Predicting State Transitions in Autonomous Nonlinear Bistable Systems With Hidden Stochasticity

Léopold Van Brandt and Jean-Charles Delvenne

***Abstract*—Bistable autonomous systems can be found in many areas of science. When the intrinsic noise intensity is large, these systems exhibits stochastic transitions from one metastable steady state to another. In electronic bistable memories, these transitions are failures, usually simulated in a Monte-Carlo fashion at a high CPU-time price. Existing closed-form formulas, relying on near-stable-steady-state approximations of the nonlinear system dynamics to estimate the mean transition time, have turned out inaccurate. Our contribution is twofold. From a unidimensional stochastic model of overdamped autonomous systems, we propose an extended Eyring-Kramers analytical formula accounting for both nonlinear drift and state-dependent white noise variance, rigorously derived from Itô stochastic calculus. We also adapt it to practical system engineering situations where the intrinsic noise sources are hidden and can only be inferred from the fluctuations of observables measured in steady states. First numerical trials on an industrial electronic case study suggest that our approximate prediction formula achieve remarkable accuracy, outperforming previous non-Monte-Carlo approaches.**

***Index Terms*—Stochastic systems, stability of nonlinear systems, autonomous systems, bistability, metastable states, first-passage time.**

I. Introduction

BISTABLE autonomous systems are ubiquitous in many scientific fields such as biology [2], [3], climatology [4], [5], chemical reaction systems [6], and electronics. Our chief example in this letter is Static Random Access Memory (SRAM) bitcells (Fig. 1a), embedded in manufactured chips for temporary information storage [1], [7], [8]. Generally, bistable systems admit, by definition, two stable steady states, sometimes referred to as "metastable" points [6], [9], [10] insofar as *stochastic transitions* from one state to the other are observed due to random noise intrinsic to the dynamics [10], [11] (Fig. 1b).

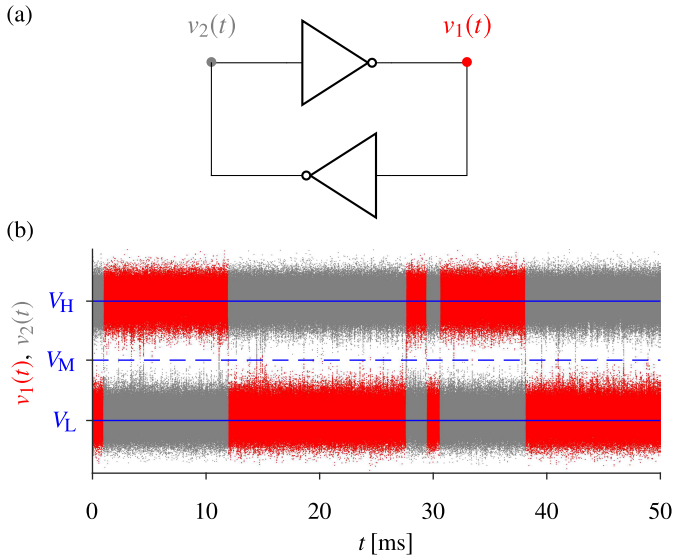


Fig. 1. (a) SRAM bitcell retaining a logical 0 or 1 encoded on two complementary node voltages (v_2 and v_1) as low and high levels V_L and V_H. The retained state is stabilised by a feedback loop implemented by two cross-coupled inverters. An inverter is a nonlinear time-invariant system producing a high V_H (resp. low V_L) output when its input is maintained at constant low V_L (resp. high V_H), yet with internal dynamics and intrinsic noise. (b) Transient noise simulation at supply voltage $V_{DD} = 70$mV (adapted from [1]). Intrinsic noise-induced stochastic state transitions (bit flips $V_L \leftrightarrow V_H$) are observed. V_M denotes the threshold voltage corresponding to the unstable state. For the illustrated case, the bistable system is symmetrical in the sense that two inverters are identical, making the two steady states equiprobable and the transitions $V_L \leftrightarrow V_H$ rates equal.

The problem addressed in this letter is to find out the mean time to switch from one stable state to another, driven by random noise. This time (defined rigorously later) is the mean transition time (*MTT*). Characterising the statistics of the transition time, notably the *MTT*, is indeed crucial for many encountered bistable systems, especially SRAM bitcells in which the state transitions mean loss of the originally retained bit and hence are failures [1], [7], [8], [11]. The naive approach is to empirically estimate the *MTT* from

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dynamic Monte-Carlo simulation, whose major weakness is its computational intensity (since statistical relevance requires repeating or lengthening transient experiment like Fig. 1b). As reviewed in Section II, several alternative approaches, all faster but with different accuracy/insight trade-offs, have been proposed in electronics in response to the practical need for efficient prediction of SRAM bitcell reliability. At the same time, the calculation of the *MTT* is a classical problem in chemical physics [9], [12], mathematically formalised as a *first-passage-time* (FPT) problem [13], [14]. The celebrated *Eyring–Kramers law* was first derived from unidimensional stochastic calculus in [15] and generalised only in the early 2000s (see for instance [9], [16]). In the present work, we aim at explaining how it can be suitably extended and applied to accurately predict the *MTT* in SRAM bitcells, from cheap deterministic (non-Monte-Carlo) circuit simulations. The framework is the stochastic nonlinear model previously developed in [8].

Importantly, we emphasise that, while working under the convenient white Gaussian noise assumption, the variance of the noise inherent to the dynamics generally depends on the state of the system. It cannot therefore be merely considered as constant for accurate *MTT* predictions. Furthermore, this variance may not be explicitly known in an analytical form, or even measurable in non steady states for many physical systems, when the different intrinsic noise sources of various origins are partially unknown or hidden. A simple example is Johnson-Nyquist noise due to the random thermal agitation of the charge carriers in dissipative electronic devices [17], [18]. The celebrated fluctuation–dissipation theorem [19] applies only for a linear resistor, but not for arbitrary nonlinear devices operating out-of-equilibrium [20]. More generally, models describing the variations of the noise statistics, notably the variance, with the state are not straightforwardly available for complex nonlinear systems operating far-from equilibrium. They are sometimes hidden in the industrial model of the system components (e.g., transistors within the inverter blocks of the SRAM bitcell in Fig. 1a). This is why in this letter, we also adapt our predictive developments to the generic situation where the noise intensity can only be extracted, from fluctuations of observables, at steady states of the system.

The rest of this letter is structured as follows. Section II further describes the SRAM bitcell, exemplifying bistable systems. We provide a reminder of the equations governing its nonlinear dynamics (Section II-A), present a general unidimensional stochastic nonlinear model of bistable systems (Section II-B), and review previous attempts of numerical or analytical predictions of the *MTT*, relying on near-steady-state approximations (Section II-C). Notably, Section II-D reintroduces Kramers formula following [9], which still assumes constant noise intensity. Our main result is derived and detailed in Section III: from Itô stochastic calculus (Section III-A), we address the more general case of state-dependent noise intensity and provide an extended explicit formula to compute the *MTT* through deterministic numerical integration (Section III-B). We also propose an approximate extended Eyring-Kramers formula involving only the noise variance extracted at the stable and unstable steady states (Section III-C). Section IV compares and discusses the accuracy of different closed-form formulas applied to SRAM bitcells, under the practical constraint of partially observable stochasticity. Section V finally draws the conclusion and opens perspectives.

II. Previous Works

A. SRAM Bitcells Exemplifying Multidimensional Autonomous Nonlinear Bistable Systems

In retention mode, an SRAM bitcell is undriven and isolated from peripheral circuitry in order to retain a bit of data. It can be assimilated to a cross-coupled inverter pair (Fig. 1a), a bistable electronic circuit implemented with four transistors.

The conventional state-space representation involves two node voltages $\boldsymbol{v} = (v_1, v_2)$ as states [21]. The *stochastic state-space* representation of the SRAM bitcell of Fig. 1a can be written as an Itô's *stochastic differential equation* (SDE):

$$\boldsymbol{C}(\boldsymbol{v})\ \mathrm{d}\boldsymbol{v} = \boldsymbol{I}(\boldsymbol{v})\ \mathrm{d}t + \boldsymbol{s}(\boldsymbol{v})\ \mathrm{d}\boldsymbol{W}(t). \tag{1}$$

In (1), the vector $\boldsymbol{I}(\boldsymbol{v}) = (I_1(v_1, v_2), I_2(v_1, v_2))$ denotes the mean currents flowing out of the inverters, a nonlinear function of both state variables v_1 and v_2. The matrix $\boldsymbol{C}(\boldsymbol{v})$ has the dimensions of a capacitance and relates the voltage variations to the currents. The vector $\boldsymbol{W}(t) = (W_1(t), W_2(t))$ contains two standard Wiener processes, i.e., the integrals of independent white Gaussian noise processes, and is multiplied by the square matrix $\boldsymbol{s}(\boldsymbol{v})$, here modelling the thermal noise of the transistors.

Because matrix $\boldsymbol{C}$ is known to be invertible (as diagonally dominant) from physical considerations, we may transform (1) into

$$\begin{aligned}\mathrm{d}\boldsymbol{v} &= \boldsymbol{C}(\boldsymbol{v})^{-1}\boldsymbol{I}(\boldsymbol{v})\ \mathrm{d}t + \boldsymbol{C}(\boldsymbol{v})^{-1}\boldsymbol{s}(\boldsymbol{v})\ \mathrm{d}\boldsymbol{W}(t)\\ &\equiv \boldsymbol{h}(\boldsymbol{v})\ \mathrm{d}t + \boldsymbol{\sigma}(\boldsymbol{v})\ \mathrm{d}\boldsymbol{W}(t).\end{aligned} \tag{2}$$

to retrieve the canonical form of state-space representation of nonlinear dynamical systems, if needed.

The bistable system has two stable steady states ($(V_\mathrm{L}, V_\mathrm{H})$ and $(V_\mathrm{H}, V_\mathrm{L})$ with voltage levels shown in Fig. 1b) and one unstable state ($v_1 = v_2 = V_\mathrm{M}$) [3], [5], [6], [7], [8]. For the special case illustrated in Fig. 1, the two inverters are identical, hence the roles of v_1 and v_2 are interchangeable in (1) (e.g., $I_1(v_1, v_2) = I_2(v_2, v_1)$). Consequently, the two steady states are equiprobable, and the transitions rates are equal. In that situation, the bistable system is said symmetrical and only one single *MTT* must be predicted. Nevertheless, all the methodology that follows is intended to be perfectly general, also covering the asymmetrical case with two different transition rates.

Although the *MTT* can be estimated from Monte-Carlo simulation on a digital twin (Fig. 1(b), performed on an industrial circuit simulator), previous works [7], [8] showed that this approach is prohibitively expensive. Relying on the formulation (1) yet with constant capacitances $\boldsymbol{C}$, reference [11] developed an accelerated noise simulator, based on an incremental algorithm with successive thresholds, for fast SRAM reliability assessment. Its difficult extension to other electronic memory technologies, even more to other bistable

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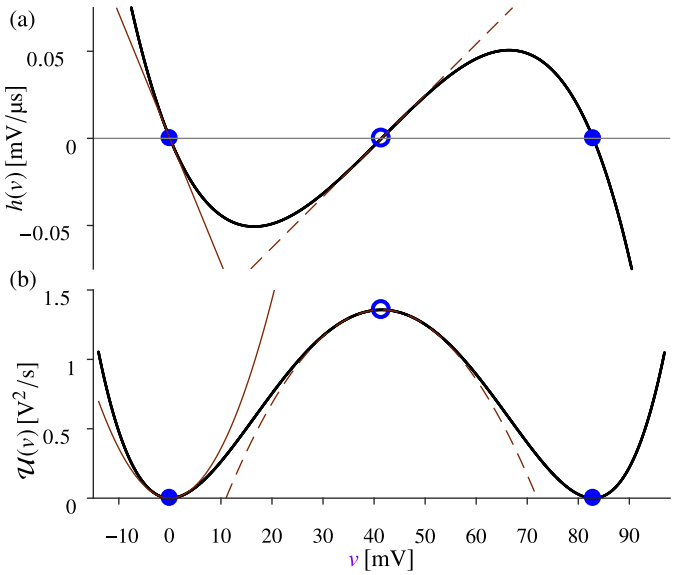


Fig. 2. (a) Deterministic drift term in (3), deriving from (b) quasi potential $\mathcal{U}(v) = -\int_0^v h(v')\,\mathrm{d}v'$. Near-stable-steady-state (full blue dots) approximations inherent to [14], [22]'s formulas are shown in full brown lines. Eyring-Kramers formula also exploits the behaviour near the unstable point (open blue dot and dashed brown curve). Illustrated case: same as Fig. 1(b) (SRAM bitcell at $V_{DD} = 70$ mV). Extracted infinitesimal variance at steady states: $\sigma_0 = 258\ \mathrm{V}^2/\mathrm{s}$ (stable) and $\sigma_M = 105\ \mathrm{V}^2/\mathrm{s}$ (unstable).

systems of different fields is however a major limitation calling for more general non-Monte-Carlo approaches.

B. Unidimensional Stochastic Nonlinear Model

Upon observation of transient experiences like Fig. 1(a) and of time-scale separation, we have proposed to reduce the dimensionality of system (2) by introducing an unidimensional (1D) reaction coordinate [7], [8], (e.g., $v = (v_2 - v_1)/\sqrt{2}$ is adequate for the case illustrated in Fig. 1) The vocable reaction coordinate comes from chemistry [23]. This is equivalent to studying the nonlinear system dynamics in the main direction dominating the transition time, described by the unidimensional Itô's SDE

$$\mathrm{d}v = h(v)\ \mathrm{d}t + \sigma(v)\ \mathrm{d}W(t). \tag{3}$$

Equation (3) models an overdamped single-state system with *drift-diffusion* dynamics, like a Brownian particle [9], [16].

The nonlinear drift function $h(v)$ (Fig. 2(a)) is understood as describing the natural dynamics of the system in absence of noise, here cheaply extracted from a deterministic dynamic simulation [8]. It derives from a scalar *quasi potential* $\mathcal{U}(v)$ [9], [16] (Fig. 2(b)): $h(v) \equiv -\,\mathrm{d}\mathcal{U}/\,\mathrm{d}v$. The stable steady states ($v = 0$ and $v = 2\Delta v$) correspond to the valleys of $\mathcal{U}(v)$, whereas the hill characterises the unstable point ($v = \Delta v$, labelled M).

In (3), the diffusion is the white Gaussian noise $\sigma(v)W(t)$: $\sigma^2(v)$ is the infinitesimal variance, or variance per unit time (i.e., in V^2/s) and depends on the state v for most nonlinear systems; $W(t)$ is the Wiener process (in $\mathrm{s}^{1/2}$). All previous works reviewed below coarsely assumed that $\sigma^2(v)$ is *constant*, i.e., that it does *not* vary with v throughout the state transition and equals to $\sigma_0^2 = \sigma^2(0)$, its stable steady-state value at $v = 0$ [8].

C. Near-Stable-Steady-State Approximate Formulas

Former closed-form analytical formulas for the *MTT* [14], [22] further rely on linearisation of the system dynamics ($h(v)$) and are thereby highly inaccurate, with discrepancies of several orders of magnitude reported for the *MTT* in SRAM bitcells [7], [8].

Siegert [13] derived the first passage time (FPT) probability of a stationary 1D Markov process and provided a recurrence formula for the moments of the FPT when the probability distribution satisfies a Fokker-Plank equation. Nobile et al. [14] applied [13]'s formula to the special case of an Ornstein-Uhlenbeck process, defined by (3) with linearised drift term $h(v) \approx -v(t)/\tau_0$ (full brown line in Fig. 2(b)), with

$$\frac{1}{\tau_0} = -\frac{\mathrm{d}h}{\mathrm{d}v}(0) = \frac{\mathrm{d}^2\mathcal{U}}{\mathrm{d}v^2}(0) \tag{4}$$

the time constant of the system linearised at $v = 0$. The *MTT* is twice the mean FPT [7], i.e., the mean time to reach $v = \Delta v$ for the first time starting from $v = 0$:

$$MTT \approx 2\,\tau_0 \left(\sqrt{\pi} \int_0^{\Delta v/(\sigma_0\sqrt{\tau_0})} \mathrm{d}u\ \exp\left(u^2\right) + \int_0^{\Delta v/(\sigma_0\sqrt{\tau_0})} \mathrm{d}u\ \exp\left(u^2\right) \mathrm{erf}(u) \right). \tag{5}$$

Kish's formula [22],

$$MTT \approx \sqrt{3}\,\pi\ \tau_0\ \exp\left(\frac{\Delta v^2/(2\tau_0)}{\frac{1}{2}\sigma_0^2} \right), \tag{6}$$

is sometimes used in electronics [24], and can be regarded as a simplified version of (5). Within the exponential, $\Delta v^2/(2\tau_0)$ is interpreted as the quasi-potential barrier evaluated at the unstable point ($v = \Delta v$) under the coarse near-stable-steady-state harmonic (parabolic) approximation (Fig. 2(b)), equivalent to linearised drift. The barrier or the drift is strongly overestimated, and so is the *MTT*. Both (5) and (6), relying on a near-stable-steady-state approximation, are inappropriate for describing the true nonlinear SRAM dynamics as previously concluded in [7], [8] and here observed in Fig. 2.

D. Classical Eyring-Kramers Formula: Constant σ^2

We have highlighted that the near-stable-steady-state approximation inherent to [14], [22] is twofold: the deterministic drift is linearised around $v = 0$ only; the noise is assumed constant and also extracted at $v = 0$ only. Dynamic simulations invalidate these coarse simplifications, which Eyring-Kramers formula [15] partially alleviates by also taking into account the drift (quasi-potential) behaviour of the system near the *unstable* point (dashed brown lines in Fig. 2) However, it still assumes constant noise $\sigma(v) \equiv \sigma$.

In our notations, Eyring-Kramers formula is written as [9]

$$\begin{aligned} MTT &\approx 2\pi\sqrt{\tau_0\,\tau_M}\ \exp\left(\frac{\mathcal{U}(\Delta v) - \mathcal{U}(0)}{\frac{1}{2}\sigma^2} \right) \\ &= 2\pi\sqrt{\tau_0\,\tau_M}\ \exp\left(-\frac{\int_0^{\Delta v} h(v')\ \mathrm{d}v'}{\frac{1}{2}\sigma^2} \right). \end{aligned} \tag{7}$$

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We explain how the new quantities introduced in (7) are defined and extracted.

$\mathcal{U}(\Delta v) - \mathcal{U}(0)$ is the *true* quasi-potential barrier (Fig. 2(b)) whose crossing by the random diffusion process leads to a state transition. Setting $\mathcal{U}(0) = 0$ as arbitrary potential reference, $\mathcal{U}(\Delta v)$ is obtained by numerically integrating $-h(v)$.

$$\frac{1}{\tau_{\mathrm{M}}} = \frac{\mathrm{d}h}{\mathrm{d}v}(\Delta v) = \left|\frac{\mathrm{d}^2\mathcal{U}}{\mathrm{d}v^2}(\Delta v)\right| \tag{8}$$

is the time constant of the system linearised at the unstable point ($v = \Delta v$), straightforwardly computed from either $h(v)$ or $\mathcal{U}(v)$. Hence, a new effective time constant of the diffusion process, defined as a geometrical mean $\tau \equiv \sqrt{\tau_0 \tau_{\mathrm{M}}}$, appears in (7) and accounts for the drift behaviour in both the near stable- and unstable-steady-state regions.

The proof of (7) is based on rigorous stochastic calculations and can be found for instance in [9], [16].

Unfortunately, the classical formula (7) still assumes constant noise variance σ^2. Using $\sigma = \sigma_0$, like in (5), and (6), would amount to estimating the fluctuations near the stable steady state only. *A priori*, there is no physical or theoretical argument to justify favouring the unstable steady state by choosing $\sigma = \sigma_{\mathrm{M}}$ instead. These two exclusive choices will be shown to be inaccurate, when attempting to apply (7) to nonlinear stochastic bistable systems such as SRAM bitcells with varying substantially $\sigma(v)$ all along the state transition from $v = 0$ to $v = \Delta v$ (extracted numerical values are provided in the caption of Fig. 2).

III. Stochastic Formulation for Varying Noise Intensity

A. Itô Drift-Diffusion Process

Itô's lemma (see [25, Th. 1.1] or [26, Th. 6]) may be regarded as the chain rule for stochastic calculus, non trivial for the case of varying $\sigma(v)$. His formula describes a new drift-diffusion process after performing a change of variable $f(v)$ (of our choice), on which we will here leverage to incorporate the varying $\sigma(v)$ within Kramers framework:

$$\mathrm{d}f = \left[h(v)\frac{\mathrm{d}f}{\mathrm{d}v} + \frac{\sigma^2(v)}{2}\frac{\mathrm{d}^2 f}{\mathrm{d}v^2}\right]\mathrm{d}t + \sigma(v)\frac{\mathrm{d}f}{\mathrm{d}v}\,\mathrm{d}W(t). \tag{9}$$

Next, we let

$$\sigma(v)\frac{\mathrm{d}f}{\mathrm{d}v} = 1 \tag{10}$$

as definition of $f(v)$ to obtain

$$\mathrm{d}f = \underbrace{\left[h(v)\frac{\mathrm{d}f}{\mathrm{d}v}(v) + \frac{\sigma^2(v)}{2}\frac{\mathrm{d}^2 f}{\mathrm{d}v^2}(v)\right]}_{\equiv \tilde{h}(f)}\mathrm{d}t + \mathrm{d}W(t). \tag{11}$$

B. Extended Eyring-Kramers Formula: $\sigma^2(v)$

Within the formulation (11) in variable f, the noise variance is now unity because the variations of noise intensity with v are absorbed in the drift term of increased complexity compared to (3), as consequence if Itô's calculus rule. 7 applies, however with the modified drift function $\tilde{h}(f)$ as appears in (11):

$$\begin{aligned} MTT &\approx 2\pi\sqrt{\tau_0\,\tau_{\mathrm{M}}}\,\exp\left(-2\int_{f(0)}^{f(\Delta v)}\tilde{h}(f')\;\mathrm{d}f'\right) \\ &= 2\pi\sqrt{\tau_0\,\tau_{\mathrm{M}}}\,\exp\left(-2\int_0^{\Delta v}\tilde{h}(f(v'))\,\frac{\mathrm{d}f}{\mathrm{d}v}(v')\;\mathrm{d}v'\right) \\ &= 2\pi\sqrt{\tau_0\,\tau_{\mathrm{M}}}\,\exp\left(-2\int_0^{\Delta v}\left[h(v')\frac{\mathrm{d}f}{\mathrm{d}v}(v') \right.\right. \\ &\qquad \left.\left. + \frac{\sigma^2(v')}{2}\frac{\mathrm{d}^2 f}{\mathrm{d}v^2}(v')\right]\right)\frac{\mathrm{d}f}{\mathrm{d}v}(v')\;\mathrm{d}v'. \end{aligned} \tag{12}$$

From (10), we easily compute

$$\frac{\mathrm{d}f}{\mathrm{d}v}(v) = \frac{1}{\sigma(v)} \quad \text{and} \quad \frac{\mathrm{d}^2 f}{\mathrm{d}v^2}(v) = -\frac{\frac{\mathrm{d}\sigma}{\mathrm{d}v}(v)}{\sigma^2(v)} \tag{13}$$

and (12) becomes, after splitting the two terms within the exponential

$$\begin{aligned} MTT[t] \;\approx 2\pi\sqrt{\tau_0\,\tau_{\mathrm{M}}}\,&\exp\left(-2\int_0^{\Delta v}\frac{h(v')}{\sigma^2(v')}\;\mathrm{d}v'\right) \\ &\cdot \exp\Bigg(\underbrace{-2\int_0^{\Delta v}\frac{1}{2}\frac{\frac{\mathrm{d}\sigma}{\mathrm{d}v}(v')}{\sigma(v')}\;\mathrm{d}v'}_{-\ln\left(\frac{\sigma(\Delta v)}{\sigma(0)}\right)}\Bigg). \end{aligned}$$

The general Eyring-Kramers formula, now accounting for both nonlinear drift and state-dependent white noise variance, finally reads:

$$MTT \approx 2\pi\sqrt{\tau_0\,\tau_{\mathrm{M}}}\,\frac{\sigma_0}{\sigma_{\mathrm{M}}}\,\exp\left(-2\int_0^{\Delta v}\frac{h(v')}{\sigma^2(v')}\;\mathrm{d}v'\right), \tag{14}$$

with $\sigma_0 \equiv \sigma(0)$ and $\sigma_{\mathrm{M}} \equiv \sigma(\Delta v)$.

C. Approximate Extended Eyring-Kramers Formula: Near-Steady-States σ_0 and σ_{M}

The noise can sometimes only be caracterised through the random time fluctuations of observable states of the system, and hence the variance extracted at the steady states only. That is, one cannot generally assume that the noise intensity $\sigma(v)$ as appears in (14) is available for all v, only that σ_0 and σ_{M} can be inferred from the steady-state fluctuations of v. The link between the intrinsic noise ($\sigma(v)w(t)$) and the state ($v(t)$) is, at equilibrium, given by the fluctuation–dissipation theorem [19].

To address this situation where the noise source is partially hidden, we also propose an approximate version of (14). Considering that only σ_0^2 and σ_{M}^2 are known, we assume a linear interpolation of $1/\sigma^2(v)$ between $v = 0$ and $v = \Delta v$:

$$\frac{1}{\sigma(v)^2} \approx \frac{1}{\sigma_0^2} + \left(\frac{1}{\sigma_{\mathrm{M}}^2} - \frac{1}{\sigma_0^2}\right)\frac{v}{\Delta v}. \tag{15}$$

This choice, as well as being natural and very convenient from an analytical perspective given the integral in (14), is physically sound since the intensity of physical noise sources evolves smoothly and gradually with the state. Should we have, beyond σ_0^2 and σ_{M}^2, some additional information about $1/\sigma^2(v)$, notably its convexity, a parabolic approximation

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could for instance be more accurate. Substituting (15) into (14) leads, after integration by parts (Appendix), to our main result:

$$MTT \approx 2\pi\sqrt{\tau_0\,\tau_M}\,\frac{\sigma_0}{\sigma_M}\,\exp\left(\frac{\mathcal{U}(\Delta v)}{\frac{1}{2}\sigma_M^2}\right)\exp\left(-2\left(\frac{1}{\sigma_M^2}-\frac{1}{\sigma_0^2}\right)\overline{\mathcal{U}}\right), \quad (16)$$

where

$$\overline{\mathcal{U}} \equiv \frac{1}{\Delta v}\int_0^{\Delta v}\mathcal{U}(v')\;\mathrm{d}v'. \quad (17)$$

We refer to (16) as the *approximate extended Eyring-Kramers formula* since it does incorporate the noise behaviour near both the stable and the unstable steady states, unlike the classical formula (7) restricted to a single constant σ.

We interpret (16). Beyond the time constant prefactor $2\pi/\sqrt{\tau_0\,\tau_M}$ as present in the classical Eyring-Kramers formula, three factors appear:

- σ_0/σ_M results from the additional drift term in Itô's formula. It is expected to have a negligible impact on the order of magnitude of the *MTT*, compared to the exponentials that follow.
- The factor

$$\exp\left(\frac{\mathcal{U}(\Delta v)}{\frac{1}{2}\sigma_M^2}\right)$$

  is nothing but Eyring-Kramers exponential factor, however with $\sigma = \sigma_M$ used instead of σ_0.
- The factor

$$\exp\left(-2\left(\frac{1}{\sigma_M^2}-\frac{1}{\sigma_0^2}\right)\overline{\mathcal{U}}\right)$$

  may be regarded as a factor correcting for the fact that the noise variance is not constant throughout the state transition. It is trivially unity when $\sigma(v) = \sigma_0 = \sigma_M$ = constant. If $\sigma_M < \sigma_0$, the argument within the exponential is *positive* and the factor is larger than 1, thereby compensating the excessive first exponential involving a too small σ. Conversely, if $\sigma_M > \sigma_0$, the correcting factor is smaller than 1.

As will be evidenced in Section IV, we harness this extended law (16) to predict the *MTT* in SRAM bitcells with much greater accuracy than the former analytical formulas reviewed in Section II. Whenever $\sigma(v)^2$ is explicitly available or characterisable for other stochastic bistable systems, the authors naturally encourage the use of the most general formula (14).

IV. Predictions and Discussion

In this section, we assess the respective accuracies of the different closed-form formulas for the *MTT* (Fig. 3) and point out the origins of the discrepancies. We consider three different SRAM bitcells, distinguished by different values of the supply voltage (V_{DD}) parameter whose dominant effect is to reduce the distance Δv between stable steady state and unstable state and hence to lower the *MTT*. References values for the *MTT* are provided by massive and accurate dynamic Monte-Carlo simulations from [1], based on a multidimensional model similar in mind to (1).

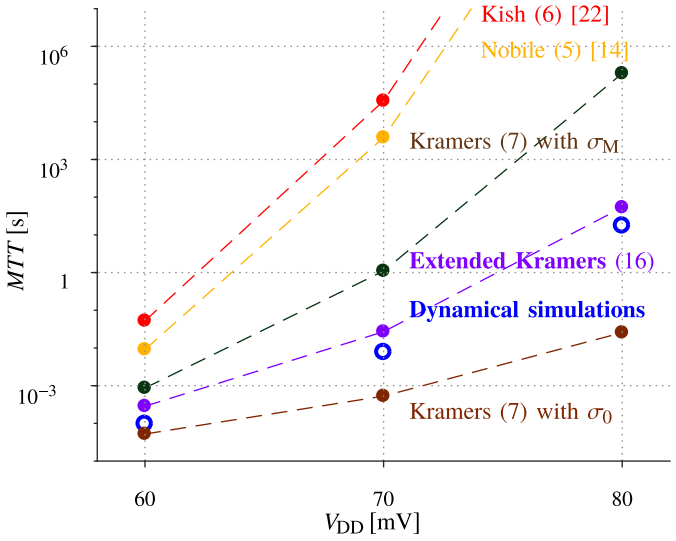


Fig. 3. Comparison between the *MTT* estimated empirically from expensive transient noise simulations [1], and from predictions of analytical near-equilibrium formulas and of the extended Eyring-Kramers formula.

As reviewed in Section II-C, [14], [22] assume both linearised drift and constant σ_0^2 related to the stable steady-state τ_0 only) and are totally inadequate to describe the SRAM dynamics with strongly nonlinear and non-monotonic drift function $h(v)$ (Fig. 2(a)) and intrinsic noise variance which depends on the time-evolving state v for nonlinear electronic devices. The discrepancy can reach orders of magnitude, as seen in Fig. 3 and already reported for another case study in [7], [8].

The classical Eyring-Kramers formula (Section II-D) is rewarded for its nonlinear treatment of the drift $h(v)$ by slightly improved accuracy. However, the noise is either under- or over-estimated, respectively when σ_M or σ_0 is used in (7), causing the *MTT* to be either over- or under-estimated compared to Monte-Carlo.

Our extended Eyring-Kramers formula was constructed in Section III-B to remedy these important shortcomings. It achieves, at least for the three investigated cases, a remarkable accuracy when compared to Monte-Carlo. Importantly, the trend *MTT* versus V_{DD} seems properly captured by our model, as a result of also taking into account the drift-diffusion behaviours near the unstable point. The residual discrepancy noticed in Fig. 3 is likely attributed to the linear interpolation of $1/\sigma^2(v)$, which could implicitly inflate or reduce the actual noise intensity depending on whether the true hidden $\sigma^2(v)$ is a concave or convex function. We may indeed observe that, for the case illustrated through this letter, (16) systematically overestimates the *MTT*, thereby suggesting that the true $1/\sigma^2(v)$ is some nonlinear function slightly overestimated by the linear interpolation.

V. Conclusion and Perspectives

Efficient and accurate prediction methodology of the mean transition time in bistable autonomous systems finds applications in various fields. Here, we have focused on SRAM bitcells as integrated in electronic chips for data retention.

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The stochastic nonlinear dynamical model may be thought as formally describing the overdamped motion of a Brownian subjected to drift (deriving from a potential) and diffusion (noise intrinsic to dissipative devices). Hence, apart from the notations, the extended Eyring-Kramers formulas for the *MTT* proposed in this letter, either the general (14) or the approximate (16), can be straightforwardly applied to a wide class of drift-diffusion problems as referenced in the introduction. Whereas (14) address the most general case of arbitrary non-constant white noise variance, (16) is valuable when the stochasticity of the system is partially hidden and can only be identified near the stable steady state and the unstable state. Our first numerical trials give faith in the efficiency and accuracy of this approach, which is still awaiting further validation across more case studies.

Appendix

Derivation of (16)

We expand (14) assuming that $1/\sigma^2(v)$ is linearly interpolated between $v = 0$ and $v = \Delta v$ according to (15):

$$MTT \approx 2\pi\sqrt{\tau_0\,\tau_{\mathrm{M}}}\,\frac{\sigma_0}{\sigma_{\mathrm{M}}}\,\exp\left(-2\left[\int_0^{\Delta v} h(v')\,\frac{1}{\sigma_0^2}\;\mathrm{d}v' + \int_0^{\Delta v} h(v')\left(\frac{1}{\sigma_{\mathrm{M}}^2}-\frac{1}{\sigma_0^2}\right)\frac{v}{\Delta v}\;\mathrm{d}v'\right]\right). \tag{18}$$

The first integral within the exponential is exactly the same as (7) with $\sigma = \sigma_0$ and $\mathcal{U}(0) \equiv 0$ by convention:

$$\exp\left(-2\int_0^{\Delta v} h(v')\,\frac{1}{\sigma_0^2}\;\mathrm{d}v'\right) = \exp\left(\frac{\mathcal{U}(\Delta v)}{\frac{1}{2}\sigma_0^2}\right) \tag{19}$$

We integrate the second term by parts, remembering that $h(v) = -\,\mathrm{d}\mathcal{U}/\,\mathrm{d}v$:

$$\begin{aligned}\int_0^{\Delta v} h(v')\,v\;\mathrm{d}v' &= \left[-\mathcal{U}(v)\,v\right]_0^{\Delta v} - \int_0^{\Delta v} -\mathcal{U}(v')\;\mathrm{d}v' \\ &= -\mathcal{U}(\Delta v)\Delta v + \int_0^{\Delta v}\mathcal{U}(v')\;\mathrm{d}v' \\ &= -\big(\mathcal{U}(\Delta v) - \overline{\mathcal{U}}\big)\Delta v\end{aligned} \tag{20}$$

where the average $\overline{\mathcal{U}}$ is defined in (17).

Inserting (19),(20) in (18) yields

$$\begin{aligned}MTT &\approx 2\pi\sqrt{\tau_0\,\tau_{\mathrm{M}}}\,\frac{\sigma_0}{\sigma_{\mathrm{M}}}\,\exp\left(\frac{\mathcal{U}(\Delta v)}{\frac{1}{2}\sigma_0^2}\right) \\ &\quad\cdot\exp\left(+2\left(\frac{1}{\sigma_{\mathrm{M}}^2}-\frac{1}{\sigma_0^2}\right)\big(\mathcal{U}(\Delta v)-\overline{\mathcal{U}}\big)\right). \\ &= 2\pi\sqrt{\tau_0\,\tau_{\mathrm{M}}}\,\frac{\sigma_0}{\sigma_{\mathrm{M}}}\,\exp\left(\frac{\mathcal{U}(\Delta v)}{\frac{1}{2}\sigma_{\mathrm{M}}^2}\right) \\ &\quad\cdot\exp\left(-2\left(\frac{1}{\sigma_{\mathrm{M}}^2}-\frac{1}{\sigma_0^2}\right)\overline{\mathcal{U}}\right)\Bigg).\end{aligned} \tag{21}$$

which is (16).

References

[1] L. Van Brandt, F. Silveira, J.-C. Delvenne, and D. Flandre, "On noise-induced transient bit flips in subthreshold SRAM," *Solid-State Electron.*, vol. 208, Oct. 2023, Art. no. 108715.

[2] A. Shilnikov, R. L. Calabrese, and G. Cymbalyuk, "Mechanism of bistability: Tonic spiking and bursting in a neuron model," *Phys. Rev. E, Stat. Phys. Plasmas Fluids Relat. Interdiscip. Top.*, vol. 71, no. 5, 2005, Art. no. 056214.

[3] A. Goldbeter, "Dissipative structures in biological systems: Bistability, oscillations, spatial patterns and waves," *Philosoph. Trans. Roy. Soc., A, Math., Phys. Eng. Sci.*, vol. 376, no. 2124, 2018, Art. no. 20170376.

[4] V. N. Livina, F. Kwasniok, and T. M. Lenton, "Potential analysis reveals changing number of climate states during the last 60 kyr," *Clim. Past*, vol. 6, no. 1, pp. 77–82, 2010.

[5] M. Hirota, M. Holmgren, E. H. Van Nes, and M. Scheffer, "Global resilience of tropical forest and savanna to critical transitions," *Science*, vol. 334, no. 6053, pp. 232–235, 2011.

[6] M. Vellela and H. Qian, "Stochastic dynamics and non-equilibrium thermodynamics of a bistable chemical system: The Schlögl model revisited," *J. Roy. Soc. Interface*, vol. 6, no. 39, pp. 925–940, 2009.

[7] L. Van Brandt, J.-C. Delvenne, and D. Flandre, "Variability-aware noise-induced dynamic instability of ultra-low-voltage SRAM bitcells," in *Proc. IEEE LASCAS*, Punta del Este, Uruguay, 2024, pp. 1–5.

[8] L. Van Brandt, D. Flandre, and J.-C. Delvenne, "Stochastic nonlinear dynamical modelling of SRAM bitcells in retention mode," in *Proc. IEEE EDTM*, Bangalore, India, 2024, pp. 1–4.

[9] N. Berglund, "Kramers' law: Validity, derivations and generalisations," 2011, *arXiv:1106.5799*.

[10] N. Freitas, K. Proesmans, and M. Esposito, "Reliability and entropy production in nonequilibrium electronic memories," *Phys. Rev. E, Stat. Phys. Plasmas Fluids Relat. Interdiscip. Top.*, vol. 105, no. 3, 2022, Art. no. 034107.

[11] E. Rezaei, M. Donato, W. R. Patterson, A. Zaslavsky, and R. I. Bahar, "Fundamental thermal limits on data retention in low-voltage CMOS latches and SRAM," *IEEE Trans. Device Mater. Rel.*, vol. 20, no. 3, pp. 488–497, Sep. 2020.

[12] H. Weidenmüller and Z. Jing-Shang, "Stationary diffusion over a multidimensional potential barrier: A generalization of Kramers' formula," *J. Statist. Phys.*, vol. 34, pp. 191–201, Jan. 1984.

[13] A. J. Siegert, "On the first passage time probability problem," *Phys. Rev.*, vol. 81, no. 4, p. 617, 1951.

[14] A. Nobile, L. Ricciardi, and L. Sacerdote, "Exponential trends of Ornstein–Uhlenbeck first-passage-time densities," *J. Appl. Probab.*, vol. 22, no. 2, pp. 360–369, 1985.

[15] H. A. Kramers, "Brownian motion in a field of force and the diffusion model of chemical reactions," *Physica*, vol. 7, no. 4, pp. 284–304, 1940.

[16] F. Bouchet and J. Reygner, "Generalisation of the eyring–kramers transition rate formula to irreversible diffusion processes," *Annales Henri Poincaré*, vol. 17, no. 12, 2016, pp. 3499–3532.

[17] J. B. Johnson, "Thermal agitation of electricity in conductors," *Phys. Rev.*, vol. 32, no. 1, p. 97, 1928.

[18] H. Nyquist, "Thermal agitation of electric charge in conductors," *Phys. Rev.*, vol. 32, no. 1, p. 110, 1928.

[19] R. Kubo, "The fluctuation-dissipation theorem," *Rep. Progr. Phys.*, vol. 29, no. 1, p. 255, 1966.

[20] L. Van Brandt and J.-C. Delvenne, "Noise–dissipation relation for nonlinear electronic circuits," *Appl. Phys. Lett.*, vol. 122, no. 26, 2023, Art. no. 263507.

[21] B. Zhang, A. Arapostathis, S. Nassif, and M. Orshansky, "Analytical modeling of SRAM dynamic stability," in *Proc. IEEE/ACM Int. Conf. Comput.-Aided Design*, 2006, pp. 315–322.

[22] L. B. Kish, "End of Moore's law: Thermal (noise) death of integration in micro and nano electronics," *Phys. Lett. A*, vol. 305, nos. 3–4, pp. 144–149, 2002.

[23] J. Rogal, "Reaction coordinates in complex systems-a perspective," *Eur. Phys. J. B*, vol. 94, pp. 1–9, Nov. 2021.

[24] F. Veirano, F. Silveira, and L. Naviner, "Minimum operating voltage due to intrinsic noise in subthreshold digital logic in nanoscale CMOS," *J. Low Power Electron.*, vol. 12, no. 1, pp. 74–81, 2016.

[25] K. Itô, "Stochastic differential equations in a differentiable manifold," *Nagoya Math. J.*, vol. 1, pp. 35–47, 1950.

[26] K. Itô, "On a formula concerning stochastic differentials," *Nagoya Math. J.*, vol. 3, pp. 55–65, Oct. 1951.

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