

Almost sure Stability of Stochastic Switched Systems: Graph lifts-based Approach

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Abstract—In this paper, we develop tools to establish almost sure stability of stochastic switched systems whose switching signal is constrained by an automaton. After having provided the necessary generalizations of existing results in the setting of stochastic graphs, we provide a characterization of almost sure stability in terms of multiple Lyapunov functions. We introduce the concept of *lifts*, providing formal expansions of stochastic graphs, together with the guarantee of conserving the underlying probability framework. We show how these techniques, firstly introduced in the deterministic setting, provide hierarchical methods in order to compute tight upper bounds for the almost sure decay rate. The theoretical developments are finally illustrated via a numerical example.

I. INTRODUCTION

Switched linear systems provide a fruitful mathematical model for a large class of physical systems and have been the center of intense research in the last decades, for an overview, see [1]. In this framework, given M linear subdynamics $A_1, \dots, A_M \in \mathbb{R}^{n \times n}$, we consider the system

$$x(k+1) = A_{\sigma(k)}x(k), \quad k \in \mathbb{N}, \quad (1)$$

where $\sigma : \mathbb{N} \rightarrow \{1, \dots, M\}$, the *switching signal*, selects, at each instant of time, which subsystems the solution will follow. In some situations, we only have partial information on the reasonable/suitable signals $\sigma : \mathbb{N} \rightarrow \{1, \dots, M\}$, and thus the jumps among the subsystems are modeled as a *stochastic process*, providing, at each instant of time, the *probability* of having followed a particular switching policy. In this case, the arising stochastic system takes the name of *discrete-time (Markov) jump linear system* or *stochastic switched system*, and the earliest works studying stability in this setting can be found in [2], [3], [4]. The case of stochastic switching modeled by an i.i.d. (independent and identically distributed) sequence or by a Markov chain process is studied in [5], [6], [7], [8], [9].

Since we consider stochasticity in system (1), several *stability notions* can be defined and tackled. The case of *worst case* stability is studied in [10], [11], providing a graph-constrained adaptation of the concept of *joint spectral radius* (JSR), see [12] for a monograph. The case of *second moment stability*, (i.e. studying the convergence of the mean of the squared norm of solutions), is tackled in [5], [6], [13].

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In these works, it is proven that the second moment stability can be ensured, without conservatism, by semidefinite optimization, via a quadratic Lyapunov functions approach. The case of q -moment stability is tackled in [5], [14], [15], providing again a characterization in terms of a “spectral quantity” (q -radius) and/or in terms of Lyapunov functions.

In some situations, worst case or q -moment stability notions are restrictive for the considered systems, see the discussion provided in [5]; for that reason, recent research focused in studying *almost sure stability*: system (1) is said to be almost surely stable if the solutions are converging to zero, for almost all switching sequences (with respect to the underlying probability measure). This problem is studied for example in [16], [8], [9], [7], [17], [18], and a common tool in this analysis is provided by the (*maximal*) *Lyapunov exponent*, which provides a characterization of almost sure stability: indeed, the Lyapunov exponent can be seen, roughly speaking, as a probabilistic counterpart of the (logarithm of the) JSR.

In this manuscript, we provide novel techniques to estimate the Lyapunov exponent, proposing new sufficient conditions for the almost sure stability of (1). Our ideas rely on a language-theoretic interpretation of Markov chains, and are inspired by techniques introduced in [10] for the deterministic case. We consider formal expansions of Markov chains, called *lifts*, which, while not modifying the behavior of the stochastic system, simplify the task of providing upper bounds to the Lyapunov exponent. This requires to introduce a flexible stochastic model, generalizing the Markov chains framework, and we thus consider the *stochastic graphs* formalism, see [19]. Then, our approach makes use of the concept of *multiple Lyapunov functions*, introduced in [20], [7] in the i.i.d. case, and adapted here in this general context. We prove that, increasing the dimension (in terms of node/edges) of the lift of the Markov chain, the corresponding estimation of the Lyapunov exponent is asymptotically exact.

The manuscript is organized as follows: in Section II we recall the necessary preliminaries of probability theory, while in Section III we provide the extension of results involving almost sure stability, the probabilistic spectral radius and multiple Lyapunov functions. In Section IV we define the main concepts of our work, the *lifts* of stochastic graphs, and we provide our main results, which are then illustrated in Section V with the help of a numerical example.

II. PRELIMINARIES

In this section we introduce the necessary notation from probability theory and stochastic switched systems.

A. Shift Space and Ergodicity

Given $M \in \mathbb{N}$, consider the finite set $\langle M \rangle := \{1, \dots, M\}$, and the *one-sided Bernoulli space* defined by $\Sigma_M := \{\sigma = (\sigma_0, \sigma_1, \sigma_2, \dots) \mid \forall k \in \mathbb{N}, \sigma_k \in \langle M \rangle\}$. We consider the *left-shift operator* $\ell : \Sigma_M \rightarrow \Sigma_M$ defined by $\ell(\sigma) = (\sigma_1, \sigma_2, \dots)$. The *Borel σ -algebra* of Σ_M , denoted by $\mathcal{B}(\Sigma_M)$, is generated by the *cylinders* of the form $[i_{k-1}, \dots, i_0] := \{\sigma \in \Sigma_M \mid \sigma_0 = i_0, \dots, \sigma_{k-1} = i_{k-1}\}$, where $k \in \mathbb{N}$ is the length of the cylinder. There is an identification between cylinders of length k and elements of $\langle M \rangle^k$ and thus we also denote with $\hat{i} = (i_{k-1}, \dots, i_0) \in \langle M \rangle^k$ a generic cylinder $[\hat{i}]$ of length k . A measure μ on $(\Sigma_M, \mathcal{B}(\Sigma_M))$ is said to be *shift-invariant* if $\mu(\ell^{-1}(C)) = \mu(C)$ for all $C \in \mathcal{B}(\Sigma_M)$. A shift-invariant measure μ is *ergodic* with respect to ℓ if for all $C \in \mathcal{B}(\Sigma_M)$ such that $\ell^{-1}(C) = C$ we have $(\mu(C) = 0 \text{ or } \mu(C) = 1)$. For an overview of this topic see [21, Chapter 1].

B. Stochastic Graphs and Stochastic Switched Systems

In the following, given $M \in \mathbb{N}$, we introduce the structure used to define probability measures on $(\Sigma_M, \mathcal{B}(\Sigma_M))$.

Definition 1. Given $M \in \mathbb{N}$, a *stochastic graph* $\mathcal{G} = (S, E, p)$ on $\langle M \rangle$ is defined by

- 1) A finite set S , the set of nodes;
- 2) The set $E \subset S \times S \times \langle M \rangle$ of directed, labeled edges;
- 3) A function $p : E \rightarrow (0, 1]$, where, given $e \in E$, $p(e)$ is the *probability* associated with $e \in E$.

We denote the generic edge by $e = (a, b, i)$; $i =: \text{lab}(e) \in \langle M \rangle$ is the *label* of e , $a =: \text{st}(e) \in S$ and $b =: \text{end}(e) \in S$ are the *starting* and *ending* nodes of e , respectively. The probability $p(e)$, when needed, is also denoted by $p_{a,b,i}$. We require that

$$\sum_{b \in S, i \in \langle M \rangle} p_{a,b,i} = 1, \quad \forall a \in S.$$

When needed for notational simplicity, we set $p_{a,b,i} = 0$ for every $(a, b, i) \notin E$. For every $K \in \mathbb{N}$, by $\text{Path}^K(\mathcal{G})$ we denote the set of paths in $\mathcal{G}$ of length K . Given $\bar{q} \in \text{Path}^K(\mathcal{G})$, $\bar{q} = (e_{j_1}, \dots, e_{j_K})$, we define $p(\bar{q}) := p(e_{j_1}) \cdots p(e_{j_K})$. We denote by $\text{st}(\bar{q}) \in S$ the *starting node* of $\bar{q} \in \text{Path}^K(\mathcal{G})$, and we say that $\hat{i} = (i_{K-1}, \dots, i_0) \in \langle M \rangle^K$ is the *label* of $\bar{q} = (e_{j_1}, \dots, e_{j_K}) \in \text{Path}^K(\mathcal{G})$ and we write $\text{lab}(\bar{q}) = \hat{i}$ if $\text{lab}(e_{j_1}) = i_0, \dots, \text{lab}(e_{j_K}) = i_{K-1}$.

We define Ξ_S the set of probability distributions on S , and we can identify $\Xi_S = \{\xi \in \mathbb{R}_+^{|S|} \mid \sum_{j=1}^{|S|} \xi_j = 1\}$. Given any $\xi \in \Xi_S$ and any stochastic graph $\mathcal{G}$ on $\langle M \rangle$, we can define a probability measure on $(\Sigma_M, \mathcal{B}(\Sigma_M))$ as clarified in what follows; for every $k \in \mathbb{N}$, we first consider a probability measure on $S \times \langle M \rangle^k$, denoted by $\mathbb{P}_{\mathcal{G}, \xi}$ (without making k explicit, for simplicity) defined recursively as follows:

$$\begin{cases} \mathbb{P}_{\mathcal{G}, \xi}(s, \emptyset) := \xi(s), \quad \forall s \in S, \\ \mathbb{P}_{\mathcal{G}, \xi}(s, i) := \sum_{a \in S} \xi(a) p_{a,s,i}, \quad \forall (s, i) \in S \times \langle M \rangle, \\ \mathbb{P}_{\mathcal{G}, \xi}(s, \hat{i}) := \sum_{a \in S} \mathbb{P}_{\mathcal{G}, \xi}(a, \hat{i}^-) p_{a,s,\hat{i}_f}, \quad \forall s \in S, \hat{i} \in \langle M \rangle^k \end{cases} \quad (2)$$

where, given $\hat{i} = (i_{k-1}, \dots, i_0) \in \langle M \rangle^k$, $\hat{i}^- := (i_{k-2}, \dots, i_0) \in \langle M \rangle^{k-1}$ denotes the *predecessor* of $\hat{i}$ and $\hat{i}_f := i_{k-1}$ is the *final label* of $\hat{i}$. Intuitively, $\mathbb{P}_{\mathcal{G}, \xi}(s, \hat{i})$ denotes the probability of “being” in the node $s \in S$ after having followed a path labeled by the (multi-)index $\hat{i} \in \langle M \rangle^k$, given an initial probability measure ξ . Finally, we introduce $\mu_{\mathcal{G}, \xi}$ on $(\Sigma_M, \mathcal{B}(\Sigma_M))$ by defining it on the set of cylinders of Σ_M , i.e., $\forall \hat{i} \in \langle M \rangle^k, \forall k \in \mathbb{N}$ we set

$$\mu_{\mathcal{G}, \xi}(\hat{i}) := \sum_{s \in S} \mathbb{P}_{\mathcal{G}, \xi}(s, \hat{i}). \quad (3)$$

Another possible definition of (3), is obtained setting

$$\mu_{\mathcal{G}, \xi}(\hat{i}) := \sum_{s \in S} \xi(s) \sum_{\substack{\bar{q} \in \text{Path}^k(\mathcal{G}) \\ \text{st}(\bar{q}) = s, \text{lab}(\bar{q}) = \hat{i}}} p(\bar{q}), \quad (4)$$

for all $\hat{i} \in \langle M \rangle^k$ and for all $k \in \mathbb{N}$. It is easy to see that (3) and (4) are equivalent and in the following we may use both, depending on the convenience. With this definition, for any stochastic graph $\mathcal{G}$ and any $\xi \in \Xi_S$, we have that $(\Sigma_M, \mathcal{B}(\Sigma_M), \mu_{\mathcal{G}, \xi})$ is a *well-defined probability space*.

Remark 1 (Choice of the model). For more details regarding the stochastic graph formalism, see [19, Definition 2.3.14] and references therein. The case of finite Markov chain is recovered by the setting in Definition 1; given a stochastic matrix $P = (p_{ij}) \in \mathbb{R}_{\geq 0}^{M \times M}$, the state transition matrix associated to a time homogeneous Markov chain, we can define the corresponding stochastic graph by $S := \langle M \rangle$, $E = \{(i, j, j) \mid (i, j) \in \langle M \rangle^2\}$ and $p_{i,j,j} = p_{i,j}$, for all $(i, j) \in \langle M \rangle^2$. For a graphical representation, see Figure 1a. It can be seen that *every* stochastic graph can be rewritten, paying the price of “enlarging” the alphabet and/or the node set, as an ordinary Markov chain, see [19]. We carry out the analysis in the setting of general stochastic graphs, since it will be crucial, in the following sections, when considering “expanded” version of a given graph/Markov chain.

In the next developments, given a stochastic graph $\mathcal{G}$, we consider the following crucial property.

Assumption 1. The considered stochastic graph $\mathcal{G}$ is strongly connected, i.e., for all $a, b \in S$ there exists a path in $\mathcal{G}$ starting at a and arriving at b .

Given $\mathcal{G} = (S, E, p)$ a stochastic graph on $\langle M \rangle$, let us consider the stochastic matrix $P_{\mathcal{G}} \in \mathbb{R}_{\geq 0}^{|S| \times |S|}$ defined by

$$p_{a,b} := \sum_{i \in \langle M \rangle} p_{a,b,i}. \quad (5)$$

If $\mathcal{G}$ satisfies Assumption 1, by the Perron-Frobenius Theorem, we can consider $\xi_{\mathcal{G}} \in \Xi_S$ as the *unique* measure such that $\xi_{\mathcal{G}}^\top P_{\mathcal{G}} = \xi_{\mathcal{G}}^\top$, which satisfies $\xi_{\mathcal{G},j} > 0$ for any $j \in \{1, \dots, |S|\}$. This unique measure $\xi_{\mathcal{G}}$ is referred to as the *invariant measure* of the stochastic graph $\mathcal{G}$. We recall the following important properties that we use in what follows.

Lemma 1. Consider a stochastic graph $\mathcal{G}$ on $\langle M \rangle$ satisfying Assumption 1. Then, the measure $\mu_{\mathcal{G}, \xi_{\mathcal{G}}}$ on $(\Sigma_M, \mathcal{B}(\Sigma_M))$ is shift-invariant and ergodic and, for any $\xi \in \Xi_S$, the measure

$\mu_{\mathcal{G},\xi}$ is absolutely continuous¹ with respect to $\mu_{\mathcal{G},\xi_{\mathcal{G}}}$.

The first part holds by ergodic theory, see for example [21, Chapter 1], while the absolute continuity of $\mu_{\mathcal{G},\xi}$ for any $\xi \in \Xi_S$ follows by the fact that $\xi_{\mathcal{G}} > 0$ (component-wise). Now that the stochastic setting is well defined, we introduce the class of systems we study in what follows.

Definition 2 ((Stochastic) Switched Systems). Let us consider $M, n \in \mathbb{N}$, $\mathcal{A} := \{A_1, \dots, A_M\} \subset \mathbb{R}^{n \times n}$, and a stochastic graph $\mathcal{G}$ on $\langle M \rangle$. We consider the *discrete-time switched system* $\mathcal{S}(\mathcal{A}, \mathcal{G})$ defined by

$$x(k+1) = A_{\sigma_k} x(k), \quad k \in \mathbb{N}, \quad (6)$$

where $\sigma \in \Sigma_M$ is also called the *switching sequence*. Given $\xi \in \Xi_S$, we consider the probability space $(\Sigma_M, \mathcal{B}(\Sigma_M), \mu_{\mathcal{G},\xi})$ and the asymptotic behavior of systems (6) is then *studied with respect to this measure*.

Given $x_0 \in \mathbb{R}^n$ and $\sigma \in \Sigma_M$, we denote by $x(k, x_0, \sigma)$ the *solution* of (6), starting at x_0 with respect to the signal σ , evaluated at time $k \in \mathbb{N}$. Similarly, given $\mathcal{A} = \{A_1, \dots, A_M\} \subset \mathbb{R}^{n \times n}$, for any $k \in \mathbb{N}$, given $\hat{i} = (i_{k-1}, \dots, i_0) \in \langle M \rangle^k$ we use the notation $A(\hat{i}) = A_{i_{k-1}} \cdots A_{i_0}$, and given $\sigma \in \Sigma_M$, $A^k(\sigma) = A_{\sigma_{k-1}} \cdots A_{\sigma_0}$.

III. ALMOST SURE STABILITY AND PROBABILISTIC SPECTRAL RADIUS

We now introduce the considered stability notion and the corresponding spectral characterization.

Definition 3. Consider $M, n \in \mathbb{N}$, $\mathcal{A} = \{A_1, \dots, A_M\} \subset \mathbb{R}^{n \times n}$ and a stochastic graph $\mathcal{G}$ on $\langle M \rangle$. Given $\Phi \subseteq \Xi_S$ system (6) is *uniformly almost surely asymptotically stable with respect to Φ* if, for any $x_0 \in \mathbb{R}^n$, any $\xi \in \Phi$ we have

$$\mu_{\mathcal{G},\xi} \left(\left\{ \sigma \in \Sigma_M \mid \lim_{k \rightarrow \infty} |x(k, x_0, \sigma)| = 0 \right\} \right) = 1. \quad (7)$$

In the case $\Phi = \Xi_S$, the term “with respect to Φ ” is omitted.

It turns out that this stability notion, (as the “deterministic” one, introduced in [7]) can be characterized by studying a corresponding probabilistic spectral radius.

Definition 4 (Probabilistic Spectral Radius). Consider $M, n \in \mathbb{N}$, $\mathcal{A} = \{A_1, \dots, A_M\} \subset \mathbb{R}^{n \times n}$, a stochastic graph $\mathcal{G}$ on $\langle M \rangle$ and $\xi \in \Xi_S$. Given any operator matrix norm $\|\cdot\|$, we define

$$\rho_0(\mathcal{A}, \mathcal{G}, \xi) := \limsup_{k \rightarrow \infty} \left[\prod_{\hat{i} \in \langle M \rangle^k} \|A(\hat{i})\|^{\mu_{\mathcal{G},\xi}(\hat{i})} \right]^{\frac{1}{k}}. \quad (8)$$

The quantity $\rho_0(\mathcal{A}, \mathcal{G}, \xi)$ is referred to as the *probabilistic spectral radius of $\mathcal{A}$ induced by $\mathcal{G}$ and $\xi \in \Xi_S$* .

The subscript 0 in $\rho_0(\mathcal{A}, \mathcal{G}, \xi)$ is motivated by the fact that this radius can be seen as the limit, for q going to 0, of the *q-spectral radius*, i.e. the quantity characterizing the

¹Given 2 measures μ, ν (on a generic measurable space), μ is absolutely continuous with respect to ν is $\nu(C) = 0$ implies $\mu(C) = 0$.

stability of the q -moment of solutions, see [22], [5]. From now on we study stability with respect to the whole set Ξ_S ; under Assumption 1 this is not restrictive: using Lemma 1 it can be shown that, for any $\xi \in \Xi_S$, we have

$$\sup_{\xi \in \Xi_S} \rho_0(\mathcal{A}, \mathcal{G}, \xi) = \rho_0(\mathcal{A}, \mathcal{G}, \xi_{\mathcal{G}}), \quad (9)$$

see [5, Lemma 2.3]. We have the following relation between almost sure stability and probabilistic spectral radius.

Proposition 1. Consider $M, n \in \mathbb{N}$, $\mathcal{A} = \{A_1, \dots, A_M\} \subset \mathbb{R}^{n \times n}$ and a stochastic graph $\mathcal{G}$ on $\langle M \rangle$ satisfying Assumption 1. Then, the system $\mathcal{S}(\mathcal{A}, \mathcal{G})$ is *uniformly almost surely asymptotically stable if and only if*

$$\rho_0(\mathcal{A}, \mathcal{G}, \xi_{\mathcal{G}}) < 1.$$

Sketch of the Proof. The proof can be found in [5] for the case of Markov jump linear systems (i.e. for stochastic graphs arising from strongly connected Markov chains), and the ideas can be adapted in this context, *mutatis mutandis*. The peculiarity here is that we consider the probabilistic spectral radius as defined in (8). Instead, most of the literature concerning stability of stochastic switched systems (cfr. [5], [16], [17], [23] and references therein) introduce the (maximal) *Lyapunov Exponent* $\lambda_0(\mathcal{A}, \mathcal{G}, \xi) := \limsup_{k \rightarrow \infty} \frac{1}{k} \mathbb{E}_{\mathcal{G},\xi} (\log \|A^k(\sigma)\|)$ to characterize almost sure stability. Since it holds that $e^{\lambda_0(\mathcal{A}, \mathcal{G}, \xi)} = \rho_0(\mathcal{A}, \mathcal{G}, \xi)$, we can apply the same arguments in our case. $\square$

The reason to consider the probabilistic spectral radius (instead of the Lyapunov exponent) is two-fold: from one side, this allows to draw a parallel with the definition and estimation techniques for the JSR (see [18]), and, on the other hand, it allows us to simplify the following notation.

A. Lyapunov Multi Functions for Almost-Sure Stability

To give a “computable” characterization of the probabilistic spectral radius, we define the space of candidate Lyapunov functions.

Definition 5. We say that f is a *candidate Lyapunov function* (and we write $f \in \mathcal{F}_n$) if $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is continuous, positive definite, and positively homogeneous².

Definition 6 (Lyapunov Multi-Functions). Consider $\mathcal{A} = \{A_1, \dots, A_N\} \subset \mathbb{R}^{n \times n}$, a stochastic graph $\mathcal{G} = (S, E, p)$ and a scalar $\rho \geq 0$. A *Lyapunov multi function (LMF)* for $\mathcal{S}(\mathcal{A}, \mathcal{G})$ w.r.t. ρ is a set of $|S|$ functions $F := \{f_a \in \mathcal{F}_n \mid a \in S\}$ such that, $\forall x \in \mathbb{R}^n, \forall a \in S$,

$$\prod_{i \in \langle M \rangle} \prod_{b \in S} (f_b(A_i x))^{p_{a,b,i}} \leq \rho f_a(x). \quad (10)$$

Proposition 2. Consider $\mathcal{A} = \{A_1, \dots, A_N\} \subset \mathbb{R}^{n \times n}$, a stochastic graph $\mathcal{G}$, it holds that

$$\sup_{\xi \in \Xi_S} \rho_0(\mathcal{A}, \mathcal{G}, \xi) \leq \inf \{ \rho \geq 0 \mid \exists F \subset \mathcal{F}_n, \text{ LMF for } \mathcal{S}(\mathcal{A}, \mathcal{G}) \text{ w.r.t. } \rho \}. \quad (11)$$

²A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is positively homogeneous if $f(ax) = af(x)$, for any $a \in \mathbb{R}_+$ and any $x \in \mathbb{R}^n$.

If $\mathcal{G}$ satisfies Assumption 1 the equality holds, i.e.,

$$\rho_0(\mathcal{A}, \mathcal{G}, \xi_{\mathcal{G}}) = \inf \{ \rho \geq 0 \mid \exists F \subset \mathcal{F}_n, \text{ LMF for } \mathcal{S}(\mathcal{A}, \mathcal{G}) \text{ w.r.t. } \rho \}. \quad (12)$$

The proof can be found in [24]. In Proposition 2 we consider the infimum over *all* the possible candidate Lyapunov functions in $(\mathcal{F}_n)^{|S|}$. Since, usually, it is practical to restrict the search to a particular subset (e.g. quadratic norms, SOS-polynomials, etcetera), we provide the following definition.

Definition 7. Consider $\mathcal{A} = \{A_1, \dots, A_N\} \subset \mathbb{R}^{n \times n}$ and a stochastic graph $\mathcal{G}$. Consider a subclass of candidate Lyapunov functions $\mathcal{V} \subset \mathcal{F}_n$, we define

$$\rho_{0,\mathcal{V}}(\mathcal{A}, \mathcal{G}) := \inf \{ \rho > 0 \mid \exists F \subset \mathcal{V} \text{ LMF for } \mathcal{S}(\mathcal{A}, \mathcal{G}) \text{ w.r.t. } \rho \}. \quad (13)$$

In Proposition 2 we have proven that $\rho_{0,\mathcal{F}_n}(\mathcal{G}, \mathcal{A}) = \sup_{\xi \in \Xi_S} \rho_0(\mathcal{G}, \mathcal{A}, \xi)$, and thus, for any $\mathcal{V} \subset \mathcal{F}_n$,

$$\sup_{\xi \in \Xi_S} \rho_0(\mathcal{G}, \mathcal{A}, \xi) \leq \rho_{0,\mathcal{V}}(\mathcal{G}, \mathcal{A}). \quad (14)$$

IV. APPROXIMATION OF THE PROBABILISTIC SPECTRAL RADIUS: GRAPH-BASED LIFTS

In the following, we define some “expansion techniques” for the computation of the probabilistic spectral radius.

A. The Step Lift

We introduce a formal expansion of stochastic graph which, intuitively, induces the same stochastic framework on Σ_M , while focusing on sub-words of the form $\hat{i} = (i_{K-1}, \dots, i_0) \in \langle M \rangle^K$ of arbitrary length $K \in \mathbb{N}$.

Definition 8 (The Step Lift). Let us consider $M \in \mathbb{N}$, a stochastic graph $\mathcal{G} = (S, E, p)$ on $\langle M \rangle$ and a set of matrices $\mathcal{A} = \{A_1, \dots, A_M\}$. Given the system $\mathcal{S}(\mathcal{A}, \mathcal{G})$ and any integer $K \geq 1$, the K -step lift of $\mathcal{S}(\mathcal{A}, \mathcal{G})$ denoted by $\mathcal{L}^K \mathcal{S}(\mathcal{A}, \mathcal{G})$ is a stochastic system defined by a stochastic graph on $\langle M \rangle^K$, $\mathcal{G}^K = (S^K, E^K, p^K)$ and a set of matrices $\mathcal{A}^K$ defined as follows:

- 1) $S^K \equiv S$,
- 2) For any “candidate edge” for $\mathcal{G}^K$, $(a, b, \hat{i}) \in S \times S \times \langle M \rangle^K$, we inductively define its probability weight by

$$p_{a,b,\hat{i}}^K := \sum_{c \in S} p_{a,c,\hat{i}-}^{K-1} p_{c,b,\hat{i}_f}.$$

By convention, if $p_{a,b,\hat{i}} = 0$ then $e = (a, b, \hat{i}) \notin E^K$.

- 3) $\mathcal{A}^K := \{A(\hat{i}) \mid \hat{i} \in \langle M \rangle^K\}$, where, given $\hat{i} = (i_{K-1}, \dots, i_0)$ we recall that $A(\hat{i}) = A_{i_{K-1}} \cdots A_{i_0}$.

It is clear that $\mathcal{L}^1 \mathcal{S}(\mathcal{A}, \mathcal{G}) = \mathcal{S}(\mathcal{A}, \mathcal{G})$. In the following statement we collect the relations between the probability measures induced by $\mathcal{G}$ and $\mathcal{G}^K$, respectively.

Lemma 2. Consider a stochastic graph $\mathcal{G}$ on $\langle M \rangle$ and $K \in \mathbb{N}$; it holds that $P_{\mathcal{G}^K} = P_{\mathcal{G}}^K$, where $P_{\mathcal{G}}$ and $P_{\mathcal{G}^K}$ are defined as in (5); this implies $\xi_{\mathcal{G}} = \xi_{\mathcal{G}^K}$. Moreover, for any $k \in \mathbb{N}$, $K \in \mathbb{N} \setminus \{0\}$, any $\xi \in \Xi_S$ and any $\hat{i} \in \langle M \rangle^{kK}$ we have

$$\mu_{\mathcal{G},\xi}(\hat{i}) = \mu_{\mathcal{G}^K,\xi}(\hat{i}). \quad (15)$$

Proof. The first part follows from the definition in (5) and from Item 2 of Definition 8. Equation (15) is a consequence of $P_{\mathcal{G}^K} = P_{\mathcal{G}}^K$, once recalled (2) and (3). $\square$

Example 1. Consider the stochastic graph $\mathcal{H}$ in Figure 1a. It represents the case of a (strongly connected) Markov chain, with transition matrix given by $P = P_{\mathcal{H}} = \begin{bmatrix} 1/3 & 2/3 \\ 1/4 & 3/4 \end{bmatrix}$ and $\xi_{\mathcal{H}} = [3/11, 8/11]^T$. The corresponding step lift of degree 2, denoted by $\mathcal{H}^2$ is depicted in Figure 1b. Recalling the definition of $P_{\mathcal{G}}$ given by (5) is easy to see that $P_{\mathcal{H}^2} = \begin{bmatrix} 5/18 & 13/18 \\ 13/48 & 35/48 \end{bmatrix} = P_{\mathcal{H}}^2$, as predicted by Lemma 2.

Theorem 1 (Properties of Step Lift). Consider $M \in \mathbb{N}$, a stochastic graph $\mathcal{G}$ on $\langle M \rangle$ and $\mathcal{A} = \{A_1, \dots, A_M\} \subset \mathbb{R}^{n \times n}$. For any $K \in \mathbb{N} \setminus \{0\}$, and any $\xi \in \Xi_S$ it holds that

$$\rho_0(\mathcal{A}^K, \mathcal{G}^K, \xi) = [\rho_0(\mathcal{A}, \mathcal{G}, \xi)]^K. \quad (16)$$

For any (non-empty) subclass of candidate functions $\mathcal{V} \subset \mathcal{F}_n$, we have

$$\sup_{\xi \in \Xi} \rho_0(\mathcal{A}, \mathcal{G}, \xi) \leq \sqrt[K]{\rho_{0,\mathcal{V}}(\mathcal{A}^K, \mathcal{G}^K)} \leq \rho_{0,\mathcal{V}}(\mathcal{A}, \mathcal{G}), \quad (17)$$

and moreover, if $\mathcal{G}$ satisfies Assumption 1, we have

$$\rho_0(\mathcal{A}, \mathcal{G}, \xi_{\mathcal{G}}) = \lim_{K \rightarrow +\infty} \sqrt[K]{\rho_{0,\mathcal{V}}(\mathcal{A}^K, \mathcal{G}^K)}. \quad (18)$$

The proof is reported in [24]. We note here that in Lemma 2 and Theorem 1, Assumption 1 is not required, except for the convergence property in (18).

B. The Path Lift

In this subsection we propose another lift, defining an augmented graph which, intuitively, adds memory to the framework, considering paths of given length as new states.

Definition 9 (The Path Lift). Consider $M \in \mathbb{N}$, a stochastic graph $\mathcal{G} = (S, E, p)$ on $\langle M \rangle$ and $\mathcal{A} = \{A_1, \dots, A_M\} \subset \mathbb{R}^{n \times n}$. Given any integer $R \geq 1$, the *path lift of degree R* of $\mathcal{S}(\mathcal{A}, \mathcal{G})$ denoted by $\mathcal{L}_R \mathcal{S}(\mathcal{A}, \mathcal{G})$ is a stochastic system composed by a stochastic graph on $\langle M \rangle$, $\mathcal{G}_R = (S_R, E_R, p_R)$ defined recursively as follows:

- 1) For any path of length R , $\bar{q} = (e_1, \dots, e_R) \in \text{Path}^R(\mathcal{G})$ in $\mathcal{G}$, add a node $s_{\bar{q}} \in S_R$;
- 2) For each path $\bar{r} = (e_1, e_2, \dots, e_R, e_{R+1})$ of length $R+1$ in $\mathcal{G}$, with $i \in \langle M \rangle$ the label of e_{R+1} , add in E_R the edge $(s_{\bar{q}_1}, s_{\bar{q}_2}, i)$, where $\bar{q}_1 = (e_1, \dots, e_R)$ and $\bar{q}_2 = (e_2, \dots, e_{R+1})$;
- 3) For any path $\bar{r} = (e_1, e_2, \dots, e_R, e_{R+1}) \in E_R$ of length $R+1$, set $p_R(\bar{r}) = p(e_{R+1})$.

Given any $\xi \in \Xi_S$, we consider the *path lift measure of degree R* $\xi_R \in \Xi_{S_R}$ defined as follows: for every $s_{\bar{q}} \in S_R$ with $\bar{q} = (e_1, \dots, e_R)$ and $e_1 = (a, b, i) \in E$, define:

$$\xi_R(s_{\bar{q}}) := \xi(a)p(e_1) \cdots p(e_R). \quad (19)$$

From Definition 9 we have that, for any $R \in \mathbb{N}$, $\mathcal{G}_{R+1} = (\mathcal{G}_R)_1$ and, similarly, for any $\xi \in \Xi_S$, $\xi_{R+1} = (\xi_R)_1$ i.e. the path lift of degree $R+1$ is the path lift (of degree 1) of the

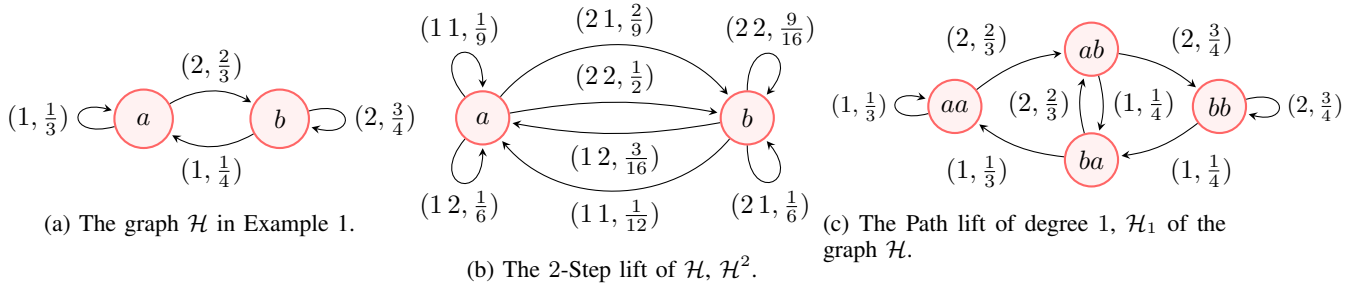


Fig. 1: A stochastic graph and its 2-Step Lift and Path Lift of degree 1.

path lift of degree R . It can be seen that, given any stochastic graph $\mathcal{G}$ and any $\xi \in \Xi_S$ and any $R \in \mathbb{N}$, $\mathcal{G}_R$ and $\xi_R \in \Xi_{S_R}$ introduced in Definition 9 are a well-defined stochastic graph and a probability measure, respectively. Moreover, if $\mathcal{G}$ satisfies Assumption 1, so does $\mathcal{G}_R$.

Lemma 3. Consider a stochastic graph $\mathcal{G}$ satisfying Assumption 1, and consider $\xi_{\mathcal{G}}$ its invariant measure. Then, for every $R \in \mathbb{N}$, we have that $(\xi_{\mathcal{G}})_R = \xi_{\mathcal{G}_R}$, i.e. the path lift measure of degree R of $\xi_{\mathcal{G}}$ is the invariant measure of $\mathcal{G}_R$.

Proof. Consider any stochastic graph $\mathcal{G} = (S, E, p)$. Since, for any $R \in \mathbb{N}$ we have that $\mathcal{G}_{R+1} = (\mathcal{G}_R)_1$ it suffices to prove the claim for $R = 1$. We want to prove that $(\xi_{\mathcal{G}})_1 = \xi_{\mathcal{G}_1}$; since $\xi_{\mathcal{G}}$ is the invariant measure of $\mathcal{G}$, recalling (5), we have that $\xi_{\mathcal{G}}(r) = \sum_{s \in S} \xi_{\mathcal{G}}(s) \sum_{i \in \langle M \rangle} p_{s,r,i}$, $\forall r \in S$. Consider any $e = (a, b, i) \in \mathcal{G}$, we have that

$$\begin{aligned} (\xi_{\mathcal{G}})_1(e) &= \xi_{\mathcal{G}}(a)p(e) = \sum_{s \in S} \xi_{\mathcal{G}}(s) \sum_{i \in \langle M \rangle} p_{s,a,i} p(e) \\ &= \sum_{s \in S} \sum_{i \in \langle M \rangle} \xi_{\mathcal{G}}(s) p_{s,a,i} p(e) \\ &= \sum_{\substack{f \in E, \\ \text{end}(f)=\text{st}(e)}} (\xi_{\mathcal{G}})_1(f) p(e) = \sum_{f \in E} (\xi_{\mathcal{G}})_1(f) p_1((f, e)), \end{aligned}$$

recalling that, by Item 3 of Definition 9, we have $p_1((f, e)) = p(e)$. We have thus proven that $(\xi_{\mathcal{G}})_1$ is Lyapunov for $\mathcal{G}_1$, and by uniqueness of invariant measure, we conclude. $\square$

Example 2. Consider again the stochastic graph $\mathcal{H}$ in Figure 1a. The corresponding path lift of degree 1, $\mathcal{H}_1$ is represented in Figure 1c. It can be seen that (considering the lexicographic order on the nodes), we have

$$P_{\mathcal{H}_1} = \begin{bmatrix} 1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 1/4 & 3/4 \\ 1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 1/4 & 3/4 \end{bmatrix},$$

and computing, we obtain $\xi_{\mathcal{H}_1} = [1/11 \ 2/11 \ 2/11 \ 6/11]^\top$, which is equal to $(\xi_{\mathcal{H}})_1$, as proven in Lemma 3.

Now we can prove the main result of this subsection, establishing relations between the probability measure on $(\Sigma_M, \mathcal{B}(\Sigma_M))$ induced by a stochastic $\mathcal{G}$ and $\xi \in \Xi_S$ and by $\mathcal{G}_R$, its path lift of degree R , and the corresponding ξ_R .

Theorem 2 (Properties path lift of degree R). Consider any stochastic graph $\mathcal{G}$, any distribution $\xi \in \Xi_S$ and any $R \in \mathbb{N}$.

For all $k \in \mathbb{N}$, for all $\hat{i} \in \langle M \rangle^k$, we have

$$\begin{aligned} \mu_{\mathcal{G}_R, \xi_R}(\hat{i}) &= \mu_{\mathcal{G}, \xi}(\ell^{-R}(\hat{i})) \\ &= \sum_{\hat{j} \in \ell^{-R}(\hat{i})} \mu_{\mathcal{G}, \xi}(\hat{j}), \quad \forall \hat{i} \in \langle M \rangle^k, \forall k \in \mathbb{N}. \end{aligned} \quad (20)$$

If $\mathcal{G}$ satisfies Assumption 1, we have

$$\mu_{\mathcal{G}_R, \xi_R} = \mu_{\mathcal{G}, \xi_{\mathcal{G}}}, \quad (21)$$

and, for any $\mathcal{A} = \{A_1 \dots, A_M\} \subset \mathbb{R}^{n \times n}$ and any $\mathcal{V} \subset \mathcal{F}_n$,

$$\rho_0(\mathcal{A}, \mathcal{G}, \xi_{\mathcal{G}}) = \rho_0(\mathcal{A}, \mathcal{G}_R, \xi_{\mathcal{G}_R}) \leq \rho_{0, \mathcal{V}}(\mathcal{A}, \mathcal{G}_R) \leq \rho_{0, \mathcal{V}}(\mathcal{A}, \mathcal{G}). \quad (22)$$

The proof can be found in [24].

V. NUMERICAL EXAMPLE

In this section, we consider a positive stochastic switched system as in (1), denoted by $\mathcal{S}(\mathcal{H}, \mathcal{A})$, with the stochastic graph $\mathcal{H}$ on $\langle 2 \rangle$ in Figure 1a, already studied in Examples 1 and 2, while the set of positive matrices $\mathcal{A} = \{A_1, A_2\} \subset \mathbb{R}_{\geq 0}^{2 \times 2}$ is defined by

$$A_1 := \begin{bmatrix} 0.5 & 1 \\ 0 & 0.5 \end{bmatrix}, \quad A_2 := \begin{bmatrix} 1 & 0 \\ 0.1 & 1 \end{bmatrix}.$$

The same set of matrices was studied in [16, Example 3.2] (with a different underlying irreducible Markov chain). We note that system $\mathcal{S}(\mathcal{H}, \mathcal{A})$ is *first moment unstable* (or *unstable in mean*) i.e., there exists $x_0 \in \mathbb{R}^n$ for which $\lim_{k \rightarrow \infty} \mathbb{E}_{\mathcal{H}, \xi_{\mathcal{H}}}(|x(k, x_0, \sigma)|) > 0$, i.e., the expected norm with respect to the measure $\mu_{\mathcal{H}, \xi_{\mathcal{H}}}$ does not asymptotically converge to 0. Indeed, by positivity of A_1 and A_2 and applying [14, Theorem 2.4], $\mathcal{S}(\mathcal{H}, \mathcal{A})$ is unstable in mean since the “averaged” matrix

$$B = \xi_{\mathcal{H}}(a)A_1 + \xi_{\mathcal{H}}(b)A_2 = \frac{3}{11}A_1 + \frac{8}{11}A_2 = \begin{bmatrix} 19/22 & 3/11 \\ 4/55 & 19/22 \end{bmatrix}$$

is Schur unstable, i.e. $\rho(B) > 1$. It can be also shown that the sufficient and necessary LMI conditions for *mean square stability* illustrated in [6] are infeasible for $\mathcal{S}(\mathcal{H}, \mathcal{A})$. In what

TABLE I: Numerical upper bounds of the probabilistic spectral radius obtained with different lifts and different candidate functions templates.

Lift $\mathcal{G}$:	$\mathcal{H}$	$\mathcal{H}^2$	$\mathcal{H}_1$
$\rho_{0, \mathcal{Q}}(\mathcal{A}, \mathcal{G})$	1.002	0.896	0.998
$\rho_{0, \mathcal{D}}(\mathcal{A}, \mathcal{G})$	1.169	1.045	1.036

follows we show that $\mathcal{S}(\mathcal{H}, \mathcal{A})$ is almost sure stable, using the ideas developed in Section IV.

First, we consider, as set of candidate Lyapunov functions, the set $\mathcal{Q} \subset \mathcal{F}_2$ of *quadratic norms*, defined by $\mathcal{Q} := \{f_Q : \mathbb{R}_{\geq 0}^2 \rightarrow \mathbb{R} \mid Q \succ 0\}$, with $f_Q(x) := \sqrt{x^\top Q x}$. To compute an upper bound of the minimal ρ satisfying (10) we use the techniques presented in [16, Corollary 2.2.]. In this case, simply considering the original Markov chain $\mathcal{H}$, we are unable to provide a stability certificate, since we have obtained $\rho_{0, \mathcal{Q}}(\mathcal{A}, \mathcal{H}) \leq 1.002$. Instead, considering the lifts $\mathcal{H}^2$ and $\mathcal{H}_1$ depicted in Figures 1, we obtain certificates of almost sure stability, since we have that $\rho_{0, \mathcal{Q}}(\mathcal{A}, \mathcal{H}^2) < 1$ and $\rho_{0, \mathcal{Q}}(\mathcal{A}, \mathcal{H}_1) < 1$. The obtained numeric values are reported in Table I.

For the sake of completeness, we report that we have also considered the set of candidate Lyapunov functions $\mathcal{D} \subset \mathcal{F}_2$ of *dual copositive norms* defined by $\mathcal{D} := \{f_v^* : \mathbb{R}_{\geq 0}^2 \rightarrow \mathbb{R} \mid v \in \mathbb{R}_{> 0}^2\}$, where, given $v \in \mathbb{R}_{> 0}^2$, we define $f_v^*(x) := \max_{i \in \{1, 2\}} \left\{ \frac{x_i}{v_i} \right\}$, $\forall x \in \mathbb{R}_{\geq 0}^2$. These functions are valid norms in this case, since A_1 and A_2 are positive, for further discussion regarding copositive norms, we refer to [25, Section 4]. This family of functions was chosen for optimization purposes: indeed, inequalities of the form (10) can be rewritten as follows

$$\prod_{i \in \langle M \rangle} \prod_{b \in S} (f_{v_b}^*(A_i x))^{p_{a,b,i}} \leq \rho f_{v_a}^*(x), \quad \forall x \in \mathbb{R}_{\geq 0}^n, \quad \text{iff}$$

$$\prod_{i \in \langle M \rangle} \prod_{b \in S} (f_{v_b}^*(A_i v_a))^{p_{a,b,i}} \leq \rho,$$

simplifying the minimization of the parameter ρ (the upper bound for the probabilistic spectral radius), see [25, Section 4] for the details. In Table I we collect the values of the obtained upper bounds on $\rho_0(\mathcal{A}, \mathcal{H}, \xi_{\mathcal{H}})$.

Concluding, we underline how, as proven in Theorem 1 and Theorem 2, the 2-steps lift $\mathcal{H}_2$ and the path lift of degree 1, $\mathcal{H}_1$, provide a better over approximation of $\rho_0(\mathcal{A}, \mathcal{H}, \xi_{\mathcal{H}})$ with respect to the original graph-based conditions imposed on $\mathcal{H}$, for these 2 different sets of candidate functions, $\mathcal{Q}$ and $\mathcal{D}$.

VI. CONCLUSION

In this work, we generalized stochastic stability notions, and related algorithms, from arbitrarily switching systems to systems whose switching signal is ruled by a Markov Chain (aka MJLS). Inspired by techniques developed for deterministic switched systems, we presented some numerical schemes to provide tight upper bounds for the probabilistic spectral radius. These techniques rely on formal expansions of the underlying stochastic graph, called *lifts*. Future research will investigate the numerical aspects related with our approximation technique, and the application of the proposed scheme for more general stochastic systems settings.

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