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SEMIGROUP GRADED ALGEBRAS AND GRADED PI-EXPONENT

BY

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ABSTRACT

Let S be a semigroup. We study the structure of graded-simple S-graded algebras A and the exponential rate $\mathrm{PIexp}^{S\text{-gr}}(A) := \lim_{n\to\infty} \sqrt[n]{c_n^{S\text{-gr}}(A)}$ of growth of codimensions $c_n^{S\text{-gr}}(A)$ of their graded polynomial identities. This is of great interest since such algebras can have non-integer $\mathrm{PIexp}^{S\text{-gr}}(A)$ despite being finite dimensional and associative. In addition, such algebras can have a non-trivial Jacobson radical $J(A)$. All this is in strong contrast with the case when S is a group since in the group case $J(A)$ is trivial, $\mathrm{PIexp}^{S\text{-gr}}(A)$ is always integer and, if the base field is algebraically closed, then $\mathrm{PIexp}^{S\text{-gr}}(A)$ equals $\dim A$. Without any restrictions on the base field F, we classify graded-simple S-graded algebras A for a class of semigroups S which is complementary to the class of groups. We explicitly describe the structure of $J(A)$ showing that $J(A)$ is built up of pieces of a maximal S-graded semisimple subalgebra of A which turns out to be simple. When F is algebraically closed, we get an upper bound for $\overline{\lim}_{n\to\infty} \sqrt[n]{c_n^{S\text{-gr}}}(A)$. If $A/J(A) \cong M_2(F)$ and S is a right zero band, we show that this upper bound is sharp and $\mathrm{PIexp}^{S\text{-gr}}(A)$ indeed exists. In particular, we present an infinite family of graded-simple algebras A with arbitrarily large non-integer $\mathrm{PIexp}^{S\text{-gr}}(A)$.

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1. Introduction

It is well known that the study of polynomial identities of an associative algebra A over a field F of characteristic 0 is equivalent to the study of multilinear identities. If one denotes by P_n the space of F-multilinear polynomials in the non-commuting variables $x_1, \dots, x_n$, then the nth codimension of A, denoted $c_n(A)$, is $\dim(P_n/P_n \cap \mathrm{Id}(A))$, where $\mathrm{Id}(A)$ denotes the ideal of all identities of A. In 1972, Regev [30] proved that if A satisfies a non-trivial polynomial identity, then $(c_n(A))_n$ is exponentially bounded. In recent years the asymptotic behaviour of the sequence $(c_n(A))_n$ has been investigated extensively and this has led to classification results of several varieties of algebras. Giambruno and Zaicev [16] (in case A is finitely generated, see [15]) proved the following fundamental result for an algebra A satisfying a polynomial identity: the limit $\lim_{n\to\infty} \sqrt[n]{c_n(A)}$ exists and it is an integer, which is called the PI-exponent. This confirmed a conjecture of Amitsur.

In recent years, this problem has been investigated for several classes of rings that have some refined information, such as being graded by a group. To do so, one has to give appropriate definitions for the identities and notions considered. In the case of algebras A graded by a (semi)group S, one obtains a sequence $(c_n^{S\text{-gr}}(A))_n$ of S-graded codimensions. Aljadeff, Giambruno and La Mattina [1, 2, 13] proved that if an associative PI-algebra is graded by a finite group, then also the graded-PI exponent exists and it is an integer, i.e., the graded analog of Amitsur's conjecture holds for group graded algebras.

The first author, in [20, Theorem 1] and [22, Theorem 3], proved the same for finite dimensional associative and Lie algebras graded by any group. It is well known that in the case of non-associative algebras non-integer exponents can arise. The first example of an infinite dimensional Lie algebra with a non-integer ordinary PI-exponent was constructed by S. P. Mishchenko and M. V. Zaicev in [26]. We refer to [27] for a complete proof and to [8] for some recent developments. In [14] Giambruno, Mishchenko and Zaicev constructed an infinite family of non-associative algebras with an arbitrary exponent.

Recently, the first author [21] constructed the first example of a finite dimensional associative algebra over a field of characteristic 0 with a non-integer exponent of some kind of polynomial identities. The example is a finite dimensional algebra A that is graded by a finite semigroup S with two elements that is not a group and the identities considered are the S-graded polynomial identities. Equivalently, A is either the direct sum of two left (or right) ideals or it is the direct sum of a two-sided ideal and a subring. This naturally leads to the investigation of the graded PI-exponent for algebras A graded by an arbitrary finite semigroup S. In case A is S-graded-simple, such a ring turns out to be a generalized matrix ring, i.e., $A = \bigoplus_{1\leqslant i\leqslant n,\, 1\leqslant j\leqslant m} A_{ij}$, a direct sum of additive subgroups A_{ij} such that $A_{ij}A_{kl} \subseteq A_{il}$ (note that $A_{ij}A_{kl}$ is not necessarily equal to $\{0\}$, even if $j \neq k$). In this paper we investigate the graded PI-exponent in this larger context of finite dimensional algebras A graded by a finite semigroup S, with A being graded-simple. Note that since in our case the graded PI-exponent can be a non-integer, we have to develop methods of study that are different from those used in the group graded case.

Recall that Bahturin, Zaicev and Sehgal in [7] have described such algebras in the case when S is a finite group and F is algebraically closed; it turns out that such algebras are matrix algebras over a finite dimensional graded skew field. Hence, for our investigations and to reduce the problem to the group graded case, it is crucial first to describe the structure of S-graded-simple algebras in the case when the maximal subgroups of S are trivial. Without any restrictions on the base field F, we construct an S-graded subalgebra B such that B is simple and $A = B \oplus J(A)$, a direct sum as vector spaces. (In particular, we provide a constructive proof of the Mal'cev–Wedderburn theorem.) Furthermore $J(A)$ has a very specific decomposition as a direct sum of simple B-modules. Although the homogeneous part of $J(A)$ is trivial, these summands are strongly related to the description of the homogeneous components of A. Earlier results on semigroup graded rings that satisfy a polynomial identity have been obtained by Kelarev [24] and Clase and Jespers [9, 10]. A characterization of semigroup graded rings that are simple has received a lot of attention; see, for example, [5, 4, 6, 31, 28]. For more results on graded rings we refer to [25]. In the second part of the paper, the obtained information on graded-simple algebras is used to give an upper bound for graded codimensions in the case $A/J(A) \cong M_n(F)$, a matrix algebra. Moreover, if $n = 2$, then we are able to calculate the graded PI-exponent and hence we obtain many examples of algebras with a non-integer graded PI-exponent. It turns out that for any positive integer m, the number $1 + m + \sqrt{m}$ is the graded PI-exponent of a graded-simple algebra.

The outline of the paper is as follows. In Section 2 we first show that the graded simplicity can easily be reduced to rings graded by finite simple semigroups and thus to generalized matrix rings. Hence, from now on we assume that graded rings are generalized matrix rings, i.e., rings graded by a 0-simple finite semigroup with trivial maximal subgroups. Second, we prove a criterion for two graded-simple rings to be graded isomorphic; this is done in terms of their factor rings by their respective Jacobson radical (Theorem 2.3). In Section 3 we state some background on left ideals in finite dimensional simple algebras. In Section 4 we prove important properties describing the structure of finite dimensional graded-simple algebras (Theorem 4.2 and Theorem 4.7). In Section 5 we give a full description of finite dimensional graded-simple algebras. In Section 6 we give the necessary background on graded polynomial identities and graded codimensions. In Section 7 we introduce polynomial H-identities for algebras with a generalized H-action where H is a unital associative algebra. Polynomial H-identities provide a powerful tool to study S-graded polynomial identities since graded codimensions equal H-codimensions when H is the algebra dual to the semigroup bialgebra of S. In Section 8 we provide a brief outline of the rest of the paper where we deal with graded codimensions. In Section 9 we prove an upper bound for graded codimensions of semigroup graded-simple algebras A over a field F of characteristic 0 with $A/J(A) \cong M_k(F)$ (Theorem 9.8). In Sections 10 and 11 we prove the existence of the graded PI-exponent for finite dimensional S-graded-simple algebras A over a field F of characteristic 0 with $A/J(A) \cong M_2(F)$ and S being a right zero band. Also an explicit formula is obtained. For some algebras the graded PI-exponent equals $\dim A$ (Theorem 10.5) and for some algebras the graded PI-exponent is strictly less than $\dim A$ (Theorem 11.5). The latter is in contrast to the group graded case for which the growth rate of such algebras always equals $\dim(A)$ provided F is algebraically closed.

Throughout this paper all rings are associative but do not necessarily have a unit element, except when explicitly mentioned otherwise. Also, we use the abbreviations

$$\mathbb{N} = \{n \in \mathbb{Z} \mid n > 0\} \quad \text{and} \quad \mathbb{Z}_+ = \{n \in \mathbb{Z} \mid n \geqslant 0\}.$$

If $k \in \mathbb{N}$ and R is a unital ring, then the (i, j) matrix unit of $M_k(R)$ is denoted by e_{ij}. If W is a subset of a left module V over a ring Λ, we denote the Λ-linear span of W in V by $\langle W\rangle_\Lambda$. If W is a subset of a semigroup S, then by $\langle W\rangle$ (without any subscripts) we denote the subsemigroup of S generated by W. Given $a \in \mathbb{R}$, denote by $[a]$ its floor $\max\{n \in \mathbb{Z} \mid n \leqslant a\}$.

2. General reduction

Let S be an arbitrary semigroup and R an **S-graded ring**, that is

$$R = \bigoplus_{s\in S} R^{(s)},$$

a direct sum of additive subgroups $R^{(s)}$ of R such that $R^{(s)}R^{(t)} \subseteq R^{(st)}$, for all $s, t \in S$. One says that R is an (S-)**graded-simple** ring if $R^2 \neq 0$ and $\{0\}$ and R are the only homogeneous ideals of R. Recall that an additive subgroup P of R is said to be **homogeneous** or **graded** if $P = \bigoplus_{s\in S}(R^{(s)} \cap P)$. Without loss of generality, we may replace S by the semigroup generated by $\mathrm{supp}(R) = \{s \in S \mid R^{(s)} \neq 0\}$. Note that if S has a zero element θ, then $R^{(\theta)}$ is a homogeneous ideal of R, and thus if $R^{(\theta)} \neq 0$ and R is graded-simple, then $R^{(\theta)} = R$ and $\mathrm{supp}(R) = \{\theta\}$, a trivial graded ring. So, in order to investigate graded-simple rings, without loss of generality, we may assume that if $\theta \in S$, then $R^{(\theta)} = 0$. Replacing S by $S^0 := S \cup \{\theta\}$, if necessary, we assume that S has a zero element, $R^{(\theta)} = 0$ and $S = \langle \mathrm{supp}(R)\rangle \cup \{\theta\}$. This assumption will be used implicitly throughout the paper without mention.

From now on we assume that R is S-graded-simple and $S = \langle \mathrm{supp}(R)\rangle \cup \{\theta\}$. Note that if I is an ideal of S, then $R_I = \bigoplus_{s\in I} R^{(s)}$ is an ideal of R. Hence, by the graded-simplicity, either $R_I = 0$ or $R_I = R$. The latter implies that $I = S$. Thus, if I is a proper ideal of S, then $I \cap \mathrm{supp}(R) = \varnothing$. Therefore, we may replace S by the Rees factor semigroup S/I and thus we may assume that S itself does not have proper ideals and $S^2 \neq \{\theta\}$. Such a semigroup S is called a **0-simple semigroup**. If $T = S \setminus \{\theta\}$ is a semigroup, then we may replace S by the simple semigroup T.

Therefore, if an S-graded ring R is S-graded-simple, then without loss of generality we may assume that S is a 0-simple semigroup.

Recall that there are three types of 0-simple semigroups:

(1) S is **completely 0-simple**, i.e., S is 0-simple and contains a non-zero primitive idempotent;
(2) S is 0-simple and contains a non-zero idempotent, but does not contain primitive idempotents (recall that in this case each idempotent is the identity element of a subsemigroup isomorphic to the bicylic semigroup [11, Theorem 2.54] and, in particular, S is an infinite semigroup);
(3) S does not have non-zero idempotents and again S is infinite.

Let G be a group, let I, J be sets, and let $P = (p_{ji})$ be a $J \times I$ matrix with entries in $G^0 := G \cup \{\theta\}$. Let $\mathcal{M}(G^0; I, J; P)$ denote the set

$$\{(g, i, j) \mid i \in I,\ j \in J,\ g \in G^0\}$$

with all elements (θ, i, j) being identified with the zero element θ. One has the following associative multiplication on $\mathcal{M}(G^0; I, J; P)$:

$$(g, i, j)(h, k, \ell) = (gp_{jk}h, i, \ell).$$

The semigroup $\mathcal{M}(G^0; I, J; P)$ is called the **Rees $I \times J$ matrix semigroup over the group with zero G^0 with sandwich matrix P.**

Recall that by the Rees theorem [11, Theorem 3.5] every completely 0-simple semigroup S is isomorphic to $\mathcal{M}(G^0; I, J; P)$ for a maximal subgroup G of S, sets I, J and a matrix P such that every row and every column of P has at least one nonzero element.

If we assume that S is finite, then the S-graded simplicity of R implies that S is non-nilpotent and contains a nonzero primitive idempotent. In other words, for finite semigroups we may restrict our consideration to gradings by completely 0-simple semigroups

$$S = \mathcal{M}(G^0, n, m; P) = \{(g, i, j) \mid 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m,\ g \in G^0\}$$

where G is a group, n and m are positive integers, and P is an $n \times m$ matrix with entries in G^0. Note that R is also graded by the semigroup

$$S' = \mathcal{M}(\{e\}^0, n, m; P'),$$

where the (i, j) component of P' is e if $p_{ij} \neq 0$ and θ otherwise.

In this paper we are focusing on this type of grading and we give a complete classification of when such graded rings are graded simple (so we will deal with the case that the maximal subgroups are trivial). Recall that group-graded rings (over an algebraically closed field) that are graded-simple have been described by Bahturin, Sehgal and Zaicev in [5, 7]. Combining this result with our classification might lead to a solution of the general problem (i.e., with maximal subgroups not necessarily trivial).

The following elementary result shows that the difference between being simple and S-graded-simple lies in the annihilators of the ring R being nonzero. We call a ring **faithful** if its left and right annihilators are trivial, i.e., if $aR = 0$ or $Ra = 0$ for some $a \in R$, then $a = 0$.

Proposition 2.1: *Let R be a ring graded by a finite 0-simple semigroup S with trivial maximal subgroups. Then, R is simple if and only if R is S-graded-simple and R is faithful.*

Proof. The necessity of the conditions is obvious. Suppose R is S-graded-simple and R is faithful. Since S is a finite 0-simple semigroup with trivial maximal subgroups, $S \cong \mathcal{M}(\{e\}^0; I, J; P)$. If L is a nonzero ideal of R, then RLR is an S-homogeneous ideal of R. Since R is faithful, L is nonzero. Hence R is simple. ∎

The following easy example shows that the faithfulness condition can not be removed.

Recall that a semigroup T is a **right zero band** if $st = t$ for any $s, t \in T$. Let $T = \{e, f\}$ be a right zero band of two elements. Clearly, T^0 is a 0-simple semigroup and the semigroup algebra FT, where F is a field, is graded-simple. However, this algebra is not simple as it contains the proper two-sided ideal $F(e - f)$. Note that $(e - f)FT = 0$ and thus FT is not faithful.

Denote the Jacobson radical of a ring R by $J(R)$. We finish the general part by showing that if $R \neq J(R)$ and R is S-graded-simple, then $J(R)$ does not contain any specific information concerning the S-grading and even the structure of R.

First we show that a non-trivial ideal cannot contain homogeneous elements.

Lemma 2.2: *Let $I \neq R$ be a two-sided ideal of an S-graded-simple ring $R = \bigoplus_{s \in S} R^{(s)}$ for some semigroup S. Then*

$$R^{(s)} \cap I = 0$$

for all $s \in S$.

Proof. Suppose $r \in R^{(s)} \cap I$ for some $s \in T$. Then the smallest two-sided ideal I_0 containing r is homogeneous. Since $I_0 \subseteq I \subsetneqq R$, we get $I_0 = 0$ and $r = 0$. ∎

Recall that a homomorphism $\varphi \colon R_1 \to R_2$ of S-graded rings R_1 and R_2 is **graded** if $\varphi(R_1^{(s)}) \subseteq R_2^{(s)}$ for all $s \in S$. Two S-graded rings R_1 and R_2 are **isomorphic as graded rings** if there exists a graded isomorphism $R_1 \to R_2$. In this case we say that the gradings on R_1 and R_2 are **isomorphic**.

The theorem below will be used later in the case when $R/J(R)$ is simple.

Theorem 2.3: *Let S be a semigroup and let $R_i = \bigoplus_{s\in S} R_i^{(s)}$, $i = 1,2$, be two S-graded-simple rings, $R_i \neq J(R_i)$ for both $i = 1,2$. If there exists a ring isomorphism $\bar\varphi\colon R_1/J(R_1) \to R_2/J(R_2)$ such that $\bar\varphi(\pi_1(R_1^{(s)})) = \pi_2(R_2^{(s)})$ for every $s \in S$ where $\pi_i\colon R_i \to R_i/J(R_i)$, $i = 1,2$, are the natural epimorphisms, then there exists an isomorphism $\varphi\colon R_1 \to R_2$ of graded rings such that $\pi_2\varphi = \bar\varphi\pi_1$.*

Conversely, if $\varphi\colon R_1 \to R_2$ is an isomorphism of graded rings, we can define the ring isomorphism $\bar\varphi\colon R_1/J(R_1) \to R_2/J(R_2)$ by $\bar\varphi(\pi_1(a)) = \pi_2\varphi(a)$ for all $a \in R_1$ and get $\bar\varphi(\pi_1(R_1^{(s)})) = \pi_2(R_2^{(s)})$ for every $s \in S$.

Proof. Suppose that there exists such an isomorphism $\bar\varphi$. Lemma 2.2 implies that

$$\pi_i|_{R_i^{(s)}}\colon R_i^{(s)} \to \pi_i(R_i^{(s)})$$

is an isomorphism of additive groups for every $s \in S$ and $i = 1,2$. Define $\varphi\colon R_1 \to R_2$ by

$$\varphi(r) := (\pi_2|_{R_2^{(s)}})^{-1}\bar\varphi\pi_1(r) \quad \text{for } r \in R_1^{(s)} \text{ and } s \in S$$

and extend it additively. Clearly, $\varphi(R_1^{(s)}) = R_2^{(s)}$ and φ is a graded surjective homomorphism of additive groups. Moreover $\pi_2\varphi = \bar\varphi\pi_1$ holds.

Suppose $\varphi(\sum_{s\in S} r^{(s)}) = 0$ for some $r^{(s)} \in R_1^{(s)}$ and $s \in S$. Since φ is graded, we have $\varphi(r^{(s)}) = 0$ for every $s \in S$. Hence $\pi_1(r^{(s)}) = 0$ and thus $r^{(s)} = 0$, since by Lemma 2.2 we have $R_1^{(s)} \cap J(R_1) = 0$ for every $s \in S$. Therefore, φ is a bijection.

Now we prove that φ is an isomorphism of rings. Indeed, suppose $r^{(s)} \in R_1^{(s)}$ and $r^{(t)} \in R_1^{(t)}$. Then

$$\pi_2\varphi(r^{(s)}r^{(t)}) = \bar\varphi\pi_1(r^{(s)}r^{(t)}) = \bar\varphi\pi_1(r^{(s)})\bar\varphi\pi_1(r^{(t)}) = \pi_2(\varphi(r^{(s)})\varphi(r^{(t)})).$$

Since both $\varphi(r^{(s)}r^{(t)})$ and $\varphi(r^{(s)})\varphi(r^{(t)})$ belong to $R_2^{(st)}$ and $\pi_2|_{R_2^{(st)}}$ is an isomorphism, we get

$$\varphi(r^{(s)}r^{(t)}) = \varphi(r^{(s)})\varphi(r^{(t)}) \quad \text{for all } r^{(s)} \in R_1^{(s)} \text{ and } r^{(t)} \in R_1^{(t)}$$

and the first assertion is proved.

The second assertion is obvious since the Jacobson radical is stable under isomorphisms. ∎

Remark 2.4: Clearly, Theorem 2.3 holds not only for the Jacobson radical, but for any radical. (See the general definition, e.g., in [25, p. 66].)

3. Left ideals of matrix algebras

Here we state some propositions which turn out to be very useful in order to classify all possible finite dimensional T-graded-simple algebras for some (right) zero band T.

These results are known; however, for the reader's convenience, we include their proofs.

Lemma 3.1: *Let F be a field and let $k \in \mathbb{N}$. Consider the natural $M_k(F)$-action on the coordinate space F^k by linear operators. Then there exists a one-to-one correspondence between left ideals I in $M_k(F)$ and subspaces $W \subseteq F^k$ such that*

$$(1) \qquad I = \operatorname{Ann} W := \{a \in M_k(F) \mid aW = 0\}, \quad W = \bigcap_{a\in I} \ker a,$$

and

$$\dim I = k(k - \dim W).$$

Moreover, if $I_1 = \operatorname{Ann} W_1$ and $I_2 = \operatorname{Ann} W_2$, then

$$I_1 + I_2 = \operatorname{Ann}(W_1 \cap W_2) \quad \text{and} \quad I_1 \cap I_2 = \operatorname{Ann}(W_1 + W_2).$$

Proof. Let W be a subspace of F^k. Let $w_{k+1-\dim W}, \dots, w_k$ be a basis in W. Choose $w_1, \dots, w_{k-\dim W} \in F^k$ such that $w_1, \dots, w_k$ is a basis in F^k. Then $\operatorname{Ann} W$ consists of all $a \in M_k(F)$ that have matrices in the basis $w_1, \dots, w_k$ with zeros in the last $\dim W$ columns. Note that $\bigcap_{a\in \operatorname{Ann} W} \ker a = W$ and $\dim \operatorname{Ann} W = k(k - \dim W)$.

Let $I \subseteq M_k(F)$ be a left ideal. Since I is a left ideal in the semisimple artinian algebra $M_k(F)$, by [23, Theorem 1.4.2], there exists an idempotent $e \in I$ such that $I = M_k(F)e$. Thus $I(\ker e) = 0$. Note that e is acting on F^k as a projection. Hence

$$F^k = \ker e \oplus \operatorname{im} e.$$

We choose a basis in F^k that is the union of bases in $\operatorname{im} e$ and $\ker e$. Then the matrix of e in this basis is $\left(\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and I contains all operators with zeros in the last $\dim \ker e$ columns. Thus $\bigcap_{a\in I} \ker a = \ker e$ and $\operatorname{Ann} \ker e = I$. Together with the first paragraph this implies that (1) is indeed a one-to-one correspondence.

Suppose $I_1 = \operatorname{Ann} W_1$ and $I_2 = \operatorname{Ann} W_2$. Then

$$I_1 \cap I_2 = \operatorname{Ann} W_1 \cap \operatorname{Ann} W_2 = \operatorname{Ann}(W_1 + W_2).$$

Moreover, $(I_1+I_2)(W_1\cap W_2)=0$ and $I_1+I_2\subseteq \operatorname{Ann}(W_1\cap W_2)$. Now

$$\begin{aligned}\dim(I_1+I_2) &= \dim I_1+\dim I_2-\dim(I_1\cap I_2)\\ &= k(2k-\dim W_1-\dim W_2)-k(k-\dim(W_1+W_2))\\ &= k(k-(\dim W_1+\dim W_2-\dim(W_1+W_2)))\\ &= \dim \operatorname{Ann}(W_1\cap W_2)\end{aligned}$$

implies the lemma. ∎

Theorem 3.2: *Let $k,s\in\mathbb{N}$ and let F be a field. Assume I_i are left ideals of $M_k(F)$ such that $M_k(F)=\bigoplus_{i=1}^s I_i$. Suppose $\dim I_i=n_ik$, $n_i\in\mathbb{Z}_+$. Then there exists $P\in \mathrm{GL}_k(F)$ such that $P^{-1}I_iP$ consists of all matrices with zeros in all columns except those that have numbers*

$$1+\sum_{j=1}^{i-1}n_j,\ 2+\sum_{j=1}^{i-1}n_j,\dots,n_i+\sum_{j=1}^{i-1}n_j.$$

Proof. Consider the standard action of $M_k(F)$ on the coordinate space F^k. By Lemma 3.1, $I_i=\operatorname{Ann}V_i$ for some $V_i\subseteq F^k$. Applying the duality from Lemma 3.1 to $M_k(F)=\bigoplus_{i=1}^s I_i$, we get $\bigcap_{i=1}^s V_i=0$ and

$$V_i+\bigcap_{\substack{j=1,\\ j\neq i}}^{s} V_j=F^k\quad\text{for all } 1\leqslant i\leqslant s.$$

Denote

$$W_i=\bigcap_{\substack{j=1,\\ j\neq i}}^{s} V_j.$$

Then $F^k=V_i\oplus W_i$. Note that

$$\operatorname{Ann}W_i=\bigoplus_{\substack{j=1,\\ j\neq i}}^{s} I_j.$$

Since $\bigcap_{\substack{j=1,\\ j\neq i}}^{s}\operatorname{Ann}W_j=I_i$, we have $V_i=\bigoplus_{\substack{j=1,\\ j\neq i}}^{s} W_j$.

Now, choose a basis in F^k that is a union of bases in W_i. Denote the transition matrix from the standard basis to this basis by $P\in\mathrm{GL}_k(F)$. Then each $P^{-1}I_iP$ consists of all matrices with zeros in all columns except those that correspond to W_i. ∎

LEMMA 3.3: *Let I be a minimal left ideal of $M_k(F)$ where $k \in \mathbb{N}$, F is a field. Then there exist $\mu_j \in F$, $1 \leqslant j \leqslant k$, such that*

$$I = \left\langle \sum_{j=1}^{k} \mu_j e_{ij} \mid 1 \leqslant i \leqslant k \right\rangle_F .$$

Proof. Let $a = \sum_{i,j=1}^{k} \mu_{ij} e_{ij} \in I \backslash \{0\}$. Since $\sum_{\ell=1}^{k} e_{\ell\ell} a = a$, we have $e_{\ell\ell} a \neq 0$ for some $1 \leqslant \ell \leqslant k$. Define $\mu_j := \mu_{\ell j}$ for all $1 \leqslant j \leqslant k$. Then

$$\left\langle \sum_{j=1}^{k} \mu_j e_{ij} \mid 1 \leqslant i \leqslant k \right\rangle_F = \langle e_{i\ell} a \mid 1 \leqslant i \leqslant k \rangle_F$$

is a left ideal contained in I. Since I is a minimal left ideal, we get the lemma. ∎

LEMMA 3.4: *Let D be a finite dimensional division algebra over a field F and let $k \in \mathbb{N}$. Let I and V be, respectively, a left and a right ideal of $M_k(D)$. Then*

$$\dim_F(VI) = \frac{\dim_F V \dim_F I}{k^2 \dim D}.$$

Proof. Note that

$$I \cong \underbrace{M_k(D)e_{11} \oplus \cdots \oplus M_k(D)e_{11}}_{\dim_F I/(k \dim_F D)} \quad \text{and} \quad V \cong \underbrace{e_{11}M_k(D) \oplus \cdots \oplus e_{11}M_k(D)}_{\dim_F V/(k \dim_F D)}$$

as, respectively, left and right $M_k(D)$-modules. Hence

$$\begin{aligned} \dim_F(VI) &= \frac{\dim_F I}{k \dim_F D} \dim_F(VM_k(D)e_{11}) \\ &= \frac{\dim_F V \dim_F I}{k^2 (\dim_F D)^2} \dim_F(e_{11}M_k(D)e_{11}) \\ &= \frac{\dim_F V \dim_F I}{k^2 \dim D}. \end{aligned}$$ ∎

4. Graded-simple algebras

Throughout this section A is a finite dimensional S-graded F-algebra where F is a field and

$$S = \mathcal{M}(\{e\}^0, n, m; P) = \langle \mathrm{supp}(A) \rangle \cup \{\theta\}$$

is a finite completely 0-simple semigroup having trivial maximal subgroups. Denote the homogeneous component corresponding to (e, i, j) by A_{ij}. Then

$$A = \bigoplus_{\substack{1 \leqslant i \leqslant n, \\ 1 \leqslant j \leqslant m}} A_{ij}$$

and

$$A_{ij} A_{k\ell} \subseteq A_{i\ell}.$$

In particular, each homogeneous component A_{ij} is a subalgebra of A. If $p_{jk} = 0$ where $(p_{jk})_{j,k} := P$, then $A_{ij} A_{k\ell} = 0$.

Note that A is $\mathcal{M}(\{e\}^0, n, m; P)$-graded-simple for some matrix P if and only if A is $\mathcal{M}(\{e\}^0, n, m; P')$-graded-simple, where P' is the matrix with all the entries being equal to e.

We begin with some basic observations.

Lemma 4.1: *The following properties hold for an S-graded-simple algebra A:*

(1) $A_{ij} \cap J(A) = 0$ *for all* i, j;

(2) *if* $I \subseteq A$ *is a set, then* AIA *is a homogeneous ideal (and thus* AIA *equals either* 0 *or* A*);*

(3) $AJ(A)A = 0$.

Proof. Part (1) is a direct consequence of Lemma 2.2, Part (2) is obvious and Part (3) is a direct consequence of (2). ∎

If $n = 1$, i.e., S has only one row, then all the graded components are left ideals and thus $J(A)A$ is homogeneous. So, if A is graded-simple, then $J(A)A = 0$. Hence in this case one can reformulate Theorem 4.7 in a simpler form.

Define the left ideals

$$L_i := \bigoplus_{k=1}^{n} A_{ki}$$

and the right ideals

$$R_i := \bigoplus_{k=1}^{m} A_{ik}.$$

Then

$$L_j \cap R_i = A_{ij}.$$

Denote by 1_R the identity element of a ring R (if 1_R exists).

The following theorem is a graded version of the Mal'cev–Wedderburn theorem. It is shown that there exist orthogonal "column" (respectively, "row") homogeneous idempotents that define a semisimple complement of the radical. This result is a first step towards the classification of S-graded-simple algebras.

THEOREM 4.2: *Let $A = \bigoplus_{i,j} A_{ij}$ be a finite dimensional S-graded F-algebra over a field F such that $AJ(A)A = 0$. Then, there exist orthogonal idempotents $f_1, \dots, f_m$ and orthogonal idempotents $f'_1, \dots, f'_n$ (some of them could be zero) such that*

$$B = \bigoplus_{i,j} f'_i A f_j = \bigoplus_{i,j} (B \cap A_{ij})$$

is an S-graded maximal semisimple subalgebra of A, $f'_i \in B \cap R_i$ for $1 \leqslant i \leqslant n$, $f_j \in B \cap L_j$ for $1 \leqslant j \leqslant m$, $\sum_{i=1}^n f'_i = \sum_{j=1}^m f_j = 1_B$, and $A = B \oplus J(A)$ (direct sum of subspaces).

Proof. We write $\bar{X}$ for the image of a subset X of A in the algebra $A/J(A)$ under the natural epimorphism $A \to A/J(A)$.

Note that $\bar{A} = \sum_{j=1}^m \bar{L}_j$. Since $\bar{A} = A/J(A)$ is semisimple and completely reducible as a left $A/J(A)$-module, there exist left ideals $\tilde{L}_i \subseteq \bar{L}_i$ complementary to $\bar{L}_i \bigcap \sum_{j=1}^{i-1} \bar{L}_j$ in $\bar{L}_i$. Clearly, $\bar{A} = \bigoplus_{i=1}^m \tilde{L}_i$. The decomposition $1_{\bar{A}} = \sum_{i=1}^m \bar{\omega}_i$ of the identity element of $\bar{A}$ yields orthogonal idempotents $\bar{\omega}_i \in \tilde{L}_i$. The idempotents $\bar{\omega}_i$ can be lifted to homogeneous idempotents $\omega_i \in L_i$ of A using the natural epimorphisms $\pi|_{L_i} \colon L_i \to L_i/L_i \cap J(A)$ since $J(A)$ is nilpotent. The idempotents $\omega_1, \dots, \omega_m$ are orthogonal too since $\omega_i \omega_j = \omega_i(\omega_i \omega_j)\omega_j \in AJ(A)A = 0$.

Analogously, one gets orthogonal idempotents $\omega'_1, \dots, \omega'_n \in A$, $\omega'_i \in R_i$ for $1 \leqslant i \leqslant n$, such that $\sum_{i=1}^n \bar{f}'_i = 1_{\bar{A}}$. Define now

$$B = \bigoplus_{\substack{1 \leqslant i \leqslant n, \\ 1 \leqslant j \leqslant m}} \omega_i^{'} A \omega_j.$$

Note that $\omega_i^{'} A \omega_j \subseteq A_{ij}$ and B is an S-graded subalgebra of A. Suppose

$$a = \sum_{\substack{1 \leqslant i \leqslant n, \\ 1 \leqslant j \leqslant m}} \omega_i^{'} a_{ij} \omega_j \in J(A) \quad \text{for some } a_{ij} \in A.$$

Then $AJ(A)A = 0$ implies $\omega_i^{'} a_{ij} \omega_j = \omega_i^{'} a \omega_j = 0$ for all i, j. Hence $a = 0$ and $J(A) \cap B = 0$. Moreover,

$$\bar{B} = 1_{\bar{A}} \bar{A} 1_{\bar{A}} = \bar{A}.$$

Hence B is an S-graded maximal semisimple subalgebra of A and $A = B \oplus J(A)$ (direct sum of subspaces).

Decomposing 1_B with respect to the left ideals $\bigoplus_{i=1}^{n} \omega_i' A \omega_j$, $1 \leqslant j \leqslant m$, and with respect to the right ideals $\bigoplus_{j=1}^{m} \omega_i' A \omega_j$, $1 \leqslant i \leqslant n$, we get orthogonal idempotents $f_i \in B \cap L_i$ for $1 \leqslant i \leqslant n$, and orthogonal idempotents $f_j' \in B \cap R_j$ for $1 \leqslant j \leqslant m$ such that $\sum_{i=1}^{n} f_i' = \sum_{j=1}^{m} f_j = 1_B$. Then

$$B = 1_B B 1_B = \bigoplus_{\substack{1\leqslant i\leqslant n,\\ 1\leqslant j\leqslant m}} f_i' B f_j = \bigoplus_{\substack{1\leqslant i\leqslant n,\\ 1\leqslant j\leqslant m}} f_i' A f_j$$

since $A = B \oplus J(A)$ and $AJ(A)A = 0$. ∎

Example 4.3 below shows that the gradings on different B in Theorem 4.7 can be non-isomorphic.

Example 4.3: Let F be a field, let I be the left $M_2(F)$-module isomorphic to $\langle e_{12}, e_{22}\rangle_F$, and let $\varphi\colon I \xrightarrow{\sim} \langle e_{12}, e_{22}\rangle_F$ be the corresponding isomorphism. Let $A = M_2(F) \oplus I$ (direct sum of $M_2(F)$-modules) where $IM_2(F) = I^2 = 0$. Define on A the following T_3-grading:

$$A^{(e_1)} = (M_2(F), 0) \quad \text{and} \quad A^{(e_2)} = \{(\varphi(a), a) \mid a \in I\}.$$

Then the algebra A is T_3-graded-simple and both

$$B_1 = A^{(e_1)} \quad \text{and} \quad B_2 = \langle (e_{11}, 0), (e_{21}, 0)\rangle_F \oplus A^{(e_2)}$$

are graded maximal semisimple subalgebras of A. However, $B_1 \not\cong B_2$ as graded algebras.

Now we present a finite dimensional S-graded non-graded-simple algebra that does not have an S-graded maximal semisimple subalgebra complementary to the radical. So, in Theorem 4.2, the assumption $AJ(A)A = 0$ is essential.

Example 4.4: Let $R = F[X]/(X^2)$ and let $A = M_2(R)$. Put $v_1 = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$, $v_2 = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$, $w_1 = \left(\begin{smallmatrix}1 & X\end{smallmatrix}\right)$, $w_2 = \left(\begin{smallmatrix}0 & 1\end{smallmatrix}\right)$. Consider the following F-subspaces of A:

$$\begin{aligned} A_{11} &= Rv_1w_1 = R\left(\begin{smallmatrix}1 & X\\ 0 & 0\end{smallmatrix}\right), & A_{12} &= Rv_1w_2 = R\left(\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right),\\ A_{21} &= Rv_2w_1 = R\left(\begin{smallmatrix}0 & 0\\ 1 & X\end{smallmatrix}\right), & A_{22} &= Rv_2w_2 = R\left(\begin{smallmatrix}0 & 0\\ 0 & 1\end{smallmatrix}\right).\end{aligned}$$

Then $A = A_{11} \oplus A_{12} \oplus A_{21} \oplus A_{22}$ is an S-grading for $S = \mathcal{M}(\{e\}^0, 2, 2, \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right))$. However, there does not exist an S-graded maximal semisimple subalgebra B of A such that $A = B \oplus J(A)$.

Proof. First we notice that $A = A_{11} \oplus A_{12} \oplus A_{21} \oplus A_{22}$ is indeed an S-grading since

$$(a\, v_i w_j)(b\, v_\ell w_k) = ab(w_j v_\ell)\, v_i w_k$$

for all $a, b \in R$ and $1 \leqslant i, j, k, \ell \leqslant 2$ since $w_j v_\ell$ is a 1×1 matrix which can be identified with the corresponding element of the field F.

Clearly,

$$J(A) = \begin{pmatrix} (X) & (X) \\ (X) & (X) \end{pmatrix}$$

and

$$A/J(A) \cong M_2(F).$$

Fix the following bases in the homogeneous components:

$$A_{11} = \langle \left(\begin{smallmatrix} 1 & X \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} X & 0 \\ 0 & 0 \end{smallmatrix}\right) \rangle_F, \qquad A_{12} = \langle \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & X \\ 0 & 0 \end{smallmatrix}\right) \rangle_F,$$
$$A_{21} = \langle \left(\begin{smallmatrix} 0 & 0 \\ 1 & X \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ X & 0 \end{smallmatrix}\right) \rangle_F, \qquad A_{22} = \langle \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & X \end{smallmatrix}\right) \rangle_F.$$

Now it is clear that $J(A)$ is a homogeneous ideal and $A/J(A) \cong M_2(F)$ is an S-graded algebra too.

Suppose that there exists a S-graded maximal semisimple subalgebra

$$B = \bigoplus_{1 \leqslant i,j \leqslant 2} B_{ij}$$

such that $A = B \oplus J(A)$ and $B_{ij} \subseteq A_{ij}$. Then there exists a graded isomorphism $\varphi\colon B \to M_2(F)$.

In particular, $B_{ii} = \langle b_{ii} \rangle_F$ where $\varphi(b_{ii}) = e_{ii}$ and $b_{ii}^2 = b_{ii}$ for $i = 1, 2$. Hence $b_{11} = (1 + \alpha X)\left(\begin{smallmatrix} 1 & X \\ 0 & 0 \end{smallmatrix}\right)$ for some $\alpha \in F$ and $b_{22} = (1 + \beta X)\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ for some $\beta \in F$. (In fact, if $\operatorname{char} F \neq 2$, then $\alpha = \beta = 0$.) Then

$$b_{11} b_{22} = (1 + (\alpha + \beta)X)\left(\begin{smallmatrix} 0 & X \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & X \\ 0 & 0 \end{smallmatrix}\right) \in J(A)$$

and we get a contradiction. ∎

Theorem 4.2 describes the semisimple part of an S-graded-simple algebra. We proceed with the description of the radical and hence we obtain a characterization of the finite dimensional S-graded-simple algebras. In Section 5 we will show that this description delivers a complete classification.

For $r \in A$, we denote $x - xr$ (respectively $x - rx$) by $x(1 - r)$ (respectively, $(1 - r)x$), even if A does not contain unity.

LEMMA 4.5: *Suppose A is S-graded-simple and let*

$$A = B \oplus J(A)$$

(direct sum of subspaces) be the decomposition from Theorem 4.2. Then the following properties hold:

(1) $J(A)^2 A = AJ(A)^2 = 0$;
(2) B *is a simple subalgebra;*
(3) $A = A1_B A$;
(4) $J(A) = (1-1_B)A1_B \oplus 1_B A(1-1_B) \oplus J(A)^2$ *(direct sum of subspaces);*
(5) $J(A)^2 = (1-1_B)A1_B A(1-1_B) = (1-1_B)A(1-1_B)$.

Proof. Then Part (3) is a direct consequence of Part (2) of Lemma 4.1. Let f be a primitive central idempotent of B. Then $A = AfA$ and, by Part (3) of Lemma 4.1,

$$\begin{aligned} B =& 1_B(B \oplus J(A))1_B \\ =& 1_B A 1_B = 1_B A f A 1_B \\ =& 1_B A 1_B f 1_B A 1_B \\ =& BfB = Bf = fB. \end{aligned}$$

Hence f is the identity of B and thus $f = 1_B$. Therefore, B is simple and we get Part (2).

Using $B = 1_B A 1_B$ and the Pierce decomposition with respect to the idempotent 1_B, we get

$$J(A) = (1-1_B)A1_B \oplus 1_B A(1-1_B) \oplus (1-1_B)A(1-1_B) \text{ (direct sum of subspaces).}$$

Part (3) implies

$$(1-1_B)A(1-1_B) = (1-1_B)A1_B A(1-1_B) \subseteq J(A)^2.$$

Hence

$$(1-1_B)A(1-1_B)J(A) \subseteq J(A)^3 = 0, \quad J(A)(1-1_B)A(1-1_B) \subseteq J(A)^3 = 0.$$

Since

$$1_B A(1-1_B)A1_B \subseteq 1_B J(A) 1_B \subseteq AJ(A)A = 0,$$

we get $J(A)^2 \subseteq (1-1_B)A(1-1_B)$ and Propositions (1), (4) and (5) follow. ∎

We remark that if S has only one row, then

$$a - 1_B a \subseteq J(A) \cap A_{1i} = 0$$

for every $a \in A_{1i}$ and $1 \leqslant i \leqslant m$. Thus in this case 1_B acts as a left identity on $J(A)$.

It is also interesting to note that condition (1) of Lemma 4.1 and condition (2) of Lemma 4.5, together with $A^2 = A$, are equivalent to the graded S-simplicity.

PROPOSITION 4.6: *Suppose that the base field F is perfect, $A/J(A)$ is a simple algebra, $A^2 = A$, and $A_{ij} \cap J(A) = 0$ for all $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$. Then A is S-graded-simple.*

Proof. Let I be a nonzero two-sided homogeneous ideal of A. Denote by $\pi\colon A \twoheadrightarrow A/J(A)$ the natural epimorphism. Then $\pi(I) \neq 0$. Since $A/J(A)$ is simple, we get $\pi(I) = A/J(A)$ and $A = I + J(A)$. By the Wedderburn–Mal'cev theorem, there exists a maximal semisimple subalgebra $B \subseteq I$ such that $I = B \oplus J(I)$ (direct sum of subspaces). Recall that $J(I) = J(A) \cap I$. Thus $A = B \oplus J(A)$. Note that

$$\pi(A(1 - 1_B)A) = 0.$$

Hence $A(1 - 1_B)A \subseteq J(A)$. Since $A(1 - 1_B)A$ is a graded ideal, we get $A(1 - 1_B)A = 0$ and $ab = a1_B b \in I$ for all $a, b \in A$. Thus $A = A^2 \subseteq I$ and $I = A$. ∎

From Lemma 4.5 we have

$$\begin{aligned} J(A) =& 1_B A(1 - 1_B) \oplus (1 - 1_B)A1_B \oplus J(A)^2 \\ =& \sum_{j=1}^{m} 1_B L_j (1 - 1_B) \oplus \sum_{i=1}^{n} (1 - 1_B) R_i 1_B \oplus J(A)^2. \end{aligned}$$

For $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$, put

$$J_{ij}^{10} := f_i' L_j (1 - 1_B) \quad \text{and} \quad J_{ij}^{01} := (1 - 1_B) R_i f_j.$$

Also, put

$$J_{*j}^{10} := \sum_{1 \leqslant i \leqslant n} J_{ij}^{10} = 1_B L_j (1 - 1_B) \quad \text{and} \quad J_{i*}^{01} := \sum_{1 \leqslant j \leqslant m} J_{ij}^{01} = (1 - 1_B) R_i 1_B.$$

We will show that these subspaces form the buildings blocks of $J(A)$.

Theorem 4.7: *Let A be a finite dimensional S-graded-simple F-algebra. Let B and let $f_1,\dots,f_m$, $f'_1,\dots,f'_n$ be, respectively, a graded subalgebra and orthogonal idempotents from Theorem 4.2.*

Then each J^{10}_{*j} is a left B-submodule of $J(A)$ and $J^{10}_{*j} = \bigoplus_{i=1}^n J^{10}_{ij}$. Also, each J^{01}_{i*} is a right B-submodule of $J(A)$ and $J^{01}_{i*} = \bigoplus_{j=1}^m J^{01}_{ij}$. Moreover,

$$J(A) = \bigoplus_{i=1}^n J^{01}_{i*} \oplus \bigoplus_{j=1}^m J^{10}_{*j} \oplus J(A)^2 \quad \textit{and} \quad J(A)^2 = \bigoplus_{i=1}^n \bigoplus_{j=1}^m J^{01}_{i*} J^{10}_{*j},$$

direct sums of subspaces.

In addition, there exists an F-linear map

$$\varphi\colon \bigoplus_{i=1}^n J^{01}_{i*} \oplus \bigoplus_{j=1}^m J^{10}_{*j} \to B$$

defined by

$$\varphi(a - 1_B a 1_B) = 1_B a 1_B - f'_i a f_j, \quad \textit{for } a \in f'_i A_{ij} + A_{ij} f_j, \tag{2}$$

and such that $\varphi|_{\bigoplus_{j=1}^m J^{10}_{*j}}$ is a homomorphism of left B-modules,

$$J^{10}_{*j} \cap \ker\varphi = 0, \quad \varphi(J^{10}_{*j}) \cap Bf_j = \varphi(J^{10}_{*j}) f_j = 0, \quad \textit{for every } 1 \leqslant j \leqslant m, \tag{3}$$

$\varphi|_{\bigoplus_{i=1}^n J^{01}_{i*}}$ is a homomorphism of right B-modules,

$$J^{01}_{i*} \cap \ker\varphi = 0, \quad \varphi(J^{01}_{i*}) \cap f'_i B = f'_i \varphi(J^{01}_{i*}) = 0, \quad \textit{for every } 1 \leqslant i \leqslant n. \tag{4}$$

Moreover,

$$\begin{aligned} A_{ij} =& f'_i B f_j \oplus \{\varphi(v) + v \mid v \in J^{10}_{ij} \oplus J^{01}_{ij}\} \\ &\oplus \langle \varphi(v)\varphi(w) + v\varphi(w) + \varphi(v)w + vw \mid v \in J^{01}_{i*},\ w \in J^{10}_{*j} \rangle_F \end{aligned} \tag{5}$$

(direct sum of subspaces), for all $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$.

If $s \in \mathbb{N}$, $v_\ell \in J^{01}_{i*}$ and $w_\ell \in J^{10}_{*j}$ for $1 \leqslant \ell \leqslant s$, then $\sum_{\ell=1}^s v_\ell w_\ell = 0$ if and only if $\sum_{\ell=1}^s \varphi(v_\ell)\varphi(w_\ell) = 0$.

Furthermore, $B \cong M_k(D)$ for some $k \in \mathbb{N}$ and a division algebra D and

$$\dim_F \bigoplus_{i=1}^n J^{01}_{i*} \leqslant (n-1)\dim_F B = (n-1)k^2 \dim_F D, \tag{6}$$

$$\dim_F \bigoplus_{j=1}^m J^{10}_{*j} \leqslant (m-1)\dim_F B = (m-1)k^2 \dim_F D, \tag{7}$$

$$\dim_F J(A) \leqslant (nm-1)\dim_F B = (|S|-1)\dim_F B = (|S|-1)k^2 \dim_F D. \tag{8}$$

Proof. By Lemma 4.5,

$$\begin{aligned} J(A) =&1_B A(1-1_B)\oplus(1-1_B)A1_B\oplus J(A)^2\\ =&\sum_{j=1}^{m}1_B L_j(1-1_B)\oplus\sum_{i=1}^{n}(1-1_B)R_i1_B\oplus J(A)^2. \end{aligned}$$

Note that if $\sum_{j=1}^m 1_B a_j(1-1_B)=0$ for some $a_j\in L_j$, then

$$\sum_{j=1}^{m}1_B a_j 1_B=\sum_{j=1}^{m}1_B a_j.$$

Since $1_BA1_B=B$ is a graded subalgebra, we get $1_Ba_j\in B\cap L_j$, $1_Ba_j1_B=1_Ba_j$ and all $1_Ba_j(1-1_B)=0$. Hence the sum $\bigoplus_{j=1}^m 1_BL_j(1-1_B)$ is direct. Analogously, the sum $\bigoplus_{i=1}^n(1-1_B)R_i1_B$ is direct too and

$$\begin{aligned} J(A) =&\bigoplus_{j=1}^{m}1_B L_j(1-1_B)\oplus\bigoplus_{i=1}^{n}(1-1_B)R_i1_B\oplus J(A)^2\\ =&\bigoplus_{j=1}^{m}J^{10}_{*j}\oplus\bigoplus_{i=1}^{n}J^{01}_{i*}\oplus J(A)^2. \end{aligned}$$

Using Part (5) of Lemma 4.5, we get

$$J(A)^2=\sum_{i=1}^{n}\sum_{j=1}^{m}(1-1_B)R_i1_B1_BL_j(1-1_B)=\sum_{i=1}^{n}\sum_{j=1}^{m}J^{01}_{i*}J^{10}_{*j}. \tag{9}$$

Now we show that φ can be defined by (2). First, $f'_iL_j\subseteq A_{ij}$ and thus $f'_iL_j=f'_iA_{ij}$, $R_if_j\subseteq A_{ij}$ and thus

$$\begin{aligned} R_if_j =&A_{ij}f_j,\\ J^{10}_{ij} =&f'_iL_j(1-1_B)=\{a-1_Ba1_B\mid a\in f'_iA_{ij}\},\\ J^{01}_{ij} =&(1-1_B)R_if_j=\{a-1_Ba1_B\mid a\in A_{ij}f_j\}. \end{aligned}$$

If $a-1_Ba1_B=0$ for some $a\in A_{ij}$, then

$$a\in B\cap A_{ij}=f'_iBf_j\quad\text{and}\quad 1_Ba1_B-f'_iaf_j=0.$$

Hence (2) can indeed be used to define φ on J^{10}_{ij} and J^{01}_{ij} and the definition is consistent. We extend φ on $\bigoplus_{i=1}^n J^{01}_{i*}\oplus\bigoplus_{j=1}^m J^{10}_{*j}$ by the linearity.

Note that if $a\in f'_iA_{ij}=f'_iL_j$, then we have $a-1_Ba1_B=a(1-1_B)$ and $\varphi(a(1-1_B))=a(1_B-f_j)$. By the linearity, this formula holds for every

$a \in 1_B L_j$. Hence $\varphi|_{J^{10}_{*j}}$ is a homomorphism of left B-modules. By the linearity, $\varphi|_{\bigoplus_{j=1}^m J^{10}_{*j}}$ is a homomorphism of left B-modules too. Analogously, $\varphi((1-1_B)a) = (1_B - f'_i)a$ for all $a \in R_i 1_B$ and $\varphi|_{\bigoplus_{i=1}^n J^{01}_{i*}}$ is a homomorphism of right B-modules.

Suppose $\varphi(1_B a(1-1_B)) = 0$ for some $a \in L_j$. Then $1_B a(1_B - f_j) = 0$, $1_B a f_j = 1_B a 1_B$ and $f'_i a f_j = f'_i a 1_B$ for all $1 \leqslant i \leqslant n$. Hence

$$f'_i a(1-1_B) = f'_i a - f'_i a f_j \in A_{ij} \cap J(A) = 0.$$

Thus $1_B a(1-1_B) = 0$ and $1_B L_j(1-1_B) \cap \ker\varphi = J^{10}_{*j} \cap \ker\varphi = 0$. Analogously,

$$(1-1_B)R_i 1_B \cap \ker\varphi = J^{01}_{i*} \cap \ker\varphi = 0.$$

Moreover

$$\varphi(J^{10}_{*j}) \cap Bf_j = \varphi(1_B L_j(1-1_B)) \cap Bf_j \subseteq \varphi(1_B L_j(1-1_B))f_j = 0$$

for every $1 \leqslant j \leqslant m$ and (3) is proved. Analogously,

$$\varphi(J^{01}_{i*}) \cap f'_i B = \varphi((1-1_B)R_i 1_B) \cap f'_i B \subseteq f'_i \varphi((1-1_B)R_i 1_B) = 0$$

for every $1 \leqslant i \leqslant n$ and (4) is proved.

By Part (2) of Lemma 4.5, we have $B \cong M_k(D)$ for some $k \in \mathbb{N}$ and a division algebra D. Now

$$\begin{aligned}\dim_F \bigoplus_{i=1}^n J^{01}_{i*} =& \sum_{i=1}^n \dim_F \varphi(J^{01}_{i*}) \leqslant \sum_{i=1}^n (\dim_F B - \dim_F f'_i B)\\ =&(n-1)\dim_F B = (n-1)k^2 \dim_F D\end{aligned}$$

and we get (6). Analogously we obtain (7).

Now we prove (5). Let

$$\begin{aligned}\tilde{A}_{ij} =&(f'_i B f_j \oplus \{\varphi(v) + v \mid v \in J^{10}_{ij} \oplus J^{01}_{ij}\})\\ &+ \langle \varphi(v)\varphi(w) + v\varphi(w) + \varphi(v)w + vw \mid v \in J^{01}_{i*},\ w \in J^{10}_{*j}\rangle_F.\end{aligned}$$

We first show that $\tilde{A}_{ij} = A_{ij}$. To do so, note that if $a \in f'_i L_j$, then

$$\varphi(a(1-1_B)) + a(1-1_B) = a(1_B - f_j) + a(1-1_B) = a - af_j \in A_{ij}.$$

Therefore $\varphi(w) + w \in A_{ij}$ for every $w \in J^{10}_{ij}$. Analogously, $\varphi(v) + v \in A_{ij}$ for every $v \in J^{01}_{ij}$. Hence $\tilde{A}_{ij} \subseteq A_{ij}$.

Obviously, $B \subseteq \bigoplus_{i=1}^n \bigoplus_{j=1}^m \tilde{A}_{ij}$ and therefore $1_B \in \bigoplus_{i=1}^n \bigoplus_{j=1}^m \tilde{A}_{ij}$. By Lemma 4.5,

$$1_B \bigoplus_{i=1}^n \bigoplus_{j=1}^m \tilde{A}_{ij}(1-1_B) = \bigoplus_{j=1}^m J_{*j}^{10}$$

and

$$(1-1_B) \bigoplus_{i=1}^n \bigoplus_{j=1}^m \tilde{A}_{ij} 1_B = \bigoplus_{i=1}^n J_{i*}^{01}.$$

Hence

$$\bigoplus_{i=1}^n J_{i*}^{01} \oplus \bigoplus_{j=1}^m J_{*j}^{10} \subseteq \bigoplus_{i=1}^n \bigoplus_{j=1}^m \tilde{A}_{ij}.$$

In addition, (9) implies

$$(1-1_B) \bigoplus_{i=1}^n \bigoplus_{j=1}^m \tilde{A}_{ij}(1-1_B) = J^2(A)$$

and $J(A)^2 \subseteq \bigoplus_{i=1}^n \bigoplus_{j=1}^m \tilde{A}_{ij}$. Hence $\bigoplus_{i=1}^n \bigoplus_{j=1}^m \tilde{A}_{ij} = A$ and $\tilde{A}_{ij} = A_{ij}$. Equality (5) will follow from the fact that the sum in the definition of $\tilde{A}_{ij}$ is direct. We prove the last fact below.

Suppose $s \in \mathbb{N}$ and $v_\ell \in J_{i*}^{01}$ and $w_\ell \in J_{*j}^{10}$ for $1 \leqslant \ell \leqslant s$. Assume $\sum_{\ell=1}^s \varphi(v_\ell)\varphi(w_\ell) = 0$. Since $\varphi|_{\bigoplus_{j=1}^m J_{i*}^{01}}$ is a homomorphism of right B-modules, we get that $\varphi(\sum_{\ell=1}^s v_\ell \varphi(w_\ell)) = 0$ and thus, by (3), $\sum_{\ell=1}^s v_\ell\varphi(w_\ell) = 0$. Analogously, $\sum_{\ell=1}^s \varphi(v_\ell)w_\ell = 0$. Hence

$$\sum_{\ell=1}^s v_\ell w_\ell = \sum_{\ell=1}^s (\varphi(v_\ell)+v_\ell)(\varphi(w_\ell)+w_\ell) \in A_{ij} \cap J(A) = 0.$$

Conversely, suppose $\sum_{\ell=1}^s v_\ell w_\ell = 0$ for some $v_\ell \in J_{i*}^{01}$ and $w_\ell \in J_{*j}^{10}$, $1 \leqslant \ell \leqslant s$, $s \in \mathbb{N}$. Let

$$a = \sum_{\ell=1}^s (\varphi(v_\ell)+v_\ell)(\varphi(w_\ell)+w_\ell),$$
$$b = \sum_{\ell=1}^s (\varphi(\varphi(v_\ell)w_\ell)+\varphi(v_\ell)w_\ell) + \sum_{\ell=1}^s (\varphi(v_\ell\varphi(w_\ell))+v_\ell\varphi(w_\ell)) - \sum_{\ell=1}^s \varphi(v_\ell)\varphi(w_\ell).$$

Then $a-b=\sum_{\ell=1}^{s} v_\ell w_\ell = 0$. Thus $b=a\in A_{ij}$. However

$$\begin{aligned} b =& \sum_{q=1}^{n}\sum_{\ell=1}^{s}(f'_q\varphi(\varphi(v_\ell)w_\ell)+f'_q\varphi(v_\ell)w_\ell) \\ &+\sum_{r=1}^{m}\sum_{\ell=1}^{s}(\varphi(v_\ell\varphi(w_\ell))f_r+v_\ell\varphi(w_\ell)f_r)-\sum_{q=1}^{n}\sum_{r=1}^{m}\sum_{\ell=1}^{s}f'_q\varphi(v_\ell)\varphi(w_\ell)f_r. \end{aligned}$$

Taking the homogeneous component of b, corresponding to A_{ij}, i.e., the summand with $q=i$ and $r=j$, we obtain $a=b=0$ since by (3) and (4) we have

$$f'_i\varphi(v_\ell)=\varphi(w_\ell)f_j=0.$$

The projection of a on B with the kernel $J(A)$ yields $\sum_{\ell=1}^{s}\varphi(v_\ell)\varphi(w_\ell)=0$. Hence

$$\sum_{\ell=1}^{s}\varphi(v_\ell)\varphi(w_\ell)=\sum_{\ell=1}^{s}v_\ell\varphi(w_\ell)=\sum_{\ell=1}^{s}\varphi(v_\ell)w_\ell=0.$$

Now we are ready to prove that the sum in the definition of $\tilde{A}_{ij}$ is direct.

Suppose

$$\begin{aligned} &\sum_{\ell=1}^{s}(\varphi(v_\ell)\varphi(w_\ell)+v_\ell\varphi(w_\ell)+\varphi(v_\ell)w_\ell+v_\ell w_\ell) \\ &\qquad \in f'_iBf_j\oplus\{\varphi(v)+v \mid v\in J^{10}_{ij}\oplus J^{01}_{ij}\}\subseteq B\oplus\bigoplus_{j=1}^{m}J^{10}_{*j}\oplus\bigoplus_{i=1}^{n}J^{01}_{i*} \end{aligned}$$

for some $v_\ell\in J^{01}_{i*}$ and $w_\ell\in J^{10}_{*j}$. Since $\sum_{\ell=1}^{s}v_\ell w_\ell\in J(A)^2$ and

$$\sum_{\ell=1}^{s}(\varphi(v_\ell)\varphi(w_\ell)+v_\ell\varphi(w_\ell)+\varphi(v_\ell)w_\ell)\in B\oplus\bigoplus_{j=1}^{m}J^{10}_{*j}\oplus\bigoplus_{i=1}^{n}J^{01}_{i*},$$

we have $\sum_{\ell=1}^{s}v_\ell w_\ell=0$ and, by the previous remarks,

$$\sum_{\ell=1}^{s}\varphi(v_\ell)\varphi(w_\ell)=\sum_{\ell=1}^{s}v_\ell\varphi(w_\ell)=\sum_{\ell=1}^{s}\varphi(v_\ell)w_\ell=0$$

and $\sum_{\ell=1}^{s}(\varphi(v_\ell)\varphi(w_\ell)+v_\ell\varphi(w_\ell)+\varphi(v_\ell)w_\ell+v_\ell w_\ell)=0$.

In particular, the sum in the definition of $\tilde{A}_{ij}$ is direct and the proof of (5) is complete.

Now we prove that the sum $J(A)^2=\sum_{i=1}^{n}\sum_{j=1}^{m}J^{01}_{i*}J^{10}_{*j}$ is direct. Indeed, suppose $\sum_{i=1}^{n}\sum_{j=1}^{m}u_{ij}=0$ for some $u_{ij}\in J^{01}_{i*}J^{10}_{*j}$. By (5), $u_{ij}=a_{ij}-v_{ij}$

where $a_{ij} \in A_{ij}$ and v_{ij} is a linear combination of homogeneous elements from B and homogeneous elements from

$$\{\varphi(v) + v \mid v \in J_{i*}^{01} \oplus J_{*j}^{10}\}.$$

Now $\sum_{i=1}^{n} \sum_{j=1}^{m} (a_{ij} - v_{ij}) = 0$ implies that each a_{ij} is a linear combination of elements from $f_i' B f_j$ and elements from $\{\varphi(v) + v \mid v \in J_{ij}^{01} \oplus J_{ij}^{10}\}$. Thus $u_{ij} = (1 - 1_B) u_{ij} (1 - 1_B) = (1 - 1_B)(a_{ij} - v_{ij})(1 - 1_B) = 0$ and the sum $J(A)^2 = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} J_{i*}^{01} J_{*j}^{10}$ is indeed direct.

Only (8) still has to be proved. Note that Lemma 3.4 implies

$$\begin{aligned}
\dim_F J(A)^2 &= \sum_{i=1}^{n} \sum_{j=1}^{m} \dim_F J_{i*}^{01} J_{*j}^{10} = \sum_{i=1}^{n} \sum_{j=1}^{m} \dim_F \varphi(J_{i*}^{01}) \varphi(J_{*j}^{10}) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{\dim_F \varphi(J_{i*}^{01}) \dim_F \varphi(J_{*j}^{10})}{k^2 \dim_F D} \\
&\leqslant \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{(\dim_F B - \dim_F f_i' B)(\dim_F B - \dim_F B f_j)}{k^2 \dim_F D} \\
&= \frac{(\dim_F B)^2 (n-1)(m-1)}{k^2 \dim_F D} \\
&= (n-1)(m-1) \dim_F B.
\end{aligned}$$

Finally, (8) follows at once from (6) and (7). ∎

As an example, we conclude this section with a specific class of algebras for which we give an explicit description of the graded Mal'cev–Wedderburn decomposition. Let R be a finite dimensional F-algebra with identity element and let P be an $m \times n$ matrix with entries in R such that each row and each column contains at least one invertible element in R. The Munn algebra $A := \mathcal{M}(R, n, m, P)$ is, by definition, the F-vector space of all $n \times m$ matrices over R with multiplication defined by

$$DE := D \circ P \circ E,$$

for $D, E \in \mathcal{M}(R, n, m, P)$, and where $D \circ P$ is the usual matrix multiplication. Clearly, $A = \bigoplus_{i,j} R e_{ij}$ and it is an S-graded-simple algebra, where S is the completely 0-simple semigroup $S = \mathcal{M}(\{e\}^0, n, m, P')$, with P' the $n \times m$ matrix with e in every entry of P'.

By Lemma 4.1 and Theorem 4.2, there exists an S-graded maximal semisimple subalgebra B such that $A = B \oplus J(A)$. In case $R = F$, one can give an explicit description of B. Indeed, let k denote the rank of P. Reindexing if needed, we may assume that the first k rows and the first k columns are F-linearly independent. Let $B = \{(a_{ij}) \in A \mid a_{ij} = 0 \text{ for } i > k \text{ or } j > k\}$. Note that B can be identified with $\mathcal{M}(F, k, k, Q)$ where the sandwich matrix Q consists of the first k rows and columns of P. Clearly, B is a graded subalgebra and the mapping $N \mapsto N \circ Q$ is an algebra isomorphism $B \to M_k(F)$. Thus $B \cong M_k(F)$ is a simple algebra. Furthermore, $A/J(A) \cong M_k(F) \cong B$ (Proposition 23 in Chapter 5 of [29]) and thus $A = B \oplus J(A)$ and A is Wedderburn–Mal'cev graded. Also recall that $J(A) = \{N \in A \mid P \circ N \circ P = 0\}$ (see, e.g., Corollary 15 in Chapter 5 of [29]).

5. Existence theorems for graded-simple algebras

In Theorem 4.2 and Theorem 4.7 we obtained a description of an S-graded-simple algebra A. It is shown that A has a graded Mal'cev–Wedderburn decomposition $B + J(A)$ and that $J(A)$ is roughly the direct sum of left and right B-modules that are isomorphic to left and right ideals of B and that also satisfy some other restrictions. To complete the description, we now show that any such collection of left and right ideals of a finite dimensional simple algebra B yields an S-graded-simple algebra A with B as a maximal semsimple graded subalgebra. We now formulate this in precise detail.

Let $k, m, n \in \mathbb{N}$ and let D be a division algebra. Put $B \cong M_k(D)$ and assume $f_1, \ldots, f_n \in B$ and $f'_1, \ldots, f'_n \in B$ are two sets of idempotents (some of them could be zero) such that $\sum_{i=1}^n f'_i = \sum_{j=1}^m f_j = 1_B$ and so that the idempotents in each set are pairwise orthogonal.

Let $J^{10}_{*1}, \ldots, J^{10}_{*m}$ and $J^{01}_{1*}, \ldots, J^{01}_{n*}$ be, respectively, left and right B-modules such that there exist embeddings

$$\varphi\colon J^{10}_{*j} \hookrightarrow B \quad \text{and} \quad \varphi\colon J^{01}_{i*} \hookrightarrow B$$

which are, respectively, left and right B-module homomorphisms and we have

$$\varphi(J^{10}_{*j}) f_j = 0 \quad \text{and} \quad f'_i \varphi(J^{01}_{i*}) = 0.$$

(For convenience, we denote both the maps by the same letter φ and we also assume that the linear map φ is defined on the additive group $\bigoplus_{i=1}^n J^{01}_{i*} \oplus \bigoplus_{j=1}^m J^{10}_{*j}$.) Define additive groups J_{ij} as isomorphic copies of $\varphi(J^{01}_{i*})\varphi(J^{10}_{*j}) \subseteq B$ for $1 \leqslant i \leqslant n$,

$1 \leqslant j \leqslant m$. Let

$$\Theta_{ij}\colon \varphi(J_{i*}^{01})\varphi(J_{*j}^{10}) \to J_{ij}$$

be the corresponding linear isomorphisms and let

$$\mu\colon J_{i*}^{01} \times J_{*j}^{10} \to J_{ij}$$

be the bilinear map defined by

$$\mu(v,w) := \Theta_{ij}(\varphi(v)\varphi(w)),$$

for $v \in J_{i*}^{01}$ and $w \in J_{*j}^{10}$. We extend μ by linearity to the map

$$\mu\colon \bigoplus_{i=1}^{n} J_{i*}^{01} \times \bigoplus_{j=1}^{m} J_{*j}^{10} \to \bigoplus_{i=1}^{n}\bigoplus_{j=1}^{m} J_{ij}.$$

Let Q denote the $n \times m$ matrix with each entry being equal to e.

Theorem 5.1: *The additive group*

$$A = B \oplus \bigoplus_{i=1}^{n} J_{i*}^{01} \oplus \bigoplus_{j=1}^{m} J_{*j}^{10} \oplus \bigoplus_{i=1}^{n}\bigoplus_{j=1}^{m} J_{ij}$$

is a $\mathcal{M}(\{e\}^0, n, m; Q)$-graded-simple ring for the multiplication defined by

$$(b_1, v_1, w_1, u_1)(b_2, v_2, w_2, u_2) = (b_1 b_2, v_1 b_2, b_1 w_2, \mu(v_1, w_2)),$$

for $b_1, b_2 \in B$, $v_1, v_2 \in \bigoplus_{i=1}^n J_{i*}^{01}$, $w_1, w_2 \in \bigoplus_{j=1}^m J_{*j}^{10}$, $u_1, u_2 \in \bigoplus_{i=1}^n \bigoplus_{j=1}^m J_{ij}$. The homogeneous components are

$$\begin{aligned} A_{ij} =& (f_i' B f_j, 0, 0, 0) \oplus \{(\varphi(v), v, 0, 0) \mid v \in J_{i*}^{01} f_j\} \oplus \{(\varphi(w), 0, w, 0) \mid w \in f_i' J_{*j}^{10}\} \\ & \oplus \langle (\varphi(v)\varphi(w), v\varphi(w), \varphi(v)w, \mu(v,w)) \mid v \in J_{i*}^{01},\ w \in J_{*j}^{10} \rangle_{\mathbb{Z}}. \end{aligned}$$

Proof. Making use of all the assumptions, direct computations show that the multiplication defines a graded ring structure and also that $A = \bigoplus_{i=1}^n \bigoplus_{j=1}^m A_{ij}$

Clearly,

$$J(A) = \left(0, \bigoplus_{i=1}^{n} J_{i*}^{01}, \bigoplus_{j=1}^{m} J_{*j}^{10}, \bigoplus_{i=1}^{n}\bigoplus_{j=1}^{m} J_{ij}\right),$$

since the third power of the right-hand side is zero.

Note that for every $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$ we have

$$\begin{aligned}
1_B(f'_i B f_j, 0,0,0)1_B =&(f'_i B f_j, 0,0,0),\\
1_B\{(\varphi(v), v, 0, 0) \mid v \in J^{01}_{i*} f_j\}1_B \subseteq&\Big(\bigoplus_{\ell\neq i} f'_\ell B f_j, 0,0,0\Big),\\
1_B\{(\varphi(w), 0, w, 0) \mid w \in f'_i J^{10}_{*j}\}1_B \subseteq&\Big(\bigoplus_{r\neq j} f'_i B f_r, 0,0,0\Big),\\
1_B\langle(\varphi(v)\varphi(w), v\varphi(w), \varphi(v)w, \mu(v,w)) \mid& v \in J^{01}_{i*},\ w \in J^{10}_{*j}\rangle_{\mathbb{Z}} 1_B\\
\subseteq&\Big(\bigoplus_{\substack{\ell\neq i,\\ r\neq j}} f'_\ell B f_r, 0,0,0\Big).
\end{aligned}$$

Thus, if $1_B a 1_B = 0$ for some $a \in A_{ij}$, then $a = 0$. Hence $A_{ij} \cap J(A) = 0$.

Suppose I is a graded two-sided ideal of A. Let $a \in I$, $a \neq 0$, be a homogeneous element. By the previous comments, $a = (b, u, v, w)$ with $b \neq 0$. Hence $(1_B, 0,0,0)a(1_B,0,0,0) = (b,0,0,0) \in I$. Since B is a simple ring, $(B,0,0,0) \subseteq I$. Thus $(1_B,0,0,0)A \subseteq I$ and $A(1_B,0,0,0) \subseteq I$. Since

$$A = (B,0,0,0) + (1_B,0,0,0)A + A(1_B,0,0,0) + (1_B,0,0,0)A^2(1_B,0,0,0),$$

we get $I = R$, and A is graded-simple. ∎

In case all modules involved, for example J^{01}_{i*} and J_{ij}, are left and right F-vector spaces on which the left and right F-structure is compatible and all maps involved are (left and right) F-linear, then the construction in Theorem 5.1 yields an S-graded algebra. In this case, if B is a finite dimensional algebra over F, then a left B-module embedding $\varphi\colon J^{10}_{*j} \to \bigoplus_{\substack{r=1,\\ r\neq j}}^m Bf_r$ exists if and only if $\dim_F J^{10}_{*j} \leqslant \dim_F B - \dim_F(Bf_j)$. A right B-module embedding

$$\varphi\colon J^{01}_{*i} \to \bigoplus_{\ell\neq i} f'_\ell B$$

exists if and only if

$$\dim_F J^{01}_{i*} \leqslant \dim_F B - \dim_F(f'_i B).$$

Theorem 2.3 shows that the grading on an algebra A is completely defined by the images of the graded components in $A/J(A)$. We show that every such decomposition determines some S-grading.

THEOREM 5.2: *Let D be a division algebra over a field F and let $B \cong M_k(D)$. Suppose $B = \sum_{i=1}^n \sum_{j=1}^m B_{ij}$, a sum of subspaces B_{ij} of B, with $B_{ij}B_{\ell r} \subseteq B_{ir}$, for all $1 \leqslant i, \ell \leqslant n$, $1 \leqslant j, r \leqslant m$. Let $P = (p_{ij})_{i,j}$ be an $n \times m$ matrix where $p_{ij} \in \{0, e\}$ such that $B_{ij}B_{\ell r} = 0$ for every (j, ℓ) with $p_{j\ell} = 0$. Then there exists an $\mathcal{M}(\{e\}^0, n, m; P)$-graded algebra $A = \bigoplus_{i=1}^n \bigoplus_{j=1}^m A_{ij}$ and a surjective algebra homomorphism $\psi\colon A \to B$ such that $\ker\psi = J(A)$ and $\psi(A_{ij}) = B_{ij}$, or all $1 \leqslant i, \ell \leqslant n$, $1 \leqslant j, r \leqslant m$.*

Proof. Let $\bar{L}_j := \bigoplus_{i=1}^n B_{ij}$ for $1 \leqslant j \leqslant m$ and $\bar{R}_i := \bigoplus_{j=1}^m B_{ij}$ for $1 \leqslant i \leqslant n$. Note that $\bar{L}_1, \dots, \bar{L}_m$ are left ideals and $\bar{R}_1, \dots, \bar{R}_n$ are right ideals. Moreover, since $B \cong M_k(D)$ is semisimple, B is completely reducible as a left and a right B-module. Define $\tilde{L}_j$ as a complementary left B-submodule to $\bar{L}_j \cap \sum_{\ell=1}^{j-1} \bar{L}_\ell$ in $\bar{L}_j$, $1 \leqslant j \leqslant m$. Analogously, define $\tilde{R}_i$ as a complementary right B-submodule to $\bar{R}_i \cap \sum_{\ell=1}^{i-1} \bar{R}_\ell$ in $\bar{R}_i$, $1 \leqslant i \leqslant n$. Then $\bigoplus_{\ell=1}^i \tilde{R}_\ell = \sum_{\ell=1}^i \bar{R}_\ell$ for $1 \leqslant i \leqslant n$ and $\bigoplus_{\ell=1}^j \tilde{L}_\ell = \sum_{\ell=1}^j \bar{L}_\ell$ for $1 \leqslant j \leqslant m$. In particular,

$$B = \bigoplus_{i=1}^n \tilde{R}_i = \bigoplus_{j=1}^m \tilde{L}_j.$$

Decomposing 1_B into the sum of elements of, respectively, $\tilde{L}_1, \dots, \tilde{L}_m$ and $\tilde{R}_1, \dots, \tilde{R}_n$, we find two sets of orthogonal idempotents $f_1, \dots, f_m$ and $f'_1, \dots, f'_n$ such that $\tilde{L}_j = Bf_j$, $\tilde{R}_i = Bf'_i$, $\sum_{i=1}^n f'_i = \sum_{j=1}^m f_j = 1_B$.

Let $W^{10}_{*j} = \bar{L}_j(1 - f_j) \subseteq \bar{L}_j$ for $1 \leqslant j \leqslant m$ and $W^{01}_{i*} = (1 - f'_i)\bar{R}_i \subseteq \bar{R}_i$ for $1 \leqslant i \leqslant n$. Then $\bar{L}_j = \bar{L}_j f_j \oplus W^{10}_{*j}$ and $\bar{R}_i = f'_i\bar{R}_i \oplus W^{01}_{i*}$ (direct sums of, respectively, left and right ideals). Denote by J^{10}_{*j} the isomorphic copy of W^{10}_{*j} and by J^{01}_{i*} the isomorphic copy of W^{01}_{i*}. Denote the corresponding left B-module isomorphisms $\varphi\colon J^{10}_{*j} \to W^{10}_{*j}$ and right B-module isomorphisms $\varphi\colon J^{01}_{i*} \to W^{01}_{i*}$ by the same letter φ. Now extend φ to an F-linear map

$$\varphi\colon \bigoplus_{i=1}^n J^{01}_{i*} \oplus \bigoplus_{j=1}^m J^{10}_{*j} \to \sum_{i=1}^n W^{01}_{i*} + \sum_{j=1}^m W^{10}_{*j} \subseteq B.$$

Let $A = \bigoplus_{i=1}^n \bigoplus_{j=1}^m A_{ij}$ be the ring constructed in Theorem 5.1. Note that this then will be an algebra by the remark given after the proof of the theorem. We claim that A satisfies all the conditions of Theorem 5.2. Indeed, define ψ as the projection of

$$A = B \oplus \bigoplus_{i=1}^n J^{01}_{i*} \oplus \bigoplus_{j=1}^m J^{10}_{*j} \oplus \bigoplus_{i=1}^n \bigoplus_{j=1}^m J_{ij}$$

on B with the kernel

$$J(A) = \bigoplus_{i=1}^{n} J_{i*}^{01} \oplus \bigoplus_{j=1}^{m} J_{*j}^{10} \oplus \bigoplus_{i=1}^{n}\bigoplus_{j=1}^{m} J_{ij}.$$

Since B is semisimple, there exist idempotents e_i' and e_j such that $\bar{R}_i = e_i'B$ and $\bar{L}_j = Be_j'$. Then

$$\begin{aligned}\psi(A_{ij}) =& f_i'Bf_j' + \varphi(J_{i*}^{01} f_j) + \varphi(f_i' J_{*j}^{10}) + \varphi(J_{i*}^{01})\varphi(J_{*j}^{10})\\ =& f_i'Bf_j + (1-f_i')\bar{R}_i f_j + f_i'\bar{L}_j(1-f_j) + (1-f_i')\bar{R}_i\bar{L}_j(1-f_j)\\ =& f_i'\bar{R}_i f_j + (1-f_i')\bar{R}_i f_j + f_i'\bar{R}_i\bar{L}_j(1-f_j) + (1-f_i')\bar{R}_i\bar{L}_j(1-f_j)\\ =& \bar{R}_i f_j + \bar{R}_i\bar{L}_j(1-f_j) = \bar{R}_i\bar{L}_j = e_i'Be_j = \bar{R}_i \cap \bar{L}_j\\ =& B_{ij}.\end{aligned}$$

Moreover, $p_{j\ell} = 0$ always implies $B_{ij}B_{\ell r} = 0$, $\psi(A_{ij}A_{\ell r}) = 0$ and $A_{ij}A_{\ell r} = 0$. The last equality follows from $(\ker\psi) \cap A_{ir} = J(A) \cap A_{ir} = 0$. ∎

Note that any simple algebra $B = M_k(D)$ can be decomposed into the sum of B_{ij}'s (as needed in Theorem 5.2) by taking a collection $\bar{L}_1, \ldots, \bar{L}_m$ and a collection $\bar{R}_1, \ldots, \bar{R}_n$ of, respectively, left and right ideals of B with

$$B = \sum_{i=1}^{n} \bar{R}_i = \sum_{j=1}^{m} \bar{L}_j$$

and define $B_{ij} = \overline{R}_i \cap \overline{L}_j$.

6. Graded polynomial identities, their codimensions and cocharacters

In the remainder of the paper we study numeric characteristics of graded polynomial identities in semigroup graded algebras.

Let T be a semigroup and let F be a field. Denote by $F\langle X^{T\text{-gr}}\rangle$ the free T-graded associative algebra over F on the countable set

$$X^{T\text{-gr}} := \bigcup_{t\in T} X^{(t)},$$

$X^{(t)} = \{x_1^{(t)}, x_2^{(t)}, \ldots\}$, i.e., the algebra of polynomials in non-commuting variables from $X^{T\text{-gr}}$. The indeterminates from $X^{(t)}$ are said to be homogeneous of degree t. The T-degree of a monomial $x_{i_1}^{(t_1)} \cdots x_{i_s}^{(t_s)} \in F\langle X^{T\text{-gr}}\rangle$ is defined to be $t_1 t_2 \cdots t_s$, as opposed to its total degree, which is defined to be s. Denote by

$F\langle X^{T\text{-gr}}\rangle^{(t)}$ the subspace of the algebra $F\langle X^{T\text{-gr}}\rangle$ spanned by all the monomials having T-degree t. Notice that

$$F\langle X^{T\text{-gr}}\rangle^{(t)}F\langle X^{T\text{-gr}}\rangle^{(h)} \subseteq F\langle X^{T\text{-gr}}\rangle^{(th)},$$

for every $t, h \in T$. It follows that

$$F\langle X^{T\text{-gr}}\rangle = \bigoplus_{t\in T} F\langle X^{T\text{-gr}}\rangle^{(t)}$$

is a T-grading. Let $f = f(x_{i_1}^{(t_1)}, \dots, x_{i_s}^{(t_s)}) \in F\langle X^{T\text{-gr}}\rangle$. We say that f is a **graded polynomial identity** of a T-graded algebra $A = \bigoplus_{t\in T} A^{(t)}$ and write $f \equiv 0$ if $f(a_{i_1}^{(t_1)}, \dots, a_{i_s}^{(t_s)}) = 0$ for all $a_{i_j}^{(t_j)} \in A^{(t_j)}$, $1 \leqslant j \leqslant s$. The set $\mathrm{Id}^{T\text{-gr}}(A)$ of graded polynomial identities of A is a graded ideal of $F\langle X^{T\text{-gr}}\rangle$.

Example 6.1: Consider the multiplicative semigroup $T = \mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ and the T-grading $\mathrm{UT}_2(F) = \mathrm{UT}_2(F)^{(\bar{0})} \oplus \mathrm{UT}_2(F)^{(\bar{1})}$ on the algebra $\mathrm{UT}_2(F)$ of upper triangular 2×2 matrices over a field F defined by

$$\mathrm{UT}_2(F)^{(\bar{0})} = \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix} \quad \text{and} \quad \mathrm{UT}_2(F)^{(\bar{1})} = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}.$$

We have

$$[x^{(\bar{0})}, y^{(\bar{0})}] := x^{(\bar{0})}y^{(\bar{0})} - y^{(\bar{0})}x^{(\bar{0})} \in \mathrm{Id}^{T\text{-gr}}(\mathrm{UT}_2(F))$$

and $x^{(\bar{1})}y^{(\bar{1})} \in \mathrm{Id}^{T\text{-gr}}(\mathrm{UT}_2(F))$.

Let $P_n^{T\text{-gr}} := \langle x_{\sigma(1)}^{(t_1)} x_{\sigma(2)}^{(t_2)} \cdots x_{\sigma(n)}^{(t_n)} \mid t_i \in T, \sigma \in S_n \rangle_F \subset F\langle X^{T\text{-gr}}\rangle$, $n \in \mathbb{N}$. Then the number

$$c_n^{T\text{-gr}}(A) := \dim\left(\frac{P_n^{T\text{-gr}}}{P_n^{T\text{-gr}} \cap \mathrm{Id}^{T\text{-gr}}(A)}\right)$$

is called the nth **codimension of graded polynomial identities** or the nth **graded codimension** of A.

If $T = \{e\}$ is the trivial group, we get ordinary polynomial identities

$$\mathrm{Id}(A) := \mathrm{Id}^{\{e\}\text{-gr}}(A),$$

the space of ordinary multilinear polynomials $P_n := P_n^{\{e\}\text{-gr}}$ and ordinary codimensions $c_n(A) := c_n^{\{e\}\text{-gr}}(A)$.

The proposition provides a relation between the ordinary and the graded codimensions.

PROPOSITION 6.2: *Let A be a T-graded algebra over a field F for some semigroup T (not necessarily finite). Then $c_n(A) \leqslant c_n^{T\text{-gr}}(A)$. If T is finite, then $c_n^{T\text{-gr}}(A) \leqslant |T|^n c_n(A)$ for all $n \in \mathbb{N}$.*

Proof. Let $t_1, \dots, t_n \in T$. Denote by $P_{t_1,\dots,t_n}$ the vector space of multilinear T-graded polynomials in $x_1^{(t_1)}, \dots, x_n^{(t_n)}$. Then $P_n^{T\text{-gr}} = \bigoplus_{t_1,\dots,t_n \in T} P_{t_1,\dots,t_n}$. Let $\bar f_1, \dots, \bar f_{c_n(A)}$ be a basis in $\frac{P_n}{P_n \cap \mathrm{Id}(A)}$ where $f_i \in P_n$. Then for every $\sigma \in S_n$ there exist $\alpha_{\sigma,i} \in F$ such that

$$x_{\sigma(1)} \cdots x_{\sigma(n)} - \sum_{i=1}^{c_n(A)} \alpha_{\sigma,i} f_i(x_1, \dots, x_n) \in \mathrm{Id}(A).$$

Then for every $t_1, \dots, t_n \in T$ we have

$$x_{\sigma(1)}^{(t_1)} \cdots x_{\sigma(n)}^{(t_n)} - \sum_{i=1}^{c_n(A)} \alpha_{\sigma,i} f_i(x_1^{(t_1)}, \dots, x_n^{(t_n)}) \in \mathrm{Id}^{T\text{-gr}}(A)$$

and

$$\frac{P_n^{T\text{-gr}}}{P_n^{T\text{-gr}} \cap \mathrm{Id}^{T\text{-gr}}(A)} = \langle \bar f_i(x_1^{(t_1)}, \dots, x_n^{(t_n)}) \mid 1 \leqslant i \leqslant c_n(A),\ t_1, \dots, t_n \in T \rangle_F.$$

This implies the upper bound.

In order to get the lower bound, for a given n-tuple $(t_1, \dots, t_n) \in T^n$ we consider the map

$$\varphi_{t_1,\dots,t_n} \colon P_n \to \frac{P_n^{T\text{-gr}}}{P_n^{T\text{-gr}} \cap \mathrm{Id}^{T\text{-gr}}(A)}$$

where $\varphi_{t_1,\dots,t_n}(f) = f(x_1^{(t_1)}, \dots, x_n^{(t_n)})$ for $f = f(x_1, \dots, x_n) \in P_n$. Note that $f(x_1, \dots, x_n) \equiv 0$ is an ordinary polynomial identity if and only if

$$f(x_1^{(t_1)}, \dots, x_n^{(t_n)}) \equiv 0$$

is a graded polynomial identity for every $t_1, \dots, t_n \in T$. In other words,

$$P_n \cap \mathrm{Id}(A) = \bigcap_{(t_1,\dots,t_n) \in T^n} \ker \varphi_{t_1,\dots,t_n}.$$

Since P_n is a finite dimensional vector space, there exists a finite subset $\Lambda \subseteq T^n$ such that $P_n \cap \mathrm{Id}(A) = \bigcap_{(t_1,\dots,t_n) \in \Lambda} \ker \varphi_{t_1,\dots,t_n}$.

Consider the diagonal embedding

$$P_n \hookrightarrow P_n^{T\text{-gr}} = \bigoplus_{t_1,\dots,t_n \in T} P_{t_1,\dots,t_n}$$

where the image of $f(x_1,\dots,x_n) \in P_n$ equals $\sum_{(t_1,\dots,t_n)\in\Lambda} f(x_1^{(t_1)},\dots,x_n^{(t_n)})$. Then our choice of Λ implies that the induced map $\frac{P_n}{P_n\cap \mathrm{Id}(A)} \hookrightarrow \frac{P_n^{T\text{-gr}}}{P_n^{T\text{-gr}}\cap \mathrm{Id}^{T\text{-gr}}(A)}$ is an embedding and the lower bound follows. ∎

The next proposition provides an upper bound for graded codimensions in terms of $\dim A$.

Proposition 6.3: *Let A be a finite dimensional T-graded algebra over a field F for some semigroup T (not necessarily finite). Then $c_n^{T\text{-gr}}(A) \leqslant (\dim A)^{n+1}$ for all $n \in \mathbb{N}$.*

Proof. This is a direct consequence of [21, Lemma 1] and [18, Lemma 4]. ∎

The analog of Amitsur's conjecture for graded codimensions can be formulated as follows.

Conjecture: *The limit* $\mathrm{PIexp}^{T\text{-gr}}(A) := \lim_{n\to\infty} \sqrt[n]{c_n^{T\text{-gr}}(A)}$ *exists.*

Remark 6.4: The original Amitsur's conjecture was formulated for ordinary polynomial identities and asserted that the exponent of codimension growth is integer. However, the recent examples [21, Theorems 3–5] show that, in the semigroup graded case, the graded PI-exponent could be non-integer.

7. Polynomial H-identities and their codimensions

In our case, instead of working with graded codimensions directly, it is more convenient to replace the grading with the corresponding dual structure and study the asymptotic behaviour of polynomial H-identities for a suitable associative algebra H.

Let H be an arbitrary associative algebra with 1 over a field F. We say that an associative algebra A is an algebra with a **generalized H-action** if A is endowed with a homomorphism $H \to \mathrm{End}_F(A)$ and for every $h \in H$ there exist $h_i', h_i'', h_i''', h_i'''' \in H$, $1 \leqslant i \leqslant s$, $s \in \mathbb{N}$, such that

$$(10) \qquad h(ab) = \sum_{i=1}^{s} ((h_i'a)(h_i''b) + (h_i'''b)(h_i''''a)) \quad \text{for all } a, b \in A.$$

Remark 7.1: We use the term "generalized H-action" in order to distinguish from the case when an algebra is an H-module algebra for some Hopf algebra H which is a particular case of the generalized H-action.

Let $F\langle X\rangle$ be the free associative algebra without 1 on the set

$$X := \{x_1, x_2, x_3, \dots\}.$$

Then $F\langle X\rangle = \bigoplus_{n=1}^{\infty} F\langle X\rangle^{(n)}$ where $F\langle X\rangle^{(n)}$ is the linear span of all monomials of total degree n. Consider the algebra

$$F\langle X|H\rangle := \bigoplus_{n=1}^{\infty} H^{\otimes n} \otimes F\langle X\rangle^{(n)}$$

with the multiplication $(u_1 \otimes w_1)(u_2 \otimes w_2) := (u_1 \otimes u_2) \otimes w_1 w_2$ for all $u_1 \in H^{\otimes j}$, $u_2 \in H^{\otimes k}$, $w_1 \in F\langle X\rangle^{(j)}$, $w_2 \in F\langle X\rangle^{(k)}$. We use the notation

$$x_{i_1}^{h_1} x_{i_2}^{h_2} \cdots x_{i_n}^{h_n} := (h_1 \otimes h_2 \otimes \cdots \otimes h_n) \otimes x_{i_1} x_{i_2} \cdots x_{i_n}.$$

Here $h_1 \otimes h_2 \otimes \cdots \otimes h_n \in H^{\otimes n}$, $x_{i_1} x_{i_2} \cdots x_{i_n} \in F\langle X\rangle^{(n)}$.

Note that if $(\gamma_\beta)_{\beta \in \Lambda}$ is a basis in H, then $F\langle X|H\rangle$ is isomorphic to the free associative algebra over F with free formal generators $x_i^{\gamma_\beta}$, $\beta \in \Lambda$, $i \in \mathbb{N}$. We refer to the elements of $F\langle X|H\rangle$ as **associative H-polynomials**. Note that here we do not consider any H-action on $F\langle X|H\rangle$.

Let A be an associative algebra with a generalized H-action. Any map $\psi\colon X \to A$ has the unique homomorphic extension $\bar{\psi}\colon F\langle X|H\rangle \to A$ such that

$$\bar{\psi}(x_i^h) = h\psi(x_i)$$

for all $i \in \mathbb{N}$ and $h \in H$. An H-polynomial $f \in F\langle X|H\rangle$ is an **H-identity** of A if

$$\bar{\psi}(f) = 0$$

for all maps $\psi\colon X \to A$. In other words, $f(x_1, x_2, \dots, x_n)$ is an H-identity of A if and only if $f(a_1, a_2, \dots, a_n) = 0$ for any $a_i \in A$. In this case we write $f \equiv 0$. The set $\mathrm{Id}^H(A)$ of all H-identities of A is an ideal of $F\langle X|H\rangle$.

We denote by P_n^H the space of all multilinear H-polynomials in $x_1, \dots, x_n$, $n \in \mathbb{N}$, i.e.,

$$P_n^H = \langle x_{\sigma(1)}^{h_1} x_{\sigma(2)}^{h_2} \cdots x_{\sigma(n)}^{h_n} \mid h_i \in H, \sigma \in S_n \rangle_F \subset F\langle X|H\rangle.$$

Then the number

$$c_n^H(A) := \dim\left(\frac{P_n^H}{P_n^H \cap \mathrm{Id}^H(A)}\right)$$

is called the nth **codimension of polynomial H-identities** or the nth **H-codimension** of A.

For an arbitrary semigroup T one can consider the **semigroup algebra** FT over a field F which is the vector space with the formal basis $(t)_{t\in T}$ and the multiplication induced by the one in T.

Consider the vector space $(FT)^*$ dual to FT. Then $(FT)^*$ is an algebra with the multiplication defined by $(hw)(t) = h(t)w(t)$ for $h, w \in (FT)^*$ and $t \in T$. The identity element is defined by $1_{(FT)^*}(t) = 1$ for all $t \in T$. In other words, $(FT)^*$ is the algebra dual to the coalgebra FT.

Let $\Gamma\colon A = \bigoplus_{t\in T} A^{(t)}$ be a grading on an algebra A. We have the following natural $(FT)^*$-action on A: $ha^{(t)} = h(t)a^{(t)}$ for all $h \in (FT)^*$, $a^{(t)} \in A^{(t)}$ and $t \in T$.

Remark 7.2: If T is a finite group, then A is an FT-comodule algebra for the Hopf algebra FT and an $(FT)^*$-module algebra for the Hopf algebra $(FT)^*$.

For every $t \in T$ define $h_t \in (FT)^*$ by

$$h_t(g) := \begin{cases} 0 & \text{if } g \neq t, \\ 1 & \text{if } g = t \end{cases} \quad \text{for } g \in T.$$

If A is finite dimensional, the set $\operatorname{supp}\Gamma := \{t \in T \mid A^{(t)} \neq 0\}$ is finite and

$$h_t(ab) = \sum_{\substack{g,w\in \operatorname{supp}\Gamma,\\ gw=t}} h_g(a)h_w(b) \quad \text{for all } a, b \in A.$$

Note that $ha = \sum_{t\in\operatorname{supp}\Gamma} h(t)h_t a$ for all $a \in A$ and $h \in (FT)^*$. By linearity, we get (10). Therefore, A is an algebra with a generalized $(FT)^*$-action.

The lemma below plays the crucial role in the passage from graded polynomial identities to polynomial H-identities.

Lemma 7.3 ([21, Lemma 1]): *Let A be a finite dimensional algebra over a field F graded by a semigroup T. Then $c_n^{T\text{-gr}}(A) = c_n^{(FT)^*}(A)$ for all $n \in \mathbb{N}$.*

One of the main tools in the investigation of polynomial identities is provided by the representation theory of symmetric groups. The symmetric group S_n acts on the space $\frac{P_n^H}{P_n^H \cap \operatorname{Id}^H(A)}$ by permuting the variables. Irreducible FS_n-modules are described by partitions $\lambda = (\lambda_1, \dots, \lambda_s) \vdash n$ and their Young diagrams D_λ. The character $\chi_n^H(A)$ of the FS_n-module $\frac{P_n^H}{P_n^H \cap \operatorname{Id}^H(A)}$ is called the nth

cocharacter of polynomial H-identities of A. We can rewrite it as a sum

$$\chi_n^H(A) = \sum_{\lambda \vdash n} m(A, H, \lambda)\chi(\lambda)$$

of irreducible characters $\chi(\lambda)$. Let $e_{T_\lambda} = a_{T_\lambda} b_{T_\lambda}$ and $e^*_{T_\lambda} = b_{T_\lambda} a_{T_\lambda}$, where

$$a_{T_\lambda} = \sum_{\pi \in R_{T_\lambda}} \pi \quad \text{and} \quad b_{T_\lambda} = \sum_{\sigma \in C_{T_\lambda}} (\operatorname{sign} \sigma)\sigma,$$

be Young symmetrizers corresponding to a Young tableau T_λ. Then

$$M(\lambda) = FS_n e_{T_\lambda} \cong FS_n e^*_{T_\lambda}$$

is an irreducible FS_n-module corresponding to a partition $\lambda \vdash n$. We refer the reader to [3, 12, 17] for an account of S_n-representations and their applications to polynomial identities.

8. Main steps of the proof of the lower and upper bounds for graded codimensions

Here we give a brief overview of Sections 9–11.

Let $A = \bigoplus_{t \in T} A^{(t)}$ be a finite dimensional T-graded algebra over a field F of characteristic 0 for some semigroup T such that $A/J(A) \cong M_k(F)$ for some $k \in \mathbb{N}$ and $A^{(t)} \cap J(A) = 0$ for all $t \in T$. (For example, F is algebraically closed and A is a finite dimensional T-graded-simple algebra for some $T = \mathcal{M}(\{e\}^0, n, m; P)$.)

As mentioned before, since $\frac{P_n^{T\text{-gr}}(A)}{P_n^{T\text{-gr}}(A) \cap \operatorname{Id}^{T\text{-gr}}(A)}$ is an FS_n-module, it can be decomposed into the sum of simple FS_n-modules $M(\lambda)$. Thus,

$$c_n^{T\text{-gr}}(A) = \sum_{\lambda \vdash n} m(A, (FT)^*, \lambda) \dim M(\lambda)$$

where $m(A, (FT)^*, \lambda)$ is the multiplicity of $M(\lambda)$ in $\frac{P_n^{T\text{-gr}}(A)}{P_n^{T\text{-gr}}(A) \cap \operatorname{Id}^{T\text{-gr}}(A)}$. A hook with the edge in (i, j) of a Young tableau T_λ consists of all the boxes to the right and below (i, j) together with the box (i, j) itself. The number of boxes in the (i, j)-hook is called its length and is denoted by h_{ij}.

Let $\lambda = (\lambda_1, \dots, \lambda_q) \vdash n$. Recall that by the hook and the Stirling formula we have

$$\begin{aligned}\dim M(\lambda) =& \frac{n!}{\prod_{i,j} h_{ij}} \\ \leqslant& \frac{C\sqrt{n}(\frac{n}{e})^n}{\sqrt{\lambda_1 \cdots \lambda_q}(\frac{\lambda_1}{e})^{\lambda_1} \cdots (\frac{\lambda_q}{e})^{\lambda_q}} \\ =& \frac{C\sqrt{n}}{\sqrt{\lambda_1 \cdots \lambda_q}}\Big(\frac{1}{(\frac{\lambda_1}{n})^{\frac{\lambda_1}{n}} \cdots (\frac{\lambda_q}{n})^{\frac{\lambda_q}{n}}}\Big)^n\end{aligned} \tag{11}$$

for some constant $C \in \mathbb{R}$. Now we compute the exponent in 3 steps.

(a) First, the multiplicities are polynomially bounded, i.e., there exist $a, b \in \mathbb{R}$ such that $\sum_{\lambda \vdash n} m(A, (FT)^*, \lambda) \leqslant an^b$. (See [19, Theorem 5].)

(b) Now (11) implies

$$\overline{\lim_{n\to\infty}} \sqrt[n]{c_n^{T\text{-gr}}(A))} \leqslant \sup_{\substack{\lambda \vdash n, \\ m(A,(FT)^*,\lambda) \neq 0}} \Phi\Big(\frac{\lambda_1}{n_1}, \dots, \frac{\lambda_q}{n_q}\Big)$$

where $\Phi(x_1, \dots, x_q) = \frac{1}{x_1^{x_1} \cdots x_q^{x_q}}$.

Let $f \in P_n^{(FT)^*}, \lambda \vdash n$ for some $n \in \mathbb{N}$ and $r = \dim A$. In Lemma 9.2 we prove that if

$$\sum_{i=1}^{r} \gamma_i \lambda_i \geqslant k \quad \text{or} \quad \lambda_{r+1} > 0, \tag{12}$$

for some specific numbers γ_i defined in Section 9, then $e^*_{T_\lambda} f \in \mathrm{Id}^{(FT)^*}(A)$ for any Young tableau T_λ of shape λ. In particular, $m(A, (FT)^*, \lambda) = 0$ for all partitions satisfying the conditions (12) and we may assume that Φ is defined in r variables.

Furthermore, now we can restrict ourselves to the partitions

$$\lambda = (\lambda_1, \dots, \lambda_r) \vdash n$$

such that $\sum_{i=1}^{r} \gamma_i \frac{\lambda_i}{n} \leqslant \frac{k}{n}$. In particular, for n large enough, we may consider Φ on the following area:

$$\begin{aligned}\Omega := \{(\alpha_1, \alpha_2, \dots, \alpha_r) \in \mathbb{R}^r \mid & \alpha_1 \geqslant \cdots \geqslant \alpha_r \geqslant 0, \alpha_1 + \alpha_2 + \cdots + \alpha_r = 1, \\ & \gamma_1\alpha_1 + \cdots + \gamma_r\alpha_r \leqslant 0\}.\end{aligned}$$

Formula (11) now yields the upper bound

$$\dim M(\lambda) \leqslant \max_{(\alpha_1,\dots,\alpha_r)\in\Omega} \Phi(\alpha_1,\dots,\alpha_r) =: d$$

for irreducible modules $M(\lambda)$ with $m(A,(FT)^*,\lambda) \neq 0$. Combined with the polynomial bound on the multiplicities, we get the upper bound

$$\overline{\lim_{n\to\infty}} \sqrt[n]{c_n^{T\text{-gr}}(A)} \leqslant d.$$

This step will be achieved in Theorem 9.8 and the formula for d is proved in Lemma 9.7.

The idea to use the function Φ on some region Ω was introduced in [14, 27]. However, in order to find an appropriate region Ω, i.e., a region which does not contain most partitions λ with $m(A,(FT)^*,\lambda) = 0$, new techniques are needed. For this we develop a special method in Section 9.

(c) Since $c_n^{T\text{-gr}}(A) \geqslant \dim M(\lambda)$ for all irreducible modules appearing in the decomposition, it is sufficient to find a partition μ such that $m(A,(FT)^*,\mu) \neq 0$ and

$$\dim(M_\mu) \geqslant \frac{n!}{n^{r(r-1)}\mu_1!\cdots\mu_r!} \geqslant C_1 n^{B_1}\Big(\frac{1}{(\frac{\mu_1}{n})^{\frac{\mu_1}{n}}\cdots(\frac{\mu_r}{n})^{\frac{\mu_r}{n}}}\Big)^n \approx C_1 n^{B_1} d^n$$

for some constants $B_1, C_1 \in \mathbb{R}$ in order to get the needed lower bound.

For this we construct for every $n \geqslant n_0$ a multilinear polynomial $f \in P_n^{T\text{-gr}}$ such that $e^*_{T_\mu} f \notin \mathrm{Id}^{T\text{-gr}}(A)$ for some tableau T_μ of a desired shape μ. In order to get $e^*_{T_\mu} f \notin \mathrm{Id}^{T\text{-gr}}(A)$, we construct f alternating in the variables of some specific sets.

Note that no general method is known to construct such non-vanishing polynomials. In Section 10 and Section 11 we have to do this manually.

The polynomial f is constructed in Lemmas 10.4 and 11.4. The proof heavily uses the description of left ideals in $M_2(F)$ and we restrict ourselves to zero bands. The exact exponent is computed in Theorem 10.5 and Theorem 11.5. We get $\mathrm{PIexp}^{T\text{-gr}}(A) = \dim A$ and

$$\mathrm{PIexp}^{T\text{-gr}}(A) = |T_0| + 2|T_1| + 2\sqrt{(|T_1| + |\bar{t}_0|)(|T_0| + |T_1| - |\bar{t}_0|)} < \dim A,$$

respectively. The numbers $|T_0|, |T_1|, |\bar{t}_0|$ are defined in the beginning of Section 10. In particular, any number $m + 1 + \sqrt{m}$ for any $m \in \mathbb{N}$ can be realized as the T-graded PI-exponent of some T-graded-simple finite dimensional algebra.

9. Upper bound for $(FT)^*$-codimensions of T-graded-simple algebras

Let $A = \bigoplus_{t\in T} A^{(t)}$ be a finite dimensional T-graded algebra over a field F of characteristic 0 for some semigroup T such that $A/J(A) \cong M_k(F)$ for some $k \in \mathbb{N}$ and $A^{(t)} \cap J(A) = 0$ for all $t \in T$. In this section we prove an upper bound for T-graded codimensions of A.

For every $t \in T$ fix a basis $\mathcal{B}^{(t)}$ in $A^{(t)}$. Then $\mathcal{B} = \bigcup_{t\in T} \mathcal{B}^{(t)}$ is a basis in A. Fix also some isomorphism $\psi\colon A/J(A) \to M_k(F)$. Denote by $\pi\colon A \to A/J(A)$ the natural epimorphism. Define the function $\theta\colon \mathcal{B} \to \mathbb{Z}$ by

$$\theta(a) = \min\{i - j \mid \alpha_{ij} \neq 0,\ 1 \leqslant i, j \leqslant k\}$$

if $\psi\pi(a) = \sum_{1\leqslant i,j\leqslant k} \alpha_{ij} e_{ij}$, $\alpha_{ij} \in F$.

The observation below plays a central role in the section.

Lemma 9.1: *Let $f \in P_n^{(FT)^*}$ for some $n \in \mathbb{N}$ and let $a_i \in \mathcal{B}$, $1 \leqslant i \leqslant n$. If $f(a_1, \dots, a_n) \neq 0$, then*

$$1 - k \leqslant \sum_{i=1}^{n} \theta(a_i) \leqslant k - 1.$$

Proof. Note that f is a linear combination of monomials $x^{h_1}_{\sigma(1)} x^{h_2}_{\sigma(2)} \cdots x^{h_n}_{\sigma(n)}$, $h_i \in (FT)^*$, $\sigma \in S_n$. Denote by $\Gamma\colon A = \bigoplus_{t\in T} A^{(t)}$ the T-grading on A. Since $\operatorname{supp}\Gamma$ is finite, we may assume that f is a linear combination of monomials $x^{h_{t_1}}_{\sigma(1)} x^{h_{t_2}}_{\sigma(2)} \cdots x^{h_{t_n}}_{\sigma(n)}$, $t_i \in \operatorname{supp}\Gamma$, $\sigma \in S_n$. (See the definition of h_t in Section 7.) Since all a_i are homogeneous, the value of $x^{h_{t_1}}_{\sigma(1)} x^{h_{t_2}}_{\sigma(2)} \cdots x^{h_{t_n}}_{\sigma(n)}$ equals $a_{\sigma(1)} \cdots a_{\sigma(n)}$ if $a_{\sigma(i)} \in A^{(t_i)}$ for all $1 \leqslant i \leqslant n$ and 0 otherwise. However $a_{\sigma(1)} \cdots a_{\sigma(n)}$ is again a homogeneous element. Recall that $J(A) \cap A^{(t)} = 0$ for every $t \in T$. Thus $a_{\sigma(1)} \cdots a_{\sigma(n)} \neq 0$ if and only if

$$\psi\pi(a_{\sigma(1)} \cdots a_{\sigma(n)}) = \psi\pi(a_{\sigma(1)})\psi\pi(a_{\sigma(2)}) \cdots \psi\pi(a_{\sigma(n)}) \neq 0.$$

Now we notice that $e_{i_1 j_1} e_{i_2 j_2} \dots e_{i_n j_n} \neq 0$ for some $1 \leqslant i_\ell, j_\ell \leqslant k$ only if $j_1 = i_2, j_2 = i_3, \dots, j_{n-1} = i_n$, and, in particular,

$$1 - k \leqslant \sum_{\ell=1}^{n} (i_\ell - j_\ell) = i_1 - j_k \leqslant k - 1.$$

Therefore, $a_{\sigma(1)} \cdots a_{\sigma(n)} \neq 0$ only if $1 - k \leqslant \sum_{\ell=1}^{n} \theta(a_i) \leqslant k - 1$. ∎

Let $r := \dim A$. Define

$$\beta_\ell := \min\left\{\sum_{i=1}^{\ell}\theta(a_i) \mid a_i \in \mathcal{B},\ a_i \neq a_j \text{ for } i \neq j\right\},$$

$\gamma_\ell := \beta_\ell - \beta_{\ell-1}$, $1 \leqslant \ell \leqslant r$, $\beta_0 := 0$. Without loss of generality, we may assume that $\mathcal{B} = (a_1, \dots, a_r)$ where

$$\theta(a_1) \leqslant \theta(a_2) \leqslant \cdots \leqslant \theta(a_r).$$

Then $\beta_\ell = \sum_{i=1}^{\ell}\theta(a_i)$ and $\gamma_\ell = \theta(a_\ell)$. In particular,

$$1 - k = \gamma_1 \leqslant \gamma_2 \leqslant \cdots \leqslant \gamma_r.$$

The equality $\gamma_1 = 1 - k$ follows from the fact that e_{1k} has the minimal value of $(i - j)$ among all matrix units e_{ij} and the matrix unit e_{1k} must appear with a nonzero coefficient in the decomposition of $\varphi\pi(a)$ for some $a \in \mathcal{B}$.

Now we prove the main inequality for $(FT)^*$-cocharacters of A.

Lemma 9.2: *Let $f \in P_n^{(FT)^*}$ and $\lambda \vdash n$ for some $n \in \mathbb{N}$. If $\sum_{i=1}^{r}\gamma_i\lambda_i \geqslant k$ or $\lambda_{r+1} > 0$, then $e^*_{T_\lambda} f \in \operatorname{Id}^{(FT)^*}(A)$ for any Young tableau T_λ of shape λ.*

Proof. Note that for each column of T_λ, the polynomial $e^*_{T_\lambda} f = b_{T_\lambda} a_{T_\lambda} f$ is alternating in the variables with indices from that column. Another remark is that, in order to determine whether a multilinear polynomial is a polynomial $(FT)^*$-identity of A, it is sufficient to substitute only elements from $\mathcal{B}$. If we substitute two coinciding elements for the variables of the same set of alternating variables, we get zero. Thus, if $\lambda_{r+1} > 0$, then the height of the first column is greater than or equal to $(r + 1)$ and at least two elements coincide. Therefore, $e^*_{T_\lambda} f \in \operatorname{Id}^{(FT)^*}(A)$.

Suppose $\sum_{i=1}^{r}\gamma_i\lambda_i \geqslant k$. We can rewrite this inequality in the form

$$\sum_{i=1}^{r}(\beta_i - \beta_{i-1})\lambda_i = \sum_{i=1}^{r}\beta_i(\lambda_i - \lambda_{i+1}) \geqslant k. \tag{13}$$

(We may assume that $\lambda_{r+1} = 0$.) Note that $(\lambda_i - \lambda_{i+1})$ equals the number of columns of height i in T_λ. Suppose $b_1, \dots, b_n \in \mathcal{B}$ are substituted for $x_1, \dots, x_n$. By the remark above, we may assume that for the variables of each column different basis elements are substituted. By the definition of β_i, $\sum_{i=1}^{n}\theta(b_i) \geqslant \sum_{i=1}^{r}\beta_i(\lambda_i - \lambda_{i+1})$. Combining with (13), we get $\sum_{i=1}^{n}\theta(b_i) \geqslant k$. Now Lemma 9.1 implies $(e^*_{T_\lambda} f)(b_1, \dots, b_n) = 0$ and $e^*_{T_\lambda} f \in \operatorname{Id}^{(FT)^*}(A)$. ∎

Let $\Phi(\alpha_1, \dots, \alpha_r) = \frac{1}{\alpha_1^{\alpha_1}\alpha_2^{\alpha_2}\cdots\alpha_r^{\alpha_r}}$. This function becomes continuous in the region $\alpha_1, \dots, \alpha_r \geqslant 0$ if we define $0^0 := 1$.

The lemma below shows the importance of the function Φ in our investigation.

Lemma 9.3: *Let*

$$\Omega := \{(\alpha_1, \alpha_2, \dots, \alpha_r) \in \mathbb{R}^r \mid \alpha_1 \geqslant \cdots \geqslant \alpha_r \geqslant 0,\ \alpha_1 + \alpha_2 + \cdots + \alpha_r = 1,$$
$$\gamma_1\alpha_1 + \cdots + \gamma_r\alpha_r \leqslant 0\}.$$

Then $\overline{\lim}_{n\to\infty} \sqrt[n]{c_n^{T\text{-gr}}(A)} \leqslant \max_{x\in\Omega} \Phi(x)$.

Proof. This is a consequence of Lemmas 7.3, 9.2 and [21, Lemma 2] (or the original paper [27]). ∎

The rest of the section is devoted to the calculation of the maximum of Φ. We begin with the most simple region.

Lemma 9.4: *Let $r \in \mathbb{N}$ and*

$$\Omega_0 := \{(\alpha_1, \alpha_2, \dots, \alpha_r) \in \mathbb{R}^r \mid \alpha_1, \dots, \alpha_r \geqslant 0,\ \alpha_1 + \alpha_2 + \cdots + \alpha_r = 1\}.$$

Then $\max_{x\in\Omega_0} \Phi(x) = r$ *and* $\operatorname{argmax}_{x\in\Omega_0} \Phi(x) = (\frac{1}{r}, \frac{1}{r}, \dots, \frac{1}{r})$.

Proof. We prove the lemma by induction on r. The case $r = 1$ is trivial. Assume $r \geqslant 2$. First, we can express $\alpha_1 = 1 - \sum_{i=2}^r \alpha_i$ in terms of $\alpha_2, \alpha_3, \dots, \alpha_r$, and study $\Phi_1(\alpha_2, \dots, \alpha_r) = \frac{1}{(1-\sum_{i=2}^r \alpha_i)^{(1-\sum_{i=2}^r \alpha_i)}\alpha_2^{\alpha_2}\cdots\alpha_r^{\alpha_r}}$ on

$$\tilde{\Omega}_0 := \left\{(\alpha_2, \dots, \alpha_r) \in \mathbb{R}^{r-1} \mid \alpha_2, \dots, \alpha_r \geqslant 0,\ 1 - \sum_{i=2}^r \alpha_i \geqslant 0\right\}.$$

Note that Φ_1 is continuous on the compact set $\tilde{\Omega}_0$ and differentiable at all inner points of $\tilde{\Omega}_0$. Thus Φ_1 can reach its extremal values only at inner critical points of Φ_1 or on $\partial\tilde{\Omega}_0$. By the induction assumption, $\Phi_1(x) \leqslant r - 1$ for all $x \in \partial\tilde{\Omega}_0$. Consider

$$\frac{\partial\Phi_1}{\partial\alpha_\ell}(\alpha_2, \dots, \alpha_r) = \left(\ln\left(1 - \sum_{i=2}^r \alpha_i\right) - \ln\alpha_\ell\right)\Phi_1(\alpha_2, \dots, \alpha_r).$$

Then $\frac{\partial\Phi}{\partial\alpha_\ell}(\alpha_2, \dots, \alpha_r) = 0$ for all $2 \leqslant \ell \leqslant r$ only for

$$\alpha_2 = \cdots = \alpha_r = 1 - \sum_{i=2}^r \alpha_i = \frac{1}{r}.$$

Since $\Phi(\frac{1}{r}, \frac{1}{r}, \dots, \frac{1}{r}) = r > r-1$, we get the lemma. ∎

The positive root ζ of the polynomial P, defined in the lemma below, will be used in the calculation of the upper bound of codimensions. Here we study the basic properties of P.

Lemma 9.5: *Let $r \in \mathbb{N}$ and $\gamma_i \in \mathbb{Z}$, $1 \leqslant i \leqslant r$. Suppose*

$$\gamma_1 \leqslant \gamma_2 \leqslant \dots \leqslant \gamma_r, \tag{14}$$

$\gamma_1 < 0$. Consider the equation

$$P(\zeta) := \sum_{i=1}^{r} \gamma_i \zeta^{\gamma_i - \gamma_1} = 0 \tag{15}$$

where ζ is the unknown variable. If $\sum_{i=1}^r \gamma_i \geqslant 0$, then (15) has the only root $\zeta > 0$. Moreover $\zeta \in (0;1]$. If $\sum_{i=1}^r \gamma_i < 0$, then $P(y) < 0$ for all $y \in [0;1]$.

Proof. If $\gamma_r \leqslant 0$, then all nonzero coefficients of P are negative and $P(y) < 0$ for all $y \in [0;1]$.

Suppose $\gamma_r > 0$. Inequality (14) implies that there is only one sign difference in the signs of coefficients of P. Therefore, by Descartes' rule of signs, (15) has the unique positive root ζ. Define $m \in \mathbb{N}$ by

$$\gamma_1 = \dots = \gamma_m < \gamma_{m+1}.$$

Note that $P(0) = m\gamma_1 < 0$ and $P(1) = \sum_{i=1}^r \gamma_i$. Therefore, if $\sum_{i=1}^r \gamma_i \geqslant 0$, we have $\zeta \in (0;1]$. If $P(1) = \sum_{i=1}^r \gamma_i < 0$, then $\zeta > 1$ and $P(y) < 0$ for all $y \in [0;1]$. ∎

It turns out that the root ζ of P is the extremal point of the function Ψ defined below. This will be used in the calculation of the maximum of Φ on our region Ω.

Lemma 9.6: *Let $r \in \mathbb{N}$ and $\gamma_i \in \mathbb{Z}$, $1 \leqslant i \leqslant r$. Suppose $\gamma_1 \leqslant \gamma_2 \leqslant \dots \leqslant \gamma_r$, $\gamma_1 < 0$. Denote $\Psi(y) = \sum_{i=1}^r y^{\gamma_i}$. Then:*

(1) *if $\sum_{i=1}^r \gamma_i \geqslant 0$, then $\min_{y \in (0;1]} \Psi(y) = \Psi(\zeta)$ where $\zeta \in (0;1]$ is the positive root of (15);*
(2) *if $\sum_{i=1}^r \gamma_i \leqslant 0$, then $\min_{y \in (0;1]} \Psi(y) = r$.*

Proof. Note that $\Psi'(y) = \sum_{i=1}^r \gamma_i y^{\gamma_i - 1}$. Thus $\Psi'(y)$ has the same sign on $(0;1]$ as $P(y) = \sum_{i=1}^r \gamma_i y^{\gamma_i - \gamma_1}$. Also $\lim_{y \to 0+} \Psi(y) = +\infty$. Lemma 9.5 implies that if $\sum_{i=1}^r \gamma_i \geqslant 0$, then $\min_{y\in(0;1]} \Psi(y) = \Psi(\zeta)$, and if $\sum_{i=1}^r \gamma_i \leqslant 0$, then $\min_{y\in(0;1]} \Psi(y) = \Psi(1) = r$. (In the case $\sum_{i=1}^r \gamma_i = 0$, we have $\zeta = 1$ and $\Psi(\zeta) = r$.) ∎

Now we are ready to calculate the maximum of Φ. For our convenience, we replace our region Ω with a larger region $\tilde\Omega$ and show that the maximum on both regions is the same.

Lemma 9.7: *Let $r \in \mathbb{N}$ and $\gamma_i \in \mathbb{Z}$, $1 \leqslant i \leqslant r$. Suppose $\gamma_1 \leqslant \gamma_2 \leqslant \cdots \leqslant \gamma_r$, $\gamma_1 < 0$, $\sum_{i=1}^r \gamma_i \geqslant 0$. Let*

$$\Omega := \{(\alpha_1, \alpha_2, \ldots, \alpha_r) \in \mathbb{R}^r \mid \alpha_1 \geqslant \cdots \geqslant \alpha_r \geqslant 0,\ \alpha_1 + \alpha_2 + \cdots + \alpha_r = 1, \\ \gamma_1\alpha_1 + \cdots + \gamma_r\alpha_r \leqslant 0\}$$

and let

$$\tilde\Omega := \{(\alpha_1, \alpha_2, \ldots, \alpha_r) \in \mathbb{R}^r \mid \alpha_1, \ldots, \alpha_r \geqslant 0,\ \alpha_1 + \alpha_2 + \cdots + \alpha_r = 1, \\ \gamma_1\alpha_1 + \cdots + \gamma_r\alpha_r \leqslant 0\}.$$

Then $\max_{x\in\Omega} \Phi(x) = \max_{x\in\tilde\Omega} \Phi(x) = \sum_{i=1}^r \zeta^{\gamma_i}$ where $\zeta \in (0;1]$ is the positive root of (15).

Proof. Like in Lemma 9.4, we use induction on r. The conditions $\gamma_1 < 0$ and $\sum_{i=1}^r \gamma_i \geqslant 0$ imply that $r \geqslant 2$. We will not prove the induction base $r = 2$ separately, but the base will follow from the arguments below since for $r = 2$ we will not use the induction assumption.

Again, we express $\alpha_1 = 1 - \sum_{i=2}^r \alpha_i$ in terms of $\alpha_2, \alpha_3, \ldots, \alpha_r$, and study

$$\Phi_1(\alpha_2, \ldots, \alpha_r) = \frac{1}{(1 - \sum_{i=2}^r \alpha_i)^{(1-\sum_{i=2}^r \alpha_i)}\alpha_2^{\alpha_2} \cdots \alpha_r^{\alpha_r}}$$

on

$$\Omega_1 := \left\{(\alpha_2, \ldots, \alpha_r) \in \mathbb{R}^{r-1} \mid \alpha_2, \ldots, \alpha_r \geqslant 0,\ 1 - \sum_{i=2}^r \alpha_i \geqslant 0, \right. \\ \left. \gamma_1 + \sum_{i=2}^r (\gamma_i - \gamma_1)\alpha_i \leqslant 0 \right\}.$$

Now the proof of Lemma 9.4 implies that the only critical point of Φ_1 is $(\frac{1}{r}, \ldots, \frac{1}{r})$. This point belongs to Ω_1 if and only if $\sum_{i=1}^r \gamma_i \leqslant 0$. If indeed $\sum_{i=1}^r \gamma_i = 0$, then by Lemma 9.4 we have $\max_{x\in\Omega} \Phi(x) = r$. Since in this case $\zeta = 1$, the lemma is proved.

Suppose $\sum_{i=1}^r \gamma_i > 0$. Then the continuous function Φ_1 reaches its maximum on $\partial\Omega_1$. Note that $\partial\Omega_1 = \Omega_2 \cup \bigcup_{i=1}^r \Omega_{1i}$ where

$$\Omega_{11} = \Big\{(\alpha_2,\dots,\alpha_r) \in \mathbb{R}^{r-1} \mid \alpha_2,\dots,\alpha_r \geqslant 0,\ 1-\sum_{i=2}^r \alpha_i = 0,$$
$$\gamma_1 + \sum_{i=2}^r (\gamma_i-\gamma_1)\alpha_i \leqslant 0V\Big\},$$
$$\Omega_{1\ell} = \Big\{(\alpha_2,\dots,\alpha_r) \in \mathbb{R}^{r-1} \mid \alpha_2,\dots,\alpha_r \geqslant 0,\ \alpha_\ell = 0,\ 1-\sum_{i=2}^r \alpha_i \geqslant 0,$$
$$\gamma_1 + \sum_{i=2}^r (\gamma_i-\gamma_1)\alpha_i \leqslant 0\Big\} \quad \text{for } 2 \leqslant \ell \leqslant r,$$
$$\Omega_2 = \Big\{(\alpha_2,\dots,\alpha_r) \in \mathbb{R}^{r-1} \mid \alpha_2,\dots,\alpha_r \geqslant 0,\ 1-\sum_{i=2}^r \alpha_i \geqslant 0,$$
$$\gamma_1 + \sum_{i=2}^r (\gamma_i-\gamma_1)\alpha_i = 0\Big\}.$$

We claim that

$$\max_{x\in\bigcup_{i=1}^r \Omega_{1i}} \Phi_1(x) < \sum_{j=1}^r \zeta^{\gamma_j} \tag{16}$$

where $\zeta \in (0;1]$ is the positive root of (15). Indeed, switching to the variables $\alpha_1,\dots,\hat\alpha_i,\dots,\alpha_r$, we get $\max_{x\in\Omega_{1i}} \Phi_1(x) = \max_{x\in\Omega'_i} \Phi(x)$, $1 \leqslant i \leqslant r$, where

$$\begin{aligned}\Omega'_i = \{(\alpha_1,\dots,\hat\alpha_i,\dots,\alpha_r) \in \mathbb{R}^{r-1} \mid{}& \alpha_1,\dots,\hat\alpha_i,\dots,\alpha_r \geqslant 0,\\ &\alpha_1+\dots+\hat\alpha_i+\dots+\alpha_r = 1,\\ &\gamma_1\alpha_1+\dots+\widehat{\gamma_i\alpha_i}+\dots+\gamma_r\alpha_r \leqslant 0\}.\end{aligned}$$

(For convenience, we denote the function $\Phi(\theta_1,\dots,\theta_m) = \frac{1}{\theta_1^{\theta_1}\cdots\theta_m^{\theta_m}}$ by the same letter Φ for all i.)

If $i=1$ and $\gamma_2 \geqslant 0$, then

$$\begin{aligned}\Omega'_i = \Omega'_1 = \{(\alpha_2,\dots,\alpha_r) \in \mathbb{R}^{r-1} \mid{}& \alpha_2,\dots,\alpha_r \geqslant 0,\ \alpha_2+\dots+\alpha_r = 1,\\ &\alpha_\ell = \alpha_{\ell+1} = \dots = \alpha_r = 0\}\end{aligned}$$

where the number $2 \leqslant \ell \leqslant r$ is defined by the equality $\gamma_2 = \dots = \gamma_{\ell-1} = 0$ and the inequality $\gamma_\ell > 0$. Then Lemma 9.4 implies $\max_{x\in\Omega'_1} \Phi(x) = \ell - 2$. Since $\sum_{j=1}^r \zeta^{\gamma_j} > \sum_{j=2}^{\ell-1} \zeta^{\gamma_j} = \ell-2$, we get $\max_{x\in\Omega_{11}} \Phi_1(x) < \sum_{j=1}^r \zeta^{\gamma_j}$.

Suppose that either $i \geqslant 2$ or $\gamma_2 < 0$, and $\sum_{\substack{\ell=1,\\ \ell\neq i}}^{r} \gamma_\ell \geqslant 0$. Then we apply the induction assumption for $(r-1)$. We have $\max_{x\in\Omega_i'} \Phi(x) = \sum_{\substack{\ell=1,\\ \ell\neq i}}^{r} (\zeta')^{\gamma_\ell}$ where $\sum_{\substack{\ell=1,\\ \ell\neq i}}^{r} \gamma_\ell(\zeta')^{\gamma_\ell-\gamma_1} = 0$ if $i>1$ and $\sum_{\ell=2}^{r} \gamma_\ell(\zeta')^{\gamma_\ell-\gamma_2} = 0$ if $i=1$. By Lemma 9.6,

$$(17)\qquad \max_{x\in\Omega_{1i}} \Phi_1(x) = \max_{x\in\Omega_i'} \Phi(x) = \min_{y\in(0;1]} \sum_{\substack{\ell=1,\\ \ell\neq i}}^{r} y^{\gamma_\ell} < \min_{y\in(0;1]} \sum_{\ell=1}^{r} y^{\gamma_\ell} = \sum_{j=1}^{r} \zeta^{\gamma_j}.$$

Suppose that either $i \geqslant 2$ or $\gamma_2 < 0$, and $\sum_{\substack{\ell=1,\\ \ell\neq i}}^{r} \gamma_\ell < 0$. Then

$$\left(\frac{1}{r-1}, \frac{1}{r-1}, \dots, \frac{1}{r-1}\right) \in \Omega_i'$$

and by Lemma 9.4 we have $\max_{x\in\Omega_i'} \Phi(x) = r-1$. Again, by Lemma 9.6, we get (17). Therefore (16) is proved.

We claim that $\max_{x\in\Omega_2} \Phi_1(x) = \sum_{i=1}^{r} \zeta^{\gamma_i}$ where $\zeta \in (0;1]$ is the positive root of (15).

If $r=2$, then $\Omega_2 = \{-\frac{\gamma_1}{\gamma_2-\gamma_1}\}$, $\zeta = (-\frac{\gamma_1}{\gamma_2})^{\frac{1}{\gamma_2-\gamma_1}}$,

$$\begin{aligned}\Phi_1\Big(-\frac{\gamma_1}{\gamma_2-\gamma_1}\Big) =& \Big(\frac{\gamma_2}{\gamma_2-\gamma_1}\Big)^{-\frac{\gamma_2}{\gamma_2-\gamma_1}} \Big(-\frac{\gamma_1}{\gamma_2-\gamma_1}\Big)^{\frac{\gamma_1}{\gamma_2-\gamma_1}}\\ =&(\gamma_2-\gamma_1)\gamma_2^{-\frac{\gamma_2}{\gamma_2-\gamma_1}}(-\gamma_1)^{\frac{\gamma_1}{\gamma_2-\gamma_1}}\\ =&\Big(-\frac{\gamma_1}{\gamma_2}\Big)^{\frac{\gamma_1}{\gamma_2-\gamma_1}} + \Big(-\frac{\gamma_1}{\gamma_2}\Big)^{\frac{\gamma_2}{\gamma_2-\gamma_1}}\\ =&\zeta^{\gamma_1}+\zeta^{\gamma_2}.\end{aligned}$$

Therefore we may assume $r \geqslant 3$. Define $1 \leqslant m < r$ by $\gamma_1 = \dots = \gamma_m < \gamma_{m+1}$. Then for all $(\alpha_2, \dots, \alpha_r) \in \Omega_2$ we have $\gamma_1 + \sum_{i=m+1}^{r} (\gamma_i - \gamma_1)\alpha_i = 0$ and

$$(18)\qquad \alpha_{m+1} = -\frac{1}{\gamma_{m+1}-\gamma_1}\Big(\gamma_1 + \sum_{i=m+2}^{r} (\gamma_i-\gamma_1)\alpha_i\Big).$$

We express α_{m+1} and notice that $\max_{x\in\Omega_2}\Phi_1(x)=\max_{x\in\Omega_3}\Phi_2(x)$ where

$$\Phi_2(\alpha_2,\dots,\alpha_m,\alpha_{m+2},\dots,\alpha_r)=\Phi(\alpha_1,\dots,\alpha_r),$$
$$\Omega_3:=\{(\alpha_2,\dots,\alpha_m,\alpha_{m+2},\dots,\alpha_r)\in\mathbb{R}^{r-2}\mid \alpha_1,\dots,\alpha_r\geqslant 0\},$$
$$\alpha_1=1-\sum_{\substack{i=2,\\ i\neq m+1}}^{r}\alpha_i+\frac{1}{\gamma_{m+1}-\gamma_1}\left(\gamma_1+\sum_{i=m+2}^{r}(\gamma_i-\gamma_1)\alpha_i\right)$$

and α_{m+1} is defined by (18).

Consider

$$\begin{aligned}&\frac{\partial\Phi_2}{\partial\alpha_i}(\alpha_2,\dots,\alpha_m,\alpha_{m+2},\dots,\alpha_r)\\ &\quad=\Big((-\ln\alpha_1-1)\frac{\partial\alpha_1}{\partial\alpha_i}-\ln\alpha_i-1+(-\ln\alpha_{m+1}-1)\frac{\partial\alpha_{m+1}}{\partial\alpha_i}\Big)\\ &\qquad\times\Phi_2(\alpha_2,\dots,\alpha_m,\alpha_{m+2},\dots,\alpha_r).\end{aligned}$$

Let $2\leqslant i\leqslant m$. Then $\frac{\partial\alpha_1}{\partial\alpha_i}=-1$, $\frac{\partial\alpha_{m+1}}{\partial\alpha_i}=0$, and

$$\frac{\partial\Phi_2}{\partial\alpha_i}(\alpha_2,\dots,\alpha_m,\alpha_{m+2},\dots,\alpha_r)=(\ln\alpha_1-\ln\alpha_i)\Phi_2(\alpha_2,\dots,\alpha_m,\alpha_{m+2},\dots,\alpha_r).$$

Let $m+2\leqslant i\leqslant r$. Then $\frac{\partial\alpha_1}{\partial\alpha_i}=\frac{\gamma_i-\gamma_{m+1}}{\gamma_{m+1}-\gamma_1}$, $\frac{\partial\alpha_{m+1}}{\partial\alpha_i}=\frac{\gamma_1-\gamma_i}{\gamma_{m+1}-\gamma_1}$, and

$$\begin{aligned}&\frac{\partial\Phi_2}{\partial\alpha_i}(\alpha_2,\dots,\alpha_m,\alpha_{m+2},\dots,\alpha_r)\\ &\quad=\Big(-\frac{\gamma_i-\gamma_{m+1}}{\gamma_{m+1}-\gamma_1}\ln\alpha_1-\ln\alpha_i-\frac{\gamma_1-\gamma_i}{\gamma_{m+1}-\gamma_1}\ln\alpha_{m+1}\Big)\\ &\qquad\times\Phi_2(\alpha_2,\dots,\alpha_m,\alpha_{m+2},\dots,\alpha_r).\end{aligned}$$

Therefore, if $(\alpha_2,\dots,\alpha_m,\alpha_{m+2},\dots,\alpha_r)\in\Omega_3$ is a critical point for Φ_2, we have

$$\frac{\partial\Phi_2}{\partial\alpha_i}(\alpha_2,\dots,\alpha_m,\alpha_{m+2},\dots,\alpha_r)=0$$

for all $2\leqslant i\leqslant m$ and $m+2\leqslant i\leqslant r$ which is equivalent to

$$\begin{cases}\alpha_i=\alpha_1 & \text{for } 2\leqslant i\leqslant m,\\ \alpha_i=\alpha_1^{\left(\frac{\gamma_{m+1}-\gamma_i}{\gamma_{m+1}-\gamma_1}\right)}\alpha_{m+1}^{\left(\frac{\gamma_i-\gamma_1}{\gamma_{m+1}-\gamma_1}\right)} & \text{for } m+2\leqslant i\leqslant r,\\ \alpha_1=1-\sum_{\substack{i=2,\\ i\neq m+1}}^{r}\alpha_i+\frac{1}{\gamma_{m+1}-\gamma_1}(\gamma_1+\sum_{i=m+2}^{r}(\gamma_i-\gamma_1)\alpha_i), & \\ \alpha_{m+1}=-\frac{1}{\gamma_{m+1}-\gamma_1}(\gamma_1+\sum_{i=m+2}^{r}(\gamma_i-\gamma_1)\alpha_i) & \end{cases}$$

and

$$\begin{cases} \alpha_i = \alpha_1 & \text{for } 2 \leqslant i \leqslant m, \\ \alpha_i = \alpha_1 \left(\frac{\alpha_{m+1}}{\alpha_1}\right)^{\frac{\gamma_i - \gamma_1}{\gamma_{m+1}-\gamma_1}} & \text{for } m+2 \leqslant i \leqslant r, \\ \alpha_1 = 1 - \sum_{\substack{i=2, \\ i \neq m+1}}^r \alpha_i + \frac{1}{\gamma_{m+1}-\gamma_1}\left(\gamma_1 + \sum_{i=m+2}^r (\gamma_i - \gamma_1)\alpha_i\right), \\ \alpha_{m+1} = -\frac{1}{\gamma_{m+1}-\gamma_1}\left(\gamma_1 + \sum_{i=m+2}^r (\gamma_i - \gamma_1)\alpha_i\right). \end{cases}$$

Note that since we are looking for inner critical points of Φ_2 on $\Omega_3 \subset \mathbb{R}^{r-2}$, we may assume that all $\alpha_i > 0$.

Performing equivalent transformations, we get

$$(19) \qquad \begin{cases} \alpha_i = \alpha_1 \left(\frac{\alpha_{m+1}}{\alpha_1}\right)^{\frac{\gamma_i - \gamma_1}{\gamma_{m+1}-\gamma_1}} \quad \text{for } 1 \leqslant i \leqslant r, \\ \sum_{i=1}^r \alpha_i = 1, \\ \sum_{i=1}^r \gamma_i \alpha_i = 0. \end{cases}$$

Now we introduce an additional variable $\zeta := \left(\frac{\alpha_{m+1}}{\alpha_1}\right)^{\frac{1}{\gamma_{m+1}-\gamma_1}}$ and get

$$(20) \qquad \begin{cases} \zeta = \left(\frac{\alpha_{m+1}}{\alpha_1}\right)^{\frac{1}{\gamma_{m+1}-\gamma_1}}, \\ \alpha_i = \alpha_1 \zeta^{\gamma_i - \gamma_1} \qquad \text{for } 1 \leqslant i \leqslant r, \\ \alpha_1 \sum_{i=1}^r \zeta^{\gamma_i - \gamma_1} = 1, \\ \sum_{i=1}^r \gamma_i \zeta^{\gamma_i - \gamma_1} = 0. \end{cases}$$

Now the first equation is the consequence of the second one for $i = m+1$. Thus the original system is equivalent to

$$(21) \qquad \begin{cases} \alpha_i = \frac{\zeta^{\gamma_i - \gamma_1}}{\sum_{i=1}^r \zeta^{\gamma_i - \gamma_1}} \qquad \text{for } 1 \leqslant i \leqslant r, \\ \sum_{i=1}^r \gamma_i \zeta^{\gamma_i - \gamma_1} = 0. \end{cases}$$

By Lemma 9.5, the last equation has the unique solution $\zeta \in (0; 1]$. Thus

$$(\alpha_2, \ldots, \alpha_m, \alpha_{m+2}, \ldots, \alpha_r)$$

defined by (21) is the unique inner critical point of Φ_2. Using (19) and (20), we get

$$\begin{aligned}\Phi_2(\alpha_2,\ldots,\alpha_m,\alpha_{m+2},\ldots,\alpha_r) =&\Phi(\alpha_1,\ldots,\alpha_r)\\ =&\frac{1}{\alpha_1^{\alpha_1}\alpha_2^{\alpha_2}\cdots\alpha_r^{\alpha_r}}\\ =&\frac{1}{\alpha_1^{\alpha_1+\cdots+\alpha_r}\zeta^{\alpha_1(\gamma_1-\gamma_1)}\ldots\zeta^{\alpha_r(\gamma_r-\gamma_1)}}\\ =&\frac{1}{\alpha_1\zeta^{-\gamma_1}}\\ =&\sum_{i=1}^r\zeta^{\gamma_i}.\end{aligned}$$

Note that the values of Φ_2 on $\partial\Omega_3$ equal the values of Φ_1 at the corresponding points of $\bigcup_{i=1}^r\Omega_{1i}$. Therefore,

$$\max_{x\in\tilde\Omega}\Phi(x)=\max_{x\in\Omega_1}\Phi_1(x)=\max_{x\in\Omega_3}\Phi_2(x)=\sum_{i=1}^r\zeta^{\gamma_i}.$$

Since (21) implies $\alpha_1\geqslant\alpha_2\geqslant\cdots\geqslant\alpha_r$, we get

$$\max_{x\in\Omega}\Phi(x)=\max_{x\in\tilde\Omega}\Phi(x)=\sum_{i=1}^r\zeta^{\gamma_i}. \qquad \blacksquare$$

Now we immediately get the desired upper bound.

Theorem 9.8: *Let $A=\bigoplus_{t\in T}A^{(t)}$ be a finite dimensional T-graded algebra over a field F of characteristic 0 for some semigroup T such that*

$$A/J(A)\cong M_k(F)$$

for some $k\in\mathbb{N}$ and $A^{(t)}\cap J(A)=0$ for all $t\in T$. Let γ_ℓ, $1\leqslant\ell\leqslant r$, $r:=\dim A$, be the numbers defined before Lemma 9.2. Suppose $\sum_{i=1}^r\gamma_i\geqslant 0$. Let ζ be the positive root of (15). Then

$$\overline{\lim_{n\to\infty}}\sqrt[n]{c_n^{T\text{-gr}}(A)}\leqslant\sum_{i=1}^r\zeta^{\gamma_i}.$$

Proof. This is a consequence of Lemmas 9.3 and 9.7. $\blacksquare$

Remark 9.9: If A is a finite dimensional T-graded-simple algebra over an algebraically closed field F of characteristic 0 for a completely 0-simple semigroup T with trivial maximal subgroups (e.g., T is a right zero band), then by Lemmas

2.2 and 4.5 the algebra A satisfies the conditions of Theorem 9.8 and the upper bound above holds.

10. The case $A/J(A) \cong M_2(F)$ and $\mathrm{PIexp}^{T\text{-gr}}(A) = \dim A$

Let $A = \bigoplus_{t\in T} A^{(t)}$ be a finite dimensional T-graded-simple algebra over a field F of characteristic 0 for some right zero band T. In the next two sections we calculate $\mathrm{PIexp}^{T\text{-gr}}(A)$ in the case $A/J(A) \cong M_2(F)$.

Let $I_t := \pi(A^{(t)})$ for $t \in T$ where $\pi\colon A \to A/J(A)$ is the natural epimorphism.

Note that since $\dim A < +\infty$, only a finite number of I_t are nonzero. Let

$$T_0 := \{t \in T \mid \dim I_t = 2\} \quad \text{and} \quad T_1 = \{t \in T \mid I_t = A/J(A)\}.$$

We have $I_t = 0$ for all $t \notin T_0 \sqcup T_1$. Moreover, $A^{(t)} \cap \ker \pi = 0$ for all $t \in T$ implies

$$r := \dim A = 2|T_0| + 4|T_1|.$$

Define $t_1 \sim t_2$ if $I_{t_1} = I_{t_2}$. Since $A/J(A) \cong M_2(F)$, all irreducible $A/J(A)$-modules are two-dimensional and isomorphic to each other. Thus

$$A/J(A) = I_{t_1} \oplus I_{t_2}$$

for all $I_{t_1} \neq I_{t_2}$.

Now we show that if the cardinalities of equivalence classes satisfy some kind of triangle inequality, then we can combine the elements into pairs and, possibly, a triple such that the elements inside each pair or triple are non-equivalent.

Lemma 10.1: *Let T_0 be a finite non-empty set with an equivalence relation $\sim$. Suppose*

$$|\bar t_0| \leqslant \sum_{\substack{\bar t\in T_0/\sim,\\ \bar t\neq \bar t_0}} |\bar t| \quad \text{for all } \bar t_0 \in T_0/\sim. \tag{22}$$

Then we can choose $\{t_1, \dots, t_{|T_0|}\} = T_0$ such that

1. *if $2 \mid |T_0|$, we have $t_{2i-1} \not\sim t_{2i}$ for all $1 \leqslant i \leqslant \frac{|T_0|}{2}$;*
2. *if $2 \nmid |T_0|$, we have $t_{2i-1} \not\sim t_{2i}$ for all $1 \leqslant i \leqslant \frac{|T_0|-1}{2}$ and $t_{|T_0|-2}$, $t_{|T_0|-1}$, $t_{|T_0|}$ are pairwise non-equivalent.*

Remark 10.2: Note that if (22) does not hold, then there exists an equivalence class $\bar t_0 \in T_0/\sim$ such that $|\bar t_0| \geqslant \frac{|T_0|}{2}$.

Proof of Lemma 10.1. We prove by induction on $|T_0|$. Note that (22) implies $|T_0/{\sim}| \geqslant 2$.

Suppose $|T_0/{\sim}| = 2$. Then (22) implies $|\bar{t}_1| = |\bar{t}_2|$ where $T_0/{\sim} = \{\bar{t}_1, \bar{t}_2\}$. We can define $\{t_1, \dots, t_{|T_0|}\} = T_0$ by

$$\{t_1, t_3, \dots, t_{|T_0|-1}\} := \bar{t}_1 \quad \text{and} \quad \{t_2, t_4, \dots, t_{|T_0|}\} := \bar{t}_2,$$

and the lemma is proved.

Suppose $|T_0| = 3$. Then all the elements of T_0 are pairwise non-equivalent and we again get the lemma.

Now we assume that $|T_0| > 3$. Choose the classes $\bar{t}_1$ and $\bar{t}_2$ with the maximal number of elements. Choose some $t_1 \in \bar{t}_1$ and $t_2 \in \bar{t}_2$. Note that for the set $T_0 \backslash \{t_1, t_2\}$ and the same equivalence relation $\sim$, we still have (22). By the induction assumption we can choose $\{t_3, \dots, t_{|T|}\} = T_0 \backslash \{t_1, t_2\}$ such that $t_1, \dots, t_{|T_0|}$ satisfy the conditions of the lemma. ∎

Inequality (22) will be used to distinguish between the cases when

$$\operatorname{PIexp}^{T\text{-gr}}(A) = \dim A$$

and when

$$\operatorname{PIexp}^{T\text{-gr}}(A) < \dim A.$$

First, we need the following technical lemma.

Lemma 10.3: *Let $\alpha, \beta, \tilde{\alpha}, \tilde{\beta} \in F$. Then*

(1) $[\alpha e_{11} + \beta e_{12}, \alpha e_{21} + \beta e_{22}] = \begin{pmatrix} \alpha\beta & \beta^2 \\ -\alpha^2 & -\alpha\beta \end{pmatrix} = \begin{pmatrix} \beta & 0 \\ -\alpha & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix}$;

(2)
$$\begin{aligned}[t][\alpha e_{11} + \beta e_{12}, &\alpha e_{21} + \beta e_{22}][\tilde{\alpha} e_{11} + \tilde{\beta} e_{12}, \tilde{\alpha} e_{21} + \tilde{\beta} e_{22}] \\ &+ [\tilde{\alpha} e_{11} + \tilde{\beta} e_{12}, \tilde{\alpha} e_{21} + \tilde{\beta} e_{22}][\alpha e_{11} + \beta e_{12}, \alpha e_{21} + \beta e_{22}] \\ &= -\begin{vmatrix} \alpha & \beta \\ \tilde{\alpha} & \tilde{\beta} \end{vmatrix}^2 (e_{11} + e_{22}).\end{aligned}$$

Proof. The first equality is verified by explicit calculations. In order to prove the second one we notice that

$$\begin{aligned}[\alpha e_{11} + \beta e_{12}, &\alpha e_{21} + \beta e_{22}][\tilde\alpha e_{11} + \tilde\beta e_{12}, \tilde\alpha e_{21} + \tilde\beta e_{22}] \\ &= \begin{pmatrix} \beta & 0 \\ -\alpha & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde\beta & 0 \\ -\tilde\alpha & 0 \end{pmatrix} \begin{pmatrix} \tilde\alpha & \tilde\beta \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \beta & 0 \\ -\alpha & 0 \end{pmatrix} \begin{pmatrix} \alpha\tilde\beta - \beta\tilde\alpha & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde\alpha & \tilde\beta \\ 0 & 0 \end{pmatrix} \\ &= \begin{vmatrix} \alpha & \beta \\ \tilde\alpha & \tilde\beta \end{vmatrix} \begin{pmatrix} \beta & 0 \\ -\alpha & 0 \end{pmatrix} \begin{pmatrix} \tilde\alpha & \tilde\beta \\ 0 & 0 \end{pmatrix} = \begin{vmatrix} \alpha & \beta \\ \tilde\alpha & \tilde\beta \end{vmatrix} \begin{pmatrix} \tilde\alpha\beta & \beta\tilde\beta \\ -\alpha\tilde\alpha & -\alpha\tilde\beta \end{pmatrix}. \end{aligned}$$ ∎

Now we can prove the existence of a multilinear polynomial $(FT)^*$-non-identity with sufficiently many alternations.

LEMMA 10.4: *Let $T_0, T_1 \subseteq T$ and $\sim$ be, respectively, the subsets and the equivalence relation defined at the beginning of Section 10. Suppose also that* (22) *holds or $T_0 = \varnothing$. Then there exist a number $n_0 \in \mathbb{N}$ such that for every $n \geqslant n_0$ there exist disjoint subsets $X_1, \dots, X_{2k} \subseteq \{x_1, \dots, x_n\}$, $k = [\frac{n-n_0}{2\dim A}]$, $|X_1| = \dots = |X_{2k}| = \dim A$ and a polynomial $f \in P_n^H \setminus \mathrm{Id}^H(A)$ alternating in the variables of each set X_j.*

Proof. Note that

$$f_0(x_1, \dots, x_4, y_1, \dots, y_4) = \sum_{\sigma,\rho \in S_4} \operatorname{sign}(\sigma\rho) x_{\sigma(1)}\, y_{\rho(1)}\, x_{\sigma(2)} x_{\sigma(3)} x_{\sigma(4)}\, y_{\rho(2)} y_{\rho(3)} y_{\rho(4)}$$

is a polynomial non-identity for $M_2(F)$ and its values are proportional to the identity matrix. (See, e.g., [17, Theorem 5.7.4].)

Let

$$f_{t_1,t_2}(x_{t_1,1}, x_{t_1,2}, x_{t_2,1}, x_{t_2,2}) := [x^{h_{t_1}}_{t_1,1}, x^{h_{t_1}}_{t_1,2}][x^{h_{t_2}}_{t_2,1}, x^{h_{t_2}}_{t_2,2}] + [x^{h_{t_2}}_{t_2,1}, x^{h_{t_2}}_{t_2,2}][x^{h_{t_1}}_{t_1,1}, x^{h_{t_1}}_{t_1,2}]$$

and

$$\begin{aligned} f_{t_1,t_2,t_3}&(x_{t_1,1}, x_{t_1,2}, x_{t_2,1}, x_{t_2,2}, x_{t_3,1}, x_{t_3,2}) \\ :=&[x^{h_{t_1}}_{t_1,1}, x^{h_{t_1}}_{t_1,2}][x^{h_{t_3}}_{t_3,1}, x^{h_{t_3}}_{t_3,2}][x^{h_{t_2}}_{t_2,1}, x^{h_{t_2}}_{t_2,2}] - [x^{h_{t_2}}_{t_2,1}, x^{h_{t_2}}_{t_2,2}][x^{h_{t_3}}_{t_3,1}, x^{h_{t_3}}_{t_3,2}][x^{h_{t_1}}_{t_1,1}, x^{h_{t_1}}_{t_1,2}]. \end{aligned}$$

Let $\{t_1, \dots, t_{|T_0|}\} = T_0$ be the elements from Lemma 10.1.

If $2 \mid |T_0|$, then we define

$$\begin{aligned} f_1 = & z_1 \cdots z_{n-(\dim A)2k} \prod_{i=1}^{k} \Bigg(\prod_{t \in T_1} f_0(x_{i,t,1}^{h_t}, \dots, x_{i,t,4}^{h_t}, y_{i,t,1}^{h_t}, \dots, y_{i,t,4}^{h_t}) \Bigg) \\ & \times \Bigg(\prod_{\ell=1}^{\frac{|T_0|}{2}} f_{t_{2\ell-1},t_{2\ell}}(x_{i,t_{2\ell-1},1}, x_{i,t_{2\ell-1},2}, x_{i,t_{2\ell},1}, x_{i,t_{2\ell},2}) \\ & \qquad \times f_{t_{2\ell-1},t_{2\ell}}(y_{i,t_{2\ell-1},1}, y_{i,t_{2\ell-1},2}, y_{i,t_{2\ell},1}, y_{i,t_{2\ell},2}) \Bigg). \end{aligned}$$

If $2 \nmid |T_0|$, then we define

$$\begin{aligned} f_1 = & z_1 \cdots z_{n-(\dim A)2k} \prod_{i=1}^{k} \Bigg(\prod_{t \in T_1} f_0(x_{i,t,1}^{h_t}, \dots, x_{i,t,4}^{h_t}, y_{i,t,1}^{h_t}, \dots, y_{i,t,4}^{h_t}) \Bigg) \\ & \times \Bigg(\prod_{\ell=1}^{\frac{|T_0|-3}{2}} f_{t_{2\ell-1},t_{2\ell}}(x_{i,t_{2\ell-1},1}, x_{i,t_{2\ell-1},2}, x_{i,t_{2\ell},1}, x_{i,t_{2\ell},2}) \\ & \qquad \times f_{t_{2\ell-1},t_{2\ell}}(y_{i,t_{2\ell-1},1}, y_{i,t_{2\ell-1},2}, y_{i,t_{2\ell},1}, y_{i,t_{2\ell},2}) \Bigg) \\ & \times f_{t_{|T_0|-2},t_{|T_0|-1},t_{|T_0|}}(x_{i,t_{|T_0|-2},1}, x_{i,t_{|T_0|-2},2}; \\ & \qquad x_{i,t_{|T_0|-1},1}, x_{i,t_{|T_0|-1},2}; x_{i,t_{|T_0|},1}, x_{i,t_{|T_0|},2}) \\ & \times f_{t_{|T_0|-2},t_{|T_0|-1},t_{|T_0|}}(y_{i,t_{|T_0|-2},1}, y_{i,t_{|T_0|-2},2}; \\ & \qquad y_{i,t_{|T_0|-1},1}, y_{i,t_{|T_0|-1},2}; y_{i,t_{|T_0|},1}, y_{i,t_{|T_0|},2}). \end{aligned}$$

We claim that $f_1 \notin \mathrm{Id}^{(FT)^*}(A)$. If $2 \mid |T_0|$, then we take any isomorphism $\psi\colon A/J(A) \mathrel{\widetilde{\rightarrow}} M_2(F)$. If $2 \nmid |T_0|$, we define ψ as follows. First, we notice that $t_{|T_0|-2} \nsim t_{|T_0|-1}$ implies $A/J(A) = I_{t_{|T_0|-2}} \oplus I_{t_{|T_0|-1}}$. By Theorem 3.2, there exists an isomorphism $\psi\colon A/J(A) \mathrel{\widetilde{\rightarrow}} M_2(F)$ such that $\psi(I_{t_{|T_0|-2}}) = \langle e_{11}, e_{21}\rangle_F$ and $\psi(I_{t_{|T_0|-1}}) = \langle e_{12}, e_{22}\rangle_F$. If $2 \nmid |T_0|$, then we take this isomorphism ψ.

Lemma 3.3 implies that for every $t \in T_0$ there exist $\alpha_t, \beta_t \in F$ such that

$$\psi(I_t) = \langle \alpha_t e_{i1} + \beta_t e_{i2} \mid i = 1, 2\rangle_F.$$

If $2 \nmid |T_0|$, by our choice of ψ, we may assume that

$$(\alpha_{t_{|T_0|-2}}, \beta_{t_{|T_0|-2}}) = (1, 0) \quad \text{and} \quad (\alpha_{t_{|T_0|-1}}, \beta_{t_{|T_0|-1}}) = (0, 1).$$

Note that $I_{t_1} = I_{t_2}$ if and only if the rows $(\alpha_{t_1}, \beta_{t_1})$ and $(\alpha_{t_2}, \beta_{t_2})$ are proportional.

Fix some element $e \in A$ such that $\psi\pi(e)$ is the identity matrix. We substitute $z_1 = \cdots = z_{n-(\dim A)2k} = e$,

$$\begin{aligned} x_{i,t,1} =& y_{i,t,1} = (\pi|_{A^{(t)}})^{-1}\psi^{-1}(e_{11}), \quad x_{i,t,2} = y_{i,t,2} = (\pi|_{A^{(t)}})^{-1}\psi^{-1}(e_{12}),\\ x_{i,t,3} =& y_{i,t,3} = (\pi|_{A^{(t)}})^{-1}\psi^{-1}(e_{21}), \quad x_{i,t,4} = y_{i,t,4} = (\pi|_{A^{(t)}})^{-1}\psi^{-1}(e_{22}) \end{aligned}$$

for all $t \in T_1$ and $1 \leqslant i \leqslant k$ and

$$\begin{aligned} x_{i,t,1} =& y_{i,t,1} = (\pi|_{A^{(t)}})^{-1}\psi^{-1}(\alpha_t e_{11} + \beta_t e_{12}),\\ x_{i,t,2} =& y_{i,t,2} = (\pi|_{A^{(t)}})^{-1}\psi^{-1}(\alpha_t e_{21} + \beta_t e_{22}) \end{aligned}$$

for all $t \in T_0$ and $1 \leqslant i \leqslant k$.

In order to show that f_1 does not vanish under this evaluation, we apply $\psi\pi$ to the result of the substitution. The value of $f_{t_{2\ell-1},t_{2\ell}}$ is nonzero since by Lemma 10.3,

$$\begin{aligned} &[\alpha_{t_{2\ell-1}}e_{11}+\beta_{t_{2\ell-1}}e_{12}, \alpha_{t_{2\ell-1}}e_{21}+\beta_{t_{2\ell-1}}e_{22}][\alpha_{t_{2\ell}}e_{11}+\beta_{t_{2\ell}}e_{12}, \alpha_{t_{2\ell}}e_{21}+\beta_{t_{2\ell}}e_{22}]\\ &+[\alpha_{t_{2\ell}}e_{11}+\beta_{t_{2\ell}}e_{12}, \alpha_{t_{2\ell}}e_{21}+\beta_{t_{2\ell}}e_{22}][\alpha_{t_{2\ell-1}}e_{11}+\beta_{t_{2\ell-1}}e_{12}, \alpha_{t_{2\ell-1}}e_{21}+\beta_{t_{2\ell-1}}e_{22}]\\ &= -\begin{vmatrix} \alpha_{t_{2\ell-1}} & \beta_{t_{2\ell-1}} \\ \alpha_{t_{2\ell}} & \beta_{t_{2\ell}} \end{vmatrix}^2 (e_{11}+e_{22}) \neq 0. \end{aligned}$$

The polynomial $f_{t_{|T_0|-2},t_{|T_0|-1},t_{|T_0|}}$ does not vanish under this evaluation since, by Lemma 10.3,

$$\begin{aligned} [e_{11}, e_{21}]&[\alpha_{t_{|T_0|}}e_{11} + \beta_{t_{|T_0|}}e_{12}, \alpha_{t_{|T_0|}}e_{21} + \beta_{t_{|T_0|}}e_{22}][e_{12}, e_{22}]\\ &- [e_{12}, e_{22}][\alpha_{t_{|T_0|}}e_{11} + \beta_{t_{|T_0|}}e_{12}, \alpha_{t_{|T_0|}}e_{21} + \beta_{t_{|T_0|}}e_{22}][e_{11}, e_{21}]\\ =& -\alpha_{t_{|T_0|}}\beta_{t_{|T_0|}}(e_{11} + e_{22}) \neq 0. \end{aligned}$$

Thus $f_1 \notin \mathrm{Id}^{(FT)^*}(A)$. Now we define $f = \mathrm{Alt}_1 \cdots \mathrm{Alt}_{2k} f_1$ with Alt_i the operator of alternation on the set X_i, where

$$\begin{aligned} X_{2i-1} =& \{x_{i,t,j} \mid t \in T,\ 1 \leqslant j \leqslant 2 \text{ for } t \in T_0,\ 1 \leqslant j \leqslant 4 \text{ for } t \in T_1\},\\ X_{2i} =& \{y_{i,t,j} \mid t \in T,\ 1 \leqslant j \leqslant 2 \text{ for } t \in T_0,\ 1 \leqslant j \leqslant 4 \text{ for } t \in T_1\}, \end{aligned}$$

$1 \leqslant i \leqslant 2k$. Note that f does not vanish under the same substitution that we used for f_1 since if the alternation replaces x_{i_1,t_1,j_1} with x_{i_2,t_2,j_2} for $t_1 \neq t_2$, then the value of $x^{h_{t_1}}_{i_2,t_2,j_2}$ is zero and the corresponding item vanishes.

For our convenience, we rename the variables of f to be $x_1, \ldots, x_n$. Then f satisfies all the conditions of the lemma. ∎

Now we are ready to prove that in the case when (22) holds,

$$\mathrm{PIexp}^{T\text{-gr}}(A) = \dim A.$$

THEOREM 10.5: *Let A be a finite dimensional T-graded-simple algebra over a field F of characteristic 0 for a right zero band T. Suppose $A/J(A) \cong M_2(F)$. Let $T_0, T_1 \subseteq T$ and $\sim$ be, respectively, the subsets and the equivalence relation defined at the beginning of Section 10. Suppose also that (22) holds or $T_0 = \varnothing$. Then there exist $C > 0$, $r \in \mathbb{R}$, such that*

$$Cn^r(\dim A)^n \leqslant c_n^{T\text{-gr}}(A) \leqslant (\dim A)^{n+1}.$$

In particular, $\mathrm{PIexp}^{T\text{-gr}}(A) = \dim A$.

Proof. Note that by Lemma 7.3, $c_n^{T\text{-gr}}(A) = c_n^{(FT)^*}(A)$ for all $n \in \mathbb{N}$. The upper bound follows from Proposition 6.3. In order to get the lower bound, we repeat literally the proof of [18, Lemma 11 and Theorem 5] using Lemma 10.4 instead of [18, Lemma 10]. ∎

11. The case $A/J(A) \cong M_2(F)$ and $\mathrm{PIexp}^{T\text{-gr}}(A) < \dim A$

Let A be a finite dimensional T-graded-simple algebra over a field F of characteristic 0 for a right zero band T. Suppose $A/J(A) \cong M_2(F)$. Let $T_0, T_1 \subseteq T$ and $\sim$ be, respectively, the subsets and the equivalence relation defined at the beginning of Section 10.

Suppose that $T_0 \neq \varnothing$ and the inequality (22) does not hold in A. This is equivalent to the existence of $t_0 \in T_0$ such that $|\bar{t}_0| > \frac{|T_0|}{2}$. Using Theorem 3.2, we fix an isomorphism $\psi\colon A/J(A) \to M_2(F)$ such that $\psi(I_{t_0}) = \langle e_{11}, e_{21}\rangle_F$. By Lemma 3.3, for every $t \in T_0$ one can choose $\alpha_t, \beta_t \in F$ such that

$$(\alpha_t e_{i1} + \beta_t e_{i2} \mid 1 \leqslant i \leqslant 2)$$

is a basis of $\psi(I_t)$. We may assume that $(\alpha_t, \beta_t) = (1, 0)$ for $t \sim t_0$. Note that $\beta_t \neq 0$ if $I_t \neq I_{t_0}$.

Now we fix the basis

$$((\pi|_{A^{(t)}})^{-1}\psi^{-1}(\alpha_t e_{i1} + \beta_t e_{i2}) \mid 1 \leqslant i \leqslant 2)$$

in $A^{(t)}$ for each $t \in T_0$ and

$$((\pi|_{A^{(t)}})^{-1}\psi^{-1}(e_{ij}) \mid 1 \leqslant i, j \leqslant 2)$$

in $A^{(t)}$ for each $t \in T_1$. We define the basis $\mathcal{B}$ in A as the union of the bases in $A^{(t)}$ chosen above.

Now we calculate the numbers γ_i introduced at the beginning of Section 9. We notice that

$$\theta((\pi|_{A^{(t)}})^{-1}\psi^{-1}(\alpha_t e_{i1} + \beta_t e_{i2})) = i - 2$$

for $1 \leqslant i \leqslant 2$ and $t \in T_0$, $t \nsim t_0$. Then

$$(\gamma_1, \dots, \gamma_r) = (\underbrace{-1, \dots, -1}_{|T_0|+|T_1|-|\bar{t}_0|}, \underbrace{0, \dots, 0}_{|T_0|+2|T_1|}, \underbrace{1, \dots, 1}_{|T_1|+|\bar{t}_0|}) \tag{23}$$

and the graded cocharacter of A satisfies the inequality from Lemma 9.2.

Below we prove three lemmas which enable us to choose elements $b_1, \dots, b_m$ that we will substitute for the variables corresponding to the numbers in a column of a given Young diagram. Here it is important to control the sum $\sum_{j=1}^m \theta(b_i)$.

Lemma 11.1: *Let $1 \leqslant m \leqslant r$. Then*

$$m - \sum_{j=1}^m \gamma_j \leqslant 3|T_0| + 4|T_1| - 2|\bar{t}_0|.$$

Proof. If

$$m \leqslant |T_0| + |T_1| - |\bar{t}_0|,$$

then $\gamma_j = -1$ for $1 \leqslant j \leqslant m$ and $\sum_{j=1}^m \gamma_j = -m$. Hence

$$m - \sum_{j=1}^m \gamma_j = 2m \leqslant 2|T_0| + 2|T_1| - 2|\bar{t}_0| \leqslant 3|T_0| + 4|T_1| - 2|\bar{t}_0|.$$

If

$$|T_0| + |T_1| - |\bar{t}_0| \leqslant m \leqslant 2|T_0| + 3|T_1| - |\bar{t}_0|,$$

then $\gamma_j = 0$ for $|T_0| + |T_1| - |\bar{t}_0| < j \leqslant m$ and $\sum_{j=1}^m \gamma_j = -(|T_0| + |T_1| - |\bar{t}_0|)$. Hence

$$\begin{aligned} m - \sum_{j=1}^m \gamma_j =& m + (|T_0| + |T_1| - |\bar{t}_0|) \\ \leqslant& 3|T_0| + 4|T_1| - 2|\bar{t}_0|. \end{aligned}$$

If

$$m \geqslant 2|T_0| + 3|T_1| - |\bar{t}_0|,$$

then $\gamma_j = 1$ for $j > 2|T_0| + 3|T_1| - |\bar{t}_0|$ implies

$$m - \sum_{j=1}^{m} \gamma_j = 3|T_0| + 4|T_1| - 2|\bar{t}_0|. \qquad \blacksquare$$

Lemma 11.2: *Let $\sum_{j=1}^m \gamma_j > 0$ for some $m \in \mathbb{N}$. Then there exist $b_1, \dots, b_m \in \mathcal{B}$, $b_i \neq b_j$ for $i \neq j$, such that $\sum_{j=1}^m \theta(b_j) = \sum_{j=1}^m \gamma_j$ and*

- *if $\{b_1, \dots, b_m\} \cap A^{(t)} = \{b_i\}$ for some $1 \leqslant i \leqslant m$ and $t \in T_0 \sqcup T_1$, then $t \in T_0$, $t \sim t_0$ and $b_i = (\pi|_{A^{(t)}})^{-1}\psi^{-1}(e_{11})$;*
- *if $\{b_1, \dots, b_m\} \cap A^{(t)} = \{b_i, b_j\}$ for some $1 \leqslant i, j \leqslant m$ and $t \in T_0 \sqcup T_1$, then either $\theta(b_i) \neq 0$ or $\theta(b_j) \neq 0$.*

Proof. By the definition of γ_i (see the beginning of Section 9), there exist $b_1, \dots, b_m \in \mathcal{B}$, $b_i \neq b_j$ for $i \neq j$, such that $\sum_{j=1}^m \theta(b_j) = \sum_{j=1}^m \gamma_j$ and $\sum_{j=1}^m \theta(a_j) \geqslant \sum_{j=1}^m \gamma_j$ for all $a_1, \dots, a_m \in \mathcal{B}$ where $a_i \neq a_j$ for $i \neq j$. Since $\sum_{j=1}^m \gamma_j > 0$, the minimality of $\sum_{j=1}^m \theta(b_j)$ implies that the set $\{b_1, \dots, b_m\}$ contains all elements $b \in \mathcal{B}$ with $\theta(b) \leqslant 0$. Now the choice of $\mathcal{B}$ implies the lemma. $\blacksquare$

Lemma 11.3: *Let $\sum_{j=1}^m \gamma_j \leqslant q \leqslant 0$ for some $m \in \mathbb{N}$, $q \in \mathbb{Z}$. Then there exist $b_1, \dots, b_m \in \mathcal{B}$, $b_i \neq b_j$ for $i \neq j$, such that $\sum_{j=1}^m \theta(b_j) = q$ and*

- *if $\{b_1, \dots, b_m\} \cap A^{(t)} = \{b_i\}$, $\theta(b_i) = 0$, for some $1 \leqslant i \leqslant m$ and $t \in T_0 \sqcup T_1$, then $t \in T_0$, $t \sim t_0$ and $b_i = (\pi|_{A^{(t)}})^{-1}\psi^{-1}(e_{11})$;*
- *if $\{b_1, \dots, b_m\} \cap A^{(t)} = \{b_i, b_j\}$ for some $1 \leqslant i, j \leqslant m$ and $t \in T_0 \sqcup T_1$, then either $\theta(b_i) \neq 0$ or $\theta(b_j) \neq 0$.*

Proof. Recall that

$$\begin{aligned}
|\{b \in \mathcal{B} \mid \theta(b) = -1\}| &= |T_0| + |T_1| - |\bar{t}_0|, \\
|\{b \in \mathcal{B} \mid \theta(b) = 0\}| &= |T_0| + 2|T_1|, \\
|\{b \in \mathcal{B} \mid \theta(b) = 1\}| &= |T_1| + |\bar{t}_0|.
\end{aligned}$$

Let $\{t_1, \dots, t_{|T_1|}\} := T_1$, $\{\tilde{t}_1, \dots, \tilde{t}_{|T_0|-|\bar{t}_0|}\} := T_0 \backslash \bar{t}_0$, $\{\hat{t}_1, \dots, \hat{t}_{|\bar{t}_0|}\} := \bar{t}_0$.

Now we consider two main cases:

1. Suppose $m < 2(|T_0| + |T_1| - |\bar{t}_0|) + q$. Let $\ell = [\frac{m+q}{2}]$. Note that

$$\ell - q \leqslant \frac{m-q}{2} < |T_0| + |T_1| - |\bar{t}_0|$$

and

$$\ell \leqslant \ell - q < |T_0| + |T_1| - |\bar{t}_0| < |T_1| + |\bar{t}_0|.$$

These inequalities imply that below we have enough elements from $(T_0 \backslash \bar{t}_0) \sqcup T_1$ and $\bar{t}_0 \sqcup T_1$, respectively.

Suppose first that $m = 2\ell - q$. If $\ell \leqslant |T_1| + q$, then we take

$$\begin{aligned}\{b_1, \dots, b_m\} = &\{(\pi|_{A^{(t_i)}})^{-1}\psi^{-1}(e_{12}) \mid 1 \leqslant i \leqslant \ell - q\} \\ &\cup \{(\pi|_{A^{(t_i)}})^{-1}\psi^{-1}(e_{21}) \mid 1 \leqslant i \leqslant \ell\}.\end{aligned}$$

If $|T_1| + q < \ell \leqslant |T_1|$, then we take

$$\begin{aligned}\{b_1, \dots, b_m\} = &\{(\pi|_{A^{(t_i)}})^{-1}\psi^{-1}(e_{12}) \mid 1 \leqslant i \leqslant |T_1|\} \\ &\cup \{(\pi|_{A^{(\tilde{t}_i)}})^{-1}\psi^{-1}(\alpha_{\tilde{t}_i} e_{11} + \beta_{\tilde{t}_i} e_{12}) \mid 1 \leqslant i \leqslant \ell - q - |T_1|\} \\ &\cup \{(\pi|_{A^{(t_i)}})^{-1}\psi^{-1}(e_{21}) \mid 1 \leqslant i \leqslant \ell\}.\end{aligned}$$

If $\ell > |T_1|$, then we take

$$\begin{aligned}\{b_1, \dots, b_m\} = &\{(\pi|_{A^{(t_i)}})^{-1}\psi^{-1}(e_{12}) \mid 1 \leqslant i \leqslant |T_1|\} \\ &\cup \{(\pi|_{A^{(\tilde{t}_i)}})^{-1}\psi^{-1}(\alpha_{\tilde{t}_i} e_{11} + \beta_{\tilde{t}_i} e_{12}) \mid 1 \leqslant i \leqslant \ell - q - |T_1|\} \\ &\cup \{(\pi|_{A^{(t_i)}})^{-1}\psi^{-1}(e_{21}) \mid 1 \leqslant i \leqslant |T_1|\} \\ &\cup \{(\pi|_{A^{(\hat{t}_i)}})^{-1}\psi^{-1}(e_{21}) \mid 1 \leqslant i \leqslant \ell - |T_1|\}.\end{aligned}$$

If $m = 2\ell - q + 1$, then in each of the three cases above we add the element $(\pi|_{A^{(\hat{t}_1)}})^{-1}\psi^{-1}(e_{11})$.

2. Suppose $m \geqslant 2(|T_0|+|T_1|-|\bar{t}_0|)+q$.

By Lemma 11.1,

$$m-\sum_{j=1}^{m}\gamma_j \leqslant 3|T_0|+4|T_1|-2|\bar{t}_0|.$$

Hence

$$m-q-2(|T_0|+|T_1|-|\bar{t}_0|) \leqslant |T_0|+2|T_1|$$

and we can choose

$$0\leqslant k,\ell\leqslant |T_1|, \quad 0\leqslant s\leqslant |\bar{t}_0|, \quad 0\leqslant u\leqslant |T_0|-|\bar{t}_0|,$$

such that $2(|T_0|+|T_1|-|\bar{t}_0|)+q+k+\ell+s+u=m$.

Note that

$$(|T_0|+|T_1|-|\bar{t}_0|)+q \geqslant (|T_0|+|T_1|-|\bar{t}_0|)+\sum_{j=1}^{m}\gamma_j \geqslant 0.$$

If $(|T_0|+|T_1|-|\bar{t}_0|)+q \leqslant |T_1|$, we define

$$\begin{aligned}\{b_1,\dots,b_m\} =& \{(\pi|_{A^{(t_i)}})^{-1}\psi^{-1}(e_{12}) \mid 1\leqslant i\leqslant |T_1|\}\\ &\cup\{(\pi|_{A^{(t_i)}})^{-1}\psi^{-1}(e_{11}) \mid 1\leqslant i\leqslant k\}\\ &\cup\{(\pi|_{A^{(t_i)}})^{-1}\psi^{-1}(e_{21}) \mid 1\leqslant i\leqslant (|T_0|+|T_1|-|\bar{t}_0|)+q\}\\ &\cup\{(\pi|_{A^{(t_i)}})^{-1}\psi^{-1}(e_{22}) \mid 1\leqslant i\leqslant \ell\}\\ &\cup\{(\pi|_{A^{(\tilde{t}_i)}})^{-1}\psi^{-1}(\alpha_{\tilde{t}_i}e_{11}+\beta_{\tilde{t}_i}e_{12}) \mid 1\leqslant i\leqslant |T_0|-|\bar{t}_0|\}\\ &\cup\{(\pi|_{A^{(\tilde{t}_i)}})^{-1}\psi^{-1}(\alpha_{\tilde{t}_i}e_{21}+\beta_{\tilde{t}_i}e_{22}) \mid 1\leqslant i\leqslant u\}\\ &\cup\{(\pi|_{A^{(\hat{t}_i)}})^{-1}\psi^{-1}(e_{11}) \mid 1\leqslant i\leqslant s\}.\end{aligned}$$

If $(|T_0|+|T_1|-|\bar{t}_0|)+q > |T_1|$, we define

$$\begin{aligned}\{b_1,\dots,b_m\} =& \{(\pi|_{A^{(t_i)}})^{-1}\psi^{-1}(e_{12}) \mid 1\leqslant i\leqslant |T_1|\}\\ &\cup\{(\pi|_{A^{(t_i)}})^{-1}\psi^{-1}(e_{11}) \mid 1\leqslant i\leqslant k\}\\ &\cup\{(\pi|_{A^{(t_i)}})^{-1}\psi^{-1}(e_{21}) \mid 1\leqslant i\leqslant |T_1|\}\\ &\cup\{(\pi|_{A^{(t_i)}})^{-1}\psi^{-1}(e_{22}) \mid 1\leqslant i\leqslant \ell\}\\ &\cup\{(\pi|_{A^{(\tilde{t}_i)}})^{-1}\psi^{-1}(\alpha_{\tilde{t}_i}e_{11}+\beta_{\tilde{t}_i}e_{12}) \mid 1\leqslant i\leqslant |T_0|-|\bar{t}_0|\}\\ &\cup\{(\pi|_{A^{(\tilde{t}_i)}})^{-1}\psi^{-1}(\alpha_{\tilde{t}_i}e_{21}+\beta_{\tilde{t}_i}e_{22}) \mid 1\leqslant i\leqslant u\}\\ &\cup\{(\pi|_{A^{(\hat{t}_i)}})^{-1}\psi^{-1}(e_{11}) \mid 1\leqslant i\leqslant s\}\\ &\cup\{(\pi|_{A^{(\hat{t}_i)}})^{-1}\psi^{-1}(e_{21}) \mid 1\leqslant i\leqslant (|T_0|-|\bar{t}_0|)+q\}.\end{aligned}$$ ∎

In the lemma below we present a graded polynomial non-identity having sufficiently many alternations. This non-identity will generate an FS_n-submodule that will have the dimension great enough to prove the lower bound for graded codimensions.

Lemma 11.4: *Let $\lambda = (\lambda_1, \dots, \lambda_r) \vdash n$ for some $n \in \mathbb{N}$. If $\sum_{i=1}^r \gamma_i \lambda_i \leqslant 0$ and $2 \mid \lambda_i$ for $i \geqslant 2$, then there exists a Young tableau T_λ of shape λ and an $(FT)^*$-polynomial $f \in P_n^{(FT)^*}$ such that $b_{T_\lambda} f := \sum_{\sigma \in C_{T_\lambda}} (\operatorname{sign} \sigma) \sigma f \notin \operatorname{Id}^{(FT)^*}(A)$.*

Proof. Let μ_i be the number of boxes in the ith column of the tableau T_λ and let $m_i := \sum_{j=1}^{\mu_i} \gamma_j$, $1 \leqslant i \leqslant \lambda_1$.

Note that since $(\lambda_i - \lambda_{i+1})$ equals the number of columns of height i (here $\lambda_{r+1} := 0$) and λ_1 equals the number of all columns, the inequality $\sum_{i=1}^r \gamma_i \lambda_i \leqslant 0$ can be rewritten as

$$\sum_{i=1}^{r} \left(\sum_{j=1}^{i} \gamma_j - \sum_{j=1}^{i-1} \gamma_j \right) \lambda_i = \sum_{i=1}^{r} \left(\sum_{j=1}^{i} \gamma_j \right) (\lambda_i - \lambda_{i+1}) = \sum_{i=1}^{\lambda_1} m_i \leqslant 0. \tag{24}$$

By Lemma 9.1, if the sum of the values of θ on basis elements substituted for the variables of a multilinear $(FT)^*$-polynomial is less than (-1) or greater than 1, then the $(FT)^*$-polynomial vanishes. We will choose elements $b_{itj} \in A^{(t)} \cap \mathcal{B}$ such that the sum of the values of θ on them equals 0. If $m_i > 0$ for some i, we have to make the sum of values of θ for some other columns negative.

Define the number $1 \leqslant \ell \leqslant \lambda_1$ by the conditions $m_\ell > 0$ and $m_{\ell+1} \leqslant 0$. Since $2 \mid \lambda_i$ for all $i \geqslant 2$, we have $m_{2j-1} = m_{2j}$ for $1 \leqslant j \leqslant \frac{\lambda_2}{2}$. Hence $2 \mid \ell$ and $2 \mid \sum_{i=1}^{\ell} m_i$. Recall that $m_i = -1$ for $\lambda_2 + 1 \leqslant i \leqslant \lambda_1$. By (24),

$$2 \sum_{i=1}^{\lambda_2/2} m_{2i} - (\lambda_1 - \lambda_2) = \sum_{i=1}^{\lambda_1} m_i \leqslant 0$$

and

$$\sum_{i=1}^{\lambda_2/2} m_{2i} - \left[\frac{\lambda_1 - \lambda_2}{2} \right] \leqslant 0.$$

Thus one can choose integers N and q_{2i} for $\ell < 2i \leqslant \lambda_2$, such that

$$0 \leqslant N \leqslant \left[\frac{\lambda_1 - \lambda_2}{2} \right], \quad m_{2i} \leqslant q_{2i} \leqslant 0 \quad \text{and} \quad \sum_{i=1}^{\ell/2} m_{2i} + \sum_{i=\ell/2+1}^{\lambda_2} q_{2i} - N = 0.$$

Define $q_{2i-1} := q_{2i}$ for $\frac{\ell}{2} < i \leqslant \frac{\lambda_2}{2}$, $q_i := -1$ for $\lambda_2 + 1 \leqslant i \leqslant \lambda_2 + 2N$, $q_i := 0$ for $\lambda_2 + 2N + 1 \leqslant i \leqslant \lambda_1$. Then

$$\sum_{i=1}^{\ell} m_i + \sum_{i=\ell+1}^{\lambda_1} q_i = 0. \tag{25}$$

Now we use Lemma 11.2 for every $1 \leqslant i \leqslant \ell$ (there $m = \mu_i$) and Lemma 11.3 for every $\ell + 1 \leqslant i \leqslant \lambda_1$ (there $m = \mu_i$ and $q = q_i$). We sort the obtained elements b_j in accordance with the homogeneous components $A^{(t)}$ they belong to. For a fixed $1 \leqslant i \leqslant \lambda_1$ we get elements $b_{itj} \in A^{(t)} \cap \mathcal{B}$ where $t \in T_0 \sqcup T_1$, $1 \leqslant j \leqslant n_{it}$, the total number of b_{itj} equals $n_{it} \geqslant 0$,

$$\begin{aligned} \sum_{t \in T_0 \sqcup T_1} n_{it} &= \mu_i, && \\ b_{it_1 j_1} &\neq b_{it_2 j_2} && \text{if } (t_1, j_1) \neq (t_2, j_2), \\ \sum_{t \in T_0 \sqcup T_1} \sum_{j=1}^{n_{it}} \theta(b_{itj}) &= m_i && \text{if } m_i > 0 \end{aligned}$$

and

$$m_i \leqslant \sum_{t \in T_0 \sqcup T_1} \sum_{j=1}^{n_{it}} \theta(b_{itj}) = q_i \leqslant 0 \qquad \text{if } m_i \leqslant 0.$$

By the virtue of (25),

$$\sum_{i=1}^{\lambda_1} \sum_{t \in T_0 \sqcup T_1} \sum_{j=1}^{n_{it}} \theta(b_{itj}) = 0. \tag{26}$$

Since $q_{2i-1} = q_{2i}$ and $\mu_{2i-1} = \mu_{2i}$ if $2i \leqslant \lambda_2$, we may assume that $n_{2i-1,t} = n_{2i,t}$, $b_{2i-1,t,j} = b_{2i,t,j}$ for all $1 < 2i \leqslant \lambda_2$, $t \in T_0 \sqcup T_1$, $1 \leqslant j \leqslant n_{2i,t}$.

We will substitute elements b_{itj} for the variables with the indices from the ith column.

Denote by W_{-1} the set of all pairs (i, t) where $1 \leqslant i \leqslant \lambda_1$, $t \in T_0 \sqcup T_1$, such that $\theta(b_{itj}) = -1$ for some $1 \leqslant j \leqslant n_{ti}$ and $\theta(b_{itj}) \leqslant 0$ for all $1 \leqslant j \leqslant n_{ti}$. Denote by W_1 the set of all pairs (i, t) where $1 \leqslant i \leqslant \lambda_1$, $t \in T_0 \sqcup T_1$, such that $\theta(b_{itj}) = 1$ for some $1 \leqslant j \leqslant n_{ti}$ and $\theta(b_{itj}) \geqslant 0$ for all $1 \leqslant j \leqslant n_{ti}$. Let

$$W_1^{(i)} := \{t \in T_0 \sqcup T_1 \mid (i, t) \in W_1\}$$

for $1 \leqslant i \leqslant \lambda_1$ and

$$W_0^{(i)} := \{t \in T_0 \sqcup T_1 \mid (i, t) \notin W_{-1} \sqcup W_1, n_{it} > 0\}.$$

By (26), $|W_{-1}| = |W_1|$. Therefore, there exist maps $\varkappa\colon W_1 \to \{1,\dots,\lambda_1\}$ and $\rho\colon W_1 \to T_0 \sqcup T_1$ such that $(i,t) \mapsto (\varkappa(i,t),\rho(i,t))$ is a bijection $W_1 \to W_{-1}$.

Define polynomials f_{it} and $\tilde f_{it}$, $1 \leqslant i \leqslant \lambda_2$, $t \in T_0 \sqcup T_1$, as follows.

If $n_{it} = 1$, then

$$f_{it}(x_1) = x_1.$$

If $n_{it} = 2$, then

$$f_{it}(x_1, x_2) = x_1x_2 - x_2x_1.$$

If $n_{it} = 3$, then

$$f_{it}(x_1, x_2, x_3) = \sum_{\sigma\in S_3} \operatorname{sign}(\sigma) x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}.$$

If $1 \leqslant n_{it} \leqslant 3$, then

$$\tilde f_{it}(x_1,\dots,x_{n_{it}}; y_1,\dots,y_{n_{it}}) = f_{it}(x_1,\dots,x_{n_{it}}) f_{it}(y_1,\dots,y_{n_{it}}).$$

If $n_{it} = 4$, then

$$\begin{aligned}&\tilde f_{it}(x_1,\dots,x_4,y_1,\dots,y_4)\\ &\quad= \sum_{\sigma,\tau\in S_4} \operatorname{sign}(\sigma\tau) x_{\sigma(1)}\ y_{\tau(1)}\, x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(4)}\, y_{\tau(2)}y_{\tau(3)}y_{\tau(4)}.\end{aligned}$$

Recall this is a polynomial non-identity for $M_2(F)$ and its values are proportional to the identity matrix. (See, e.g., [17, Theorem 5.7.4].)

Let $X_i := \{x_{itj} \mid 1 \leqslant j \leqslant n_{it},\ t \in T_0 \sqcup T_1\}$, $1 \leqslant i \leqslant \lambda_1$. Denote by Alt_i the operator of alternation in the variables from X_i.

Define

$$\begin{aligned}
f := {}& \operatorname{Alt}_1 \operatorname{Alt}_2 \cdots \operatorname{Alt}_{\lambda_1}\\
&\times \prod_{i=1}^{\lambda_2/2} \Bigg(\prod_{t\in W_0^{(2i-1)}} \tilde f_{2i-1,t}(x^{h_t}_{2i-1,t,1},\dots,x^{h_t}_{2i-1,t,n_{2i-1,t}};\ x^{h_t}_{2i,t,1},\dots,x^{h_t}_{2i,t,n_{2i,t}})\\
&\qquad \times \Bigg(\prod_{t\in W_1^{(2i-1)}} f_{\varkappa(2i-1,t)\rho(2i-1,t)}(x^{h_{\rho(2i-1,t)}}_{\varkappa(2i-1,t)\rho(2i-1,t)1},\dots,\\
&\qquad\qquad x^{h_{\rho(2i-1,t)}}_{\varkappa(2i-1,t)\rho(2i-1,t)n_{\varkappa(2i-1,t)\rho(2i-1,t)}})\\
&\qquad\quad \times f_{2i-1,t}(x^{h_t}_{2i-1,t,1},\dots,x^{h_t}_{2i-1,t,n_{2i-1,t}})\\
&\qquad\quad \times f_{\varkappa(2i,t)\rho(2i,t)}(x^{h_{\rho(2i,t)}}_{\varkappa(2i,t)\rho(2i,t)1},\dots,x^{h_{\rho(2i,t)}}_{\varkappa(2i,t)\rho(2i,t)n_{\varkappa(2i,t)\rho(2i,t)}})\\
&\qquad\quad \times f_{2i,t}(x^{h_t}_{2i,t,1},\dots,x^{h_t}_{2i,t,n_{2i,t}})\Bigg)\Bigg) \prod_{\substack{i=\lambda_2+1,\\ t\in W_0^{(i)}}}^{\lambda_1} x^{h_t}_{it1}.
\end{aligned}$$

Note that by Lemmas 11.2 and 11.3, if $(i,t) \in W_{-1}$, then

$$\{\psi\pi(b_{it1}), \dots, \psi\pi(b_{itn_{it}})\}$$

coincides with one of the following sets: $\{e_{12}\}$, $\{\alpha_t e_{11} + \beta_t e_{12}\}$, $\{e_{12}, e_{22}\}$, $\{e_{11}, e_{12}\}$, $\{\alpha_t e_{11} + \beta_t e_{12}, \alpha_t e_{21} + \beta_t e_{22}\}$, $\{e_{11}, e_{12}, e_{22}\}$.

If $t \in W_0^{(i)}$, then $\{\psi\pi(b_{it1}), \dots, \psi\pi(b_{itn_{it}})\}$ coincides with one of the following sets: $\{e_{11}\}$, $\{e_{12}, e_{21}\}$, $\{e_{12}, e_{22}, e_{21}\}$, $\{e_{11}, e_{12}, e_{21}\}$, $\{e_{11}, e_{12}, e_{22}, e_{21}\}$.

If $t \in W_1^{(i)}$, then $\{\psi\pi(b_{it1}), \dots, \psi\pi(b_{itn_{it}})\}$ coincides with one of the following sets: $\{e_{21}\}$, $\{e_{22}, e_{21}\}$, $\{e_{21}, e_{11}\}$, $\{e_{22}, e_{21}, e_{11}\}$.

By Lemma 10.3 and the remarks above, the image of the value of f under the substitution $x_{itj} = b_{itj}$, $1 \leqslant i \leqslant \lambda_1$, $t \in T_0 \sqcup T_1$, $1 \leqslant j \leqslant n_{it}$, does not vanish under the homomorphism $\psi\pi$ since if the alternation replaces x_{i_1,t_1,j_1} with x_{i_2,t_2,j_2} for $t_1 \neq t_2$, then the value of $x_{i_2,t_2,j_2}^{h_{t_1}}$ is zero and the corresponding item vanishes. For our convenience, we rename the variables of f to be $x_1, \dots, x_n$. Then f satisfies all the conditions of the lemma. ∎

Now we are ready to calculate $\mathrm{PIexp}^{T\text{-gr}}(A)$ in the case when (22) does not hold.

Theorem 11.5: *Let A be a finite dimensional T-graded-simple algebra over a field F of characteristic 0 for a right zero band T. Suppose $A/J(A) \cong M_2(F)$. Let $T_0, T_1 \subseteq T$ and $\sim$ be, respectively, the subsets and the equivalence relation defined at the beginning of Section 10. Suppose also that $|\bar{t}_0| > \frac{|T_0|}{2}$ for some $\bar{t}_0 \in T_0/\sim$. Then*

$$\begin{aligned}\mathrm{PIexp}^{T\text{-gr}}(A) =& |T_0| + 2|T_1| + 2\sqrt{(|T_1| + |\bar{t}_0|)(|T_0| + |T_1| - |\bar{t}_0|)} \\ <& 2|T_0| + 4|T_1| = \dim A.\end{aligned}$$

Proof. Recall that at the beginning of Section 11 we chose the basis $\mathcal{B}$ in A. Equation (23) implies that

$$0 < \zeta = \sqrt{\frac{|T_0| + |T_1| - |\bar{t}_0|}{|T_1| + |\bar{t}_0|}} < 1$$

is the root of (15).

Let

$$\Omega = \left\{ (\alpha_1, \dots, \alpha_r) \in \mathbb{R}^r \mid \sum_{i=1}^r \alpha_i = 1,\, \alpha_1 \geqslant \alpha_2 \geqslant \dots \geqslant \alpha_r \geqslant 0, \sum_{i=1}^r \gamma_i \alpha_i \leqslant 0 \right\}.$$

By Lemma 9.7,

$$(27)\qquad \begin{aligned} d := \max_{x\in\Omega} \Phi(x) = & \sum_{i=1}^{r} \zeta^{\gamma_i} \\ = & |T_0| + 2|T_1| + 2\sqrt{(|T_1| + |\bar{t}_0|)(|T_0| + |T_1| - |\bar{t}_0|)}. \end{aligned}$$

Denote by $(\alpha_1, \dots, \alpha_r) \in \Omega$ such a point that $\Phi(\alpha_1, \dots, \alpha_r) = d$.

For every $n \in \mathbb{N}$ define $\mu \vdash n$ by

$$\mu_i = 2[\frac{\alpha_i n}{2}]$$

for $2 \leqslant i \leqslant r$ and

$$\mu_1 = n - \sum_{i=2}^{r} \mu_i.$$

By (19), $\sum_{i=1}^{r} \gamma_i \alpha_i = 0$. Since

$$\gamma_1 = \dots = \gamma_{|T_0|+|T_1|-|\bar{t}_0|} = -1,$$
$$\gamma_{|T_0|+|T_1|-|\bar{t}_0|+1} = \dots = \gamma_{2|T_0|+3|T_1|-|\bar{t}_0|} = 0,$$

and

$$\gamma_{2|T_0|+3|T_1|-|\bar{t}_0|+1} = \dots = \gamma_r = 1,$$

By (20),

$$\alpha_1 = \dots = \alpha_{|T_0|+|T_1|-|\bar{t}_0|},$$
$$\alpha_{|T_0|+|T_1|-|\bar{t}_0|+1} = \dots = \alpha_{2|T_0|+3|T_1|-|\bar{t}_0|},$$
$$\alpha_{2|T_0|+3|T_1|-|\bar{t}_0|+1} = \dots = \alpha_r,$$

and we have

$$\begin{aligned} \alpha_1 n - 2 \leqslant & \mu_2 = \dots = \mu_{|T_0|+|T_1|-|\bar{t}_0|} \leqslant \alpha_1 n, \\ \alpha_{|T_0|+|T_1|-|\bar{t}_0|+1} n - 2 \leqslant & \mu_{|T_0|+|T_1|-|\bar{t}_0|+1} = \dots = \mu_{2|T_0|+3|T_1|-|\bar{t}_0|} \\ \leqslant & \alpha_{|T_0|+|T_1|-|\bar{t}_0|+1} n, \\ \alpha_r n - 2 \leqslant & \mu_{2|T_0|+3|T_1|-|\bar{t}_0|+1} = \dots = \mu_r \leqslant \alpha_r n. \end{aligned}$$

Now $\sum_{i=1}^{r} \alpha_i = 1$ implies

$$\alpha_1 n \leqslant \mu_1 \leqslant \alpha_1 n + 2r.$$

Note that

$$\begin{aligned}
\sum_{i=1}^{r}\gamma_i\mu_i &= -\left(n-\sum_{i=2}^{r}\mu_i\right)-\sum_{i=2}^{|T_0|+|T_1|-|\bar{t}_0|}\mu_i+\sum_{i=2|T_0|+3|T_1|-|\bar{t}_0|+1}^{r}\mu_i\\
&=\left(\sum_{i=|T_0|+|T_1|-|\bar{t}_0|+1}^{2|T_0|+3|T_1|-|\bar{t}_0|}\mu_i\right)+2\left(\sum_{i=2|T_0|+3|T_1|-|\bar{t}_0|+1}^{r}\mu_i\right)-n\\
&\leqslant n\left(\sum_{i=|T_0|+|T_1|-|\bar{t}_0|+1}^{2|T_0|+3|T_1|-|\bar{t}_0|}\alpha_i\right)+2n\left(\sum_{i=2|T_0|+3|T_1|-|\bar{t}_0|+1}^{r}\alpha_i\right)-n\sum_{i=1}^{r}\alpha_i\\
&=n\sum_{i=1}^{r}\gamma_i\alpha_i=0.
\end{aligned}\tag{28}$$

Analogously,

$$\begin{aligned}
\sum_{i=1}^{r}\gamma_i\mu_i &=\left(\sum_{i=|T_0|+|T_1|-|\bar{t}_0|+1}^{2|T_0|+3|T_1|-|\bar{t}_0|}\mu_i\right)+2\left(\sum_{i=2|T_0|+3|T_1|-|\bar{t}_0|+1}^{r}\mu_i\right)-n\\
&\geqslant n\left(\sum_{i=|T_0|+|T_1|-|\bar{t}_0|+1}^{2|T_0|+3|T_1|-|\bar{t}_0|}\alpha_i\right)+2n\left(\sum_{i=2|T_0|+3|T_1|-|\bar{t}_0|+1}^{r}\alpha_i\right)-n\sum_{i=1}^{r}\alpha_i-4r\\
&=n\sum_{i=1}^{r}\gamma_i\alpha_i-4r=-4r.
\end{aligned}\tag{29}$$

By (28) and Lemma 11.4, $b_{T_\mu}f\notin \mathrm{Id}^{(FT)^*}(A)$ for some $f\in P_n^{(FT)^*}$. Let $\bar f$ be the image of f in $\frac{P_n^{(FT)^*}}{P_n^{(FT)^*}\cap\,\mathrm{Id}^{(FT)^*}(A)}$. Consider $FS_nb_{T_\mu}\bar f\subseteq\frac{P_n^{(FT)^*}}{P_n^{(FT)^*}\cap\,\mathrm{Id}^{(FT)^*}(A)}$. Since all S_n-representations over fields of characteristic 0 are completely reducible, we have

$$FS_nb_{T_\mu}\bar f\cong FS_ne_{T_{\lambda^{(1)}}}\oplus\cdots\oplus FS_ne_{T_{\lambda^{(s)}}}$$

for some $\lambda^{(i)}\vdash n$ and some Young tableaux $T_{\lambda^{(i)}}$ of shape $\lambda^{(i)}$, $1\leqslant i\leqslant s$, $s\in\mathbb{N}$. In particular, $e^*_{T_{\lambda^{(1)}}}FS_nb_{T_\mu}\neq 0$.

Now we notice that $e^*_{T_{\lambda^{(1)}}}FS_nb_{T_\mu}\neq 0$ implies $a_{T_{\lambda^{(1)}}}\sigma b_{T_\mu}=\sigma a_{\sigma^{-1}T_{\lambda^{(1)}}}b_{T_\mu}\neq 0$ for some $\sigma\in S_n$. Since $a_{\sigma^{-1}T_{\lambda^{(1)}}}$ is the operator of symmetrization in the numbers from the rows of the Young tableau $\sigma^{-1}T_{\lambda^{(1)}}$ and b_{T_μ} is the operator of alternation in the numbers from the columns of the Young tableau T_μ, all numbers from the first row of $\sigma^{-1}T_{\lambda^{(1)}}$ must be in different columns of T_μ. Thus $(\lambda^{(1)})_1\leqslant\mu_1$. Moreover, all numbers from each of the first μ_r columns of T_μ

must be in different rows of $\sigma^{-1}T_{\lambda^{(1)}}$. Since, by Lemma 9.2, $(\lambda^{(1)})_{r+1}=0$, we have $(\lambda^{(1)})_r \geqslant \mu_r$.

Thus (29) implies

$$\begin{aligned}\sum_{i=1}^{r}\gamma_i(\lambda^{(1)})_i \geqslant&\sum_{i=1}^{r}\gamma_i(\lambda^{(1)})_i-\sum_{i=1}^{r}\gamma_i\mu_i-4r\\ =&\sum_{i=1}^{|T_0|+|T_1|-|\bar t_0|}(\mu_i-(\lambda^{(1)})_i)+\sum_{i=2|T_0|+3|T_1|-|\bar t_0|+1}^{r}((\lambda^{(1)})_i-\mu_i)-4r\\ \geqslant&\sum_{i=1}^{|T_0|+|T_1|-|\bar t_0|}(\mu_1-(\lambda^{(1)})_i)+\sum_{i=2|T_0|+3|T_1|-|\bar t_0|+1}^{r}((\lambda^{(1)})_i-\mu_r)-6r-2r^2.\end{aligned}$$

Since by Lemma 9.2 we have $\sum_{i=1}^{r}\gamma_i(\lambda^{(1)})_i\leqslant 1$ and both $(\mu_1-(\lambda^{(1)})_i)$ and $((\lambda^{(1)})_i-\mu_r)$ are nonnegative, we get $\mu_1-(2r^2+6r+1)\leqslant(\lambda^{(1)})_i\leqslant\mu_1$ for all $1\leqslant i\leqslant |T_0|+|T_1|-|\bar t_0|$ and $\mu_r\leqslant(\lambda^{(1)})_i\leqslant\mu_r+(2r^2+6r+1)$ for all $2|T_0|+3|T_1|-|\bar t_0|+1\leqslant i\leqslant r$.

Recall that $a_{\sigma^{-1}T_{\lambda^{(1)}}}b_{T_\mu}\neq 0$ implies that all numbers from each column of T_μ are in different rows of $\sigma^{-1}T_{\lambda^{(1)}}$. Applying this to the first $\mu_{2|T_0|+3|T_1|-|\bar t_0|}$ columns, we obtain that in the last $|\bar t_0|+|T_1|+1$ rows of $T_{\lambda^{(1)}}$ we have at least $\sum_{i=2|T_0|+3|T_1|-|\bar t_0|}^{r}\mu_i$ boxes and

$$\sum_{i=2|T_0|+3|T_1|-|\bar t_0|}^{r}(\lambda^{(1)})_i\geqslant\sum_{i=2|T_0|+3|T_1|-|\bar t_0|}^{r}\mu_i=(|\bar t_0|+|T_1|)\mu_r+\mu_{2|T_0|+3|T_1|-|\bar t_0|}.$$

Thus

$$\begin{aligned}\lambda^{(1)}_{2|T_0|+3|T_1|-|\bar t_0|}\geqslant&(|\bar t_0|+|T_1|)\mu_r+\mu_{2|T_0|+3|T_1|-|\bar t_0|}-\sum_{i=2|T_0|+3|T_1|-|\bar t_0|+1}^{r}(\lambda^{(1)})_i\\ \geqslant&\mu_{2|T_0|+3|T_1|-|\bar t_0|}-(2r^3+6r^2+r)\end{aligned}$$

Let $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_r)$ where

$$\lambda_i=\begin{cases}\mu_1-(2r^2+6r+1) & \text{for } 1\leqslant i\leqslant |T_0|+|T_1|-|\bar t_0|,\\ \mu_{2|T_0|+3|T_1|-|\bar t_0|}-(2r^3+6r^2+r) & \text{for } |T_0|+|T_1|-|\bar t_0|+1\\ & \qquad\leqslant i\leqslant 2|T_0|+3|T_1|-|\bar t_0|,\\ \mu_r & \text{for } 2|T_0|+3|T_1|-|\bar t_0|+1\leqslant i\leqslant r.\end{cases}$$

Let $n_1=\sum_{i=1}^{r}\lambda_i$. Note that $n-(2r^4+6r^3+r^2)\leqslant n_1\leqslant n$.

For every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for every $n \geqslant n_0$ we have $\lambda_1 \geqslant \dots \geqslant \lambda_r$ and $\Phi(\frac{\lambda_1}{n_1}, \dots, \frac{\lambda_r}{n_1}) > d - \varepsilon$. Since D_λ is a subdiagram of $D_{\lambda^{(1)}}$, we have $c_n^{(FT)^*}(A) \geqslant \dim M_\lambda$ and by the hook and the Stirling formulas, there exist $C_1 > 0$ and $r_1 \in \mathbb{R}$ such that we have

$$\begin{aligned} c_n^{(FT)^*}(A) \geqslant \dim M(\lambda) = \frac{n_1!}{\prod_{i,j} h_{ij}} &\geqslant \frac{n_1!}{(\lambda_1 + r - 1)! \cdots (\lambda_r + r - 1)!} \\ &\geqslant \frac{n_1!}{n_1^{r(r-1)} \lambda_1! \cdots \lambda_r!} \geqslant \frac{C_1 n_1^{r_1} (\frac{n_1}{e})^{n_1}}{(\frac{\lambda_1}{e})^{\lambda_1} \cdots (\frac{\lambda_r}{e})^{\lambda_r}} \\ &\geqslant C_1 n_1^{r_1} \Bigg(\frac{1}{(\frac{\lambda_1}{n_1})^{\frac{\lambda_1}{n_1}} \cdots (\frac{\lambda_r}{n_1})^{\frac{\lambda_r}{n_1}}} \Bigg)^{n_1} \\ &\geqslant C_1 n_1^{r_1} (d - \varepsilon)^{n - (2r^4 + 6r^3 + r^2)}. \end{aligned} \tag{30}$$

Hence $\underline{\lim}_{n\to\infty} \sqrt[n]{c_n^{(FT)^*}(A)} \geqslant d - \varepsilon$. Since $\varepsilon > 0$ is arbitrary,

$$\underline{\lim}_{n\to\infty} \sqrt[n]{c_n^{(FT)^*}(A)} \geqslant d.$$

By Lemma 7.3 and (27),

$$\underline{\lim}_{n\to\infty} \sqrt[n]{c_n^{T\text{-gr}}(A)} \geqslant |T_0| + 2|T_1| + 2\sqrt{(|T_1| + |\bar{t}_0|)(|T_0| + |T_1| - |\bar{t}_0|)}.$$

Theorem 9.8 and (27) imply

$$\overline{\lim}_{n\to\infty} \sqrt[n]{c_n^{T\text{-gr}}(A)} \leqslant |T_0| + 2|T_1| + 2\sqrt{(|T_1| + |\bar{t}_0|)(|T_0| + |T_1| - |\bar{t}_0|)}.$$

The condition $|\bar{t}_0| > \frac{|T_0|}{2}$ implies $|T_1| + |\bar{t}_0| > |T_0| + |T_1| - |\bar{t}_0|$,

$$2\sqrt{(|T_1| + |\bar{t}_0|)(|T_0| + |T_1| - |\bar{t}_0|)} < (|T_1| + |\bar{t}_0|) + (|T_0| + |T_1| - |\bar{t}_0|)$$

and

$$\begin{aligned} \lim_{n\to\infty} \sqrt[n]{c_n^{T\text{-gr}}(A)} =& |T_0| + 2|T_1| + 2\sqrt{(|T_1| + |\bar{t}_0|)(|T_0| + |T_1| - |\bar{t}_0|)} \\ <& 2|T_0| + 4|T_1| = \dim A. \qquad \blacksquare \end{aligned}$$

Remark 11.6: A finite dimensional T-graded-simple algebra A with any given $|T_0|$, $|T_1|$, $\frac{|T_0|}{2} < |\bar{t}_0| \leqslant |T_0|$ exists by Proposition 5.2.

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