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VIETORIS ENDOFUNCTOR FOR CLOSED RELATIONS AND ITS DE VRIES DUAL

MARCO ABBADINI, GURAM BEZHANISHVILI, AND LUCA CARAI

Abstract. We generalize the classic Vietoris endofunctor to the category of compact Hausdorff spaces and closed relations. The lift of a closed relation is done by generalizing the construction of the Egli–Milner order. We describe the dual endofunctor on the category of de Vries algebras and subordinations. This is done in several steps, first by generalizing the known endofunctor on the category of boolean algebras and boolean homomorphisms, then lifting it up to $\mathsf{S5}$-subordination algebras, and finally using MacNeille completions to further lift it to de Vries algebras. Among other things, this yields a generalization of Johnstone's pointfree construction of the Vietoris endofunctor to the category of compact regular frames and preframe homomorphisms.

1. Introduction

Hyperspace constructions are among classic constructions in topology [21] with numerous applications [23]. One of the most famous is taking the Vietoris space $\mathbb{V}(X)$ of a compact Hausdorff space X. It is well known

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[30, Sec. III.4] that this defines an endofunctor $\mathbb{V}$ on the category KHaus of compact Hausdorff spaces and continuous functions.

In recent years there has been some work on generalizing KHaus to the category $\mathsf{KHaus}^\mathsf{R}$ of compact Hausdorff spaces and closed relations ([44], [33], [39], [12], [1]). One advantage of $\mathsf{KHaus}^\mathsf{R}$ is that it is more symmetric. In particular, $\mathsf{KHaus}^\mathsf{R}$ is a dagger category (see, e.g., [12, Rem. 3.7]). It is then natural to try to generalize the Vietoris endofunctor to an endofunctor $\mathbb{V}^\mathsf{R}\colon \mathsf{KHaus}^\mathsf{R} \to \mathsf{KHaus}^\mathsf{R}$. This is one of the aims of the present paper.

If $f\colon X \to Y$ is a continuous function between compact Hausdorff spaces, then its lift $\mathbb{V}(f)\colon \mathbb{V}(X) \to \mathbb{V}(Y)$ is defined by $\mathbb{V}f(F) = f[F]$, where F is a closed subset of X and $f[F]$ is the f-image of F in Y. We observe that if $R \subseteq X \times Y$ is a closed relation, then the lift $\mathbb{V}^\mathsf{R}(R)$ can be defined by generalizing the well-known Egli–Milner order, which originates in domain theory (see, e.g., [43, p. 224]). More precisely, if $F \subseteq X$ and $G \subseteq Y$ are closed sets, then we define $\mathbb{V}^\mathsf{R}(R)$ by

$$F \ \mathbb{V}^\mathsf{R}(R) \ G \iff G \subseteq R[F] \text{ and } F \subseteq R^{-1}[G],$$

where $R[F]$ is the R-image of F in Y and $R^{-1}[G]$ is the R-inverse image of G in X. We show that this defines an endofunctor $\mathbb{V}^\mathsf{R}\colon \mathsf{KHaus}^\mathsf{R} \to \mathsf{KHaus}^\mathsf{R}$, which restricts to the Vietoris endofunctor $\mathbb{V}\colon \mathsf{KHaus} \to \mathsf{KHaus}$ (see section 2).

In pointfree topology there is a well-known duality between KHaus and the category KRFrm of compact regular frames [29] (see also [6] and [30, Sec. III.1]), known as Isbell duality. It is obtained by associating with each compact Hausdorff space X the frame $\mathcal{O}(X)$ of opens of X. In [30, Sec. III.4], Peter T. Johnstone described an endofunctor $\mathbb{J}\colon \mathsf{KRFrm} \to \mathsf{KRFrm}$ dual to $\mathbb{V}\colon \mathsf{KHaus} \to \mathsf{KHaus}$ (see also [31]).

Isbell duality extends to a duality between $\mathsf{KHaus}^\mathsf{R}$ and the category $\mathsf{KRFrm}^\mathsf{P}$ of compact regular frames and preframe homomorphisms ([44], [33]). While it is possible to extend Johnstone's endofunctor to an endofunctor $\mathbb{J}^\mathsf{P}\colon \mathsf{KRFrm}^\mathsf{P} \to \mathsf{KRFrm}^\mathsf{P}$ that is dual to $\mathbb{V}^\mathsf{R}$, the construction is less elegant. Indeed, for $L \in \mathsf{KRFrm}$, the frame $\mathbb{J}(L)$ is constructed as the quotient of the free frame over the set $\{\Box_a, \Diamond_a \mid a \in L\}$ by appropriate relations [30, p. 112]. Then $\mathbb{J}(L)$ is a compact regular frame and the functoriality of $\mathbb{J}$ follows since every frame homomorphism preserves arbitrary joins and finite meets [30, pp. 113–115]. Since $\mathsf{KRFrm}^\mathsf{P}$ is a wide subcategory of KRFrm, to define $\mathbb{J}^\mathsf{P}(L)$ we do not need to change the construction of $\mathbb{J}(L)$, but the functoriality of $\mathbb{J}^\mathsf{P}$ becomes a non-trivial issue because preframe homomorphisms do not preserve arbitrary joins (see Remark 8.15). We instead take a different route.

There is another duality for KHaus that is closely related to Isbell duality. It is obtained by associating with each compact Hausdorff space X the boolean frame $\mathcal{RO}(X)$ of regular opens of X equipped with the proximity relation given by $U \prec V$ if and only if $\mathsf{cl}(U) \subseteq V$. This yields a duality between KHaus and the category of pairs $(B, \prec)$ where B is a boolean frame and $\prec$ is a proximity relation on B. These pairs are known as de Vries algebras, the resulting category is denoted by DeV, and the duality between KHaus and DeV is known as de Vries duality [18] (see also [8]).

Since both KRFrm and DeV are dually equivalent to KHaus, the two categories are equivalent, and the equivalence is obtained through the booleanization $\mathfrak{B}\colon \mathsf{KRFrm} \to \mathsf{DeV}$ [9]. The endofunctor $\mathbb{J}\colon \mathsf{KRFrm} \to \mathsf{KRFrm}$ then gives rise to an endofunctor on DeV that is dual to the Vietoris endofunctor on KHaus. A direct pointfree construction of this endofunctor, without utilizing $\mathbb{J}$, remained an open problem [10, p. 375]. The approach we develop in this paper, among other things, leads to a solution of this problem (see Remark 8.10).

In [1], we extended de Vries duality to $\mathsf{KHaus}^{\mathsf{R}}$. For this we worked with subordination relations, which originated in [11]. We introduced the category BA^{S} whose objects are boolean algebras and whose morphisms are subordination relations between them (see section 3). Let $\mathsf{Stone}^{\mathsf{R}}$ be the full subcategory of $\mathsf{KHaus}^{\mathsf{R}}$ consisting of Stone spaces. Then Stone duality extends to an equivalence between BA^{S} and $\mathsf{Stone}^{\mathsf{R}}$ ([17], [36], [1]. Using the machinery of allegories [22] yields an equivalence between $\mathsf{KHaus}^{\mathsf{R}}$ and the category whose objects are pairs (B, S) where B is a boolean algebra and S is a subordination relation on B satisfying axioms that generalize the axioms of an $\mathsf{S5}$-modality, which plays a prominent role in modal logic [14]. Because of this connection, we termed such pairs (B, S) $\mathsf{S5}$-subordination algebras and denoted the resulting category by $\mathsf{SubS5}^{\mathsf{S}}$ [1, p. 8]. The category $\mathsf{DeV}^{\mathsf{S}}$ is a full subcategory of $\mathsf{SubS5}^{\mathsf{S}}$ consisting of de Vries algebras. It turns out that $\mathsf{DeV}^{\mathsf{S}}$ is equivalent to $\mathsf{KHaus}^{\mathsf{R}}$ and, hence, to $\mathsf{SubS5}^{\mathsf{S}}$ [1, p. 12]. In fact, the equivalence between $\mathsf{SubS5}^{\mathsf{S}}$ and $\mathsf{DeV}^{\mathsf{S}}$ is obtained by generalizing the well-known MacNeille construction to $\mathsf{S5}$-subordination algebras [2].

In this paper, we define an endofunctor on $\mathsf{SubS5}^{\mathsf{S}}$ which is dual to the Vietoris endofunctor $\mathbb{V}^{\mathsf{R}}$ on $\mathsf{KHaus}^{\mathsf{R}}$. This we do as follows. In [47], the authors define the endofunctor $\mathbb{K}$ on the category BA of boolean algebras that is dual to the Vietoris endofunctor $\mathbb{V}$ on the category Stone of Stone spaces. One of our main technical results lifts this endofunctor to the endofunctor $\mathbb{K}^{\mathsf{S}}$ on BA^{S} (see section 5). In section 7, we show that $\mathbb{K}^{\mathsf{S}}$ is equivalent to $\mathbb{V}^{\mathsf{R}}$ on $\mathsf{Stone}^{\mathsf{R}}$. Finally, in section 8, we lift $\mathbb{K}^{\mathsf{S}}$ to an endofunctor on $\mathsf{SubS5}^{\mathsf{S}}$ which is equivalent to $\mathbb{V}^{\mathsf{R}}$ on $\mathsf{KHaus}^{\mathsf{R}}$. We then show that composing the MacNeille completion functor with this endofunctor yields an endofunctor $\mathbb{L}^{\mathsf{S}}$ on $\mathsf{DeV}^{\mathsf{S}}$ which is also equivalent to $\mathbb{V}^{\mathsf{R}}$.

This resolves the problem mentioned above in the more general setting of the category $\mathsf{SubS5}^{\mathsf{S}}$ and its full subcategory $\mathsf{DeV}^{\mathsf{S}}$. In [3], we show how this yields a solution of the problem for the category DeV by utilizing the technique developed in [1].

2. The Vietoris Endofunctor on $\mathsf{KHaus}^{\mathsf{R}}$

We start by recalling the classic Vietoris construction.

Definition 2.1. Let X be a compact Hausdorff space and $\mathcal{O}(X)$ the frame of opens of X. The *Vietoris space* of X is the space $\mathbb{V}(X)$ of closed subsets of X topologized by the subbasis $\{\Box_U \mid U \in \mathcal{O}(X)\} \cup \{\Diamond_U \mid U \in \mathcal{O}(X)\}$, where

$$\Box_U = \{F \in \mathbb{V}(X) \mid F \subseteq U\} \text{ and } \Diamond_U = \{F \in \mathbb{V}(X) \mid F \cap U \neq \varnothing\}.$$

For a continuous function $f\colon X \to Y$ between compact Hausdorff spaces, define the function $\mathbb{V}(f)\colon \mathbb{V}(X) \to \mathbb{V}(Y)$ by $\mathbb{V}(f)[F] = f[F]$ for each $F \in \mathbb{V}(X)$. It is well known (see, e.g., [30, p. 112]) that this defines an endofunctor on KHaus which we denote by $\mathbb{V}$.

The following useful lemma is well known (see [30, p. 112]).

Lemma 2.2. *Let $X \in \mathsf{KHaus}$ and $\mathcal{U} \subseteq \mathcal{O}(X)$. Then*

$$\begin{array}{ll} \Box_{\bigcap \mathcal{U}} = \bigcap\{\Box_U \mid U \in \mathcal{U}\} & \mathcal{U} \text{ finite},\\ \Box_{\bigcup \mathcal{U}} = \bigcup\{\Box_U \mid U \in \mathcal{U}\} & \mathcal{U} \text{ directed},\\ \Diamond_{\bigcup \mathcal{U}} = \bigcup\{\Diamond_U \mid U \in \mathcal{U}\} & \mathcal{U} \text{ arbitrary},\\ \Box_{U\cup V} \subseteq \Box_U \cup \Diamond_V, & \\ \Box_U \cap \Diamond_V \subseteq \Diamond_{U\cap V}. & \end{array}$$

As a simple consequence of Lemma 2.2, we obtain the following lemma.

Lemma 2.3. *The following is a basis for $\mathbb{V}(X)$:*

$$\{\Box_U \cap \Diamond_{V_1} \cap \dots \cap \Diamond_{V_n} \mid U, V_1, \dots, V_n \in \mathcal{O}(X) \text{ and } V_1, \dots, V_n \subseteq U\}.$$

We recall that a relation $R\colon X \to Y$ between compact Hausdorff spaces is *closed* if it is a closed subset of $X \times Y$. Let $\mathsf{KHaus}^{\mathsf{R}}$ be the category of compact Hausdorff spaces and closed relations between them. Identity morphisms in $\mathsf{KHaus}^{\mathsf{R}}$ are identity relations and composition is relation composition. Our goal is to lift $\mathbb{V}$ to an endofunctor on $\mathsf{KHaus}^{\mathsf{R}}$. For this, we generalize the definition of the Egli–Milner order [43, p. 224].

Definition 2.4. Let $R\colon X \to Y$ be a closed relation between compact Hausdorff spaces. Define $\mathcal{R}_\Box$, $\mathcal{R}_\Diamond$, and $\mathcal{R}_{EM}$ from $\mathbb{V}(X)$ to $\mathbb{V}(Y)$ as follows:

(1) $F\ \mathcal{R}_\Box\ G$ iff $G \subseteq R[F]$,
(2) $F\ \mathcal{R}_\Diamond\ G$ iff $F \subseteq R^{-1}[G]$,
(3) $\mathcal{R}_{EM} = \mathcal{R}_\Box \cap \mathcal{R}_\Diamond$.

In order to show that these relations are closed, we utilize the following lemma (see, e.g., [11, Lem. 2.12]).

Lemma 2.5. *For a relation $R\colon X \to Y$ between compact Hausdorff spaces, the following are equivalent:*

(1) *R is closed,*
(2) *$R[F]$ is closed for each closed $F \subseteq X$ and $R^{-1}[G]$ is closed for each closed $G \subseteq Y$,*
(3) *$(x, y) \notin R$ implies the existence of two open neighborhoods U_x and U_y such that $R[U_x] \cap U_y = \varnothing$.*

In order to simplify notation (see Proposition 2.6 and Proposition 7.1), for $E \subseteq Z$, we write $-E$ to denote the complement of E in Z.

Proposition 2.6. *The relations $\mathcal{R}_\Box$, $\mathcal{R}_\Diamond$, and $\mathcal{R}_{EM}$ are closed.*

Proof. To see that $\mathcal{R}_\Box$ is a closed relation, it is enough to show that Lemma 2.5(3) is satisfied. Let $(F, G) \notin \mathcal{R}_\Box$. Then $G \not\subseteq R[F]$. Therefore, there is $x \in G$ such that $x \notin R[F]$. Since $R[F]$ is closed, there are disjoint open sets U and V such that $x \in U$ and $R[F] \subseteq V$. Thus, $F \subseteq -R^{-1} - V$, so $F \in \Box_{-R^{-1}-V}$. Also, $G \cap U \neq \varnothing$, so $G \in \Diamond_U$. We show that $\mathcal{R}_\Box[\Box_{-R^{-1}-V}] \cap \Diamond_U = \varnothing$. If $K \in \mathcal{R}_\Box[\Box_{-R^{-1}-V}] \cap \Diamond_U$, then $K \cap U \neq \varnothing$ and there is $H \in \Box_{-R^{-1}-V}$ such that $H\ \mathcal{R}_\Box\ K$. Therefore, $H \subseteq -R^{-1} - V$ and $K \subseteq R[H]$. From $H \subseteq -R^{-1} - V$ it follows that $R[H] \subseteq V$, so $K \subseteq V$, which together with $K \cap U \neq \varnothing$ implies that U and V are not disjoint, a contradiction. Thus, $\mathcal{R}_\Box$ is closed.

To see that $\mathcal{R}_\Diamond$ is a closed relation, we again show that Lemma 2.5(3) is satisfied. Let $(F, G) \notin \mathcal{R}_\Diamond$. Then $F \not\subseteq R^{-1}[G]$. Therefore, there is $x \in F$ such that $x \notin R^{-1}[G]$. Since $R^{-1}[G]$ is closed, there are disjoint open U and V such that $x \in U$ and $R^{-1}[G] \subseteq V$. Thus, $F \cap U \neq \varnothing$ and $G \subseteq -R - V$. This implies that $F \in \Diamond_U$ and $G \in \Box_{-R-V}$. We show that $\mathcal{R}_\Diamond[\Diamond_U] \cap \Box_{-R-V} = \varnothing$. If $K \in \mathcal{R}_\Diamond[\Diamond_U] \cap \Box_{-R-V}$, then $K \subseteq -R - V$ and there is $H \in \Diamond_U$ such that $H\ \mathcal{R}_\Diamond\ K$. Therefore, $R^{-1}[K] \subseteq V$, $H \cap U \neq \varnothing$, and $H \subseteq R^{-1}[K]$. Thus, $H \subseteq V$ and $H \cap U \neq \varnothing$, which contradicts that U and V are disjoint. Consequently, $\mathcal{R}_\Diamond$ is closed.

Finally, since both $\mathcal{R}_\Box$ and $\mathcal{R}_\Diamond$ are closed relations, so is their intersection; hence, $\mathcal{R}_{EM}$ is a closed relation. $\Box$

Definition 2.7. For a morphism $R\colon X \to Y$ in $\mathsf{KHaus}^{\mathsf{R}}$, define

$$\mathbb{V}^{\mathsf{R}}_{\Box}(R) = \mathcal{R}_{\Box},\ \mathbb{V}^{\mathsf{R}}_{\Diamond}(R) = \mathcal{R}_{\Diamond},\ \text{and } \mathbb{V}^{\mathsf{R}}(R) = \mathcal{R}_{EM}.$$

We next show that $\mathbb{V}^{\mathsf{R}}_{\Box}$ and $\mathbb{V}^{\mathsf{R}}_{\Diamond}$ are semi-functors, while $\mathbb{V}^{\mathsf{R}}$ is an endofunctor on $\mathsf{KHaus}^{\mathsf{R}}$. We recall that, given two categories C and D, a *semi-functor* $F\colon \mathsf{C} \to \mathsf{D}$ is an assignment that maps objects of C to objects of D, morphisms of C to morphisms of D, and preserves composition (see, e.g., [26], [28]). The notion of a semi-functor is weaker than that of a functor since identity morphisms are not required to be preserved.

Theorem 2.8.

(1) $\mathbb{V}^{\mathsf{R}}_{\Box}, \mathbb{V}^{\mathsf{R}}_{\Diamond}\colon \mathsf{KHaus}^{\mathsf{R}} \to \mathsf{KHaus}^{\mathsf{R}}$ *are semi-functors.*
(2) $\mathbb{V}^{\mathsf{R}}$ *is an endofunctor on* $\mathsf{KHaus}^{\mathsf{R}}$ *which restricts to the Vietoris endofunctor* $\mathbb{V}$ *on* KHaus.
(3) *Each of the three restricts to* $\mathsf{Stone}^{\mathsf{R}}$.

Proof. (1) That $\mathbb{V}^{\mathsf{R}}_{\Box}$ and $\mathbb{V}^{\mathsf{R}}_{\Diamond}$ are well defined follows from Proposition 2.6. Let $R_1\colon X_1 \to X_2$ and $R_2\colon X_2 \to X_3$ be morphisms in $\mathsf{KHaus}^{\mathsf{R}}$, and let $F \subseteq X_1$, $G \subseteq X_2$, and $H \subseteq X_3$ be closed subsets. If $F\ \mathbb{V}^{\mathsf{R}}_{\Box}(R_1)\ G$ and $G\ \mathbb{V}^{\mathsf{R}}_{\Box}(R_2)\ H$, then $G \subseteq R_1[F]$ and $H \subseteq R_2[G]$. Therefore,

$$H \subseteq R_2[G] \subseteq R_2[R_1[F]] = (R_2 \circ R_1)[F],$$

and thus, $F\ \mathbb{V}^{\mathsf{R}}_{\Box}(R_2 \circ R_1)\ H$. Conversely, if $F\ \mathbb{V}^{\mathsf{R}}_{\Box}(R_2 \circ R_1)\ H$, then $H \subseteq R_2[R_1[F]]$. Let $G = R_1[F]$. Then G is a closed subset of X_2 such that $G \subseteq R_1[F]$ and $H \subseteq R_2[G]$. Therefore, $F\ \mathbb{V}^{\mathsf{R}}_{\Box}(R_1)\ G$ and $G\ \mathbb{V}^{\mathsf{R}}_{\Box}(R_2)\ H$. Thus, $\mathbb{V}^{\mathsf{R}}_{\Box}(R_2 \circ R_1) = \mathbb{V}^{\mathsf{R}}_{\Box}(R_2) \circ \mathbb{V}^{\mathsf{R}}_{\Box}(R_1)$. The proof for $\mathbb{V}^{\mathsf{R}}_{\Diamond}$ is similar.

(2) That $\mathbb{V}^{\mathsf{R}}$ is well defined follows from Proposition 2.6. It is an immediate consequence of the definition of $\mathcal{R}_{EM}$ that $\mathbb{V}^{\mathsf{R}}$ preserves identity relations. If $F\ \mathbb{V}^{\mathsf{R}}(R_1)\ G$ and $G\ \mathbb{V}^{\mathsf{R}}(R_2)\ H$, then

$$G \subseteq R_1[F],\ F \subseteq R_1^{-1}[G],\ H \subseteq R_2[G],\ \text{and } G \subseteq R_2^{-1}[H].$$

The same argument as in (1) yields that $F\ \mathbb{V}^{\mathsf{R}}(R_2 \circ R_1)\ H$. Conversely, let $F \subseteq X_1$ and $H \subseteq X_3$ be closed subsets such that $F\ \mathbb{V}^{\mathsf{R}}(R_2 \circ R_1)\ H$. Then $H \subseteq R_2[R_1[F]]$ and $F \subseteq R_1^{-1}[R_2^{-1}[H]]$. Since R_1 and R_2 are closed relations, $G \coloneqq R_1[F] \cap R_2^{-1}[H]$ is a closed subset of X_2. We show that $F\ \mathbb{V}^{\mathsf{R}}(R_1)\ G$ and $G\ \mathbb{V}^{\mathsf{R}}(R_2)\ H$. The definition of G yields that $G \subseteq R_1[F]$ and $G \subseteq R_2^{-1}[H]$. Since $H \subseteq R_2[R_1[F]]$, for each $z \in H$ there are $x \in F$ and $y \in X_2$ such that $x\ R_1\ y$ and $y\ R_2\ z$. Therefore, $y \in R_1[F] \cap R_2^{-1}[H] = G$. Thus, $H \subseteq R_2[G]$. Similarly, $F \subseteq R_1^{-1}[R_2^{-1}[H]]$ implies $F \subseteq R_1^{-1}[G]$. Consequently, $F\ \mathbb{V}^{\mathsf{R}}(R_1)\ G$ and $G\ \mathbb{V}^{\mathsf{R}}(R_2)\ H$. This shows that $\mathbb{V}^{\mathsf{R}}$ preserves compositions and, hence, is an endofunctor on $\mathsf{KHaus}^{\mathsf{R}}$.

If $f\colon X_1 \to X_2$ is a continuous function, then $F \subseteq f^{-1}[G]$ if and only if $f[F] \subseteq G$. Therefore, $F\ \mathbb{V}^{\mathsf{R}}(f)\ G$ if and only if $G = f[F]$, and so $\mathbb{V}^{\mathsf{R}}(f)$ is the function that maps $F \in \mathbb{V}(X_1)$ to $f[F] \in \mathbb{V}(X_2)$. Thus, $\mathbb{V}^{\mathsf{R}}$ extends the Vietoris endofunctor on KHaus.

(3) It is well known (see [38, Sec. 4]) that if X is a Stone space, then $\mathbb{V}(X)$ is also a Stone space. Consequently, $\mathbb{V}^{\mathsf{R}}_{\Box}$, $\mathbb{V}^{\mathsf{R}}_{\Diamond}$, and $\mathbb{V}^{\mathsf{R}}$ restrict to $\mathsf{Stone}^{\mathsf{R}}$. ☐

Remark 2.9. The semi-functors $\mathbb{V}^{\mathsf{R}}_{\Box}$ and $\mathbb{V}^{\mathsf{R}}_{\Diamond}$ do not preserve identity. Indeed, if R is the identity relation on X, then $F\ \mathcal{R}_{\Box}\ G$ if and only if $G \subseteq F$ and $F\ \mathcal{R}_{\Diamond}\ G$ if and only if $F \subseteq G$. Therefore, if $X \neq \varnothing$, then neither $\mathcal{R}_{\Box}$ nor $\mathcal{R}_{\Diamond}$ is the identity on $\mathbb{V}(X)$ in $\mathsf{KHaus}^{\mathsf{R}}$.

We recall that a *dagger* on a category C is a contravariant functor $(-)^{\dagger}\colon \mathsf{C} \to \mathsf{C}$ that is the identity on objects and the composition $(-)^{\dagger} \circ (-)^{\dagger}$ is the identity functor on C (see, e.g., [27, p. 74]). It is well known and easy to see that mapping a closed relation $R\colon X \to Y$ to its converse, which is clearly a closed relation, defines a dagger on $\mathsf{KHaus}^{\mathsf{R}}$. We conclude the section by observing that $\mathbb{V}^{\mathsf{R}}_{\Box}$ and $\mathbb{V}^{\mathsf{R}}_{\Diamond}$ are definable from each other using $(-)^{\dagger}$, and that $\mathbb{V}^{\mathsf{R}}$ commutes with $(-)^{\dagger}$.

Proposition 2.10.

(1) $\mathbb{V}^{\mathsf{R}}_{\Box} \circ (-)^{\dagger} = (-)^{\dagger} \circ \mathbb{V}^{\mathsf{R}}_{\Diamond}$ *and* $\mathbb{V}^{\mathsf{R}}_{\Diamond} \circ (-)^{\dagger} = (-)^{\dagger} \circ \mathbb{V}^{\mathsf{R}}_{\Box}$.
(2) $\mathbb{V}^{\mathsf{R}} \circ (-)^{\dagger} = (-)^{\dagger} \circ \mathbb{V}^{\mathsf{R}}$.

Proof. Since $\mathbb{V}^{\mathsf{R}}_{\Box}$, $\mathbb{V}^{\mathsf{R}}_{\Diamond}$, and $\mathbb{V}^{\mathsf{R}}$ coincide on objects and $(-)^{\dagger}$ fixes the objects, we only need to show that the compositions agree on the morphisms.

(1) We only prove the first equality since the second is proved similarly. Let $R\colon X \to Y$ be a morphism in $\mathsf{KHaus}^{\mathsf{R}}$. For $F \in \mathbb{V}(X)$ and $G \in \mathbb{V}(Y)$, we have

$$\begin{aligned} G\ \mathbb{V}^{\mathsf{R}}_{\Box}(R^{\dagger})\ F &\iff F \subseteq R^{\dagger}[G] \iff F \subseteq R^{-1}[G] \\ &\iff F\ \mathbb{V}^{\mathsf{R}}_{\Diamond}(R)\ G \iff G\ \mathbb{V}^{\mathsf{R}}_{\Diamond}(R)^{\dagger}\ F. \end{aligned}$$

(2) Let $R\colon X \to Y$ be a morphism in $\mathsf{KHaus}^{\mathsf{R}}$. If $F \in \mathbb{V}(X)$ and $G \in \mathbb{V}(Y)$, then (1) implies

$$\begin{aligned} G\ \mathbb{V}^{\mathsf{R}}(R^{\dagger})\ F &\iff G\ \mathbb{V}^{\mathsf{R}}_{\Box}(R^{\dagger})\ F \text{ and } G\ \mathbb{V}^{\mathsf{R}}_{\Diamond}(R^{\dagger})\ F \\ &\iff G\ \mathbb{V}^{\mathsf{R}}_{\Diamond}(R)^{\dagger}\ F \text{ and } G\ \mathbb{V}^{\mathsf{R}}_{\Box}(R)^{\dagger}\ F \\ &\iff F\ \mathbb{V}^{\mathsf{R}}_{\Diamond}(R)\ G \text{ and } F\ \mathbb{V}^{\mathsf{R}}_{\Box}(R)\ G \\ &\iff G\ \mathbb{V}^{\mathsf{R}}(R)^{\dagger}\ F. \end{aligned}$$

☐

Remark 2.11. We point out that Proposition 2.10(1) simplifies the proof of Proposition 2.6. Indeed, if $R\colon X \to Y$ is a closed relation, then $\mathbb{V}^{\mathsf{R}}_{\Diamond}(R) = \mathbb{V}^{\mathsf{R}}_{\Box}(R^{\dagger})^{\dagger}$, and so is closed. Thus, it is enough to prove that $\mathbb{V}^{\mathsf{R}}_{\Box}(R)$ is closed.

3. Closed Relations and Subordinations

In this section, we recall the definition of subordination relations and the results connecting them to closed relations between compact Hausdorff spaces. Subordinations on boolean algebras were introduced in [11]. They are closely related to precontact relations ([19], [20]) and quasi-modal operators [16].

Definition 3.1 ([11, Def. 2.3])**.** We call a relation $S\colon A \to B$ between boolean algebras a *subordination relation* if it satisfies the following axioms, where $a, b \in A$ and $c, d \in B$:

(S1) $0 \mathrel{S} 0$ and $1 \mathrel{S} 1$,
(S2) $a, b \mathrel{S} c$ implies $(a \vee b) \mathrel{S} c$,
(S3) $a \mathrel{S} c, d$ implies $a \mathrel{S} (c \wedge d)$,
(S4) $a \leq b \mathrel{S} c \leq d$ implies $a \mathrel{S} d$.

Boolean algebras and subordination relations between them form a category, which we denote by BA^{S}. The identity on $B \in \mathsf{BA}^{\mathsf{S}}$ is the order $\leq$ on B and the composition is relation composition. We have the following generalization of Stone duality, which was obtained in [1, Cor. 2.6] by utilizing a result of Sergio A. Celani [17, Thm. 4]. It is also a consequence of a more general result of Alexander Kurz, Andrew Moshier, and Achim Jung [36, Thm. 5.5.9].

Theorem 3.2. *The categories* $\mathsf{Stone}^{\mathsf{R}}$ *and* BA^{S} *are equivalent.*

Remark 3.3. The functors $\mathsf{Clop}\colon \mathsf{Stone}^{\mathsf{R}} \to \mathsf{BA}^{\mathsf{S}}$ and $\mathsf{Uf}\colon \mathsf{BA}^{\mathsf{S}} \to \mathsf{Stone}^{\mathsf{R}}$ yielding the above equivalence are defined as follows. On objects, they act as the clopen and ultrafilter functors of Stone duality. On morphisms, they act as follows. If $R\colon X \to Y$ is a closed relation between Stone spaces, then $\mathsf{Clop}(R)$ is the subordination relation $S_R\colon \mathsf{Clop}(X) \to \mathsf{Clop}(Y)$ given by

$$U \mathrel{S_R} V \iff R[U] \subseteq V.$$

If $S\colon A \to B$ is a subordination relation between boolean algebras, then $\mathsf{Uf}(S)$ is the closed relation $R_S\colon \mathsf{Uf}(A) \to \mathsf{Uf}(B)$ given by

$$x \mathrel{R_S} y \iff S[x] \subseteq y.$$

Definition 3.4 ([1, p. 7])**.** An $\mathsf{S5}$-*subordination space* is a pair (X, E) where X is a Stone space and E is a closed equivalence relation on X. A closed relation $R \subseteq X_1 \times X_2$ between $\mathsf{S5}$-subordination spaces (X_1, E_1)

and (X_2, E_2) is *compatible* if $R \circ E_1 = R = E_2 \circ R$. We let $\mathsf{StoneE}^\mathsf{R}$ be the category of $\mathsf{S5}$-subordination spaces and compatible closed relations.

Theorem 3.5 ([1, Cor. 3.12])**.** *The categories* $\mathsf{StoneE}^\mathsf{R}$ *and* $\mathsf{KHaus}^\mathsf{R}$ *are equivalent.*

Remark 3.6. The equivalence in Theorem 3.5 is established by the functor $\mathcal{Q}\colon \mathsf{StoneE}^\mathsf{R} \to \mathsf{KHaus}^\mathsf{R}$ that maps $(X, E) \in \mathsf{StoneE}^\mathsf{R}$ to the quotient space X/E and a morphism $R\colon (X_1, E_1) \to (X_2, E_2)$ in $\mathsf{StoneE}^\mathsf{R}$ to the closed relation $\mathcal{Q}(R) \coloneqq \pi_2 \circ R \circ \pi_1^\dagger$, where $\pi_1\colon X_1 \to X_1/E_1$ and $\pi_2\colon X_2 \to X_2/E_2$ are the projection maps. A quasi-inverse of $\mathcal{Q}$ is defined by utilizing the well-known Gleason cover construction (see, e.g., [30, Sec. III.3]). For a compact Hausdorff space X, let $\mathcal{G}(X) = (\widehat{X}, E)$ where $g_X\colon \widehat{X} \to X$ is the Gleason cover of X and $x \mathrel{E} y$ if and only if $g_X(x) = g_X(y)$. For a closed relation $R\colon X \to Y$, let $\mathcal{G}(R)\colon \mathcal{G}(X) \to \mathcal{G}(Y)$ be the relation given by $\mathcal{G}(R) \coloneqq g_Y^\dagger \circ R \circ g_X$. This defines the Gleason cover functor $\mathcal{G}\colon \mathsf{KHaus}^\mathsf{R} \to \mathsf{StoneE}^\mathsf{R}$, which is a quasi-inverse of $\mathcal{Q}$ [1, Thm. 4.6].

Definition 3.7.

(1) We say that a subordination S on a boolean algebra B is an $\mathsf{S5}$-*subordination* if S satisfies the following axioms where $a, b \in B$:
  (S5) $a \mathrel{S} b$ implies $a \leq b$,
  (S6) $a \mathrel{S} b$ implies $\neg b \mathrel{S} \neg a$,
  (S7) $a \mathrel{S} b$ implies there is $c \in B$ such that $a \mathrel{S} c$ and $c \mathrel{S} b$.

(2) An $\mathsf{S5}$-*subordination algebra* is a pair (B, S) where B is a boolean algebra and S is an $\mathsf{S5}$-subordination on B.

(3) A subordination $T \subseteq B_1 \times B_2$ between $\mathsf{S5}$-subordination algebras (B_1, S_1) and (B_2, S_2) is *compatible* if $T \circ S_1 = T = S_2 \circ T$.

(4) Let $\mathsf{SubS5}^\mathsf{S}$ be the category of $\mathsf{S5}$-subordination algebras and compatible subordinations.

De Vries algebras are special $\mathsf{S5}$-subordination algebras.

Definition 3.8.

(1) A *de Vries algebra* is an $\mathsf{S5}$-subordination algebra (B, S) such that B is complete and S satisfies the axiom:
  (S8) $b \neq 0$ implies that there is $a \neq 0$ such that $a \mathrel{S} b$.

(2) Let DeV^S be the full subcategory of $\mathsf{SubS5}^\mathsf{S}$ consisting of de Vries algebras.

Theorem 3.9 ([1, Cor. 3.14, 4.7])**.** *The categories* $\mathsf{KHaus}^\mathsf{R}$, $\mathsf{StoneE}^\mathsf{R}$, $\mathsf{SubS5}^\mathsf{S}$, *and* DeV^S *are equivalent.*

The equivalence of $\mathsf{StoneE}^{\mathsf{R}}$ and $\mathsf{SubS5}^{\mathsf{S}}$ was obtained by utilizing the machinery of allegories and splitting equivalences. As we observed in the previous section, $\mathsf{KHaus}^{\mathsf{R}}$ is a dagger category and $\mathbb{V}^{\mathsf{R}}$ commutes with the dagger. In addition, ordering the hom-sets by inclusion turns $\mathsf{KHaus}^{\mathsf{R}}$ into an order-enriched category (meaning that each hom-set is partially ordered and composition preserves the order; see, e.g., [41, p. 105]).

An *allegory* ([22], [32]) is an order-enriched dagger category such that

(1) each hom-set has binary meets,
(2) $(-)^\dagger$ preserves the order on the hom-sets,
(3) the modular law holds: $gf \wedge h \leq (g \wedge hf^\dagger)f$ for all $f\colon C \to D$, $g\colon D \to E$, and $h\colon C \to E$.

We already saw that $\mathbb{V}^{\mathsf{R}}$ commutes with the dagger, and it is straightforward to see that $\mathbb{V}^{\mathsf{R}}$ preserves inclusions of relations. Therefore, $\mathbb{V}^{\mathsf{R}}$ is a morphism of ordered categories with involution (see [45], [37]). However, as we will see in the next example, $\mathbb{V}^{\mathsf{R}}$ does not preserve the meet and, hence, is not a morphism of allegories (see [30], [1]).

Example 3.10. Since $\mathbb{V}^{\mathsf{R}}$ preserves inclusions of relations, for any pair of closed relations $R_1, R_2\colon X \to Y$, we have $\mathbb{V}^{\mathsf{R}}(R_1 \cap R_2) \subseteq \mathbb{V}^{\mathsf{R}}(R_1) \cap \mathbb{V}^{\mathsf{R}}(R_2)$. We show that the other inclusion does not hold in general. Let X be a finite discrete space with at least two elements. We let $R_1\colon X \to X$ be the identity and $R_2\colon X \to X$ its complement. Clearly, X is compact Hausdorff and R_1 and R_2 are closed. Moreover, $R_1 \cap R_2 = \varnothing$. Thus, $X\ \mathbb{V}^{\mathsf{R}}(R_1 \cap R_2)\ X$ does not hold. However, $X = R_1^{-1}[X] = R_1[X]$ and $X = R_2^{-1}[X] = R_2[X]$ because X has at least two elements. Therefore, $X\ (\mathbb{V}^{\mathsf{R}}(R_1) \cap \mathbb{V}^{\mathsf{R}}(R_2))\ X$ holds.

Our goal is to describe an endofunctor on $\mathsf{SubS5}^{\mathsf{S}}$ that corresponds to the Vietoris endofunctor $\mathbb{V}^{\mathsf{R}}$ on $\mathsf{KHaus}^{\mathsf{R}}$ and then lift it to an endofunctor on $\mathsf{DeV}^{\mathsf{S}}$. For this, we first generalize the functor $\mathbb{K}\colon \mathsf{BA} \to \mathsf{BA}$ dual to $\mathbb{V}\colon \mathsf{Stone} \to \mathsf{Stone}$ (see the Introduction).

4. The Endofunctor $\mathbb{K}$ on BA

We recall the definition of the endofunctor $\mathbb{K}$ on BA. There are several equivalent approaches to defining $\mathbb{K}$. We will follow the one given in [47], which, in turn, relies on [31].

Definition 4.1. For $B \in \mathsf{BA}$, let $X = \{\Box_a \mid a \in B\} \cup \{\Diamond_a \mid a \in B\}$ be a set of symbols and define $\mathbb{K}(B)$ as the quotient of the free boolean algebra over X by the relations

$$\begin{aligned}
&\Box_1 = 1, && \Diamond_0 = 0,\\
&\Box_{a\wedge b} = \Box_a \wedge \Box_b, && \Diamond_{a\vee b} = \Diamond_a \vee \Diamond_b,\\
&\Box_{a\vee b} \leq \Box_a \vee \Diamond_b, && \Box_a \wedge \Diamond_b \leq \Diamond_{a\wedge b}.
\end{aligned}$$

With a small abuse of notation, we write $\Box_a$ and $\Diamond_a$ also for the equivalence classes $[\Box_a]$ and $[\Diamond_a]$ in $\mathbb{K}(B)$.

Remark 4.2.

(1) The last two inequalities in Definition 4.1 can be replaced by the equation $\Box_a = \neg\Diamond_{\neg a}$, which is equivalent to $\Diamond_a = \neg\Box_{\neg a}$. Indeed, the two inequalities imply

$$1 = \Box_1 = \Box_{a\vee\neg a} \leq \Box_a \vee \Diamond_{\neg a} \quad \text{and} \quad \Box_a \wedge \Diamond_{\neg a} \leq \Diamond_{a\wedge\neg a} = \Diamond_0 = 0.$$

Therefore, $\neg\Box_a = \Diamond_{\neg a}$ and, hence, $\Box_a = \neg\Diamond_{\neg a}$. Conversely, since

$$\Box_{a\vee b} \wedge \Box_{\neg b} = \Box_{a\wedge\neg b} \leq \Box_a \quad \text{and} \quad \Diamond_b \leq \Diamond_{b\vee\neg a} = \Diamond_{a\wedge b} \vee \Diamond_{\neg a},$$

it follows that

$$\Box_{a\vee b} \leq \Box_a \vee \neg\Box_{\neg b} \quad \text{and} \quad \neg\Diamond_{\neg a} \wedge \Diamond_b \leq \Diamond_{a\wedge b}.$$

Thus,

$$\Box_{a\vee b} \leq \Box_a \vee \Diamond_b \quad \text{and} \quad \Box_a \wedge \Diamond_b \leq \Diamond_{a\wedge b}.$$

(2) Equivalently, $\mathbb{K}(B)$ can be defined as the quotient of the free boolean algebra over the set $\{\Box_a \mid a \in B\}$ by the relations $\Box_1 = 1$ and $\Box_{a\wedge b} = \Box_a \wedge \Box_b$ (see [35, Rem. 3.13] and [47, Rem. 1]). Alternatively, $\mathbb{K}(B)$ can be defined as the quotient of the free boolean algebra over the set $\{\Diamond_a \mid a \in B\}$ by the relations $\Diamond_0 = 0$ and $\Diamond_{a\vee b} = \Diamond_a \vee \Diamond_b$ (see [4, Sec. 7] and [13, Def. 2.4]).

Definition 4.3. Let A and B be boolean algebras and let $\alpha\colon A \to B$ be a boolean homomorphism. The map $\mathbb{K}(\alpha)\colon \mathbb{K}(A) \to \mathbb{K}(B)$ is the unique boolean homomorphism satisfying $\mathbb{K}(f)(\Box_a) = \Box_{\alpha(a)}$ and $\mathbb{K}(f)(\Diamond_a) = \Diamond_{\alpha(a)}$.

It is straightforward to see that this defines an endofunctor on BA (see [47, p. 122]).

Proposition 4.4. $\mathbb{K}$ *is an endofunctor on* BA.

As suggested by the notation, there is a close connection between $\mathbb{K}(B)$ and modal operators on B. We recall that a *modal operator* on B is a function $\Box\colon B \to B$ satisfying $\Box 1 = 1$ and $\Box(a\wedge b) = \Box a \wedge \Box b$ (equivalently, $\Box$ preserves finite meets). A *modal algebra* is a pair $(B, \Box)$ where $\Box$ is a modal operator on B. Let MA be the category of modal algebras and modal homomorphisms (i.e., boolean homomorphisms preserving $\Box$).

Remark 4.5. Modal algebras can alternatively be defined as pairs $(B, \Diamond)$ where $\Diamond 0 = 0$ and $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$ (equivalently, $\Diamond$ preserves finite joins). The two operators $\Box$ and $\Diamond$ are interdefinable by the well-known identities $\Diamond a = \neg\Box\neg a$ and $\Box a = \neg\Diamond\neg a$.

The next result is well known; see [47, Fact 3]. For the definition of algebras for an endofunctor, see, e.g., [5, Def. 5.37].

Theorem 4.6.

(1) *For a boolean algebra B, there is a bijection between modal operators on B and boolean homomorphisms $\mathbb{K}(B) \to B$.*
(2) *This bijection extends to an isomorphism between* MA *and the category* $\mathsf{Alg}(\mathbb{K})$ *of algebras for the endofunctor* $\mathbb{K}\colon \mathsf{BA} \to \mathsf{BA}$.

Remark 4.7.

(1) The bijection of Theorem 4.6(1) is obtained by associating to each modal operator $\Box$ on B the boolean homomorphism $\alpha\colon \mathbb{K}(B) \to B$ defined by $\alpha(\Box_a) = \Box a$ and to each boolean homomorphism $\alpha\colon \mathbb{K}(B) \to B$ the modal operator $\Box$ given by $\Box a = \alpha(\Box_a)$.
(2) An analogous bijection holds if we replace $\Box$ by $\Diamond$.

The next result is well known (see [4], [35], or [47]). For the definition of coalgebras for an endofunctor, see, e.g., [46, Def. 140].

Theorem 4.8. *The following diagram is commutative up to natural isomorphism.*

$$\begin{tikzcd}
\mathsf{Stone} \arrow[r, shift left, "\mathsf{Clop}"] \arrow[d, "\mathbb{V}"'] & \mathsf{BA} \arrow[l, shift left, "\mathsf{Uf}"] \arrow[d, "\mathbb{K}"] \\
\mathsf{Stone} \arrow[r, shift left, "\mathsf{Clop}"] & \mathsf{BA} \arrow[l, shift left, "\mathsf{Uf}"]
\end{tikzcd}$$

In other words, there are natural isomorphisms $\mathsf{Clop} \circ \mathbb{V} \simeq \mathbb{K} \circ \mathsf{Clop}$ *and* $\mathsf{Uf} \circ \mathbb{K} \simeq \mathbb{V} \circ \mathsf{Uf}$. *Consequently, Stone duality extends to a dual equivalence between* $\mathsf{Alg}(\mathbb{K})$ *and the category* $\mathsf{Coalg}(\mathbb{V})$ *of coalgebras for the Vietoris endofunctor* $\mathbb{V}\colon \mathsf{Stone} \to \mathsf{Stone}$.

5. The Semi-Functors $\mathbb{K}^{\mathsf{S}}_{\Box}$ and $\mathbb{K}^{\mathsf{S}}_{\Diamond}$

In this section, we describe how to lift a subordination $S\colon A \to B$ to the subordinations $\mathcal{S}_\Box, \mathcal{S}_\Diamond\colon \mathbb{K}(A) \to \mathbb{K}(B)$, which are the algebraic counterparts of $\mathcal{R}_\Box$ and $\mathcal{R}_\Diamond$. In the next section, we use $\mathcal{S}_\Box$ and $\mathcal{S}_\Diamond$ to define $\mathcal{S}$, which is the algebraic counterpart of $\mathcal{R}_{EM}$. To define $\mathcal{S}_\Box$ and $\mathcal{S}_\Diamond$, we first introduce disjunctive and conjunctive normal forms for elements of $\mathbb{K}(B)$.

Definition 5.1. Let $x \in \mathbb{K}(B)$.

(1) We say that x is in *disjunctive normal form* if it is written as a finite join
$$x = \bigvee_{i=1}^{n} (\Box_{a_i} \wedge \Diamond_{b_{i1}} \wedge \cdots \wedge \Diamond_{b_{in_i}}),$$
where $0 \neq b_{ij} \leq a_i$ for each i and j.

(2) We say that x is in *conjunctive normal form* if it is written as a finite meet
$$x = \bigwedge_{i=1}^{n} (\Diamond_{c_i} \vee \Box_{d_{i1}} \vee \cdots \vee \Box_{d_{in_i}}),$$
where $c_i \leq d_{ij} \neq 1$ for each i and j.

Since we allow $n = 0$, the empty join and meet yield disjunctive and conjunctive normal forms for 0 and 1, respectively.

Lemma 5.2. *Any element of $\mathbb{K}(B)$ can be written in disjunctive and conjunctive normal forms.*

Proof. Let $x \in \mathbb{K}(B)$. We only prove that x can be written in disjunctive normal form because the proof for the conjunctive normal form is similar. Since $\mathbb{K}(B)$ is generated by the elements of the form $\Box_a$ and $\Diamond_b$, we may write $x \in \mathbb{K}(B)$ as a finite join of finite meets of $\Box_a$ and $\Diamond_b$, or their negations (see, e.g., [42, p. 14]). As $\neg\Box_a = \Diamond_{\neg a}$ and $\neg\Diamond_b = \Box_{\neg b}$, we have
$$x = \bigvee_{i=1}^{n} (\Box_{a_{i1}} \wedge \cdots \wedge \Box_{a_{ik_i}} \wedge \Diamond_{b_{i1}} \wedge \cdots \wedge \Diamond_{b_{in_i}}).$$
Let $a_i = a_{i1} \wedge \cdots \wedge a_{ik_i}$. It follows from the definition of $\mathbb{K}(B)$ that
$$\Box_{a_i} = \Box_{a_{i1}} \wedge \cdots \wedge \Box_{a_{ik_i}}.$$
Therefore,
$$x = \bigvee_{i=1}^{n} (\Box_{a_i} \wedge \Diamond_{b_{i1}} \wedge \cdots \wedge \Diamond_{b_{in_i}}).$$
The definition of $\mathbb{K}(B)$ also yields that $\Box_a \wedge \Diamond_b = \Box_a \wedge \Diamond_{a \wedge b}$ for each $a, b \in B$. Thus, by replacing each b_{ij} with $a_i \wedge b_{ij}$, we may assume that $b_{ij} \leq a_i$ for each i and j. Since $\Diamond 0 = 0$, we can suppress every disjunct $\Box_{a_i} \wedge \Diamond_{b_{i1}} \wedge \cdots \wedge \Diamond_{b_{in_i}}$ with $b_{ij} = 0$ for some j. $\square$

The next definition is motivated by Proposition 7.1.

Definition 5.3. Let $S\colon A \to B$ be a subordination, $x \in \mathbb{K}(A)$, and $y \in \mathbb{K}(B)$.

(1) We set $x \mathrel{\mathcal{S}_\Box} y$ if it is possible to write x in disjunctive normal form
$$x = \bigvee_{i=1}^{n} (\Box_{a_i} \wedge \Diamond_{b_{i1}} \wedge \cdots \wedge \Diamond_{b_{in_i}})$$
and y in conjunctive normal form
$$y = \bigwedge_{j=1}^{m} (\Diamond_{c_j} \vee \Box_{d_{j1}} \vee \cdots \vee \Box_{d_{jm_j}})$$
so that, for each $i \leq n$ and $j \leq m$, there exists $k \leq m_j$ with $a_i \mathrel{S} d_{jk}$.

(2) We set $x \mathrel{\mathcal{S}_\Diamond} y$ if it is possible to write x in disjunctive normal form
$$x = \bigvee_{i=1}^{n} (\Box_{a_i} \wedge \Diamond_{b_{i1}} \wedge \cdots \wedge \Diamond_{b_{in_i}})$$
and y in conjunctive normal form
$$y = \bigwedge_{j=1}^{m} (\Diamond_{c_j} \vee \Box_{d_{j1}} \vee \cdots \vee \Box_{d_{jm_j}})$$
so that, for each $i \leq n$ and $j \leq m$, there exists $l \leq n_i$ such that $b_{il} \mathrel{S} c_j$.

The following technical lemma characterizes the order on $\mathbb{K}(B)$ in terms of normal forms.

Lemma 5.4. *Let $x, y \in \mathbb{K}(B)$ be written in disjunctive and conjunctive normal form, respectively:*
$$x = \bigvee_{i=1}^{n} (\Box_{a_i} \wedge \Diamond_{b_{i1}} \wedge \cdots \wedge \Diamond_{b_{in_i}}), \quad y = \bigwedge_{j=1}^{m} (\Diamond_{c_j} \vee \Box_{d_{j1}} \vee \cdots \vee \Box_{d_{jm_j}}).$$
Then $x \leq y$ if and only if, for each $i \leq n$ and $j \leq m$, there exists $k \leq m_j$ such that $a_i \leq d_{jk}$ or there exists $l \leq n_i$ such that $b_{il} \leq c_j$.

Proof. The inequality
$$\bigvee_{i=1}^{n} (\Box_{a_i} \wedge \Diamond_{b_{i1}} \wedge \cdots \wedge \Diamond_{b_{in_i}}) \leq \bigwedge_{j=1}^{m} (\Diamond_{c_j} \vee \Box_{d_{j1}} \vee \cdots \vee \Box_{d_{jm_j}})$$
holds if and only if, for each i and j,
$$\Box_{a_i} \wedge \Diamond_{b_{i1}} \wedge \cdots \wedge \Diamond_{b_{in_i}} \leq \Diamond_{c_j} \vee \Box_{d_{j1}} \vee \cdots \vee \Box_{d_{jm_j}}.$$
Thus, it is sufficient to show that for a, $b_1, \dots, b_n$, c, $d_1, \dots, d_m \in B$ such that $b_i \leq a$ and $c \leq d_j$ for every i and j, the following two conditions are equivalent:

(1) $\Box_a \wedge \Diamond_{b_1} \wedge \cdots \wedge \Diamond_{b_n} \leq \Diamond_c \vee \Box_{d_1} \vee \cdots \vee \Box_{d_m}$,
(2) there exists j such that $a \leq d_j$ or there exists i such that $b_i \leq c$.

(2) $\Rightarrow$ (1). It follows from Definition 4.1 that $a \leq d_j$ implies $\Box_a \leq \Box_{d_j}$ and $b_i \leq c$ implies $\Diamond_{b_i} \leq \Diamond_c$. Thus, (1) holds.

(1) $\Rightarrow$ (2). Suppose (1) holds, so

$$\Box_a \wedge \Diamond_{b_1} \wedge \cdots \wedge \Diamond_{b_n} \leq \Diamond_c \vee \Box_{d_1} \vee \cdots \vee \Box_{d_m}.$$

Let $A = \{0, 1\}$ be the two-element boolean algebra. Define two functions $\Diamond, \Box\colon B \to A$ by

$$\Diamond e = \begin{cases} 0 & \text{if } e \wedge (a \wedge \neg c) = 0, \\ 1 & \text{if } e \wedge (a \wedge \neg c) \neq 0 \end{cases}$$

and

$$\Box e = \begin{cases} 1 & \text{if } \neg e \wedge (a \wedge \neg c) = 0, \\ 0 & \text{if } \neg e \wedge (a \wedge \neg c) \neq 0 \end{cases}$$

for each $e \in B$. It is straightforward to see that $\Diamond$ preserves finite joins and that $\Box e = \neg \Diamond \neg e$ for each $e \in B$. Therefore, there is a unique boolean homomorphism $\alpha\colon \mathbb{K}(B) \to A$ such that $\alpha(\Box_e) = \Box e$ and $\alpha(\Diamond_e) = \Diamond e$ for each $e \in B$. Therefore,

$$\alpha(\Box_a \wedge \Diamond_{b_1} \wedge \cdots \wedge \Diamond_{b_n}) = \Box a \wedge \Diamond b_1 \wedge \cdots \wedge \Diamond b_n,$$
$$\alpha(\Diamond_c \vee \Box_{d_1} \vee \cdots \vee \Box_{d_m}) = \Diamond c \vee \Box d_1 \vee \cdots \vee \Box d_m.$$

Because $\Box_a \wedge \Diamond_{b_1} \wedge \cdots \wedge \Diamond_{b_n} \leq \Diamond_c \vee \Box_{d_1} \vee \cdots \vee \Box_{d_m}$ and α is order preserving,

$$\Box a \wedge \Diamond b_1 \wedge \cdots \wedge \Diamond b_n \leq \Diamond c \vee \Box d_1 \vee \cdots \vee \Box d_m.$$

We have $\Box a = 1$ since $\neg a \wedge (a \wedge \neg c) = 0$ and $\Diamond c = 0$ since $c \wedge (a \wedge \neg c) = 0$. Thus,

$$\Diamond b_1 \wedge \cdots \wedge \Diamond b_n \leq \Box d_1 \vee \cdots \vee \Box d_m.$$

Consequently, $\Diamond b_1 \wedge \cdots \wedge \Diamond b_n = 0$ or $\Box d_1 \vee \cdots \vee \Box d_m = 1$. In the former case, there is $i \leq n$ such that $\Diamond b_i = 0$, so $b_i \wedge (a \wedge \neg c) = 0$ and, hence, $b_i \wedge \neg c = 0$ (since $b_i \leq a$). Therefore, $b_i \leq c$. In the latter case, there is $j \leq m$ such that $\Box d_j = 1$, so $\neg d_j \wedge (a \wedge \neg c) = 0$ and, hence, $\neg d_j \wedge a = 0$ (since $c \leq d_j$). Thus, $a \leq d_j$. $\Box$

The next proposition says that the definitions of $\mathcal{S}_\Box$ and $\mathcal{S}_\Diamond$ are independent of the normal forms used to write x and y. This fact will be useful in proving that $\mathcal{S}_\Box$ and $\mathcal{S}_\Diamond$ are subordinations in Theorem 5.6.

Proposition 5.5. *Let $S\colon A \to B$ be a subordination. If $x \in \mathbb{K}(A)$ and $y \in \mathbb{K}(B)$ are written in disjunctive and conjunctive normal form*

$$x = \bigvee_{i=1}^{n} (\Box_{a_i} \wedge \Diamond_{b_{i1}} \wedge \cdots \wedge \Diamond_{b_{in_i}}), \quad y = \bigwedge_{j=1}^{m} (\Diamond_{c_j} \vee \Box_{d_{j1}} \vee \cdots \vee \Box_{d_{jm_j}}),$$

then

(1) *$x \mathrel{\mathcal{S}_\Box} y$ if and only if, for all i and j, there exists $k \leq m_j$ such that $a_i \mathrel{S} d_{jk}$.*
(2) *$x \mathrel{\mathcal{S}_\Diamond} y$ if and only if, for all i and j, there exists $l \leq n_i$ such that $b_{il} \mathrel{S} c_j$.*

Proof. We only prove (1) since the proof of (2) is similar. The right-to-left implication follows immediately from the definition of $\mathcal{S}_\Box$. To prove the left-to-right implication, assume that $x \mathrel{\mathcal{S}_\Box} y$. The definition of $\mathcal{S}_\Box$ implies that it is possible to write x and y in disjunctive and conjunctive normal form, respectively:

$$x = \bigvee_{r=1}^{t} (\Box_{e_r} \wedge \Diamond_{f_{r1}} \wedge \cdots \wedge \Diamond_{f_{rt_r}}), \quad y = \bigwedge_{s=1}^{u} (\Diamond_{g_s} \vee \Box_{h_{s1}} \vee \cdots \vee \Box_{h_{su_s}}),$$

so that, for any r and s, there exists $q \leq u_s$ with $e_r \mathrel{S} h_{sq}$. Fix $i \leq n$ and $j \leq m$. We show that there is $k \leq m_j$ such that $a_i \mathrel{S} d_{jk}$. Clearly,

$$\begin{aligned} &\Box_{a_i} \wedge \Diamond_{b_{i1}} \wedge \cdots \wedge \Diamond_{b_{in_i}} \leq \bigvee_{r=1}^{t} (\Box_{e_r} \wedge \Diamond_{f_{r1}} \wedge \cdots \wedge \Diamond_{f_{rt_r}}), \\ &\bigwedge_{s=1}^{u} (\Diamond_{g_s} \vee \Box_{h_{s1}} \vee \cdots \vee \Box_{h_{su_s}}) \leq \Diamond_{c_j} \vee \Box_{d_{j1}} \vee \cdots \vee \Box_{d_{jm_j}}. \end{aligned} \tag{1}$$

Thus,

$$\Box_{a_i} \wedge \Diamond_{b_{i1}} \wedge \cdots \wedge \Diamond_{b_{in_i}} \leq \bigvee_{r=1}^{t} (\Box_{e_r} \wedge \Diamond_{f_{r1}} \wedge \cdots \wedge \Diamond_{f_{rt_r}}) \leq \Diamond_0 \vee \Box_{e_1} \vee \cdots \vee \Box_{e_t}.$$

We claim that there is $r' \leq t$ such that $a_i \leq e_{r'}$. If $e_r = 1$ for some r, then we can take $r' = r$. Otherwise, $e_r \neq 1$ for every r and, hence, the expression $\Diamond_0 \vee \Box_{e_1} \vee \cdots \vee \Box_{e_t}$ is in conjunctive normal form (with a unique conjunct). Therefore, since $a_i \neq 0$, Lemma 5.4 implies that there is $r' \leq t$ such that $a_i \leq e_{r'}$, proving our claim.

By our assumption, for each $s \le u$, there exists q_s such that $e_{r'} \; S \; h_{sq_s}$. Consequently, $e_{r'} \; S \; \bigwedge_{s=1}^{u} h_{sq_s}$. Equation (1) yields

$$\begin{aligned}\Box_{\bigwedge_{s=1}^{u} h_{sq_s}} &= \Box_{h_{1q_1}} \wedge \cdots \wedge \Box_{h_{uq_u}} \\ &\le \bigwedge_{s=1}^{u} (\Diamond_{g_s} \vee \Box_{h_{s1}} \vee \cdots \vee \Box_{h_{su_s}}) \\ &\le \Diamond_{c_j} \vee \Box_{d_{j1}} \vee \cdots \vee \Box_{d_{jm_j}}.\end{aligned}$$

By Lemma 5.4, there exists $k \le m_j$ such that $\bigwedge_{s=1}^{u} h_{sq_s} \le d_{jk}$. Thus, we have found $k \le m_j$ such that

$$a_i \le e_{r'} \; S \; \bigwedge_{s=1}^{u} h_{sq_s} \le d_{jk},$$

and so $a_i \; S \; d_{jk}$. $\square$

Theorem 5.6. *If $S\colon A \to B$ is a subordination, then $\mathcal{S}_\Box$ and $\mathcal{S}_\Diamond$ are subordinations.*

Proof. We only prove that $\mathcal{S}_\Box$ is a subordination because the proof for $\mathcal{S}_\Diamond$ is similar. For this we must show that $\mathcal{S}_\Box$ satisfies the axioms (S1)–(S4) of Definition 3.1.

(S1) Write 0 in disjunctive normal form as the empty join and in conjunctive normal form as $0 = \Diamond_0$. Then $0 \; \mathcal{S}_\Box \; 0$ holds trivially. That $1 \; \mathcal{S}_\Box \; 1$ is proved similarly.

(S2) Suppose $x \; \mathcal{S}_\Box \; z$ and $y \; \mathcal{S}_\Box \; z$. Write x and y in disjunctive normal form and z in conjunctive normal form:

$$x = \bigvee_{i=1}^{n} (\Box_{a_i} \wedge \Diamond_{b_{i1}} \wedge \cdots \wedge \Diamond_{b_{in_i}}), \quad y = \bigvee_{j=1}^{m} (\Box_{c_j} \wedge \Diamond_{d_{j1}} \wedge \cdots \wedge \Diamond_{d_{jm_j}}),$$

$$z = \bigwedge_{r=1}^{t} (\Diamond_{e_r} \vee \Box_{f_{r1}} \vee \cdots \vee \Box_{f_{rt_r}}).$$

By Proposition 5.5(1), for all i and r, there exists $k \le t_r$ such that $a_i \; S \; f_{rk}$ and for all j and r, there exists $l \le t_r$ such that $c_j \; S \; f_{rl}$. Since $x \vee y$ can be written in disjunctive normal form as

$$x \vee y = \bigvee_{i=1}^{n} (\Box_{a_i} \wedge \Diamond_{b_{i1}} \wedge \cdots \wedge \Diamond_{b_{in_i}}) \vee \bigvee_{j=1}^{m} (\Box_{c_j} \wedge \Diamond_{d_{j1}} \wedge \cdots \wedge \Diamond_{d_{jm_j}}),$$

it follows from the definition of $\mathcal{S}_\Box$ that $(x \vee y) \; \mathcal{S}_\Box \; z$.

The proof of (S3) is similar to that of (S2).

(S4) Let $x \leq y \mathrel{\mathcal{S}_\Box} z \leq w$. By the definition of $\mathcal{S}_\Box$, we can write x and y in disjunctive normal form and z and w in conjunctive normal form

$$x = \bigvee_{i=1}^{n} (\Box_{a_i} \wedge \Diamond_{b_{i1}} \wedge \cdots \wedge \Diamond_{b_{in_i}}), \quad y = \bigvee_{r=1}^{t} (\Box_{e_r} \wedge \Diamond_{f_{r1}} \wedge \cdots \wedge \Diamond_{f_{rt_r}}),$$
$$z = \bigwedge_{s=1}^{u} (\Diamond_{g_s} \vee \Box_{h_{s1}} \vee \cdots \vee \Box_{h_{su_s}}), \quad w = \bigwedge_{j=1}^{m} (\Diamond_{c_j} \vee \Box_{d_{j1}} \vee \cdots \vee \Box_{d_{jm_j}}),$$

so that, for all r and s, there exists $q \leq u_s$ with $e_r \mathrel{S} h_{sq}$. By arguing as in the proof of Proposition 5.5, we obtain that, for any i and j, there exists $k \leq m_j$ such that $a_i \mathrel{S} d_{jk}$. Therefore, $x \mathrel{\mathcal{S}_\Box} w$ by Proposition 5.5(1). $\Box$

We are ready to define the semi-functors $\mathbb{K}^{\mathsf{S}}_\Box, \mathbb{K}^{\mathsf{S}}_\Diamond \colon \mathsf{BA}^{\mathsf{S}} \to \mathsf{BA}^{\mathsf{S}}$.

Definition 5.7. For a boolean algebra B, let $\mathbb{K}^{\mathsf{S}}_\Box(B) = \mathbb{K}^{\mathsf{S}}_\Diamond(B) = \mathbb{K}(B)$. For a morphism S in BA^{S}, let $\mathbb{K}^{\mathsf{S}}_\Box(S) = \mathcal{S}_\Box$ and $\mathbb{K}^{\mathsf{S}}_\Diamond(S) = \mathcal{S}_\Diamond$.

Theorem 5.8. $\mathbb{K}^{\mathsf{S}}_\Box, \mathbb{K}^{\mathsf{S}}_\Diamond \colon \mathsf{BA}^{\mathsf{S}} \to \mathsf{BA}^{\mathsf{S}}$ *are semi-functors.*

Proof. We show that $\mathbb{K}^{\mathsf{S}}_\Box$ is a semi-functor. That $\mathbb{K}^{\mathsf{S}}_\Diamond$ is a semi-functor is proved similarly. Let $S \colon B_1 \to B_2$ and $T \colon B_2 \to B_3$ be subordinations. Suppose that $x \in \mathbb{K}^{\mathsf{S}}(B_1)$ and $y \in \mathbb{K}^{\mathsf{S}}(B_3)$ are written in disjunctive and conjunctive normal form, respectively:

$$x = \bigvee_{i=1}^{n} (\Box_{a_i} \wedge \Diamond_{b_{i1}} \wedge \cdots \wedge \Diamond_{b_{in_i}}), \quad y = \bigwedge_{j=1}^{m} (\Diamond_{c_j} \vee \Box_{d_{j1}} \vee \cdots \vee \Box_{d_{jm_j}}).$$

We show that $x \mathrel{\mathbb{K}^{\mathsf{S}}_\Box(T \circ S)} y$ if and only if $x \left(\mathbb{K}^{\mathsf{S}}_\Box(T) \circ \mathbb{K}^{\mathsf{S}}_\Box(S)\right) y$. To prove the left-to-right implication, suppose that $x \mathrel{\mathbb{K}^{\mathsf{S}}_\Box(T \circ S)} y$, which means that for every $i \leq n$ and $j \leq m$, there exists $k_j \leq m_j$ such that $a_i \mathrel{(T \circ S)} d_{jk_j}$. Thus, there is $f_{ij} \in B_2$ such that $a_i \mathrel{S} f_{ij} \mathrel{T} d_{jk_j}$, and so

$$\Box_{a_i} \mathrel{\mathbb{K}^{\mathsf{S}}_\Box(S)} \Box_{f_{ij}} \mathrel{\mathbb{K}^{\mathsf{S}}_\Box(T)} \Box_{d_{jk_j}}.$$

It follows from the properties of subordinations that, for each i, we have

$$\Box_{a_i} \mathrel{\mathbb{K}^{\mathsf{S}}_\Box(S)} \left(\bigwedge_{j=1}^{m} \Box_{f_{ij}} \right) \mathrel{\mathbb{K}^{\mathsf{S}}_\Box(T)} \left(\bigwedge_{j=1}^{m} \Box_{d_{jk_j}} \right)$$

and, hence,

$$x \leq \left(\bigvee_{i=1}^{n} \Box_{a_i} \right) \mathrel{\mathbb{K}^{\mathsf{S}}_\Box(S)} \left(\bigvee_{i=1}^{n} \bigwedge_{j=1}^{m} \Box_{f_{ij}} \right) \mathrel{\mathbb{K}^{\mathsf{S}}_\Box(T)} \left(\bigwedge_{j=1}^{m} \Box_{d_{jk_j}} \right) \leq y.$$

Let $z = \bigvee_{i=1}^{n} \bigwedge_{j=1}^{m} \Box_{f_{ij}} \in \mathbb{K}^{\mathsf{S}}(B_2)$. Then $x \mathrel{\mathbb{K}^{\mathsf{S}}_\Box(S)} z \mathrel{\mathbb{K}^{\mathsf{S}}_\Box(T)} y$. This shows that $\mathbb{K}^{\mathsf{S}}_\Box(T \circ S) \subseteq \mathbb{K}^{\mathsf{S}}_\Box(T) \circ \mathbb{K}^{\mathsf{S}}_\Box(S)$. To prove the other inclusion, suppose that $x\ \left(\mathbb{K}^{\mathsf{S}}_{\Box}(T) \circ \mathbb{K}^{\mathsf{S}}_{\Box}(S)\right)\ y$ and that x and y are written in disjunctive and conjunctive normal form as above. Then there is $z \in \mathbb{K}^{\mathsf{S}}(B_2)$ such that $x\ \mathbb{K}^{\mathsf{S}}_{\Box}(S)\ z\ \mathbb{K}^{\mathsf{S}}_{\Box}(T)\ y$. Write z in conjunctive normal form

$$z = \bigwedge_{r=1}^{t} (\Diamond_{e_r} \vee \Box_{f_{r1}} \vee \cdots \vee \Box_{f_{rt_r}}).$$

Fix $i \leq n$ and $j \leq m$. Since $x\ \mathbb{K}^{\mathsf{S}}_{\Box}(S)\ z$, for each $r \leq t$, there is $l_r \leq t_r$ such that $a_i\ S\ f_{rl_r}$. Because

$$\Box_{\bigwedge_{r=1}^{t} f_{rl_r}} \leq z\ \mathbb{K}^{\mathsf{S}}_{\Box}(T)\ y,$$

there is $s \leq m_j$ such that $\left(\bigwedge_{r=1}^{t} f_{rl_r}\right) T\ d_{js}$. Then

$$a_i\ S\ \left(\bigwedge_{k=1}^{t} f_{kl_k}\right) T\ d_{js},$$

and so $a_i\ (T \circ S)\ d_{js}$. Therefore, $x\ \mathbb{K}^{\mathsf{S}}_{\Box}(T \circ S)\ y$. This shows that $\mathbb{K}^{\mathsf{S}}_{\Box}$ is a semi-functor. $\square$

6. The Endofunctor $\mathbb{K}^{\mathsf{S}}$ on BA^{S}

In this section, we utilize $\mathcal{S}_{\Box}$ and $\mathcal{S}_{\Diamond}$ defined in the previous section to lift a subordination $S\colon A \to B$ to the subordination $\mathcal{S}\colon \mathbb{K}(A) \to \mathbb{K}(B)$, which is the algebraic counterpart of $\mathcal{R}_{EM}$. This allows us to extend the endofunctor $\mathbb{K}\colon \mathsf{BA} \to \mathsf{BA}$ to the endofunctor $\mathbb{K}^{\mathsf{S}}\colon \mathsf{BA}^{\mathsf{S}} \to \mathsf{BA}^{\mathsf{S}}$.

For boolean algebras A and B, we denote by $\mathsf{BA}^{\mathsf{S}}(A,B)$ the poset of subordinations $S\colon A \to B$ ordered by inclusion.[1] Our goal is to define $\mathcal{S}$ as the join of $\mathcal{S}_{\Box}$ and $\mathcal{S}_{\Diamond}$ in $\mathsf{BA}^{\mathsf{S}}(\mathbb{K}(A), \mathbb{K}(B))$. For this we need to show that joins exist in $\mathsf{BA}^{\mathsf{S}}(\mathbb{K}(A), \mathbb{K}(B))$. In fact, we will show that $\mathsf{BA}^{\mathsf{S}}(A,B)$ is always a frame, where we recall that a complete lattice L is a *frame* if it satisfies $a \wedge \bigvee E = \bigvee\{a \wedge b \mid b \in E\}$ for each $a \in L$ and $E \subseteq L$ (see, e.g., [40, p. 10]).

We also recall that a complete lattice is a *coframe* if its order-dual is a frame (see, e.g., [40, p. 10]). Let X be the Stone dual of A and let Y be the Stone dual of B. It is clear that $(\mathsf{Stone}^{\mathsf{R}}(X,Y), \subseteq)$ is the poset of closed subsets of $X \times Y$, and so is a coframe. By [1, Thm. 2.14], $(\mathsf{BA}^{\mathsf{S}}(A,B), \subseteq)$ is dually isomorphic to $(\mathsf{Stone}^{\mathsf{R}}(\mathsf{Uf}(A), \mathsf{Uf}(B)), \subseteq)$. Therefore, $\mathsf{BA}^{\mathsf{S}}(A,B)$ is a frame. However, this proof uses Stone duality. As promised in [1, Rem. 2.15], we give a choice free proof of this result.

[1] This is in contrast to [1], where $\mathsf{BA}^{\mathsf{S}}(A,B)$ was ordered by reverse inclusion to guarantee that BA^{S} was an allegory.

Theorem 6.1. $\mathsf{BA}^{\mathsf{S}}(A,B)$ *is a frame, where meets are given by intersections and the join of* $\{S_\alpha\} \subseteq \mathsf{BA}^{\mathsf{S}}(A,B)$ *is given by* $x\,(\bigvee S_\alpha)\,y$ *if and only if there exist finite subsets* $F \subseteq A$ *and* $G \subseteq B$ *such that* $x = \bigvee F$, $y = \bigwedge G$, *and for all* $a \in F$ *and* $b \in G$, *there is* α *with* $a \, S_\alpha \, b$.

Proof. It is straightforward to see that the intersection of a family of subordinations is a subordination. Therefore, $\mathsf{BA}^{\mathsf{S}}(A,B)$ is a complete lattice where meets are given by intersections. Let $\{S_\alpha\} \subseteq \mathsf{BA}^{\mathsf{S}}(A,B)$. Define $S\colon A \to B$ by $x \, S \, y$ if and only if there exist finite subsets $F \subseteq A$ and $G \subseteq B$ such that $x = \bigvee F$, $y = \bigwedge G$, and for all $a \in F$ and $b \in G$, there is α with $a \, S_\alpha \, b$. It is clear that S contains S_α for each α. We show that S is a subordination.

(S1) Since $0 = \bigvee \varnothing$ and $1 = \bigwedge \varnothing$, it trivially holds that $0 \, S \, 0$ and $1 \, S \, 1$.

(S2) Suppose that $x \, S \, y$ and $x' \, S \, y$. Then there exist $F, F' \subseteq A$ and $G, G' \subseteq B$ such that $x = \bigvee F$, $x' = \bigvee F'$, $y = \bigwedge G = \bigwedge G'$, and for all $a \in F$ and $b \in G$, there exists α with $a \, S_\alpha \, b$ and for all $a' \in F'$ and $b' \in G'$, there exists β with $a' \, S_\beta \, b'$. Since B is a boolean algebra, by distributivity, we obtain that $y = \bigwedge G''$, where $G'' = \{b \vee b' \mid b \in G \text{ and } b' \in G'\}$. Let $F'' = F \cup F'$. Then $x \vee x' = \bigvee F''$. Let $a'' \in F''$ and $b'' \in G''$. Then $b'' = b \vee b'$ with $b \in G$ and $b' \in G'$. Therefore, if $a'' \in F$, there is α such that $a'' \, S_\alpha \, b \leq b \vee b'$. If $a'' \in F'$, there is β such that $a'' \, S_\beta \, b' \leq b \vee b'$. In either case, $a'' \, S \, b''$. Thus, $x \, S \, y$.

(S3) is proved similarly to (S2).

(S4) Suppose $x' \leq x \, S \, y \leq y'$. Then $x = \bigvee F$, $y = \bigwedge G$, and for all $a \in F$ and $b \in G$, there is α with $a \, S_\alpha \, b$. Let $F' = \{a \wedge x' \mid a \in F\}$ and $G' = \{b \vee y' \mid b \in G\}$. Then $x' = \bigvee F'$ and $y' = \bigwedge G'$. Moreover, if $a' \in F'$ and $b' \in G'$, there exist $a \in F$ and $b \in G$ such that $a' = a \wedge x'$ and $b' = b \vee y'$. Therefore, $a' = a \wedge x \leq a \, S_\alpha \, b \leq b \vee y' = b'$. Thus, $x' \, S \, y'$.

It remains to prove that S is the least upper bound of $\{S_\alpha\}$. Let $T\colon A \to B$ be such that $S_\alpha \subseteq T$ for each α and let $x \, S \, y$. Then $x = \bigvee F$, $y = \bigwedge G$ for some finite F and G such that for all $a \in F$ and $b \in G$ there is α with $a \, S_\alpha \, b$. Therefore, $a \, T \, b$. Since T satisfies (S2) and (S3), it follows that $x = (\bigvee F) \, T \, (\bigwedge G) = y$. Thus, $S \subseteq T$.

It is left to show that $T \cap \bigvee S_\alpha = \bigvee (T \cap S_\alpha)$ for each $T \in \mathsf{BA}^{\mathsf{S}}(A,B)$ and $\{S_\alpha\} \subseteq \mathsf{BA}^{\mathsf{S}}(A,B)$. The right-to-left inclusion is clear since BA^{S} is a complete lattice. To show the left-to-right inclusion, let $x \in A$ and $y \in B$ such that $x \, (T \cap \bigvee S_\alpha) \, y$. Then $x \, T \, y$ and $x \,(\bigvee S_\alpha)\, y$. So there exist finite subsets $F \subseteq A$ and $G \subseteq B$ such that $x = \bigvee F$, $y = \bigwedge G$, and for all $a \in F$ and $b \in G$, there is α with $a \, S_\alpha \, b$. For all $a \in F$ and $b \in G$, we have $a \leq x \, T \, y \leq b$, which implies that $a \, T \, b$. Therefore, for all $a \in F$

and $b \in G$, there is α with $a\ (T \cap S_\alpha)\ b$ and, hence, $x \bigvee(T \cap S_\alpha)\ y$. Thus, $\mathsf{BA}^{\mathsf{S}}(A, B)$ is a frame. $\square$

Definition 6.2. Let $S\colon A \to B$ be a subordination between boolean algebras. We define $\mathcal{S}$ to be the join $\mathcal{S}_\Box \vee \mathcal{S}_\Diamond$ in $\mathsf{BA}^{\mathsf{S}}(\mathbb{K}(A), \mathbb{K}(B))$.

We thus have the following description of $\mathcal{S}$, which is an immediate consequence of Definition 5.3, Theorem 5.6, and Theorem 6.1.

Corollary 6.3. *The relation $\mathcal{S}$ is a subordination and $x\ \mathcal{S}\ y$ if and only if it is possible to write x and y in disjunctive and conjunctive normal form, respectively:*

$$x = \bigvee_{i=1}^{n} (\Box_{a_i} \wedge \Diamond_{b_{i1}} \wedge \cdots \wedge \Diamond_{b_{in_i}}), \quad y = \bigwedge_{j=1}^{m} (\Diamond_{c_j} \vee \Box_{d_{j1}} \vee \cdots \vee \Box_{d_{jm_j}})$$

so that, for all i and j, there exists $k \leq m_j$ with $a_i\ S\ d_{jk}$ or there exists $l \leq n_i$ with $b_{il}\ S\ c_j$.

The description of $\mathcal{S}$ given in Corollary 6.3 is similar to Definition 5.3. In Proposition 5.5, we showed that the definitions of $\mathcal{S}_\Box$ and $\mathcal{S}_\Diamond$ are independent of the normal forms used to represent elements. The same is true for $\mathcal{S}$ and the proof is similar, so we skip it.

Proposition 6.4. *Let $S\colon A \to B$ be a subordination. If $x \in \mathbb{K}(A)$ and $y \in \mathbb{K}(B)$ are written in disjunctive and conjunctive normal form, respectively:*

$$x = \bigvee_{i=1}^{n} (\Box_{a_i} \wedge \Diamond_{b_{i1}} \wedge \cdots \wedge \Diamond_{b_{in_i}}), \quad y = \bigwedge_{j=1}^{m} (\Diamond_{c_j} \vee \Box_{d_{j1}} \vee \cdots \vee \Box_{d_{jm_j}}),$$

then $x\ \mathcal{S}\ y$ if and only if, for all i and j, there exists $k \leq m_j$ with $a_i\ S\ d_{jk}$ or there exists $l \leq n_i$ with $b_{il}\ S\ c_j$.

Definition 6.5. For a boolean algebra B, let $\mathbb{K}^{\mathsf{S}}(B) = \mathbb{K}(B)$; and for a morphism S in BA^{S}, let $\mathbb{K}^{\mathsf{S}}(S) = \mathcal{S}$.

Theorem 6.6. *$\mathbb{K}^{\mathsf{S}}$ is an endofunctor on BA^{S}.*

Proof. We show that $\mathbb{K}^{\mathsf{S}}$ preserves composition; that is,

$$\mathbb{K}^{\mathsf{S}}(T \circ S) = \mathbb{K}^{\mathsf{S}}(T) \circ \mathbb{K}^{\mathsf{S}}(S).$$

We first show the left-to-right inclusion. It follows from Definition 6.2 that $\mathbb{K}^{\mathsf{S}}_\Box(S), \mathbb{K}^{\mathsf{S}}_\Diamond(S) \subseteq \mathbb{K}^{\mathsf{S}}(S)$ and $\mathbb{K}^{\mathsf{S}}_\Box(T), \mathbb{K}^{\mathsf{S}}_\Diamond(T) \subseteq \mathbb{K}^{\mathsf{S}}(T)$. Therefore,

$$(\mathbb{K}^{\mathsf{S}}_\Box(T) \circ \mathbb{K}^{\mathsf{S}}_\Box(S)) \subseteq (\mathbb{K}^{\mathsf{S}}(T) \circ \mathbb{K}^{\mathsf{S}}(S))$$

and

$$(\mathbb{K}^{\mathsf{S}}_\Diamond(T) \circ \mathbb{K}^{\mathsf{S}}_\Diamond(S)) \subseteq (\mathbb{K}^{\mathsf{S}}(T) \circ \mathbb{K}^{\mathsf{S}}(S)).$$

Since $\mathbb{K}^{\mathsf{S}}_{\Box}$ and $\mathbb{K}^{\mathsf{S}}_{\Diamond}$ are semi-functors, we have

$$\begin{aligned}\mathbb{K}^{\mathsf{S}}(T \circ S) &= \mathbb{K}^{\mathsf{S}}_{\Box}(T \circ S) \vee \mathbb{K}^{\mathsf{S}}_{\Diamond}(T \circ S)\\ &= (\mathbb{K}^{\mathsf{S}}_{\Box}(T) \circ \mathbb{K}^{\mathsf{S}}_{\Box}(S)) \vee (\mathbb{K}^{\mathsf{S}}_{\Diamond}(T) \circ \mathbb{K}^{\mathsf{S}}_{\Diamond}(S)).\end{aligned}$$

Thus, $\mathbb{K}^{\mathsf{S}}(T \circ S) \subseteq \mathbb{K}^{\mathsf{S}}(T) \circ \mathbb{K}^{\mathsf{S}}(S)$. To prove the other inclusion, let $x \in \mathbb{K}^{\mathsf{S}}(B_1)$ and $y \in \mathbb{K}^{\mathsf{S}}(B_3)$ be written in disjunctive and conjunctive normal form as above, and assume that $x \left(\mathbb{K}^{\mathsf{S}}(T) \circ \mathbb{K}^{\mathsf{S}}(S)\right) y$. Then there exists $z \in \mathbb{K}^{\mathsf{S}}(B_2)$ such that $x \; \mathbb{K}^{\mathsf{S}}(S) \; z \; \mathbb{K}^{\mathsf{S}}(T) \; y$. Write z in conjunctive normal form

$$z = \bigwedge_{r=1}^{t} (\Diamond_{e_r} \vee \Box_{f_{r1}} \vee \cdots \vee \Box_{f_{rt_r}}).$$

Fix $i \leq n$ and $j \leq m$. We can reorder the conjuncts in the conjunctive normal form of z in such a way that there is $t' \leq t$ such that $r \leq t'$ implies that there is $l_r \leq t_r$ with $a_i \; S \; f_{rl_r}$ and that $r > t'$ implies that there is $s \leq n_i$ such that $b_{is} \; S \; e_r$. We also allow $t' = 0$ for the case in which the first condition never holds. Let $f = \bigwedge_{r=1}^{t'} f_{rl_r}$. Then

$$\begin{aligned}\Box_f \wedge \Diamond_{(e_{t'+1} \wedge f)} &\wedge \cdots \wedge \Diamond_{(e_t \wedge f)}\\ &= \Box_{f_{1l_1}} \wedge \cdots \wedge \Box_{f_{t'l_{t'}}} \wedge \Diamond_{(e_{t'+1} \wedge f)} \wedge \cdots \wedge \Diamond_{(e_t \wedge f)}\\ &\leq z \; \mathbb{K}^{\mathsf{S}}(T) \; y \leq \Diamond_{c_j} \vee \Box_{d_{j1}} \vee \cdots \vee \Box_{d_{jm_j}}.\end{aligned}$$

Thus, there exists $k \leq m_j$ such that $f \; T \; d_{jk}$ or there exists $r > t'$ such that $(e_r \wedge f) \; T \; c_j$. In the first case, $a_i \; S \; \bigwedge_{r=1}^{t'} f_{rl_r} = f \; T \; d_{jk}$ and, hence, $a_i \; (T \circ S) \; d_{jk}$. In the second case, there is $s \leq n_i$ such that $b_{is} \; S \; e_r$. Since x is written in conjunctive normal form, $b_{is} \leq a_i \; S \; f$ and, hence, $b_{is} \; S \; f$. Thus, $b_{is} \; S \; (e_r \wedge f) \; T \; c_j$, which implies that $b_{is} \; (T \circ S) \; c_j$. In either case, we have that $x \; \mathbb{K}^{\mathsf{S}}(T \circ S) \; y$.

Recall from section 3 that the identity on $B \in \mathsf{BA}^{\mathsf{S}}$ is the partial order $\leq$. By Lemma 5.4, $\mathbb{K}^{\mathsf{S}}(\leq)$ is the partial order $\leq$ on $\mathbb{K}(B)$. Therefore, $\mathbb{K}^{\mathsf{S}}$ preserves identities, hence is an endofunctor on BA^{S}. $\Box$

The dagger $(-)^\dagger \colon \mathsf{KHaus}^{\mathsf{R}} \to \mathsf{KHaus}^{\mathsf{R}}$ that we introduced in the paragraph preceding Proposition 2.10 clearly restricts to $\mathsf{Stone}^{\mathsf{R}}$. The corresponding dagger $(-)^\dagger$ on BA^{S} is given by mapping a subordination $S \colon A \to B$ to the subordination $S^\dagger \colon B \to A$ defined by $b \; S^\dagger \; a$ if and only if $\neg a \; S \; \neg b$ (see [1, Thm. 2.14]). Since $(-)^\dagger \colon \mathsf{BA}^{\mathsf{S}}(A, B) \to \mathsf{BA}^{\mathsf{S}}(B, A)$ preserves inclusion, it gives an order-isomorphism between $\mathsf{BA}^{\mathsf{S}}(A, B)$ and $\mathsf{BA}^{\mathsf{S}}(B, A)$. We conclude this section by observing that $\mathbb{K}^{\mathsf{S}}_{\Box}$ and $\mathbb{K}^{\mathsf{S}}_{\Diamond}$ are definable from each other using $(-)^\dagger$ and that $\mathbb{K}^{\mathsf{S}}$ commutes with $(-)^\dagger$.

Proposition 6.7.

(1) $\mathbb{K}^{\mathsf{S}}_{\Box} \circ (-)^{\dagger} = (-)^{\dagger} \circ \mathbb{K}^{\mathsf{S}}_{\Diamond}$ *and* $\mathbb{K}^{\mathsf{S}}_{\Diamond} \circ (-)^{\dagger} = (-)^{\dagger} \circ \mathbb{K}^{\mathsf{S}}_{\Box}$.
(2) $\mathbb{K}^{\mathsf{S}} \circ (-)^{\dagger} = (-)^{\dagger} \circ \mathbb{K}^{\mathsf{S}}$.

Proof. Since $\mathbb{K}^{\mathsf{S}}_{\Box}$, $\mathbb{K}^{\mathsf{S}}_{\Diamond}$, and $\mathbb{K}^{\mathsf{S}}$ coincide on objects and $(-)^{\dagger}$ fixes the objects, we only need to show that the compositions agree on the morphisms.

(1) We only prove the first equality because the second is proved similarly. Let $S\colon A \to B$ be a morphism in BA^{S}. Suppose that $x \in \mathbb{K}(A)$ and $y \in \mathbb{K}(B)$ are written in disjunctive and conjunctive normal form, respectively:

$$x = \bigvee_{i=1}^{n} (\Box_{a_i} \wedge \Diamond_{b_{i1}} \wedge \cdots \wedge \Diamond_{b_{in_i}}),$$
$$y = \bigwedge_{j=1}^{m} (\Diamond_{c_j} \vee \Box_{d_{j1}} \vee \cdots \vee \Box_{d_{jm_j}}).$$

As $\neg\Box_a = \Diamond_{\neg a}$ and $\neg\Diamond_b = \Box_{\neg b}$, we obtain that $\neg x$ and $\neg y$ can be written in conjunctive and disjunctive normal form, respectively:

$$\neg x = \bigwedge_{i=1}^{n} (\Diamond_{\neg a_i} \vee \Box_{\neg b_{i1}} \vee \cdots \vee \Box_{\neg b_{in_i}}),$$
$$\neg y = \bigvee_{j=1}^{m} (\Box_{\neg c_j} \wedge \Diamond_{\neg d_{j1}} \wedge \cdots \wedge \Diamond_{\neg d_{jm_j}}).$$

By Proposition 5.5, $x \; \mathbb{K}^{\mathsf{S}}_{\Box}(S^{\dagger}) \; y$ if and only if, for all i and j, there exists k such that $a_i \; S^{\dagger} \; d_{jk}$, which means that $\neg d_{jk} \; S \; \neg a_i$. Applying Proposition 5.5 again, this is equivalent to $\neg y \; \mathbb{K}^{\mathsf{S}}_{\Diamond}(S) \; \neg x$, which means that $x \; \mathbb{K}^{\mathsf{S}}_{\Diamond}(S)^{\dagger} \; y$.

(2) Since $(-)^{\dagger}\colon \mathsf{BA}^{\mathsf{S}}(A,B) \to \mathsf{BA}^{\mathsf{S}}(B,A)$ is an order-isomorphism, we have $(S_1 \vee S_2)^{\dagger} = S_1^{\dagger} \vee S_2^{\dagger}$ for each $S_1, S_2 \in \mathsf{BA}^{\mathsf{S}}(A,B)$. Therefore, by (1), for each subordination $S\colon A \to B$, we have

$$\begin{aligned} \mathbb{K}^{\mathsf{S}}(S^{\dagger}) &= \mathbb{K}^{\mathsf{S}}_{\Box}(S^{\dagger}) \vee \mathbb{K}^{\mathsf{S}}_{\Diamond}(S^{\dagger}) = \mathbb{K}^{\mathsf{S}}_{\Diamond}(S)^{\dagger} \vee \mathbb{K}^{\mathsf{S}}_{\Box}(S)^{\dagger} \\ &= (\mathbb{K}^{\mathsf{S}}_{\Diamond}(S) \vee \mathbb{K}^{\mathsf{S}}_{\Box}(S))^{\dagger} = \mathbb{K}^{\mathsf{S}}(S)^{\dagger}. \end{aligned}$$ □

Remark 6.8. By Proposition 6.7(2), $\mathbb{K}^{\mathsf{S}}$ commutes with the dagger on BA^{S}. It follows from Corollary 6.3 that $\mathbb{K}^{\mathsf{S}}$ preserves inclusions of subordinations. Thus, $\mathbb{K}^{\mathsf{S}}$ is a morphism of order-enriched categories with involution. However, like $\mathbb{V}^{\mathsf{R}}$, the functor $\mathbb{K}^{\mathsf{S}}$ is not a morphism of allegories. This follows from Example 3.10, Theorem 3.2, and Theorem 7.3(3).

7. Connecting $\mathbb{K}^{\mathsf{S}}$ and $\mathbb{V}^{\mathsf{R}}$

In this section, we show that, under the equivalence between $\mathsf{Stone}^{\mathsf{R}}$ and BA^{S}, the endofunctor $\mathbb{V}^{\mathsf{R}}$ on $\mathsf{Stone}^{\mathsf{R}}$ corresponds to the endofunctor $\mathbb{K}^{\mathsf{S}}$ on BA^{S}. Analogous results are obtained for $\mathbb{V}^{\mathsf{R}}_{\Box}$ and $\mathbb{K}^{\mathsf{S}}_{\Box}$, as well as for $\mathbb{V}^{\mathsf{R}}_{\Diamond}$ and $\mathbb{K}^{\mathsf{S}}_{\Diamond}$.

Proposition 7.1. *Suppose that $R\colon X \to Y$ is a closed relation between Stone spaces. Let $\mathcal{U} \in \mathsf{Clop}(\mathbb{V}(X))$ and $\mathcal{W} \in \mathsf{Clop}(\mathbb{V}(Y))$ be written in disjunctive and conjunctive normal form, respectively:*

$$\mathcal{U} = \bigcup_{i=1}^{n}(\Box_{U_i} \cap \Diamond_{V_{i1}} \cap \cdots \cap \Diamond_{V_{in_i}}), \quad \mathcal{W} = \bigcap_{j=1}^{m}(\Diamond_{W_j} \cup \Box_{Z_{j1}} \cup \cdots \cup \Box_{Z_{jm_j}})$$

with clopen sets $\varnothing \neq V_{il} \subseteq U_i$ for each i and l, and $W_j \subseteq Z_{jk} \neq Y$ for each j and k.

1. *$\mathcal{U}\ S_{\mathcal{R}_\Box}\ \mathcal{W}$ iff $\forall\, i, j\, \exists\, k \leq m_j : U_i\ S_R\ Z_{jk}$,*
2. *$\mathcal{U}\ S_{\mathcal{R}_\Diamond}\ \mathcal{W}$ iff $\forall\, i, j\, \exists\, l \leq n_i : V_{il}\ S_R\ W_j$,*
3. *$\mathcal{U}\ S_{\mathcal{R}_{EM}}\ \mathcal{W}$ iff*

$$\forall\, i, j\, ((\exists\, k \leq m_j : U_i\ S_R\ Z_{jk}) \vee (\exists\, l \leq n_i : V_{il}\ S_R\ W_j)).$$

Proof. (1) Since $S_{\mathcal{R}_\Box}$ is a subordination, it is enough to prove the case $n = m = 1$, i.e., that

$$(\Box_U \cap \Diamond_{V_1} \cap \cdots \cap \Diamond_{V_p})\ S_{\mathcal{R}_\Box}\ (\Diamond_W \cup \Box_{Z_1} \cup \cdots \cup \Box_{Z_q}) \text{ iff } \exists\, k \leq q : U\ S_R\ Z_k,$$

under the assumption that $\varnothing \neq V_i \subseteq U$ and $W \subseteq Z_j \neq Y$.

To prove the right-to-left implication, suppose that $U\ S_R\ Z_k$. Then $R[U] \subseteq Z_k$. Let $F \in \Box_U \cap \Diamond V_1 \cap \cdots \cap \Diamond V_p$ and $G \in \mathbb{V}(Y)$ with $F\ \mathcal{R}_\Box\ G$, so $G \subseteq R[F]$. Since $F \in \Box_U$, we have $F \subseteq U$, and hence,

$$G \subseteq R[F] \subseteq R[U] \subseteq Z_k.$$

Thus, $G \in \Box_{Z_k} \subseteq \Diamond_W \cup \Box_{Z_1} \cup \cdots \cup \Box_{Z_q}$, and so

$$(\Box_U \cap \Diamond_{V_1} \cap \cdots \cap \Diamond_{V_p})\ S_{\mathcal{R}_\Box}\ (\Diamond_W \cup \Box_{Z_1} \cup \cdots \cup \Box_{Z_q}).$$

To prove the other implication, suppose that $U \not S_R\ Z_k$ for each $k \leq q$, so there are $x_k \in U$ and $y_k \in -Z_k$ such that $x_k\ R\ y_k$. Then

$$\{x_1, \ldots, x_q\} \cup V_1 \cup \cdots \cup V_p \in \Box_U \cap \Diamond_{V_1} \cap \cdots \cap \Diamond_{V_p},$$
$$\{y_1, \ldots, y_q\} \notin \Diamond_W \cup \Box_{Z_1} \cup \cdots \cup \Box_{Z_q}, \text{ and}$$
$$\{y_1, \ldots, y_q\} \subseteq R[\{x_1, \ldots, x_q\}] \subseteq R[\{x_1, \ldots, x_q\} \cup V_1 \cup \cdots \cup V_p].$$

Therefore, $\{x_1, \ldots, x_q\} \cup V_1 \cup \cdots \cup V_n\ \mathcal{R}_\Box\ \{y_1, \ldots, y_q\}$, proving the failure of

$$(\Box_U \cap \Diamond_{V_1} \cap \cdots \cap \Diamond_{V_p})\ S_{\mathcal{R}_\Box}\ (\Diamond_W \cup \Box_{Z_1} \cup \cdots \cup \Box_{Z_q}).$$

(2) Since $S_{\mathcal{R}_\Diamond}$ is a subordination, it is sufficient to prove the case $n = m = 1$, i.e., that

$$(\Box_U \cap \Diamond_{V_1} \cap \cdots \cap \Diamond_{V_p})\ S_{\mathcal{R}_\Diamond}\ (\Diamond_W \cup \Box_{Z_1} \cup \cdots \cup \Box_{Z_q}) \text{ iff } \exists\, l \leq p : V_l\ S_R\ W,$$

under the assumption that $\varnothing \neq V_i \subseteq U$ and $W \subseteq Z_j \neq Y$.

For the right-to-left implication, suppose $V_l\ S_R\ W$. Then $R[V_l] \subseteq W$. Let $F \in \Box_U \cap \Diamond_{V_1} \cap \cdots \cap \Diamond_{V_p}$ and $G \in \mathbb{V}(Y)$ with $F\ \mathcal{R}_\Diamond\ G$, so $F \subseteq R^{-1}[G]$. Since $F \in \Diamond_{V_l}$, we have $F \cap V_l \neq \varnothing$. Therefore, $R^{-1}[G] \cap V_l \neq \varnothing$, and hence, $G \cap R[V_l] \neq \varnothing$. Thus, $G \cap W \neq \varnothing$. Consequently, $G \in \Diamond_W$, which implies that $G \in \Diamond_W \cup \Box_{Z_1} \cup \cdots \cup \Box_{Z_q}$. This shows that

$$(\Box_U \cap \Diamond_{V_1} \cap \cdots \cap \Diamond_{V_p})\ S_{\mathcal{R}_\Diamond}\ (\Diamond_W \cup \Box_{Z_1} \cup \cdots \cup \Box_{Z_q}).$$

For the left-to-right implication, suppose that $V_l\ \not{S}_R\ W$ for each $l \leq p$, so there are $x_l \in V_l$ and $y_l \in -W$ such that $x_l\ R\ y_l$. Then

$$\begin{aligned}
&\{x_1, \dots, x_p\} \in \Box_U \cap \Diamond_{V_1} \cap \cdots \cap \Diamond_{V_p},\\
&\{y_1, \dots, y_p\} \cup -Z_1 \cup \cdots \cup -Z_q \notin \Diamond_W \cup \Box_{Z_1} \cup \cdots \cup \Box_{Z_q}, \text{ and}\\
&\{x_1, \dots, x_p\} \subseteq R^{-1}[\{y_1, \dots, y_p\}] \subseteq R^{-1}[\{y_1, \dots, y_p\} \cup -Z_1 \cup \cdots \cup -Z_q],
\end{aligned}$$

which implies that $\{x_1, \dots, x_p\}\ \mathcal{R}_\Diamond\ \{y_1, \dots, y_p\} \cup -Z_1 \cup \cdots \cup -Z_q$, proving the failure of

$$(\Box_U \cap \Diamond_{V_1} \cap \cdots \cap \Diamond_{V_p})\ S_{\mathcal{R}_\Diamond}\ (\Diamond_W \cup \Box_{Z_1} \cup \cdots \cup \Box_{Z_q}).$$

(3) Since $S_{\mathcal{R}_{\mathrm{EM}}}$ is a subordination, it is enough to prove the case $n = m = 1$, i.e., that

$$\begin{aligned}
(\Box_U \cap \Diamond_{V_1} \cap \cdots \cap \Diamond_{V_p})\ & S_{\mathcal{R}_{\mathrm{EM}}} (\Diamond_W \cup \Box_{Z_1} \cup \cdots \cup \Box_{Z_q})\\
&\text{iff } \exists\, k : U\ S_R\ Z_k \text{ or } \exists\, l : V_l\ S_R\ W,
\end{aligned}$$

under the assumption that $\varnothing \neq V_i \subseteq U$ and $W \subseteq Z_j \neq Y$.

For the right-to-left implication, suppose that $U\ S_R\ Z_k$ for some $k \leq q$ or $V_l\ S_R\ W$ for some $l \leq p$. Then, by (1) and (2),

$$(\Box_U \cap \Diamond_{V_1} \cap \cdots \cap \Diamond_{V_p})\ S_{\mathcal{R}_\Box}\ (\Diamond_W \cup \Box_{Z_1} \cup \cdots \cup \Box_{Z_q})$$

or

$$(\Box_U \cap \Diamond_{V_1} \cap \cdots \cap \Diamond_{V_p})\ S_{\mathcal{R}_\Diamond}\ (\Diamond_W \cup \Box_{Z_1} \cup \cdots \cup \Box_{Z_q}).$$

Thus,

$$(\Box_U \cap \Diamond_{V_1} \cap \cdots \cap \Diamond_{V_p})\ S_{\mathcal{R}_{\mathrm{EM}}}\ (\Diamond_W \cup \Box_{Z_1} \cup \cdots \cup \Box_{Z_q}).$$

For the other implication, suppose that for each $k \leq q$, there are $x_k \in U$ and $y_k \in -Z_k$ such that $x_k\ R\ y_k$, and that for each $l \leq p$, there are $x'_l \in V_l$

and $y'_l \in -W$ such that $x'_l \, R \, y'_l$. Therefore,

$$\begin{aligned}
&\{x_1, \dots, x_q, x'_1, \dots, x'_p\} \in \Box_U \cap \Diamond_{V_1} \cap \dots \cap \Diamond_{V_p}, \\
&\{y_1, \dots, y_q, y'_1, \dots, y'_p\} \notin \Diamond_W \cup \Box_{Z_1} \cup \dots \cup \Box_{Z_q}, \\
&\{y_1, \dots, y_q, y'_1, \dots, y'_p\} \subseteq R[\{x_1, \dots, x_q, x'_1, \dots, x'_p\}], \text{ and} \\
&\{x_1, \dots, x_q, x'_1, \dots, x'_p\} \subseteq R^{-1}[\{y_1, \dots, y_q, y'_1, \dots, y'_p\}].
\end{aligned}$$

Thus, $\{x_1, \dots, x_q, x'_1, \dots, x'_p\} \; \mathcal{R}_{\mathrm{EM}} \; \{y_1, \dots, y_q, y'_1, \dots, y'_p\}$, proving the failure of

$$(\Box_U \cap \Diamond_{V_1} \cap \dots \cap \Diamond_{V_p}) \; S_{\mathcal{R}_{\mathrm{EM}}} \; (\Diamond_W \cup \Box_{Z_1} \cup \dots \cup \Box_{Z_q}). \qquad \Box$$

We next recall the notions of a natural transformation and natural isomorphism between semi-functors.

Definition 7.2 ([28, Sec. 2.2]). Let D and E be categories and $F, G, H : \mathsf{D} \to \mathsf{E}$ semi-functors.

(1) A *natural transformation* (called "semi natural transformation" in [28, Def. 2.4]) α from F to G is a function which assigns to each object $D \in \mathsf{D}$ a morphism $\alpha_D \colon F(D) \to G(D)$ in E such that for every morphism $f \colon D \to D'$ in D, the following square commutes

$$\begin{tikzcd}
F(D) \arrow[r, "\alpha_D"] \arrow[d, "F(f)"'] & G(D) \arrow[d, "G(f)"] \\
F(D') \arrow[r, "\alpha_{D'}"] & G(D'),
\end{tikzcd}$$

and for every $D \in \mathsf{D}$, the following diagram commutes (under the hypotheses above, the commutativity of one triangle implies the commutativity of the other triangle).

$$\begin{tikzcd}
F(D) \arrow[r, "\alpha_D"] \arrow[d, "F(\mathrm{id}_D)"'] \arrow[dr, "\alpha_D"] & G(D) \arrow[d, "G(\mathrm{id}_D)"] \\
F(D) \arrow[r, "\alpha_D"'] & G(D)
\end{tikzcd}$$

(2) The composite $\beta \circ \alpha \colon F \to H$ of two natural transformations $\alpha \colon F \to G$ and $\beta \colon G \to H$ is the natural transformation defined by $(\beta \circ \alpha)_D = \beta_D \circ \alpha_D$ for $D \in \mathsf{D}$.

(3) The identity natural transformation $1_F : F \to F$ on F has components $(1_F)_D \coloneqq F(\mathrm{id_D})$. (Note that, in general, $(1_F)_D \neq \mathrm{id}_{F(D)}$.)
(4) F and G are *naturally isomorphic* (denoted $F \simeq G$) if there are natural transformations $\alpha\colon F \to G$ and $\beta\colon G \to F$ such that $\alpha \circ \beta = 1_G$ and $\beta \circ \alpha = 1_F$.

Theorem 7.3. *We have the following natural isomorphisms:*

(1) $\mathsf{Clop} \circ \mathbb{V}^{\mathsf{R}}_{\Box} \simeq \mathbb{K}^{\mathsf{S}}_{\Box} \circ \mathsf{Clop}$ *and* $\mathsf{Uf} \circ \mathbb{K}^{\mathsf{S}}_{\Box} \simeq \mathbb{V}^{\mathsf{R}}_{\Box} \circ \mathsf{Uf}$.
(2) $\mathsf{Clop} \circ \mathbb{V}^{\mathsf{R}}_{\Diamond} \simeq \mathbb{K}^{\mathsf{S}}_{\Diamond} \circ \mathsf{Clop}$ *and* $\mathsf{Uf} \circ \mathbb{K}^{\mathsf{S}}_{\Diamond} \simeq \mathbb{V}^{\mathsf{R}}_{\Diamond} \circ \mathsf{Uf}$.
(3) $\mathsf{Clop} \circ \mathbb{V}^{\mathsf{R}} \simeq \mathbb{K}^{\mathsf{S}} \circ \mathsf{Clop}$ *and* $\mathsf{Uf} \circ \mathbb{K}^{\mathsf{S}} \simeq \mathbb{V}^{\mathsf{R}} \circ \mathsf{Uf}$.

Therefore, the following three diagrams commute up to natural isomorphism.

$$\begin{tikzcd}
\mathsf{Stone}^{\mathsf{R}} \arrow[r, shift left, "\mathsf{Clop}"] \arrow[d, "\mathbb{V}^{\mathsf{R}}_{\Box}"'] & \mathsf{BA}^{\mathsf{S}} \arrow[l, shift left, "\mathsf{Uf}"] \arrow[d, "\mathbb{K}^{\mathsf{S}}_{\Box}"] \\
\mathsf{Stone}^{\mathsf{R}} \arrow[r, shift left, "\mathsf{Clop}"] & \mathsf{BA}^{\mathsf{S}} \arrow[l, shift left, "\mathsf{Uf}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{Stone}^{\mathsf{R}} \arrow[r, shift left, "\mathsf{Clop}"] \arrow[d, "\mathbb{V}^{\mathsf{R}}_{\Diamond}"'] & \mathsf{BA}^{\mathsf{S}} \arrow[l, shift left, "\mathsf{Uf}"] \arrow[d, "\mathbb{K}^{\mathsf{S}}_{\Diamond}"] \\
\mathsf{Stone}^{\mathsf{R}} \arrow[r, shift left, "\mathsf{Clop}"] & \mathsf{BA}^{\mathsf{S}} \arrow[l, shift left, "\mathsf{Uf}"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathsf{Stone}^{\mathsf{R}} \arrow[r, shift left, "\mathsf{Clop}"] \arrow[d, "\mathbb{V}^{\mathsf{R}}"'] & \mathsf{BA}^{\mathsf{S}} \arrow[l, shift left, "\mathsf{Uf}"] \arrow[d, "\mathbb{K}^{\mathsf{S}}"] \\
\mathsf{Stone}^{\mathsf{R}} \arrow[r, shift left, "\mathsf{Clop}"] & \mathsf{BA}^{\mathsf{S}} \arrow[l, shift left, "\mathsf{Uf}"]
\end{tikzcd}$$

Proof. (1) We only prove that $\mathsf{Clop} \circ \mathbb{V}^{\mathsf{R}}_{\Box} \simeq \mathbb{K}^{\mathsf{S}}_{\Box} \circ \mathsf{Clop}$; since Clop and Uf are quasi-inverses, it follows that $\mathsf{Uf} \circ \mathbb{K}^{\mathsf{S}}_{\Box} \simeq \mathbb{V}^{\mathsf{R}}_{\Box} \circ \mathsf{Uf}$. By [47, Fact 1], for each $X \in \mathsf{Stone}$, there is a boolean isomorphism $f_X : \mathsf{Clop}(\mathbb{V}(X)) \to \mathbb{K}(\mathsf{Clop}(X))$, which maps, for every clopen U of X, the clopen $\Box_U$ of $\mathbb{V}(X)$ to the formal symbol $\Box_U \in \mathbb{K}(\mathsf{Clop}(X))$, and the clopen $\Diamond_U$ of $\mathbb{V}(X)$ to the formal symbol $\Diamond_U \in \mathbb{K}(\mathsf{Clop}(X))$.[2] For each $X \in \mathsf{Stone}^{\mathsf{R}}$, we define two relations $\alpha_X : \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(X)) \to \mathbb{K}^{\mathsf{S}}_{\Box}(\mathsf{Clop}(X))$ and $\beta_X : \mathbb{K}^{\mathsf{S}}_{\Box}(\mathsf{Clop}(X)) \to \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(X))$ by setting

$$\mathcal{U}\ \alpha_X\ f_X(\mathcal{V}) \iff f_X(\mathcal{U})\ \beta_X\ \mathcal{V} \iff \mathcal{U}\ \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(\mathrm{id}_X))\ \mathcal{V}$$

[2] We have a conflict of notation here since, on the one hand, $\Box_U$ denotes a clopen subset of $\mathbb{V}(X)$ and, on the other hand, a formal symbol in $\mathbb{K}(\mathsf{Clop}(X))$. This will come up again in Claim 7.4, but nowhere else.

for each $\mathcal{U}, \mathcal{V} \in \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(X))$. For the sake of completeness, we give its explicit formulation:

$$\begin{aligned}\mathcal{U}\ \alpha_X\ f_X(\mathcal{V}) &\iff f_X(\mathcal{U})\ \beta_X\ \mathcal{V}\\ &\iff \forall F \in \mathcal{U}\ \forall G \in \mathbb{V}(X)\,(G \subseteq F \Rightarrow G \in \mathcal{V}).\end{aligned}$$

We prove that α and β are natural transformations. Since $\mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(1_X))$ is a subordination and f_X a boolean isomorphism, α_X and β_X are subordinations. Let $R\colon X \to Y$ be a morphism in $\mathsf{Stone}^{\mathsf{R}}$. We prove that the following square commutes.

$$\begin{array}{ccc}\mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(X)) & \xrightarrow{\alpha_X} & \mathbb{K}^{\mathsf{S}}_{\Box}(\mathsf{Clop}(X))\\ {\scriptstyle \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(R))}\big\downarrow & & \big\downarrow{\scriptstyle \mathbb{K}^{\mathsf{S}}_{\Box}(\mathsf{Clop}(R))}\\ \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(Y)) & \xrightarrow{\alpha_Y} & \mathbb{K}^{\mathsf{S}}_{\Box}(\mathsf{Clop}(Y))\end{array}$$

For this we require the following claim.

Claim 7.4. *For $\mathcal{U} \in \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(X))$ and $\mathcal{W} \in \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(Y))$, we have*

$$\mathcal{U}\ \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(R))\ \mathcal{W} \iff f_X(\mathcal{U})\ \mathbb{K}^{\mathsf{S}}_{\Box}(\mathsf{Clop}(R))\ f_Y(\mathcal{W}).$$

Proof of claim. Write $\mathcal{U}$ and $\mathcal{W}$ in disjunctive and conjunctive normal form, respectively:

$$\mathcal{U} = \bigcup_{i=1}^{n}(\Box_{U_i} \cap \Diamond_{V_{i1}} \cap \cdots \cap \Diamond_{V_{in_i}}), \quad \mathcal{W} = \bigcap_{j=1}^{m}(\Diamond_{W_j} \cup \Box_{Z_{j1}} \cup \cdots \cup \Box_{Z_{jm_j}})$$

with clopen sets $\varnothing \neq V_{il} \subseteq U_i$ for each i and l, and $W_j \subseteq Z_{jk} \neq Y$ for each j and k. Since $\mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(R)) = S_{\mathcal{R}_\Box}$, Proposition 7.1(1) yields

$$\mathcal{U}\ \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(R))\ \mathcal{W} \iff \forall\, i, j\, \exists\, k \leq m_j : U_i\ S_R\ Z_{jk}.$$

On the other hand, since f_X is a boolean homomorphism,

$$f_X(\mathcal{U}) = \bigvee_{i=1}^{n}(\Box_{U_i} \wedge \Diamond_{V_{i1}} \wedge \cdots \wedge \Diamond_{V_{in_i}})$$

and

$$f_X(\mathcal{W}) = \bigwedge_{j=1}^{m}(\Diamond_{W_j} \vee \Box_{Z_{j1}} \vee \cdots \vee \Box_{Z_{jm_j}}).$$

Since $\mathbb{K}^{\mathsf{S}}_{\Box}(\mathsf{Clop}(R)) = \mathbb{K}^{\mathsf{S}}_{\Box}(S_R)$, Proposition 5.5(1) yields

$$f_X(\mathcal{U})\ \mathbb{K}^{\mathsf{S}}_{\Box}(\mathsf{Clop}(R))\ f_X(\mathcal{W}) \iff \forall\, i, j\, \exists\, k \leq m_j : U_i\ S_R\ Z_{jk}.$$

Thus, $\mathcal{U}\ \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(R))\ \mathcal{W}$ if and only if $f_X(\mathcal{U})\ \mathbb{K}^{\mathsf{S}}_{\Box}(\mathsf{Clop}(R))\ f_Y(\mathcal{W})$. $\Box$

To prove the commutativity of the square, let $\mathcal{U} \in \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(X))$ and $\mathcal{W} \in \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(Y))$. On the one hand,

$$\begin{aligned}
&\mathcal{U}\ (\alpha_Y \circ \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(R)))\ f_Y(\mathcal{W}) \\
&\iff \exists\, \mathcal{Z} \in \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(Y)) : \mathcal{U}\ \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(R))\ \mathcal{Z}\ \alpha_Y\ f_Y(\mathcal{W}) \\
&\iff \exists\, \mathcal{Z} \in \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(Y)) : \mathcal{U}\ \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(R))\ \mathcal{Z}\ \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(\mathrm{id}_X))\ \mathcal{W} \\
&\iff \mathcal{U}\ (\mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(\mathrm{id}_X)) \circ \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(R)))\ \mathcal{W} \\
&\iff \mathcal{U}\ \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(\mathrm{id}_X \circ R))\ \mathcal{W} \\
&\iff \mathcal{U}\ \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(R))\ \mathcal{W}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\mathcal{U}\ (\mathbb{K}^{\mathsf{S}}_{\Box}(\mathsf{Clop}(R)) \circ \alpha_X)\ f_Y(\mathcal{W}) \\
&\iff \exists\, \mathcal{V} \in \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(X)) : \mathcal{U}\ \alpha_X\ f_X(\mathcal{V})\ \mathbb{K}^{\mathsf{S}}_{\Box}(\mathsf{Clop}(R))\ f_Y(\mathcal{W}) \\
&\iff \exists\, \mathcal{V} \in \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(X)) : \mathcal{U}\ \alpha_X\ f_X(\mathcal{V}),\ \mathcal{V}\ \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(R))\ \mathcal{W} \text{ by Claim 7.4} \\
&\iff \exists\, \mathcal{V} \in \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(X)) : \mathcal{U}\ \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(\mathrm{id}_X))\ \mathcal{V}\ \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(R))\ \mathcal{W} \\
&\iff \mathcal{U}\ (\mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(R)) \circ \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(\mathrm{id}_X)))\ \mathcal{W} \\
&\iff \mathcal{U}\ \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(R \circ \mathrm{id}_X))\ \mathcal{W} \\
&\iff \mathcal{U}\ \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(R))\ \mathcal{W}.
\end{aligned}$$

Thus, $\alpha_Y \circ \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(R)) = \mathbb{K}^{\mathsf{S}}_{\Box}(\mathsf{Clop}(R)) \circ \alpha_X$. Moreover, when $X = Y$ and $R = \mathrm{id}_X$, the calculation in the second to last display above and the definition of α_X give

$$\begin{aligned}
\mathcal{U}\ (\alpha_X \circ \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(\mathrm{id}_X)))\ f_X(\mathcal{V}) &\iff \mathcal{U}\ \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(\mathrm{id}_X))\ \mathcal{V} \\
&\iff \mathcal{U}\ \alpha_X\ f_X(\mathcal{V}).
\end{aligned}$$

Therefore, the following triangle commutes for every $X \in \mathsf{Stone}^{\mathsf{R}}$.

$$\begin{tikzcd}
\mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(X)) \arrow[d, "\mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(\mathrm{id}_X))"'] \arrow[dr, "\alpha_X"] & \\
\mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(X)) \arrow[r, "\alpha_X"'] & \mathbb{K}^{\mathsf{S}}_{\Box}(\mathsf{Clop}(X))
\end{tikzcd}$$

Thus, α is a natural transformation. That β is a natural transformation is proved similarly.

Let $X \in \mathsf{Stone}^{\mathsf{R}}$. We prove that $\beta_X \circ \alpha_X = \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(\mathrm{id}_X))$. For $\mathcal{U}, \mathcal{V} \in \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(X))$, we have

$$\begin{aligned}
&\mathcal{U}\ (\beta_X \circ \alpha_X)\ \mathcal{V} \\
&\quad\iff \exists\, \mathcal{W} \in \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(X)) : \mathcal{U}\ \alpha_X\ f_X(\mathcal{W})\ \beta_X\ \mathcal{V} \\
&\quad\iff \exists\, \mathcal{W} \in \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(X)) : \mathcal{U}\ \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(\mathrm{id}_X))\ \mathcal{W}\ \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(\mathrm{id}_X))\ \mathcal{V} \\
&\quad\iff \mathcal{U}\ \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(\mathrm{id}_X)) \circ \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(\mathrm{id}_X))\ \mathcal{V} \\
&\quad\iff \mathcal{U}\ \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(\mathrm{id}_X \circ \mathrm{id}_X))\ \mathcal{V} \\
&\quad\iff \mathcal{U}\ \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(\mathrm{id}_X))\ \mathcal{V}.
\end{aligned}$$

This proves that $\beta_X \circ \alpha_X = \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(\mathrm{id}_X))$. We next prove that $\alpha_X \circ \beta_X = \mathbb{K}^{\mathsf{S}}_{\Box}(\mathsf{Clop}(\mathrm{id}_X))$. For $\mathcal{U}, \mathcal{V} \in \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(X))$, we have

$$\begin{aligned}
&f_X(\mathcal{U})\ (\alpha_X \circ \beta_X)\ f_X(\mathcal{V}) \\
&\iff \exists\, \mathcal{W} \in \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(X)) : f_X(\mathcal{U})\ \beta_X\ \mathcal{W}\ \alpha_X\ f_X(\mathcal{V}) \\
&\iff \exists\, \mathcal{W} \in \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(X)) : \mathcal{U}\ \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(\mathrm{id}_X))\ \mathcal{W}\ \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(\mathrm{id}_X))\ f_X(\mathcal{V}) \\
&\iff \mathcal{U}\ \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(\mathrm{id}_X \circ \mathrm{id}_X))\ \mathcal{V} \\
&\iff \mathcal{U}\ \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Box}(\mathrm{id}_X))\ \mathcal{V} \\
&\iff f_X(\mathcal{U})\ \mathbb{K}^{\mathsf{S}}_{\Box}(\mathsf{Clop}(\mathrm{id}_X))\ f_X(\mathcal{V}) \quad \text{by Claim 7.4.}
\end{aligned}$$

Therefore, $\alpha_X \circ \beta_X = \mathbb{K}^{\mathsf{S}}_{\Box}(\mathsf{Clop}(\mathrm{id}_X))$. Thus, α and β yield a natural isomorphism between $\mathsf{Clop} \circ \mathbb{V}^{\mathsf{R}}_{\Box}$ and $\mathbb{K}^{\mathsf{S}}_{\Box} \circ \mathsf{Clop}$.

(2) and (3) are proved similarly. For the sake of completeness, we describe the natural isomorphisms. For (2), the natural isomorphism is given by the natural transformations $\alpha \colon \mathsf{Clop} \circ \mathbb{V}^{\mathsf{R}}_{\Diamond} \to \mathbb{K}^{\mathsf{S}}_{\Diamond} \circ \mathsf{Clop}$ and $\beta \colon \mathbb{K}^{\mathsf{S}}_{\Diamond} \circ \mathsf{Clop} \to \mathsf{Clop} \circ \mathbb{V}^{\mathsf{R}}_{\Diamond}$ defined by

$$\begin{aligned}
\mathcal{U}\ \alpha_X\ f_X(\mathcal{V}) \iff f_X(\mathcal{U})\ \beta_X\ \mathcal{V} &\iff \mathcal{U}\ \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}_{\Diamond}(\mathrm{id}_X))\ \mathcal{V} \\
&\iff \forall F \in \mathcal{U}\ \forall G \in \mathbb{V}(X)\ (G \subseteq F \Rightarrow G \in \mathcal{V}).
\end{aligned}$$

For (3), the natural transformations $\alpha \colon \mathsf{Clop} \circ \mathbb{V}^{\mathsf{R}} \to \mathbb{K}^{\mathsf{S}} \circ \mathsf{Clop}$ and $\beta \colon \mathbb{K}^{\mathsf{S}} \circ \mathsf{Clop} \to \mathsf{Clop} \circ \mathbb{V}^{\mathsf{R}}$ witnessing the natural isomorphism are defined by

$$\mathcal{U}\ \alpha_X\ f_X(\mathcal{V}) \iff f_X(\mathcal{U})\ \beta_X\ \mathcal{V} \iff \mathcal{U}\ \mathsf{Clop}(\mathbb{V}^{\mathsf{R}}(\mathrm{id}_X))\ \mathcal{V} \iff \mathcal{U} \subseteq \mathcal{V}.$$

□

8. Extending $\mathbb{K}^{\mathsf{S}}$ to $\mathsf{SubS5}^{\mathsf{S}}$, $\mathsf{DeV}^{\mathsf{S}}$, and $\mathsf{KRFrm}^{\mathsf{P}}$

In this section, we lift the endofunctor $\mathbb{K}^{\mathsf{S}}$ on BA^{S} to an endofunctor on $\mathsf{SubS5}^{\mathsf{S}}$. We then use MacNeille completions of $\mathsf{S5}$-subordination algebras developed in [2] to obtain an endofunctor $\mathbb{L}^{\mathsf{S}}$ on $\mathsf{DeV}^{\mathsf{S}}$ and prove that $\mathbb{L}^{\mathsf{S}}$ is dual to the Vietoris endofunctor $\mathbb{V}^{\mathsf{R}}$ on $\mathsf{KHaus}^{\mathsf{R}}$. We conclude by utilizing ideal completions of $\mathsf{S5}$-subordination algebras to obtain an endofunctor on $\mathsf{KRFrm}^{\mathsf{P}}$, thus generalizing Johnstone's construction of the Vietoris endofunctor on KRFrm ([30, p. 112]).

We recall (see [1, Rem. 3.11]) that the equivalence between $\mathsf{SubS5}^{\mathsf{S}}$ and $\mathsf{StoneE}^{\mathsf{R}}$ is obtained by lifting the functors Uf and Clop establishing the equivalence between BA^{S} and $\mathsf{Stone}^{\mathsf{R}}$. We next lift $\mathbb{V}^{\mathsf{R}}\colon \mathsf{Stone}^{\mathsf{R}} \to \mathsf{Stone}^{\mathsf{R}}$ to an endofunctor on $\mathsf{StoneE}^{\mathsf{R}}$ and $\mathbb{K}^{\mathsf{S}}\colon \mathsf{BA}^{\mathsf{S}} \to \mathsf{BA}^{\mathsf{S}}$ to an endofunctor on $\mathsf{SubS5}^{\mathsf{S}}$. We slightly abuse notation and denote the lift of a functor by the same letter.

Definition 8.1.

(1) Define $\mathbb{V}^{\mathsf{R}}\colon \mathsf{StoneE}^{\mathsf{R}} \to \mathsf{StoneE}^{\mathsf{R}}$ by sending an object (X, E) to $(\mathbb{V}^{\mathsf{R}}(X), \mathbb{V}^{\mathsf{R}}(E))$ and a morphism R to $\mathbb{V}^{\mathsf{R}}(R)$.

(2) Define $\mathbb{K}^{\mathsf{S}}\colon \mathsf{SubS5}^{\mathsf{S}} \to \mathsf{SubS5}^{\mathsf{S}}$ by sending an object (B, S) to $(\mathbb{K}^{\mathsf{S}}(B), \mathbb{K}^{\mathsf{S}}(S))$ and a morphism T to $\mathbb{K}^{\mathsf{S}}(T)$.

To show that $\mathbb{V}^{\mathsf{R}}$ and $\mathbb{K}^{\mathsf{S}}$ are well defined, we require the following two lemmas.

Lemma 8.2.

(1) *If* $(X, E) \in \mathsf{StoneE}^{\mathsf{R}}$, *then* $\mathbb{V}^{\mathsf{R}}(E)$ *is a closed equivalence relation on* $\mathbb{V}^{\mathsf{R}}(X)$ *and* $\mathbb{V}^{\mathsf{R}}(X)/\mathbb{V}^{\mathsf{R}}(E) \cong \mathbb{V}^{\mathsf{R}}(X/E)$.

(2) *If* R *is a morphism in* $\mathsf{StoneE}^{\mathsf{R}}$, *then* $\mathbb{V}^{\mathsf{R}}(\mathcal{Q}(R)) = \mathcal{Q}(\mathbb{V}^{\mathsf{R}}(R))$.

Proof. (1) Since $E = E^{\dagger}$, it follows from Proposition 2.6 that $\mathbb{V}^{\mathsf{R}}(E)$ is a closed equivalence relation. Therefore, if $\pi\colon X \to X/E$ is the quotient map, then for $F, G \in \mathbb{V}^{\mathsf{R}}(X)$, we have

$$\begin{aligned} F \; \mathbb{V}^{\mathsf{R}}(E) \; G &\iff E[F] = E[G] \iff \pi^{-1}\pi[F] = \pi^{-1}\pi[G] \\ &\iff \pi[F] = \pi[G]. \end{aligned}$$

Thus, $\mathbb{V}^{\mathsf{R}}(E)$ is the kernel of $\mathbb{V}(\pi)$ and, hence, $\mathbb{V}^{\mathsf{R}}(X)/\mathbb{V}^{\mathsf{R}}(E)$ is homeomorphic to $\mathbb{V}^{\mathsf{R}}(X/E)$.

(2) Since $\mathbb{V}^{\mathsf{R}}$ is a functor and commutes with the dagger, we have

$$\mathcal{Q}(\mathbb{V}^{\mathsf{R}}(R)) = \mathbb{V}^{\mathsf{R}}(\pi) \circ \mathbb{V}^{\mathsf{R}}(R) \circ \mathbb{V}^{\mathsf{R}}(\pi)^{\dagger} = \mathbb{V}^{\mathsf{R}}(\pi \circ R \circ \pi^{\dagger}) = \mathbb{V}^{\mathsf{R}}(\mathcal{Q}(R)). \quad \square$$

Lemma 8.3. *Let* S *be an* $\mathsf{S5}$*-subordination on a boolean algebra* B.

(1) $\mathcal{S}_{\Box}$ *and* $\mathcal{S}_{\Diamond}$ *satisfy* (S5) *and* (S7).

(2) $\mathcal{S}$ *is an* $\mathsf{S5}$*-subordination.*

Proof. (1) That $\mathcal{S}_{\Box}$ and $\mathcal{S}_{\Diamond}$ satisfy (S5) is an immediate consequence of Definition 5.3 and Lemma 5.4. We prove that $\mathcal{S}_{\Box}$ satisfies (S7). Since $\mathbb{K}^{\mathsf{S}}_{\Box}$ preserves inclusion of subordinations and composition,

$$\mathcal{S}_{\Box} \circ \mathcal{S}_{\Box} = \mathbb{K}^{\mathsf{S}}_{\Box}(S) \circ \mathbb{K}^{\mathsf{S}}_{\Box}(S) = \mathbb{K}^{\mathsf{S}}_{\Box}(S \circ S) \subseteq \mathbb{K}^{\mathsf{S}}_{\Box}(S) = \mathcal{S}_{\Box}.$$

Thus, $\mathcal{S}_\Box$ satisfies (S7). That $\mathcal{S}_\Diamond$ satisfies (S7) is proved similarly.

(2) By (1), to show that $\mathcal{S}$ satisfies (S5) and (S7), it is sufficient to prove that if $S, T \in \mathsf{BA}^\mathsf{S}(B, B)$ satisfy (S5) and (S7), then so does their join. Since (S5) holds for S and T, we have that S and T are contained in $\leq$, where $\leq$ is the partial order on B. Thus, $S \vee T$ is contained in $\leq$, and so $S \vee T$ satisfies (S5). Since S and T satisfy (S7), $S \subseteq S \circ S$ and $T \subseteq T \circ T$. Thus,

$$S \vee T \subseteq (S \circ S) \vee (T \circ T) \subseteq (S \vee T) \circ (S \vee T),$$

where the last inclusion follows from the fact that both $S \circ S$ and $T \circ T$ are contained in $(S \vee T) \circ (S \vee T)$ because $S, T \subseteq S \vee T$. Therefore, $S \vee T$ satisfies (S7).

It remains to show that $\mathcal{S}$ satisfies (S6). We have that $S^\dagger = S$ because S satisfies (S6). By Proposition 6.7, $\mathcal{S}^\dagger = \mathbb{K}^\mathsf{S}(S)^\dagger = \mathbb{K}^\mathsf{S}(S^\dagger) = \mathbb{K}^\mathsf{S}(S) = \mathcal{S}$. Thus, $\mathcal{S}$ satisfies (S6). $\square$

Theorem 8.4. *$\mathbb{V}^\mathsf{R} \colon \mathsf{StoneE}^\mathsf{R} \to \mathsf{StoneE}^\mathsf{R}$ and $\mathbb{K}^\mathsf{S} \colon \mathsf{SubS5}^\mathsf{S} \to \mathsf{SubS5}^\mathsf{S}$ are well-defined functors and the following diagram commutes up to natural isomorphism.*

$$\begin{tikzcd}
\mathsf{KHaus}^\mathsf{R} \arrow[r, shift left, "\mathcal{G}"] \arrow[d, "\mathbb{V}^\mathsf{R}"'] & \mathsf{StoneE}^\mathsf{R} \arrow[l, shift left, "\mathcal{Q}"] \arrow[r, shift left, "\mathsf{Clop}"] \arrow[d, "\mathbb{V}^\mathsf{R}"] & \mathsf{SubS5}^\mathsf{S} \arrow[l, shift left, "\mathsf{Uf}"] \arrow[d, "\mathbb{K}^\mathsf{S}"] \\
\mathsf{KHaus}^\mathsf{R} \arrow[r, shift left, "\mathcal{G}"] & \mathsf{StoneE}^\mathsf{R} \arrow[l, shift left, "\mathcal{Q}"] \arrow[r, shift left, "\mathsf{Clop}"] & \mathsf{SubS5}^\mathsf{S} \arrow[l, shift left, "\mathsf{Uf}"]
\end{tikzcd}$$

Proof. That $\mathbb{V}^\mathsf{R}$ is well defined follows from Lemma 8.2(1) and that $\mathbb{K}^\mathsf{S}$ is well defined from Lemma 8.3(2). By Lemma 8.2, $\mathbb{V}^\mathsf{R} \circ \mathcal{Q} = \mathcal{Q} \circ \mathbb{V}^\mathsf{R}$, and since $\mathcal{G}$ is a quasi-inverse of $\mathcal{Q}$ (see Remark 3.6), the left square commutes. Finally, because $\mathbb{V}^\mathsf{R}$ and $\mathbb{K}^\mathsf{S}$ are defined componentwise (see Definition 8.1), Theorem 7.3(3) yields commutativity of the right square. $\square$

To obtain an endofunctor on DeV^S, we recall the definition of the MacNeille completion of an $\mathsf{S5}$-subordination algebra. For a subset E of a poset, we write E^u for the set of upper bounds and E^ℓ for the set of lower bounds of E.

Definition 8.5. Let (B, S) be an $\mathsf{S5}$-subordination algebra.

(1) [2, Def. 3.1] An ideal I of B is a *round ideal* if, for each $a \in I$, there is $b \in I$ such that $a \mathrel{S} b$.
(2) [2, Thm. 4.7] A round ideal I is *normal* if $I = S^{-1}[(S[I^u])^\ell]$.

For $(B,S) \in \mathsf{SubS5^S}$, let $\mathcal{NI}(B,S)$ be the set of normal round ideals of (B,S) ordered by inclusion. Following [2, Def. 4.6], we call $\mathcal{NI}(B,S)$ the *MacNeille completion* of (B,S).

Theorem 8.6 ([2, Prop. 4.4, Thm. 4.13])**.** *If (B,S) is an* $\mathsf{S5}$*-subordination algebra, then $\mathcal{NI}(B,S)$ is a de Vries algebra. Moreover, the assignment $(B,S) \mapsto \mathcal{NI}(B,S)$ extends to a functor $\mathbb{M}\colon \mathsf{SubS5^S} \to \mathsf{DeV^S}$, which is an equivalence whose quasi-inverse is the inclusion $\Delta\colon \mathsf{DeV^S} \to \mathsf{SubS5^S}$.*

Definition 8.7. Define $\mathbb{L}^\mathsf{S}\colon \mathsf{DeV^S} \to \mathsf{DeV^S}$ to be the composite $\mathbb{M}\circ\mathbb{K}^\mathsf{S}\circ\Delta$.

Clearly, $\mathbb{L}^\mathsf{S}$ is a well-defined endofunctor on $\mathsf{DeV^S}$. Moreover, since $\mathbb{M}\colon \mathsf{SubS5^S} \to \mathsf{DeV^S}$ is an equivalence whose quasi-inverse is $\Delta\colon \mathsf{DeV^S} \to \mathsf{SubS5^S}$, we have the following theorem.

Theorem 8.8. *The following diagram commutes up to a natural isomorphism.*

$$\begin{tikzcd}
\mathsf{SubS5^S} \arrow[r, "\mathbb{M}"] \arrow[d, "\mathbb{K}^\mathsf{S}"'] & \mathsf{DeV^S} \arrow[d, "\mathbb{L}^\mathsf{S}"] \\
\mathsf{SubS5^S} \arrow[r, "\mathbb{M}"'] & \mathsf{DeV^S}
\end{tikzcd}$$

We recall from [1, Def. 4.10] that the functor $\mathcal{D}\colon \mathsf{KHaus^R} \to \mathsf{DeV^S}$ sends each compact Hausdorff space X to the de Vries algebra $\mathcal{RO}(X)$ of its regular opens and a closed relation $R\colon X_1 \to X_2$ to the compatible subordination $\mathcal{D}(R)\colon \mathcal{RO}(X_1) \to \mathcal{RO}(X_2)$ defined by

$$U\ \mathcal{D}(R)\ V \iff R[\mathsf{cl}(U)] \subseteq V.$$

By [2, Thm. 7.8], $\mathbb{M}\circ\mathsf{Clop}$ is naturally isomorphic to $\mathcal{D}\circ\mathcal{Q}$. Therefore, combining Theorem 8.4 and Theorem 8.8, we obtain Theorem 8.9.

Theorem 8.9. *The following diagram commutes up to natural isomorphism.*

$$\begin{tikzcd}
\mathsf{KHaus^R} \arrow[d, "\mathbb{V}^\mathsf{R}"] \arrow[rrr, bend left, "\mathcal{D}"] & \mathsf{StoneE^R} \arrow[l, "\mathcal{Q}"'] \arrow[r, "\mathsf{Clop}"] \arrow[d, "\mathbb{V}^\mathsf{R}"] & \mathsf{SubS5^S} \arrow[r, "\mathbb{M}"] \arrow[d, "\mathbb{K}^\mathsf{S}"] & \mathsf{DeV^S} \arrow[d, "\mathbb{L}^\mathsf{S}"] \\
\mathsf{KHaus^R} \arrow[rrr, bend right, "\mathcal{D}"'] & \mathsf{StoneE^R} \arrow[l, "\mathcal{Q}"'] \arrow[r, "\mathsf{Clop}"] & \mathsf{SubS5^S} \arrow[r, "\mathbb{M}"] & \mathsf{DeV^S}
\end{tikzcd}$$

Remark 8.10. We thus obtain that the Vietoris endofunctor $\mathbb{V}^{\mathsf{R}}$ on $\mathsf{KHaus}^{\mathsf{R}}$ can be dually described either as the endofunctor $\mathbb{K}^{\mathsf{S}}$ on $\mathsf{SubS5}^{\mathsf{S}}$ or as the endofunctor $\mathbb{L}^{\mathsf{S}}$ on $\mathsf{DeV}^{\mathsf{S}}$. The usefulness of the latter description stems from the fact that, unlike in $\mathsf{SubS5}^{\mathsf{S}}$, isomorphisms in $\mathsf{DeV}^{\mathsf{S}}$ are given by structure-preserving bijections [1, Thm. 5.4].

In [10, p. 375], it was left as an open problem to give a direct description of an endofunctor on DeV that is dual to the Vietoris endofunctor $\mathbb{V}$ on KHaus. Our result above solves a similar problem by showing that it is the endofunctor $\mathbb{L}^{\mathsf{S}}$ that is dual to the Vietoris endofunctor $\mathbb{V}^{\mathsf{R}}$ on $\mathsf{KHaus}^{\mathsf{R}}$. In [3], we show how to modify our result to obtain a solution of [10, p. 375].

We conclude the paper by developing the dual endofunctor $\mathbb{V}^{\mathsf{R}}$ for $\mathsf{KRFrm}^{\mathsf{P}}$. For this, we recall from [2, Thm. 3.5] that associating with each $\mathsf{S5}$-subordination algebra (B, S) the frame $\mathcal{RI}(B, S)$ of round ideals of (B, S) defines a contravariant functor $\mathbb{I}\colon \mathsf{SubS5}^{\mathsf{S}} \to \mathsf{KRFrm}^{\mathsf{P}}$. To describe its quasi-inverse, we recall that for a frame L, the *pseudocomplement* of $a \in L$ is given by $a^* = \bigvee\{x \in L \mid a \wedge x = 0\}$. The *well-inside relation* on L is then defined by $a \prec b$ if and only if $a^* \vee b = 1$. The *booleanization* of L is the boolean frame $\mathfrak{B}L = \{a \in L \mid a = a^{**}\}$ (see, e.g., [7]), and restricting $\prec$ to $\mathfrak{B}L$ yields a de Vries algebra $(\mathfrak{B}L, \prec)$ [9]. This defines a contravariant functor $\mathfrak{B}\colon \mathsf{KRFrm}^{\mathsf{P}} \to \mathsf{DeV}^{\mathsf{S}}$ [2, Prop. 4.2], and we have the following theorem.

Theorem 8.11 ([2, Thm. 4.18(1)])**.** *The functors $\mathbb{I}$ and $\Delta \circ \mathfrak{B}$ form a dual equivalence between $\mathsf{SubS5}^{\mathsf{S}}$ and $\mathsf{KRFrm}^{\mathsf{P}}$.*

Definition 8.12. Define $\mathbb{J}^{\mathsf{P}}\colon \mathsf{KRFrm}^{\mathsf{P}} \to \mathsf{KRFrm}^{\mathsf{P}}$ to be the composition $\mathbb{I} \circ \mathbb{K}^{\mathsf{S}} \circ \Delta \circ \mathfrak{B}$.

Clearly, $\mathbb{J}^{\mathsf{P}}$ is a well-defined endofunctor on $\mathsf{KRFrm}^{\mathsf{P}}$. Moreover, since $\mathbb{I}\colon \mathsf{SubS5}^{\mathsf{S}} \to \mathsf{KRFrm}^{\mathsf{P}}$ is an equivalence with $\Delta \circ \mathfrak{B}\colon \mathsf{KRFrm}^{\mathsf{P}} \to \mathsf{SubS5}^{\mathsf{S}}$ as quasi-inverse, we obtain Theorem 8.13.

Theorem 8.13. *The following diagram commutes up to natural isomorphism.*

$$\begin{tikzcd}
\mathsf{SubS5}^{\mathsf{S}} \arrow[r, "\mathbb{I}"] \arrow[d, "\mathbb{K}^{\mathsf{S}}"'] & \mathsf{KRFrm}^{\mathsf{P}} \arrow[d, "\mathbb{J}^{\mathsf{P}}"] \\
\mathsf{SubS5}^{\mathsf{S}} \arrow[r, "\mathbb{I}"] & \mathsf{KRFrm}^{\mathsf{P}}
\end{tikzcd}$$

For a compact Hausdorff space X, let $\mathcal{O}(X)$ be the compact regular frame of opens of X. For a closed relation $R\colon X \to Y$, let $\Box_R\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ be given by $\Box_R U = -R^{-1} - U$. This defines a contravariant functor

$\mathcal{O}\colon \mathsf{KHaus}^{\mathsf{R}} \to \mathsf{KRFrm}^{\mathsf{P}}$, which is a dual equivalence ([44], [33]). By [2, Thm. 7.4], $\mathcal{O}\circ\mathcal{Q}$ is naturally isomorphic to $\mathbb{I}\circ\mathsf{Clop}$. Thus, putting Theorem 8.4 and Theorem 8.13 together yields the next theorem.

Theorem 8.14. *The following diagram commutes up to natural isomorphism.*

$$\begin{tikzcd}
\mathsf{KHaus}^{\mathsf{R}} \arrow[d, "\mathbb{V}^{\mathsf{R}}"] \arrow[rrr, bend left, "\mathcal{O}"] & \mathsf{StoneE}^{\mathsf{R}} \arrow[l, "\mathcal{Q}"'] \arrow[r, "\mathsf{Clop}"] \arrow[d, "\mathbb{V}^{\mathsf{R}}"] & \mathsf{SubS5}^{\mathsf{S}} \arrow[r, "\mathbb{I}"] \arrow[d, "\mathbb{K}^{\mathsf{S}}"] & \mathsf{KRFrm}^{\mathsf{P}} \arrow[d, "\mathbb{J}^{\mathsf{P}}"] \\
\mathsf{KHaus}^{\mathsf{R}} \arrow[rrr, bend right, "\mathcal{O}"'] & \mathsf{StoneE}^{\mathsf{R}} \arrow[l, "\mathcal{Q}"'] \arrow[r, "\mathsf{Clop}"] & \mathsf{SubS5}^{\mathsf{S}} \arrow[r, "\mathbb{I}"] & \mathsf{KRFrm}^{\mathsf{P}}
\end{tikzcd}$$

Remark 8.15. We thus obtain an endofunctor on $\mathsf{KRFrm}^{\mathsf{P}}$ that is dual to $\mathbb{V}^{\mathsf{R}}$. This extends Johnstone's endofunctor $\mathbb{J}$ on KRFrm that is dual to the Vietoris endofunctor $\mathbb{V}$ on KHaus. Alternatively, one could generalize Johnstone's construction directly, but the difficulty would be in lifting the morphisms. Indeed, Johnstone writes each element of $\mathbb{J}(L)$ as a join of finite meets of the generators $\{\Box_a, \Diamond_a \mid a \in L\}$. Since every frame homomorphism $f\colon L \to M$ preserves arbitrary joins and finite meets, it lifts to a frame homomorphism $\mathbb{J}(f)\colon \mathbb{J}(L) \to \mathbb{J}(M)$. On the other hand, preframe homomorphisms only preserve directed joins, so the above lift could be modified as follows. Write each element as a directed join of finite meets of finite joins of the generators $\{\Box_a, \Diamond_a \mid a \in L\}$. Then the hard part is in proving that the natural lift of $f\colon L \to M$ to $\mathbb{J}^{\mathsf{P}}(f)\colon \mathbb{J}(L) \to \mathbb{J}(M)$ is well defined. This issue disappears in our alternate construction of $\mathbb{J}^{\mathsf{P}}$ as the composite $\mathbb{I} \circ \mathbb{K}^{\mathsf{S}} \circ \Delta \circ \mathfrak{B}$.

Note Added in Proof

- Theorem 2.8(2) was proved earlier by Alexandre Goy, Daniela Petrişan, and Marc Aiguier [25] using different methods; see also [24]. We thank Quentin Aristote and Daniela Petrişan for pointing this out to us.
- Lemma 5.4 is essentially due to Jan Cederquist and Thierry Coquand [15]; see also the paper by Tatsuji Kawai [34].

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