CORE DISCUSSION PAPER
2000/47

DYNAMIC KNAPSACK SETS AND CAPACITATED LOT-SIZING

Marko LOPARIC[1] Hugues MARCHAND[2]
and Laurence A. WOLSEY[3]

September 2000

Abstract

A dynamic knapsack set is a natural generalization of the 0-1 knapsack set with a continuous variable studied recently. For dynamic knapsack sets a large family of facet-defining inequalities, called dynamic knapsack inequalities, are derived by fixing variables to one and then lifting. Surprisingly such inequalities have the simultaneous lifting property, and for small instances provide a significant proportion of all the facet-defining inequalities.

We then consider single-item capacitated lot-sizing problems, and propose the joint study of three related sets. The first models the discrete lot-sizing problem, the second the continuous lot-sizing problem with Wagner-Whitin costs, and the third the continuous lot-sizing problem with arbitrary costs. The first set that arises is precisely a dynamic knapsack set, the second an intersection of dynamic knapsack sets, and the unrestricted problem can be viewed as both a relaxation and a restriction of the second. It follows that the dynamic knapsack inequalities and their generalizations provide strong valid inequalities for all three sets.

Some limited computation results are reported as an initial test of the effectiveness of these inequalities on capacitated lot-sizing problems.

Keywords: Knapsack sets, valid inequalities, simultaneous lifting, lot-sizing, Wagner-Whitin costs.

[1]CORE, Louvain-la-Neuve, Belgium. E-mail: loparic@core.ucl.ac.be
[2]London School of Economics, London, UK.
[3]CORE and INMA, Louvain-la-Neuve, Belgium. E-mail: wolsey@core.ucl.ac.be

Research carried out with financial support of the project TMR-DONET nr. ERB FMRX-CT98-0202 of the European Union.
This paper presents research results of the Belgian Program on Interuniversity Poles of Attraction initiated by the Belgian State, Prime Minister's Office, Science Policy Programming. The scientific responsibility is assumed by the authors.

1 Introduction

Here we study sets of the form

$$X_{\leq} = \{(w,\tau) \in B^n \times R^1_+ : \sum_{j:j\geq t} a_j w_j \leq \sum_{j:j\geq t} d_j + \tau \text{ for } t \in N\},$$

and

$$X_{\geq} = \{(w,\sigma) \in B^n \times R^1_+ : \sigma + \sum_{j:j\leq t} a_j w_j \geq \sum_{j:j\leq t} d_j \text{ for } t \in N\}$$

where $N = \{1,\ldots,n\}, a \in R^n_+$ and $d \in R^n$. $X_{\leq}$ and $X_{\geq}$ are called *dynamic knapsack sets* as they generalize the knapsack set with a single continuous variable studied in [9].

Such sets arise naturally in studying lot-sizing problems with demands d_t and capacities a_t in periods $t = 1,\ldots,n$. In the discrete lot-sizing problem [4] in which production in period t is either 0 or a_t, the feasible region can be written as

$$X^{DLS} = \{(y,s) \in B^n \times R^{n+1}_+ : s_{t-1} + a_t y_t = d_t + s_t \text{ for } t \in N\}$$

where variable s_t denotes the stock at the end of period t and $y_t = 1$ if there is a set-up (production of a_t) in period t. Note that there is a one to one correspondence between this set and both the sets $X_{\leq}$ and $X_{\geq}$. It suffices to view s_{t-1} as the slack variable in constraint t of $X_{\leq}$ and $\tau = s_n$, or to take s_t to be the surplus variable in constraint t of $X_{\geq}$ and $\sigma = s_0$.

In the capacitated lot-sizing problem [11] where x_t denotes the amount between 0 and a_t that can be produced in period t if there is a set-up ($y_t = 1$), the feasible region can be written as

$$X^{CLS} = \{(x,y,s) \in R^n_+ \times B^n \times R^{n+1}_+ : s_{t-1} + x_t = d_t + s_t, x_t \leq a_t y_t \text{ for } t \in N\}.$$

X^{CLS} can be relaxed to give a set of the form $X_{\geq}$ or $X_{\leq}$. On the other hand with Wagner-Whitin costs [12, 14], once the set-up periods are specified, production takes place as late as possible. Now the capacitated lot-sizing problem can be modelled with the set

$$X^{WW} = \{(y,s) \in B^n \times R^{n+1}_+ : s_{k-1} + \sum_{j=k}^{i} a_j y_j \geq \sum_{j=k}^{i} d_j \text{ for } i = k,\ldots,n, k \in N\}.$$

This can be viewed as the intersection of n dynamic knapsack sets $X^k_{\geq}$ for $k \in N$, and also as a relaxation of X^{CLS}.

In Section 2 we derive a family of valid inequalities for dynamic knapsack sets in the form $X_{\geq}$. Surprisingly these inequalities have the simultaneous lifting property [15, 7]. Combined with complementation, this procedure allows us to explain 21 of the 22 nontrivial facet-defining inequalities for an instance with $n = 6$.

In the next section we use the results of Section 2 to derive valid inequalities for X^{DLS}, X^{CLS} and X^{WW} respectively. For small instances a significant fraction of the facet-defining inequalities can be derived in this way, which suggests that these inequalities might be computationally useful. This possibility is explored in Section 4.

2 Valid Inequalities for Dynamic Knapsack Sets

Here we introduce new notation. Throughout we deal with vectors $a^j \in R^n$ where $a^j = a_j(\sum_{i=j}^n e^i)$ for $j \in N$ with $a_j \geq 0$ where e^i denotes the i^{th} unit vector. We let $d_{ij} \equiv \sum_{t=i}^j d_t$. We now consider the set

$$Y_N(d) = \{(y,\sigma) \in B^n \times R^1_+ : \sigma \underline{1} + \sum_{j\in N} a^j y_j \geq \tilde{d}\}$$

where $\tilde{d}_i = d_{1i}$ and $\underline{1} = (1,\ldots,1)^T$. Note that constraint t takes the form

$$\sigma + \sum_{j=1}^t a_j y_j \geq d_{1t}.$$

Below we derive a facet-defining inequality for $Y_N(d)$.

To avoid trivial instances we suppose that $d_{1i} > 0$ for some i. Let $\bar{d}_{1i} = \max\{\max_{j\leq i} d_{1i}, 0\}$, so now setting $\bar{d}_i = \bar{d}_{1i} - \bar{d}_{1,i-1}$ we have a modified nonzero demand vector $\bar{d} \geq 0$.

Observation 1 *The inequality $\sigma + \sum_{j=1}^t a_j y_j \geq \bar{d}_{1t}$ is valid for $Y_N(d)$ because if $\bar{d}_{1t} > 0$, then $\bar{d}_{1t} = d_{1i}$ for some $i \leq t$, and so $\sigma + \sum_{j=1}^t a_j y_j \geq \sigma + \sum_{j=1}^i a_j y_j \geq d_{1i} = \bar{d}_{1t}$.*

Note that if $\bar{d}_t > 0$, then $\bar{d}_{1t} = d_{1t}$.

Proposition 2.1 *The inequality*

$$\sigma + \sum_{j=1}^t \min[a_j, \bar{d}_{jt}] y_j \geq \bar{d}_{1t} \tag{1}$$

is valid for conv($Y_N(d)$) for $t = 1,\ldots,n$, and facet-defining when $t = n$.

Proof. First we show validity. Consider a point $(y^*,\sigma^*) \in Y_N(d)$. If $y^*_j = 0$ for all $j \leq t$ with $a_j > \bar{d}_{jt}$, the inequality is valid as

$$\sigma^* + \sum_{j=1}^t \min[a_j, \bar{d}_{jt}] y^*_j = \sigma^* + \sum_{j=1}^t a_j y^*_j \geq \bar{d}_{1t}.$$

Otherwise let $k = \min\{i : y^*_i = 1 \text{ and } a_i > \bar{d}_{i\tau}\}$. Using Observation 1 with $t = k-1$,

$$\sigma^* + \sum_{j=1}^t \min[a_j, \bar{d}_{jt}] y^*_j \geq \sigma^* + \sum_{j=1}^{k-1} a_j y^*_j + \bar{d}_{kt} \geq \bar{d}_{1,k-1} + \bar{d}_{kt} = \bar{d}_{1t}.$$

Now we take $t = n$ and let $\tau = \max\{t : \bar{d}_{1t} = \bar{d}_{1n}\}$. Consider now the $n+1$ points specified as follows:

(y^0, σ^0) with $\sigma^0 = \bar{d}_{1\tau}$ and $y^0 = 0$,

(y^i, σ^i) with $\sigma^i = \bar{d}_{1\tau} - \min[a_i, \bar{d}_{i\tau}]$ and $y^i = e^i$ for $i = 1, \ldots, \tau$, and

(y^i, σ^i) with $\sigma^i = \bar{d}_{1\tau} - \min[a_1, \bar{d}_{1\tau}]$ and $y^i = e^1 + e^i$ for $i = \tau + 1, \ldots, n$.

It is easily verified that these points lie in $Y_N(d)$, satisfy the inequality at equality and are affinely independent.

∎

Now we will derive other valid inequalities for $Y_N(d)$ by setting some variables to one thus modifying the demands, generating the basic inequality and then lifting back the variables that have been fixed.

Setting $y_j = 1$ for $j \in T$, the new demand vector is $\delta = d - \sum_{j \in T} a_j e^j$, and the basic inequality we start from is

$$\sigma + \sum_{j \in N \setminus T} \min[a_j, \bar{\delta}_{jn}] y_j \geq \bar{\delta}_{1n}.$$

We now define the function

$$\psi_T(h) = \min\{\sigma + \sum_{j \in N \setminus T} \min[a_j, \bar{\delta}_{jn}] y_j - \bar{\delta}_{1n} : (y, \sigma) \in Y_{N \setminus T}(\delta + h)\}.$$

Observation 2 *A necessary condition for*

$$\sigma + \sum_{j \in N \setminus T} \min[a_j, \bar{\delta}_{jn}] y_j + \sum_{j \in T} \alpha_j y_j \geq \bar{\delta}_{1n} + \sum_{j \in T} \alpha_j$$

to be valid for $Y_N(d)$ is that $\alpha_j \leq \psi_T(a_j e^j)$ for all $j \in T$.

Proof. By definition of ψ_T and for any $j \in T$, there exists a point $(y^*, \sigma^*) \in Y(\delta + a_j e^j)$ for which $\sigma^* + \sum_{j \in N \setminus T} \min[a_j, \bar{\delta}_{jn}] y^*_j - \bar{\delta}_{1n} = \psi_T(a_j e^j)$. Now if $\tilde{y}$ is defined by $\tilde{y}_i = y^*_i$ for $i \in N \setminus T, \tilde{y}_i = 1$ for $i \in T \setminus \{j\}, \tilde{y}_j = 0$, $(\tilde{y}, \sigma^*) \in Y_N(d)$. If the inequality is valid for $Y_N(d)$, $(\tilde{y}, \sigma^*)$ must satisfy the inequality giving

$$\sigma^* + \sum_{j \in N \setminus T} \min[a_j, \bar{\delta}_{jn}] y^*_j + \sum_{i \in T \setminus \{j\}} \alpha_i \geq \bar{\delta}_{1n} + \sum_{i \in T} \alpha_i,$$

or $\psi_T(a_j e^j) \geq \alpha_j$ for $j \in T$.

∎

Theorem 2.2 *The inequality*

$$\sigma + \sum_{j \in N \setminus T} \min[a_j, \bar{\delta}_{jn}] y_j + \sum_{j \in T} \psi_T(a_j e^j) y_j \geq \bar{\delta}_{1n} + \sum_{j \in T} \psi_T(a_j e^j) \qquad (2)$$

is valid and facet-defining for $Y_N(d)$.

Proof. The important step is to show that the inequality is valid. In the Lemma in Appendix 1 we show that $\psi_T(\sum_{j\in S} a_j e^j) \geq \sum_{j\in S} \psi_T(a_j e^j)$ for all $S \subseteq T$.

Consider $(y,\sigma) = (y^{N\setminus T}, y^T, \sigma) \in Y_N(d)$. Then $(y^{N\setminus T}, \sigma) \in Y_{N\setminus T}(\delta + \sum_{j\in T} a_j e^j(1 - y_j^T))$, and so by definition of ψ_T, $\sigma + \sum_{j\in N\setminus T} \min[a_j, \bar{\delta}_{jn}] y_j^{N\setminus T} - \bar{\delta}_{1n} \geq \psi_T(\sum_{j\in T} a_j e^j(1 - y_j^T))$.

Taking $S = \{j \in T : y_j = 0\}$, $\sigma + \sum_{j\in N\setminus T} \min[a_j, \bar{\delta}_{jn}] y_j^{N\setminus T} - \bar{\delta}_{1n} \geq \psi_T(\sum_{j\in S} a_j e^j) \geq \sum_{j\in S} \psi_T(a_j e^j) = \sum_{j\in T} \psi_T(a_j e^j)(1 - y_j^T)$, where the first inequality is by definition of ψ_T and the second from the Lemma. Hence the inequality is valid.

To obtain $n+1$ tight affinely independent points, it suffices to take the $|N \setminus T| + 1$ points from the proof of Proposition 2.1 and the $|T|$ points from the proof of Observation 2. ∎

3 Dynamic Knapsack Inequalities and Lot-Sizing

3.1 Discrete Lot-Sizing

Consider the discrete lot-sizing set

$$X^{DLS} = \{(y,s) \in B^n \times R_+^{n+1} : s_{t-1} + a_t y_t = d_t + s_t \text{ for } t \in N\}.$$

As it can be rewritten in the form $Y_N(d)$ by setting $\sigma = s_0$, we can apply Theorem 2.2 directly so the dynamic knapsack inequalities (2) are valid for X^{DLS}.

Observation 3 (Complementation). *After setting $\bar{y}_j = 1 - y_j$ for all $j \in N$, X^{DLS} can be rewritten as*

$$\bar{X}^{DLS} = \{(z', s') \in B^n \times R_+^{n+1} : s'_{t-1} + a'_t z'_t = d'_t + s'_t \text{ for } t \in N\}$$

where $s'_i = s_{n-i}$, $z'_i = \bar{y}_{n-i+1}$, $d'_i = a_{n-i+1} - d_{n-i+1}$ and $a'_i = a_{n-i+1}$ for $i \in N$.

Obviously Theorem 2.2 can also be applied to the complemented set $\bar{X}^{DLS}$.

Example 1 Take $n = 6$, $d = (2,3,2,0,3,1)$ and $a = (7,6,5,5,3,2)$. If $T = N$, $\bar{d} = (11,9,6,4,4,1)$. Proposition 2.1 gives the basic inequality

$$s_0 + 7y_1 + 6y_2 + 5y_3 + 4y_4 + 3y_5 + 1y_6 \geq 11.$$

Now applying Theorem 2.2 with $T = \{1,5\}$, we obtain $\delta = (-5,3,2,0,0,1)$, so $\tilde{\delta} = (-5,-2,0,0,0,1)$, $\bar{\delta}_{1\cdot} = (0,0,0,0,0,1)$ and $\bar{\delta}_{\cdot n} = (1,1,1,1,1,1)$.

$$
\begin{array}{rlllllllllllll}
\psi_T(a_1e^1) = & \min s_0 & + & 1y_2 & + & 1y_3 & + & 1y_4 & + & 1y_6 & - & 1 & \\
& s_0 & & & & & & & & & \geq & -5 & +7 \\
& s_0 & + & 6y_2 & & & & & & & \geq & -2 & +7 \\
& s_0 & + & 6y_2 & + & 5y_3 & & & & & \geq & 0 & +7 \\
& s_0 & + & 6y_2 & + & 5y_3 & + & 5y_4 & & & \geq & 0 & +7 \\
& s_0 & + & 6y_2 & + & 5y_3 & + & 5y_4 & & & \geq & 0 & +7 \\
& s_0 & + & 6y_2 & + & 5y_3 & + & 5y_4 & + & 2y_6 & \geq & 1 & +7 \\
& & & s_0 & \geq & 0 & ,y & \in & B^4, & & & &
\end{array}
$$

with optimal solution $y_2 = 1, y_3 = y_4 = y_6 = 0, s_0 = 2$ and value 2. Similarly $\psi_T(a_5e^5) = 0$, and by Theorem 2.2 the inequality

$$s_0 + 2y_1 + 1y_2 + 1y_3 + 1y_4 + 0y_5 + 1y_6 \geq 3. \tag{3}$$

is valid and facet-defining for $\text{conv}(X^{DLS})$. Altogether 12 nontrivial facets can be generated in this way from Theorem 2.2, and another 9 can be obtained from the set obtained by complementation described in Observation 3. Thus only one of the 22 nontrivial facets of $\text{conv}(X^{DLS})$ cannot be explained by simultaneous lifting. See the Appendix 2 for a complete list of the inequalities.

3.2 Capacitated Lot-Sizing

Consider the capacitated lot-sizing set

$$X^{CLS} = \{(x, y, s) \in R^n_+ \times B^n \times R^{n+1}_+ : s_{t-1} + x_t = d_t + s_t, x_t \leq a_t y_t \text{ for } t \in N\}.$$

Theorem 3.1 *If $s_0 + \sum_{j \in N} \pi_j y_j \geq \pi_0$ is a valid inequality for $X_\geq$, and $S \subseteq N$,*

$$s_0 + \sum_{j \in S} x_j + \sum_{j \in N \setminus S} \pi_j y_j \geq \pi_0$$

is a valid inequality for X^{CLS}.

Proof. Consider a point $(x^*, y^*, s^*) \in X^{CLS}$. Now contruct the point $(\tilde{x}, \tilde{y}, \tilde{s})$ defined by $\tilde{x}_j = x^*_j, \tilde{y}_j = y^*_j$ for $j \in N \setminus S$, $\tilde{x}_j = \tilde{y}_j = 0$ for $j \in S$, $\tilde{s}_0 = s^*_0 + \sum_{j \in S} x^*_j$, and $\tilde{s}_t = \tilde{s}_{t-1} + \tilde{x}_t - d_t$ for $t = 1, \ldots, n$. $(\tilde{x}, \tilde{y}, \tilde{s}) \in X^{CLS}$ as all production in the periods in S has been inserted into the initial stock. This implies that $\tilde{s}_0 + \sum_{j \leq t} a_j \tilde{y}_j \geq d_{1t}$ for $t \in N$, and so $(\tilde{s}_0, \tilde{y}) \in X_\geq$. So by hypothesis $\tilde{s}_0 + \sum_{j \in N} \pi_j \tilde{y}_j \geq \pi_0$, or in other words $s^*_0 + \sum_{j \in S} x^*_j + \sum_{j \in N \setminus S} \pi_j y^*_j \geq \pi_0$, and so the inequality is valid for X^{CLS}. ∎

Corollary The inequality

$$s_0 + \sum_{j \in N \setminus (T \cup S)} \min[a_j, \bar{\delta}_{jn}] y_j + \sum_{j \in T \setminus S} \psi_T(a_j e^j) y_j + \sum_{j \in S} x_j \geq \bar{\delta}_{1n} + \sum_{j \in T} \psi_T(a_j e^j) \tag{4}$$

is valid for X^{CLS} for all $S \subseteq N$.

Example 1 (cont). For demands d and capacities a as in Example 1, it follows, using inequality (3) and Theorem 3.1, that the inequalities

$$s_0 + z_1 + z_2 + z_3 + z_4 + z_6 \geq 3$$

are valid for X^{CLS} where $z_1 \in \{2y_1, x_1\}, z_2 \in \{1y_2, x_2\}, z_3 \in \{1y_3, x_3\}, z_4 \in \{1y_4, x_4\}$ and $z_6 \in \{1y_6, x_6\}$. In fact all 8 inequalities with $z_1 = 2y_1$ and $z_2 = 1y_2$ can be shown to be facet-defining. In total 100 nontrivial facets of conv(X^{CLS}) out of a total of 182 can be obtained by applying the Corollary to the 21 facets of X^{DLS} generated above.

3.3 Capacitated Lot-Sizing with Wagner-Whitin Costs

Consider again the capacitated lot-sizing problem

$$\min\{p'x + h's + fy : (x, y, s) \in X^{CLS}\}.$$

When the production and storage costs satisfy $h'_0 \geq 0, p'_t + h'_t \geq p'_{t+1}$ for $t = 1, \ldots, n-1$, the problem is said to have *Wagner-Whitin costs*. In this case the objective function can be rewritten as $\sum_{t=0}^{n} h_t s_t + \sum_{t=1}^{n} f_t y_t + K$ where $h_0 = h'_0 \geq 0$, $h_t = h'_t + p'_t - p'_{t+1} \geq 0$ for $t \in N$ and $K = \sum_{t=1}^{n} p'_t d_t$, and the problem can be formulated as

$$\begin{gathered}\min \textstyle\sum_{t=0}^{n} h_t s_t + \sum_{t=1}^{n} f_t y_t \\ s_{k-1} + \textstyle\sum_{j=k}^{t} a_j y_j \geq d_{kt} \text{ for } t = k, \ldots, n, k \in N \\ s \in R_+^{n+1}, y \in B^n,\end{gathered}$$

where $x_t = d_t + s_t - s_{t-1}$ for all t. Note that the feasible region is of the form $X^{WW} = \cap_{i=1}^{n} X_{\geq}^{i}$, where, for each $i = 1, \ldots, n$,

$$X_{\geq}^{i} = \{(s_{i-1}, y, \ldots, y_n) \in R_+^1 \times B^{n-i+1} : s_{i-1} + \sum_{j=i}^{t} a_j y_j \geq d_{it} \text{ for } t = i, \ldots, n\}$$

is a dynamic knapsack set.

Surprisingly, for small values of n, nearly all the facets of conv(X^{WW}) are facets of conv$(X_{\geq}^{i})$ for some i. In particular for the instance of Example 1, all the facet-defining inequalities are of this type. However the instance with $n = 6, d = (1, 3, 1, 2, 1, 3), a = (7, 3, 5, 4, 4, 3)$ has the facet-defining inequality

$$s_0 + 5y_1 + s_1 + 4y_2 + 8y_3 + 6y_4 + 6y_5 + 4y_6 \geq 17$$

which is clearly not of this type.

4 Computation

Given that Theorem 2.2 apparently provides a large proportion of the facet-defining inequalities of conv(X^{DLS}), that Theorem 2.2 applied to the n distinct sets $X^k_{\geq}$ gives a large number of facets of conv(X^{WW}), and Theorems 2.2 and 4.1 together lead to a large number of facets of conv(X^{CLS}), it is natural to ask whether the family of dynamic knapsack inequalities described above are effective in solving, or nearly solving capacitated lot-sizing instances.

Here we restrict our attention to the general lot-sizing problem X^{CLS}, and limit our investigation to two issues: i) the derivation of a heuristic separation algorithm in order to choose a lifting set T for which the dynamic knapsack inequality (2) is often violated, and ii) a comparison of the effectiveness of the dynamic knapsack inequalities generated in comparison with other cutting plane systems designed at least in part to be effective in solving lot-sizing problems. Questions of computational efficiency, though important, are left for further study.

Below we describe the test instances, the separation heuristic selected, the branch-and-cut systems with which comparisons are made, and finally the computational results.

The Problem Instances. We have chosen to carry out tests on the most general problem X^{CLS} formulated in the form: $\min\{hs + fy + px : s_{t-1} + x_t = d_t + s_t, x_t \leq a_t y_t \text{ for } t \in N, (x, y, s) \in R^n_+ \times B^n \times R^{n+1}_+\}$ presented in Section 1. We have generated 6 sets of instances with three distinct average capacity levels and two distinct average ratios between fixed cost and capacity. The capacities and fixed costs are randomly chosen with uniform distribution. The capacities are within 25% of the mean and the fixed cost within 10% of the product of the capacity and the ratio parameter. The mean capacity of sets 1 and 4 is 30, of sets 2 and 5 is 40, and of sets 3 and 6 is 50. For sets 1, 2 and 3 we fixed the ratio parameter at 50 and the initial stock cost h_0 at 60, while for sets 4, 5 and 6 the ratio parameter is 30 and h_0 is 40. All instances cover 30 time periods. The stock costs are $h_t = 10$ for $t = 1, \dots, n$. The unit production cost p_t varies uniformly in the range $[0 - 20]$, and the demands d_t in the range $[1 - 19]$.

The Separation Heuristic. Given a fractional point (x^*, y^*, s^*), the problem is to find an inequality of the form (4) cutting off the point. The first step is to find an inequality of the form (2). The most important decision is the choice of the lifting set T, and a secondary question of interest is whether it is really important to calculate the lifting coefficients ψ_T exactly or not.

Based on some limited experimentation, the candidate set T is chosen as follows. Let $V^1 = \{j : y^*_j = 1\}$ and $V^0 = \{j : y^*_j = 0\}$. It can then be argued that it suffices to choose T such that $V^1 \subseteq T \subseteq N \setminus V^0$. Starting with $T = V^1$, we then greedily augment T taking up to k new variables at each step, i.e. consider all sets $U \subseteq N \setminus T$ with up to k elements, and find U for which the violation with $T' = T \cup U$ is the greatest. If the violation is greater than with T, set $T \leftarrow T'$ and repeat. The k-greedy procedure is also repeated with the complemented instance (Observation 3). Finally to derive an inequality for X^{CLS} as opposed to an inequality for X^{DLS}, terms of type $\alpha_j y_j$ are replaced by terms x_j if the violation is increased, both before and after calculating lifting coefficients. The heuristic has been used for values of $k = 1$ and 2.

In practice the decomposition of the fractional solution (x^*, y^*, s^*) into regeneration intervals $[t, l]$ is used both to speed up the calculations and to limit the choice of sets T. Specifically if $s^*_{t-1} = s^*_l = 0, s^*_t, \ldots, s^*_{l-1} > 0$ and $y^*_t, \ldots, y^*_l$ is not integral, we limit the search for a violated inequality to the interval $[t, l]$.

It is not known whether the exact calculation of the lifting function ψ_T is NP-hard or not. For the testing here, we have just used enumeration to obtain the exact values of the coefficients ψ_T.

Alternative Cutting Plane Systems. We have compared the dynamic knapsack cuts with cutting planes from three other systems. $mp-opt$ (XPRESS)[16] is a state-of-the-art commercial mixed integer programming system generating mixed integer rounding inequalities and Gomory mixed integer cuts, along with various row aggregation procedures that have in part been motivated by lot-sizing and fixed charge network flow problems [10]. $bc-opt$ [3] is a prototype branch-and-cut system including flow cover inequalities, and also path inequalities [13] that were explicitly designed for uncapacitated lot-sizing problems, and have been modified slightly to take capacities into account. $bc-prod$ [2] is a prototype branch-and-cut system specifically designed for lot-sizing problems. In addition to the routines of $bc-opt$, it has efficient separation routines for constant capacity lot-sizing, that have been adapted in ad hoc fashion to treat varying capacities.

Results. In Table 1 five different approaches are compared. Columns 1 and 2 show results with dynamic knapsack inequalities with lifting sets T generated using the heuristic discussed above. Columns 1 and 2 show results with exact lifting and with $k = 2$ and $k = 1$ respectively, whereas Column 3 shows results with no lifting ($\alpha_j = 0$ for all $j \in T$) and $k = 2$. Columns 4, 5 and 6 show results for $bc-prod$, $bc-opt$ and $mp-opt$ respectively.

In each case we have generated inequalities at the top node of the tree, and measured how much of the duality gap is closed by the cuts. Specifically for each instance we calculate $\frac{XLP-LP}{IP-LP} \times 100$, where LP is the initial linear programming lower bound, XLP the lower bound obtained after adding the cuts, and IP is the optimal value. For each problem class and column, we show the mean value over ten randomly generated instances.

The performance in terms of time is indicated in the last line of the table. The average time needed for each method to completely solve the instances in each set is divided by the average time needed by bc-prod. These values are approximate since the systems were run in different machines. The heuristic results and the mp-opt solutions were obtained using a Sun Sparc Ultra 60 of 250 Mhz and the bc-prod and bc-prod results with a PC of 400 Mhz running Windows NT. To obtain realistic time comparison, the Sun Sparc times were divided by 1.6. The actual average time taken by bc-prod is 4.6 seconds.

set	k2-lift	k1-lift	k2-nolift	bc-prod	bc-opt	mp-opt
1	95.98	89.68	77.02	81.23	69.89	77.71
2	95.04	89.87	83.06	85.99	79.90	78.37
3	96.68	91.99	85.50	88.56	83.34	76.25
4	97.41	91.94	84.89	87.57	79.07	84.61
5	98.64	94.94	89.02	92.25	88.09	84.97
6	98.77	95.69	92.68	92.92	90.00	82.99
time	52	2.8	0.99	1	0.85	0.47

Table 1: Gap reduction with 3 heuristics, and bc-prod, bc-opt and mp-opt

Inspection of Table 1 leads us to the following observations:

- The best results using dynamic knapsack cuts are obtained using $k = 2$ and lifting (heuristic k2-lift). Weaker results are obtained with the two other heuristics (k1-lift and k2-nolift).
- The k2-lift heuristic is able to reduce the gap much more than the other systems. The average gap reduction is always higher than 95%. The same reduction has been achieved in all other data sets we have looked at. However in our implementation this heuristic is very time consuming (on average 52 times slower than bc-prod).
- Comparing results of the heuristics k2-lift and k2-nolift shows that the lifting procedure considerably increases the strength of the cuts.
- The gap reduction of heuristic k1-lift is slightly greater than that of bc-opt, which in turn is slightly greater than heuristic k2-nolift. On other instances similar behaviour has been observed with few fluctuations. The solution time of k1-lift and k2-nolift is of the same order of magintude as that of bc-prod.
- The gap reduction of bc-prod is greater than that of bc-opt and mp-opt.
- The gap reduction is smaller on instances more tightly constrained and with higher setup costs.

5 Conclusions

Both examples in Section 2 and the computational results in Section 3 suggest that dynamic knapsack inequalities can become an effective tool for the solution of lot-sizing problems with varying capacities. At present k1-lift is close to being a practical and effective separation heuristic. Ideally we would like to speed up the k2-lift heuristic sufficiently for it to become the separation routine of choice.

The simultaneous lifting property for the dynamic knapsack inequality is surprising to us, especially as the function ψ_T is not superadditive. Some better understanding of this aspect, and the complexity of the calculation of ψ_T merit further investigation.

The above results also suggest that further progress may be made by studying the polyhedral structure of the three different lot-sizing sets X^{DLS}, X^{CLS} and X^{WW} simultaneously. One intriguing question concerns the constant capacity Wagner-Whitin case. Here a complete description of $\text{conv}(X^{WW})$ is obtained by applying the mixing procedure [12, 6] to the inequalities given in Observation 1, but no link with the lifted inequalities (2) is apparent. Further computational tests might be useful to investigate these interactions.

References

[1] I. Barany, T.J. Van Roy and L.A. Wolsey, Uncapacitated lot sizing : the convex hull of solutions, *Mathematical Programming Study* **22**, 32-43 (1984).

[2] G. Belvaux and L.A. Wolsey, $bc - prod$: A Specialized Branch-and-Cut System for Lot-Sizing Problems, *Management Science* , (2000).

[3] C. Cordier, H. Marchand, R. Laundy and L.A. Wolsey, $bc - opt$: A Branch-and-Cut Code for Mixed Integer Programs, *Mathematical Programming* **86**, 335-354 (1999).

[4] B. Fleischmann, The Discrete Lot-Sizing and Scheduling Problem, *European Journal of Operations Research* **44**, 337-348 (1990).

[5] K. Fukuda and A. Prodon, Double Description method revisited, Combinatorics and Computer Science, Lectue Notes in Computer Science no. 1120, Springer Verlag (1996).

[6] O. Günlük and Y. Pochet, Mixing mixed integer inequalities, Core DP 9911, Université Catholique de Louvain, Louvain-la-Neuve, 1999.

[7] Z.Gu, G.L. Nemhauser and M.W.P. Savelsbergh, Sequence Independent Lifting, LEC-95-08, Georgia Institute of Technology (1995).

[8] H. Marchand, A Polyhedral study of the mixed knapsack set and its use to solve mixed integer programs, Ph.D. Thesis, Faculté des Sciences Appliquées, Université Catholique de Louvain, (1998).

[9] H. Marchand and L.A. Wolsey, The 0-1 knapsack problem with a single continuous variable, *Mathematical Programming* **85**, 15-33 (1999).

[10] H. Marchand and L.A. Wolsey, Aggregation and Mixed Integer Rounding to solve MIPs, *Operations Research* (to appear).

[11] Y. Pochet, Valid inequalities and separation for capacitated economic lot-sizing, *Operations Research Letters* **7**, 109-116, 1988.

[12] Y. Pochet and L.A. Wolsey, Polyhedra for Lot-Sizing with Wagner-Whitin Costs, *Mathematical Programming* **67**, 297-324 (1994).

[13] T.J. Van Roy and L.A. Wolsey, Solving Mixed 0-1 Problems by Automatic Reformulation, *Operations Research* **35**, 45-57 (1987).

[14] H.M. Wagner and T.M. Whitin, Dynamic Version of the Economic Lot Size Model, *Management Science* **5**, 89-96 (1958).

[15] L.A. Wolsey, Valid Inequalities and Superadditivity for 0/1 Integer Programs, *Mathematics of Operations Research* **2**, 66-77 (1977).

[16] XPRESS-MP Extended Modelling and Optimisation Subroutine Library, Reference Manual, Release 11, Dash Associates, Blisworth House, Blisworth, Northants NN73BX, UK (1999).

Appendix 1.

Let $\psi_T(h) = \min\{\sigma + \sum_{j\in N} \min[a_j, \bar{\delta}_{jn}]y_j - \bar{\delta}_{1n} : (\sigma, y) \in Y_{N\setminus T}(\delta + h)\}$.

Lemma 5.1 *$\psi_T(\sum_{j\in S} g_j e^j) \geq \sum_{j\in S} \psi_T(g_j e^j)$ for all $S \subseteq T$.*

Proof. Let $l = \max\{j : j \in S\}$. We will show that $\psi_T(u) + \psi_T(v) \leq \psi_T(w)$ where $w = u + v, u = \sum_{j\in S\setminus\{l\}} g_j e^j$ and $v = g_l e^l$. Let (σ^w, y^w) be an optimal solution of $\psi_T(w)$. Clearly there exists such a solution with $y^w_j = 0$ whenever $a_j \leq \bar{\delta}_{jn}$. We will use this solution to construct a feasible solution of $Y_{N\setminus T}(\delta + u)$ and $Y_{N\setminus T}(\delta + v)$. Let $N^w = \{j \in N \setminus T : y^w_j = 1\}$.

We first define a solution $(\sigma^u, y^u) \in Y_{N\setminus T}(\delta + u)$. Choose a subset $N^u \subseteq N^w$ and set $y_j = 1$ for $j \in N^u$, $y^u_j = 0$ otherwise. Then choose σ to be minimal so that $(\sigma, y) \in Y_{N\setminus T}(\delta + u)$. Among all such points choose any point with $q^u_n = 0$ and y maximal where $q^u_i = \sigma^u + \sum_{j\leq i} a_j y^u_j - \delta_{1i} - u_{1i}$. Observe that such a point exists, as $(\sigma, y) = (\delta_{1n} + u_{1n}, 0)$ is a point of $Y_{N\setminus T}(\delta + u)$ with $q^u_n = 0$. Call such a point (σ^u, y^u).

Now define $(\sigma^v, y^v) \in Y_{N\setminus T}(\delta + v)$ by setting $y^v_j = 1$ for $j \in N^w \setminus N^u$ and $y^v_j = 0$ otherwise and taking σ^v to be minimal.

Observation 4 *By construction of (σ^u, y^u), $a_j > \min(\min_{i<j} q^u_i, \sigma^u)$ for all $j \in N^v$. Otherwise the point $(\sigma^u - a_j, y^u + e^j)$ is such that $q^u_n = 0$ and therefore (σ^u, y^u) is not maximal.*

There are two cases to consider.

Case 1. σ^v attains its value in row $t \geq l$.

Note that in this case $u_{1t} = u_{1n}$. Thus

$$\begin{aligned} \sigma^w &\geq \delta_{1t} + w_{1t} - \textstyle\sum_{j\leq t} a_j y^w_j \\ &= \delta_{1t} + v_{1t} - \textstyle\sum_{j\leq t} a_j y^v_j + u_{1t} - \sum_{j\leq t} a_j y^u_j \\ &\geq \sigma^v + \delta_{1n} + u_{1n} - \textstyle\sum_{j\leq n} a_j y^u_j - \delta_{1n} \\ &= \sigma^v + \sigma^u - \delta_{1n}. \end{aligned}$$

Therefore

$$\begin{aligned} \psi_T(w) &= \sigma^w + \textstyle\sum_{j\in N\setminus T} \min[a_j, \bar{\delta}_{jn}]y^w_j - \delta_{1n} \\ &\geq \sigma^u + \textstyle\sum_{j\in N\setminus T} \min[a_j, \bar{\delta}_{jn}]y^u_j - \delta_{1n} \\ &\quad + \sigma^v + \textstyle\sum_{j\in N\setminus T} \min[a_j, \bar{\delta}_{jn}]y^v_j - \delta_{1n} \\ &\geq \psi_T(u) + \psi_T(v). \end{aligned}$$

Case 2a. σ^v attains its value in row $t < l$.

First observe that if $y^v_j = 0$ for all $j \geq t + 1$,

$$\sigma^v = \delta_{1t} - \sum_{j\leq t} a_j y^v_j < \delta_{1n} + v_{1n} - \sum_{j\in N\setminus T} a_j y^v_j,$$

a contradiction. So let $j^* = \min_{j \geq t+1}\{j : y_j^v = 1\}$.

Claim. $v_{1t} - \sum_{j \leq t} a_j y_j^v \leq v_{1i} - \sum_{j \leq i} a_j y_j^v$ for $i \in \{1, \ldots, j^* - 1\}$.

This is a direct consequence of $v_{1i} \geq v_{1t} = 0$, and $\sum_{j \leq i} a_j y_j^v \leq \sum_{j \leq t} a_j y_j^v$ for all $i \in \{1, \ldots, j^* - 1\}$.

Now for $i \in \{1, \ldots, j^* - 1\}$,

$$\begin{aligned} \sigma^w &\geq \delta_{1i} + w_{1i} - \textstyle\sum_{j \leq i} a_j y_j^w \\ &= \delta_{1i} + u_{1i} - \textstyle\sum_{j \leq i} a_j y_j^u + v_{1i} - \sum_{j \leq i} a_j y_j^v \\ &= \sigma^u - q_i^u + v_{1i} - \textstyle\sum_{j \leq i} a_j y_j^v \\ &\geq \sigma^u - q_i^u + v_{1t} - \textstyle\sum_{j \leq t} a_j y_j^v \\ &= \sigma^u - q_i^u + \sigma^v - \delta_{1t} \\ &\geq \sigma^u - q_i^u + \sigma^v + \bar{\delta}_{j^*n} - \delta_{1n} \\ &\geq \sigma^u + \sigma^v + \min[a_j, \bar{\delta}_{j^*n}] - \delta_{1n} - q_i^u \end{aligned}$$

As $v_{1t} - \sum_{j \leq t} a_j y_j^v \leq 0$, we also have that

$$\sigma^w \geq v_{1t} - \sum_{j \leq t} a_j y_j^v + \delta_{1t} - \delta_{1t}$$

$$\geq \sigma^v + \min[a_j, \bar{\delta}_{j^*n}] - \delta_{1n}$$

$$= \sigma^u + \sigma^v + \min[a_j, \bar{\delta}_{j^*n}] - \delta_{1n} - \sigma^u.$$

Therefore we have that

$$\sigma^w \geq \sigma^u + \sigma^v - \delta_{1n} + \min[a_j, \bar{\delta}_{j^*n}] - \min[\sigma^u, \min_{i<j^*} q_i^u].$$

Case 2b. $\sigma^v = 0$.
First observe that if $y_j^v = 0$ for all $j \in N \setminus T$, then

$$\sigma^v = 0 < \delta_{1n} + v_{1n} - \sum_{j \in N \setminus T} a_j y_j^v,$$

a contradiction. So let $j^* = \min_{j \in N \setminus T}\{j : y_j^v = 1\}$.
If all $i < j^*$, we have that

$$\begin{aligned} \sigma^w &\geq \delta_{1i} + w_{1i} - \textstyle\sum_{j \leq i} a_j y_j^w \\ &\geq \delta_{1i} + u_{1i} - \textstyle\sum_{j \leq i} a_j y_j^u \\ &= \sigma^u - q_i^u + \delta_{1n} - \delta_{1n} + \sigma^v \\ &\geq \sigma^u + \sigma^v + \min[a_{j^*}, \bar{\delta}_{j^*n}] - \delta_{1n} - q_i^u. \end{aligned}$$

As $\sigma^w \geq 0$ and $\sigma^v = 0$, we have also that

$$\sigma^w \geq \sigma^v + \min[a_{j^*}, \bar{\delta}_{j^*n}] - \delta_{1n}.$$

Therefore we have again that

$$\sigma^w \geq \sigma^u + \sigma^v + \min[a_j, \bar{\delta}_{j^*n}] - \delta_{1n} - \min[\sigma^u, \min_{i<j^*} q_i^u].$$

Finally in cases 2a and 2b, let $\Delta = \min[\min_{i<j^*} q_i^u, \sigma^u]$. By Observation 5, $a_{j^*} > \Delta$ and therefore the point $(\tilde{\sigma}^u, \tilde{y}^u) = (\sigma^u - \Delta, y^u + e^{j^*}) \in Y_{N \setminus T}(\delta + u)$.

Thus

$$\begin{array}{c}
\psi_T(w) = \sigma^w + \sum_{j \in N \setminus T} \min[a_j, \bar{\delta}_{jn}] y_j^w - \delta_{1n} \\
\geq \sigma^v + \sum_{j \in N \setminus T} \min[a_{j^*}, \bar{\delta}_{j^*n}] y_j^v - \delta_{1n} \\
+\sigma^u + \sum_{j \in N \setminus T} \min[a_{j^*}, \bar{\delta}_{j^*n}] y_j^u - \delta_{1n} + \min[a_{j^*}, \bar{\delta}_{j^*n}] - \min[\sigma^u, \min_{i<j^*} q_i^u] \\
= \sigma^v + \sum_{j \in N \setminus T} \min[a_{j^*}, \bar{\delta}_{j^*n}] y_j^v - \delta_{1n} \\
+\tilde{\sigma}^u + \sum_{j \in N \setminus T} \min[a_{j^*}, \bar{\delta}_{j^*n}] \tilde{y}_j^u - \delta_{1n} \\
\geq \psi_T(v) + \psi_T(u).
\end{array}$$

∎

Appendix 2.

Here we list the nontrivial facet defining inequalities for the instance of X^{DLS} in Example 1. The first five are basic inequalities. The next 16 are obtained by the lifting procedure and possibly complementation. Only the last is not obtained in this way. The inequalities were generated with the code of [5].

$$
\begin{array}{ccccccccccccccc}
s_0 & + & 7y_1 & + & 6y_2 & + & 5y_3 & + & 4y_4 & + & 3y_5 & + & 1y_6 & \geq & 11 \\
s_0 & + & 7y_1 & + & 6y_2 & + & 5y_3 & + & 3y_4 & + & 3y_5 & + & 0y_6 & \geq & 10 \\
s_0 & + & 7y_1 & + & 5y_2 & + & 2y_3 & + & 0y_4 & + & 0y_5 & + & 0y_6 & \geq & 7 \\
s_0 & + & 5y_1 & + & 3y_2 & + & 0y_3 & + & 0y_4 & + & 0y_5 & + & 0y_6 & \geq & 5 \\
s_0 & + & 2y_1 & + & 0y_2 & + & 0y_3 & + & 0y_4 & + & 0y_5 & + & 0y_6 & \geq & 2 \\
\hline
s_0 & + & 2y_1 & + & 1y_2 & + & 1y_3 & + & 1y_4 & + & 0y_5 & + & 1y_6 & \geq & 3 \\
s_0 & + & 5y_1 & + & 3y_2 & + & 3y_3 & + & 3y_4 & + & 3y_5 & + & 1y_6 & \geq & 8 \\
s_0 & + & 5y_1 & + & 4y_2 & + & 4y_3 & + & 4y_4 & + & 3y_5 & + & 1y_6 & \geq & 9 \\
s_0 & + & 5y_1 & + & 4y_2 & + & 1y_3 & + & 1y_4 & + & 0y_5 & + & 1y_6 & \geq & 6 \\
s_0 & + & 6y_1 & + & 4y_2 & + & 3y_3 & + & 3y_4 & + & 3y_5 & + & 1y_6 & \geq & 9 \\
s_0 & + & 6y_1 & + & 4y_2 & + & 1y_3 & + & 1y_4 & + & 1y_5 & + & 1y_6 & \geq & 7 \\
s_0 & + & 6y_1 & + & 5y_2 & + & 4y_3 & + & 4y_4 & + & 3y_5 & + & 1y_6 & \geq & 10 \\
s_0 & + & 7y_1 & + & 6y_2 & + & 3y_3 & + & 1y_4 & + & 0y_5 & + & 1y_6 & \geq & 8 \\
s_0 & + & 7y_1 & + & 6y_2 & + & 3y_3 & + & 2y_4 & + & 1y_5 & + & 1y_6 & \geq & 9 \\
s_0 & + & 7y_1 & + & 5y_2 & + & 2y_3 & + & 1y_4 & + & 1y_5 & + & 1y_6 & \geq & 8 \\
s_0 & + & 7y_1 & + & 5y_2 & + & 4y_3 & + & 3y_4 & + & 3y_5 & + & 1y_6 & \geq & 10 \\
s_0 & + & 7y_1 & + & 5y_2 & + & 4y_3 & + & 2y_4 & + & 2y_5 & + & 0y_6 & \geq & 9 \\
s_0 & + & 5y_1 & + & 4y_2 & + & 3y_3 & + & 3y_4 & + & 3y_5 & + & 0y_6 & \geq & 8 \\
s_0 & + & 5y_1 & + & 3y_2 & + & 2y_3 & + & 2y_4 & + & 2y_5 & + & 0y_6 & \geq & 7 \\
s_0 & + & 4y_1 & + & 3y_2 & + & 3y_3 & + & 3y_4 & + & 3y_5 & + & 0y_6 & \geq & 7 \\
s_0 & + & 4y_1 & + & 2y_2 & + & 2y_3 & + & 2y_4 & + & 2y_5 & + & 0y_6 & \geq & 6 \\
\hline
s_0 & + & 6y_1 & + & 5y_2 & + & 2y_3 & + & 2y_4 & + & 1y_5 & + & 1y_6 & \geq & 8
\end{array}
$$

Appendix 3

We present in the next three tables more details about the results obtained for each randomly generated instance. In the first table we show the value of the linear program relaxation (lp) and the optimal value (ip). In the other two tables we show the gap (as explained in Section), number of nodes of the branch-and-bound tree and the time needed for the optimization (in seconds).

problem	lp	ip	problem	lp	ip
cls1-0	18680.4	22121.0	cls4-0	10403.0	13112.0
cls1-1	15478.6	18522.0	cls4-1	11339.0	14030.0
cls1-2	19221.9	22592.0	cls4-2	10388.5	13007.0
cls1-3	20254.6	23436.0	cls4-3	10694.4	13493.0
cls1-4	16388.8	19174.0	cls4-4	10641.0	12894.0
cls1-5	15594.1	18795.0	cls4-5	12008.4	15036.0
cls1-6	15258.7	18635.0	cls4-6	12416.4	15165.0
cls1-7	13200.2	15972.0	cls4-7	13318.7	16037.0
cls1-8	17566.8	20460.0	cls4-8	10866.6	13402.0
cls1-9	18381.4	21417.0	cls4-9	11085.3	13573.0
cls2-0	17609.5	22752.0	cls5-0	9325.9	13396.0
cls2-1	17575.4	22062.0	cls5-1	10607.2	14073.0
cls2-2	17742.7	21891.0	cls5-2	9652.4	12770.0
cls2-3	14456.5	18749.0	cls5-3	13477.1	16796.0
cls2-4	18207.8	21737.0	cls5-4	11210.7	15099.0
cls2-5	16682.1	21167.0	cls5-5	12214.1	16319.0
cls2-6	15802.0	19733.0	cls5-6	12882.5	16840.0
cls2-7	19305.0	23044.0	cls5-7	13045.8	16708.0
cls2-8	15926.8	20390.0	cls5-8	12002.3	15623.0
cls2-9	21101.0	25410.0	cls5-9	9142.9	12546.0
cls3-0	15842.9	20806.0	cls6-0	10598.4	15951.0
cls3-1	18650.2	23978.0	cls6-1	10409.1	15444.0
cls3-2	18385.6	23449.0	cls6-2	10523.3	15304.0
cls3-3	16355.0	22226.0	cls6-3	12519.2	17468.0
cls3-4	17154.5	23160.0	cls6-4	13100.6	18334.0
cls3-5	14241.0	19515.0	cls6-5	11132.3	15067.0
cls3-6	19245.7	25338.0	cls6-6	12287.9	17556.0
cls3-7	17427.9	22380.0	cls6-7	11182.7	15735.0
cls3-8	19207.6	24187.0	cls6-8	12511.0	17061.0
cls3-9	16336.9	21701.0	cls6-9	12425.9	17464.0

problem	k2-lift			k1-lift			k2-nolift			bc-prod			bc-opt			mp-opt		
	gap	nodes	time	gap	nodes	time	gap	nodes	time	gap	nodes	time	gap	nodes	time	gap	nodes	time
cls1-0	93.2	60	475.0	86.5	686	17.5	74.1	1897	8.1	81.2	896	5.9	75.2	2603	3.9	75.2	916	4.1
cls1-1	97.4	29	393.7	93.8	94	17.6	80.6	388	10.9	80.2	517	4.6	78.2	1259	2.3	70.6	681	3.9
cls1-2	92.9	300	367.4	85.8	737	18.8	68.2	2899	11.6	78.9	830	5.0	73.9	4627	5.9	63.2	2102	7.6
cls1-3	96.2	51	330.2	90.9	240	13.1	76.8	1451	10.2	79.5	916	5.7	75.0	3109	3.8	70.4	565	3.3
cls1-4	97.4	5	256.0	95.5	24	8.9	84.9	124	14.1	82.0	170	3.3	81.8	148	0.6	75.1	149	2.3
cls1-5	99.9	3	453.0	94.8	140	14.2	85.5	328	9.8	87.4	225	4.6	75.5	4368	5.5	77.5	418	3.0
cls1-6	88.4	320	359.5	82.7	1027	14.9	67.8	3057	12.7	76.0	1301	6.8	73.8	4731	6.5	55.5	3795	10.9
cls1-7	100.0	1	103.3	97.8	12	21.2	88.6	35	3.4	90.9	122	3.7	90.4	58	0.5	82.4	82	2.5
cls1-8	99.2	15	322.5	88.8	135	13.1	79.1	336	6.8	82.2	842	4.6	81.5	1138	1.8	65.3	520	2.7
cls1-9	95.2	34	523.0	80.2	1334	12.9	64.6	3120	13.2	74.0	1625	5.9	71.8	2254	3.6	63.7	2328	7.3
cls2-0	98.2	47	624.5	92.3	146	22.2	84.8	1567	15.0	86.5	737	5.6	73.6	4759	7.8	81.9	432	4.4
cls2-1	95.0	85	206.8	93.8	79	20.9	75.3	832	10.4	81.2	462	4.3	75.7	2248	3.3	72.0	1478	6.4
cls2-2	99.7	7	502.0	96.7	38	23.5	87.7	133	6.9	87.5	116	3.6	81.2	843	1.2	81.1	226	3.2
cls2-3	96.6	67	606.4	89.2	176	28.1	85.3	343	3.0	90.5	83	4.4	81.2	584	1.7	87.9	221	4.0
cls2-4	100.0	1	145.5	97.7	17	14.1	94.3	28	4.7	94.2	113	4.4	90.5	142	0.5	85.7	172	2.3
cls2-5	83.1	1061	145.2	79.4	1458	15.0	76.1	1615	8.0	80.4	717	4.7	71.2	5430	7.5	74.9	1498	6.1
cls2-6	94.3	30	892.1	89.3	202	8.7	88.4	125	10.2	88.7	253	4.6	79.8	443	1.0	80.1	276	3.6
cls2-7	93.5	76	132.1	88.9	107	14.0	79.7	452	2.6	82.5	229	4.0	79.7	370	1.0	78.5	300	3.3
cls2-8	97.1	25	637.7	84.8	209	18.8	83.1	620	6.1	85.8	173	5.0	76.3	1704	2.3	81.7	301	4.1
cls2-9	92.9	182	167.7	86.6	237	14.4	75.9	1148	5.4	82.6	1271	5.8	74.5	1360	2.0	75.2	738	4.0
cls3-0	97.3	11	562.9	94.7	64	33.4	85.1	124	4.0	87.3	89	4.7	76.2	384	0.9	80.9	135	3.1
cls3-1	96.2	176	448.4	92.8	112	37.0	80.7	808	6.7	83.9	767	5.2	73.5	1034	1.4	80.3	696	4.0
cls3-2	97.4	32	541.7	91.6	97	22.3	82.2	487	6.8	84.7	173	4.3	76.1	1083	2.1	76.5	790	4.2
cls3-3	93.5	81	524.2	85.3	186	32.0	84.2	404	7.1	86.5	229	5.7	72.0	1812	3.1	81.9	447	5.2
cls3-4	97.6	69	517.3	91.9	324	43.6	83.7	819	4.2	90.3	293	4.6	75.2	3106	4.8	85.9	449	4.5
cls3-5	95.6	15	627.1	88.8	285	15.8	91.1	43	16.1	90.0	72	5.8	73.0	667	1.0	82.4	120	4.3
cls3-6	94.8	83	614.4	84.8	713	24.1	84.7	976	11.6	86.1	643	5.4	75.5	3017	3.8	82.2	812	7.0
cls3-7	98.5	14	716.3	97.3	20	29.3	93.4	109	4.8	94.1	54	4.2	76.0	1074	2.0	88.7	91	4.1
cls3-8	99.9	3	553.3	98.0	13	16.8	79.2	261	4.0	91.6	85	5.2	83.1	260	0.7	84.8	131	3.2
cls3-9	96.0	27	397.2	94.7	53	38.1	90.7	116	4.1	91.1	103	6.2	81.9	547	1.6	89.8	107	3.8

problem	k2-lift			k1-lift			k2-nolift			bc-prod			bc-opt			mp-opt		
	gap	nodes	time	gap	nodes	time	gap	nodes	time	gap	nodes	time	gap	nodes	time	gap	nodes	time
cls4-0	100.0	1	100.9	92.6	142	6.5	94.1	64	7.1	97.2	109	4.3	93.0	102	2.7	89.3	244	0.7
cls4-1	100.0	1	156.3	93.1	37	9.2	76.5	359	4.1	83.3	159	4.1	69.6	309	2.7	85.8	369	1.3
cls4-2	94.9	47	133.4	85.2	175	6.1	82.9	180	5.4	86.4	186	4.0	79.3	294	2.8	81.9	706	1.2
cls4-3	99.7	7	501.4	96.6	85	15.0	92.1	136	5.5	89.9	186	4.2	83.4	215	3.1	86.6	256	0.8
cls4-4	100.0	1	258.7	98.7	12	26.2	83.8	171	4.7	89.4	122	3.1	81.0	180	2.3	85.0	138	0.6
cls4-5	91.0	108	66.9	88.0	253	7.9	81.3	971	9.9	79.8	606	5.1	72.4	761	3.6	72.6	1599	3.0
cls4-6	93.6	47	98.2	80.7	394	4.3	73.9	428	8.2	77.7	574	3.8	63.7	740	3.0	81.8	863	1.7
cls4-7	98.1	15	51.9	95.6	45	11.3	84.7	285	3.7	89.2	156	3.8	75.5	488	2.6	91.7	130	0.5
cls4-8	99.2	6	312.6	94.4	37	12.0	90.1	128	8.2	91.4	70	3.5	85.7	145	2.6	88.5	229	0.9
cls4-9	97.6	32	176.6	94.5	59	17.2	89.5	147	3.0	91.4	168	3.9	87.1	167	2.8	82.9	534	1.6
cls5-0	97.2	56	471.9	96.8	61	43.1	91.4	180	7.4	89.5	123	4.3	86.2	169	3.3	83.8	672	1.3
cls5-1	100.0	1	48.6	100.0	1	16.8	98.5	27	3.4	98.9	22	4.1	97.2	23	3.1	90.3	114	0.6
cls5-2	99.5	33	295.1	98.8	31	13.2	93.5	30	2.2	95.2	53	4.0	91.2	80	2.8	88.9	127	0.6
cls5-3	98.9	5	178.7	94.4	161	21.8	85.7	144	4.2	91.8	355	3.9	85.7	417	3.3	85.1	601	1.2
cls5-4	100.0	1	124.7	94.4	106	17.8	94.5	35	7.4	96.0	11	5.0	88.2	42	2.7	86.8	157	0.7
cls5-5	99.5	22	567.6	94.0	189	28.6	89.0	308	11.3	89.9	337	5.0	85.4	336	3.8	76.5	1918	3.2
cls5-6	98.6	40	588.0	89.8	391	18.6	87.2	481	8.8	88.2	337	5.0	85.8	446	3.4	82.6	2016	3.3
cls5-7	100.0	1	156.2	99.5	7	20.2	76.4	1138	5.1	92.6	283	4.9	89.5	245	3.2	86.7	365	1.1
cls5-8	92.7	155	386.9	81.7	630	16.2	77.1	885	9.1	83.6	390	4.1	77.3	543	4.0	81.8	690	1.4
cls5-9	100.0	1	147.4	100.0	1	28.3	96.9	13	6.4	96.8	35	3.9	94.4	34	3.6	87.2	142	0.7
cls6-0	98.7	6	506.4	95.5	166	28.7	92.0	260	7.9	93.4	43	4.9	90.7	69	4.4	78.2	1844	3.7
cls6-1	99.3	7	421.3	97.3	9	26.6	93.0	109	6.2	92.6	183	4.3	90.3	117	4.1	82.0	569	1.5
cls6-2	98.5	61	1149.7	92.8	75	35.1	91.0	228	11.5	90.8	264	5.7	87.5	502	5.2	81.9	1321	2.5
cls6-3	97.9	63	531.4	96.5	61	50.5	90.4	469	5.9	90.4	267	5.5	89.0	143	4.2	79.2	1798	3.0
cls6-4	98.5	13	431.2	92.4	162	14.4	87.0	243	8.5	89.9	354	4.9	83.9	501	4.3	76.0	2284	3.8
cls6-5	100.0	1	78.7	100.0	1	12.0	100.0	1	3.6	100.0	1	2.9	96.7	7	3.9	92.7	46	0.5
cls6-6	98.5	36	493.0	95.2	210	37.7	90.5	475	9.8	90.9	384	5.4	88.2	195	4.4	84.5	1324	2.2
cls6-7	99.0	13	562.1	97.7	51	28.8	94.5	71	4.7	95.8	46	4.9	92.1	49	3.5	87.7	665	1.5
cls6-8	99.3	13	517.2	94.2	51	30.2	94.1	33	7.3	91.6	68	3.9	88.7	171	3.7	82.8	237	0.9
cls6-9	98.0	15	962.8	95.3	146	36.7	94.3	105	8.7	93.8	91	4.0	92.9	116	3.9	84.9	644	1.1