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DISCUSSION PAPER

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Modeling Poset Convex Subsets

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Abstract

A subset S of a poset (partially ordered set) is *convex* if and only if S contains every poset element which is between any two elements in S. Poset convex subsets arise in applications that involve precedence constraints, such as in project scheduling, production planning, and assembly line balancing. We give a strongly polynomial time algorithm which, given a poset and element weights (of arbitrary sign), finds a convex subset with maximum total weight. This algorithm relies on a reduction to a maximum weight filter (or closure) problem in a poset about twice the size of the given poset; the latter problem is well-solved as a minimum s-t cut problem. We also use this reduction to construct a compact, ideal extended formulation for the convex hull C_P of the characteristic vectors of all convex subsets in poset P. We define a class of *alternating inequalities* that are valid for C_P and admit a linear time separation algorithm based on Dynamic Programming (DP). Furthermore, whenever the point to separate is actually in C_P the associated DP value functions induce a feasible solution to the extended formulation. This implies that the alternating inequalities and nonnegative inequalities suffice to describe C_P. We conclude by showing that this polyhedral description is minimal, and thus also admits a linear time separation algorithm.

Keywords: partial order, convex subsets, extended formulation, convex hull, separation algorithms, dynamic programming

AMS 2010 Mathematics Subject Classification: 06A06, 90C27, 90C57

1 Introduction

We consider the following question:

Polyhedral Description: Given a finite poset $P = (V, \leq_P)$, determine an explicit, finite system of linear inequalities describing the convex hull of the characteristic vectors of all (poset) convex subsets in P.

A subset $S \subseteq V$ of elements of a poset (partially ordered set) $P = (V, \leq_P)$ is *convex* if and only if it contains every poset element that is between any two elements in S, i.e., for every $t, v \in S$ and $u \in V$, $t \leq_P u \leq_P v$ implies $u \in S$ (e.g., [3]). Recall that the *characteristic vector* of a subset $S \subseteq V$ is the vector $\chi^S \in \mathbb{R}^V$ with components $\chi^S_u = 1$ if $u \in S$, and 0 otherwise. Let $C_P \subseteq \mathbb{R}^V$ denote the convex hull of the characteristic vectors of all convex subsets in P.

Using suitable decision variables and linear inequalities to represent subsets that are "convex", according to various notions of convexity, are modeling decisions that arise when developing integer programming models for problems of selecting a single subset, or of partitioning a given set into subsets, that should satisfy certain "shape requirements". Examples, involving various notions of "convex" subsets, include political districting, spatial planning, police quadrant design, forest planning, and statistical clustering, see [15] for details and references. Shape and related structural constraints may often be captured by appropriate notions from *abstract convexity* (see, e.g., Duchet's survey [4] and van de Vels's extensive monograph [19] on abstract convexity). Note that poset convexity corresponds to *geodesic* (shortest path) *convexity* [12] in the Hasse diagram (transitive reduction) of P in which every arc has zero length. Poset convexity requirements arise in applications that involve precedence constraints, such as in project scheduling, production planning, and assembly line balancing. Indeed, the present polyhedral description problem was posed to the first author by Frédéric Meunier [11] in relation with an assembly line balancing problem that will be outlined below.

Our main result (Theorem 8 below) is a complete answer to the Polyhedral Description question, giving the minimal system of linear inequalities defining the convex hull C_P. This system consists of all nonnegativity constraints and *alternating inequalities*, a family of linear inequalities generalizing those introduced in [6, 9, 16] for special cases. In order to prove this main result, we consider three related problems as follows.

As is often the case in combinatorial optimization, the number of linear inequalities in the minimal system defining C_P is exponential in the input size of the poset P. A common computational strategy to deal with such a large number of inequalities is a cutting plane approach, which relies on the ability to solve the following

Separation Problem: Given a finite poset $P = (V, \leq_P)$ and a vector $\bar{x} \in \mathbb{R}^V$, decide whether $\bar{x} \in C_P$ and, if not, produce a linear inequality that is satisfied by the characteristic vectors of all convex subsets in P and is violated by $\bar{x}$.

A closely related problem, given the polynomial equivalence of separation and optimization [7], is the following:

Optimization Problem: Given a finite poset $P = (V, \leq_P)$ and a weight vector $w \in \mathbb{R}^V$, find a convex subset $S^* \subseteq V$ with maximum total weight $w(S^*) = \sum_{u \in S^*} w(u)$.

Note that $S^* = V$ (resp., $S^* = \emptyset$) is optimum when w is nonnegative (resp., nonpositive), so the nontrivial instances arise when the element weights $w(u)$ have arbitrary sign.

The *subcontractor work package design* problem in project scheduling is a direct application of this Optimization Problem: a *project* P consists of a finite set V of activities subject to technological (and logical) precedence constraints $\leq_P$, whereby $u \leq_P v$ represents the constraint that activity u must be completed before activity v can start. (Thus the Hasse diagram of poset P is the *activity-on-nodes* network representation of the project.) A subset S of activities may be assigned to a subcontractor provided the latter can perform all of them without interruption, i.e., provided there are no activities t, u, v such that $t \leq_P u \leq_P v$, t and v are assigned to the subcontractor but u is not; in other words, if and only if S is a convex subset in P, called a *work package*. The subcontractor may have better equipment, workforce or experience in dealing with certain activities. Thus let $w(u)$ denote the benefit from subcontracting activity u, for example $w(u)$ may be the difference between the direct cost of u to the project owner and the price charged for u by the subcontractor. Determining a work package S^* to assign to the subcontractor so as to achieve maximum total benefit $w(S^*)$, is thus an instance of the poset convex optimization problem.

In contrast to one-time projects, assembly line balancing arises in repetitive manufacturing environments. In the problem considered by El Lemdani [10], a set V of tasks, partially ordered by technological precedence constraints, must be partitioned and assigned to given workstations such that, as in the subcontractor work package design problem, the subset of tasks assigned to each workstation may be performed without interruption, i.e., is a convex subset. There are additional constraints within and between workstations. The column generation approach applied to a set partitioning formulation requires finding maximum weight convex task subsets where the task weights are derived from task data and dual variables (from a linear programming relaxation). For this, El Lemdani [10] uses the optimization algorithm described below (communicated to him by the first author).

When dealing with a large (exponential) system of inequalities such as that describing C_P, an alternative to the separation approach is to introduce additional (auxiliary) variables and inequalities that may capture the underlying combinatorial structure more concisely. An *extended formulation* of a polyhedron $C \subseteq \mathbb{R}^V$ is a polyhedron $Q \in \mathbb{R}^V \times \mathbb{R}^Y$ in a higher dimensional space such that C is the projection of Q onto the coordinate subspace $\mathbb{R}^V$ of the *natural variables.* It is *compact* if it is defined by a number $|Y|$ of additional variables and a number of linear inequalities that are polynomial in the input size describing C. Here, the polyhedron is defined by the poset $P = (V, \leq_P)$, for which we can take as input size that of its Hasse diagram, i.e., the total number $|V|$ of elements, plus the total number of irredundant relations in P. The variables and inequalities in a compact extended formulation may be used in integer programming formulations of more complicated problems involving the polyhedron C or the underlying combinatorial structure. This leads to the following question:

Extended Formulation: Given a finite poset P, determine a compact extended formulation of the convex hull C_P of the characteristic vectors of all convex subsets in P.

We settle all these four questions and problems. In Section 2 we show that the maximum-weight poset convex subset problem in a poset P can be solved in polynomial time by reduction to a well-solved problem, the maximum-weight *filter* (or *closure*) problem, in a poset of about twice the size of P. This reduction leads in Section 3 to a compact extended formulation for poset convex subsets. This extended formulation is actually of linear size, with three bounded variables and one equation per poset element, and two inequalities per irredundant relation in P. It is also “ideal” in the sense of [20], in that its extreme points are integer, and are precisely the characteristic vectors of all poset convex subsets. In Section 4 we introduce a class of *alternating inequalities* that are valid for poset convex subsets, and for which the separation problem can be solved in linear time by Dynamic Programming (DP). In Section 5 we show that when the point $\bar{x}$ to be separated is actually in C_P the components of the corresponding DP value functions define a feasible solution to the extended formulation. This allows us to conclude that the alternating inequalities, together with nonnegativity, define a complete polyhedral description for poset convex subsets.

2 The Maximum-Weight Poset Convex Subset Problem

The notion of convex subsets in posets is closely related to the notions of ideals and filters in posets, and closures in digraphs. Recall (e.g., [3]) that a subset $I \subseteq V$ of a poset is an *ideal* (a *down-set*; a *decreasing* set; a *lower ideal*) if and only if it contains every poset element lower than (a *predecessor*, or *ancestor*, of) every element in I; i.e., whenever $u \leq_P v$, $v \in I$ implies $u \in I$. Dually (in the poset sense), $F \subseteq V$ is a *filter* (an *up-set*; an *increasing*, or *hereditary*, set; an *upper ideal*) if and only if it contains every poset element higher than (a *successor*, or *descendant*, of) every element in F; i.e., whenever $u \leq_P v$, $u \in F$ implies $v \in F$. Note that F is a filter if and only if its complement $V \setminus F$ is an ideal.

Lemma 1. *For any poset $P = (V, \leq_P)$ and subset $S \subseteq V$ the following are equivalent:*

(i) S is poset convex;

(ii) $S = I \cap F$ for some ideal I and filter F in P;

(iii) $S = F \setminus \tilde{F}$ for some filters F and $\tilde{F}$ in P.

(iv) $S = F \setminus \tilde{\tilde{F}}$ for some filters $\tilde{\tilde{F}} \subseteq F$ in P.

Remark. By complementation, or poset duality, statements *(iii)* and *(iv)* above are also equivalent to

(iii') *$S = \tilde{I} \setminus I$ for some ideals I and $\tilde{I}$ in P;* and

(iv') *$S = \tilde{\tilde{I}} \setminus I$ for some ideals $I \subseteq \tilde{\tilde{I}}$ in P.*

Proof of Lemma 1. (i) $\Rightarrow$ (ii) By transitivity of $\leq_P$, the set $I = \{v \in V : v \leq_P s \text{ for some } s \in S\}$ is an ideal, whereas $F = \{u \in V : t \leq_P u \text{ for some } t \in S\}$ is a filter. Since $S \subseteq I$ and $S \subseteq F$, we have $S \subseteq I \cap F$. Conversely, for every $u \in I \cap F$ there exist $t, s \in S$ such that $t \leq_P u \leq_P s$, and thus, since S is convex, $u \in S$. This shows that $I \cap F \subseteq S$, and therefore $S = I \cap F$.

(ii) $\Rightarrow$ *(iii)* The complement $\tilde{F} = V \setminus I$ is a filter, and $F \setminus \tilde{F} = F \cap I = S$.

(iii) $\Rightarrow$ *(iv)* The intersection $\tilde{\tilde{F}} = F \cap \tilde{F}$ is a filter contained in F, and $S = F \setminus \tilde{F} = F \setminus \tilde{\tilde{F}}$.

(iv) $\Rightarrow$ *(i)* For all $t, s \in S = F \setminus \tilde{\tilde{F}}$ and $u \in V$ such that $t \leq_P u \leq_P s$, the conditions $t \in F$ and $t \leq_P u$ imply $u \in F$, whereas $s \in V \setminus \tilde{\tilde{F}}$, an ideal, and $u \leq_P s$ imply $u \in V \setminus \tilde{\tilde{F}}$; therefore $u \in F \setminus \tilde{\tilde{F}} = S$. This shows that S is poset convex. $\square$

An associated optimization problem is:

Maximum Weight Filter Problem: Given a finite poset $P = (V, \leq_P)$ and a weight vector $w \in \mathbb{R}^V$, find a filter $S^* \subseteq V$ with maximum total weight $w(S^*)$.

The maximum weight *ideal* problem is similarly defined. By complementation, these two problems are equivalent. As for the maximum weight poset convex problem, these problems are trivial if all weights $w(u)$ have the same sign (all nonnegative, or all nonpositive). The maximum weight ideal (or filter) problem is equivalent to the problem of minimizing a modular function over a finite distributive lattice, see, e.g., [5, Section 7.1(b)]. The maximum weight filter problem is also equivalent to the *maximum weight closure problem* in acyclic digraphs (directed graphs), and it is well-solved by reduction to a minimum s-t cut problem on an acyclic network of about the same size as the Hasse diagram of the poset. Indeed, a *closure* (e.g., [13], [1, Section 19.2], [8]) in an acyclic digraph $G = (V, A)$ is a filter in the associated poset $(V, \leq_G)$ wherein $u \leq_G v$ if and only if there exists a directed path from u to v in G. The bipartite digraph version of these problems was introduced by Rhys [17] to model the optimum selection of shared fixed costs in freight handling terminals and, more generally, the selection of indivisible activities and facilities to maximize the excess of benefits over costs. Rhys also showed a reduction to the minimum s-t cut problem in an associated network. Balinski [2] noted a simple and direct connection between selections and such s-t cuts. Picard [13] generalized selections in bipartite digraphs to closures in general digraphs, for which he extended Rhys and Balinski's reduction to s-t cuts. He also noted an application to open pit mine design. Further applications of maximum weight closures (and poset ideals and filters) are described in Picard and Queyranne [14], Ahuja et al. [1] and Hochbaum [8].

For the sake of completeness we outline this reduction of the maximum weight filter problem to a minimum s-t cut problem. Given poset $P = (V, \leq_P)$ and weight vector $w \in \mathbb{R}^V$, construct a digraph $\hat{G} = (\hat{V}, \hat{A})$ with node set $\hat{V} = V \cup \{s, t\}$, where s and t are two new nodes representing a source and a sink, respectively, and three types of arcs, $\hat{A} = A_s \cup A_P \cup A_t$ with capacities defined as follows. There is a source arc $(s, u) \in A_s$ with capacity $c(s, u) = w(u)$ for every $u \in V$ with $w(u) > 0$; a sink arc $(v, t) \in A_t$ with capacity $c(v, t) = -w(v)$ for every $v \in V$ with $w(v) < 0$; and a precedence arc $(u, v) \in A_P$ with infinite capacity[1], $c(u, v) = +\infty$, for every irredundant[2] relation $u <_P v$ (i.e., every pair $(u, v) \in V^2$ such that v *covers* u in P, see, e.g., [3]). Recall that an s-t cut $(C, \hat{V} \setminus C)$ is defined by any subset $C \subset \hat{V}$ such that $s \in C$ and $t \notin C$, and its capacity is $c(C, \hat{V} \setminus C) = \sum\{c(a, b) : (a, b) \in \hat{A},\ a \in C \text{ and } b \notin C\}$. Note that an s-t cut $(C, \hat{V} \setminus C)$ has finite capacity if and only if $S = C \setminus \{s\}$ is a filter in P, defining a one-to-one correspondence between finite-capacity s-t cuts and filters in P, and that $c(C, \hat{V} \setminus C) = w^+(V) - w(S)$ where $S = C \setminus \{s\}$ and the constant $w^+(V) = \sum_{u \in V} \max\{0, w(u)\}$ is independent of S. Thus minimum capacity s-t cuts correspond precisely to maximum weight filters.

We now present a reduction, inspired by the equivalences in Lemma 1, of the maximum weight poset convex subset problem (MWPCSP) to the maximum weight filter problem (MWFP) in a poset about twice the size of P and with similar weights. To a given MWPCSP instance (P, w) where $P = (V, \leq_P)$, associate the MWFP instance defined by the poset $Q = (U, \leq_Q)$ made of two copies $P' = (V', \leq_{P'})$ and $P'' = (V'', \leq_{P''})$ of P, i.e., $U = V' \cup V''$, in which each element $v \in V$ has two copies $v' \in V'$ and $v'' \in V''$ such that $v' \leq_Q v''$ and every relation $u \leq_P v$ induces the relations $u' \leq_Q v'$, $u' \leq_Q v''$ and $u'' \leq_Q v''$ in Q. Note that the Hasse diagram of Q consists of the two copies in P' and P'' of the Hasse diagram of P, connected by the relations $v' \leq_Q v''$ for all $v \in V$. The weights are $w_Q(v') = -w(v)$ et $w_Q(v'') = w(v)$.

Lemma 2. *Given a finite poset $P = (V, \leq_P)$ and a weight vector $w \in \mathbb{R}^V$, a subset $T \subseteq U$ is a maximum weight filter in the associated poset $Q = (V' \cup V'', \leq_Q)$ for the weights w_Q defined above, if and only if the set*

$$S = \{v \in V : v' \notin T \text{ and } v'' \in T\} \tag{1}$$

[1]Note that any finite values $c(u, v) > \sum_{u \in V} |w(u)|$ would suffice.

[2]Precedence arcs (u, v) for redundant (i.e., implied by transitivity) relations $u <_P v$ may also be included, if convenient, so the effort to compute the transitive reduction, or Hasse diagram, of P is actually not needed.

is a maximum weight convex subset for the original weights w in poset P.

Proof. Let $w^*(P)$ and $w^*_Q(Q)$ denote the maximum weight of a convex subset in P and of a filter in Q, respectively. Let S^* be a maximum weight convex subset for instance (P, w). By Lemma 1, $S^* = F \setminus \tilde{\tilde{F}}$ for some filters $\tilde{\tilde{F}} \subseteq F$ in P. Let T' be the copy of $\tilde{\tilde{F}}$ in V' and T'' that of F in V''. Since $\tilde{\tilde{F}}$ and F are filters in P and $\tilde{\tilde{F}} \subseteq F$, it follows that $T = T' \cup T''$ is a filter in Q, $S^* = \{v \in V : v' \notin T \text{ and } v'' \in T\}$ as in (1), and $w_Q(T) = w(S^*) = w^*(P)$. Therefore $w^*_Q(Q) \geq w_Q(T) = w^*(P)$.

Conversely, let T be a maximum weight filter in Q for the weights w_Q and S as defined in (1). Since $\{v \in V : v' \in T\} \subseteq \{v \in V : v'' \in T\}$, we have $w(S) = w_Q(T) = w^*_Q(Q)$. This implies $w(S) = w_Q(T)$. Since V'' is a filter in Q, $F'' = T \cap V''$ is also a filter in Q, and $F = \{v \in V : v'' \in T\}$ is a filter in P. Likewise, V' and $U \setminus T$ are ideals in Q, so $I' = V' \cap (U \setminus T)$ is also an ideal in Q, and thus $I = \{v \in V : v' \notin T\}$ is an ideal in P. By Lemma 1, $S = I \cap F$ is convex in P. This implies $w^*(P) \geq w(S) = w^*_Q(Q)$. Therefore $w^*(P) = w^*_Q(Q)$ and Lemma 2 follows. □

Since the minimum capacity s-t cut problem can be solved in strongly polynomial time (see, e.g., [1, 18] for references) we have:

Theorem 3. *The Maximum-Weight Poset Convex Subset Problem is solvable in strongly polynomial time.*

This solves the Optimization Problem posed in the Introduction.

3 Extended Formulation

It is well known (e.g., [13, 14, 8]) that digraph closures, and thus poset filters, admit a compact formulation which is "ideal" in the sense of [20]: the convex hull of the characteristic vectors of filters in a poset P is the polytope

$$D_P = \left\{ y \in \mathbb{R}^V : 0 \leq y \leq 1, \text{ and } y_u \leq y_v \text{ for all } u <_P v \right\}. \tag{2}$$

This follows from the reduction to s-t cuts described above or, more directly, from the total unimodularity (dual network flow structure) of the constraint matrix in (2).

To apply this to poset Q of Lemma 2, write any vector $y \in \mathbb{R}^U$ as $y = (y', y'')$ where $y'_v = y_{v'}$ and $y''_v = y_{v''}$ for all $v \in V$. The convex hull of the characteristic vectors of filters in Q is thus

$$D_Q = \big\{(y', y'') \in \mathbb{R}^{V\times V} \ : \ 0 \le y' \le y'' \le 1 \tag{3}$$

$$y'_u \le y'_v \text{ and } y''_u \le y''_v \ \text{ for all } u <_P v\big\}\,. \tag{4}$$

(Referring back to the definition of Q, note that for every $u <_P v$ the inequality $y'_u \le y''_v$ is indeed implied by $y'_u \le y''_u$ in (3) and $y''_u \le y''_v$ in (4), so it need not be explicitly added to (3)–(4).)

Every convex subset S in P is the (unique) maximum weight convex subset for some weight vector w (e.g., with $w_v = 1$ if $v \in S$, and -1 otherwise). As in Lemma 2, let T be the corresponding maximum weight filter in Q, and let $y = \chi^T \in \mathbb{R}^U$ be its characteristic vector. By (1) the characteristic vector $x = \chi^S \in \mathbb{R}^V$ of S satisfies $x = y'' - y'$. Adding to (3)–(4) the variables x and the defining equations $x = y'' - y'$ does not affect the integrality of extreme points. This observation and Lemma 2 then imply:

Theorem 4. *Given a poset P, let*

$$E_P = \Big\{(x, y', y'') \in \mathbb{R}^{V\times V\times V} \ : \ x = y'' - y' \tag{5}$$

$$y'_u \le y'_v \text{ and } y''_u \le y''_v \ \text{ for all } u <_P v \tag{6}$$

$$0 \le y' \le y'' \le 1\Big\}. \tag{7}$$

All the extreme points of E_P are integer and its projection $\mathrm{proj}_x E_P \subset \mathbb{R}^V$ onto the coordinate subspace of the natural variables x is the convex hull C_P of the characteristic vectors of all convex subsets in P.

Thus the system (5)–(7) gives a compact, linear-sized extended formulation of the poset convex subsets, which is "ideal" in the sense of [20]. This answers the Extended Formulation question posed in the Introduction.

4 Alternating Inequalities and their Separation

A *chain* in a poset $P = (V, \le_P)$ is a sequence $c = (c_1, \ldots, c_{\ell(c)})$ of elements such that $c_1 <_P c_2 <_P \cdots <_P c_{\ell(c)}$. Thus a chain is (the node set) of a directed path in the associated acyclic digraph (but not necessarily in its Hasse diagram). The special case when P is itself a chain (i.e., P is totally ordered) is also a special case of structures studied in [6, 9, 16]. In this Section we consider alternating inequalities, which play an important role as well in [6, 9, 16].

The *alternating vector* $\alpha^c \in \mathbb{R}^V$ associated with chain $c = (c_1, \ldots, c_{\ell(c)})$ has components $\alpha^c_u = +1$ if $u = c_i$ for some odd i; -1 if $u = c_i$ for some even i; and 0 otherwise (i.e., if u does not belong to the chain). A chain c is *odd* if its *length* $\ell(c)$ is odd. Let $\mathrm{Odd}(P)$ denote the set of all odd chains in P.

Lemma 5. *The characteristic vector of every convex subset in a poset P satisfies the* alternating inequalities

$$\alpha^c \cdot x \leq 1 \qquad \text{for all } c \in \mathrm{Odd}(P). \tag{8}$$

Proof. Let chain $c \in \mathrm{Odd}(P)$. It suffices to consider any convex subset in P which contains at least two elements of the chain. The corresponding least and largest indices, $i(c, S) = \min\{j \in \{1, \ldots, \ell(c)\} : c_j \in S\}$ and $k(c, S) = \max\{j \in \{1, \ldots, \ell(c)\} : c_j \in S\}$, satisfy $1 \leq i(c, S) < k(c, S) \leq \ell(c)$. Since S is convex it must contain every chain element c_j between $c_{i(c,S)}$ and $c_{j(c,S)}$ and its characteristic vector x satisfies $\alpha^c \cdot x = \sum_{j=i(c,S)}^{k(c,S)} \alpha^c_{c_j} \leq 1$. □

Note that, by letting $c = (v)$, inequalities (8) include the simple upper bound constraints $x_v \leq 1$. Note also that there may be an exponential number of alternating inequalities. This naturally leads us to consider the following:

Separation Problem for the Alternating Inequalities: Given a vector $\bar{x} \in \mathbb{R}^V$, decide whether $\bar{x}$ satisfies (8) and, if not, produce a violated inequality, i.e., a chain $c \in \mathrm{Odd}(P)$ such that $\alpha^c \cdot \bar{x} > 1$.

The separation problem for the alternating inequalities may be solved by Dynamic Programming. For this, we introduce a few additional definitions. Let $\mathrm{Pred}(v) = \{u \in V : u <_P v\}$ denote the set of all (strict) *predecessors* of $v \in V$, and $\mathrm{Pred}^*(v) = \mathrm{Pred}(v) \setminus \bigcup_{u \in \mathrm{Pred}(v)} \mathrm{Pred}(u)$ that of its *immediate predecessors* (i.e., $\mathrm{Pred}^*(v)$ is the set of all elements *covered* by v). A chain c *ends at or before* v if its last element $c_{\ell(c)} \leq_P v$. Let $\mathrm{Odd}(v)$ denote the set of all odd chains in P that end at or before v. Observe that, if chain $c \in \mathrm{Odd}(v)$ then either $c_{\ell(c)} = v$ and c consists of a (possibly empty) chain of even length that ends at or before some immediate predecessor of v, followed by v itself; or else $c \in \mathrm{Odd}(u)$ for some immediate predecessor u of v. Similarly, let $\mathrm{Even}(v)$ denote the set of all chains in P, including the empty chain $c = ()$ (which is thus considered to end at or before every poset element), of even length that end at or before v.

Given a poset P and a vector $\bar{x} \in \mathbb{R}^V$, define the DP value functions F and $G : V \mapsto \mathbb{R}$ where $F(v)$ (resp., $G(v)$) is the maximum value of $\alpha^c \cdot \bar{x}$

among all chains $c \in \mathrm{Odd}(v)$ (resp., $c \in \mathrm{Even}(v)$). Thus $\bar{x}$ satisfies (8) if and only if $F(v) \leq 1$ for all $v \in V$. By the observation in the preceding paragraph, these value functions satisfy the following initial conditions and (interleaved) DP recursions:

$$F(v) = \begin{cases} \bar{x}_v & \text{if } \mathrm{Pred}^*(v) = \emptyset; \\ \max\big\{ \max\{F(u),\, G(u) + \bar{x}_v\} : u \in \mathrm{Pred}^*(v)\big\} & \text{otherwise;} \end{cases} \tag{9}$$

and

$$G(v) = \begin{cases} 0 & \text{if } \mathrm{Pred}^*(v) = \emptyset; \\ \max\big\{ \max\{G(u),\, F(u) - \bar{x}_v\} : u \in \mathrm{Pred}^*(v)\big\} & \text{otherwise.} \end{cases} \tag{10}$$

If $\bar{x}$ violates an alternating inequality (8) then $F(\bar{v}) > 1$ for some $\bar{v} \in V$ and we can trace back in the usual way an odd chain $\bar{c}$ that ends at or before $\bar{v}$ and for which $\alpha^{\bar{c}} \cdot \bar{x} = F(\bar{v}) > 1$, i.e., an alternating inequality violated by $\bar{x}$. Thus we have:

Theorem 6. *Using the DP recursions* (9)–(10)*, the separation problem for the alternating inequalities* (8) *is solvable in time linear in the size of poset P.*

5 Polyhedral Description in the Space of Natural Variables

The DP approach to the separation problem for the alternating inequalities leads to a simple proof (as in [16]) that these inequalities give a complete polyhedral description of the poset convex subsets, in the space of the natural variables x.

Lemma 7. *Given $\bar{x} \in \mathbb{R}^V$, let $\bar{y'}_v = G(v)$ and $\bar{y''}_v = F(v)$ for all $v \in V$. If $\bar{x} \geq 0$ and $\bar{y''} \leq 1$ then $(\bar{x}, \bar{y'}, \bar{y''}) \in E_P$.*

Proof. We first note a simple consequence of the DP recursions (9)–(10). For any $v \in V$ such that $\mathrm{Pred}^*(v) = \emptyset$ we trivially have $F(v) = \bar{x}_v + G(v)$. Otherwise, extracting $\bar{x}_v$ from both arguments of the inner maximum in (9), we have

$$F(v) = \bar{x}_v + \max\big\{ \max\{F(u) - \bar{x}_v,\, G(u)\} : u \in \mathrm{Pred}^*(v)\big\} = \bar{x}_v + G(v).$$

Hence the DP value functions F and G satisfy $F(v) = G(v) + \bar{x}_v$ for all $v \in V$. With the notation of the present lemma, this is $\bar{y''} - \bar{y'} = \bar{x}$, verifying equation (5). With $\bar{x} \geq 0$ this in turn implies $\bar{y''} \geq \bar{y'}$ and, with $\bar{y''} \leq 1$, all the inequalities in (7) are satisfied. For every $u <_P v$, by transitivity (9) implies $y''_v \geq y''_u$ and (10) implies $y'_v \geq y'_u$, satisfying constraints (6). $\square$

Theorem 8. *Let P be a given finite poset.*

(i) The alternating inequalities (8) *plus the non-negativity constraints $x \geq 0$ form the minimal linear inequality system (which is unique up to scalar multiples) defining the convex hull of the characteristic vectors of all convex subsets in P.*

(ii) The separation problem for this convex hull is solvable in linear time.

Proof. (i) Given any $\bar{x} \in \mathbb{R}^V$, let $\bar{y'}$ and $\bar{y''}$ be as defined in Lemma 7. If $\bar{x} \geq 0$ and $\bar{y''} \leq 1$ then, by Lemma 7 and Theorem 4, $\bar{x} \in C_P$. Otherwise, $\bar{x}$ violates an inequality that is valid for C_P. This implies that $\bar{x} \in C_P$ if and only if it satisfies $\bar{x} \geq 0$ and the alternating inequalities (8).

We now show that this defining system, $x \geq 0$ and (8), is minimal, i.e., that no inequality in this system is redundant. First, note that the empty set and all the singleton sets $\{v\}$ $(v \in V)$ are poset convex, and their characteristic vectors $\chi^{\emptyset} = 0$ and $\chi^{\{v\}} = e^v$ (the v-th unit vector) form $|V| + 1$ affinely independent points in C_P, therefore C_P is full-dimensional. Thus it suffices to show that each inequality in $x \geq 0$ and (8) uniquely defines a facet of C_P.

For every $v \in V$ the $|V|$ affinely independent vectors $\chi^{\emptyset}$ and $\chi^{\{u\}}$ for all $u \in V \setminus \{v\}$ are on the face $\{x \in C_P : x_v = 0\}$ of C_P induced by the nonnegativity inequality $x_v \geq 0$, hence this face is a facet of C_P.

Now consider any chain $c = (c_1, \ldots, c_{\ell(c)}) \in \mathrm{Odd}(P)$ and the face $Q_c = \{x \in C_P : \alpha^c \cdot x = 1\}$ of C_P induced by the corresponding alternating inequality. Since face Q_c is contained in at least one facet of C_P, let $\beta \cdot x \leq \beta_0$ denote a valid inequality for C_P that induces such a facet Q^{β} containing Q_c. Since $0 \in C_P \setminus Q_{\beta}$ we have $\beta_0 > \beta \cdot 0 = 0$ and we may, w.l.o.g., scale the inequality so $\beta_0 = 1$. We now prove that we must have $\beta = \alpha^c$. For this, let $E(c) = \{c_{2i} : 1 \leq i \leq (\ell(c) - 1)/2\} = \{c_2, c_4, \ldots, c_{\ell(c)-1}\}$ (with $E(c) = \emptyset$ if $\ell(c) = 1$) and $O(c) = \{c_{2i+1} : 0 \leq i \leq (\ell(c) - 1)/2\} = \{c_1, c_3, \ldots, c_{\ell(c)}\}$ denote the sets of *even* and *odd* elements of c, respectively, and let $V \setminus c = V \setminus (O(c) \cup E(c))$ be the set of all other poset elements. Thus $\alpha^c_u = 1$ for all $u \in O(c)$; -1 for all $u \in E(c)$; and 0 for all $u \in V \setminus c$. Note that for any $t <_P v$ the *interval* $[t, v] = \{u \in V : t \leq_P u \leq_P v\}$ is poset convex. We consider the following cases for all $u \in V$.

1. If $u \in O(c)$ then $\alpha^c \cdot \chi^{\{u\}} = 1$, hence $\chi^{\{u\}} \in Q_c \subseteq Q^\beta$ and thus $\beta_u = \beta \cdot \chi^{\{u\}} = 1 = \alpha^c_u$.

2. If $u \in V \setminus c$ is incomparable with at least one $v \in O(c)$ (that is, satisfies neither $u \leq_P v$ nor $v \leq_P u$ for some $v \in O(c)$), then $\{u, v\}$ is a convex subset of P and $\alpha^c \cdot \chi^{\{u,v\}} = 1$. Hence $e^u + e^v = \chi^{\{u,v\}} \in Q_c \subseteq Q^\beta$ and thus $\beta_u = \beta \cdot \chi^{\{u,v\}} - \beta \cdot e^v = 1 - 1 = 0 = \alpha^c_u$.

3. Let $\Gamma(c)$ denote the set of all $u \in V \setminus c$ that are comparable with all $v \in O(c)$. Thus for every $u \in \Gamma(c)$ the set $O(c) \cup \{u\}$ forms a chain in P. The sets $B_1 = \{u \in V : u <_P c_1\}$ and $A_k = \{u \in V : c_{\ell(c)} <_P u\}$, where $k = \lceil \ell(c)/2 \rceil$, are contained in $\Gamma(c)$. If $u \in \Gamma(c) \setminus (B_1 \cup A_k)$ then $c_{2i-1} <_P u <_P c_{2i+1}$ for some $i \in \{1, \dots, k-1\}$. Let $\Gamma_i(c) = \Gamma(c) \cap [c_{2i-1}, c_{2i+1}]$, $A_i = \{u \in \Gamma_i(c) : c_{2i} \notin [c_{2i-1}, u]\}$ and $B_{i+1} = \{u \in \Gamma_i(c) : c_{2i} \notin [u, c_{2i+1}]\}$, so $\Gamma_i(c) = A_i \cup B_{i+1}$ (note, however, that A_i and B_{i+1} are not necessarily disjoint) and $\Gamma(c) = \bigcup_{i=1}^{k} (A_i \cup B_i)$. We prove that $\beta_u = 0$ for all $u \in A_i$ by induction on the distance d (i.e., the shortest length of a maximal chain) from $s = c_{2i-1}$ to u in P. For the base case, if $d = 1$, i.e., u is an immediate successor of s, then $\{s, u\}$ is poset convex and $\alpha^c \cdot \chi^{\{s,u\}} = 1$. Hence $\chi^{\{s,u\}} \in Q^\beta$ and $\beta_u = \beta \cdot \chi^{\{s,u\}} - \beta \cdot e^s = 0$. In the inductive step, assume that $\beta_t = 0$ for all $t \in A_i$ at distance at most $d - 1 \geq 1$ from s, and consider $u \in A_i$ at distance d from s. Thus all $t \in [s, u] \setminus \{s, u\}$ have $\beta_t = 0$. Since $[s, u]$ is poset convex and $\alpha^c \cdot \chi^{[s,u]} = 1$, we have $\chi^{[s,u]} \in Q^\beta$ and $\beta_u = \beta \cdot \chi^{[s,u]} - \beta \cdot e^s = 0$, as claimed. A dual argument applies to each B_i to show that $\beta_u = 0$ for all $u \in B_i$. Therefore $\beta_u = 0$ for all $u \in \Gamma(c)$.

4. Otherwise, $u = c_{2i} \in E(c)$. Then, letting $s = c_{2i-1}$ and $v = c_{2i+1}$, the interval $[s, v]$ is poset convex and $\alpha^c \cdot \chi^{[s,v]} = 1$, hence $\chi^{[s,v]} \in Q^\beta$ and $\beta_u = \beta \cdot \chi^{[s,v]} - \beta \cdot e^s - \beta \cdot e^v = -1 = \alpha^c_u$.

This completes the proof that every alternating inequality induces a distinct facet of C_P. Since the facets induced by the nonnegativity inequalities are all distinct, and distinct from those induced by the alternating inequalities (the former contain the origin 0, whereas the latter do not), this implies that these inequalities, $x \geq 0$ and (8), form the (unique up to scalar multiples) minimal linear system defining C_P.

Part *(ii)* now follows from *(i)* and Theorem 6. $\square$

In summary, we have answered all the questions posed in the Introduction. Theorem 8 fully answers the *Polyhedral Description* question and, with

Theorem 6, yields a linear time algorithm for the *Separation Problem.* We proved Theorem 8 by means of Theorem 4, which answers the *Extended Formulation* question with a linear-sized extended formulation which is “ideal” in the sense of [20], itself a consequence of the reduction in Lemma 2 which also yields a strongly polynomial time algorithm for the *Optimization Problem.*

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