Two-step nilpotent Leibniz algebras

Gianmarco La Rosa, Manuel Mancini

Department of Mathematics and Computer Science
University of Palermo, Italy

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Part 1: Classification of two-step nilpotent Leibniz algebras

- Let $\mathbb{F}$ be a field with $char(\mathbb{F}) \neq 2$. A *left Leibniz algebra* over $\mathbb{F}$ is a vector space $\mathfrak{g}$ over $\mathbb{F}$ endowed of a bilinear map (called *commutator* or *bracket*) $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which satisfies the *left Leibniz identity*

$$[x,[y,z]] = [[x,y],z] + [y,[x,z]] \ \ \forall x,y,z \in \mathfrak{g}.$$

- We can define a *right Leibniz algebra*, using the right Leibniz identity

$$[[x,y],z] = [[x,z],y] + [x,[y,z]] \ \ \forall x,y,z \in \mathfrak{g}.$$

- A Leibniz algebra that is both left and right is called *symmetric Leibniz algebra*.
- We assume that $\dim_{\mathbb{F}} \mathfrak{g} < \infty$.

- The subalgebra

$$\mathrm{Leib}(\mathfrak{g}) = \mathrm{Span}_{\mathbb{F}} \{[x, x] \mid x \in \mathfrak{g}\},$$

is the smallest bilateral ideal of $\mathfrak{g}$ such that $\mathfrak{g}/\,\mathrm{Leib}(\mathfrak{g})$ is a Lie algebra.

- The *left* and the *right center* of $\mathfrak{g}$ are

$$\mathrm{Z}_l(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x, \mathfrak{g}] = 0\}, \;\; \mathrm{Z}_r(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [\mathfrak{g}, x] = 0\}.$$

- The *center of* $\mathfrak{g}$ is $\mathrm{Z}(\mathfrak{g}) = \mathrm{Z}_l(\mathfrak{g}) \cap \mathrm{Z}_r(\mathfrak{g})$.
- If $\mathfrak{g}$ is symmetric, $\mathrm{Z}_l(\mathfrak{g})$ and $\mathrm{Z}_r(\mathfrak{g})$ are bilateral ideals.

Definition

Let $\mathfrak{g}$ be a left Leibniz algebra and let

$$\mathfrak{g}^{(0)} = \mathfrak{g},\ \ \mathfrak{g}^{(k+1)} = [\mathfrak{g}, \mathfrak{g}^{(k)}],\ \ \forall k \geq 0,$$

be the *lower central series of* $\mathfrak{g}$. $\mathfrak{g}$ is *$n-$step nilpotent* if $\mathfrak{g}^{(n-1)} \neq 0$ and $\mathfrak{g}^{(n)} = 0$.

Proposition

If $\mathfrak{g}$ is a left two-step nilpotent Leibniz algebra, then $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}] \subseteq \mathrm{Z}(\mathfrak{g})$ and $\mathfrak{g}$ is symmetric.

Proposition

If $\mathfrak{g}$ is a left nilpotent Leibniz algebra with $\dim_{\mathbb{F}} \mathfrak{g}^{(1)} = 1$, then $\mathfrak{g}^{(1)} \subseteq \mathrm{Z}(\mathfrak{g})$ and $\mathfrak{g}$ is symmetric.

- Let $\mathfrak{g}$ be a two-step nilpotent Leibniz algebra with $[\mathfrak{g}, \mathfrak{g}] = Span_{\mathbb{F}}\{z_1, \ldots, z_t\}$, then

$$[x, y] = \phi_1(x, y)z_1 + \cdots + \phi_t(x, y)z_t, \ \ \forall x, y \in \mathfrak{g}$$

  where $\phi_i : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$ is a bilinear form, $\forall i = 1, \ldots, t$.

- Our purpose is to reduce such a given bilinear form to a suitable canonical representation.

- Let $\mathfrak{g}$ be a nilpotent Leibniz algebra with $[\mathfrak{g}, \mathfrak{g}] = \mathbb{F}z$ and

$$[x, y] = \phi(x, y)z, \;\; \forall x, y \in \mathfrak{g}$$

- If $\mathfrak{g}$ is not a Lie algebra, then $0 \neq \mathrm{Leib}(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}] \subseteq \mathrm{Z}(\mathfrak{g})$.
- $\phi = \phi^- + \phi^+$, where

$$\phi^- = \frac{\phi - \phi^t}{2}, \;\; \phi^+ = \frac{\phi + \phi^t}{2}.$$

- We use the simultaneous reduction to canonical form of a pair of bilinear forms.

Definition

A Kronecker module is a quadruple (V_1, V_2, f_1, f_2), where V_1, V_2 are vector spaces over $\mathbb{F}$ and $f_1, f_2 : V_1 \to V_2$ are linear maps.

- Given a pair of bilinear forms $\phi^-, \phi^+ : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$ as above, we define the Kronecker module $\left(\mathfrak{g}, \mathfrak{g}^*, \overline{\phi^-}, \overline{\phi^+}\right)$, where $\overline{\phi^-}(x)$ and $\overline{\phi^+}(y)$ are defined by

$$\overline{\phi^-}(x) : z \mapsto \phi^-(x, z), \quad \overline{\phi^+}(y) : z \mapsto \phi^+(y, z), \ \forall z \in \mathfrak{g}.$$

- $\mathfrak{g}$ is *decomposable* if $\mathfrak{g} = U \oplus U^\perp$.
- We can assume that $\mathfrak{g}$ is indecomposable.

The indecomposable modules $\left(\mathfrak{g}, \mathfrak{g}^*, \overline{\phi^-}, \overline{\phi^+}\right)$ turn out to be one of the following three pairs

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \begin{pmatrix} 0 & A \\ A^t & 0 \end{pmatrix} \tag{1}$$

$$\begin{pmatrix} 0 & A \\ -A^t & 0 \end{pmatrix}, \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \tag{2}$$

$$\begin{pmatrix} 0 & J_1 \\ -J_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & J_2 \\ J_2^t & 0 \end{pmatrix} \tag{3}$$

where $A \in \mathrm{M}_n(\mathbb{F})$, and J_1, J_2 are the matrices associated with the linear maps $F_1, F_2 : \mathbb{F}^n \to \mathbb{F}^{n+1}$ defined by

$$F_1(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0),$$

$$F_2(x_1, \ldots, x_n) = (0, x_1, \ldots, x_n).$$

Proposition

Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be Leibniz algebras of dimension n with one-dimensional commutator ideals $[\mathfrak{g}_1, \mathfrak{g}_1] = \mathbb{F}z_1$ and $[\mathfrak{g}_2, \mathfrak{g}_2] = \mathbb{F}z_2$, and let ϕ_1 and ϕ_2 be the bilinear forms associated with $\mathfrak{g}_1$ and $\mathfrak{g}_2$ respectively. One can fix bases of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ such that, if Φ_1 and Φ_2 are the matrices of ϕ_1 and ϕ_2 respectively, then $\mathfrak{g}_1$ is isomorphic to $\mathfrak{g}_2 \iff \Phi_1$ is congruent to Φ_2.

If $X \in \mathrm{GL}_n(\mathbb{F})$, then

$$\begin{pmatrix} X & 0 \\ 0 & \left(X^{-1}\right)^t \end{pmatrix}$$

induces a change of basis that transforms (1) and (2) respectively in

$$\left(\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \begin{pmatrix} 0 & XAX^{-1} \\ \left(XAX^{-1}\right)^t & 0 \end{pmatrix} \right),$$

$$\left(\begin{pmatrix} 0 & XAX^{-1} \\ -\left(XAX^{-1}\right)^t & 0 \end{pmatrix}, \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \right),$$

so we can always reduce A in its rational canonical form.

- If $A \in \mathrm{GL}_n(\mathbb{F})$, then the second canonical pair is equivalent to the first one:

$$\begin{pmatrix} A^{-1} & 0 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} 0 & A \\ -A^t & 0 \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & I_n \end{pmatrix}^t = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$
$$\begin{pmatrix} A^{-1} & 0 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & I_n \end{pmatrix}^t = \begin{pmatrix} 0 & A^{-1} \\ \left(A^{-1}\right)^t & 0 \end{pmatrix}.$$

- If $A \notin \mathrm{GL}_n(\mathbb{F})$, then $A \sim J_0$ and we have a unique Kronecker module of type (2) up to isomorphisms.

Definition

Let $f(x) \in \mathbb{F}[x]$ be a monic irreducible polynomial. Let $n \in \mathbb{N}$ and let $A = C(f(x)^n)$. The *Heisenberg* Leibniz algebra $\mathfrak{l}_{2n+1}^{A}$ is the $(2n+1)$-dimensional indecomposable Leibniz algebra with associated Kronecker module of type (1).

- If $A = (a_{ij}) \in \mathrm{M}_n(\mathbb{F})$ and $\{e_1, \ldots, e_n, f_1, \ldots, f_n, h\}$ is a basis of $\mathfrak{l}_{2n+1}^{A}$, then the non-trivial commutators are

$$[e_i, f_j] = (\delta_{ij} + a_{ij})\, h, \ \ [f_j, e_i] = (-\delta_{ij} + a_{ij})\, h, \ \forall i, j = 1, \ldots, n,$$

- If $(a_{ij}) = 0$, then we obtain the classical Heisenberg algebra $\mathfrak{h}_{2n+1}$.

Definition

Let $n \in \mathbb{N}$ and let $A = C(x^n)$. The *Kronecker* Leibniz algebra $\mathfrak{k}_n$ is the $(2n+1)$-dimensional indecomposable Leibniz algebra with associated Kronecker module of type (2).

Definition

The *Dieudonné* Leibniz algebra $\mathfrak{d}_n$ is the $(2n+2)-$dimensional Leibniz algebra with associated Kronecker module of type (3).

The case $\mathbb{F} = \mathbb{C}$

Let $n \in \mathbb{N}$ and let $a \in \mathbb{C}$. Then

$$C((x-a)^n) \sim J_a = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 1 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 1 & a \end{pmatrix}$$

Thus $\mathfrak{l}^{A}_{2n+1} \cong \mathfrak{l}^{J_a}_{2n+1}$ and the non-trivial Leibniz brackets are

$$[e_i, f_i] = (1+a)h,\ [f_i, e_i] = (-1+a)h\ \forall i = 1, \dots, n$$

$$[e_i, f_{i-1}] = h,\ [f_{i-1}, e_i] = h,\ \forall i = 2, \dots, n$$

where $\{e_1, \dots, e_n, f_1, \dots, f_n, h\}$ is a basis of $\mathfrak{l}^{J_a}_{2n+1}$.

Proposition

Let $a \in \mathbb{C}$. The Heinseberg Leibniz algebras $\mathfrak{l}_{2n+1}^{J_a}$ and $\mathfrak{l}_{2n+1}^{J_{-a}}$ are isomorphic.

Proof.

- $\mathfrak{l}_{2n+1}^{J_a} \cong \mathfrak{l}_{2n+1}^{-J_a^t}$ via the linear map φ defined by

$$\varphi(e_i) = f_i', \ \ \varphi(f_i) = e_i', \ \ \varphi(h) = -h', \ \ \forall i = 1, \cdots, n$$

- $-J_a^t \sim J_{-a} \Rightarrow \mathfrak{l}_{2n+1}^{J_a} \cong \mathfrak{l}_{2n+1}^{J_{-a}}$.

$\square$

Proposition

Let $a, a' \in \mathbb{C}$. Then $\mathfrak{l}_3^a \cong \mathfrak{l}_3^{a'} \iff a' = \pm a$.

Derivations of two-step nilpotent Leibniz algebras

Definition

Let $\mathfrak{g}$ be a Leibniz algebra over $\mathbb{F}$. A *derivation* of $\mathfrak{g}$ is a linear map $d : \mathfrak{g} \to \mathfrak{g}$ such that

$$d([x,y]) = [d(x), y] + [x, d(y)]\,, \ \ \forall x, y \in \mathfrak{g}.$$

- With the usual bracket $[d_1, d_2] = d_1 \circ d_2 - d_2 \circ d_1$, the set $\mathrm{Der}(\mathfrak{g})$ is a Lie algebra and the set $\mathrm{Inn}(\mathfrak{g})$ of all inner derivations of $\mathfrak{g}$ is an ideal of $\mathrm{Der}(\mathfrak{g})$.
- We obtain a class of Lie algebras with trivial center and abelian ideal of inner derivations.
- We show that every *almost inner derivation* of a nilpotent Leibniz algebra with one-dimensional commutator ideal, with three exceptions, is an inner derivation.

Derivations of $\mathfrak{l}_{2n+1}^{J_a}$

The Lie algebra of derivations of $\mathfrak{l}_{2n+1}^{J_a}$ is represented, with respect to the basis $\{e_1, f_1, \ldots, e_n, f_n, z\}$, by the set of $(2n+1) \times (2n+1)$ matrices of type

$$\left(\begin{array}{ccccccc}
X & -\widetilde{M}_1 & -\widetilde{M}_2 & -\widetilde{M}_3 & \cdots & -\widetilde{M}_{n-1} & 0 \\
M_1 & X & -\widetilde{M}_1 & -\widetilde{M}_2 & \cdots & -\widetilde{M}_{n-2} & 0 \\
M_2 & M_1 & X & -\widetilde{M}_1 & \cdots & -\widetilde{M}_{n-3} & 0 \\
M_3 & M_2 & M_1 & X & \cdots & -\widetilde{M}_{n-4} & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
M_{n-1} & M_{n-2} & M_{n-3} & M_{n-4} & \cdots & X & 0 \\
\hline
v_1 & v_2 & v_3 & v_4 & \cdots & v_n & \mathrm{tr}(X)
\end{array}\right)$$

where

$$X = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, \quad M_i = \begin{pmatrix} 0 & 0 \\ 0 & \alpha_i \end{pmatrix}, \quad \widetilde{M}_i = \begin{pmatrix} \alpha_i & 0 \\ 0 & 0 \end{pmatrix}, \quad \forall i = 1, \ldots, n-1,$$

$v_k = (a_k, b_k) \in \mathbb{C}^2$, $\forall k = 1, \ldots, n$, and $\mathrm{tr}(X) = x + y$, if $a \neq 0$;

by the matrices of type

$$\left(\begin{array}{cccccccccccc}
x & c_2 & -\alpha_1 & 0 & -\alpha_2 & c_4 & \cdots & -\alpha_{n-2} & c_n & -\alpha_{n-1} & 0 & 0 \\
0 & y & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & x & -c_4 & -\alpha_1 & 0 & \cdots & -\alpha_{n-3} & 0 & -\alpha_{n-2} & 0 & 0 \\
0 & \alpha_1 & 0 & y & 0 & 0 & \cdots & 0 & 0 & -b_{n+2} & 0 & 0 \\
0 & c_4 & 0 & 0 & x & c_6 & \cdots & -\alpha_{n-4} & 0 & -\alpha_{n-3} & 0 & 0 \\
0 & \alpha_2 & 0 & \alpha_1 & 0 & y & \cdots & b_{n+2} & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & c_n & 0 & 0 & 0 & 0 & \cdots & x & 0 & -\alpha_1 & 0 & 0 \\
0 & \alpha_{n-2} & 0 & \alpha_{n-3} & b_{n+2} & \alpha_{n-4} & \cdots & b_{2n-2} & y & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & x & 0 & 0 \\
0 & \alpha_{n-1} & -b_{n+2} & \alpha_{n-2} & 0 & \alpha_{n-3} & \cdots & 0 & \alpha_1 & b_{2n} & y & 0 \\
\hline
a_1 & b_1 & a_2 & b_2 & a_3 & b_3 & \cdots & a_{n-1} & b_{n-1} & a_n & b_n & x+y
\end{array}\right)$$

if $a = 0$ and n is even;

by the set of matrices of the form

$$\left(\begin{array}{cccccccccccc}
x & c_2 & -\alpha_1 & 0 & -\alpha_2 & c_4 & \cdots & -\alpha_{n-2} & 0 & -\alpha_{n-1} & c_{n+1} & 0 \\
0 & y & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & b_{n+1} & 0 & 0 \\
0 & 0 & x & -c_4 & -\alpha_1 & 0 & \cdots & -\alpha_{n-3} & -c_{n+1} & -\alpha_{n-2} & 0 & 0 \\
0 & \alpha_1 & 0 & y & 0 & 0 & \cdots & -b_{n+1} & 0 & 0 & 0 & 0 \\
0 & c_4 & 0 & 0 & x & c_6 & \cdots & -\alpha_{n-4} & 0 & -\alpha_{n-3} & 0 & 0 \\
0 & \alpha_2 & 0 & \alpha_1 & 0 & y & \cdots & 0 & 0 & b_{n+3} & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \\
0 & 0 & 0 & -c_{n+1} & 0 & 0 & \cdots & x & 0 & -\alpha_1 & 0 & 0 \\
0 & \alpha_{n-2} & -b_{n+1} & \alpha_{n-3} & 0 & \alpha_{n-4} & \cdots & -b_{2n-2} & y & 0 & 0 & 0 \\
0 & c_{n+1} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & x & 0 & 0 \\
b_{n+1} & \alpha_{n-1} & 0 & \alpha_{n-2} & b_{n+3} & \alpha_{n-3} & \cdots & 0 & \alpha_1 & b_{2n} & y & 0 \\
\hline
a_1 & b_1 & a_2 & b_2 & a_3 & b_3 & \cdots & a_{n-1} & b_{n-1} & a_n & b_n & x+y
\end{array}\right)$$

if $a = 0$ and n is odd.

The Lie algebra $\mathrm{Der}(\mathfrak{l}_{2n+1}^{J_a})$, with $a \neq 0$

- $\dim_{\mathbb{F}} \mathrm{Der}(\mathfrak{l}_{2n+1}^{J_a}) = 3n + 1$. A basis over $\mathbb{F}$ is

$$\{x, y, \alpha_1, \ldots, \alpha_{n-1}, A_1, \ldots, A_n, B_1, \ldots, B_n\}$$

  and the non-trivial commutators are

$$[x, B_i] = B_i,\ [y, A_i] = A_i,\ \ \forall i = 1, \ldots, n$$

$$[\alpha_i, B_k] = -B_{k-2i},\ \ \forall k > 2i, \quad [\alpha_i, A_k] = A_{k+2i},\ \ \forall k + 2i \leq 2n;$$

- $\dim_{\mathbb{F}}[\mathrm{Der}(\mathfrak{l}_{2n+1}^{J_a}), \mathrm{Der}(\mathfrak{l}_{2n+1}^{J_a})] = 2n$ and a basis is

$$\{A_1, \ldots A_n, B_1, \ldots, B_n\};$$

The Lie algebra $\mathrm{Der}(\mathfrak{l}_{2n+1}^{J_a})$, with $a \neq 0$

- $[\mathrm{Der}(\mathfrak{l}_{2n+1}^{J_a}), \mathrm{Der}(\mathfrak{l}_{2n+1}^{J_a})]$ is an abelian Lie algebra;
- $\mathrm{Der}(\mathfrak{l}_{2n+1}^{J_a})$ is a two-step solvable Lie algebra;
- $\mathrm{Der}(\mathfrak{l}_{2n+1}^{J_a})$ is not nilpotent and its nilradical is the ideal

$$N = \mathrm{Span}_{\mathbb{F}}\{\alpha_1, \ldots, \alpha_{n-1}, A_1, \ldots, A_n, B_1, \ldots, B_n\};$$

- $Z(\mathrm{Der}(\mathfrak{l}_{2n+1}^{J_a})) = 0$.

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Almost inner derivations

Definition

A derivation d of a left Leibniz algebra $\mathfrak{g}$ is an *almost inner derivation* if $d(x) \in [\mathfrak{g}, x]$, for every $x \in \mathfrak{g}$.

- The set of all almost inner derivations of $\mathfrak{g}$ forms a Lie subalgebra of $\mathrm{Der}(\mathfrak{g})$, denoted by $\mathrm{AIDer}(\mathfrak{g})$, containing the ideal $\mathrm{Inn}(\mathfrak{g})$;
- Every almost inner derivations of a Lie algebra with one-dimensional commutator ideal is an inner derivation *(D. Burde, K. Dekimpe and B. Verbeke*, 2017).

Proposition (*G. L., M. M.*, 2022)

Let $\mathfrak{g}$ be a nilpotent Leibniz algebra with one-dimensional commutator ideal $[\mathfrak{g}, \mathfrak{g}] = \mathbb{F}z$, such that $\mathfrak{g} \neq \mathfrak{l}_{2n+1}^{J_{\pm 1}}$ and $\mathfrak{g} \neq \mathfrak{d}_n$. Then every almost inner derivation $d \in \mathrm{AIDer}(\mathfrak{g})$ is of the form $\mathrm{ad}_x = [x, -]$, for some $x \in \mathfrak{g}$.

Almost inner derivations of $\mathfrak{l}_{2n+1}^{J_{\pm 1}}$

- For the Heisenberg Leibniz algebras $\mathfrak{l}_{2n+1}^{J_a}$ with $a = \pm 1$ and for the Dieudonné Leibniz algebra $\mathfrak{d}_n$ it is possible to define an almost inner derivation d which is not inner.
- Every almost inner derivation of $\mathfrak{l}_{2n+1}^{J_a}$ is of the form

$$\left(\begin{array}{c|c} & 0 \\ 0 & \vdots \\ & 0 \\ \hline a_1 \; b_1 \ldots a_n \; b_n & 0 \end{array}\right)$$

meanwhile for the inner derivations we have that $a_1 = 0$, if $a = 1$ and $b_n = 0$, if $a = -1$;

Almost inner derivations of $\mathfrak{d}_n$

- $AIDer(\mathfrak{d}_n)$ consists of the matrices of the type

$$\left(\begin{array}{c|c} \multirow{3}{*}{0} & 0 \\ & \vdots \\ & 0 \\ \hline a_1 \ a_2 \dots a_{2n+1} & 0 \end{array}\right)$$

and an example of almost inner but non-inner derivation is given by the linear map $d \in \mathrm{gl}(\mathfrak{d}_n)$ defined by $d(e_{n+1}) = z$.

Part 2: *Coquecigrue Problem* for nilpotent real Leibniz algebras

- It is the problem, formulated by J.-L. Loday, of finding a generalization to Leibniz algebras of Lie's third theorem, which associates a Lie local group with any real or complex Lie algebra.
- Given any Leibniz algebra $\mathfrak{g}$, one wants to find a manifold endowed with a smooth map which plays the role of Ad for Lie groups, and such that the tangent space at a distinguished point, endowed with the differential ad of Ad, is a Leibniz algebra isomorphic to $\mathfrak{g}$.
- In the special case of Lie algebras $\mathfrak{g}$, one wants to obtain the usual simply connected Lie group associated with $\mathfrak{g}$.
- We assume now that $\mathbb{F} = \mathbb{R}$.

Definition

A (left) *rack* is a set X with a binary operation $\triangleright : X \times X \to X$ which is (left) autodistributive

$$x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)\,, \quad \forall x, y, z \in X$$

and such that $x \triangleright - : X \to X$ is a bijection $\forall x \in X$.
A rack is *pointed* if $\exists\, 1 \in X$ such that $1 \triangleright x = x$ and $x \triangleright 1 = 1$, $\forall x \in X$.
A rack $(X, \triangleright)$ is a *quandle* if $x \triangleright x = x$, $\forall x \in X$.

Example

Every group endowed with the conjugation is a pointed quandle.

Definition

A *Lie rack* is a pointed rack $(X, \triangleright, 1)$ such that X is a smooth manifold, $\triangleright$ is a smooth map and $x \triangleright -$ is a diffeomorphism $\forall x \in X$.

The tangent space of a Lie rack

Proposition (*M.-K. Kinyon*, 2007)

If X is a Lie rack, then the bracket

$$[x, y] = \mathrm{ad}_x(y),$$

where

- $\mathrm{ad} : T_1 X \to \mathfrak{gl}(T_1 X)$ *is the differential of* $\Phi : X \mapsto \mathrm{GL}(T_1 X)$;
- $\Phi(x) = T_1 \phi(x)$, $\forall x \in X$;
- $\phi(x) = x \rhd -$, $\forall x \in X$

defines on $T_1 X$ a Leibniz algebra structure.

- M.-K. Kinyon solves the *coquecigrue problem* for the class of *split Leibniz algebras*, which are Leibniz algebras $\mathfrak{g}$ with a bilateral ideal $\mathrm{Leib}(\mathfrak{g}) \subseteq \mathfrak{a} \subseteq \mathrm{Z}_l(\mathfrak{g})$ and with a Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{h} \oplus_s \mathfrak{a}$$

  and

$$[(x,a),(y,b)] = ([x,y], \rho_x(b)), \ \ \forall (x,a),(y,b) \in \mathfrak{h} \oplus \mathfrak{a}$$

  where $\rho : \mathfrak{h} \times \mathfrak{a} \to \mathfrak{a}$ is the action on the $\mathfrak{h}$-module $\mathfrak{a}$.

Proposition (*M.-K. Kinyon*, 2007)

Let $\mathfrak{g}$ be a split Leibniz algebra. Then there exists a Lie rack X such that T_1X is isomorphic to $\mathfrak{g}$.

The local integration of a Leibniz algebra

- S. Covez gives a local solution to the coquegigrue problem: he shows how to integrate every Leibniz algebra into a Lie local rack.
- The central point of his result is to see every Leibniz algebra $\mathfrak{g}$ as an abelian extension with kernel $Z_l(\mathfrak{g})$ and to integrate explicitly the corresponding Leibniz algebra 2-cocycle into a Lie local rack 2-cocycle.

- We associate with $\mathfrak{g}$ the short exact sequence

$$0 \to \mathrm{Z}_l(\mathfrak{g}) \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{g}_0 \to 0$$

where $\mathfrak{g}_0 = \mathfrak{g}/\,\mathrm{Z}_l(\mathfrak{g})$.

- $\mathrm{Z}_l(\mathfrak{g})$ is a $\mathfrak{g}_0-$module $\Rightarrow \exists\, \omega \in \mathrm{ZL}^2\left(\mathfrak{g}_0, \mathrm{Z}_l(\mathfrak{g})\right)$ such that

$$\mathfrak{g} = \mathfrak{g}_0 \oplus_\omega \mathrm{Z}_l(\mathfrak{g})$$

with

$$[(x,a)\,,(y,b)] = \Big([x,y]_{\mathfrak{g}_0}\,, \rho_x(b) + \omega(x,y)\Big)\,,$$

where $\rho : \mathfrak{g}_0 \times \mathrm{Z}_l(\mathfrak{g}) \to \mathrm{Z}_l(\mathfrak{g})$ is the action induced by the $\mathfrak{g}_0$-module structure of $\mathrm{Z}_l(\mathfrak{g})$.

Theorem (*S. Covez*, 2013)

Every Leibniz algebra $\mathfrak{g} = \mathfrak{g}_0 \oplus_\omega \mathfrak{a}$ *can be integrated into a Lie local rack of the form*

$$G_0 \times_f \mathfrak{a}$$

with operation defined by

$$(g, a) \rhd (h, b) = \left(ghg^{-1}, \phi_g(b) + f(g, h)\right), \ \ \forall (g, a), (h, b) \in G_0 \times \mathfrak{a}$$

and unit $(1, 0)$, *where* G_0 *is a Lie group such that* $\mathrm{Lie}(G_0) = \mathfrak{g}_0$, ϕ *is the exponentiation of the action* ρ, $f : G_0 \times G_0 \to \mathfrak{a}$ *is the Lie local racks* $2-$ *cocycle defined by*

$$f(g, h) = \int_{\gamma_h} \left(\int_{\gamma_g} \tau^2(\omega)^{eq} \right)^{eq}, \ \ \forall g, h \in G_0$$

and $\tau^2(\omega) \in \mathrm{ZL}^1(\mathfrak{g}_0, Hom(\mathfrak{g}_0, \mathfrak{a}))$ *is defined by* $\tau^2(\omega)(x)(y) = \omega(x, y)$, $\forall x, y \in \mathfrak{g}_0$.

The case of quandles

Theorem (*G. L., M. M.*, 2022)

Let R be a Lie rack integrating a Leibniz algebra $\mathfrak{g}$. R is a quandle if and only if $\mathfrak{g}$ is a Lie algebra. In particular $R = \mathrm{Conj}(G)$, where $\mathrm{Lie}(G) = \mathfrak{g}$.

$\mathrm{G}{=}\mathrm{G}_0 \times_F \mathrm{Z}_l(\mathfrak{g})$ is the Lie group with operation

$$(g, a)(h, b) = (gh, a + b + F(g, h)), \ \ \forall (g, a), (h, b) \in G_0 \times \mathrm{Z}_l(\mathfrak{g})$$

where $F : G_0 \times G_0 \to \mathrm{Z}_l(\mathfrak{g})$ is a Lie group $2-$cocycle such that

$$f(g, h) = F(g, h) - F(g, g^{-1}) + F(gh, g^{-1}), \quad \forall g, h \in G_0.$$

Global integration of nilpotent Leibniz algebras

Theorem (*G. L., M. M.*, 2022)

Every nilpotent real Leibniz algebra $\mathfrak{g}$ *has a global integration into a Lie rack.*

Sketch of the proof

- $\mathfrak{g} = \mathfrak{g}_0 \oplus_\omega \mathrm{Z}_l(\mathfrak{g})$
- $\mathfrak{g}$ is nilpotent $\Rightarrow \mathfrak{g}_0$ is a nilpotent Lie algebra and

$$\rho_x(a) = \mathrm{ad}_x(a), \ \ \forall x \in \mathfrak{g}_0, a \in \mathrm{Z}_l(\mathfrak{g}),$$

  can be represented as strictly lower triangular matrix.
- If G_0 is the simply connected Lie group integrating $\mathfrak{g}_0$, then $\phi_{\exp(x)} = \exp(\rho_x) \in \mathrm{Aut}(\mathrm{Z}_l(\mathfrak{g}))$ is a unitriangular matrix whose entries are polynomial expressions.
- Fixed $\gamma_g(s) = g^s, \gamma_h(t) = h^t, \forall g, h \in G_0, \forall s, t \in [0,1]$, we have that

$$f(g,h) = \int_{\gamma_h} \left(\int_{\gamma_g} \tau^2(\omega)^{eq} \right)^{eq} \quad \forall g, h \in G_0$$

  is everywhere defined because it involves the integration of matrices with polynomial entries in $\mathbb{R}[s]$ and $\mathbb{R}[t]$.

Integration of two-step nilpotent Leibniz algebras

Theorem (*G. L., M. M.*, 2022)

Let $(\mathfrak{g}, [-,-])$ *be a two-step nilpotent Leibniz algebra and let* $\omega : \mathfrak{g}_0 \times \mathfrak{g}_0 \to [\mathfrak{g}, \mathfrak{g}]$, *where* $\mathfrak{g}_0 = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$, *be the Leibniz algebras* $2-$*cocycle associated with the short exact sequence*

$$0 \to [\mathfrak{g}, \mathfrak{g}] \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{g}_0 \to 0.$$

Then the multiplication

$$(x, a) \rhd (y, b) = (y, b + \omega(x, y)), \ \ \forall (x, a), (y, b) \in \mathfrak{g}_0 \times [\mathfrak{g}, \mathfrak{g}]$$

defines a Lie global rack structure on $\mathfrak{g}_0 \times [\mathfrak{g}, \mathfrak{g}]$, *such that* $T_{(0,0)}(\mathfrak{g}_0 \times_\omega [\mathfrak{g}, \mathfrak{g}], \rhd)$ *is a Leibniz algebra isomorphic to* $\mathfrak{g}$.

The effective strategy is to choose $G_0 = \mathfrak{g}_0$ as a Lie group integrating the abelian Lie algebra $\mathfrak{g}_0$. If we change the Lie group G_0, then the integration may not be global.

Example

Let $a \in \mathbb{R}$ and let $\mathfrak{g} = \mathfrak{l}_3^a = \mathrm{Span}_{\mathbb{R}}\{e, f, h\}$ be the three-dimensional Heinseberg Lebiniz algebra with non-trivial commutators

$$[e, f] = (1 + a)h, \quad [f, e] = (-1 + a)h.$$

Then $[\mathfrak{g}, \mathfrak{g}] = \mathrm{Z}(\mathfrak{g}) \cong \mathbb{R}$ and $\mathfrak{g}_0 = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \cong \mathbb{R}^2$ by $\mathbb{R}$. The corresponding Leibniz algebras 2-cocycle is

$$\omega((x, y), (x', y')) = (1 + a)xy' + (-1 + a)x'y.$$

Example

Now we can choose

$$G_0 = \mathrm{SO}(2) \times \mathrm{SO}(2) \cong \{(e^{ix}, e^{iy}) \mid x, y \in \mathbb{R}\}$$

as a Lie group integrating $\mathfrak{g}_0$. In this case a Lie local rack integrating $\mathfrak{g}$ is $(G_0 \times \mathrm{SO}(2), \rhd)$ with multiplication

$$(e^{ix}, e^{iy}, e^{iz}) \rhd (e^{ix'}, e^{iy'}, e^{iz'}) =$$

$$= (e^{ix'}, e^{iy'}, e^{i(z' + \omega((\log(e^{ix}), \log(e^{iy})), (\log(e^{ix'}), \log(e^{iy'}))))}),$$

that is defined only for $(x, y), (x', y') \in [0, 2\pi[\times [0, 2\pi[$, where we choose $[0, 2\pi[$ as the domain of the principal value of the function log.

Example

Let $A = (a_{ij})_{i,j} \in \mathrm{M}_n(\mathbb{R})$ and let $\mathfrak{g} = \mathfrak{l}^A_{2n+1}$ be the $(2n+1)-$dimensional Heisenberg Leibniz algebra with basis $\{e_1, \dots, e_n, f_1, \dots, f_n, h\}$ and bracket

$$\left[(x_1, \dots, x_n, y_1, \dots, y_n, z), (x'_1, \dots, x'_n, y'_1, \dots, y'_n, z')\right] =$$

$$= \left(0, \dots, 0, \sum_{i,j=1}^{n} (\delta_{ij} + a_{ij})\, x_i y'_j + \sum_{i,j=1}^{n} (-\delta_{ij} + a_{ij})\, x'_i y_j\right)$$

We have that $[\mathfrak{g}, \mathfrak{g}] = \mathbb{R}h \subseteq \mathrm{Z}(\mathfrak{g})$, thus the Lie global rack integrating $\mathfrak{g}$ is $R^A_{2n+1} = (\mathfrak{g}_0 \times_f \mathbb{R}h, \rhd)$ with multiplication

$$(x_1, \dots, x_n, y_1, \dots, y_n, z) \rhd (x'_1, \dots, x'_n, y'_1, \dots, y'_n, z') =$$

$$= \left(x'_1, \dots, x'_n, y'_1, \dots, y'_n, z' + \sum_{i,j=1}^{n} \left[(\delta_{ij} + a_{ij})\, x_i y'_j + (-\delta_{ij} + a_{ij})\, x'_i y_j\right]\right)$$

Definition

We define $(R^A_{2n+1}, \triangleright)$ as the *Heisenberg rack*.

- The multiplication $\triangleright$ in R^A_{2n+1} is a *perturbation* of the conjugation of the Heisenberg Lie group H_{2n+1}.
- We use the canonical matrix representation

$$H_{2n+1} = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & I_n & y^t \\ 0 & 0 & 1 \end{pmatrix} \middle| \, x, y \in \mathbb{R}^n, z \in \mathbb{R} \right\} \leq \mathrm{GL}_{n+2}(\mathbb{R})$$

The conjugation formula for two matrices in H_{2n+1} is given by

$$\begin{pmatrix}1 & x & z\\ 0 & I_n & y^t\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & x' & z'\\ 0 & I_n & y'^t\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & x & z\\ 0 & I_n & y^t\\ 0 & 0 & 1\end{pmatrix}^{-1} =$$

$$= \begin{pmatrix}1 & x' & z' + \sum_{i=1}^{n}(x_i y_i' - y_i x_i')\\ 0 & I_n & y'^t\\ 0 & 0 & 1\end{pmatrix}$$

With the same representation, the multiplication $\triangleright$ of R^{A}_{2n+1} turns into

$$\begin{pmatrix} 1 & x & z \\ 0 & I_n & y^t \\ 0 & 0 & 1 \end{pmatrix} \triangleright \begin{pmatrix} 1 & x' & z' \\ 0 & I_n & y'^t \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & x' & z' + \sum_{i,j=1}^{n} \left[(\delta_{ij} + a_{ij})\, x_i y'_j + (\delta_{ij} - a_{ij})\, x'_i y_j \right] \\ 0 & I_n & y'^t \\ 0 & 0 & 1 \end{pmatrix},$$

hence for $A = 0_{n\times n}$, it holds $R^{0_{n\times n}}_{2n+1} = \mathrm{Conj}(H_{2n+1})$.

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Thank you for your attention!