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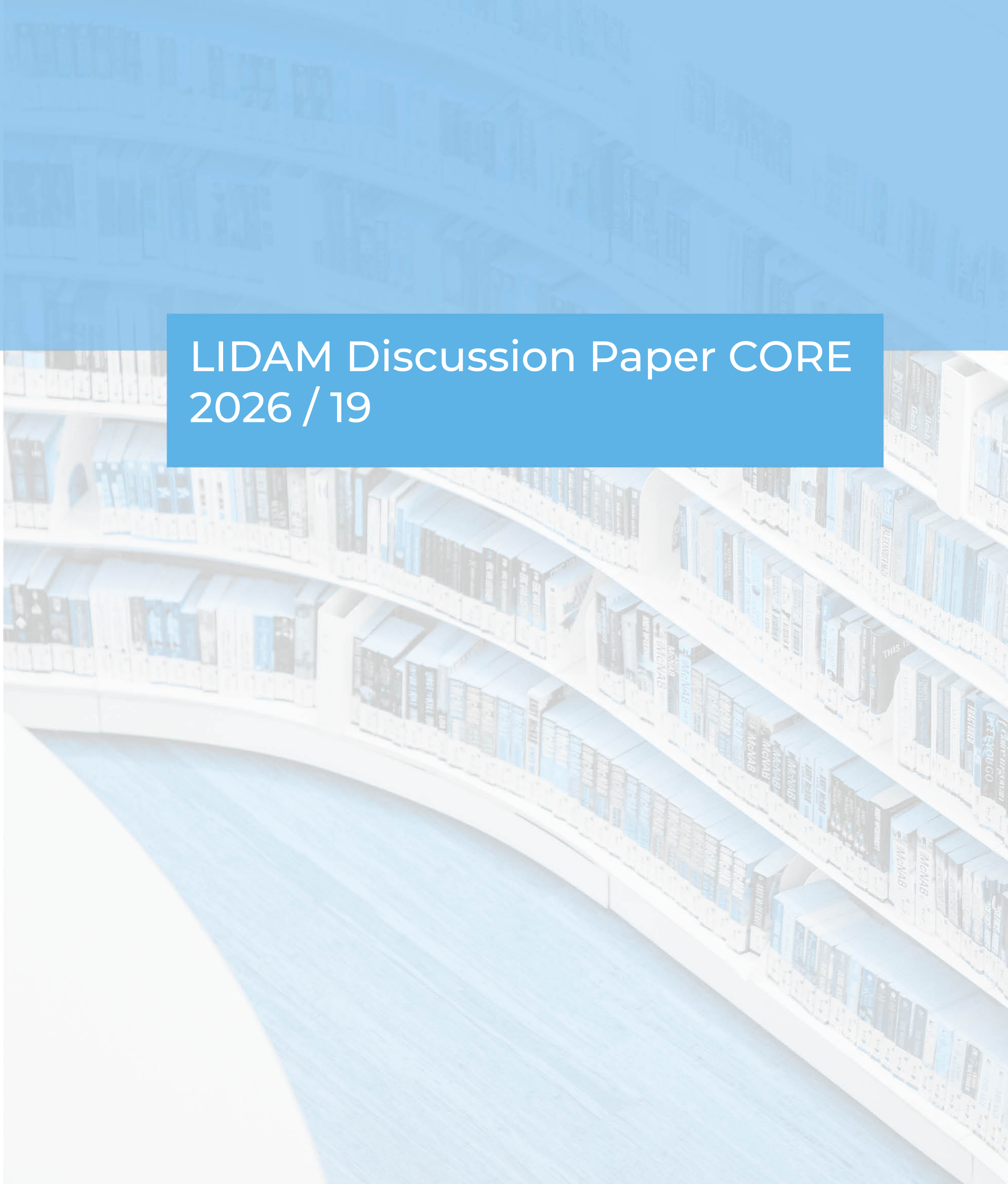

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Fertility, Opportunity, and Intergenerational Mobility

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Abstract

Can demographic change affect intergenerational mobility even when fertility declines uniformly across communities? We show that it can when opportunities are unequal. We embed a quantity–quality fertility trade-off in a model of unequal opportunity. Under equal opportunity, fertility affects aggregate human capital but not intergenerational persistence. Under unequal opportunity, by contrast, fertility decline increases the intergenerational elasticity of income by transforming fertility differences into differences in human-capital transmission across communities. The effect is amplified by community inequality and applies equally to changes in child-rearing costs and endogenous fertility gaps. A US calibration shows that the mechanism is quantitatively meaningful and strengthens sharply with community inequality.

JEL Classification: I22; J13; J62

Keywords: Intergenerational mobility; fertility; quantity-quality trade-off; unequal opportunity; segregation; inequality

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1 Introduction

Global fertility has halved over the past sixty years, from roughly five children per woman in 1960 to 2.3 today, and the fall has been broad-based: the decline has now spread to low- and middle-income countries and, within countries, to the poorest and least-educated (Fernández-Villaverde and Norrick, 2026), driven in significant part by a shift in the value that men and women attach to family life relative to alternative sources of fulfilment (Fernández *et al.*, 2026). The fall has also been uneven: US women without a college degree continue to have substantially more children than college graduates, and this gap has widened steadily since the 1970s (Jones and Tertilt, 2008; Baudin *et al.*, 2015). Across the OECD, fertility has fallen fastest among higher-income and more educated groups (Skirbekk, 2008; Doepke *et al.*, 2023). Recent work in the "new economics of fertility" (Doepke *et al.*, 2023) links this uneven decline to intensive parenting and status competition: Mahler *et al.* (2025) argue that parents compete to raise their children's relative performance, which pushes up the effective cost of children and reduces family size, while Kim *et al.* (2024) show, in a calibrated model of South Korea, that fertility would be about 28 percent higher without this "education rat race." Over the same period, intergenerational mobility has flattened in the United States (Chetty *et al.*, 2014) and depends heavily on community-level opportunity: neighbourhood exposure has causal effects on children's adult outcomes (Chetty and Hendren, 2018), and community social capital is one of the strongest place-level correlates of mobility, on par with parental economic resources (Chetty *et al.*, 2022).

These two facts have been studied separately, but they are connected — a point already recognized by nineteenth-century demographers: "For people who want to go up, many children make inconvenient luggage [...] a large seed dilutes parental resources and therefore complicates or aggravates the social situation in the next generation" (Dumont, 1890, pp. 73–91); Banks (1954) developed the same idea for the English middle classes. If richer families invest more per child precisely because they have fewer, and if those children disproportionately access higher-opportunity communities, the fertility decline may have reinforced intergenerational persistence in ways that standard mobility models, which treat fertility as exogenous, cannot capture. This paper formalises that mechanism.

This paper asks: when parents choose both the number of children and how much to invest in them, how does neighbourhood and community segregation shape fertility decisions, and how do those fertility decisions feed back into intergenerational income persistence? We use "opportunity" broadly throughout, in the sense of Chetty *et al.* (2014) and Chetty and Hendren (2018): the local return to parental investment in a child's growing-up environment, which combines school quality, social capital, peer effects, role-model exposure, and other neighbourhood-level inputs. School quality is the most measurable component, but it is not the only one.

Two literatures speak to parts of this question but not to their interaction. The quantity-quality (QQ) trade-off literature (Becker and Lewis, 1973; Galor and Moav, 2002; De la Croix and Doepke, 2003) shows that differential fertility across income groups can sustain inequality through human capital accumulation, and finds empirical support for negative effects of family size on child outcomes (Rosenzweig and Wolpin, 1980; Li *et al.*, 2008; Cáceres-Delpiano, 2006). The intergenerational mobility literature (Becker and Tomes, 1979, 1986; Solon, 1999, 2004) models how parental investment transmits economic status across generations; Arenas and Hindriks (2021) extend this framework to incorporate unequal school opportunity, identifying a fundamental trade-off between efficiency and mobility under school segregation. But the QQ literature abstracts from school quality heterogeneity, and the mobility literature treats fertility as exogenous. No existing model asks what happens when the school environment that shapes mobility also shapes fertility—and when that fertility response feeds back into mobility.

The first result is the irrelevance of family size under equal opportunity. When opportunity is equal across communities, fertility is neutral for social mobility. A decline in family size, or a fall in per-child child-rearing costs, raises average human capital but leaves the intergenerational elasticity of income (IGE) unchanged: it is a pure level shift. The intuition is straightforward: a uniform fall in fertility or child-rearing costs frees resources for all families proportionally, and with equal returns to parental investment, per-child investment rises by the same factor across the income distribution. The rich-to-poor investment ratio — and hence the IGE — therefore remains unchanged. This holds whether fertility is treated as exogenous or is allowed to respond endogenously to the return to parental investment.

The second result is amplification under unequal opportunity. As soon as returns to parental investment differ across communities, the irrelevance breaks. Under unequal opportunity, the same demographic shocks—a uniform decline in fertility, a change in the effective cost of children, or an endogenous fertility gap between advantaged and disadvantaged families—raise the

IGE. Changes in child-rearing costs may operate directly, through investment at fixed fertility, or indirectly, by inducing fertility responses as in the intensive-parenting mechanism of Doepke and Zilibotti, 2017; Mahler *et al.*, 2025). Resources freed by lower fertility or child-rearing costs generate larger human-capital gains in high-opportunity communities; because richer families are disproportionately concentrated in these communities, the shock widens differences in children's human capital. The magnitude is scaled up by a dissimilarity-weighted community-inequality amplifier that combines school inequality with residential segregation. A US calibration suggests both channels are quantitatively meaningful even at moderate levels of community inequality, and that the amplification grows substantially with community inequality.

Endogenous fertility adds a second amplification channel to the second result. Families in higher-opportunity communities face a higher return to per-child investment and optimally choose fewer, more resource-intensive children. The resulting fertility gap concentrates resources at the top of the income distribution and reinforces the assortative-matching gain that drives the efficiency–mobility trade-off.

In our model, higher child-rearing costs (modeled as a fraction of parental income per child) reduce the number of children but raise investment per child. This is in line with the Mahler *et al.* (2025) model of intensive parenting. Recent evidence (Adserà, 2017) suggests that the opportunity cost of raising children has risen disproportionately for higher-income families as wage growth has become increasingly concentrated at the top of the income distribution, contributing to widening gaps in parental investment by educational background. This trend may widen fertility differentials and strengthen the mechanism highlighted in our model. Kim *et al.* (2024) describe this mechanism as a status externality in education, whereby parents compete by increasing investments in their children's educational outcomes.

Policies that reduce disparities in child-rearing costs can therefore attenuate this fertility-driven amplification. These policies operate through a different mechanism than school policies. Desegregation weakens the correlation between parental income and access to high-quality schools, school equalization reduces disparities in school quality itself, and policies that lower child-rearing cost differentials dampen the fertility response to differences in school quality. Although all three types of policies reduce the persistence of income across generations, they do so through distinct channels.

Durlauf *et al.* (2022) classify structural theories of social mobility into five broad mechanisms: family investment, skill formation, social interactions, political economy, and aspirations. Our paper belongs to the *social* class, building on school segregation models such as Bénabou (1996), Durlauf (1996), and Arenas and Hindriks (2021). We extend this literature by introducing an endogenous quantity–quality fertility margin, through which unequal opportunity affects fertility decisions and, in turn, the composition of the next generation. This demographic channel complements existing social models of mobility.

A leading structural social model is Fogli *et al.* (2026), who show how neighbourhood human-capital spillovers and residential sorting amplify inequality and reduce mobility. Their amplification mechanism is geographic, operating through endogenous neighbourhood composition; ours is demographic, operating through endogenous fertility and the concentration of resources per child. The two mechanisms are complementary and can jointly contribute to rising intergenerational persistence.

Our mechanism is also complementary to a distinct household-economics literature that models fertility as a joint bargaining decision under limited commitment, most recently Cherchye *et al.* (2026), who use the *value of marriage* as the organising concept for cross-country fertility heterogeneity. Their object is the *micro* anatomy of the fertility choice — who bears the time cost, what is bargained, what is enforceable. Ours is the *aggregate equilibrium* consequence of that choice once returns to parental investment differ across communities. In our reduced form, the child-rearing cost parameter stands in for the equilibrium output of the collective household problem their framework analyses; the caveats we raise in Section 5 about heterogeneity in child-rearing costs across income groups are the macro-counterpart of the high-maternal-career-cost mechanism they emphasise.

The model is also consistent with a robust cross-country pattern: countries with more socio-economically segregated schools tend to exhibit lower intergenerational mobility. School segregation varies substantially across OECD countries (Gutiérrez *et al.*, 2020), and intergenerational income elasticities (Corak, 2013) tend to be higher in more segregated systems — a co-movement in the same direction as the amplification our mechanism predicts, and one that motivates the quantitative calibration in Section 5.

The paper proceeds as follows. Section 2 develops the exogenous-fertility benchmark and states the irrelevance and amplification results. Section 3 introduces endogenous fertility and derives the additional QQ-trade-off amplification under full segregation. Section 4 extends the analysis to partial segregation. Section 5 calibrates the model to US data and reports the quantitative decompositions. Section 6 concludes. Proofs are collected in Appendix B; auxiliary derivations are in Appendix A.

2 Baseline Model with exogenous fertility

We build on the regime switch approach to unequal school opportunity in the Becker–Tomes–Solon tradition (Arenas and Hindriks, 2021), in which children of different income groups attend schools of different quality, to understand how unequal school opportunity contributes to differences in social mobility and human capital. We extend the model to fertility variables: the number n of children per family and the child rearing cost α that we consider proportional to income. Each parent (generation $t-1$) allocates lifetime income y_{t-1} between own consumption C_{t-1}, investment per child I_{t-1} in each of n children's human capital, and a total child-rearing cost $\alpha n y_{t-1}$, where $\alpha > 0$ represents the child rearing unit cost (time, foregone earnings, direct outlays). The budget constraint is:

$$y_{t-1} = C_{t-1} + nI_{t-1} + \alpha n y_{t-1} \tag{1}$$

so that disposable income after child-rearing costs is $y_{t-1}(1-\alpha n)$. Human capital is produced as $h_t = \theta \log(I_{t-1}) + u_t$ where $\theta > 0$ is the school productivity parameter and $u_t = \mu + \epsilon_t + \tau_t$ is a child ability component with $\mu > 0$, ϵ_t iid mean zero, and τ_t mean zero with $cov(h_{t-1}, \tau_t) \geq 0$ (endowment heritability). Human capital translates into income as $\log(y_t) = h_t$.

Parents maximize $U_{t-1} = \log(C_{t-1}) + \log(n \cdot y_t)$, valuing total offspring income.[1] Substituting the budget constraint and the human capital production function, the parent's problem is:

$$\max_{I_{t-1}} \log(y_{t-1}(1-\alpha n) - nI_{t-1}) + \log(n) + \theta \log(I_{t-1}) + u_t$$

The first-order condition $\frac{n}{y_{t-1}(1-\alpha n) - nI_{t-1}} = \frac{\theta}{I_{t-1}}$ gives $\theta y_{t-1}(1-\alpha n) = nI_{t-1}(1+\theta)$, so the

[1]The specification $\log(ny_t) = \log(n) + \log(y_t)$ is a standard simplification in the QQ literature (Galor and Moav, 2002; De la Croix and Doepke, 2003) producing closed-form solutions for the fertility choice and ensures scale invariance: doubling all incomes does not change optimal fertility.

optimal per-child investment is:

$$I_{t-1}^{*} = \frac{1-\alpha n}{n}\frac{\theta}{1+\theta}y_{t-1} \tag{2}$$

This has the standard Becker–Tomes–Solon structure adjusted by the multiplicative factor $\frac{1-\alpha n}{n}$: the propensity to invest $\frac{1-\alpha n}{n}\frac{\theta}{1+\theta}$ is concave in θ, and total investment is divided equally among n children. The new element is the multiplicative factor $(1-\alpha n)$: child-rearing costs reduce the income available for educational investment.

Benchmark: equal school opportunity. Substituting the optimal investment into the human capital equation gives human capital per child:

$$h_t = \theta\log\left(\frac{1-\alpha n}{n}\frac{\theta}{1+\theta}y_{t-1}\right) + u_t = g(\theta) + \theta\log(\frac{1-\alpha n}{n}) + \theta\log(y_{t-1}) + u_t$$

where $g(\theta) = \theta\log(\frac{\theta}{1+\theta}) < 0$, $g'(\theta) < 0$ and $g''(\theta) > 0$ for all $\theta \in (0,1)$. Both fertility n and child-rearing costs α enter in human capital through additive intercept terms: $\theta\log(\frac{1-\alpha n}{n})$. Since it is a constant term (given exogenous n and α), the income equation $\log(y_t) = g(\theta) + \theta\log(\frac{1-\alpha n}{n}) + \theta\log(y_{t-1}) + u_t$ has slope θ independent of n and α.

Lemma 1. *Under equal school opportunity, with fertility n and child-rearing costs α homogeneous across families, the intergenerational elasticity of income*

$$\beta = \theta + \mathrm{cov}(\log y_{t-1}, \tau_t)/\mathrm{Var}(\log y_{t-1})$$

is independent of both n and α. A decline in fertility from n to $n' < n$ (at fixed α) raises average human capital per child by $\theta\log(n/n') + \theta\log\big((1-\alpha n')/(1-\alpha n)\big) > 0$, while a decline in α from α to $\alpha' < \alpha$ (at fixed n) raises it by $\theta\log\big((1-\alpha' n)/(1-\alpha n)\big) > 0$.

Fewer children raise investment per child by freeing up resources, while higher child-rearing costs reduce it by absorbing income. But both effects are pure level shifts that benefit (or harm) all families proportionally. The transmission slope θ (the rate at which parental income advantages are passed to the next generation) is unchanged by either n or α.

Unequal school opportunity. We account for both unequal school quality and an unequal probability of access to the best schools (school segregation) using a bi-variate Markov regime switch model. School inequality relates to the disparity in school quality across neighborhoods. With probability $p \geq 1/2$ a child attends the school matching her income group (above-median $\rightarrow$ high-quality, below-median $\rightarrow$ low-quality); with the complementary probability $1-p$ she attends the opposite type. There is full school segregation around the median income family when $p = 1$ and full social mix when $p = 1/2$. We normalize school quality so that the *child-weighted* average is constant at θ, regardless of possible differential fertility. Let π_H and $\pi_L = 1 - \pi_H$ denote the population shares of children from above-median and below-median families, respectively. School qualities are:

$$\theta^H = \theta + \frac{\kappa}{\pi_H}, \qquad \theta^L = \theta - \frac{\kappa}{\pi_L} \tag{3}$$

so that $\pi_H\theta^H + \pi_L\theta^L = \theta$ by construction. The parameter $\kappa > 0$ governs the overall degree of school inequality: a higher κ widens the quality gap between good and bad schools.[2]

A broader interpretation of θ^s. Although we describe the regime-switch model in terms of "school quality" to keep the language concrete, the parameter θ^s should be read as the *local return to parental investment*, an environment-specific elasticity that captures everything in a child's growing-up context that mediates how parental dollars translate into adult human capital. School quality is one component, but θ^s also embeds the social capital, peer composition, role-model exposure, neighbourhood safety, and informal networks of the community in which a child grows up. Chetty and Hendren (2018) provide causal evidence that these neighbourhood-level inputs (not just school quality) have substantial effects on children's adult outcomes, and that the effects accumulate roughly linearly in years of childhood exposure. The empirical literature on equality of opportunity (e.g. Chetty *et al.*, 2014; Corak, 2013) uses the term "opportunity" in this broad sense as well. Within our model, p is then the probability that an above-median family accesses a high-opportunity community (a good neighbourhood that bundles a good school with high social capital, etc.) rather than the narrower probability of attending a high-quality school. The full-segregation case $p = 1$ corresponds to perfect income sorting across communities; $p = \frac{1}{2}$ to a perfectly mixed neighbourhood composition. School

[2]Under equal fertility ($\pi_H = \pi_L = \frac{1}{2}$), this reduces to $\theta^H = \theta + 2\kappa$ and $\theta^L = \theta - 2\kappa$, a symmetric mean-preserving spread. When fertility is differential ($\pi_H < \frac{1}{2}$), both $\theta^H = \theta + \kappa/\pi_H$ and $\theta^L = \theta - \kappa/\pi_L$ shift upward as π_H falls and π_L rises, but the high-quality school rises further above θ than the low-quality school moves toward it, so the quality gap widens (holding the child-weighted average at θ).

policy (κ) and residential desegregation (p) are then two complementary levers, both operating on the broader opportunity gap that $\theta^H - \theta^L$ summarises. The model and all subsequent results are unchanged; the broader reading simply clarifies what the high-/low-type "quality" parameters represent in the data.

Given this bi-variate Markov regime switch model, each parent in generation $t-1$ will choose I_{t-1} conditional on the school quality available so as to maximize their preference over own consumption and child's income. Thus parental investment is $I^* = \frac{1-\alpha n}{n}\frac{\theta^s}{1+\theta^s}y_{t-1}$ for $\theta^s = \theta^L, \theta^H$. The distribution of parental investment conditional on school quality is

For $y_{t-1} \leq y_{t-1}^M$,

$$I_{t-1}^* = \begin{cases} \frac{1-\alpha n}{n}\frac{\theta^L}{1+\theta^L}y_{t-1}, & \text{with probability p.} \\ \frac{1-\alpha n}{n}\frac{\theta^H}{1+\theta^H}y_{t-1}, & \text{with probability 1-p.} \end{cases} \tag{4}$$

For $y_{t-1} > y_{t-1}^M$,

$$I_{t-1}^* = \begin{cases} \frac{1-\alpha n}{n}\frac{\theta^L}{1+\theta^L}y_{t-1}, & \text{with probability 1-p.} \\ \frac{1-\alpha n}{n}\frac{\theta^H}{1+\theta^H}y_{t-1}, & \text{with probability p.} \end{cases} \tag{5}$$

Let $\omega_h(\theta) = p\frac{\theta^H}{1+\theta^H} + (1-p)\frac{\theta^L}{1+\theta^L}$ and $\omega_l(\theta) = p\frac{\theta^L}{1+\theta^L} + (1-p)\frac{\theta^H}{1+\theta^H}$ denote the expected propensities to invest in children for the above-median and below-median income group, respectively. Then, $\omega_h(\theta) \geq \omega_l(\theta)$ for $p \geq 1/2$: the high income families display a higher propensity to invest in children if they are more likely to attend high-quality schools. This involves a complementarity between parental investment and school quality. Unequal school opportunity breaks the standard Becker–Tomes–Solon model which assumes that parents have a uniform tendency to invest in their children. When schools are of higher quality, parents are willing to invest more. High-income families increase their investment in education partly because their children attend better schools, offering in turn higher returns to those investments, while low-income parents invest less because of poor-quality schooling. This increases the parenting gap, creating a "double down" on inequality. Note also that

$$\frac{\partial \omega_h(\theta)}{\partial p} = -\frac{\partial \omega_l(\theta)}{\partial p} = \frac{\theta^H}{1+\theta^H} - \frac{\theta^L}{1+\theta^L} > 0$$

which means that unequal opportunity increases the investment gap between high income and low income families.

This investment gap prompts our analysis of the dynamic pattern of parental investment, studying the effects of unequal school opportunity on average parental investment, human capital, and intergenerational mobility.

2.1 Average parental investment

We begin with average parental investment. In the regime switch model the average investment under unequal school opportunity is given by:[3]

$$E[I_s] = \frac{1-\alpha n}{2n}\left(\omega_h(\theta)\right) E[y_{t-1}|y_{t-1} > y^M_{t-1}] + \frac{1-\alpha n}{2n}\left(\omega_l(\theta)\right) E[y_{t-1}|y_{t-1} \leq y^M_{t-1}]$$

Let $\gamma = \frac{E[y_{t-1}|y_{t-1} > y^M]}{E[y_{t-1}]}$ represent the between-group income inequality . For γ close to 1, between-group income inequality is minimum: both income groups have equal shares of total income. For γ close to 2, the between-group income inequality is maximum: total income is concentrated in the high income group. Hence $\gamma \in (1, 2)$. Using this between-group income inequality, we can rewrite the average investment under unequal school opportunity as

$$E[I_s] = \frac{1-\alpha n}{n}[\gamma\omega_h(\theta) + (2-\gamma)\omega_l(\theta)]\frac{E[y_{t-1}]}{2}$$

Proposition 1. *Under exogenous fertility rate $n > 0$ and child care cost $\alpha > 0$, unequal school opportunity with social segregation $p \geq 1/2$, school inequality $\kappa > 0$, and between-group income inequality $\gamma > 1$:*

(i) Average investment is increasing in school segregation p and in between-group income inequality γ.

(ii) Average investment is increasing in school inequality κ if and only if $p \geq p^0$ with

$$p^0 = \frac{1}{2} + \frac{2\kappa(1+\theta)}{(\gamma-1)\big((1+\theta)^2 + 4\kappa^2\big)}.$$

(iii) Average investment is decreasing in fertility n and in the child-care cost α.

[3]We drop the time index of the investment variable to ease notation and when there is no risk of confusion.

The interpretation of part (ii) on school inequality is as follows. On one hand there is the complementarity between investment and school quality. High income families invest a higher proportion of income in education because there are more likely to attend high-quality schools, whereas low-income families more likely to attend low-quality school will invest a smaller proportion of income. This effect tends to increase average investment. On the other hand, there are decreasing returns: the propensity to invest in children is a concave function of the school quality. Hence, a mean preserving increase in school inequality (higher κ) will decrease the average propensity to invest. This effect tends to lower average investment. When school segregation is sufficiently high $p > p^0$, the (first) positive effect dominates the (second) negative effect and school inequality increases average investment.

2.2 Average human capital

Given the incidence of unequal school opportunity on parental investment, what is the impact on human capital? A central effect is the efficiency gain (in terms of increased skills and knowledge) stemming from positive assortative matching of high investment in high quality schools. Under unequal school quality $\theta^s = \theta^L, \theta^H$, optimal per-child investment is $I^* = \frac{1-\alpha n}{n} \frac{\theta^s}{1+\theta^s} y_{t-1}$, and human capital is:

$$h_t = \theta^s \log(\frac{1-\alpha n}{n}) + g(\theta^s) + \theta^s \log(y_{t-1}) + u_t$$

where $g(\theta) = \theta \log(\frac{\theta}{1+\theta}) < 0$. The function $g(\theta)$ is negative, decreasing and convex, with $g'(\theta) < 0$ and $g''(\theta) > 0$ for all $\theta \in (0, 1)$.. The fertility parameters n and α enter in the human capital equation only through additive intercept terms.

The average human capital under unequal school opportunity is:

$$E[h_{t,s}] = \theta \log(\frac{1-\alpha n}{n}) + \frac{1}{2}\left(g(\theta^H) + g(\theta^L)\right) + \frac{1}{2}(p\theta^H + (1-p)\theta^L)\phi E[\log(y_{t-1})]$$

$$+\frac{1}{2}(p\theta^L + (1-p)\theta^H)(2-\phi)E[\log(y_{t-1})] + \mu$$

where $\phi = \frac{E[\log(y_{t-1})|\log(y_{t-1})>\log(y^M)]}{E[\log(y_{t-1})]} \in (1, 2)$ is the between-group log-income inequality. The fertility parameters n and α enter as an additive constant $\log(\frac{1-\alpha n}{n})$ in the average human capital, with average school quality coefficient $\theta = \frac{1}{2}(\theta^H + \theta^L)$ (note that under exogenous symmetric fertility $\pi_H = \pi_L = \frac{1}{2}$, the child-weighted mean is θ). The comparative static properties of school inequality, school segregation, between-group income inequality, and fertility parameters on average human capital are summarized in the following proposition:

Proposition 2. *(i) For any $\kappa > 0$ and $\phi > 1$, average human capital is increasing function in school segregation p; (ii) For any $\kappa > 0$ and $p > 1/2$, average human capital is increasing in between-group log-income inequality ϕ; (iii) For any $\phi > 1$ and $p > 1/2$, average human capital is increasing in school inequality κ; (iv) Average human capital is decreasing in fertility n, and child care cost α regardless of school inequality κ, segregation p, and income inequality ϕ.*

Parts (i)-(iii) suggest that average human capital always increases with unequal school opportunity because of the complementarity between school productivity and parental investment that shapes human capital. High-income families invest more in education because their children attend better schools, offering in turn higher returns to those investments, while low-income parents invest less because of poor-quality schooling, offering in turn lower returns to those investments. Part (iv) reflects the family size cost on human capital. Higher child-rearing costs reduce the income available for investment, and a larger family dilutes parental investment over more children.

2.3 Intergenerational elasticity

Under unequal school opportunity, the regime switch model involves multiple auto-regressive equations that characterize the dynamics of income transmission in different regimes.

Recall that $g(\theta) = \theta \log(\frac{\theta}{1+\theta})$, and that ϕ is the between-group log income inequality. Since $\log(y) = h$, it follows that ϕ is equivalent to the between-group human capital inequality. Substituting optimal parental investment into the earnings equation for the regime switch model gives:

for $y_{t-1} \leq y_{t-1}^M$

$$\log(y_t) = (p\theta^L + (1-p)\theta^H)\log(\frac{1-\alpha n}{n}) + pg(\theta^L) + (1-p)g(\theta^H) + (p\theta^L + (1-p)\theta^H)\log(y_{t-1}) + u_t$$

for $y_{t-1} > y^M_{t-1}$

$$\log(y_t) = ((1-p)\theta^L + p\theta^H)\log(\frac{1-\alpha n}{n}) + (1-p)g(\theta^L) + pg(\theta^H) + ((1-p)\theta^L + p\theta^H)\log(y_{t-1}) + u_t$$

Note that the within-group slope coefficients $(p\theta^L + (1-p)\theta^H)$ and $((1-p)\theta^L + p\theta^H)$ are independent of fertility parameters n and α. The fertility parameters change the intercept of the income transmission in each regime, with different intercepts for the two income groups. This income transmission with different intercepts and slopes below and above median income, can be rewritten in a single encompassing equation:

$$\log(y_t) = (p\theta^L + (1-p)\theta^H)\log(\frac{1-\alpha n}{n}) + pg(\theta^L) + (1-p)g(\theta^H) + D(p)(g(\theta^H) - g(\theta^L))Z_{t-1}$$

$$+D(p)(\theta^H - \theta^L)\log(\frac{1-\alpha n}{n})Z_{t-1} + (p\theta^L + (1-p)\theta^H)\log(y_{t-1}) + D(p)(\theta^H - \theta^L)\log(y_{t-1})Z_{t-1} + u_t$$

where $D(p) = 2p - 1$ is the dissimilarity index, and $Z_{t-1} = \mathbf{1}[y_{t-1} > y^M]$ is the income-group indicator. The slope terms $(p\theta^L + (1-p)\theta^H)\log(y_{t-1})$ and $D(p)(\theta^H - \theta^L)\log(y_{t-1})Z_{t-1}$ are independent of fertility parameters. The fertility terms $\log(\frac{1-\alpha n}{n})$ enter only the intercept.

The key results of this model with exogenous fertility and unequal school opportunity are the following. Using an omitted-variable-bias decomposition (following Solon, 2018), the estimated IGE β_s under our regime switch model is:

$$\begin{aligned}\beta_s \quad = \quad & D(p)[(\theta^H - \theta^L)\log(\frac{1-\alpha n}{n}) + g(\theta^H) - g(\theta^L)]\frac{cov(Z_{t-1}, \log(y_{t-1}))}{Var(\log(y_{t-1}))} + (p\theta^L + (1-p)\theta^H) \\ & +D(p)(\theta^H - \theta^L)\frac{cov(\log(y_{t-1})Z_{t-1}, \log(y_{t-1}))}{Var(\log(y_{t-1}))} + \frac{cov(\log(y_{t-1}), \tau_t)}{Var(\log(y_{t-1}))}\end{aligned}$$

The intercept terms involving the fertility parameters n and α shift the first line but do not affect the slope terms (second line).

Proposition 3. *Consider the exogenous-fertility regime with segregation rate $p \geq 1/2$, school inequality $\kappa > 0$, fertility rate $n \geq 1$, child-rearing cost $\alpha > 0$ with $\alpha n < 1$, and inter-group log-income inequality $\phi > 1$. Then:*

(i) Assume the within-group variance condition $\mathrm{Var}(\log y_{t-1} \,|\, y > y^M) \geq \mathrm{Var}(\log y_{t-1} \,|\, y < y^M)$.

Then β_s is increasing in p and κ whenever

$$E[\log(y_{t-1})] + \log\Big(\frac{1-\alpha n}{n}\Big) + \tfrac{1}{2}\big(g'(\theta^H) + g'(\theta^L)\big) > 0. \tag{6}$$

Under log-normal parental income $\log(y_{t-1}) \sim N(\mu, \sigma^2)$*, the within-group variance condition holds with equality and (6) becomes* $\mu + \log\left(\frac{1-\alpha n}{n}\right) + \frac{1}{2}\big(g'(\theta^H) + g'(\theta^L)\big) > 0.$

(ii) The intergenerational elasticity of income β_s is decreasing in the fertility rate n and the child-rearing cost α for all $p > 1/2$ and $\kappa > 0$.

Proof. See Appendix B. □

The key result of Part (i) is that when there is more variation in parental income at the top of the distribution than at the bottom (which holds in almost every country), the increase in income persistence among high-income families due to better school quality, together with the widening gap between top and bottom, dominates the decrease in persistence among low-income families. This decreases intergenerational mobility overall. School inequality pits steeper income transmission at the top against a negative *intercept* channel that bundles two forces working in the same direction: the fertility wedge $\log((1-\alpha n)/n) < 0$, which scales with θ^s and so pulls the high group's intercept further down, and Becker–Tomes decreasing returns $g'(\theta) < 0$, which contributes a smaller intercept at the top. Condition (6) is the requirement that mean log-income μ be large enough ($\mu > |A| + \frac{1}{2}|g'(\theta^H) + g'(\theta^L)|$) for the slope channel (which scales with $E[\log y_{t-1}]$ through the OVB moment) to dominate the combined intercept channel.[4]

An irrelevance result. Under *equal* school opportunity ($\kappa = 0$), the pooled IGE equals $\theta + cov(\tau, \log y)/Var(\log y)$, which is also independent of n and α: this is the Lemma 1 irrelevance result. A uniform fall in fertility (or in child care cost) frees resources for every family in the same proportion. Per-child investment scales up by the same factor across the income distribution

[4]When could (6) fail? Three corners are conceivable: (i) a very compressed log-income distribution, μ small relative to $|A|$, implausible for any realistic labour-income calibration, where μ is on the order of log(median earnings)$\approx$ 7–10; (ii) very high fertility, n large, which drives $A = \log((1-\alpha n)/n)$ sharply negative; (iii) a very low average school quality θ, where $g'(\theta) = \log(\theta/(1+\theta)) + 1/(1+\theta)$ blows up in magnitude (it $\to -\infty$ as $\theta \to 0$). At the Section 5 baseline ($\theta = 0.27$, $\alpha = 0.125$, $n = 2.92$, $E[\log y_{t-1}] = 10$, $\kappa = 0.006$) the LHS of (6) equals $10 - 1.53 - 0.76 = 7.71 > 0$, and it remains above 6 across the entire $(\alpha, n) \in [0.05, 0.25] \times [1.5, 3.5]$ grid we explore in robustness checks, comfortably away from all three failure corners.

given the uniform return on parental investment. Therefore the ratio of rich-child to poor-child investment is unchanged which is what drives the IGE (the slope of income transmission is the same across the) not the level.Under *unequal* school opportunity ($\kappa > 0$), however, the pooled IGE includes an OVB intercept correction that depends on $\log\left(\frac{1-\alpha n}{n}\right)$, and Part (ii) shows this correction makes β_s *decrease* in both n and α. The reason is the following. When returns to parental investment differ across communities, the same demographic shock has unequal effects. Resources freed by lower fertility or child-rearing costs generate larger human-capital gains in high-opportunity communities. Because richer families are disproportionately concentrated in these communities, the shock widens differences in children's human capital, thereby increasing the IGE.

2.4 Beyond the fertility gap: the demographic channel

Proposition 3 Part (ii) is, on its own, a substantive result. It says that under unequal opportunity ($\kappa > 0$, $p > 1/2$), the IGE rises whenever fertility falls or child-rearing costs fall, even when fertility is *uniform across income groups*. No fertility *gap* is required; what is required is a community *gap*. This subsection makes the point self-contained: it isolates the demographic channel from the QQ amplification of Sections 3–4, derives the comparative statics explicitly for both n and α, and shows in calibrated simulations that the impact of a fertility decline or a childcare-cost reduction scales up in the community gap.

The demographic comparative statics. Reading off the proof of Proposition 3 (ii) gives

$$\frac{\partial \beta_s}{\partial n} = -\, D(p)\,(\theta^H - \theta^L) \cdot \frac{1}{n(1-\alpha n)} \cdot \frac{cov(Z_{t-1}, \log y_{t-1})}{Var(\log y_{t-1})}, \tag{7}$$

$$\frac{\partial \beta_s}{\partial \alpha} = -\, D(p)\,(\theta^H - \theta^L) \cdot \frac{n}{1-\alpha n} \cdot \frac{cov(Z_{t-1}, \log y_{t-1})}{Var(\log y_{t-1})}. \tag{8}$$

Both derivatives are negative, a uniform fall in fertility, or a fall in child-rearing costs, raises the IGE. Their *magnitudes share a common amplifier*: the dissimilarity-weighted community gap $D(p)(\theta^H - \theta^L) = D(p)\kappa/(\pi_H \pi_L)$. The same community-inequality term that scales the slope channel in Part (i) also scales the demographic channel in Part (ii). When the community gap is zero, demographic forces are silent for the IGE; when it is large, every percentage-point reduction in n or α is amplified accordingly. Note also that $|\partial \beta_s / \partial \alpha| / |\partial \beta_s / \partial n| = n^2$, so at

$n \approx 2$ the IGE is roughly four times as sensitive to a one-unit change in α as to a one-unit change in n.

Joint movement. Over the past half-century the two demographic forces have moved together: fertility has fallen (from $n \approx 3.5$ to $n \approx 1.8$ in the OECD) and the policy-relevant cost of childcare per child has fallen too (parental leave, public childcare, in-kind transfers; e.g. OECD, 2019). Both reductions push the IGE in the same direction. The joint move $(\alpha, n) \mapsto (\alpha', n')$ with $\alpha' < \alpha$ and $n' < n$ raises β_s by approximately

$$\Delta\beta_s \approx -\frac{\partial\beta_s}{\partial\alpha}(\alpha - \alpha') - \frac{\partial\beta_s}{\partial n}(n - n'),$$

each term being amplified by the same community-gap factor $D(p)\kappa/(\pi_H\pi_L)$ and so reinforced when communities are unequal. *At fixed fertility, childcare-subsidy policies — often discussed as equity-enhancing — may raise income persistence in environments with substantial community inequality*, a perhaps counter-intuitive corollary that the literature on the new economics of fertility (Doepke *et al.*, 2023) has not yet drawn. Section 5.5 shows that this direction reverses once fertility responds endogenously: with endogenous fertility the same subsidies reduce persistence, while a rise in the effective cost of children — as emphasised by the intensive-parenting literature — raises it.

Calibrated illustration. Figures 1–3 report the three comparative statics for three (p, κ) cells at the US calibration ($\theta = 0.27$, $\alpha = 0.125$, $E[\log y_{t-1}] = 10$, $\sigma^2 = 0.65$, coherent $\phi = 1 + \frac{\sigma}{\mu}\sqrt{\frac{2}{\pi}} \approx 1.06$, meaning that the average log income of the upper half is about 6% higher than the population average log income). The four cells collapse to three distinct curves because the segregation–inequality combinations ($p = 1, \kappa = 0.006$) and ($p = 0.7, \kappa = 0.02$) share (approximately) the same $D(p)\kappa$ and so deliver nearly identical IGE paths, a clean illustration of the substitution between p and κ in the formula. Figure 1 shows that a uniform fertility decline from $n = 4$ to $n = 1.5$ raises β_s by $+0.058$ in the high-community-inequality cell ($\kappa = 0.02$, $p = 1$), by $+0.017$ in the mixed cell ($\kappa = 0.006$, $p = 1$), and by only $+0.007$ in the low-community-inequality cell ($\kappa = 0.006$, $p = 0.7$), a roughly $8\times$ amplification of the same demographic shock by community inequality. Figure 2 shows the analogous (smaller in absolute terms but qualitatively identical) pattern for a childcare-cost reduction from $\alpha = 0.20$ to $\alpha = 0.05$ at fixed $n = 2$.

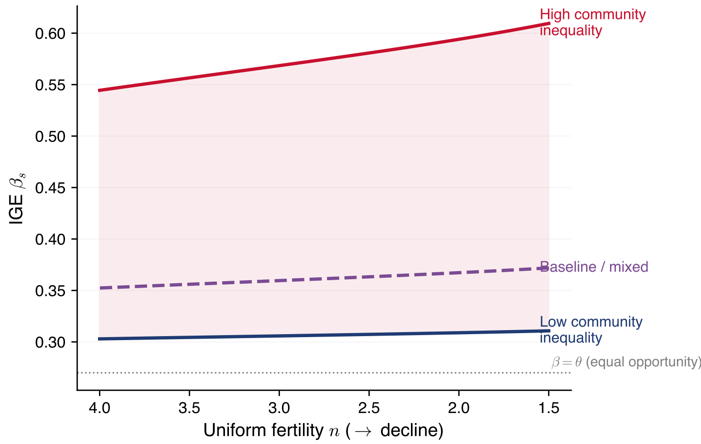

Figure 1: Beyond the fertility gap: community inequality amplifies the secular fertility decline on the IGE. β_s as a function of *uniform* fertility n (x-axis inverted so left-to-right reads as the decline), at three (p, κ) cells (high, baseline, low). Childcare cost $\alpha = 0.125$ fixed.

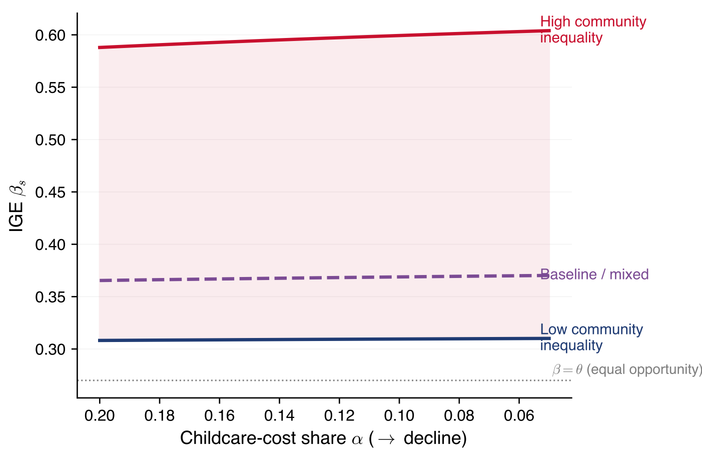

Figure 2: Childcare-cost comparative statics. β_s as a function of α at fixed uniform fertility $n = 2$. A reduction in α (left-to-right) raises the IGE, by an amount that scales with $D(p)\kappa$, the same community-gap amplifier as the fertility-decline channel.

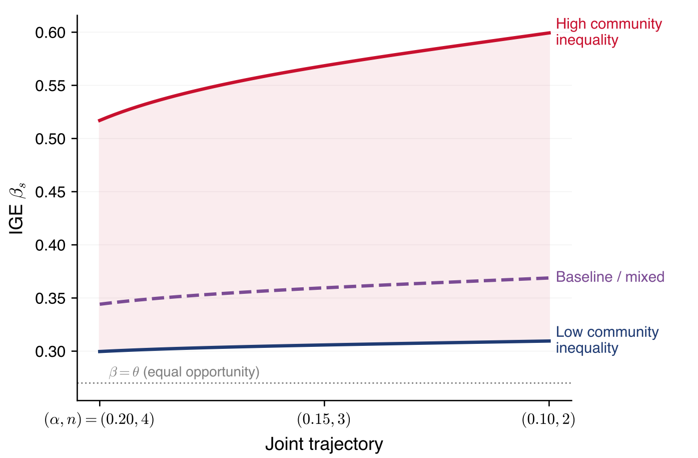

Figure 3: Joint trajectory (α, n): $(0.20, 4) \rightarrow (0.10, 2)$, representative of the half-century joint decline of fertility and childcare cost in OECD economies. Both forces push the IGE in the same direction; their joint effect is amplified by community inequality.

Figure 3 traces the IGE along a joint (α, n) trajectory from $(0.20, 4)$ to $(0.10, 2)$: in the high-community-inequality cell, β_s rises by $+0.082$ along this empirically motivated path, about $8\times$ the response in the low-community-inequality cell ($+0.010$).

3 Endogenous fertility and the QQ trade-off

We now introduce the Beckerian constraint with a child quality quantity tradeoff combined to the Galor and Moav (2002) approach linking this tradeoff to the rise in education. Our central idea is that fertility is endogenously linked to the returns on education. The works of Galor and Moav (2002) and Cinnirella and Streb (2017) suggest that the negative impact of education on fertility is stronger where the returns to education were higher. We will consider the following mechanism: high-income families have access to high-opportunity community and have fewer children; whereas low-income families have access to low-opportunity community and have more children. The fertility gap is stemming from the community gap in the return to investment.

QQ fertility trade off Under full segregation ($p = 1$), above-median families face θ^H and below-median families face θ^L. Parents jointly choose both the number of children n and investment per child I.[5] Substituting the budget constraint and the human capital and earnings equations, a parent with access to school quality $\theta^s \in (\theta^L, \theta^H)$ maximises:

$$\max_{I,n} \log(y_{t-1}(1 - \alpha n) - nI) + \log(n) + \theta^s \log(I) + u_t \tag{9}$$

The first-order condition for I gives:

$$I^*(\theta^s, n) = \frac{\theta^s}{1 + \theta^s} \cdot \frac{y_{t-1}(1 - \alpha n)}{n} \tag{10}$$

This is the exogenous-fertility investment : parents divide available (net of child cost) income across n children with propensity $\frac{\theta^s}{1+\theta^s}$, which is concave in θ^s as in the exogenous case.

Fertility choice. Substituting I^* back into the objective function, consumption simplifies to $C^* = y_{t-1}(1 - \alpha n)/(1 + \theta^s)$, and after rearrangement, the indirect utility can be written as:

$$V(n) = (1 + \theta^s) \log(y_{t-1}(1 - \alpha n)) + (1 - \theta^s) \log(n) + K(\theta^s)$$

[5]Under partial segregation ($p < 1$), fertility is linked to expected opportunity across communities: see Section 4.

where $K(\theta^s)$ collects terms independent of n. Solving the first-order condition gives:[6]

$$n^* = \frac{1-\theta^s}{2\alpha} \tag{11}$$

Plugging back this optimal fertility rate n^* into investment I^* gives

$$I^* = (\frac{\alpha\theta^s}{1-\theta^s})y_{t-1} \tag{12}$$

Compared to exogenous fertility, the child cost α now has the *opposite* effect on parental investment. Under constant fertility, higher α shifts resources away from education. Under variable fertility, higher α leads parents to substitute away from quantity toward quality, they have fewer children, but invest more per child. More fundamentally, the curvature of the propensity to invest *flips*:

	Exogenous fertility	**Endogenous fertility** ($p = 1$)
Per-child propensity I^*/y_{t-1}	$\frac{1-\alpha n}{n} \cdot \frac{\theta^s}{1+\theta^s}$	$\frac{\alpha\theta^s}{1-\theta^s}$
Curvature in θ^s	**concave**	**convex**
Mean-preserving spread of θ^s	*lowers* average investment	*raises* average investment

This is a novel insight : with constant fertility, school inequality (a mean-preserving spread of school quality) reduces average investment because of decreasing returns. With variable fertility, the fertility decline at the top (induced by higher returns to education) concentrates resources so strongly that, holding everything else constant, the same mean-preserving spread *raises* average investment. Parents with fewer children save on child cost and invest more per child, and this concentration accelerates with school quality. This is the mechanism through which wealthier parents gain an advantage over poorer parents in their children's educational attainment.

Under full segregation, above-median families face higher return to education $\theta^H = \theta + \kappa/\pi_H$ and below-median face lower return $\theta^L = \theta - \kappa/\pi_L$, inducing a fertility gap across families:

$$n_H^* = \frac{1-\theta-\kappa/\pi_H}{2\alpha}, \qquad n_L^* = \frac{1-\theta+\kappa/\pi_L}{2\alpha}$$

[6]The second-order condition $V''(n) < 0$ is satisfied.

Equilibrium. The population shares π_H and $\pi_L = 1 - \pi_H$ are themselves determined by the fertility choices. Each above-median family has n_H^* children and each below-median family has n_L^* children (with equal mass $\frac{1}{2}$ of families in each group), so $\pi_H = n_H^*/(n_H^* + n_L^*)$. Since θ^H and θ^L depend on π_H through the normalization $\theta^H = \theta + \kappa/\pi_H$, the equilibrium is a fixed point: the population shares implied by optimal fertility must be consistent with the school qualities that generated those fertility choices. Individual families take θ^H and θ^L as given (the large-population assumption), so the fixed point is:

$$\pi_H = \frac{1 - \theta - \kappa/\pi_H}{(1 - \theta - \kappa/\pi_H) + (1 - \theta + \kappa/(1 - \pi_H))} \tag{13}$$

A polynomial form of the fixed-point equation. Writing $T \equiv 1 - \theta$ and $r \equiv \pi_H$, cross-multiplying (13) and clearing denominators (full derivation in Appendix A.1) yields the cubic

$$2T\,r^3 - (3T + 2\kappa)\,r^2 + (T + 2\kappa)\,r - \kappa = 0. \tag{14}$$

The change of variable $d = 2r - 1 \in (-1, 1)$ (representing the fertility dissimilarity) recasts (14) in the more revealing form

$$T\,d\,(d^2 - 1) = 2\kappa\,(d^2 + 1). \tag{15}$$

This formulation gives the existence-and-sign result:

Proposition 4 (Strict-interior equilibrium)**.** *For any $\theta \in (0, 1)$ and $\kappa/(1 - \theta) \le 0.15$, equation (15) admits a strict-interior root $d^* \in (-1 + 2\kappa/(1 - \theta), 0)$, equivalently $\pi_H^* \in (\kappa/(1 - \theta), 1/2)$, unique on the branch continuous from $d^* = 0$ at $\kappa = 0$ (the "near-symmetric" branch, to which the small-κ expansion below applies).*[7] *In particular,*

$$\pi_H^* < \tfrac{1}{2} \qquad \textit{whenever } \kappa > 0. \tag{16}$$

Proof. See Appendix B. □

[7] A second interior root lies near the lower boundary, $\pi_H \approx \kappa/(1 - \theta)$, corresponding to a small-share equilibrium with $n_H^* \to 0$ as $\kappa \to 0$. We do not select it: the near-symmetric branch is the one continuous with the equal-opportunity benchmark ($\pi_H = 1/2$ at $\kappa = 0$) and the empirically relevant regime.

Small-κ approximation. For $2\kappa/(1-\theta) \ll 1$, expanding (15) around $d = 0$ gives

$$\pi_H^* = \frac{1}{2} - \frac{\kappa}{1-\theta} + O\left(\frac{\kappa^3}{(1-\theta)^3}\right). \tag{17}$$

The leading term $\pi_H^* \approx \frac{1}{2} - \kappa/(1-\theta)$ is highly accurate in the empirically relevant regime: at the Section 5 baseline ($\theta = 0.27$, $\kappa = 0.02$), the exact root of (14) is $\pi_H^* = 0.47244$ versus a linear approximation of 0.47260, an error of 1.6×10^{-4} consistent with the predicted $O(\kappa^3)$ remainder. Figure 4 plots $\pi_H^*(\kappa)$ for two values of θ: the slope is steeper at higher θ, so the composition shift induced by a given amount of school inequality is larger when the average return to education is higher.

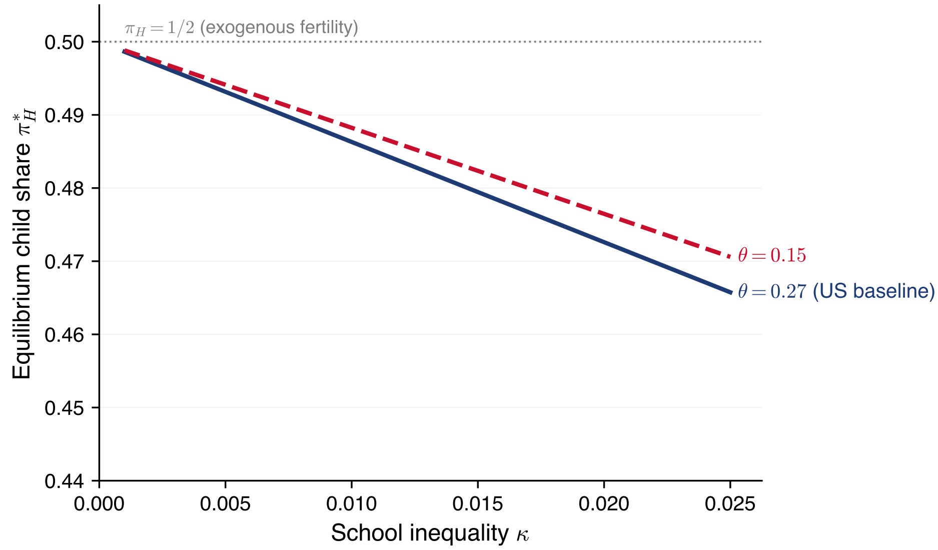

Figure 4: Equilibrium child share of above-median families, π_H^*, as a function of school inequality κ, for two values of average school quality θ, using the linear approximation $\pi_H^* \approx 1/2 - \kappa/(1-\theta)$ from (17). Full segregation, $p = 1$.

The fertility gap is:

$$n_L^* - n_H^* = \frac{1}{2\alpha}\left(\frac{\kappa}{\pi_L} + \frac{\kappa}{\pi_H}\right) = \frac{\kappa}{2\alpha\pi_H\pi_L} \tag{18}$$

The school quality gap $\theta^H - \theta^L = \kappa/\pi_H + \kappa/\pi_L = \kappa/(\pi_H\pi_L)$ is increasing with the fertility gap $\pi^H - \pi^L$. This is the endogenous amplification: differential fertility widens the school quality gap (by shifting more children to the low-quality school), which in turn widens the fertility gap further. Total parental investment with school quality $\theta^s \in (\theta^L, \theta^H)$ is:

$$n^* I^* = \frac{\theta^s}{2} y_{t-1}$$

Note that α cancels out: total parental investment depends only on the return to education θ^s and the family income y_{t-1}. Higher child-rearing costs reduce optimal fertility $n^* = \frac{1-\theta^s}{2\alpha}$ and *raise* per-child investment $I^* = \frac{\alpha\theta^s}{1-\theta^s} y_{t-1}$; the two effects exactly offset in aggregate spending. Note also that school inequality triggers intensive parenting as average family investment $E(n^* I^*) = \frac{1}{2}\big[E(\theta^s)\, E(y_{t-1}) + Cov(\theta^s, y_{t-1})\big]$.

We require the following parameter restriction to ensure interior solutions.

Assumption 1. $\theta^H < 1$ and $\alpha < \frac{1-\theta^H}{2}$ in equilibrium, ensuring $n^* = \frac{1-\theta^s}{2\alpha} > 1$ for $\theta^s \in (\theta^L, \theta^H)$. A sufficient condition is $\kappa < \frac{1-\theta}{4}$ and $\alpha < \frac{1-\theta-2\kappa}{2}$.

3.1 Average human capital

We evaluate how community inequality affects average human capital, averaging across families on each side of the median income (equal weight $\frac{1}{2}$ per group). The child-perspective average, which reweights by equilibrium fertility, is discussed informally in the conclusion.

Under full segregation with endogenous fertility, substituting $n^* = (1-\theta^s)/(2\alpha)$ and I^* into the human capital equation gives, for a child from a family with school quality θ^s:

$$h_t = \widetilde{g}(\theta^s) + \theta^s \log(y_{t-1}) + u_t \tag{19}$$

where $\widetilde{g}(\theta^s) = \theta^s \log(\frac{\alpha\theta^s}{1-\theta^s})$. This is a clean expression: the intercept $\widetilde{g}(\theta^s)$ depends only on school quality and child-rearing costs, not on n separately (since n^* has been substituted out). The slope is θ^s, the same as under exogenous fertility.

Key property: convexity of $\widetilde{g}$. The function $\widetilde{g}$ is convex in θ^s, with $\widetilde{g}''(\theta^s) = \frac{1}{\theta^s(1-\theta^s)^2} > 0$, and this convexity is strictly greater than that of the exogenous-fertility intercept $g(\theta^s)$ (where $g''(\theta^s) = \frac{1}{\theta^s(1+\theta^s)^2}$). The greater convexity of $\widetilde{g}$ is the mathematical signature of the fertility amplification: the endogenous reduction in family size at high θ^s concentrates resources and amplifies the effect of school quality on human capital.

Under school equality ($\theta^s = \theta$ for all families), taking expectations:

$$E[h_t] = \widetilde{g}(\theta) + \theta E[\log(y_{t-1})] + \mu$$

Proposition 5. *Under endogenous fertility with equal school quality ($\kappa = 0$), there is no fertility gap ($n^* = \frac{1-\theta}{2\alpha}$, $\pi^L = \pi^H = 1/2$), and the average human capital is increasing with child-rearing cost α. A higher α reduces fertility and raises investment per child $I^* = (\frac{\alpha\theta}{1-\theta})y_{t-1}$ (QQ trade-off).*

Proof. See Appendix B. □

Under school inequality, above-median children have school quality $\theta^H(\pi_H) = \theta + \kappa/\pi_H$ and below-median children have $\theta^L(\pi_L) = \theta - \kappa/\pi_L$. The human capital of a child from income group j is:

$$h_t = \widetilde{g}(\theta_j^s) + \theta_j^s \log(y_{t-1}) + u_t$$

where $\theta_H^s = \theta^H(\pi_H)$ and $\theta_L^s = \theta^L(\pi_L)$ are the equilibrium school qualities. Expected human capital for above-median families is:

$$E[h_t|y_{t-1} > y^M] = \widetilde{g}(\theta^H) + \theta^H E[\log(y_{t-1})|y_{t-1} > y^M] + \mu$$

and for below-median families:

$$E[h_t|y_{t-1} \leq y^M] = \widetilde{g}(\theta^L) + \theta^L E[\log(y_{t-1})|y_{t-1} \leq y^M] + \mu$$

Using the between-group log-income inequality $\phi = \frac{E[\log(y_{t-1})|\log(y_{t-1})>\log(y^M)]}{E[\log(y_{t-1})]} \in (1, 2)$, the average human capital under segregation is:

$$E[h_{t,s}] = \frac{1}{2}\left[\widetilde{g}(\theta^H) + \widetilde{g}(\theta^L)\right] + \frac{1}{2}\left[\theta^H\phi + \theta^L(2-\phi)\right] E[\log(y_{t-1})] + \mu$$

Under school equality ($\kappa = 0$, so $\theta^H = \theta^L = \theta$ and $\pi_H = \frac{1}{2}$), this reduces to $E[h_t] = \widetilde{g}(\theta) + \theta E[\log(y_{t-1})] + \mu$. Let $\bar{\theta} = \frac{1}{2}(\theta^H + \theta^L)$ denote the (family) average school quality and $\theta = \pi_H\theta^H + \pi_L\theta^L$ the child-weighted one. Their gap admits a clean closed form:

$$\bar{\theta} - \theta = \left(\tfrac{1}{2} - \pi_H\right)\left(\theta^H - \theta^L\right) = \frac{(\frac{1}{2} - \pi_H)\kappa}{\pi_H\pi_L} \;\geq\; 0, \tag{20}$$

with strict inequality whenever $\pi_H < \frac{1}{2}$ (rich families have fewer children). The efficiency gain from school inequality decomposes into three *unambiguously positive* terms:

$$E[h_{t,s}] - E[h_t] = \underbrace{\tfrac{1}{2}\left[\widetilde{g}(\theta^H) + \widetilde{g}(\theta^L)\right] - \widetilde{g}(\theta)}_{\text{intercept gain}} + \underbrace{\tfrac{1}{2}(\theta^H - \theta^L)(\phi - 1)E[\log y_{t-1}]}_{\text{slope gain}>0} + \underbrace{\frac{(\frac{1}{2}-\pi_H)\kappa}{\pi_H \pi_L} E[\log y_{t-1}]}_{\text{composition uplift}>0} \tag{21}$$

There are three sources of "efficiency gain". The *intercept gain* arises without segregation ($p = 1/2$) because $\widetilde{g}$ is convex ($\widetilde{g}'' > 0$): when school quality is spread randomly around median income, the average investment increases (propensity to invest is convex in school quality under endogenous fertility). The *slope gain* arises when there is segregation ($p > 1/2$) because of assortative matching: school inequality creates a positive correlation between income and school quality, so that high-income parents—who have more resources to invest—face higher returns to their investment. This channel operates identically under exogenous and endogenous fertility. The *composition uplift* arises because $\pi_H < \frac{1}{2}$ under endogenous fertility (rich families have fewer children), so the average school quality $\bar{\theta} = \frac{\theta^H + \theta^L}{2}$ exceeds the (child-weighted) average θ: $\bar{\theta} - \theta = (\frac{1}{2} - \pi^H)(\theta^H - \theta^L) > 0$. This creates an additional positive level effect on average human capital.

We can decompose $\widetilde{g} = g + d$ where $d(\theta^s) = \widetilde{g}(\theta^s) - g(\theta^s) = \theta^s \log(\frac{\alpha(1+\theta^s)}{1-\theta^s})$ so as to isolate the fertility amplification (the sign of $d(\theta^s)$ depends on α; what matters below is the *comparison* of intercept gaps across regimes, not the level of d). The intercept gain then separates into:

$$\frac{1}{2}\left[\widetilde{g}(\theta^H) + \widetilde{g}(\theta^L)\right] - \widetilde{g}(\theta) = \underbrace{\frac{1}{2}\left[g(\theta^H) + g(\theta^L)\right] - g(\theta)}_{\text{exogenous-fertility part}} + \underbrace{\frac{1}{2}\left[d(\theta^H) + d(\theta^L)\right] - d(\theta)}_{\text{fertility amplification}}$$

The *fertility amplification* $\frac{1}{2}[d(\theta^H) + d(\theta^L)] - d(\theta)$ is unambiguously positive by strict convexity of d ($d'' = \widetilde{g}'' - g'' > 0$, Lemma 3).

Convexity flip. The economic force behind the amplification is a reversal in the curvature of the per-child investment propensity. Under exogenous fertility, $I^*/y_{t-1} = \frac{\theta^s}{n(1+\theta^s)}(1 - \alpha n)$ is concave in θ^s (through the propensity $\frac{\theta^s}{1+\theta^s}$). Thus average propensity to invest decreases with school inequality. Under endogenous fertility (with full segregation), the propensity to invest becomes:

$$\frac{I^*}{y_{t-1}} = \frac{\alpha\theta^s}{1 - \theta^s}$$

which is convex in θ^s. This convexity flip is the mathematical signature of the fertility amplification.

Proposition 6. *Under full segregation ($p = 1$), school inequality $\kappa > 0$ and income inequality $\phi > 1$ with endogenous fertility*

(i) For any $\kappa > 0$, and $\phi > 1$, the average human capital exceeds the equal-opportunity level. The slope gain and composition uplift are strictly positive. The intercept gain is strictly positive whenever the fertility gap is small enough and the leading small-κ coefficient $\widetilde{g}''(\theta) + 2\widetilde{g}'(\theta)/(1-\theta)$ is positive; this coefficient is strictly positive throughout the empirically plausible range of (θ, α) .

(ii) The average human capital is increasing with income inequality ϕ and school inequality κ for small enough κ.

Proof. See Appendix B. □

The comparative statics of the exogenous-fertility benchmark carry over qualitatively for the average: school inequality raises average human capital. The new element is that the fertility channel provides an *additional* source of efficiency gain (the fertility amplification and composition uplift) that is absent under exogenous fertility.

3.2 Intergenerational elasticity with the quantity–quality trade-off

We now derive the intergenerational elasticity under endogenous fertility. We show how the QQ trade-off operates in shaping (amplifying) income persistence.

Income transmission equations. Under full segregation, a child from income group $j \in \{H, L\}$ has income:

$$\log(y_t) = \widetilde{g}(\theta^j) + \theta^j \log(y_{t-1}) + u_t \tag{22}$$

where $\theta^H = \theta + \kappa/\pi_H$ and $\theta^L = \theta - \kappa/\pi_L$ are the equilibrium school qualities and $\widetilde{g}(\theta^j) = \theta^j \log(\frac{\alpha\theta^j}{1-\theta^j})$ embeds the endogenous fertility response. Under equal opportunity ($\kappa = 0$), both groups have the same slope θ and intercept $\widetilde{g}(\theta)$, and the IGE reduces to the benchmark $\beta = \theta + cov(\log(y_{t-1}), \tau_t)/Var(\log(y_{t-1}))$, independent of α.

Decomposing the IGE: within-group and between-group components. The regime switch model generates group-specific income transmission equations with different slopes and intercepts. The pooled IGE can therefore be decomposed into a within-group component (reflecting persistence within each income group) and a between-group component (reflecting the contribution of group mean differences to the overall regression slope).[8] Let $\beta_j = \theta^j$ denote the within-group slope for group $j \in \{H, L\}$, $V_j = Var(\log y_{t-1}|j)$ the within-group variance, $\bar{x}_j = E[\log y_{t-1}|j]$ the group mean, $\bar{x}$ the overall mean, and $\tilde{n}_j$ the group's share of the population. For the IGE (equal weight per family), $\tilde{n}_j = \frac{1}{2}$. Then:

$$\beta_s = \underbrace{\sum_j \tilde{n}_j \beta_j \frac{V_j}{V}}_{\text{A: within-group}} + \underbrace{\sum_j \tilde{n}_j \frac{(\bar{x}_j - \bar{x})(\bar{y}_j - \bar{y})}{V}}_{\text{B: between-group}} \tag{23}$$

where $V = Var_{\tilde{n}}(\log y_{t-1}) = \sum_j \tilde{n}_j V_j + \sum_j \tilde{n}_j (\bar{x}_j - \bar{x})^2$ is the $\tilde{n}$-weighted total variance. The within-group term (A) is a weighted average of the group-specific slopes $\beta_j = \theta^j$, scaled by each group's share of the total variance. The between-group term (B) captures the contribution of the gap in group means to the pooled regression coefficient.[9]

For analytical tractability, we assume that $\log(y_{t-1})$ is normally distributed within each income group, so that $V_H = V_L = V = \sigma^2(1 - 2/\pi)$ (where σ^2 is the unconditional variance). This assumption is maintained throughout the IGE analysis and substantially simplifies the decomposition.

Proposition 7. *Assume the small-fertility-gap condition of Lemma 4(i), (which holds at empirically plausible parameters) and a lognormal distribution of income*

(i) The IGE β_s is increasing in school inequality κ

(ii) Endogenous fertility amplifies the between-group component of β_s, and increases IGE β_s evaluated at the reference $n = (1 - \theta)/(2\alpha)$ (the endogenous fertility under equal opportunity),

Proof. See Appendix B. □

[8] This within/between decomposition is closely related to the group-specific persistence measure in Hertz (2008), who shows that accounting for between-group effects is essential when comparing intergenerational mobility across groups with different mean incomes.

[9] The decomposition follows from the law of total covariance: $Cov(\log y_t, \log y_{t-1}) = \sum_j \tilde{n}_j Cov_j(\log y_t, \log y_{t-1}) + \sum_j \tilde{n}_j (\bar{x}_j - \bar{x})(\bar{y}_j - \bar{y})$, divided by the total variance V. See Hertz (2008).

3.3 Quantifying the mechanism

To gauge the magnitude of the channels above, we calibrate the model to US data: $\theta = 0.27$ (the "frictionless" return to education such that, with school inequality, the model reproduces the empirical IGE), $\kappa = 0.006$ (modest school inequality), $\alpha = 0.125$ (the marginal per-child cost share, consistent with the USDA range of 15–20% once shared costs across children are netted out), $p = 1$, and parental income $\log y \sim N(\mu = 10, \sigma^2 = 0.65)$ (so the median-split between-group inequality index $\phi \approx 1.06$). Full parameter justifications and additional results are in Section 5; here we report the headline numbers.

At baseline, the equilibrium child share of above-median families falls from $\pi_H = 0.50$ (exogenous fertility) to $\pi_H = 0.49$, generating a modest fertility gap of about 0.10 children ($n_L^* = 2.97$ vs. $n_H^* = 2.87$). The IGE rises from $\beta_s^{exo} = 0.362$ to $\beta_s = 0.369$, in line with US empirical estimates around 0.35–0.40 (Chetty *et al.*, 2014). As community inequality rises, both the fertility gap and the amplification grow sharply: at $\kappa = 0.02$ — roughly the top quartile of US metropolitan-area segregation — the fertility gap widens to 0.32 children ($n_L^* = 3.07$ vs. $n_H^* = 2.75$) and the IGE reaches $\beta_s = 0.60$ (from an exogenous benchmark of 0.58), an 8.8% amplification (the endogenous IGE exceeds the exogenous benchmark by 8.8%) . This pattern — a fertility gap that scales linearly with the community-inequality parameter — is the model's basic quantitative demonstration.

The numbers above are intended only as an illustration of the orders of magnitude implied by the model. A systematic calibration (with parameter sources, sensitivity to κ, p, and α, decomposition of the efficiency gain, and the IGE decomposition) is taken up in Section 5.

4 Extension to partial segregation ($\frac{1}{2} \leq p < 1$)

Sections 3.1–3.3 analysed the full-segregation case ($p = 1$), where each family knows the return on investment when choosing both fertility and investment. Under partial segregation ($p < 1$), families face uncertainty: with probability p their child attends the school matching their income group, and with probability $1 - p$ the opposite. This naturally generates a *sequential* timing structure. Fertility (a longer-term decision) is chosen before school quality is realised, based on the *expected* quality $\bar{\theta}_j$ available to income group j. Investment per child is then chosen after school quality θ^s is observed. Certainty equivalence holds: because the indirect utility function is additively separable in θ^s and linear in the term that interacts with n, optimal fertility depends only on $E[\theta^s|j] = \bar{\theta}_j$, not on higher moments of the school quality distribution.[10] At $p = 1$, $\bar{\theta}_j = \theta^j$ and the sequential model reduces to the simultaneous-choice model of Sections 3.1–3.3. At $p = 1/2$, $\bar{\theta}_H = \bar{\theta}_L$ and the fertility gap vanishes, the exogenous-fertility benchmark is recovered.

Expected school qualities. The expected school qualities for above-median and below-median families are:

$$\bar{\theta}_H = p\theta^H + (1-p)\theta^L, \qquad \bar{\theta}_L = (1-p)\theta^H + p\theta^L \tag{24}$$

Using $\theta^H = \theta + \kappa/\pi_H$ and $\theta^L = \theta - \kappa/\pi_L$, the expected quality gap is:

$$\bar{\theta}_H - \bar{\theta}_L = (2p-1)(\theta^H - \theta^L) = \frac{D(p)\kappa}{\pi_H \pi_L} \tag{25}$$

where $D(p) = 2p-1 \in [0,1]$ is the dissimilarity index. The sum $\bar{\theta}_H + \bar{\theta}_L = \theta^H + \theta^L$ is independent of p: changing segregation rotates the expected qualities around a fixed midpoint without shifting their average. The fertility gap is therefore governed by $D(p)$, which continuously interpolates between zero (at $p = 1/2$) and the full-segregation gap $\kappa/(\pi_H\pi_L)$ (at $p = 1$).

Fertility, equilibrium, and human capital. Optimal fertility is $n_j^* = (1 - \bar{\theta}_j)/(2\alpha)$, giving a fertility gap:

$$n_L^* - n_H^* = \frac{D(p)\kappa}{2\alpha\pi_H\pi_L} \tag{26}$$

This is identical to the full-segregation formula (equation 18) but scaled by $D(p)$. The population share $\pi_H = n_H^*/(n_H^* + n_L^*)$ satisfies the same fixed-point condition as before, with the small-κ approximation $\pi_H \approx \frac{1}{2} - \frac{D(p)\kappa}{1-\theta}$: the substitution $\kappa \mapsto D(p)\kappa$ from the full-segregation expansion (17).The composition shift is proportional to $D(p)\kappa$ at leading order: partial segregation attenuates the shift relative to full segregation.

[10]From the parent's problem $\max_{I,n}\ \log(y(1-\alpha n) - nI) + \log(n) + \theta^s \log(I) + u$, the FOC for I gives $I^*(\theta^s, n) = \frac{\theta^s}{1+\theta^s} \cdot \frac{y(1-\alpha n)}{n}$. Substituting back into the objective, the n-dependent terms reduce to $(1+\theta^s)\log(1-\alpha n) + (1-\theta^s)\log(n)$ up to n-independent constants. Taking expectations over θ^s therefore yields a FOC for n that is linear in $E[\theta^s]$, so the optimal n depends only on $\bar{\theta}_j = E[\theta^s \mid j]$.

A child from income group j with realised school quality θ^s has human capital:

$$h_t = \widehat{g}(\theta^s, \bar{\theta}_j) + \theta^s \log(y_{t-1}) + u_t \tag{27}$$

where the two-argument intercept function is:

$$\widehat{g}(\theta^s, \bar{\theta}) = g(\theta^s) + \theta^s \log\left(\frac{\alpha(1+\bar{\theta})}{1-\bar{\theta}}\right) \tag{28}$$

The first term, $g(\theta^s)$, is the exogenous-fertility response to *realised* school quality. The second term captures the fertility-driven resource concentration through *expected* school quality $\bar{\theta}$: families expecting higher quality choose fewer children, freeing more resources per child regardless of the realised θ^s. At $p = 1$, $\bar{\theta}_j = \theta^j$ and $\widehat{g}(\theta^s, \theta^s) = \widetilde{g}(\theta^s)$, recovering the full-segregation intercept. The per-child investment propensity decomposes as:

$$\frac{I^*}{y_{t-1}} = \underbrace{\frac{\theta^s}{1+\theta^s}}_{\substack{\text{direct response} \\ \text{(concave in } \theta^s\text{)}}} \times \underbrace{\frac{\alpha(1+\bar{\theta}_j)}{1-\bar{\theta}_j}}_{\substack{\text{fertility amplifier} \\ \text{(convex in } \bar{\theta}_j\text{)}}}$$

The first factor is the exogenous-fertility propensity, the direct response to *realised* school quality, which is concave in θ^s. The second factor captures how the parent's *fertility choice*, driven by *expected* school quality $\bar{\theta}_j$, amplifies investment through resource concentration. This factor is convex in $\bar{\theta}_j$, and it is entirely absent under exogenous fertility.

Efficiency and persistence under general p. Expected human capital for above-median families is:

$$E[h_t|H] = p\widehat{g}(\theta^H, \bar{\theta}_H) + (1-p)\widehat{g}(\theta^L, \bar{\theta}_H) + \bar{\theta}_H E[\log(y_{t-1})|H] + \mu$$

and analogously for below-median families (replacing H with L and swapping p and $1-p$). The within-group slopes are $\bar{\theta}_H$ and $\bar{\theta}_L$, identical to the exogenous-fertility case. All qualitative results from the full-segregation analysis extend, with $D(p)$ governing the strength of the fertility channel.

Proposition 8. *Under the sequential timing with endogenous fertility and partial segregation ($p \geq 1/2$):*

(i) The average human capital is increasing in school segregation p for any $\kappa > 0$, $\phi > 1$, whenever the small-fertility-gap sufficient condition of Lemma 4(i) applies to the expected qualities $(\bar{\theta}_H, \bar{\theta}_L)$.

(ii) The IGE β_s is increasing in school segregation p and school inequality κ, under the variance condition $V_H \geq V_L$ and the small-fertility-gap condition $d(\bar{\theta}_H) - d(\bar{\theta}_L) > (\bar{\theta}_H - \bar{\theta}_L)A$ of Lemma 4(i), which holds at empirically plausible parameters even when $d'(\bar{\theta}_L) < 0$.

(iii) At $p = 1$, Propositions 6–7 are recovered. At $p = 1/2$, $\bar{\theta}_H = \bar{\theta}_L$ so all families choose the same fertility $n^* = (1-\theta)/(2\alpha)$; the IGE then reduces to the exogenous-fertility benchmark of Section 2 evaluated at this common fertility level.

Proof. See Appendix B. □

The extension to general p clarifies the interaction between the two policy margins. Desegregation ($p \to 1/2$) eliminates the *expected*-quality differential even if school inequality (κ) persists, while school equalisation ($\kappa \to 0$) eliminates both the *realised* and the *expected* quality differentials even under full segregation. The fertility channel requires both $D(p) > 0$ and $\kappa > 0$: desegregation and equalisation are complementary instruments that operate through distinct margins. Since the amplification scales continuously with $D(p)$, partial desegregation yields partial attenuation, it is not an all-or-nothing instrument.

5 Calibration and simulation

This section presents a numerical calibration of the model to discipline the parameter choices using external evidence and to quantify the fertility amplification channel highlighted in Sections 3.3 and 4. It is not a structural estimation.

5.1 Parameter choices

"Frictionless" return to education ($\theta = 0.27$). Under equal opportunity, the IGE equals θ. US empirical estimates of the IGE typically lie in the range 0.34–0.40 (Solon, 1999, 2004; Chetty *et al.*, 2014), but those estimates already reflect the prevailing degree of school inequality and segregation. We choose the "frictionless" θ such that, once school inequality is added, the

model reproduces an empirical IGE in this range. With $\kappa = 0.006$ and $p = 1$, $\theta = 0.27$ delivers $\beta_s \approx 0.37$, consistent with US estimates.

School inequality ($\kappa \in [0, 0.02]$). At the baseline $\kappa = 0.006$ with equilibrium $\pi_H \approx 0.49$, the model yields $\theta^H \approx 0.28$ and $\theta^L \approx 0.26$, a quality gap of about 0.024. The high-κ scenario ($\kappa = 0.02$) yields a quality gap of about 0.08, closer to the between-school test-score dispersion documented by PISA in more segregated systems. This is consistent with the range of between-school differences in value-added documented across US school districts (Chetty *et al.*, 2014) and with the cross-country variation in between-school test score gaps reported in PISA (Jenkins *et al.*, 2008). The OECD's PISA-based review of school choice and equity (OECD, 2019) documents substantial between-school variation in student performance across OECD systems, with the between-school share of total variance in PISA reading scores exceeding 50% in several segregated systems.

Per-child rearing cost ($\alpha = 0.125$). The USDA's "Expenditures on Children by Families" report (Lino *et al.*, 2017) estimates that a middle-income family spends roughly 15–20% of household income per child on direct costs (housing, food, childcare, education); for a middle-income two-child household with pre-tax income \$59,200–\$107,400, average annual spending per child was about \$12,980, or roughly 15% of income. Once shared costs across children are netted out, the marginal per-child cost is somewhat lower. We use $\alpha = 0.125$ as a conservative baseline for the marginal per-child cost share, giving a reference optimal fertility of $n = (1 - \theta)/(2\alpha) \approx 2.92$ children — close to US fertility rates of the 1970s, from which the paper's demographic shocks are measured. De la Croix and Doepke (2003) use a similar range in their calibration of a quantity-quality growth model.

Parental income distribution. We assume $\log y_{t-1} \sim N(\mu = 10, \sigma^2 = 0.65)$, so median household earnings are about $\exp(10) \approx \$22{,}000$ (a conservative US log-median) and the median-split between-group inequality index is $\phi = 1 + \frac{\sigma}{\mu}\sqrt{\frac{2}{\pi}} \approx 1.06$. This is the coherent value implied by the log-normal median split.

Segregation probability ($p \in [1/2, 1]$). Duncan dissimilarity indices (Duncan and Duncan, 1955) for US metropolitan areas range from 0.3 to 0.7 (Reardon and Bischoff, 2011), translating to $p \approx 0.65$–0.85. We use $p = 1$ (full segregation) as the baseline and vary p in Panel B of the tables.

5.2 Baseline results

Table 1 reports the equilibrium at the baseline calibration. Under endogenous fertility, the equilibrium child share of above-median families falls to $\pi_H = 0.49$ (from 0.50 under exogenous fertility), generating a fertility gap of 0.10 children ($n_L^* = 2.97$ vs. $n_H^* = 2.87$). The fertility gap scales linearly with community inequality: at the high-κ reference ($\kappa = 0.02$, characteristic of the top quartile of US MSA segregation), the gap widens to 0.32 children, closer to the fertility differential observed in US administrative data.

Per-child investment I^*/y moves in opposite directions for the two groups when fertility is endogenised. The exogenous benchmark uses the reference $n = (1-\theta)/(2\alpha) = 2.92$ (the optimal fertility at equal opportunity θ), giving investment propensities 4.8% (high type) and 4.5% (low type). Under endogenous fertility, above-median families choose $n_H^* = 2.87 < 2.92$, freeing additional resources per child and raising the propensity from 4.8% to 4.9%. Below-median families choose $n_L^* = 2.97 > 2.92$, diluting per-child resources and pushing the investment propensity from 4.5% down to 4.4%. The high-to-low ratio thus widens from $4.8/4.5 = 1.07$ under exogenous fertility to $4.9/4.4 = 1.13$ under endogenous, about a 5% widening of the per-child investment gap at baseline; the widening rises sharply with κ. That illustrates the quantity–quality amplification in operation: the rich concentrate while the poor dilute, even when the average fertility level is essentially unchanged across the two scenarios (2.92 vs. 2.92 at baseline).

Table 1: Baseline Calibration

	Exogenous Fertility	Endogenous Fertility
Parameters		
θ	0.27	
κ	0.006	
α	0.12	
p	1.00	
Equilibrium		
π_H (high-type child share)	0.500	0.492
θ^H (high school quality)	0.282	0.282
θ^L (low school quality)	0.258	0.258
$\theta^H - \theta^L$ (quality gap)	0.024	0.024
n_H^* (high-type fertility)	2.92	2.87
n_L^* (low-type fertility)	2.92	2.97
$n_L^* - n_H^*$ (fertility gap)	0.00	0.10
I^*/y (high type)	0.048	0.049
I^*/y (low type)	0.045	0.044

Notes: Parental income follows a log-normal distribution with $\log y \sim N(\mu = 10, \sigma^2 = 0.65)$, implying the median-split between-group inequality index $\phi = 1 + \sigma\sqrt{2/\pi}/\mu \approx 1.064$. $\theta = 0.27$ is the "frictionless" return to education (the IGE under equal opportunity); with school inequality $\kappa = 0.006$ and full segregation, the model produces an IGE of about 0.37, close to US empirical estimates of 0.34–0.40 (Solon, 1999; Chetty *et al.*, 2014). $\alpha = 0.12$ from USDA child-expenditure estimates. Exogenous fertility uses $n = (1 - \theta)/(2\alpha) = 2.92$ (the optimal fertility at average school quality θ), $\pi_H = 1/2$.

5.3 Efficiency gain decomposition

Table 2 decomposes the efficiency gain into its three components (intercept gain, slope gain, composition uplift) defined in equation (21). At the baseline ($\kappa = 0.006$, $p = 1$), the slope gain of about 0.08% of the equal-opportunity human-capital benchmark $E[\log y_{t-1}] = 10$ dominates, and the fertility amplification adds about 27% on top of the exogenous-fertility benchmark. The amplification rises sharply with school inequality: at $\kappa = 0.02$, endogenous fertility raises the efficiency gain from 0.26% to 0.49% of the equal-opportunity benchmark, nearly doubling it. Panel B shows the amplification scales monotonically with the segregation parameter p (with the slope gain now correctly incorporating the dissimilarity factor $D(p) = 2p - 1$).

Equal-opportunity benchmark. To interpret the magnitude of these efficiency gains, compare them with the equal-opportunity ($\kappa = 0$) human-capital level. Under endogenous fertility with $\kappa = 0$, $E[h] = \widetilde{g}(\theta) + \theta\, E[\log y_{t-1}] + \mu$, where $\widetilde{g}(\theta) = \theta \log(\alpha\theta/(1-\theta))$ is the intercept evaluated at the equal-opportunity school quality. Using the calibration values $\theta = 0.27$, $\alpha = 0.125$, and $E[\log y_{t-1}] = 10$, we get $\widetilde{g}(\theta) = -0.83$. Imposing the steady state normalization of average child ability component $\mu = 0.83 + (1-\theta)E[\log y_{t-1}]$ so that $E[h] = E[\log y_t] = E[\log y_{t-1}] = 10$. The largest entry in Panel A of Table 2 is the endogenous total gain 0.050 log-income units at $\kappa = 0.02$, $p = 1$, i.e. a 0.50% increase in average human capital relative to the equal-opportunity benchmark. Modest in absolute size once the between-group inequality is anchored to a realistic $\phi \approx 1.06$, but doubled by the endogenous-fertility amplification relative to the exogenous benchmark.

Figure 5 plots the efficiency gain as a function of school inequality κ (with $p = 1$) and segregation p (with $\kappa = 0.05$). In both cases the shaded area is the fertility amplification, the gap between exogenous and endogenous curves. The two figures share a common y-axis, so the relative magnitudes of the κ- and p-driven channels are directly comparable; both curves converge to the same maximum gain (about 0.5% of $E[\log y]$) as $p \to 1$ and $\kappa \to 0.02$. The two channels interact multiplicatively through the dissimilarity-weighted quality gap $D(p)\kappa/(\pi_H \pi_L)$: the impact of p scales with κ and vice versa. At $\kappa = 0.02$, the IGE rises from about 0.30 at $p = 0.6$ to 0.61 at $p = 1$ (a range of about 0.31 versus only 0.08 at $\kappa = 0.006$), so segregation policy is far from neutral; its leverage simply depends on how unequal schools are to begin with.

Table 2: Efficiency Gain Decomposition (% of $E[\log y_{t-1}] = 10$)

κ	p	Intercept (exo)	Intercept (endo)	Slope gain[a]	Comp. uplift	Total (exo)	Total (endo)	Ampl. (%)
Panel A: Varying κ ($p = 1$)								
0.002	1.00	0.00	0.00	0.03	0.00	0.03	0.03	9.2
0.004	1.00	0.00	0.00	0.05	0.01	0.05	0.06	18.3
0.006	1.00	0.00	0.00	0.08	0.02	0.08	0.10	27.4
0.010	1.00	0.00	0.00	0.13	0.05	0.13	0.19	45.7
0.020	1.00	0.00	0.02	0.26	0.22	0.26	0.50	91.7
Panel B: Varying p ($\kappa = 0.020$)								
0.020	0.60	0.02	0.01	0.05	0.04	0.07	0.11	61.9
0.020	0.75	0.01	0.01	0.13	0.11	0.14	0.25	78.5
0.020	0.85	0.01	0.01	0.18	0.15	0.19	0.35	84.5
0.020	1.00	0.00	0.02	0.26	0.22	0.26	0.50	91.7

[a] Slope gain is $\frac{1}{2}D(p)(\theta^H - \theta^L)(\phi - 1)E[\log y_{t-1}]$, where $D(p) = 2p - 1$. It vanishes at $p = \frac{1}{2}$ (no segregation).

Notes: Entries expressed as a percentage of $E[\log y_{t-1}] = 10$, the normalised steady-state human-capital level. "Ampl. (%)" is the percentage increase in total efficiency gain from endogenous fertility relative to exogenous fertility. Parental income $\log y \sim N(\mu = 10, \sigma^2 = 0.65)$, so the median-split $\phi \approx 1.064$. Baseline: $\theta = 0.27$, $\alpha = 0.12$.

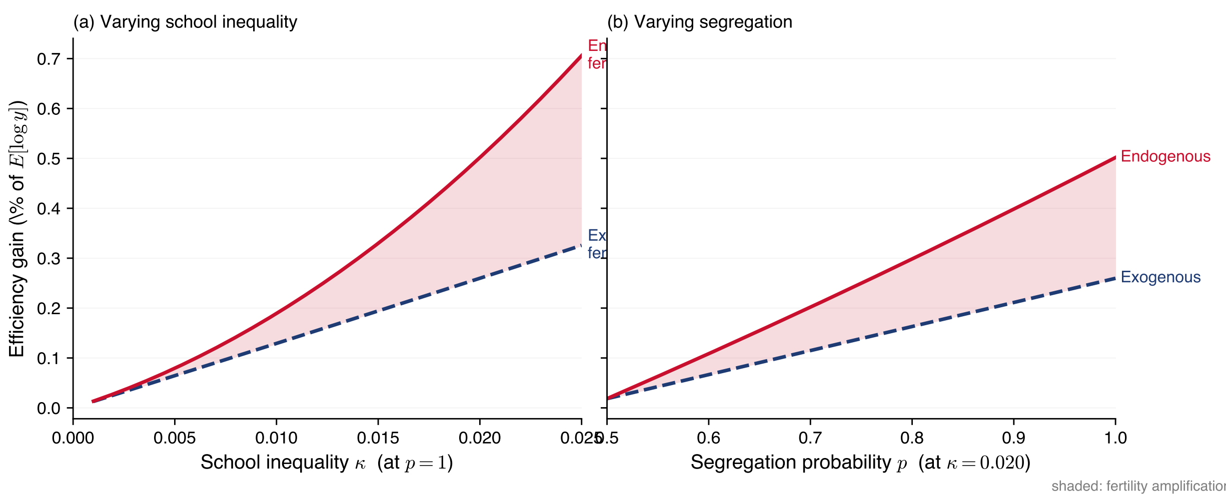

Figure 5: Efficiency gain (expressed as % of $E[\log y_{t-1}] = 10$) under exogenous and endogenous fertility. Panel (a) varies school inequality κ at full segregation ($p = 1$); Panel (b) varies segregation p at high inequality ($\kappa = 0.02$). The shaded region between the two curves is the fertility amplification. Both curves converge to the same maximum gain as $p \to 1$ and $\kappa \to 0.02$.

5.4 Intergenerational elasticity

At the baseline ($\theta = 0.27$, $\kappa = 0.006$, $\alpha = 0.125$, $p = 1$), the IGE rises from $\beta_s^{exo} = 0.362$ (exogenous fertility) to $\beta_s = 0.369$ (endogenous fertility), an increase of 0.007 points, or about 7.8% of the segregation-induced increase above the benchmark $\theta = 0.27$. The amplification grows with both κ and p: at $\kappa = 0.02$ and $p = 1$, $\beta_s = 0.602$ vs. $\beta_s^{exo} = 0.575$ (an 8.8% amplification). Table 3 reports the full (κ, p) grid around the baseline.

Table 3: Intergenerational Elasticity: Exogenous vs. Endogenous Fertility

κ	p	Exogenous[a]	Endogenous[b]	Amplif.[c] (%)
Panel A: Varying school inequality ($p = 1$)				
0.002	1.00	0.3005	0.3029	7.6
0.004	1.00	0.3311	0.3358	7.7
0.006	1.00	0.3616	0.3688	7.8
0.010	1.00	0.4227	0.4349	8.0
0.020	1.00	0.5752	0.6022	8.8
Panel B: Varying segregation ($\kappa = 0.020$)				
0.020	0.60	0.3310	0.3361	8.3
0.020	0.75	0.4226	0.4355	8.5
0.020	0.85	0.4836	0.5020	8.6
0.020	1.00	0.5752	0.6022	8.8

[a] IGE under exogenous fertility ($n = (1-\theta)/(2\alpha) = 2.92$, $\pi_H = 1/2$): the benchmark from Section 2, using the same reference n as the per-child-investment analysis of Table 1 and Lemma 4 for internal consistency across the paper.

[b] IGE under endogenous fertility, giving each family equal weight $\tilde{n}_j = 1/2$.

[c] Amplification: percentage increase in the segregation-induced IGE component, $(\beta_s^{endo} - \beta_s^{exo})/(\beta_s^{exo} - \theta) \times 100$. Baseline: $\theta = 0.27$, $\alpha = 0.12$, $\log y \sim N(\mu = 10, \sigma^2 = 0.65)$, so $\phi = 1 + \sigma\sqrt{2/\pi}/\mu \approx 1.064$.

Figure 6 plots the IGE along both amplifier dimensions — school inequality κ and segregation p — with a common y-axis so the two channels can be read side by side.

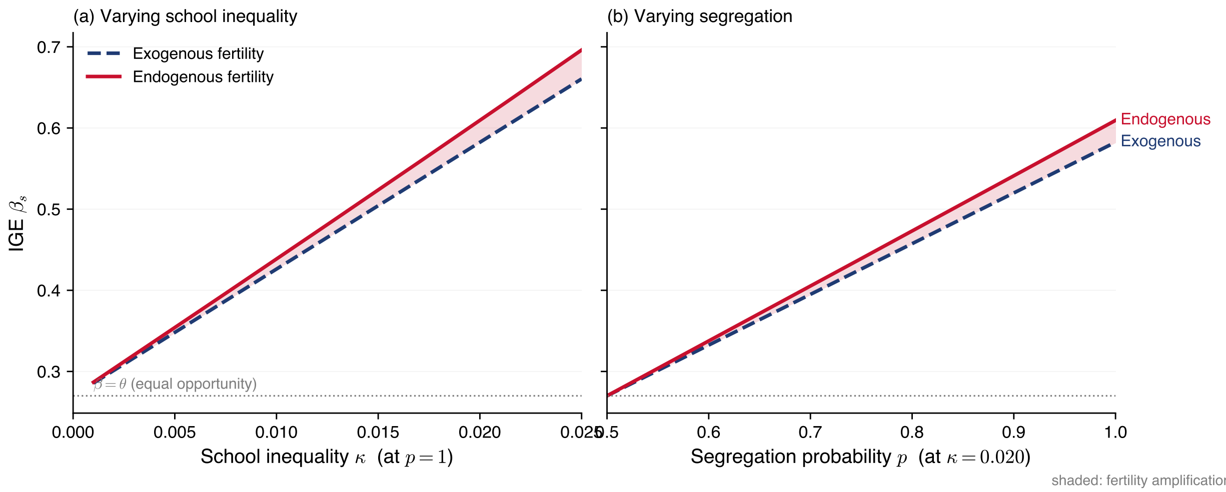

Figure 6: Intergenerational elasticity under exogenous and endogenous fertility. Panel (a) varies school inequality κ at full segregation ($p = 1$); Panel (b) varies segregation p at high inequality ($\kappa = 0.02$). The shaded region is the fertility amplification. The dotted horizontal line marks the equal-opportunity benchmark $\beta = \theta$.

5.5 Sensitivity to the child-rearing cost

A recurring theme in the recent economics of fertility (Doepke and Zilibotti, 2017; Mahler *et al.*, 2025; Kim *et al.*, 2024) is that the effective cost of raising children — the time and resources parents devote to each child — has risen over recent decades as intensive-parenting and status-competition motives have taken hold. Our model has an interpretable counterpart: the child-rearing cost share α. We ask how the model's predictions on the IGE respond to a rise in α from the baseline of 0.125 up to 0.30.

Table 4 reports the exercise at two levels of community inequality. Figure 7 plots the endogenous-fertility IGE against α across three values of κ. The exercise reveals a substantive asymmetry between the exogenous and endogenous cases that is central to the paper's message.

Under exogenous fertility, higher α modestly reduces the IGE, consistent with Proposition 3(ii). Holding n fixed at its exogenous benchmark, a rise of α from the baseline 0.125 to 0.30 lowers β_s^{exo} by about 0.008 points at $\kappa = 0.006$ and 0.025 at $\kappa = 0.02$. The direct comparative static says that higher child-rearing costs, at fixed fertility, mechanically shrink per-child investment for every family and, through the OVB intercept term, marginally attenuate the segregation-induced IGE gap.

Under endogenous fertility, higher α raises the IGE. At the same rise of α from 0.125 to 0.30, β_s^{endo} rises by about 0.010 points at $\kappa = 0.006$ and 0.035 at $\kappa = 0.02$. The sign of the total effect flips because the induced fall in equilibrium fertility ($n^* = (1 - \theta^s)/(2\alpha)$ falls with α) concentrates parental investment on fewer children. The endogenous fertility response *amplifies* the intensive-parenting mechanism: the rising cost of children does not simply erode per-child investment; it triggers a compensating fall in family size that concentrates resources at the top of the community distribution.

The community-inequality amplifier scales the endogenous response. Moving κ from 0.006 to 0.02 roughly triples the IGE response to a given change in α. This is the same $D(p)\kappa$-scaling story as for the fertility-decline channel: any demographic force is transmitted into the IGE with strength proportional to the community gap. The intensive-parenting channel is no exception. The IGE-amplification percentages in Table 4 illustrate this: at the baseline $\kappa = 0.006$, the endogenous-fertility IGE moves from small amplification at $\alpha = 0.125$ to $\sim 20\%$ at $\alpha = 0.30$; at $\kappa = 0.02$, from small amplification at $\alpha = 0.125$ to $\sim 22\%$ at $\alpha = 0.30$.

Together with the fertility-decline analysis of Section 2.4, these results extend the paper's central message beyond a fall in n. Any rise in the effective cost of children — whether driven by the rat-race dynamics of Doepke and Zilibotti (2017); Mahler *et al.* (2025), by the labour-market institutions studied by Adserà (2004), or by a policy change — feeds through into higher intergenerational persistence, provided fertility is allowed to respond. Community inequality scales the transmission: the same α shift moves the IGE about three to four times as much in a high-inequality environment as in a moderate one.

Table 4: Sensitivity to the child-rearing cost share α (baseline $\theta = 0.27$, $p = 1$)

α	$n^*_L - n^*_H$ (endo)[a]	β_s $\kappa = 0.006$ Exo[b]	Endo	Amplif.[c] (%)	β_s $\kappa = 0.020$ Exo[b]	Endo	Amplif.[c] (%)
0.100	0.12	0.3688	0.3661	-2.8	0.5993	0.5933	-1.8
0.125	0.10	0.3681	0.3688	0.7	0.5968	0.6022	1.6
0.150	0.08	0.3673	0.3709	3.8	0.5941	0.6094	4.7
0.200	0.06	0.3654	0.3743	9.3	0.5879	0.6208	10.3
0.250	0.05	0.3633	0.3770	14.7	0.5807	0.6297	15.8
0.300	0.04	0.3606	0.3792	20.5	0.5719	0.6369	21.5

[a] Equilibrium fertility gap under endogenous fertility.
[b] IGE under exogenous fertility, with n held *fixed* at $n_{\text{exo}} = 2.0$.
[c] IGE amplification: $(\beta_s^{endo} - \beta_s^{exo})/(\beta_s^{exo} - \theta) \times 100$.
Reading. With n held fixed, higher α modestly *reduces* the IGE (Prop 3(ii)). Under endogenous fertility, higher α triggers a fall in equilibrium fertility that concentrates parental investment on fewer children — and this concentration reverses the sign of the total effect, so β_s^{endo} *rises* with α. Community inequality scales the effect: at $\kappa = 0.020$, the same α shift moves β_s^{endo} about three times as much as at $\kappa = 0.006$. Parental income $\log y \sim N(\mu = 10, \sigma^2 = 0.65)$.

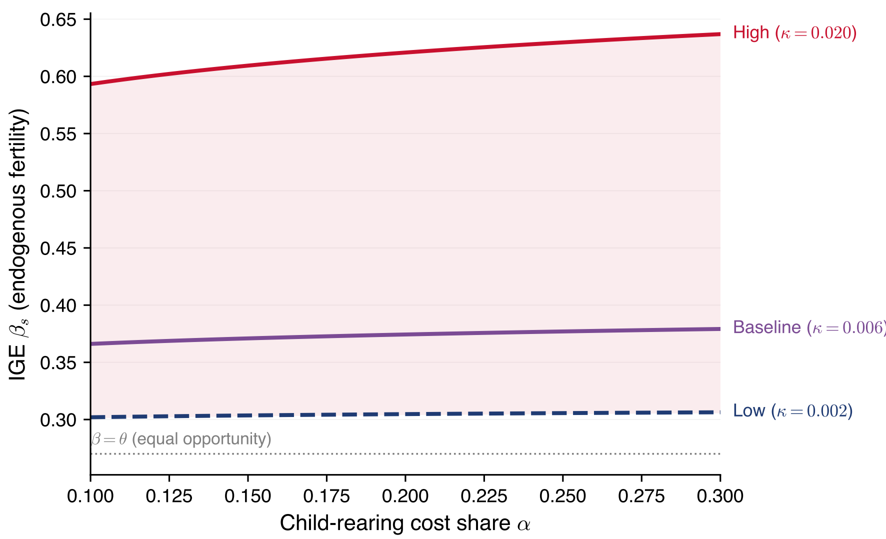

Figure 7: Endogenous-fertility IGE β_s as a function of the child-rearing cost share α, at three levels of community inequality κ. Rising α raises the IGE consistent with the intensive-parenting mechanism; the community-inequality amplifier makes the response two to three times larger at $\kappa = 0.02$ than at $\kappa = 0.006$.

5.6 Scope and caveats

Two caveats bear on the interpretation of the quantitative results reported above. First, throughout the paper each family receives equal weight in the IGE, consistent with the standard empirical convention for estimating intergenerational persistence (Solon, 1999; Chetty *et al.*, 2014). A child-weighted measure — which reweights by equilibrium fertility and thus gives more weight to larger, lower-opportunity families — delivers essentially the same IGE in our calibration: the two measures coincide at the third decimal at baseline and differ by at most 0.001 across the parameter range explored here, so the choice of weighting is quantitatively irrelevant for the reported results. Second, our model captures the intensive margin of the fertility decline (fewer children per parent) but not the extensive margin. In recent decades a substantial share of the fertility decline in advanced economies has come instead from rising childlessness (Baudin *et al.*, 2015). Reassuringly, the extensive margin points in the same direction as our mechanism: rising childlessness at higher income levels concentrates the child-bearing population toward the lower end of the distribution, widening the effective fertility gap and reinforcing the community-inequality amplifier we identify. Incorporating both the intensive and extensive margins would therefore likely strengthen, rather than overturn, our qualitative conclusions. At the same time, childless families, by definition, do not transmit income across generations, so accounting for the extensive margin also changes the population over which intergenerational persistence is measured.

6 Conclusion

This paper identifies community inequality and segregation as an amplification mechanism of demographic change for intergenerational income persistence. The analysis uncovers two complementary channels. The demographic channel operates even when fertility is uniform across income groups: a decline in fertility or in the per-child cost raises the IGE by an amount proportional to the community opportunity gap. The QQ amplification channel compounds this when fertility is endogenous: families in higher-opportunity communities optimally choose fewer, more resource-intensive children, and the resulting fertility gap widens the parenting gap further. Both channels share the same amplifier, community inequality, so that the larger the gap in local opportunities across communities, the more strongly demographic forces translate into measured income persistence. The same logic extends to the intensive-parenting mechanism of Doepke and Zilibotti (2017); Mahler *et al.* (2025); Kim *et al.* (2024): a rise in the effective cost of raising children has an ambiguous direct effect at fixed fertility but raises the IGE once fertility responds, and the response is again scaled by community inequality.

Our results have direct policy implications. Three levers reduce intergenerational persistence. Residential desegregation and community-opportunity equalisation weaken the amplification mechanism directly, at the cost of some efficiency loss, a trade-off that may help explain the public resistance to community-based policies discussed in the OECD review of school choice and equity (OECD, 2019). The third lever operates through the demographic channel: once fertility is allowed to respond, policies that lower the effective cost of raising children — uniform childcare subsidies, family transfers, or reforms that alleviate the labour-market squeeze on parents — reduce intergenerational persistence, and the reduction is larger where community inequality is greater. Symmetrically, and less intuitively, the ongoing intensive-parenting trend documented by Doepke and Zilibotti (2017); Mahler *et al.* (2025); Kim *et al.* (2024) — which raises the effective cost of children — pushes persistence upward rather than downward, again more so in high-inequality settings.

The mechanism yields a natural testable prediction. The IGE should respond more strongly to demographic shocks in more community-segregated settings: as fertility falls and childcare costs decline across the OECD, the model predicts larger IGE increases in more-segregated regions than in more-mixed ones. Several extensions of the framework are worth pursuing. Future work could estimate the model structurally, endogenise the cost of children through family and labour-market policies (Cherchye *et al.*, 2026; Mahler *et al.*, 2025; Adserà, 2004), extend to multiple generations to study whether demographic effects accumulate into a demographic multiplier, or model residential sorting explicitly (Jones, 2023) to close the feedback loop between inequality, segregation, and mobility.

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Appendix A: Auxiliary Lemmas for Section 3

A.1 Derivation of the polynomial fixed-point equation

This appendix gives the algebra behind (14) and (15). We work under full segregation ($p = 1$); the partial-segregation case shares the same first-order structure via the substitution $\kappa \mapsto D(p)\kappa$ described in Section 4, though higher-order terms in the cubic do not follow the same exact rescaling. To avoid the notational clash with the exogenous-fertility intercept term $A = \log((1-\alpha n)/n)$ used in Lemma 4 below, we write $T \equiv 1 - \theta$ in this derivation.

Setup. Optimal fertilities are $n_H^* = (1 - \theta - \kappa/\pi_H)/(2\alpha)$ and $n_L^* = (1 - \theta + \kappa/\pi_L)/(2\alpha)$. Writing $T \equiv 1 - \theta$ and using $\pi_L = 1 - \pi_H$,

$$n_H^* = \frac{T - \kappa/\pi_H}{2\alpha}, \qquad n_L^* = \frac{T + \kappa/(1 - \pi_H)}{2\alpha}.$$

The share definition $\pi_H = n_H^*/(n_H^* + n_L^*)$, with $N \equiv n_H^* + n_L^*$ and $r \equiv \pi_H$, gives $n_H^* = rN$ and $n_L^* = (1 - r)N$, hence

$$2\alpha N r = T - \kappa/r,$$
$$2\alpha N(1 - r) = T + \kappa/(1 - r).$$

Multiplying the first by r and the second by $1 - r$,

$$2\alpha N r^2 = Tr - \kappa, \qquad 2\alpha N(1 - r)^2 = T(1 - r) + \kappa. \tag{29}$$

Eliminating N. Dividing each of (29) by the appropriate square and equating gives $(Tr - \kappa)/r^2 = [T(1 - r) + \kappa]/(1 - r)^2$, hence

$$(Tr - \kappa)(1 - r)^2 = \big(T(1 - r) + \kappa\big)r^2.$$

Expanding $(1 - r)^2 = 1 - 2r + r^2$, the left-hand side equals $Tr^3 + (-2T - \kappa)r^2 + (T + 2\kappa)r - \kappa$; the right-hand side equals $(T + \kappa)r^2 - Tr^3$. Subtracting and collecting terms yields (14):

$$2Tr^3 - (3T + 2\kappa)r^2 + (T + 2\kappa)r - \kappa = 0.$$

The change of variable $d = 2r - 1$. Substituting $r = (1 + d)/2$ into (14) and simplifying (each $(1 + d)^k$ expanded; constants, d, d^2, d^3 collected) gives

$$2Td^3 - 4\kappa d^2 - 2Td - 4\kappa = 0,$$

which, after dividing by 2 and grouping, becomes (15):

$$T\,d\,(d^2 - 1) = 2\kappa\,(d^2 + 1).$$

Small-κ expansion. Treating κ as a small parameter and seeking $d = d_1\kappa + d_3\kappa^3 + \cdots$ (only odd powers appear because (15) is invariant under $(d, \kappa) \mapsto (-d, -\kappa)$), substituting and matching orders gives $d_1 = -2/T = -2/(1 - \theta)$ and $d_3 = -16/T^3 < 0$, hence $d = -2\kappa/(1 - \theta) - 16\kappa^3/(1 - \theta)^3 + O(\kappa^5)$. Returning to p,

$$\pi_H^* = \tfrac{1}{2} - \frac{\kappa}{1 - \theta} + O\left(\frac{\kappa^3}{(1 - \theta)^3}\right),$$

which is (17). The cancellation of the $O(\kappa^2)$ term (a consequence of the symmetry of (15) under sign flips) explains why the linear approximation is so accurate at empirically plausible κ: the relative error is roughly $(\kappa/(1 - \theta))^2$, of order 10^{-3} at the US baseline.

Other roots. The cubic (14) admits two further real roots in addition to the near-symmetric π_H^*: (i) a small-share root near $\kappa/(1-\theta)$, corresponding to an interior equilibrium with n_H^* close to zero (a degenerate solution branch continuous with $n_H = 0$ at $\kappa = 0$), and (ii) a root above 1, outside the feasible domain. Proposition 4 selects the near-symmetric branch (continuous with the equal-opportunity benchmark $\pi_H = 1/2$ at $\kappa = 0$) and excludes the positive-d branch directly from the sign of (15); the small-share branch is a distinct interior root that we do not analyse, and the $\pi > 1$ root is outside the feasible domain. The existence bound $\kappa/(1-\theta) \leq 0.15$ is the threshold at which the near-symmetric and small-share branches merge — above it, the cubic has no interior roots.

A.2 Log-normal moments used in the IGE derivation

The proof of Proposition 3 expresses the IGE through two regression moments,

$$B \;=\; \frac{cov(Z_{t-1}, \log y_{t-1})}{Var(\log y_{t-1})}, \qquad C \;=\; \frac{cov(\log y_{t-1} \cdot Z_{t-1}, \log y_{t-1})}{Var(\log y_{t-1})},$$

where $Z_{t-1} = \mathbf{1}[y_{t-1} > y^M]$ is the income-group indicator. This appendix provides explicit closed forms for B and C under log-normal parental income, $\log y_{t-1} \sim N(\mu, \sigma^2)$.

Closed forms. The median of a log-normal is $y^M = e^\mu$, so $P(y_{t-1} > y^M) = \frac{1}{2}$ and $E[Z_{t-1}] = \frac{1}{2}$. For a standard normal X, $E[X \mid X > 0] = \sqrt{2/\pi}$ and $Var(X \mid X > 0) = 1 - 2/\pi$ (half-normal moments). It follows that

$$E[\log y_{t-1} \mid \log y_{t-1} > \mu] = \mu + \sigma\sqrt{2/\pi}, \qquad Var(\log y_{t-1} \mid \log y_{t-1} > \mu) = \sigma^2\big(1 - 2/\pi\big), \quad (30)$$

and symmetrically below the median. Computing the unconditional moments,

$$cov(Z_{t-1}, \log y_{t-1}) = E[Z_{t-1} \log y_{t-1}] - E[Z_{t-1}]E[\log y_{t-1}] = \tfrac{1}{2}\big(\mu + \sigma\sqrt{2/\pi}\big) - \tfrac{1}{2}\mu = \tfrac{\sigma\sqrt{2/\pi}}{2},$$

so dividing by $Var(\log y_{t-1}) = \sigma^2$ gives

$$B \;=\; \frac{\sqrt{2/\pi}}{2\sigma}. \quad (31)$$

For C, use $E[(\log y_{t-1})^2 \mid \log y_{t-1} > \mu] = Var(\cdot) + E[\cdot]^2 = \sigma^2(1 - 2/\pi) + (\mu + \sigma\sqrt{2/\pi})^2 = \sigma^2 + \mu^2 + 2\mu\sigma\sqrt{2/\pi}$. Then

$$\begin{aligned} cov(\log y_{t-1} \cdot Z_{t-1}, \log y_{t-1}) &= E[Z_{t-1}(\log y_{t-1})^2] - E[Z_{t-1} \log y_{t-1}]\, E[\log y_{t-1}] \\ &= \tfrac{1}{2}\big(\sigma^2 + \mu^2 + 2\mu\sigma\sqrt{2/\pi}\big) - \tfrac{\mu}{2}\big(\mu + \sigma\sqrt{2/\pi}\big) \\ &= \tfrac{\sigma^2}{2} + \tfrac{\mu\sigma\sqrt{2/\pi}}{2}, \end{aligned}$$

so dividing by σ^2,

$$C \;=\; \tfrac{1}{2} + \frac{\mu\sqrt{2/\pi}}{2\sigma}, \qquad 2C - 1 = \frac{\mu\sqrt{2/\pi}}{\sigma}. \quad (32)$$

Refined condition, explicit form. Substituting (31) and (32) into the derivative $\partial\beta_s/\partial\kappa =$

$2D(p)\{[2A + g'(\theta^H) + g'(\theta^L)]B + (2C - 1)\}$ derived in the proof of Proposition 3, the inner bracket is positive iff

$$\frac{\mu\sqrt{2/\pi}}{\sigma} > \left|2A + g'(\theta^H) + g'(\theta^L)\right| \frac{\sqrt{2/\pi}}{2\sigma},$$

which simplifies (after multiplying through by $\sigma/\sqrt{2/\pi}$ and rearranging) to condition (6). The dependence on σ cancels: the refined condition is invariant to the variance of the log-income distribution.

Lemma 2 (Convexity ordering of Jensen gaps)**.** *Let $f, h : (a, b) \to \mathbb{R}$ be twice continuously differentiable functions with $f''(\theta) > h''(\theta) > 0$ for all $\theta \in (a, b)$. Then for any $\theta \in (a, b)$ and any $\kappa > 0$ such that $\theta \pm \kappa \in (a, b)$:*

$$\frac{1}{2}\left[f(\theta + \kappa) + f(\theta - \kappa)\right] - f(\theta) > \frac{1}{2}\left[h(\theta + \kappa) + h(\theta - \kappa)\right] - h(\theta)$$

Proof. Define $d(\theta) = f(\theta) - h(\theta)$. By assumption, $d''(\theta) = f''(\theta) - h''(\theta) > 0$ for all $\theta \in (a, b)$, so d is strictly convex on (a, b). By strict convexity, for any $\kappa > 0$:

$$\frac{1}{2}\left[d(\theta + \kappa) + d(\theta - \kappa)\right] > d(\theta)$$

Expanding:

$$\frac{1}{2}\left[f(\theta + \kappa) - h(\theta + \kappa) + f(\theta - \kappa) - h(\theta - \kappa)\right] > f(\theta) - h(\theta)$$

Rearranging gives the result. □

Lemma 3 (Convexity ordering of $\widetilde{g}$ and g)**.** *For all $\theta \in (0, 1)$, $\widetilde{g}''(\theta) > g''(\theta) > 0$, where $g(\theta) = \theta \log(\frac{\theta}{1+\theta})$ and $\widetilde{g}(\theta) = \theta \log(\frac{\alpha\theta}{1-\theta})$ with $\alpha > 0$.*

Proof. From Section 2, $g''(\theta) = \frac{1}{\theta(1+\theta)^2} > 0$. From Section 3, $\widetilde{g}''(\theta) = \frac{1}{\theta(1-\theta)^2} > 0$. The difference is:

$$\widetilde{g}''(\theta) - g''(\theta) = \frac{1}{\theta}\left(\frac{1}{(1-\theta)^2} - \frac{1}{(1+\theta)^2}\right) > 0$$

Hence $\widetilde{g}''(\theta) > g''(\theta)$ for all $\theta \in (0, 1)$. □

Lemma 4 (Sufficient conditions for $\beta_s^{endo} > \beta_s^{exo}$, revised)**.** *Let $d(\theta) = \theta \log\left(\frac{\alpha(1+\theta)}{1-\theta}\right)$ denote the fertility-amplification function and write $A = \log\left(\frac{1-\alpha n}{n}\right)$ for the exogenous-fertility intercept term (using the reference $n \geq 1$ at which β_s^{exo} is evaluated). Suppose:*

(i) $d(\theta^H) - d(\theta^L) > (\theta^H - \theta^L)A$ *(equivalently, under small* κ*,* $d'(\theta) > A$*). Since* $A = \log((1-\alpha n)/n) < 0$ *is negative and sizeable at empirically plausible parameters, this condition is much weaker than requiring* $d' \geq 0$ *and holds at the paper's US baseline with margin* 0.58 *(where* $d'(0.27) \approx -0.76 > A \approx -1.34$*);*

(ii) the within-group variance condition $Var(\log y_{t-1}|y > y^M) \geq Var(\log y_{t-1}|y < y^M)$ *holds.*

Then under full segregation ($p = 1$*) and any* $\kappa \geq 0$*,* $\beta_s^{endo} \geq \beta_s^{exo}$*, with strict inequality when* $\kappa > 0$.

Proof. At $\kappa = 0$, both expressions reduce to the equal-opportunity benchmark $\beta = \theta + cov(\tau, \log y)/Var(\log y)$ and the equality holds. For $\kappa > 0$, decompose the difference into two channels, each independently non-negative.

(a) Intercept channel. Fix the school qualities θ^H, θ^L at any common reference values (in particular, take the exogenous-fertility values $\theta \pm 2\kappa$) and compare the intercept gaps that enter the OVB term $D(p) \cdot [\cdot] \cdot B$ (with $B = cov(Z_{t-1}, \log y_{t-1})/Var(\log y_{t-1}) > 0$ by (ii) and $Z_{t-1} = \mathbf{1}[y_{t-1} > y^M]$, as in Proposition 3). The exogenous-fertility intercept gap is $(\theta^H - \theta^L)A + g(\theta^H) - g(\theta^L)$; the endogenous-fertility intercept gap is $\widetilde{g}(\theta^H) - \widetilde{g}(\theta^L)$. Their difference is

$$\widetilde{g}(\theta^H) - \widetilde{g}(\theta^L) - \left[(\theta^H - \theta^L)A + g(\theta^H) - g(\theta^L)\right] = \left[d(\theta^H) - d(\theta^L)\right] - (\theta^H - \theta^L)A.$$

Under (i), this combined difference is strictly positive for $\kappa > 0$ (the two individual terms need not be non-negative on their own; only their sum is required to be positive). Hence the intercept channel contributes strictly positively to $\beta_s^{endo} - \beta_s^{exo}$ through $B > 0$.

(b) Composition channel. Under endogenous fertility with $\kappa > 0$, the equilibrium share $\pi_H^* < \frac{1}{2}$ (inequality (16), established by the fixed-point argument in Section 3), so the school-quality gap widens: $\theta_{endo}^H - \theta_{endo}^L = \kappa/(\pi_H^* \pi_L^*) > 4\kappa = \theta_{exo}^H - \theta_{exo}^L$, since $\pi_H \pi_L < \frac{1}{4}$ whenever $\pi_H \neq \frac{1}{2}$. The OVB *slope* contribution $D(p)(\theta^H - \theta^L) \cdot C$ is therefore strictly larger under endogenous fertility (here $C = cov(\log y_{t-1} \cdot Z_{t-1}, \log y_{t-1})/Var(\log y_{t-1}) > 0$ under (ii), as established in the proof of Proposition 3). The slope channel contributes strictly positively.

Combining (a) and (b), $\beta_s^{endo} - \beta_s^{exo} > 0$ for $\kappa > 0$. □ □

Remark. *The relative importance of the two channels depends on the parameter configuration. When school inequality* κ *is small relative to* $1 - \theta$*, the composition channel (*$\pi_L - \frac{1}{2} \approx 0$*) is* *negligible and the amplification is driven almost entirely by the intercept channel—the greater convexity of $\widetilde{g}$ relative to g. When κ is large (approaching the Assumption 1 bound), the composition shift becomes substantial and can either reinforce or partially offset the intercept channel depending on the shape of the income distribution. Under condition (ii) ($Var(\log y|y > y^M) \geq Var(\log y|y < y^M)$, satisfied for log-normal), the composition shift tends to reinforce the amplification because it gives more weight to the below-median group where income variance is lower, effectively concentrating the regression on the high-variance, high-persistence upper group.*

Remark. *Lemma 4 extends to partial segregation ($p \geq 1/2$) by replacing κ with $D(p)\kappa$ in the composition shift bound (condition (iii)), and replacing the intercept gap $d(\theta^H) - d(\theta^L)$ with $d(\bar{\theta}_H) - d(\bar{\theta}_L)$ where $\bar{\theta}_H - \bar{\theta}_L = D(p)\kappa/(\pi_H\pi_L)$. Since $D(p) \leq 1$, the composition effect is weakened while the intercept channel retains its positive sign (by convexity of d), so the sufficient condition is easier to satisfy under partial segregation. At $p = 1/2$, $D(p) = 0$ and $\beta_s^{endo} = \beta_s^{exo}$ trivially.*

Appendix B: Proofs of Body Propositions

Proof of Proposition 3.

Proof. Part (i). Let $A = \log((1-\alpha n)/n)$ and write the IGE expression as

$$\beta_s = D(p)\left[(\theta^H - \theta^L)A + g(\theta^H) - g(\theta^L)\right] B + \bar{\theta}_L + D(p)(\theta^H - \theta^L)\, C + \tfrac{cov(\log y_{t-1}, \tau_t)}{Var(\log y_{t-1})},$$

where $\bar{\theta}_L = p\theta^L + (1-p)\theta^H$, $Z_{t-1} = \mathbf{1}[y_{t-1} > y^M]$, and

$$B = \frac{cov(Z_{t-1}, \log y_{t-1})}{Var(\log y_{t-1})}, \qquad C = \frac{cov(\log y_{t-1}\, Z_{t-1}, \log y_{t-1})}{Var(\log y_{t-1})}.$$

Note that $B > 0$ follows from the median split (Z_{t-1} is the above-median-income indicator, so $\text{cov}(Z_{t-1}, \log y_{t-1}) > 0$ mechanically), independently of the within-group variance condition. The within-group variance condition enters the analysis through C and through the strict positivity of the intercept-channel contribution below. Differentiating with respect to κ, using

$\partial\theta^H/\partial\kappa = 2$ and $\partial\theta^L/\partial\kappa = -2$ under exogenous symmetric fertility,

$$\frac{\partial\beta_s}{\partial\kappa} = 2D(p)\Big\{\big[2A + g'(\theta^H) + g'(\theta^L)\big]B + (2C-1)\Big\}.$$

Under log-normal $\log y_{t-1} \sim N(\mu, \sigma^2)$, Appendix A.2 derives $B = \sqrt{2/\pi}/(2\sigma)$ (eq. 31) and $2C - 1 = \mu\sqrt{2/\pi}/\sigma$ (eq. 32), so the inner bracket is positive iff $\mu + A + \frac{1}{2}(g'(\theta^H) + g'(\theta^L)) > 0$. For a general distribution satisfying the variance condition, the analogous moment ratio $E[\log y_{t-1}]$ replaces μ, giving condition (6). The argument for $\partial\beta_s/\partial p > 0$ is analogous: $\partial\beta_s/\partial p = 2[(\theta^H - \theta^L)A + g(\theta^H) - g(\theta^L)]B + (\theta^H - \theta^L)(2C-1)$, which is positive under the same condition (with the secant slope $(g(\theta^H) - g(\theta^L))/(\theta^H - \theta^L)$ in place of the average of g' at endpoints; the secant version is weaker by convexity of g, so (6) is sufficient for both).

Part (ii). Differentiating the OVB term $D(p)(\theta^H - \theta^L)\log(\frac{1-\alpha n}{n})$ with respect to n uses $\frac{\partial}{\partial n}\log\frac{1-\alpha n}{n} = -\frac{\alpha}{1-\alpha n} - \frac{1}{n} = -\frac{1}{n(1-\alpha n)}$. Hence:

$$\frac{\partial\beta_s}{\partial n} = -D(p)(\theta^H - \theta^L)\cdot\frac{1}{n(1-\alpha n)}\cdot\frac{cov(Z_{t-1}, \log(y_{t-1}))}{Var(\log(y_{t-1}))} < 0$$

for all $p > 1/2$, $\kappa > 0$ and $\alpha n < 1$ (since $cov(Z_{t-1}, \log y_{t-1}) > 0$). Analogously, $\frac{\partial}{\partial\alpha}\log\frac{1-\alpha n}{n} = -\frac{n}{1-\alpha n}$, so:

$$\frac{\partial\beta_s}{\partial\alpha} = -D(p)(\theta^H - \theta^L)\cdot\frac{n}{1-\alpha n}\cdot\frac{cov(Z_{t-1}, \log(y_{t-1}))}{Var(\log(y_{t-1}))} < 0$$

for all $p > 1/2$, $\kappa > 0$ and $\alpha n < 1$. □

Proof of Proposition 4 (Strict-interior equilibrium).

Proof. For $d \in (0,1)$, the LHS of (15) satisfies $Td(d^2-1) < 0$ (since $T > 0$, $d > 0$, $d^2 - 1 < 0$), while the RHS satisfies $2\kappa(d^2+1) > 0$. Equality is therefore impossible: no positive root exists, ruling out $\pi_H \geq 1/2$. The cubic in p has three real roots: a degenerate corner root $\pi_H \approx \kappa/(1-\theta)$ (where $n_H \to 0^+$, so the share definition is satisfied trivially), the proper interior equilibrium π_H^* near $1/2 - \kappa/(1-\theta)$, and a spurious root $\pi_H > 1$ outside the feasible domain. Only π_H^* lies strictly inside $(\kappa/(1-\theta), 1/2)$ and corresponds to an interior equilibrium with both $n_H^* > 0$ and $n_L^* > 0$. □ □

Proof of Proposition 5.

Proof. From $\widetilde{g}(\theta^s) = \theta^s \log(\frac{\alpha\theta^s}{1-\theta^s})$, differentiation with respect to α gives

$$\frac{\partial}{\partial \alpha}\left[\theta \log\left(\frac{\alpha\theta}{1-\theta}\right)\right] = \frac{\theta}{\alpha} > 0$$

□

Proof of Proposition 6.

Proof. Part (i). The slope gain is $\frac{1}{2}(\theta^H - \theta^L)(\phi - 1)E[\log y] > 0$ for $\kappa > 0$, $\phi > 1$, and $E[\log y] > 0$. The composition uplift is $(\bar{\theta} - \theta)E[\log y] > 0$ since $\bar{\theta} = \frac{1}{2}(\theta^H + \theta^L) > \theta$ whenever $\pi_H < \frac{1}{2}$, so the total gain is strictly positive. For the intercept gain, expand $\widetilde{g}(\theta^H)$ and $\widetilde{g}(\theta^L)$ around θ to second order in κ. With $\theta^H = \theta + \kappa/\pi_H$ and $\theta^L = \theta - \kappa/\pi_L$,

$$\tfrac{1}{2}\big[\widetilde{g}(\theta^H) + \widetilde{g}(\theta^L)\big] - \widetilde{g}(\theta) = \widetilde{g}'(\theta)\,(\bar{\theta} - \theta) + \tfrac{1}{4}\widetilde{g}''(\theta)\big[(\theta^H - \theta)^2 + (\theta^L - \theta)^2\big] + O(\kappa^3).$$

Under endogenous fertility with $\pi_H^* = \frac{1}{2} - \kappa/(1-\theta) + O(\kappa^3)$, $\theta^H - \theta = \kappa/\pi_H \approx 2\kappa + \frac{4\kappa^2}{1-\theta} + O(\kappa^3)$ and $\theta^L - \theta \approx -2\kappa + \frac{4\kappa^2}{1-\theta} + O(\kappa^3)$, so $\bar{\theta} - \theta \approx \frac{4\kappa^2}{1-\theta}$ (second-order in κ, not first), and $(\theta^H - \theta)^2 + (\theta^L - \theta)^2 \approx 8\kappa^2$. Both pieces of the Taylor expansion are therefore $O(\kappa^2)$, and the intercept gain reduces to

$$\tfrac{1}{2}\big[\widetilde{g}(\theta^H) + \widetilde{g}(\theta^L)\big] - \widetilde{g}(\theta) = 2\Big[\widetilde{g}''(\theta) + \tfrac{2\widetilde{g}'(\theta)}{1-\theta}\Big]\,\kappa^2 + O(\kappa^3).$$

The bracketed leading coefficient is strictly positive throughout the empirically relevant range: at the baseline $(\theta, \alpha) = (0.27, 0.125)$ it equals $\widetilde{g}''(\theta) + 2\widetilde{g}'(\theta)/(1-\theta) \approx 6.95 - 4.67 = 2.28 > 0$; at $\theta = 0.10$ it equals $\approx 5.31 > 0$. So for small κ the intercept gain is strictly positive, even when $\widetilde{g}'(\theta) < 0$ (the strict α-threshold fails). This is the weaker small-κ condition that replaces $\widetilde{g}'(\theta^L) > 0$.

Part (ii). Differentiating with respect to ϕ: $\frac{\partial E[h_{t,s}]}{\partial \phi} = \frac{1}{2}(\theta^H - \theta^L)E[\log y] > 0$ for $\kappa > 0$. Differentiating with respect to κ requires accounting for the endogenous response of $\pi_H(\kappa)$ and hence of $\theta^H(\kappa)$ and $\theta^L(\kappa)$. At fixed π_H (the partial effect), $\partial\theta^H/\partial\kappa = 1/\pi_H$ and $\partial\theta^L/\partial\kappa = -1/\pi_L$, giving positive contributions from both the convexity of $\widetilde{g}$ (intercept) and the slope and composition terms. The equilibrium feedback ($d\pi_H/d\kappa < 0$) further widens the quality gap, reinforcing the gain. $\square$

Proof of Proposition 7.

Proof. Part (i). At $\kappa = 0$, both groups have the same slope ($\theta^H = \theta^L = \theta$) and the between-group intercept gap vanishes. For $\kappa > 0$, the slopes diverge ($\theta^H > \theta^L$) and the between-group gap in child outcomes becomes positive. The variance condition $V_H \geq V_L$ ensures that the high-slope group receives at least as much within-group variance weight.

Part (ii). Under log normal distribution within each income group, the within-group component of β_s (the within-group term A) is the same under endogenous and exogenous fertility when evaluated at the same fertility reference n. The between-group term B measures the contribution of the gap in group means to the pooled IGE. The endogenous-fertility intercept is $\widetilde{g}(\theta^s)$; the exogenous-fertility intercept (at a fixed reference n) is $g(\theta^s) + \theta^s A$, where $A = \log((1 - \alpha n)/n)$ is the exogenous-fertility intercept term evaluated at the natural reference $n = (1 - \theta)/(2\alpha)$ (the optimal fertility under equal opportunity). The two intercept gaps are therefore $\widetilde{g}(\theta^H) - \widetilde{g}(\theta^L)$ (endo) and $g(\theta^H) - g(\theta^L) + (\theta^H - \theta^L)A$ (exo). Endogenous fertility amplifies the between-group OVB component (in the direction that pushes β_s up) iff the endo gap is less negative than the exo gap, i.e. $d(\theta^H) - d(\theta^L) > (\theta^H - \theta^L)\,A$. Expanding d around θ to first order in κ, $d(\theta^H) - d(\theta^L) = d'(\theta)(\theta^H - \theta^L) + O(\kappa^3)$, so the small-$\kappa$ condition is $d'(\theta) > A$. At the baseline ($\theta = 0.27$, $\alpha = 0.125$, $n = 2.92$): $d'(0.27) \approx -0.94$, $A \approx -1.53$; the condition $-0.94 > -1.53$ holds with margin 0.59. Numerically, $\widetilde{g}(\theta^H) - \widetilde{g}(\theta^L) = -0.122$ versus $g(\theta^H) - g(\theta^L) + (\theta^H - \theta^L)A = -0.168$, confirming the endo gap is less negative than the exo gap.

$\square$

Proof of Proposition 8.

Proof. Part (i). Differentiating the average human capital with respect to p, using $\frac{\partial D}{\partial p} = 2$ and the chain rule through $\bar{\theta}_H = \theta + D(p)\kappa \cdot f(\pi_H)$ and $\bar{\theta}_L = \theta - D(p)\kappa \cdot h(\pi_H)$ (where f and h are functions of the equilibrium π_H), the derivative has two positive components. The intercept channel is positive under Lemma 4(i) applied to the expected qualities $(\bar{\theta}_H, \bar{\theta}_L)$. The slope channel: $\kappa(\phi - 1)E[\log(y_{t-1})]/(\pi_H \pi_L) > 0$ for $\phi > 1$.

Part (ii). At $p = 1/2$, $\bar{\theta}_H = \bar{\theta}_L$, the between-group intercept gap vanishes, and both the within-group and between-group components of β_s equal the exogenous-fertility benchmark. For $p > 1/2$, the slopes diverge ($\bar{\theta}_H > \bar{\theta}_L$) and the fertility-driven intercept comparison $d(\bar{\theta}_H) - d(\bar{\theta}_L) > (\bar{\theta}_H - \bar{\theta}_L)A$ (Lemma 4(i) applied to expected qualities) amplifies the between-group component, as in Proposition 7. The derivative with respect to p is positive by the same argument as Part (i) applied to the OVB decomposition.

Part (iii). At $p = 1$, $D(p) = 1$ and $\bar{\theta}_j = \theta^j$, so $\widehat{g}(\theta^s, \theta^s) = \widetilde{g}(\theta^s)$ and all expressions reduce to Sections 3.1–3.3. At $p = 1/2$, $D(p) = 0$ and $\bar{\theta}_H = \bar{\theta}_L = \frac{1}{2}(\theta^H + \theta^L)$, so all families choose the same fertility and the IGE reduces to the exogenous-fertility benchmark (though human capital levels differ because the common endogenous fertility depends on the average school quality, not on θ alone). $\square$