot instability of a PSNE, as used in Matsuyama (2002) in a complex model where player's action spaces are two-dimensional.

Let T be a subset of a poset S. We say that $t \in T$ is the least (greatest) element of T if $\forall s \in T: \ t \leq s \ (t \geq s)$. We say that $s \in S$ is the upper (lower) bound of T if $\forall t \in T$ we have $t \leq s \ (t \geq s)$. The least upper bound (greatest lower bound) of T is called a *supremum* (*infimum*) and denoted $\sup(T)$ $(\inf(T))$. We say that $t \in T$ is the maximal (minimal) element of T if $\neg\exists s \in T: \ t < s \ (s < t)$.

A poset S is a *lattice* if $\forall s_1, s_2 \in S$ we have $s_1 \vee s_2 \equiv \sup(\{s_1, s_2\}) \in S$ and $s_1 \wedge s_2 \equiv \inf(\{s_1, s_2\}) \in S$. A poset S is a *complete lattice*, if $\forall T \subset S$ we have $\sup(T) \in S$ and $\inf(T) \in S$. A subset T of a lattice S is a *sublattice*, if $\forall t_1, t_2 \in T: \ t_1 \vee t_2 \in T$ and $t_1 \wedge t_2 \in T$, where $\vee$ and $\wedge$ are taken w.r.t. S. A *chain* $C \subset S$ is a totally ordered subset of S, that is, for any $s_1 \in C$ and $s_2 \in C$, $s_1 \geq s_2$ or $s_2 \geq s_1$.

If S is a lattice, a function $F : S \to \mathbb{R}$ is *supermodular* if $\forall s_1, s_2 \in S$, we have $F(s_1 \vee s_2) + F(s_1 \wedge s_2) \geq F(s_1) + F(s_2)$. If the inequality is strict for unordered elements s_1 and s_2, the function is *strictly* supermodular. A function $F : S \to \mathbb{R}$ is *quasi supermodular* if $\forall s_1, s_2 \in S, \ F(s_1) \geq (>) F(s_1 \wedge s_2) \Rightarrow F(s_1 \vee s_2) \geq (>) F(s_2)$. Obviously supermodularity implies quasi-supermodularity.

Theorem 3.1 (Topkis, 1998) *Let $F : S \longrightarrow \mathbb{R}$ be supermodular on S, then* $\arg\max_{s \in S} F(s)$ *a sublattice of S.*

Given posets S and T, we say that a function $F(s,t) : S \times T \to \mathbb{R}$ has *increasing differences* in s and t, if $\forall s_1, s_2 \in S$, and $t_1, t_2 \in T$, such that $s_1 < s_2$ and $t_1 < t_2$, we have $F(s_1, t_2) - F(s_1, t_1) \leq$

$F(s_2, t_2) - F(s_2, t_1)$. If the inequality is strict, then we say that function F has *strictly* increasing differences. (Strict) supermodularity implies (strictly) increasing differences. The opposite is true if the domain is a product of chains. The next result says that on the direct product of finitely many lattices, increasing differences together with supermodularity in each component implies overall supermodularity.

Theorem 3.2 (Topkis, 1998) *Let S_i be a lattice for $i = 1, \ldots, n$, and $S^n = \times_{i=1}^n S_i$ (thus S^n is a lattice in the product order). If $F : S^n \to \mathbb{R}$ has (strictly) increasing differences on S^n, and is (strictly) supermodular in $s_i \in S_i$ for all fixed $s_{i'} \in S_{i'}$ for all $i' \neq i$ and for each $i = 1, \ldots, n$, then $F(\cdot)$ is (strictly) supermodular on S^n.*

A function $F(s, t) : S \times T \to \mathbb{R}$, where S is a lattice and T is a poset, has the *(strict) single crossing property* if $\forall s_1, s_2 \in S$, and $t_1, t_2 \in T$, such that $s_1 \leq s_2$ and $t_1 \leq t_2$, we have $F(s_1, t_2) - F(s_1, t_1) \geq 0 \Rightarrow F(s_2, t_2) - F(s_2, t_1) \geq (>)0$. Both (strict) increasing differences and quasi supermodularity imply (strict) single crossing property. The following theorem characterizes maximizers of functions with the property of supermodularity and increasing differences.

Theorem 3.3 (Topkis's Monotonicity Theorem) *Let S be a lattice and T be a poset. Suppose $F : S \times T \to \mathbb{R}$ is supermodular in s for given t and has increasing differences in s and t. Then*

$s^*(t) \equiv \arg\max_{s\in S} F(s,t)$ *is ascending and its extremal selections* $\overline{s}^*$ *and* $\underline{s}^*$ *are increasing in* t*.*

The conclusion of this result continues to hold if supermodularity is replaced with quasi supermodularity, and increasing differences with the single crossing property (Milgrom and Shannon, 1994).

3.3 Symmetric versus asymmetric PSNE

Consider a normal-form $n-$player game given by (I, S_i, F_i), where $I = \{1, \ldots, n\}$ is the player set, S_i is player i's action set and $F_i : S^n = \times_{i=1}^n S_i \longrightarrow R$ is his payoff function. We first formulate the definition of $n-$player symmetric supermodular game, the main object of interest of this paper.[5]

Definition 3.4 *A normal-form* $n-$*player game* (I, S_i, F_i) *is a* symmetric (strictly) supermodular game *if*

i) the strategy spaces are equal, $S_1 = \ldots = S_n \triangleq S$*, and have the same partial order,*

ii) each F_i *satisfies* $F_i(s_1, \ldots, s_n) = F_{\sigma(i)}(s_{\sigma^{-1}(1)}, \ldots, s_{\sigma^{-1}(n)})$*, for all* $i \in I$*,* $s_1 \in S_1$*, …,* $s_n \in S_n$*,* $\sigma(\cdot)$ *being any permutation of* I*,*

iii) S *is a complete lattice,*

[5] Two elements of notation are needed throughout the paper. We follow the usual abuse of notation that consists of writing player i's payoff function as $F_i(s_i, \mathbf{s}_{-i})$. For extra clarity vector-valued arguments are bolded, while scalar arguments are not.

iv) each payoff F_i is upper semi continuous (usc) in own strategy[6] $s_i \in S$, *for any* $\mathbf{s}_{-i}$,

v) F_i *is (strictly) supermodular in* s_i, *for any* $\mathbf{s}_{-i}$, *and*

vi) F_i *has (strictly) increasing differences in* $(s_i, \mathbf{s}_{-i})$.

Observe that in this definition, assumptions i)-ii) simply define a symmetric game with ordered action spaces, while iii)-vi) make it a supermodular game.

If we replace supermodularity with quasi supermodularity and increasing differences with the single crossing property, we get a larger family of games that we call *ordinally supermodular games*, which were introduced by Milgrom and Shannon (1994).

Let $r_i(\mathbf{s}_{-i}) = \arg\max_{s \in S_i} F_i(s, \mathbf{s}_{-i})$, for all $\mathbf{s}_{-i} \in S^{n-1}$, be firm i's *reaction correspondence*. Then the overall best response (or best reply) correspondence is defined by

$$\begin{array}{rlcl} \mathbf{B}: & S^n & \rightarrow & 2^{S^n} \\ & (s_1, ..., s_n) & \rightarrow & (r_1(\mathbf{s}_{-1}), ..., r_n(\mathbf{s}_{-n})) \,. \end{array}$$

By Topkis's Monotonicity Theorem, in (ordinally) supermodular games, the extremal selections of the best response correspondence are increasing, since the extremal selections of every $r_i(\mathbf{s}_{-i})$ are increasing.

Remark 3.5 *Notice that if* $(s_1, \ldots, s_n)$ *is a PSNE in a game defined as in definition 3.4, then for any permutation* $\sigma(\cdot)$ *strategy profile* $(s_{\sigma(1)}, \ldots, s_{\sigma(n)})$ *is also a PSNE.*

[6] With respect to the interval topology of S.

The set of pure-strategy Nash equilibria (PSNE) of this game is non-empty. Moreover Zhou showed (Theorem 2) that the set of PSNE is also a complete lattice. Both these results hold also for ordinally supermodular games. This proposition is a corollary from Milgrom and Roberts (1990), page 1266.

Proposition 3.6 *Consider a symmetric $n-$player (ordinally) supermodular game. Then the greatest and least PSNE (w.r.t. strategies) exist and are symmetric.*

Proof. Take any asymmetric PSNE $\mathbf{s} = (s_1, \ldots, s_n)$. i.e. such that $\exists 1 \leq i < j \leq n :: s_i \neq s_j$. Then, from remark 3.5, it follows that any $\mathbf{s}_\sigma = (s_{\sigma(1)}, \ldots, s_{\sigma(n)})$ is also a PSNE (we can reorder strategies and still have a PSNE). Denote by S_σ the set of all permutations of $\mathbf{s}$. From Zhou (1994) we conclude that $\sup(S_\sigma)$ and $\inf(S_\sigma)$, taken w.r.t. set of PSNE exist and are themselves PSNEs . They are symmetric by their very construction. ■

Existence of asymmetric PSNE can be excluded only for strictly supermodular games and ordered strategies.

Proposition 3.7 *In a symmetric $n-$player strictly ordinally supermodular game, any strategy profile $(s_1, \ldots, s_n)$ that has at least two strategies unequal but ordered (i.e. $\exists 1 \leq i < j \leq n :: s_i < s_j$) cannot be a PSNE.*

Proof. Assume there is such a PSNE $(s_1, \ldots, s_n)$, where we can assume that $s_1 < s_2$ (this is without loss of generality as we can reorder elements and still have a PSNE). From the PSNE property, we get $F_i(s_1, s_2, s_3, \ldots, s_n) \geq F_i(s_2, s_2, s_3, \ldots, s_n)$. We know that $(s_2, s_1, s_3, \ldots, s_n)$ is also a PSNE, and so $F_1(s_2, s_1, s_3, \ldots, s_n) \geq F_1(s_1, s_1, s_3, \ldots, s_n)$. These two inequalities contradict the very definition of the strict single crossing property condition (as the first inequality would imply $F_1(s_1, s_1, s_3, \ldots, s_n) > F_1(s_2, s_1, s_3, \ldots, s_n)$). Hence, $(s_1, \ldots, s_n)$ cannot be a PSNE of the game. ■

From Proposition 3.7 we can deduce the following statement, proved by Amir (1996), page 145.

Corollary 3.8 *In a symmetric $n-$player strictly ordinally supermodular game, with the strategy space S being a chain, there exists no asymmetric PSNE .*

In more general strategy spaces, asymmetric PSNE cannot be excluded. We can however consider comparisons amongst equilibria with respect to payoffs. Below we formulate sets of conditions for $n-$players games that ensure that every asymmetric PSNE is Pareto dominated by a symmetric one. We say that a PSNE is (strictly) Pareto dominated by another PSNE if the latter secures (strictly) better payoffs for all players. If a PSNE (strictly) Pareto dominates every other PSNE, then this PSNE is said to be (strictly) Pareto-dominant.

One set of such conditions is an immediate corollary of Theorem 2 in Zhou (1994) and Theorem 7 in Milgrom and Roberts (1990).

Corollary 3.9 *Consider a symmetric* $n-$*player ordinally*[7] *supermodular game with strategy space* S^n*. If player's* i *payoff* $F_i(s_i, \mathbf{s}_{-i})$ *is increasing (decreasing) in* $\mathbf{s}_{-i}$ *for all* i*, then the greatest PSNE is Pareto-dominant, which is symmetric. When the single-crossing property or the monotonicity is strict, the stated Pareto dominance is actually strict.*

Such a situation is presented in example 3.10.

Example 3.10 *Consider a game as in definition 3.4 with* $n = 2$ *and* $S = \{(0,0), (1,0), (0,1), (1,1)\}^2$ *ordered coordinate-wise. The payoffs given in the normal form are strictly supermodular and monotonic.*

	(0,0)	**(1,0)**	**(0,1)**	**(1,1)**
(0,0)	$(\mathbf{60}, \mathbf{60})$	$(80, 20)$	$(80, 20)$	$(200, 0)$
(1,0)	$(20, 80)$	$(50, 50)$	$(\mathbf{205}, \mathbf{200})$	$(350, 190)$
(0,1)	$(20, 80)$	$(\mathbf{200}, \mathbf{205})$	$(50, 50)$	$(350, 190)$
(1,1)	$(0, 200)$	$(190, 350)$	$(190, 350)$	$(\mathbf{600}, \mathbf{600})$

There are four PSNEs, in bold: $((0,0), (0,0))$*,* $((1,0), (0,1))$*,* $((0,1), (1,0))$ *and* $((1,1), (1,1))$*. Two of them are asymmetric, but both are strictly Pareto dominated by* $((1,1), (1,1))$*.*

Now we consider a special kind of symmetric games, in which for any strategy profile, the payoffs of all the players are equal.[8]

[7] Milgrom and Roberts (1990) give Theorem 7 for supermodular games, but the generalization for ordinally supermodular case is straightforward.

[8] Cf. Weibull (1995). Such games are also called *partnership games* (Hofbauer and Sigmund, 1989) or *symmetric games of identical interests* (Monderer and Shapley, 1996).

This restriction enables us to provide another set of conditions to guarantee that asymmetric PSNEs are Pareto dominated.

Proposition 3.11 *Consider a symmetric* $n-$*player supermodular game. If payoffs are doubly symmetric, i.e.* $F_1(\mathbf{s}) = F_2(\mathbf{s}) = \ldots = F_n(\mathbf{s}) \equiv F(\mathbf{s})$*, for all* $\mathbf{s} \in \mathbf{S}^n$*, then any asymmetric PSNE is Pareto dominated by a symmetric PSNE. If* $F(\cdot)$ *is strictly supermodular, the dominance is strict as well.*

Proof. Let GM denote the set of global maximizers of F. If all players play a strategy profile $\mathbf{s}^*$ that belongs to GM, none of them has a unilateral incentive to deviate, since their payoffs (which are equal) correspond to global maxima of the common objective F. Hence, $\mathbf{s}^*$ is also a PSNE, and therefore $GM \subset PSNE$. Moreover, any element of GM payoff dominates all elements of $PSNE \setminus GM$.

It follows from the supermodularity of the game and the equality of all payoffs that each of them is supermodular in each $s_i, i = 1, 2, ..., n$, and has increasing differences in (s_i, s_j) for $i \neq j$. Hence, by Theorem 3.2, F is supermodular on S^n. This enables us to use Theorem 3.1 to conclude that GM is a non-empty sublattice of S^n. Take any $\mathbf{s} = (s_1, ..., s_n) \in GM$ that is asymmetric. Then, from the symmetry of the game, all the permutations of $\mathbf{s}$, i.e. all of $\mathbf{s}_\sigma = (s_{\sigma(1)}, ..., s_{\sigma(n)})$, belong to GM. The supremum and infimum of all the permutations of $\mathbf{s}$, which are clearly both symmetric, also belong to GM, since from Theorem 3.1 GM is a sublattice. Hence we conclude that amongst all PSNE having the maximal payoff, the maximal and minimal ones (in strategies) are symmetric.

Consider now a game with $F(\cdot)$ strictly supermodular. It is sufficient to show that there are no asymmetric strategy profiles in GM. Assume the contrary and take any asymmetric $\mathbf{s} = (s_1, s_2, ..., s_n) \in GM$. Then w.l.o.g we may take, say, s_1 and s_2 to be unordered, by Proposition 3.7. Then by symmetry, $(s_2, s_1, ..., s_n) \in GM$ as well. Since GM is a sublattice, $(s_1 \vee s_2, s_1 \vee s_2, ..., s_n)$ and $(s_1 \wedge s_2, s_1 \wedge s_2, ..., s_n)$ are also in GM. By strict supermodularity of F_i, we have $F(s_1 \vee s_2, s_1 \vee s_2, ..., s_n) + F(s_1 \wedge s_2, s_1 \wedge s_2, ..., s_n) > F(s_1, s_2, ..., s_n) + F(s_2, s_1, ..., s_n)$, which contradicts the fact that $(s_1, s_2, ..., s_n) \in GM$ and $(s_2, s_1, ..., s_n) \in GM$. Hence, there are only symmetric strategy profiles in GM, and therefore any asymmetric PSNE is strictly Pareto dominated by a symmetric one. ■

A potential field of application for proposition 3.11 is the theory of *matching,* introduced by Becker (1973). A recent development of this theory is Shimer and Smith (2000).

The following remarks show that the assumptions in proposition 3.11 are minimal.

Remark 3.12 *Proposition 3.11 does not rule out an asymmetric Pareto-dominated PSNE. As an example consider a double symmetric 2−player game with* $S = \{(0,0), (0,1), (1,0), (1,1), (2,2)\}$ *ordered coordinate-wise and with strictly supermodular payoffs given by the normal form given below.*

There are five PSNEs, in bold: $((0,0), (0,0))$, $((1,0), (0,1))$, $((0,1), (1,0))$ *and* $((1,1), (1,1))$ *and* $((2,2), (2,2))$. *Two of them are asymmetric, but they do not belong to* GM, *since* $GM =$

	(0,0)	**(1,0)**	**(0,1)**	**(1,1)**	**(2,2)**
(0,0)	$(\mathbf{10},\mathbf{10})$	$(2,2)$	$(2,2)$	$(0,0)$	(-13,-13)
(1,0)	$(2,2)$	$(0,0)$	$(\mathbf{3},\mathbf{3})$	$(2,2)$	(-10,-10)
(0,1)	$(2,2)$	$(\mathbf{3},\mathbf{3})$	$(0,0)$	$(2,2)$	(-10,-10)
(1,1)	$(0,0)$	$(2,2)$	$(2,2)$	$(\mathbf{10},\mathbf{10})$	(-1,-1)
(2,2)	$(-13,-13)$	$(-10,-10)$	$(-10,-10)$	$(-1,-1)$	**(0,0)**

$\{((0,0),(0,0)),\ ((1,1),(1,1))\}$.

Remark 3.13 *If we relax the assumption $F_1 = F_2 = \ldots = F_n$ in proposition 3.11 we can get an asymmetric non-Pareto-dominated PSNE. Consider a $2-$player game with $S = \{(0,0),(0,1),(1,0),(1,1)\}$ ordered coordinate-wise and strictly supermodular payoffs given by the normal form below.*

	(0,0)	**(1,0)**	**(0,1)**	**(1,1)**
(0,0)	$(\mathbf{300},\mathbf{300})$	$(260,35)$	$(40,160)$	$(5,-55)$
(1,0)	$(35,260)$	$(0,0)$	$(\mathbf{100},\mathbf{360})$	$(70,150)$
(0,1)	$(160,40)$	$(\mathbf{360},\mathbf{100})$	$(0,0)$	$(270,65)$
(1,1)	$(-55,5)$	$(150,70)$	$(65,270)$	$(\mathbf{350},\mathbf{350})$

There are four PSNEs, in bold: $((0,0),(0,0))$, $((1,0),(0,1))$, $((0,1),\ (1,0))$ and $((1,1),(1,1))$. None of the symmetric PSNEs Pareto dominates the asymmetric ones.

Remark 3.14 *Proposition 3.11 does not hold for ordinally supermodular games. The reason it that Theorem 3.2 does not hold if* *we replace increasing differences by the single crossing property. We can construct a game having asymmetric non-Pareto-dominated PSNEs. Consider a* $2-$*player game with* $S = \{(0,0), (0,1), (1,0), (1,1)\}$ *ordered coordinate-wise and ordinally supermodular payoffs given by the normal form below.*

	(0,0)	**(1,0)**	**(0,1)**	**(1,1)**
(0,0)	$(\mathbf{9}, \mathbf{9})$	$(4,4)$	$(4,4)$	$(0,0)$
(1,0)	$(4,4)$	$(-3,-3)$	$(\mathbf{10}, \mathbf{10})$	$(4,4)$
(0,1)	$(4,4)$	$(\mathbf{10}, \mathbf{10})$	$(-3,-3)$	$(4,4)$
(1,1)	$(0,0)$	$(4,4)$	$(4,4)$	$(\mathbf{9}, \mathbf{9})$

There are four PSNEs, in bold: $((0,0),(0,0))$, $((1,0),(0,1))$, $((0,1),(1,0))$, $((1,1),(1,1))$, *but the asymmetric ones:* $((1,0),(0,1))$ *and* $((0,1),(1,0))$ *have higher payoff than the symmetric ones.*

We introduce another class of functions, mentioned by Shannon (1995). A function F is said to be (*strictly*) *superjoin* if

$$F(x \vee y) \vee F(x \wedge y) \geq (>) F(x) \vee F(y).$$

The next proposition refers to $2-$player games and gives another set of conditions to maintain the result of proposition 3.11. For the proof we use the result of Echenique (2003) that, for $2-$players ordinally supermodular games with ordered strategy spaces, the set of PSNEs is a sublattice of the joint strategy space.

Proposition 3.15 *Consider a symmetric (but not necessarily doubly symmetric)* $2-$ *player ordinally supermodular game. Assume* *that the strategy space of each player is a chain (i.e. completely ordered) and that for any asymmetric strategy profile* $\mathbf{s} = (s_1, s_2)$*,* $s_1 \neq s_2$ *and its permutation* $\mathbf{s}' = (s_2, s_1)$*, the payoff function* F_i *is superjoin. Then there exists a Pareto dominant PSNE, which is symmetric.*

Proof. Take any asymmetric PSNE $\mathbf{s} = (s_1, s_2)$. Then by symmetry $\mathbf{s}' = (s_2, s_1)$ is also a PSNE. Since the set of PSNEs is a sublattice of S^2 (Echenique, 2003), we know that $(\mathbf{s} \vee \mathbf{s}')$ and $(\mathbf{s} \wedge \mathbf{s}')$, where $\vee$ and $\wedge$ are taken in S^2, belong to PSNE. $(\mathbf{s} \vee \mathbf{s}')$ and $(\mathbf{s} \wedge \mathbf{s}')$ are symmetric. The assumption $F_i(\mathbf{s} \vee \mathbf{s}') \vee F_i(\mathbf{s} \wedge \mathbf{s}') > F_i(\mathbf{s}) \vee F_i(\mathbf{s}')$, $\forall i = 1, 2$, guarantees that any asymmetric equilibrium is Pareto dominated by a symmetric one. ■

If $F(\mathbf{s})$ is the payoff function in a game and it is superjoin, it means that a certain amount of return to coordination exists in the game. Coordination pays off more than disparity, as the maximal payoff to the agent if everyone chooses identical actions, either higher levels or lower levels, is greater than the maximal payoff if only one agent chooses higher and another one chooses lower level. Such a situations related to the coordination failure models discussed e.g. in Cooper (1999).

When we relax the assumption of completely ordered strategy space we have to add one of two possible conditions, formulated in the next proposition, to obtain the same result.

Proposition 3.16 *Consider a* $2-$*player symmetric (not necessarily doubly symmetric) supermodular game.*

(C1) If for all asymmetric strategy profiles $\mathbf{s} = (s_1, s_2) \in S^2$, $\mathbf{s}^* = (s_2, s_1) \in S^2$, *s.t.* $s_1 \neq s_2$ *players' payoffs is superjoin and at least one of two conditions is satisfied :*

(C2) S *is finite;*

(C2') $F_i(s_i, s_{-i})$ *is usc in* $s_{-i}, i = 1, 2$.

Then there exists a PSNE that Pareto dominates all other PSNEs, and it is symmetric.

The proof of proposition 3.16 uses the following lemma.

Lemma 3.17 *Consider a game defined as in proposition 3.16. Let* $\bar{\mathbf{B}}(\cdot)$ *and* $\underline{\mathbf{B}}(\cdot)$ *denote the greatest and the least elements of the best-response correspondence. If for some strategy profile* $\mathbf{s}^* \in S^2$, $\mathbf{s}^* = (s^*, s^*)$, *the best response correspondence satisfies* $\bar{\mathbf{B}}(\mathbf{s}^*) \geq \mathbf{s}^*$ *then there exists a strategy profile* $\mathbf{s}^{**} \in PSNE$, $\mathbf{s}^{**} = (s^{**}, s^{**})$, *such that* $F_i(\mathbf{s}^{**}) \geq F_i(\mathbf{s}^*)$, $\forall i \in \{1, 2\}$.

Alternatively, another lemma can be formulated if we replace $\bar{\mathbf{B}}(\cdot)$ with $\underline{\mathbf{B}}(\mathbf{s}^*) \leq \mathbf{s}^*$. The result of the lemma 3.17 remains unchanged. We give a proof of lemma 3.17 only, since the proof of the latter case is analogous.

Proof. Denote $\mathbf{F}(\mathbf{s}) = [F_1(\mathbf{s}), F_2(\mathbf{s})]$. For given $\mathbf{s}^* = (s^*, s^*)$ define a poset

$$G = \left\{\mathbf{s} = (s, s) \in S^2 : \bar{\mathbf{B}}(\mathbf{s}) \geq \mathbf{s} \text{ and } \mathbf{F}(\mathbf{s}) \geq \mathbf{F}(\mathbf{s}^*) \text{ and } \mathbf{s} \notin PSNE\right\}.$$

Obviously if $\mathbf{s}^* \notin G$ then it means that $\mathbf{s}^* \in PSNE$ and letting $\mathbf{s}^{**} = \mathbf{s}^*$ we finish the proof. Hence we know that $G \neq \emptyset$. Using the Hausdorff maximal principle[9] we infer that the maximal chain $C \subset G$ exists and we define $\mathbf{c}^* = \sup(C)$, $\mathbf{c}^* = (c^*, c^*)$.

Case 1: Assume that condition C2 holds and so S^2 is finite. Then we have $\mathbf{c}^* \in C$. Define $\mathbf{s}^{**} = (s^{**}, s^{**})$ as $\mathbf{s}^{**} = \bar{\mathbf{B}}(\mathbf{c}^*)$. As $\bar{\mathbf{B}}(\mathbf{c}^*) > \mathbf{c}^*$ ($\mathbf{c}^* \notin PSNE$ by definition of G) we get that $\mathbf{s}^{**} > \mathbf{c}^*$, so $\bar{\mathbf{B}}(\mathbf{s}^{**}) \geq \bar{\mathbf{B}}(\mathbf{c}^*)$, and finally $\bar{\mathbf{B}}(\mathbf{s}^{**}) \geq \mathbf{s}^{**}$.

By definition of best response correspondence $F_1(s^{**}, c^*) \geq F_1(\mathbf{c}^*)$ and $F_2(c^*, s^{**}) \geq F_2(\mathbf{c}^*)$. From these inequalities and C1 condition applied for both players payoffs: $F_i(s^{**}, c^*) \vee F_i(c^*, s^{**}) < F_i(\mathbf{c}^*) \vee F_i(\mathbf{s}^{**})$, $i = 1, 2$, we conclude that $\mathbf{F}(\mathbf{s}^{**}) \geq \mathbf{F}(\mathbf{c}^*)$. As $\mathbf{s}^{**} \notin G$ ($\mathbf{s}^{**} > \mathbf{c}^*$ and C is maximal) we infer that $\mathbf{s}^{**} \in PSNE$, which finishes the proof.

Case 2: Assume now that C2' holds. It may happen then that $\mathbf{c}^* \notin C$. Hence $\forall \mathbf{c} \in C :\ \mathbf{c}^* > \mathbf{c}$, and so $\forall \mathbf{c} \in C :\ \bar{B}(\mathbf{c}^*) \geq \bar{B}(\mathbf{c})$. As $\forall \mathbf{c} \in C :: \bar{B}(\mathbf{c}) \geq \mathbf{c}$ we get that $\bar{B}(\mathbf{c}^*) \geq \sup(C) = \mathbf{c}^*$.

Now we'll prove that $\mathbf{F}(\mathbf{c}^*) \geq \mathbf{F}(\mathbf{s}^*)$. Assume otherwise — hence $F_1(\mathbf{c}^*) < F_1(\mathbf{s}^*)$. Then $\exists \epsilon > 0$ such that $\forall \mathbf{c} \in C :\ F_1(\mathbf{c}) - F_1(\mathbf{c}^*) > 2\epsilon$.

From the usc property of $F_1(\cdot, c^*)$ we conclude that $\exists (\hat{c}_1, \hat{c}_1) \in C$, such that $\forall s_1 \in S$, $s_1 \geq \hat{c}_1$ we have $F_1(s_1, c^*) < F_1(c^*, c^*) + \epsilon$. Continuing the construction, for any such s_1, from the usc property of the function $F_1(s_1, \cdot)$ (guaranteed by condition C2') we conclude

[9] In every poset there exists a maximal chain (w.r.t. inclusion relation).

that $\exists(\hat{c}_2, \hat{c}_2) \in C$, such that $\hat{c}_2 \geq s_1$ and $\forall s_2 \in S$, $s_2 \geq \hat{c}_2$ we get $F_1(s_1, s_2) < F_1(s_1, c^*) + \epsilon$ and so $F_1(s_1, s_2) < F_1(\mathbf{c}^*) + 2\epsilon$.

From the increasing differences property ($s_2 \geq s_1$) we conclude that $F_1(s_2, s_2) < F_1(s_2, c^*) + \epsilon$ and because $s_2 \geq \hat{c}_1$ we also get $F_1(s_2, s_2) < F_1(\mathbf{c}^*) + 2\epsilon$. It yields a contradiction and so $\mathbf{F}(\mathbf{c}^*) \geq \mathbf{F}(\mathbf{s}^*)$. We infer that $\mathbf{c}^* \in PSNE$ and taking $\mathbf{s}^{**} = \mathbf{c}^*$ we finish the proof. ■

Using the lemma 3.17 we can now prove Proposition 3.16.

Proof. If $(s_1, s_2) \in PSNE$ then also $(s_2, s_1) \in PSNE$, and obviously $F_1(s_1, s_2) = F_2(s_2, s_1)$. Because of the C1 condition one of the strategy profiles: $(s_1 \vee s_2, s_1 \vee s_2)$, $(s_1 \wedge s_2, s_1 \wedge s_2)$ must yield a strictly greater payoff for the first player than in any of the strategy profiles (s_1, s_2), (s_2, s_1) and so the payoff must also be greater for the second player (both players' payoffs are equal in symmetric strategy profiles).

Assume that $F_1(s_1 \vee s_2, s_1 \vee s_2) > \max(F_1(s_1, s_2), F_1(s_2, s_1))$ (otherwise we may use Lemma 3.17 with $\underline{\mathbf{B}}(\cdot)$ assumption). It is now easy to show (Topkis's Monotonicity Theorem) that $\bar{\mathbf{B}}(s_1 \vee s_2, s_1 \vee s_2) \geq (s_1 \vee s_2, s_1 \vee s_2)$, and so we can use Lemma 3.17 with $\mathbf{s}^* = (s_1 \vee s_2, s_2 \vee s_1)$ which finishes the proof. ■

Example 3.18 *Proposition 3.16 does not work for ordinally supermodular games. The reason for that is that usc in each strategy space does not guarantee the usc in whole strategy profile space in ordinally supermodular games. Consider a 2−player double symmetric game with ordinally supermodular payoffs and with* $S =$

$\{-1, a, b\} \cup [0,1]$. S has natural order except for the fact that a and b are unordered elements such that $-1 < a < 0$ and $-1 < b < 0$. The payoffs are given in the normal form below (in the table below only payoffs of the first player are shown, the payoffs of the second player are the same, because of double symmetry) where $h(x, y)$ is

	$\mathbf{-1}$	**a**	**b**	$\mathbf{x \in [0,1)}$	**1**
$\mathbf{-1}$	**2**	0	0	$-1-x$	-3
a	0	0	**4**	$2-x$	1
b	0	**4**	0	$2-x$	1
$\mathbf{y \in [0,1)}$	$-1-y$	$2-y$	$2-y$	$h(x,y)$	$2+y$
1	-3	1	1	$2+x$	**3**

defined (on a larger domain $[0,1] \times [0,1]$) as :

$$h(x,y) = \begin{cases} h(x,x) = 5 + x \\ h(x, \frac{x+1}{2}) = 5 + \frac{3x+1}{4} \\ h(x,1) = 2 + x \\ h(x,y) \textit{ is otherwise a piecewise linear function of } y \end{cases} \tag{3.1}$$

(obviously $h(x,y) = h(y,x)$). It can be verified that the condition $(C1)$ is satisfied, but the asymmetric equilibria (a, b) and (b, a) dominate the symmetric ones: $(-1, -1)$ and $(1, 1)$.

3.4 On the scope of our results

This section examines other classes of games, beyond those with strategic complementarities, which may enjoy similar properties in terms of the limited scope of existence of asymmetric pure-strategy equilibria and of the Pareto dominance of symmetric equilibria.

3.4.1 Submodular games

It is well known that any symmetric $2-$player *submodular* game[10] can be converted into a supermodular by reversing the order on the action space of one of the players. Existence of a PSNE follows directly from this observation for the case of $2-$player games but not for the case of $n-$player games (Vives, 1990 and Milgrom and Roberts, 1990). One might then be tempted to further argue that the conclusions presented in this paper extend to the class of $2-$player submodular games. However, it is straightforward to come up with counter-examples to establish the fact that strictly submodular games need not have any symmetric equilibria.

Consider for instance and anti-coordination game

	b	d
b	$(0,0)$	$(1,2)$
d	$(2,1)$	$(0,0)$

This is a symmetric and submodular game (the order being $d > b$). Yet, it has only asymmetric PSNEs, (b,d) and (d,b).

Another example of interest here is from the theory of Cournot duopoly with increasing returns. Amir and Lambson (2000) show that whenever the inverse demand $P(\cdot)$ and the (common) cost function $C(\cdot)$ satisfy the condition $-P'(x+y)+C''(x)<0$ for all

[10] A function $F(\cdot)$ is submodular if function $-F(\cdot)$ is supermodular. A game is submodular if in Definition 3.4 we replace supermodularity by submodularity and increasing differences by decreasing differences.

$x, y \geq 0$, the Cournot duopoly is strictly submodular (in fact satisfies the stronger property that the reaction curve r is such that $r'(y) \leq -1$) and admits at least two PSNEs, which are asymmetric and of the form $(0, x_m)$ and $(x_m, 0)$, where x_m is the optimal monopoly output. In addition, if each profit function is quasi-concave in own output, there will also be a symmetric interior PSNE. Hence, asymmetric PSNEs are not ruled out despite the strong submodularity of the game. Furthermore, when all three equilibria exist, the asymmetric PSNEs are not Pareto dominated by the symmetric one, although profits are decreasing in the other player's strategy (the monopoly profit, which is the profit of, say, firm 1 under the PSNE $(x_m, 0)$, exceeds its profit at the interior symmetric PSNE). The details are left out for the sake of brevity (see Amir and Lambson, 2000).[11]

The reason the results of this paper do not extend to $2-$player *submodular* game is that the order-reversing transformation breaks the symmetry of the game by making the two action spaces order-different (thus in the anti-coordination game, one player's space would be $\{d, b\}$ while the other's remains $\{b, d\}$, when both are ordered from left to right). This underscores the fact that the definition of symmetry of a game in our setting includes the requirement that the same action spaces be ordered in the same way (see

[11] It is well known that any $n-$player Cournot oligopoly with payoffs that are submodular in (own action, rivals' total output) has a PSNE (see e.g. Novshek, 1985). This result is due to the additional property that each payoff function depends only on own action and on the sum of rivals' actions. The results of Amir and Lambson (2000) clearly show that this class of games does not satisfy any of the properties of symmetric supermodular games.

Definition 3.4)!

Hence, while a 2−player submodular game always satisfies all the properties of a supermodular game, it need not satisfy those extra properties of symmetric supermodular games that rely on symmetry. More generally, while adopting the results of Echenique (2004), wherein the order on the action spaces is not exogenously given, would enlarge the scope of supermodular games, his redefinition of the order would generally destroy the symmetry of the game, for reasons similar to those discussed above.

3.4.2 Games with quasi-convex payoffs

The last class of games we consider is characterized by the assumptions that the game has strict strategic complementarities when the players are restricted to the two extreme actions from their completely ordered action set, and each payoff function is strictly quasi-convex in own action.

Let S be a complete chain. Player i's payoff function F_i is strictly quasi-convex in s_i for fixed $\mathbf{s}_{-i}$ if for all s_i, s_i', s_i'' in S with $s_i < s_i' < s_i''$, we have

$$F_i(s_i', \mathbf{s}_{-i}) > F_i(s_i, \mathbf{s}_{-i}) \wedge F_i(s_i'', \mathbf{s}_{-i}).$$

Denote the minimal and maximal elements of S by 0 and 1, and let $E = \{0, 1\}$. We will write $\widetilde{F}_i : E^n \longrightarrow R$ for the restriction of each payoff function F_i to the subdomain E^n.

Proposition 3.19 *Consider a symmetric $n-$player game satisfy-*

ing

(i) the action space S is a chain,

(ii) each payoff function F_i is strictly quasi-convex in s_i for fixed $\mathbf{s}_{-i}$,

(iii) each payoff function F_i has strictly increasing differences in $(s_i, \mathbf{s}_{-i}) \in E \times E^{n-1}$.

Then the PSNE set consists of either $(0, 0, .., 0), (1, 1, ..., 1)$ or both.

Proof. Consider a restricted symmetric game wherein each player's action set is E and his payoff function is $\widetilde{F}_i : E^n \longrightarrow R$. Since E is trivially a complete lattice, this game is a strictly supermodular game in view of assumption (iii). By Proposition 3.7, we conclude that the PSNE set of this restricted game consists of either $(0, 0, .., 0), (1, 1, ..., 1)$ or both.

We claim that the PSNE of the original game must be the same as the PSNE of the restricted game. For otherwise, for at least one joint strategy $(0, 0, .., 0)$ or $(1, 1, ..., 1)$, one player must have a profitable deviation that is neither 0 nor 1 (the latter are ruled out by the equilibrium property in the restricted game). But this contradicts the strict quasi-convexity of his payoff in own action (assumption (ii)). Hence, the two games must have the same equilibrium set. ■

Although the structure of the class of games considered here is not very prevalent in applications, there are some noteworthy exceptions. One of these is the model of strategic R&D model with constant returns to scale analyzed in Amir (2000).

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CHAPTER 4

MARKET TRANSPARENCY AND BERTRAND COMPETITION

4.1 Introduction

In economics, market is said to be transparent if much is known by many about what products and services are available, at what price and where. Increasing transparency is often considered as a cure for some imperfections and inefficiency of the market. From the consumers' point of view it is supposed to increase competition and their surplus, by leading to price decrease and to reduction in the price dispersion. Due to the growing popularity of the Internet, which allows for quick spreading of information, in many markets the increase in transparency can be observed.

Empirical findings provide mixed evidence on comparing prices in the Internet and in traditional retailers. For instance, Bailey (1998) shows that Internet commerce may not reduce market friction because prices are higher when consumers buy homogeneous products on the Internet, and price dispersion for homogenous products among Internet retailers is greater than the price dispersion among physical retailers. Lee et al. (2000) found that the average product price in one of the most successful electronic commerce systems (an electronic market system for used-car transactions in Japan) is much higher than in traditional, nonelectronic markets. The secondhand cars traded there are usually of much higher quality than those sold in traditional markets, but used-car prices are slightly higher than in traditional markets even for cars of similar quality. Conversely, Brynjolfsson and Smith (2000) observed that books and CDs in the internet are cheaper than in the conventional outlets. They find that prices on the Internet are 9-16% lower than prices in conventional outlets and conclude that while there is lower friction in many dimensions of Internet competition, branding, awareness, and trust remain important sources of heterogeneity among Internet retailers.

In the theoretical literature there are studies explaining the phenomenon that prices do not always go down in case of increased transparency. The main argument, recalled in a number of papers, is that increasing transparency means for the producers facilitating tacit collusion (see e.g. Mollgaard and Overgaard, 2001, Nilsson, 1999, and Schultz, 2005).

The problem of market transparency appeared in many different strands of literature. Varian (1980) showed that in case of homogenous goods and symmetric firms, the expected equilibrium profits decrease in the level of market transparency. This idea was developed in the search literature, for instance Burdett and Judd

(1983) or Stahl (1989). If the cost of searching goes down, the consumers search more and firms' competition becomes tougher. A contribution to explaining market transparency issues can be found as well in the literature on advertising, increased advertising typically leads to lower prices, see, for instance, Bester and Petrakis (1995). They consider transparency as a firms' decision variable, while in this paper it is an exogenous parameter.

Another strand of literature studies the demand side of the market in case of increasing transparency. For the Hotelling model with product differentiation, Schultz (2004) shows that it leads to less product differentiation and lower prices and profits. Moreover, welfare improves for all consumers and total surplus increases. Boone and Potters (2002) analyzed symmetric Cournot-Nash model, where goods are imperfect substitutes and consumers value variety. They found out that rising transparency may lead to an increase in total demand and also to higher prices. The level of substitutability is exogenous in their model and, when goods are perfect substitutes, the effect of increasing demand disappears.

The model presented in this paper is closely related to this last strand of literature. We deal with effects of market transparency on prices in the Bertrand duopoly model. The analysis is intuitive and simple when we consider two types of strategic interaction between firms in the industry - strategic complementarities and substitutabilities. We present also traditional comparative statics analysis, demanding additionally the regularity condition, to cover those problems, when none of these situations is the case.

In the first case, the results are close to conventional wisdom, especially, when in the same time products are substitutes. Namely, equilibrium prices and profits are always decreasing in transparency level, while the consumer's surplus is increasing. For a special case when supermodularity holds, but products are not substitutes, the result on profits is not valid anymore.

Considering price competition with strategic substitutes an ambiguity in the direction of change of prices appears. This leads to ambiguity concerning equilibrium profits' and surplus's changes caused by increasing transparency. Assuming in addition complementarity of the products, which promotes submodularity of the profits, enables us to conclude that consumer surplus increase, but firms profits remain ambiguous.

The paper is organized as follows. In Section 4.2 we present the general setup of the model of price competition in case of incomplete transparency. Further there is an overview of definitions of strategic complementarity and substitutability and results concerning these notions, useful in the paper. In Section 4.3 we study the reaction of prices to increased market transparency, we distinguish cases of strategic complementarity and substitutability. Section 4.4 is devoted to the analysis of welfare issues - changes in profits and consumers' surplus. In Section 4.5 we illustrate the results of Sections 4.3 and 4.4 by a linear example. Conclusions follow.

4.2 Setup and definitions

4.2.1 General setup

We consider a Bertrand price competition game Γ with the following characteristics. Two firms, producing different products, respectively 1 and 2, compete in prices. Both firms have constant marginal costs, respectively c_1 and c_2.

Following Schultz (2004) we consider two different types of consumers. A fraction ϕ are informed about products (both characteristics and prices) and the rest $1-\phi$ are uninformed. $\phi \in [0,1]$ measures the level of transparency of the market.

Schultz considers the Hotelling model with a continuum of consumers uniformly distributed along the interval $[0,1]$ and the demand for firm's 1 product is given by $\phi x + (1-\phi)\frac{1}{2}$. The demand for firm's 2 product is $1 - \left(\phi x + (1-\phi)\frac{1}{2}\right)$.

We generalize this approach allowing for other forms of demand functions, but retain the same way of modelling the behavior of informed versus uninformed consumers. We consider a one-shot model with exogenous heterogeneity of the products and firms deciding only on prices.

The full-information demands for goods 1 and 2 are denoted respectively $D^1(p_1, p_2)$ and $D^2(p_2, p_1)$. The uninformed consumers know only one of the products and are not aware of the other. Hence, their demands depend only on one price: $d(p_i)$, $i = 1, 2$. We assume that half of the uninformed consumers know each good.

Thus, the total demand for good i is $\phi D^i(p_i, p_j) + \frac{1-\phi}{2} d(p_i)$.

Moreover, we assume throughout that $D^i(p_i, p_j)$ and $d(p_i)$ are twice continuously differentiable. Goods are *substitutes* (*complements*) when the demand for one of them is increasing (decreasing) in the other's price, hence this implies that $D^i_j > (<) 0$.[1] Moreover, both $D^i(p_i, p_j)$ and $d(p_i)$ can be characterized by their price elasticities: $\varepsilon_D = D^i_i \frac{p_i}{D^i}$ and $\varepsilon_d = d'(p_i) \frac{p_i}{d}$.

Consider the situation if the level of market transparency is zero. Then every firm faces half of the consumers, and there is no relation between firms' pricing decision, hence

$$\pi^i(p_i) = \frac{1}{2}(p_i - c_i) \, d(p_i).$$

The solution of profit maximization problem in this case, $\mathring{p}_i$, is given by the first order condition:

$$d(\mathring{p}_i) - (\mathring{p}_i - c_i) \, d'(\mathring{p}_i) = 0$$

Notice that for any $p_i < \mathring{p}_i$, $\pi^i_i(p_i) > 0$.

When the market is perfectly transparent, i.e. all consumers are informed about prices and characteristics of both goods, the profit of firm i can be expressed by:

$$\begin{aligned} \pi^i(p_i, p_j) &= p_i D^i(p_i, p_j) - c_i D^i(p_i, p_j) \\ &= (p_i - c_i) \, D^i(p_i, p_j), i \neq j, i, j \in \{1, 2\}. \end{aligned}$$

[1] In case of many variables functions, lower subscripts denote partial derivative taken with respect to the given variable, here i.e. $D^i_j = \frac{\partial D^i}{\partial p_{,j}}$.

In case of imperfect market transparency, the profit of firm i is given by:

$$\Pi^i(p_i, p_j) = (p_i - c_i)\left(\phi D^i(p_i, p_j) + \frac{1-\phi}{2} d(p_i)\right). \qquad (4.1)$$

We restrict our consideration to prices in $[c_i, \infty), i = 1, 2$, since lower prices are dominated by pricing at marginal cost. Moreover we assume an upper bound on price, $\overline{p}_i$, such that $p_i \in P_i = [c_i, \overline{p}_i]$.

In the next subsection we provide definitions of super- and submodularity, and a number of theorems which will be useful in the remainder.

4.2.2 Useful definitions and results

A function $\Pi : X \times Y \to R$ is *supermodular* (*submodular*) if for each $x_1 > x_2$ and $y_1 > y_2$

$$\Pi(x_1, y_1) - \Pi(x_2, y_1) \geq (\leq)\, \Pi(x_1, y_2) - \Pi(x_2, y_2).$$

When the inequality is strict, we have strict supermodularity (submodularity). Supermodularity (submodularity) can be easily detected using differential characterization, namely, if $\Pi_{12}(x, y) \geq (\leq)0$, then $\Pi(x, y)$ is supermodular (submodular).

In R^2, supermodularity (submodularity) is equivalent to *nondecreasing* (*nonincreasing*) *differences* property, in different words, $\Pi(\cdot, y_1) - \Pi(\cdot, y_2)$ is an nondecreasing (nonincreasing) function.

Π has the (*dual*) *single crossing property* in (x, y) when for each $x_1 > x_2$ and $y_1 > y_2$

$$\Pi(x_1, y_2) - \Pi(x_2, y_2) \geq (\leq)0 \Rightarrow \Pi(x_1, y_1) - \Pi(x_2, y_1) \geq (\leq)0.$$

If Π is supermodular (submodular) in (x, y) then it satisfies also the (dual) single crossing property. But the reverse is not generally true. This property is ordinal, thus it is preserved by strictly monotonic transformation. Moreover, if g is strictly monotonic and $g \circ \Pi$ is supermodular (submodular), then Π has the (dual) single crossing property (see Milgrom and Shannon, 1994).

A game is supermodular if the players' strategy spaces are compact and the payoff functions are upper semi continuous in own strategies and supermodular. This kind of games is also called *games of strategic complementarity*. Intuitively, it expresses the idea that one player's marginal payoff is nondecreasing in his opponent's action, hence, their actions are complementary. These games have property of nondecreasing reaction curves defined as

$$r^i(p_j) = \arg\max_{p_i \in P_i} \{\Pi^i(p_i, p_j) : p_j \in P_j\}.$$

Analogously we define *games of strategic substitutability*, when players' payoff functions are submodular. This describes opposite situation, when one player's marginal payoff is nonincreasing in his opponent's action. These games have nonincreasing reaction curves, since if one player increases his action, the other one reacts by decreasing his.

To determine the direction of change of an equilibrium point as an exogenous parameter changes, we may use the following theorem.

Theorem 4.1 (Milgrom and Shannon, 1994) *Consider a game,* *where payoff of player i is given by $\Pi^i(x_i, x_{-i}, \phi)$, where ϕ is a parameter. Assume that for each $\phi \in [0,1]$ the game is supermodular, and Π^i satisfies the dual single crossing property in (x_i, ϕ) for each x_{-i}. Then, the maximal and minimal equilibria of the game are decreasing functions of ϕ.*

There is no dual version of this theorem for submodular games. When analogous conditions are satisfied for submodular game, the downward sloping reaction curves will both shift down, but it does not mean that both equilibrium actions decrease. It depends on the magnitude of the shifts. This is due to effects, which will be described in the next section.

4.3 Effect of transparency on prices

In this section we consider the impact of increasing market transparency on equilibrium prices in the model formulated in (4.1). We distinguish two cases, depending on character of strategic interactions between firms.

4.3.1 Strategic complementarities

Consider a Bertrand game with perfect transparency and payoffs given by $\pi^i(p_i, p_j) = (p_i - c_i)D^i(p_i, p_j)$. Vives (1990) provides condition for the Bertrand competition to be a supermodular game. For the linear costs case this condition is as follows (see also Amir and Grilo, 2003):

$$D^i_j + (p_i - c_i)D^i_{ij} \geq 0 \text{ for all } (p_i, p_j) \in P_i \times P_j.$$

It is more easily satisfied when the products are substitutes and when demand is supermodular, but none of these is a necessary condition.[2]

This condition guarantees as well supermodularity of the game with imperfect transparency.

Proposition 4.2 Π^i *defined in (4.1) is supermodular in* $(p_i, p_j) \in P_i \times P_j$ *if* D^i *satisfies*

$$D^i_j + (p_i - c_i)D^i_{ij} \geq 0. \tag{4.2}$$

Proof. The result follows directly from the cross-partial derivative of the profit function, $\Pi^i_{ij}(p_i, p_j) = \phi\left(D^i_j + (p_i - c_i)D^i_{ij}\right)$. ■

The supermodularity of the profit function is useful to establish, how the prices react to a change in the level of market transparency.

Proposition 4.3 *If* $\Pi^i, i = 1, 2,$ *is supermodular and* $|\varepsilon_D| > |\varepsilon_d|$, *then an increase in market transparency causes the extremal equilibrium prices of both goods to decrease.*

Proof. The proof uses Theorem 4.1. Since supermodularity of Π^i is assumed, now we want to check whether $\Pi^i(p_i, p_j, \phi)$ has

[2] There is another condition making Bertrand duopoly with linear cost into a game of strategic complementarities. It was given by Milgrom and Roberts (1990) and is equivalent to cross partial derivative of logarithm of the profit function being positive. In our case it is less useful, since requires imposing additional conditions on the game with imperfect transparency to secure its log-supermodularity.

the dual single crossing property in (p_i, ϕ). To this end, we consider strictly monotonic transformation of Π^i, namely $\ln \Pi^i(p_i, p_j, \phi)$, and show that this is submodular in (p_i, ϕ), so we can conclude that $\Pi^i(p_i, p_j, \phi)$ has the dual single crossing property in (p_i, ϕ). We obtain

$$\frac{\partial^2 \ln \Pi^i(p_i, p_j, \phi)}{\partial p_i \partial \phi} = \frac{1}{2} \frac{D_i^i(p_i, p_j) d(p_i) - D^i(p_i, p_j) d'(p_i)}{\left(\phi D^i(p_i, p_j) + \frac{1-\phi}{2} d(p_i)\right)^2}. \tag{4.3}$$

This is negative, whenever $D_i^i(p_i, p_j) d(p_i) - D^i(p_i, p_j) d'(p_i) < 0$. Dividing this inequality by $D^i(p_i, p_j) d(p_i)$ and multiplying by p_i gives us $\varepsilon_D - \varepsilon_d < 0$. Since both these elasticities are negative, it is sufficient that $|\varepsilon_D| > |\varepsilon_d|$, to have assumptions of Theorem 4.1 satisfied, therefore we can conclude that when ϕ goes up both prices go down. ■

The condition $|\varepsilon_D| > |\varepsilon_d|$ is quite natural since it says that the demand for good i is more sensitive to changes in price for those consumers who are aware of existing another good in the market. Reacting to the price increase they may switch to buy another good, whereas those uninformed are not aware of this possibility.

The decrease of equilibrium prices can be interpreted as consisting of two effects. There is a direct effect shifting the reaction curve down as a reaction of the player to the parameter change and an indirect effect of decreasing own price in response to the decrease in opponent's price (see Amir, 2005).

4.3.2 Strategic substitutes

Analogously to Proposition 4.2 we can formulate a condition on D^i to make the game Γ a submodular game.

Proposition 4.4 Π^i *defined like in (4.1) is submodular in* $(p_i, p_j) \in P_i \times P_j$ *if* D^i *satisfies*

$$D^i_j + (p_i - c_i) D^i_{ij} \leq 0. \tag{4.4}$$

Proof. The result follows directly from the cross-partial derivative of the profit function, $\Pi^i_{ij}(p_i, p_j) = \phi\left(D^i_j + (p_i - c_i) D^i_{ij}\right)$. ■

If condition (4.4) is satisfied we can conclude that the price competition is of strategic substitutes and hence the best replies are nonincreasing. In this case we cannot use Theorem 4.1 or any analog to state how the equilibrium prices will react to increased market transparency. From the fact that, as before, (4.3) is negative, whenever $|\varepsilon_D| > |\varepsilon_d|$, it follows that both reaction curves shift down, but it does not mean that both equilibrium prices decrease. It is possible that one of them may increase. Intuitively, it can be explained by the fact that the two effects mentioned before are conflicting now. The direct effect of the shift in the reaction curve makes the price go down but the indirect effect of adjusting to opponent's price leads in the opposite direction, since in case of strategic substitutes it is profitable to increase own action when the other player decreases his. Thus, the total effect depends on which of these two dominates. Nonetheless, for the special case of symmetric submodular game, we recover the proposition that both equilibrium prices are decreasing in ϕ.

4.3.2.1 Symmetric games

A Bertrand duopoly is symmetric if $P_i = P_j \equiv P$ and $\Pi^i(p_i, p_j) = \Pi^j(p_j, p_i)$.

Proposition 4.5 *Consider a symmetric Bertrand duopoly such that $\Pi^i(p_i, p_j)$ is strictly quasi-concave in own action, submodular and $|\varepsilon_D| > |\varepsilon_d|$. Then an increase in market transparency causes the equilibrium prices of both goods to decrease.*

Proof. Quasi-concavity of Π^i in p_i guarantees that the reaction curve of player i, $r^i(p_j) = \arg\max\{\Pi^i(p_i, p_j) : p_j \in P\}$ is continuous. Then, there must exist a symmetric equilibrium and it is unique. Let $r^i(p^*(\phi)) = p^*(\phi)$ be the equilibrium of the game.

From submodularity it follows that $r^i(p_j)$ is decreasing. Consider now $\Pi^i(p_i, p_j, \phi)$. Since $\frac{\partial^2 \ln \Pi^i(p_i,p_j,\phi)}{\partial p_i \partial \phi}$, given by (4.3) is negative, as showed in the proof of Proposition 4.3, we know that $\Pi^i(p_i, \phi)$ satisfies the dual single crossing property, and the reaction curve $r^i(p_j)$ shifts down when ϕ goes up.

Take $\phi_1 < \phi_2$. Consider situation when ϕ increases from ϕ_1 to ϕ_2 and denote $\widehat{r}^i(p_j)$ the reaction curve after the increase of ϕ. We want to show that the equilibrium prices both decrease. Since the unique equilibrium is still symmetric, we conclude that both prices change in the same direction. Proceed by contradiction and assume that $p^*(\phi_1) < p^*(\phi_2)$. Then

$$r^i(p^*(\phi_1)) = p^*(\phi_1) < p^*(\phi_2) = \widehat{r}^i(p^*(\phi_2)) \leq \widehat{r}^i(p^*(\phi_1)) \geq r^i(p^*(\phi_1))$$

where the second inequality comes from the fact that reaction curve is decreasing and the third one comes from the negative shift of the reaction curve when ϕ increases. This is a contradiction, hence we conclude that $p^*(\phi_1) > p^*(\phi_2)$. ■

4.3.2.2 Asymmetric games

To deal with asymmetric supermodular games we need to impose an additional assumption on Π^i. Since we have to relay now on traditional comparative statics method, we need Π^i to be concave, at least locally at equilibrium, throughout the rest of the paper. The analysis presented below covers as well all the cases when the strategic interactions of the game are neither strategic complements nor substitutes.

Here we investigate sufficient conditions for equilibrium prices to be decreasing in transparency level. To simplify the notation below we omit the argument of all functions, hence we denote $p_i^{*\prime}(\phi) = p_i^{*\prime}$ and $D^i(p_i^{*\prime}(\phi), p_j^{*\prime}(\phi)) = D^i$. To avoid misunderstanding and keep the notation compact we denote here $d(p_i) = d^i$ and $d(p_j) = d^j$.

Proposition 4.6 *The equilibrium price of good i is decreasing in* *the level of transparency if*

$$
\begin{aligned}
& [d^{i\prime}\,(p_i - c_i) + d^i][2D^j_j + (p_j - c_j)\,(D^j_{jj} - \frac{1-\phi}{2\phi} d^{j\prime\prime})] \quad (4.5) \\
< \; & [d^{j\prime}\,(p_j - c_j) + d^j][D^i_j + (p_i - c_i)\,D^i_{ij}].
\end{aligned}
$$

Proof. To establish this result we use the classical method of comparative statics, thus we differentiate the first order conditions of both firms with respect to ϕ and solve the system of equations to find $p_1^{*\prime}(\phi)$ and $p_2^{*\prime}(\phi)$.

The first order conditions for firms' profit maximization are following:

$$
\begin{aligned}
\phi D^1 + (p_1 - c_1)\left(\phi D^1_1 + \frac{1-\phi}{2} d^{1\prime}\right) + \frac{1-\phi}{2} d^1 &= 0 \\
\phi D^2 + (p_2 - c_2)\left(\phi D^2_2 + \frac{1-\phi}{2} d^{2\prime}\right) + \frac{1-\phi}{2} d^2 &= 0
\end{aligned}
$$

Differentiating them with respect to ϕ, we obtain following system of equations:

$$
\begin{aligned}
& p_1^{*\prime}(2\phi D^1_1 + (p_1 - c_1)\,(\phi D^1_{11} - \frac{1-\phi}{2} d^{1\prime\prime})) + \\
& + \phi p_2^{*\prime}\left(D^1_2 + (p_1 - c_1)\,D^1_{12}\right) \\
= \; & \frac{1}{2} d^1 - D^1 - (p_1 - c_1)\,(D^1_1 - \frac{1}{2} d^{1\prime}) \\
& \phi p_1^{*\prime}\left(D^2_1 + (p_2 - c_2)\,D^2_{21}\right) + \\
& + p_2^{*\prime}(2\phi D^2_2 + (p_2 - c_2)\,\phi(D^2_{22} - \frac{1-\phi}{2} d^{2\prime\prime})) \\
= \; & \frac{1}{2} d^2 - D^2 - (p_2 - c_2)\,(D^2_2 - \frac{1}{2} d^{2\prime})
\end{aligned}
$$

From first order conditions it follows that

$$\begin{aligned}
\frac{1}{2}d^1 - D^1 - (p_1 - c_1)\left(D_1^1 - \frac{1}{2}d^{1\prime}\right) &= \frac{1}{2\phi}(d^{1\prime}(p_1 - c_1) + d^1) \\
\frac{1}{2}d^2 - D^2 - (p_2 - c_2)\left(D_2^2 - \frac{1}{2}d^{2\prime}\right) &= \frac{1}{2\phi}(d^{2\prime}(p_2 - c_2) + d^2)
\end{aligned}$$

so that we replace it in the system.

The solution to this system is following:

$$\begin{aligned}
p_i^{*\prime} &= \frac{1}{2\phi}\frac{(d^{i\prime}(p_i - c_i) + d^i)(2\phi D_j^j + (p_j - c_j)(\phi D_{jj}^j - \frac{1-\phi}{2}d^{j\prime\prime}))}{k} \\
&\quad -\frac{1}{2\phi}\frac{\phi(d^{j\prime}(p_j - c_j) + d^j)(D_j^i + (p_i - c_i)D_{ij}^i)}{k}
\end{aligned}$$

where

$$\begin{aligned}
k &= (2\phi D_i^i + (p_i - c_i)(\phi D_{ii}^i - \frac{1-\phi}{2}d^{i\prime\prime})) \\
&\quad \times(2\phi D_j^j + (p_j - c_j)(\phi D_{jj}^j - \frac{1-\phi}{2}d^{j\prime\prime})) \\
&\quad -\phi^2\left(D_i^j + (p_j - c_j)D_{ji}^j\right)\left(D_j^i + (p_i - c_i)D_{ij}^i\right)
\end{aligned}$$

Since the denominator is positive for stable equilibria, we restrict our attention to the latter so that sign of $p_i^{*\prime}$ depends solely on the sign of the numerator:

$$\begin{aligned}
&sign(p_i^{*\prime}) \\
=\; &sign\{[d^{i}{}'(p_i - c_i) + d^i][2\phi D_j^j + (p_j - c_j)(\phi D_{jj}^j - \frac{1-\phi}{2}d^{j}{}'')] \\
&-\phi[d^{j}{}'(p_j - c_j) + d^j][D_j^i + (p_i - c_i)D_{ij}^i]\}.
\end{aligned}$$

For simplicity we can divide this expression by ϕ, since it is positive and we obtain condition (4.5). ■

Note the interpretation of the components of condition (4.5). Expression $d^{i\prime}(p_i - c_i) + d^i = \pi_i^i(p_i) > 0$ for all $p_i < \mathring{p}_i$, where $\mathring{p}_i$ can be interpreted as a monopoly price, if there is zero level of transparency (compare Section 4.2). The second term of the left hand side captures the concavity of the opponent's profit, hence it is always locally negative at equilibrium, and the second term of the right hand side describes the strategic complementarity or substitutability of players' actions in the game from the point of view of player i.[3]

Is is easy to observe that convexity of the demand in own actions, its submodularity and complementary character of goods are in favor to one of the prices being increasing in ϕ.

In the next section we analyze how an increase in market transparency influences firms' profits and consumers' surplus, taking into account the behavior of the equilibrium prices.

4.4 Effects of transparency on profits

In this section we study how the transparency level influences consumers and firms welfare in equilibrium.

Clearly, consumers are better off, when both prices decrease. Hence, consumers' surplus rises if the game is of strategic complementarities and if it is symmetric of strategic substitutes and if condition (4.5) is satisfied for $i = 1, 2$.

[3]Although for each demand system $D_2^1 = D_1^2$, it may happen that one of the demands is strictly supermodular, while the other is strictly submodular. The reaction curves have in such a case the opposite slopes. For an example, see Amir et al. (1999).

Consider now equilibrium profits of the firms.

Proposition 4.7 $\Pi^i(p_i^*(\phi), p_j^*(\phi))$ *is decreasing in* ϕ *if*

- *goods are substitutes and* $p_j^{*\prime}(\phi) < 0$*;*

  or

- *goods are complements and* $p_j^{*\prime}(\phi) > 0$*.*

Proof. We compute the derivative of $\Pi^i(p_i^*(\phi), p_j^*(\phi))$ to state its reactions to increasing transparency.

$$\begin{aligned}\frac{d}{d\phi}\Pi^i(p_i^*(\phi), p_j^*(\phi)) &= \Pi_i^i p_i^{*\prime} + \Pi_j^i p_j^{*\prime} \\ &= (p_i^* - c_i)\phi D_j^i p_j^{*\prime}\end{aligned}$$

since the component $\Pi_i^i p_i^{*\prime}$ is equal to zero from the first order condition of the firm. This is negative if exactly one of D_j^i and $p_j^{*\prime}$ is negative. ■

In case of complementary goods, when $D_j^i < 0$, and decreasing equilibrium prices both firms are better off. If one of the product's prices increases in ϕ, its producer is better off, contrary to his opponent.

In case of substitute goods, when $D_j^i > 0$, and both prices decreasing, we conclude that both profits decrease as well. But in general case substitute goods do not imply strategic complementarity in the Bertrand model (which guarantees that both equilibrium prices decrease with ϕ), so it may happen, for strongly supermodular demand functions to form strategic substitutes. In this case firm's profit increase in ϕ, whenever the opponent's price increases.

Hence, in case of complementary goods and both prices decreasing, both firms and consumers gain in consequence of transparency increase.

4.5 Linear example

This section contains a numerical example, based on liner demand functions, illustrating main findings of the paper.

Consider the following demand of an representative informed consumer for product i:

$$D^i(p_i, p_j) = \alpha_i - \beta_i p_i + \gamma p_j.$$

We assume that $\alpha_i > 0$. This is needed for demand function to be well-defined - otherwise the autonomous demand (when both prices are zero) would be negative. Moreover, we assume that each demand reacts more to changes of own price than to changes of the opponent's price. Hence, $|\gamma| < \beta_i, i = 1, 2$.

Goods are substitutes whenever $\gamma > 0$ and complements if $\gamma < 0$. It will be shown that also γ is responsible for character of strategic interaction between firms.

Consider also a linear form of the demand of an uninformed consumer:

$$d(p_i) = a - bp_i.$$

We assume that $a > 0$ for the same reason as before. Moreover, we assume that $|\varepsilon_D| > |\varepsilon_d|$, since the consumer who is aware of the existence of two goods, reacts more to changes in price of one of them.

Firm's i profit is given by:

$$\Pi^i(p_i, p_j) = (p_i - c_i)(\phi(\alpha_i - \beta_i p_i + \gamma p_j) + \frac{1-\phi}{2}(a - bp_i))$$

Its cross-partial derivative

$$\Pi^i_{ij}(p_i, p_j) = \gamma\phi$$

is positive whenever γ is positive. Hence, we can conclude that reaction curve of one firm to price of the other one is increasing if goods are substitutes and decreasing if they are complements.

From Proposition 4.2 follows that for positive γ both equilibrium prices are decreasing in ϕ. For negative γ, this is guaranteed only for symmetric demand functions. The numerical example below illustrates situation, when this is not the case.

Example 4.8 *Lets* $P_1 = P_2 = [0, 1]$. *Consider following parameter values:* $\alpha_1 = 2$, $\beta_1 = 3$, $\gamma = -2.99$, $c_1 = 0.05$, $a = 6$, $b = 4$, $\alpha_2 = 4$, $\beta_2 = 3$, $c_2 = 0.02$, $\phi = 0.5$. *We left to the reader checking if all the assumptions mentioned in this chapter are satisfied.*

Firms profits are given by

$$\begin{aligned}\Pi^1(p_1, p_2) &= (p_1 - 0.05)(0.5(2. = -3p_1 - 2.99p_2) + 0.25(6 - 4p_1)) \\ \Pi^2(p_1, p_2) &= (p_2 - 0.02)(0.5(4 - 3p_2 - 2.99p_1) + 0.25(6 - 4p_1))\end{aligned}$$

and reaction curves are

$$\begin{aligned}r^1(p_2) &= 0.525 - 0.299p_2 \\ r^2(p_1) &= 0.71 - 0.299p_1.\end{aligned}$$

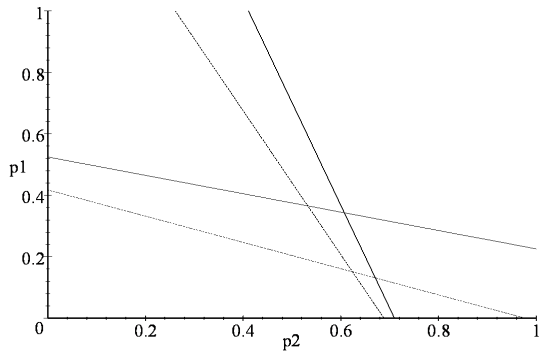

Figure 4.1: Reaction curves for $\phi = 0.5$ depicted as solid lines, reaction curves for $\phi = 0.8$ depicted as dashed lines. If ϕ increases, one of the equilibrium prices goes down, when the other goes up.

They are depicted by solid lines in figure 4.1. Equilibrium prices, given by the point where the reaction functions cross, are $p_1^* = 0.343\,41$, $p_2^* = 0.607\,32$. *Firms equilibrium profits are, respectively,* $\Pi^1(p_1^*, p_2^*) = 0.21523$ *and* $\Pi^2(p_1^*, p_2^*) = 0.86236$

When the transparency level ϕ *increases to reach* 0.8, *all the best responses of both firms decrease. New reaction curves are given by*

$$\begin{aligned}\hat{r}^1(p_2) &= 0.417\,86 - 0.427\,14 p_2 \\ \hat{r}^2(p_1) &= 0.688\,57 - 0.\,427\,14 p_1\end{aligned}$$

and are depicted by dashed lines in figure 4.1. They cross at $\hat{p}_1^* = 0.151\,35$, $\hat{p}_2^* = 0.623\,92$, *which are new equilibrium prices. Notice that the first one decreased, but the second one increased with the level of transparency.*

Since goods are complements and p_2^* *increases in* ϕ *, it follows* *from Proposition 4.7 that* $\Pi^1(p_1^*, p_2^*)$ *decreases and* $\Pi^2(p_1^*, p_2^*)$ *increases in* ϕ. *In fact* $\Pi^1(\hat{p}_1^*, \hat{p}_2^*) = 0.028763$ *and* $\Pi^2(\hat{p}_1^*, \hat{p}_2^*) = 1.0212$.

4.6 Conclusions

We analyzed market transparency issue in the context of the Bertrand price competition model. In case of strategic complementarity we directly generalize the results of Schultz (2004) in terms of equilibrium prices and consumers' surplus, but not for firms' profits. While prices are decreasing in transparency level and consumers are better off, firms are worse off only in case of substitute goods. Otherwise, one of them may gain, and even, in same cases of complementary goods, both profits can increase.

Allowing for strategic substitutability leads to ambiguous results even for prices - one of them may increase in the level of transparency. Such a situation precludes general conclusions about profits and consumers' surplus. However, when both prices decrease, consumers gain and in case of complementary goods, both firms gain as well.

We provide also a condition allowing for conclusion in cases when the strategic complementarity or substitutability is not satisfied, but it requires knowing functional forms of the demands and equilibrium prices. Its derivation relays on the regularity condition, since it comes from traditional comparative statics.

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CHAPTER 5

STRATEGIC SUBSTITUTES AND COMPLEMENTS IN COURNOT OLIGOPOLY WITH PRODUCT DIFFERENTIATION

5.1 Introduction

Cournot's model of oligopolistic competition (1838) represents the starting point of formalized economic theory and game theory. Nowadays it continues to be widely used in a number of applications in economic theory. The reason for this is its tractability, when product homogeneity can be assumed, and the fact that its properties are well established. In this paper we follow the line of literature dealing with the monotonicity of reaction correspondences, which is important in particular for studying problems of equilibrium existence and comparative statics.

Increasing reaction correspondences characterizes games with strategic complementarities, while games of strategic substitutes have downward sloping best replies. This expresses the strategic relationship among actions in this kind of games. Strategic complementarities cover situations when an increase in one player action leads to an increase in other players' marginal payoffs. Strategic substitutes describe the opposite situation, when an increase in one player action causes a decrease in other players' marginal payoffs. The former kind of games always possesses pure strategy Nash equilibria. Moreover, the set of pure strategy Nash equilibria, correlated equilibria and rationalizable strategies have identical bounds (see, Milgrom and Roberts, 1990, and Milgrom and Shannon, 1994). The latter kind of games, in the case of $2-$player game, can be converted into the game of strategic complementarities by reordering of one player's action set (Milgrom and Roberts, 1990).

To establish properties of a Cournot oligopoly with heterogeneous products several simplifying assumptions have been imposed. For instance, inverse demand function have an aggregative form with respect to other players' strategies (cf. Dubey et al., 2005). Hoernig (2003) assumes that, departing from the situation when firms produce the same quantities, when one firm deviates by raising its output, other firms adjust their outputs to the level allowing for avoiding an increase in market price. He conducts comparative statics in case of market entry and increasing number of firms, extending the results of Amir and Lambson (2000) for homogeneous products.

In this paper we provide general conditions for a Cournot oligopoly with product differentiation to have monotonic reaction correspondences. These results are a generalization of the results of Amir (1996) for homogeneous products to the case of differentiated products. We give various sufficient conditions for downward and upward sloping reaction correspondences. They allow for identifying increasing best responses even in case of inverse demand being submodular, and similarly, decreasing best responses in case of supermodular inverse demand. Examples illustrating the scope of applicability of these results are provided.

This approach gives a significant value added to the problem of equilibria existence and comparative statics. The standard approach demands profit function to be quasi-concave in own quantities. This is quite restrictive, particularly because non-concavities in costs are not uncommon and very convex demand functions cannot be ruled out (Vives, 1999). The lack of quasi-concavity of payoffs causes discontinuities in the best response correspondences of firms and makes possible the nonexistence of equilibrium. Our approach covers games with monotonic reaction correspondences and does not rely on the regularity condition.

The paper is organized as follows. Next section contains an brief overview of relevant notions from supermodular optimization and games. Section 5.3 presents main results. In Section 5.4 they are discussed. All proofs are placed in Appendix.

5.2 Supermodular games

We introduce in this section a summary of all the relevant notions and results from lattice theory, supermodular optimization and supermodular games useful in the remainder. We present them in context of real decision parameter spaces, since this is sufficient for our needs.

A function $F : X \times Y \to R$ is *supermodular* if, for all $x_1 \geq x_2, y_1 \geq y_2$

$$F(x_1, y_1) - F(x_2, y_1) \geq F(x_1, y_2) - F(x_2, y_2). \tag{5.1}$$

A function F is *submodular* whenever $-F$ is supermodular. For twice continuously differentiable functions this notion has a useful differential characterization, namely supermodularity (submodularity) is equivalent to cross-partial derivative being positive (negative).

We say that F is *log-supermodular* (*log-submodular*) if logarithm of F is supermodular (*submodular).*

Topkis (1978) showed the following theorem on monotone optimization, central to our approach.

Theorem 5.1 *If F is upper semi-continuous and supermodular (submodular), then the maximal and minimal selections of*

$$\arg\max_{x \in X}\{F(x,y), y \in Y\}$$

are non-decreasing (non-increasing).

If F is strictly supermodular (submodular), then this theorem holds for every selection of $\arg\max_{x\in X}\{F(x,y), y \in Y\}$.

The property of supermodularity is of cardinal nature, in the sense that it is not preserved by monotonic transformation. Below we provide a definition of another notion, of ordinal nature, which can be treated as a generalization of supermodularity.

A function F has the (*dual*) *single-crossing property* in (x, y) if, for all $x_1 \geq x_2, y_1 \geq y_2$

$$F(x_2, y_1) - F(x_2, y_2)(\leq) \geq 0 \Rightarrow F(x_1, y_1) - F(x_1, y_2)(\leq) \geq 0. \quad (5.2)$$

Strict (dual) single crossing property is defined by the implication with strict inequality on the right hand side.

Obviously, (5.1) implies (5.2), but the converse does not hold.

Milgrom and Shannon (1994) generalized the result of Topkis for functions possessing the single crossing property.

Theorem 5.2 *If F is upper semi-continuous and satisfies the (dual) single crossing property, then the maximal and minimal selections of*

$$\arg\max_{x \in X}\{F(x, y), y \in Y\}$$

are non-decreasing (non-increasing).

If F has strict (dual) single crossing property, then this theorem holds for every selection of $\arg\max_{x \in X}\{F(x, y), y \in Y\}$.

The single crossing property has no differential characterization like supermodularity. Milgrom and Shannon (1994) showed that this property can be tested using the Spence-Mirrlees condition defined in the following theorem.

Theorem 5.3 *Let $F : R^3 \to R$ be continuously differentiable and $F_2(a, b, s) \neq 0$.*[1] *$F(a, h(a), s)$ satisfies the single-crossing property in (a, s) for all functions $h : R \to R$ if and only if*

$$\frac{F_1(a, b, s)}{|F_2(a, b, s)|} \text{ is increasing in } s. \tag{5.3}$$

In applications, verifying (5.3) leads to the conclusion that $F(a, h(a), s)$ satisfies the single crossing property in (a, s) for suitable choice of function h (which is often an identity function, see Milgrom and Shannon, 1994, and Amir, 2005a).

A game with compact real action spaces is *supermodular* (*submodular*) if each payoff function is supermodular (submodular) and upper semi continuous in own actions.

Replacing supermodularity by the single crossing property enables to define a broader class of games: a game is *ordinally supermodular* (*submodular*) when players' action spaces are compact, their payoff functions have the (dual) single crossing property and are upper semi continuous in own actions.

Supermodularity of the payoff functions can be interpreted as a complementarity between the players' actions, namely an increase some players' actions causes an increase in the marginal payoff of the others. Hence, this kind of games are also called *games with strategic complementarities.* Submodularity expresses the opposite phenomenon - substitutability of the players actions: increasing some players actions causes a decrease in the marginal payoff of the

[1] Subscripts denote partial derivatives with respect to the certain variable, e.g. here $F_2(a, b, s) = \frac{\partial F(a,b,s)}{\partial b}$. We keep this notation throughout the paper.

others, thus this kind of games are also called *games of strategic substitutabilities.*.

Corollary 5.4 *Every $n-$player game of strategic complementarities has a pure strategy Nash equilibrium.*

There is no equivalent corollary for $n-$player games of strategic substitutes. Only for the case of two players the existence of equilibrium is guaranteed.

Corollary 5.5 *Every $2-$player game of strategic substitutes has a pure strategy Nash equilibrium.*

5.3 Conditions and examples

Consider a Cournot oligopoly game, when n firms decide simultaneously about the products quantity. Let $x_i \in X_i$ be a production quantity of firm i and $x_{-i} \in X_{-i}$ represents vector of production quantities of the other $n-1$ firms. The products are heterogenous. Denote $P^i(x_i, x_{-i})$, $i = 1, .., n$, a system of inverse demand functions describing the market. Each of the functions is twice continuously differentiable and $P^i_j(x_i, x_{-i}) = P^j_i(x_j, x_{-j})$, $i \neq j$. Then the profit of firm i is given by

$$\Pi^i(x_i, x_{-i}) = x_i P^i(x_i, x_{-i}) - C^i(x_i). \tag{5.4}$$

where $C^i(\cdot)$ is a differentiable cost function. Assume that $P^i(x_i, x_{-i})$ is decreasing in own action, and $C^i(\cdot)$ is increasing.

We say that goods i and j are (strict) *substitutes*, if demand for i (strictly) rises with the increase in the price of j. Goods i and j are (strict) *complements,* if demand for i (strictly) goes down with the increase of the price of j. It can be translated in terms of inverse demand function, so that P^i is (strictly) decreasing in x_j, if goods i and j are substitutes, and the converse holds if goods i and j are complements.

Define *reaction (best response) correspondence* of firm i as

$$r_i(x_{-i}) = \arg\max_{x_i \in X_i} \{\Pi^i(x_i, x_{-i}), x_{-i} \in X_{-i}\},$$

From Theorem 5.1 the game has increasing reaction correspondences if $\Pi^i, i = 1, .., n$, is supermodular or $\Pi^i_{ij} \geq 0 \; \forall j \neq i$, hence if each firm's revenue is supermodular (see also Novshek, 1985). For firm i it is equivalent to the following condition:

$$P^i_j(x_i, x_{-i}) - x_i P^i_{ij}(x_i, x_{-i}) \geq 0 \; \forall j \neq i.$$

But from Theorem 5.2 it follows that even weaker conditions can secure the monotonicity of the reaction correspondences. It is enough for the payoff function to satisfy the single crossing property.[2]

Vives (1999) formulated independent conditions on demand and cost function in a Cournot oligopoly with product differentiation to

[2] While ordinal complementarity is more general than cardinal complementarity, the corresponding global sufficient conditions placed on the primitives of a constrained optimization problem are generally not comparable. For the Cournot oligopoly with homogenous product this issue was studied by Amir (2005b).

be an ordinally supermodular game. We give them in the form of a theorem.

Theorem 5.6 *Assume that for $i = 1, .., n$*

1. *$P^i(\cdot)$ is log-supermodular.*

2. *C^i is strictly increasing.*

3. *Goods i and j are strict complements $\forall j \neq i$.*

Then the Cournot oligopoly, with profits given by (5.4) is an ordinally supermodular game.

The proof (not given by Vives, 1999), presented in Appendix, does not require differentiability of the inverse demand nor the cost function.

An analogous theorem can be formulated to provide conditions on a Cournot oligopoly to have decreasing reaction correspondences.

Theorem 5.7 *Assume that for $i = 1, .., n$*

1. *$P^i(\cdot)$ is log-submodular.*

2. *C^i is strictly increasing.*

3. *Goods i and j are strict substitutes $\forall j \neq i$.*

Then the Cournot oligopoly, with profits given by (5.4) is an ordinally submodular game.

To provide some intuition of the scope of duality between these theorems for a duopoly case we can use Milgrom and Roberts (1990) actions reordering argument. In case of $2-$players game, changing order of the action space of one of the players converts a submodular game into a supermodular game. Take a game Γ, the Cournot duopoly and consider situation of player 1 with $P^1(x_1, x_2)$ log-submodular and x_1, x_2 substitute goods. Reordering of the player's 1 action space creates a new game, $\widehat{\Gamma}$. This is the Cournot duopoly with $P^1(\widehat{x_1}, x_2)$. Observe that whenever $P^1(\widehat{x_1}, x_2)$ is positive, it is log-supermodular, since $P^1 P^1_{12} - P^1_1 P^1_2 \leq 0$ and $\widehat{x_1}' = -1$ implies that $\widehat{x_1}' \left(P^1 P^1_{12} - P^1_1 P^1_2\right) \geq 0$ for every $x_1 \in X_1$ and $x_2 \in X_2$. Moreover, $\widehat{x_1}$ and x_2 are complements now, since relation between them changed to the reverse.[3] Therefore, there is complete duality between these two results. This kind of duality does not work if there is more than two players in the game.

The theorems given above work for any increasing cost function. But considering a specific cost function in relation with the inverse demand we may relax the assumption of log-supermodularity of the demand. It is possible to formulate a more general condition for a specific cost function, such that it guarantees increasing best response correspondence. We provide it in the theorem below. The differentiability of $P^i(x_i, x_{-i})$ and $C^i(x_i)$ is assumed here .

[3] Now, when price of good 2 increases, the demand for $\widehat{x}$ goes up.

Theorem 5.8 *Assume that* $\forall j \neq i$

$$P^i_{ij}(x_i, x_{-i})P^i(x_i, x_{-i}) - P^i_i(x_i, x_{-i})P^i_j(x_i, x_{-i}) \geq P^i_{ij}(x_i, x_{-i})C^{i\prime}(x_i) \tag{5.5}$$

Then the Cournot oligopoly, with profits given by (5.4) is an ordinally supermodular game.

Condition (5.5) can be reformulated into

$$P^i_{ij}(x_i, x_{-i})(P^i(x_i, x_{-i}) - C^{i\prime}(x_i)) - P^i_i(x_i, x_{-i})P^i_j(x_i, x_{-i}) \geq 0. \tag{5.6}$$

It is straightforward that this condition is met, when the assumptions of Theorem 5.6 are satisfied. Moreover, for a suitable choice of C^i, this condition can be satisfied for a number of inverted demand functions, which do not satisfy log-supermodularity, as long as goods i and j remain complements. In particular, there are log-submodular functions giving rise to strategic complementarity in the Cournot model. An illustration of this possibility is provided in Example 5.10.

When we assume fixed marginal costs, condition (5.5) can be interpreted as log-supermodularity of net-of-cost inverse demand functions $P^i(x_i, x_{-i}) - c^i$, $i = 1, 2$. Moreover, this condition says that the firm's i perceived net-of-cost inverse demand elasticity is increasing in firm's j output. Indeed, elasticity of net-of-cost inverse demand is given by

$$\begin{aligned} \varepsilon(x_i, x_j) &\triangleq \frac{\partial\left(P^i(x_i, x_{-i}) - c^i\right)}{\partial x_i} \frac{x_i}{P^i(x_i, x_{-i}) - c^i} \\ &= P^i_i(x_i, x_{-i}) \frac{x_i}{P^i(x_i, x_{-i}) - c^i}. \end{aligned}$$

It is increasing in x_j whenever its derivative with respect to x_j is positive:

$$\frac{\partial \varepsilon(x_i, x_j)}{\partial x_j} = x_i \frac{P^i_{ij}(x_i, x_{-i})\left(P^i(x_i, x_{-i}) - c^i\right) - P^i_i(x_i, x_{-i}) P^i_j(x_i, x_{-i})}{\left(P^i(x_i, x_{-i}) - c^i\right)^2}.$$

It holds if and only if (5.6) holds.

Corollary 5.9 *Assume that marginal cost is constant, denoted c^i and $P^i(x_i, x_{-i}) - c^i$ is log-supermodular, $i = 1, .., n$. Then the Cournot oligopoly, with profits given by (5.4) is an ordinally supermodular game.*

We provide now an example illustrating the scope of application of Theorem 5.8 in the duopoly case. Our generalization enables to capture even situations when inverse demand, being submodular and/or log-submodular, gives rise, for specific cost functions, to an ordinally supermodular game.

Example 5.10 *Consider a 2−player game. Let*

$$P^1(x_1, x_2) = 1 + \frac{1}{(x_1 + 1)^2} + (x_2 + 1)\exp(-x_1).$$

It is easily verified that

$$\begin{aligned} P^1_1(x_1, x_2) &= \frac{-2}{(x_1 + 1)^3} - (x_2 + 1)\exp(-x_1) < 0 \\ P^1_2(x_1, x_2) &= \exp(-x_1) > 0\,. \end{aligned}$$

Hence, the goods are complements. Also

$$P^1_{12}(x_1, x_2) = -\exp(-x_1) < 0$$

and moreover

$$\left(\ln P^1(x_1,x_2)\right)_{12} = \frac{-\left(x_1^2+3x_1+4\right)x_1\left(x_1+1\right)e^{-x_1}}{\left(1+\left(x_1+1\right)^2\left(\left(x_2+1\right)e^{-x_1}+1\right)\right)^2} < 0.$$

Hence, the inverse demand is strictly submodular and strictly log-submodular, and thus cannot be log-supermodular. Therefore, the conditions of Theorem 5.6 are not satisfied. Now take the following cost function: $C^1(x_1) = 2x_1$ *and check the condition (5.6):*

$$\begin{aligned} & P^1_{12}(x_1,x_2)(P^1(x_1,x_2) - C^{1\prime}(x_1)) - P^1_1(x_1,x_2)P^1_2(x_1,x_2) \\ = \; & e^{-x_1}\left(\left(x_2+1\right)e^{-x_1} + 2x_1 + \frac{2}{\left(x_1+1\right)^3}\right) > 0 \end{aligned}$$

It is satisfied for all positive x_1 *and* x_2*. The verification that not all cost functions satisfy this condition is left to the reader.*

Concavity of firm's 1 *profit can be easily verified, hence we use first order condition to find the reaction curve.*

$$\begin{aligned} \Pi^1_1(x_1,x_2) &= -\frac{\left(x_1-1\right)\left(x_1+1\right)^3\left(x_2+1\right)e^{-x_1} + x_1\left(x_1^2+3x_1+4\right)}{\left(x_1+1\right)^3} \\ &= 0 \end{aligned}$$

It cannot be solved explicitly for x_1*, thus we present a numerical plot in figure 5.1.*

Again, the dual theorem guaranteeing decreasing best responses can be formulated, and is given below.

Theorem 5.11 *Assume that* $\forall j \neq i$

$$P^i_{ij}(x_i,x_{-i})P^i(x_i,x_{-i}) - P^i_i(x_i,x_{-i})P^i_j(x_i,x_{-i}) \leq P^i_{ij}(x_i,x_{-i})C^{i\prime}(x_i). \tag{5.7}$$

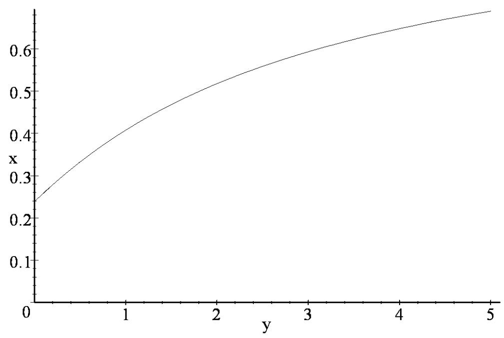

Figure 5.1: Increasing reaction function of firm 1 form example 5.10.

Then the Cournot oligopoly, with profits given by (5.4) is an ordinally submodular game.

Similarly, in the case of duopoly, one can imagine this condition as derived from the reordering approach.

This condition covers all the cases, which satisfy conditions of Theorem 5.7 and, moreover, for specific cost functions, inverted demands, which are not log-submodular. In fact, even log-supermodular inverted demands may lead to an ordinally submodular Cournot game, an illustration is provided in Example 5.13.

As before, we formulate a corollary for the case of constant marginal costs.

Corollary 5.12 *Assume that marginal cost is constant, denoted* c^i *and* $P^i(x_i, x_{-i}) - c^i$ *is log-submodular,* $i = 1, .., n$. *Then the Cournot oligopoly, with profits given by (5.4) is an ordinally sub-*

modular game.

Log-submodularity of net-of-cost inverted demand can be interpreted as its elasticity to own quantity being decreasing in x_j.

Usefulness of the Theorem 5.11 in the duopoly case is illustrated by the example below. Again, even supermodular and log-supermodular inverse demand function can give rise to an ordinally submodular game for some cost functions.

Example 5.13 *Consider a* $2-$*player game and take*

$$P^1(x_1, x_2) = \frac{e^{-x_1}}{(x_2+1)} + \frac{1}{x_1+1}.$$

It is easy to verify that

$$\begin{aligned} P_1^1(x_1, x_2) &= -\frac{e^{-x_1}}{(x_2+1)} - \frac{1}{(x_1+1)^2} < 0, \\ P_2^1(x_1, x_2) &= -\frac{e^{-x_1}}{(x_2+1)^2} < 0, \\ P_{12}^1(x_1, x_2) &= \frac{e^{-x_1}}{(x_2+1)^2} > 0, \end{aligned}$$

and

$$\left(\ln P^1(x_1, x_2)\right)_{12} = e^{-x_1} \frac{x_1}{(e^{-x_1}x_1 + e^{-x_1} + x_2 + 1)^2} > 0.$$

Therefore, the products are substitutes and inverse demand is log-supermodular. Therefore the game does not satisfy the conditions of Theorem 5.7.

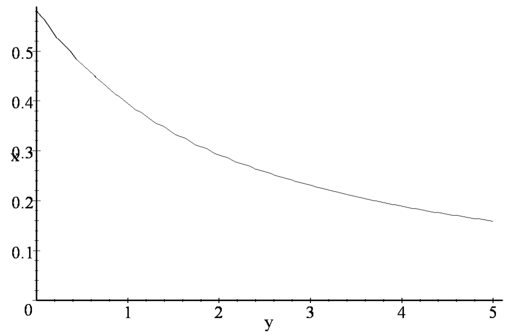

Figure 5.2: Downward sloping reaction function of firm 1 from example 5.13.

Take now the following cost function: $C^1(x_1) = \ln(x_1+1)$ *and check condition (5.7),*

$$\begin{aligned} P^1_{12}(x_1,x_2)(P^1(x_1,x_2) - C^{1\prime}(x_1)) - P^1_1(x_1,x_2)P^1_2(x_1,x_2) &= \\ -e^{-x_1}\frac{(x_2+1)(x_1+1)^2\ln(x_1+1) + (x_1+1)^2 e^{-x_1} + x_2 + 1}{(x_2+1)^3(x_1+1)^2} &< 0. \end{aligned}$$

It is satisfied for all positive x_1 *and* x_2*. Therefore, the game is ordinally submodular.*

From the first order condition of profit maximization we find best response curve. We verified that $\Pi^1(x_1, x_{12})$ *is locally concave on the best response function. It is given by an implicit formula*

$$e^{-x_1}x_1^2 - e^{-x_1}x_1 - e^{-x_1} + e^{-x_1}x_1^3 + x_2x_1 + x_1 = 0.$$

Since it cannot be explicitly solved for x_1*, we present a numerical plot in figure 5.2.*

5.4 Discussion

Games of strategic complementarities have Nash equilibria in pure strategies. Hence, when the proper conditions are satisfied for a Cournot oligopoly with differentiated products, one can be sure that equilibria exist, moreover, in terms of strategies, there exist largest and smallest equilibria.

For games of strategic substitutes there is no equivalent result. We may only guarantee the existence of pure strategy Nash equilibria in 2−player game. For $n > 2$ one can distinguished games, in which the strategies of all opponents can be aggregated into one number. For this kind of games the existence of the pure strategy Nash equilibrium was shown by Dubey et al. (2005), see also Novshek (1985). Unfortunately, not all reasonable inverse demand functions have this property (cf. Hoernig, 2003).

5.5 Appendix

Proof. (of theorem 5.6) Consider firm i. Take $x_i^1 > x_i^2, x_{-i}^1 > x_{-i}^2$. Form assumption 1 we have:

$$\ln P^i(x_i^1, x_{-i}^1) - \ln P^i(x_i^2, x_{-i}^1) \geq \ln P^i(x_i^1, x_{-i}^2) - \ln P^i(x_i^2, x_{-i}^2)$$

This implies that

$$\frac{P^i(x_i^1, x_{-i}^1)}{P^i(x_i^2, x_{-i}^1)} \geq \frac{P^i(x_i^1, x_{-i}^2)}{P^i(x_i^2, x_{-i}^2)}$$

or

$$P^i(x_i^2, x_{-i}^2)\frac{P^i(x_i^1, x_{-i}^1)}{P^i(x_i^2, x_{-i}^1)} \geq P^i(x_i^1, x_{-i}^2) \tag{5.8}$$

To prove that the game with log-supermodular inverse demand and increasing cost function is ordinally supermodular, it is enough to show that $\Pi^i(x_i, x_{-i})$ has the single crossing property. To this end we start from assuming that

$$\Pi^i(x_i^1, x_{-i}^2) \geq \Pi^i(x_i^2, x_{-i}^2).$$

Then

$$x_i^1 P^i(x_i^1, x_{-i}^2) - C^i(x_i^1) \geq x_i^2 P^i(x_i^2, x_{-i}^2) - C^i(x_i^2).$$

We can replace $P^1(x_i^1, x_{-i}^2)$ from (5.8)

$$x_i^1 P^i(x_i^2, x_{-i}^2) \frac{P^i(x_i^1, x_{-i}^1)}{P^i(x_i^2, x_{-i}^1)} - C^i(x_i^1) \geq x_i^2 P^i(x_i^2, x_{-i}^2) - C^i(x_i^2).$$

Multiplying by $\frac{P^i(x_i^2, x_{-i}^1)}{P^i(x_i^2, x_{-i}^2)}$ we get

$$x_i^1 P^i(x_i^1, x_{-i}^1) - \frac{P^i(x_i^2, x_{-i}^1)}{P^i(x_i^2, x_{-i}^2)} C^i(x_i^1) \geq x_i^2 P^i(x_i^2, x_{-i}^1) - \frac{P^i(x_i^2, x_{-i}^1)}{P^i(x_i^2, x_{-i}^2)} C^i(x_i^2).$$

$$\begin{aligned} x_i^1 P^i(x_i^1, x_{-i}^1) - x_i^2 P^i(x_i^2, x_{-i}^1) &\geq \frac{P^i(x_i^2, x_{-i}^1)}{P^i(x_i^2, x_{-i}^2)} (C^i(x_i^1) - C^i(x_i^2)) \\ &> C^i(x_i^1) - C^i(x_i^2) \end{aligned}$$

Since $P^i(x_i^2, x_{-i}^1) > P^i(x_i^2, x_{-i}^2)$ from assumption 2 and $C^i(x_i^1) > C^i(x_i^2)$ from assumption 3, it follows that

$$x_i^1 P^i(x_i^2, x_{-i}^1) - C^i(x_i^1) > x_i^2 P^i(x_i^1, x_{-i}^1) - C^i(x_i^2).$$

Hence

$$\Pi^i(x_i^1, x_{-i}^1) > \Pi^i(x_i^2, x_{-i}^1),$$

which means that (5.2) is satisfied. ■

Proof. (of Theorem 5.7) is analogous to the previous one. ■

Proof. (of Theorem 5.8) Consider firm i. Let the $F(a,b,s) = bP^i(a,s) - C^i(b)$. Then $F_1(a,b,s) = bP_1^i(a,s)$, $F_2(a,b,s) = P^i(a,s) - C^{i\prime}(b)$ and

$$\begin{aligned}
\frac{F_1(a,b,s)}{|F_2(a,b,s)|} &= \frac{bP_1^i(a,s)}{|P^i(a,s) - C^{i\prime}(b)|} \\
& \frac{\partial F_1(a,b,s)/\left|F_2(a,b,s)\right|}{\partial s} \\
&= b\frac{P_{12}^i(a,s)\left|P^i(a,s) - C^{i\prime}(b)\right| - P_1^i(a,s)P_2^i(a,s)}{\left(P^i(a,s) - C^{i\prime}(b)\right)^2}
\end{aligned}$$

Then, condition (5.3) holds when

$$P_{12}^i(a,s)\left|P^i(a,s) - C^{i\prime}(b)\right| - P_1^i(a,s)P_2^i(a,s) \geq 0.$$

Sufficient condition for this is (5.5). ■

Proof. (of Corollary 5.9) Consider firm i. Take $x_i^1 > x_i^2, x_{-i}^1 > x_{-i}^2$. Form assumption 2 we have:

$$\begin{aligned}
& \ln(P^i(x_i^1, x_{-i}^1) - c^i) - \ln(P^i(x_i^2, x_{-i}^1) - c^i) \\
\geq\ & \ln(P^i(x_i^1, x_{-i}^2) - c^i) - \ln(P^i(x_i^2, x_{-i}^2) - c^i)
\end{aligned}$$

This implies that

$$\frac{P^i(x_i^1, x_{-i}^1) - c^i}{P^i(x_i^2, x_{-i}^1) - c^i} \geq \frac{P^i(x_i^1, x_{-i}^2) - c^i}{P^i(x_i^2, x_{-i}^2) - c^i}$$

and

$$(P^i(x_i^2, x_{-i}^2) - c^i)\frac{P^i(x_i^1, x_{-i}^1) - c^i}{P^i(x_i^2, x_{-i}^1) - c^i} \geq P^i(x_i^1, x_{-i}^2) - c^i \tag{5.9}$$

To prove that the game characterized by log-supermodular net-of cost inverse demand and constant marginal cost is ordinally supermodular, it is enough to show that $\Pi^i(x_i, x_{-i})$ has the single crossing property. To this end we start from assuming that

$$\Pi^i(x_i^1, x_{-i}^2) \geq \Pi^i(x_i^2, x_{-i}^2). \tag{5.10}$$

Then

$$x_i^1(P^i(x_i^1, x_{-i}^2) - c^i) \geq x_i^2(P^i(x_i^2, x_{-i}^2) - c^i).$$

We can replace $P^i(x_i^1, x_{-i}^2) - c^i$ from (5.9)

$$\begin{aligned} x_i^1((P^i(x_i^2, x_{-i}^2) - c^i)\frac{P^i(x_i^1, x_{-i}^1) - c^i}{P^i(x_i^2, x_{-i}^1) - c^i}) &\geq x_i^2(P^i(x_i^2, x_{-i}^2) - c^i). \\ x_i^1(\left(P^i(x_i^1, x_{-i}^1) - c^i\right)\frac{P^i(x_i^2, x_{-i}^2) - c^i}{P^i(x_i^2, x_{-i}^1) - c^i}) &\geq x_i^2(P^i(x_i^2, x_{-i}^2) - c^i). \end{aligned}$$

Dividing by $\frac{P^i(x_i^2, x_{-i}^2) - c^i}{P^i(x_i^2, x_{-i}^1) - c^i}$ we get

$$x_i^1(P^i(x_i^1, x_{-i}^1) - c^i) \geq x_i^2(P^i(x_i^2, x_{-i}^1) - c^i)$$

which means that (5.10) implies

$$\Pi^i(x_i^1, x_{-i}^1) \geq \Pi^i(x_i^2, x_{-i}^1),$$

which concludes the proof. ■

Proof. (of Theorem 5.11) is analogous to the proof of Theorem 5.8. ■

Proof. (of Corollary 5.12) is analogous to the proof of Corollary 5.9. ■

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