

# Slow-Roll Inflation at N3LO

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The next generation of cosmological observations will be sensitive to small deviations from a pure power law in the primordial power spectrum of the curvature perturbations. In the context of slow-roll inflation, these deviations are expected and correspond to the so-called running of the spectral index. Their measurement would bring as much information as the discovery of deviations from scale invariance. However, robust parameter inference requires to marginalize over any possible higher order uncertainties, which have been, up to now, not fully determined.

We tackle this issue by deriving the inflationary scalar and tensor slow-roll power spectra at next-to-next-to-next to leading order (N3LO), fully expanded around an observable pivot wavenumber, for all single field inflationary models having minimal and non-minimal kinetic terms. Our result therefore encompasses string-inspired inflationary models having a varying speed of sound.

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## I. INTRODUCTION

The final release of the Planck Legacy Cosmic Microwave Background (CMB) data has made Cosmic Inflation the most favored scenario of the early universe [1–12]. Inflation implies that all forms of matter and radiation observed today, as well as their large scale structures, are the outcome of coupled quantum fluctuations of both the metric and a yet unknown scalar degree of freedom around a quasi de-Sitter empty universe. Inflation can be realized in multiple ways, ranging from modified gravity effects, multifield models, to membranes moving in higher dimensional spacetimes [13–17]. However, the Planck Legacy data come with strong constraints on various observables that could have been the signature of non-linear physics in the early universe. As of today, the CMB anisotropies show no detectable primordial non-Gaussianities, no peculiar features, and, primordial isocurvature modes are severely constrained to be small [18, 19]. In other words, the data favor the simplest vanilla inflationary models in which a single scalar field slowly rolls down a smooth potential. Obviously, this does not imply that other physics is not at work, but, it has to be subdominant. Searching for small effects will then require us to have at our disposal precise theoretical predictions for the dominant one; namely, slow-roll inflation. This is the context of this paper.

Slow-roll inflation is a landscape in itself, it is the natural outcome of many self-gravitating single field models [20–25] and hundreds of scenarios have been proposed so far [14, 15]. However, there exists a unified treatment to perturbatively derive the expected scalar and tensor power spectra. The method has been pioneered in

Ref. [1] for the tensor modes and in Refs. [26, 27] for the scalar modes, then extended to include next-to-leading order corrections in Refs [28–35]. It has also been tested with other approximation schemes and applied to various classes of models in Refs. [36–58]. The perturbative expansion uses the so-called Hubble flow functions (also abusively referred to as the slow-roll parameters when evaluated at a given time) defined by

$$\epsilon_{i+1}(N) \equiv \frac{d \ln |\epsilon_i|}{dN}, \quad \epsilon_1(N) \equiv -\frac{d \ln H}{dN}. \quad (1)$$

Here  $H = \dot{a}(t)/a(t)$  stands for the Hubble parameter,  $N = \ln(a)$  is the e-fold number and  $a(t)$  the Friedmann–Lemaître–Robertson–Walker (FLRW) scale factor during inflation. For a quasi-de Sitter expansion,  $H(N)$  is almost constant and the Hubble flow functions quantitatively encode how much inflation deviates from de Sitter. For any single field model, their functional shape can be (perturbatively) calculated from the field’s potential [29] while they are exactly determined by the equation-of-state parameter in fluid representations of single-field inflation [59, 60]. Without yet entering into details,  $H$  (leading order) fixes the amplitude of the fluctuations,  $\epsilon_1$  makes some corrections to that amplitude, sets the tensor-to-scalar ratio, and, together with  $\epsilon_2$ , fixes the spectral index (first order). At second order,  $\epsilon_3$  makes corrections to the former quantities and enters into the running of the spectral index, and so on and so forth.

Historically, the second-order corrections for the scalar spectral index were first derived in Ref. [31] but the fully expanded scalar power spectrum at second order (N2LO) was explicitly derived in Ref. [33]. The tensor-mode power spectrum at N2LO was derived soon after in Ref. [34]. For inflation with a non-minimal kinetic term, the N2LO slow-roll scalar and tensor spectra have been calculated in Refs. [56, 57]. Third-order corrections to the scalar amplitude, for a minimal kinetic term, were first derived in Ref. [40], but not fully expanded around a pivot wavenumber.

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The accuracy of the slow-roll expanded power spectra can be assessed by comparing their predictions to a full numerical integration of the linear perturbations [61–68]. Moreover, as discussed in Refs. [34, 65], performing robust data analysis of slow-roll inflation requires marginalizing over the unconstrained higher order slow-roll parameters. These tests have been done in Ref. [69] for the Planck data, and, performing inflationary Bayesian parameter estimation based on N2LO power spectra ends up being indistinguishable from an exact numerical treatment. Therefore, up to now, N2LO corrections are sufficient.

But the situation is bound to change with the next generation of CMB observations and the incoming large-scale structures surveys. Combined together, data from the CMB-S4 telescopes and from the LiteBird satellite are not only going to increase the sensitivity to  $B$ -modes, and thus to primordial gravitational waves ( $\epsilon_1$ ), but also to probe much higher multipoles [70–73]. The same remark concerns the Euclid satellite, and other ground surveys, that will be measuring the small scales in the matter power spectrum [74–76]. All these cosmological measurements are going to increase the lever arm between the large and small angular scales, and, assuming that foregrounds will be well enough understood to be removed, or fitted, one should expect the data to be sensitive to any possible running of the primordial spectral index; namely, to the third slow-roll parameter  $\epsilon_3$  [77–79]. Let us also notice that if inflation proceeds at very low-energy scales, primordial gravitational waves would be undetectable and the first parameter  $\epsilon_1$  would end up being very small [80–82]. In that situation, at N2LO, the scalar power spectrum is driven by only  $\epsilon_2$  and  $\epsilon_3$ , with  $\epsilon_2 \simeq 1 - n_s \simeq 0.04$ , and all new data will necessarily contribute to an information gain on  $\epsilon_3$  [83, 84].

For all these reasons, in this paper, we push the slow-roll calculations one step further, and derive at next-to-next-to-next-to-leading order (N3LO) both the scalar and tensor power spectrum, with minimal and non-minimal kinetic terms. Our results are derived using the Green’s function method [31] and are fully expanded around a pivot wavenumber. As such,  $\epsilon_4$  explicitly corrects the lower perturbative terms and enters into the expression of the running of the running of the spectral index. The expressions that we derive in this work will allow us to robustly search for a non-vanishing  $\epsilon_3$  in the future cosmological data.

Finally, let us clarify that the slow-roll calculations in general, and the ones presented below, are semi-classical. We are dealing with quantum fluctuations arising around a classical quasi-de Sitter spacetime. As such, one cannot go forever including higher-order semi-classical corrections, at some point the quantum backreaction is expected to dominate. The latter can actually be estimated using the stochastic inflation formalism [85, 86]. The non-perturbative quantum corrections to the scalar amplitude are given by  $\delta_{\text{qu}}\mathcal{P}_\zeta/\mathcal{P}_\zeta = \mathcal{P}_\zeta(\epsilon_1 + \epsilon_2) \simeq 8 \times 10^{-11}$  [87]. The semi-classical slow-roll corrections

at order  $p$  contain a term of order  $\epsilon_2^p$ , which, at N3LO, is  $\epsilon_2^3 \simeq 6 \times 10^{-5}$ , i.e., six orders of magnitude larger than the expected quantum backreaction<sup>1</sup>. This simple estimate suggests that semi-classical slow-roll corrections are expected to be relevant up to order six. Notice, however, that the aforementioned reasoning assumes that quantum diffusion never dominates the field dynamics in the last 60, or so, e-folds of inflation. As such, it may not apply if a phase of quantum diffusion takes place just before the end of inflation, as one that could produce primordial black holes [88–91]. In that situation, one can show that the relation between observable wavenumbers and field values at Hubble crossing become probabilistic, and the observed power spectra may even be severely modified at first order [92].

The paper is organized as follows. In Section II we briefly recap the Green’s function method and apply it to the equation of motion of primordial gravitational waves to derive the slow-roll tensor power spectrum at N3LO. At this order, a new set of definite integrals need to be calculated and some details on their derivation are given in Appendix A. In Section III, we derive the scalar power spectra, for inflation with minimal and non-minimal kinetic terms, using the mapping method introduced in Ref. [57]. The scalar results have been cross-checked using a direct calculation which is summarized in the Appendix B. For varying speed of sound models, we also explicitly recast the tensor spectrum at the same pivot as the scalar spectrum where it depends on the speed of sound  $c_s$  [56]. Finally, using the independence of the spectra with respect to the pivot scale [35], we give, in Appendix C, all the power-law indices expanded at one more order, namely N4LO. We conclude in Section IV.

## II. PRIMORDIAL GRAVITATIONAL WAVES

Within the theory of cosmological perturbations, the tensor linear perturbations around a FLRW spatially flat metric verify, in Fourier space, the equation of motion [93]

$$\mu'' + \left(k^2 - \frac{a''}{a}\right)\mu = 0, \quad (2)$$

where a prime denotes differentiation with respect to the conformal time  $\eta$ . In this equation,  $k$  stands for the comoving wavenumber and we assume a mostly positive metric signature. The mode function  $\mu(\eta, k)$  is related to the traceless and divergenceless tensor perturbations  $h_{ij}(\eta, k)$  by  $\mu(\eta, k) \equiv h_\lambda(\eta, k)a(\eta)$  where  $h_\lambda$  is one of the

<sup>1</sup> Non-perturbative quantum corrections to the spectral index can also be explicitly derived within the stochastic inflation formalism. One finds that they are accordingly reduced by one power in the slow-roll parameters. Ignoring terms containing  $\epsilon_1$  and  $\epsilon_3$ , one has  $\delta_{\text{qu}}n_s/n_s \simeq -3\mathcal{P}_\zeta\epsilon_2^2$ .

two polarization states of  $h_{ij}$  ( $\lambda = +, \times$ ). In the following, the explicit dependence in  $\lambda$  is dropped as, within General Relativity, the two states are identically created and propagated [94]. They are accounted for, as two degrees of freedom, when estimating the tensor power spectrum. Equation (2) describes a parametric oscillator evolving in a time-dependent effective potential

$$U_T(\eta) \equiv \frac{a''}{a} = \mathcal{H}^2 (2 - \epsilon_1), \quad (3)$$

where  $\mathcal{H}(\eta) = a(\eta)H(\eta)$  is the conformal Hubble parameter and use has been made of Eq. (1).

### A. Hubble flow expansion

The Hubble flow functions introduced in Eq. (1) can be used to provide a perturbative expression for the potential  $U_T$ . In the following, we choose the conformal time  $\eta$  to be negative during inflation. From the definition of  $\eta$ , one gets

$$\begin{aligned} -\eta &= \int_{\eta}^{0^-} \frac{dt}{a(t)} = \int_{a(\eta)}^{+\infty} \frac{da}{a^2 H(a)} \\ &= \frac{1}{aH} + \int_a^{+\infty} \frac{1}{a} \frac{d(H^{-1})}{da} da, \end{aligned} \quad (4)$$

where we have made a change of variable from  $t$  to  $a(t)$  and performed an integration by part. The integrand is now proportional to the derivative of the Hubble radius, i.e., a small quantity during slow-roll inflation. From Eq. (1) one has

$$\frac{d(H^{-1})}{da} = \frac{\epsilon_1}{aH}, \quad (5)$$

and Eq. (4) can be once more integrated by parts as

$$-\eta = \frac{1 + \epsilon_1}{aH} + \int_a^{+\infty} \frac{1}{a} \frac{d}{da} \left( \frac{\epsilon_1}{H} \right) da. \quad (6)$$

From Eq. (1), the Hubble flow functions verify

$$\frac{d\epsilon_i}{da} = \frac{\epsilon_i \epsilon_{i+1}}{a}, \quad (7)$$

and each successive integration by parts of Eq. (6) ends up being a perturbative expansion in the Hubble flow functions. For our purpose, this needs to be pushed up to third order and one gets

$$\begin{aligned} -\eta &= \frac{1}{\mathcal{H}} (1 + \epsilon_1 + \epsilon_1^2 + \epsilon_1 \epsilon_2 + \epsilon_1^3 + 3\epsilon_1^2 \epsilon_2 + \epsilon_1 \epsilon_2^2 + \epsilon_1 \epsilon_2 \epsilon_3) \\ &\quad + \int_a^{+\infty} \frac{1}{a} \frac{d}{da} \left[ \frac{1}{H} (\epsilon_1^3 + 3\epsilon_1^2 \epsilon_2 + \epsilon_1 \epsilon_2^2 + \epsilon_1 \epsilon_2 \epsilon_3) \right] da, \end{aligned} \quad (8)$$

where the integral in the second line is of order  $\mathcal{O}(\epsilon^4)$  and can be omitted.

From Eqs. (3) and (8), we obtain the perturbative expression for the effective potential  $U_T$

$$\begin{aligned} \eta^2 U_T(\eta) &= 2 + 3\epsilon_1 + 4\epsilon_1^2 + 4\epsilon_1 \epsilon_2 + 5\epsilon_1^3 + 14\epsilon_1^2 \epsilon_2 + 4\epsilon_1 \epsilon_2^2 \\ &\quad + 4\epsilon_1 \epsilon_2 \epsilon_3 + \mathcal{O}(\epsilon^4). \end{aligned} \quad (9)$$

Notice that all  $\epsilon_i(\eta)$  are function of  $\eta$ , and, from now on, we assume that they are all of the same order  $\mathcal{O}(\epsilon)$  and no additional assumptions will be made. Nevertheless, in order to solve Eq. (2), one needs the explicit conformal time dependence of  $U_T$ ; namely, how the flow functions  $\epsilon_i$  depend on  $\eta$ . In fact, this is already encoded in their definition. Remarking that  $N = \ln(a)$  implies

$$\frac{dN}{d \ln |\eta|} = \eta \mathcal{H}, \quad (10)$$

one has, from Eqs. (1) and (8),

$$\begin{aligned} \frac{d\epsilon_i}{d \ln |\eta|} &= \eta \mathcal{H} \epsilon_i \epsilon_{i+1} = -\epsilon_i \epsilon_{i+1} (1 + \epsilon_1 + \epsilon_1^2 + \epsilon_1 \epsilon_2 + \epsilon_1^3 \\ &\quad + 3\epsilon_1^2 \epsilon_2 + \epsilon_1 \epsilon_2^2 + \epsilon_1 \epsilon_2 \epsilon_3) + \mathcal{O}(\epsilon^6). \end{aligned} \quad (11)$$

At any order, the derivatives of all the  $\epsilon_i$  are polynomials of the  $\epsilon_i$ . One can now pick a peculiar time, say  $\eta_b$ , and Taylor expand the Hubble flow functions at  $\eta_b$ . As we will see later on, the N3LO scalar power spectrum requires determining  $\epsilon_1$  up to the fourth order. At this order, the Taylor expansion of the Hubble flow functions reads

$$\begin{aligned} \epsilon_i(\eta) &= \epsilon_{ib} - \epsilon_{ib} \epsilon_{i+1b} (1 + \epsilon_{1b} + \epsilon_{1b}^2 + \epsilon_{1b} \epsilon_{2b}) \ln \left( \frac{\eta}{\eta_b} \right) \\ &\quad + \frac{\epsilon_{ib} \epsilon_{i+1b}}{2} (\epsilon_{i+1b} + \epsilon_{i+2b} + \epsilon_{1b} \epsilon_{2b} + 2\epsilon_{1b} \epsilon_{i+1b} \\ &\quad + 2\epsilon_{1b} \epsilon_{i+2b}) \ln^2 \left( \frac{\eta}{\eta_b} \right) - \frac{\epsilon_{ib} \epsilon_{i+1b}}{6} (\epsilon_{i+1b}^2 + \epsilon_{i+2b}^2 \\ &\quad + 3\epsilon_{i+1b} \epsilon_{i+2b} + \epsilon_{i+2b} \epsilon_{i+3b}) \ln^3 \left( \frac{\eta}{\eta_b} \right) + \mathcal{O}(\epsilon^5), \end{aligned} \quad (12)$$

where we have used the shortcut notation  $\epsilon_{ib} \equiv \epsilon_i(\eta_b)$ . All these quantities are now truly slow-roll “parameters”. As a result, the effective potential  $U_T$  in Eq. (9) inherits an explicit time dependence involving up to the third power of  $\ln(\eta/\eta_b)$ . This allows us to solve Eq. (2) by the Green’s function method [31].

### B. Green’s function

Using the dimensionless positive variable  $x \equiv -k\eta$ , Eq. (2) can be recast into

$$\frac{d^2 \mu}{dx^2} + \left[ 1 - \frac{U_T(x)}{k^2} \right] \mu = 0, \quad (13)$$

which, from Eqs. (9) and (12), can be split as [31]

$$\frac{d^2 \mu}{dx^2} + \left( 1 - \frac{2}{x^2} \right) \mu = \frac{g[\ln(x)]}{x^2} \mu. \quad (14)$$

By choosing a convenient conformal time

$$\eta_b = -\frac{1}{k}, \quad (15)$$

the function  $g$  reduces to a polynomial in  $\ln(x)$ . At third order in the slow-roll parameters, it reads

$$g[\ln(x)] = g_{1b} + g_{2b} \ln(x) + g_{3b} \ln^2(x) + \mathcal{O}(\epsilon^4), \quad (16)$$

with

$$\begin{aligned} g_{1b} &= 3\epsilon_{1b} + 4\epsilon_{1b}^2 + 4\epsilon_{1b}\epsilon_{2b} + 5\epsilon_{1b}^3 + 14\epsilon_{1b}^2\epsilon_{2b} + 4\epsilon_{1b}\epsilon_{2b}^2 \\ &\quad + 4\epsilon_{1b}\epsilon_{2b}\epsilon_{3b} + \mathcal{O}(\epsilon^4), \\ g_{2b} &= -(3\epsilon_{1b}\epsilon_{2b} + 11\epsilon_{1b}^2\epsilon_{2b} + 4\epsilon_{1b}\epsilon_{2b}^2 + 4\epsilon_{1b}\epsilon_{2b}\epsilon_{3b}) \\ &\quad + \mathcal{O}(\epsilon^4), \\ g_{3b} &= \frac{3}{2}(\epsilon_{1b}\epsilon_{2b}^2 + \epsilon_{1b}\epsilon_{2b}\epsilon_{3b}) + \mathcal{O}(\epsilon^4). \end{aligned} \quad (17)$$

The left-hand side of Eq. (14) is the equation of gravitational waves (GW) propagating in a pure matter era, namely a Riccati-Bessel equation [95, 96]. As such, Eq. (14) can be formally solved using the advanced<sup>2</sup> Green's function sourced by a Dirac distribution  $\delta(x-y)$ . The boundary conditions are chosen to match positive energy waves at small scales, i.e., varying as  $e^{-ik\eta}$  for  $k\eta \rightarrow -\infty$ . From the Wronskian method, one gets

$$G_y(x) = \frac{i}{2} [u(x)\bar{u}(y) - u(y)\bar{u}(x)] \Theta(y-x), \quad (18)$$

where  $u(x)$  is the Riccati-Hankel function of order one:

$$u(x) \equiv \left(1 + \frac{i}{x}\right) e^{ix}, \quad (19)$$

and  $\bar{u}$  denotes its complex conjugate. The exact solution of Eq. (14) reads

$$\begin{aligned} \mu(x) &= \frac{i}{2} \int_x^\infty \frac{g[\ln(y)]}{y^2} \mu(y) [\bar{u}(y)u(x) - \bar{u}(x)u(y)] dy \\ &\quad + \mu_0(x), \end{aligned} \quad (20)$$

with  $\mu_0(x)$  a solution of the homogeneous equation. From Eq. (17), we see that the leading-order term in the function  $g[\ln(x)]$  is of order  $\mathcal{O}(\epsilon)$  and Eq. (20) is readily a perturbative expansion in the Hubble flow functions. Imposing the Bunch-Davies vacuum state for  $k\eta \rightarrow -\infty$ , the zero-order term is completely fixed [93]. In Planck units, one gets

$$\mu_0(x) = \frac{u(x)}{\sqrt{k}}. \quad (21)$$

Defining the rescaled mode function

$$\hat{\mu}(x) \equiv \sqrt{k}\mu(x), \quad (22)$$

we can expand it as

$$\hat{\mu}(x) = \hat{\mu}_0(x) + \hat{\mu}_1(x) + \hat{\mu}_2(x) + \hat{\mu}_3(x) + \mathcal{O}(\epsilon^4), \quad (23)$$

with  $\hat{\mu}_0(x) = u(x)$ . From Eqs. (20) and (23), one finally gets the recursive solutions

$$\hat{\mu}_1(x) = g_{1b} \int_x^\infty \frac{G_y(x)}{y^2} u(y) dy, \quad (24)$$

$$\begin{aligned} \hat{\mu}_2(x) &= g_{2b} \int_x^\infty \frac{G_y(x) \ln(y)}{y^2} u(y) dy \\ &\quad + g_{1b} \int_x^\infty \frac{G_y(x)}{y^2} \hat{\mu}_1(y) dy, \end{aligned} \quad (25)$$

and

$$\begin{aligned} \hat{\mu}_3(x) &= g_{3b} \int_x^\infty \frac{G_y(x) \ln^2(y)}{y^2} u(y) dy \\ &\quad + g_{2b} \int_x^\infty \frac{G_y(x) \ln(y)}{y^2} \hat{\mu}_1(y) dy \\ &\quad + g_{1b} \int_x^\infty \frac{G_y(x)}{y^2} \hat{\mu}_2(y) dy, \end{aligned} \quad (26)$$

which consistently ensure that  $\hat{\mu}_p = \mathcal{O}(\epsilon^p)$ .

### C. Solution at N3LO

Pushing the slow-roll expansion at N3LO not only complexifies the functional form of the  $g_{ib}$  parameters but also introduces new definite multidimensional integrals. For instance, the last term of Eq. (26) is an integral over  $\hat{\mu}_2(y)$ , which is itself defined by integrals over  $\hat{\mu}_1(y)$  and  $u(y)$ . Using the explicit form of  $u(y)$  given in Eq. (19) and recursing on integration by parts, one can show that Eqs. (24) to (26) are completely determined by a few integrals that we now enumerate.

One has first a family of one-dimensional integrals

$$F_n(x) \equiv \int_x^\infty \frac{e^{+2iy}}{y} \ln^n(y) dy, \quad (27)$$

for  $n = 0$ ,  $n = 1$  and  $n = 2$ . These then enter into the definition of three other two-dimensional integrals

$$\begin{aligned} F_{00}(x) &= \int_x^\infty \frac{e^{-2iy}}{y} F_0(y) dy, \\ F_{01}(x) &= \int_x^\infty \frac{e^{-2iy}}{y} F_1(y) dy, \\ F_{10}(x) &= \int_x^\infty \frac{e^{-2iy}}{y} \ln(y) F_0(y) dy. \end{aligned} \quad (28)$$

Finally, the third-order terms involve the three-dimensional integral

$$F_{000}(x) = \int_x^\infty \frac{e^{+2iy}}{y} F_{00}(y) dy. \quad (29)$$

<sup>2</sup> We have defined  $x = -k\eta$  and this corresponds to retarded solution in  $\eta$ .

In terms of these integrals, the explicit solution for  $\hat{\mu}_1$  reads

$$\hat{\mu}_1(x) = -\frac{[(x-i)F_0(x) - 2ie^{2ix}]g_{1b}e^{-ix}}{3x}, \quad (30)$$

while the second order term simplifies to

$$\begin{aligned} \hat{\mu}_2(x) = & \frac{g_{1b}^2}{27} \left[ 3F_{00}(x)e^{ix} + F_0(x)e^{-ix} + \frac{3iF_{00}(x)e^{ix}}{x} + \frac{5iF_0(x)e^{-ix}}{x} - \frac{2ie^{ix}}{x} \right] \\ & - \frac{g_{2b}}{9} \left[ 7F_0(x)e^{-ix} + 3F_1(x)e^{-ix} - \frac{7iF_0(x)e^{-ix}}{x} - \frac{3iF_1(x)e^{-ix}}{x} - \frac{6ie^{ix}\ln(x)}{x} - \frac{8ie^{ix}}{x} \right]. \end{aligned} \quad (31)$$

Most of the action takes place for the third order correction which, after some algebra, reads

$$\begin{aligned} \hat{\mu}_3(x) = & -\frac{g_{1b}^3}{243} \left[ 6F_{00}(x)e^{ix} + 2F_0(x)e^{-ix} + 9F_{000}(x)e^{-ix} - \frac{12iF_{00}(x)e^{ix}}{x} + \frac{10iF_0(x)e^{-ix}}{x} - \frac{9iF_{000}(x)e^{-ix}}{x} \right. \\ & \left. - \frac{4ie^{ix}}{x} \right] + \frac{g_{1b}g_{2b}}{81} \left[ 21F_{00}(x)e^{ix} + 9F_{01}(x)e^{ix} + 26F_0(x)e^{-ix} + 3F_1(x)e^{-ix} + \frac{21iF_{00}(x)e^{ix}}{x} + \frac{9iF_{01}(x)e^{ix}}{x} \right. \\ & \left. + \frac{16iF_0(x)e^{-ix}}{x} + \frac{15iF_1(x)e^{-ix}}{x} - \frac{6ie^{ix}\ln(x)}{x} - \frac{28ie^{ix}}{x} \right] + \frac{g_{1b}g_{2b}}{81} \left[ 21F_{00}(x)e^{ix} + 9F_{10}(x)e^{ix} \right. \\ & \left. + 26F_0(x)e^{-ix} + 3F_1(x)e^{-ix} + \frac{18iF_0(x)e^{-ix}\ln(x)}{x} + \frac{21iF_{00}(x)e^{ix}}{x} + \frac{9iF_{10}(x)e^{ix}}{x} - \frac{2iF_0(x)e^{-ix}}{x} \right. \\ & \left. - \frac{3iF_1(x)e^{-ix}}{x} - \frac{6ie^{ix}\ln(x)}{x} - \frac{28ie^{ix}}{x} \right] - \frac{g_{3b}}{27} \left[ 50F_0(x)e^{-ix} + 42F_1(x)e^{-ix} + 9F_2(x)e^{-ix} - \frac{18ie^{ix}\ln(x)^2}{x} \right. \\ & \left. - \frac{50iF_0(x)e^{-ix}}{x} - \frac{42iF_1(x)e^{-ix}}{x} - \frac{9iF_2(x)e^{-ix}}{x} - \frac{48ie^{ix}\ln(x)}{x} - \frac{52ie^{ix}}{x} \right]. \end{aligned} \quad (32)$$

Let us notice that the time dependence in  $x$  appears explicitly in functions such as  $\ln(x)$ , but also implicitly through the integrals introduced earlier. Most of these integrals do not have an explicit expression, but, as we show in Appendix A, they can be recursively derived from a generating functional and this will allow us to uniquely determine their super-Hubble limit. From the theory of the cosmological perturbations, we know that, after Hubble exit, for  $x \ll 1$ ,  $\mu/a$  should be conserved and can no longer depend on  $x$ . As a result, a consistency check of the whole calculation is that all the terms in  $\ln(x)$ , which are singular in the limit  $x \rightarrow 0$ , must cancel in the final result.

#### D. Power spectrum

The constancy of  $\mu/a$  after Hubble exit allows us to derive the observable power spectrum of gravitational waves generated during inflation. Taking into account the polarization degrees of freedom, one has

$$\mathcal{P}_h(k) = \frac{2k^3}{\pi^2} \lim_{x \rightarrow 0} \left| \frac{\mu}{a} \right|^2, \quad (33)$$

and we need the explicit expression of  $a(\eta)$ , at third order in the Hubble flow functions. This one can be obtained from the definition of the number of e-folds. From Eq. (10), one obtains

$$\begin{aligned} \Delta N_b \equiv N - N_b = & -\left(1 + \epsilon_{1b} + \epsilon_{1b}^2 + \epsilon_{1b}\epsilon_{2b} + \epsilon_{1b}^3 \right. \\ & \left. + 3\epsilon_{1b}^2\epsilon_{2b} + \epsilon_{1b}\epsilon_{2b}^2 + \epsilon_{1b}\epsilon_{2b}\epsilon_{3b}\right) \ln\left(\frac{\eta}{\eta_b}\right) \\ & + \frac{1}{2} \left( \epsilon_{1b}\epsilon_{2b} + 3\epsilon_{1b}^2\epsilon_{2b} + \epsilon_{1b}\epsilon_{2b}^2 + \epsilon_{1b}\epsilon_{2b}\epsilon_{3b} \right) \ln^2\left(\frac{\eta}{\eta_b}\right) \\ & - \frac{1}{6} \left( \epsilon_{1b}\epsilon_{2b}^2 + \epsilon_{1b}\epsilon_{2b}\epsilon_{3b} \right) \ln^3\left(\frac{\eta}{\eta_b}\right) + \mathcal{O}(\epsilon^4), \end{aligned} \quad (34)$$

from which we get

$$a(\eta) = a_b e^{\Delta N_b}. \quad (35)$$

As the scale factor is not a measurable quantity, we can trade  $a_b$  for  $H_b$  by remarking that  $a_b = \mathcal{H}_b/H_b$  and making use of Eq. (8). One finally gets

$$\begin{aligned}
a(\eta) = \frac{-1}{\eta H_b} & \left[ 1 + \epsilon_{1b} + \epsilon_{1b}^2 + \epsilon_{1b}\epsilon_{2b} + \epsilon_{1b}^3 + 3\epsilon_{1b}^2\epsilon_{2b} + \epsilon_{1b}\epsilon_{2b}^2 + \epsilon_{1b}\epsilon_{2b}\epsilon_{3b} - (\epsilon_{1b} + 2\epsilon_{1b}^2 + \epsilon_{1b}\epsilon_{2b} + 3\epsilon_{1b}^3 + 5\epsilon_{1b}^2\epsilon_{2b} \right. \\
& + \epsilon_{1b}\epsilon_{2b}^2 + \epsilon_{1b}\epsilon_{2b}\epsilon_{3b}) \ln\left(\frac{\eta}{\eta_b}\right) + \frac{1}{2} (\epsilon_{1b}^2 + \epsilon_{1b}\epsilon_{2b} + 3\epsilon_{1b}^3 + 6\epsilon_{1b}^2\epsilon_{2b} + \epsilon_{1b}\epsilon_{2b}^2 + \epsilon_{1b}\epsilon_{2b}\epsilon_{3b}) \ln^2\left(\frac{\eta}{\eta_b}\right) \\
& \left. - \frac{1}{6} (\epsilon_{1b}^3 + 3\epsilon_{1b}^2\epsilon_{2b} + \epsilon_{1b}\epsilon_{2b}^2 + \epsilon_{1b}\epsilon_{2b}\epsilon_{3b}) \ln^3\left(\frac{\eta}{\eta_b}\right) + \mathcal{O}(\epsilon^4) \right].
\end{aligned} \tag{36}$$

This expression suggests defining the rescaled quantity

$$\hat{a} \equiv \frac{a H_b x}{k}, \tag{37}$$

which is an explicit function of  $\ln(x)$  only. From Eq. (33), one has

$$\mathcal{P}_h(k) = \lim_{x \rightarrow 0} \frac{2H_b^2}{\pi^2} \left| \frac{x\hat{\mu}}{\hat{a}} \right|^2, \tag{38}$$

which can be evaluated using the expression  $\hat{a}$  in Eq. (36). Some intermediate steps are given in the Appendix B and, after some serious algebra, keeping only the leading-order terms of all the integrals  $F_n(x)$  and  $F_{0n}(x)$ , the divergent terms in  $\ln(x)$  all cancel. Dropping the explicit order at which the slow-roll expansion is made, the super-Hubble limit finally reads

$$\begin{aligned}
\lim_{x \rightarrow 0} \left| \frac{x\hat{\mu}}{\hat{a}} \right|^2 &= 1 - 2(1+C)\epsilon_{1b} + \frac{1}{2} (\pi^2 + 4C^2 + 4C - 6) \epsilon_{1b}^2 + \frac{1}{12} (\pi^2 - 12C^2 - 24C - 24) \epsilon_{1b}\epsilon_{2b} \\
&- \frac{1}{3} [4C^3 + 3(\pi^2 - 8)C + 14\zeta(3) - 16] \epsilon_{1b}^3 + \frac{1}{12} [24C^3 + 13\pi^2 + 2(5\pi^2 - 36)C + 36C^2 - 96] \epsilon_{1b}^2\epsilon_{2b} \\
&- \frac{1}{12} [4C^3 - \pi^2 - (\pi^2 - 24)C + 12C^2 + 8\zeta(3) + 8] (\epsilon_{1b}\epsilon_{2b}^2 + \epsilon_{1b}\epsilon_{2b}\epsilon_{3b}).
\end{aligned} \tag{39}$$

In this expression,  $\zeta(3)$  is a number given by the Riemann zeta function  $\zeta(z)$  evaluated at  $z = 3$ . Similarly,  $C$  stands for another number

$$C \equiv \gamma_E + \ln(2) - 2, \tag{40}$$

$\gamma_E$  being the Euler constant. The dependence of Eq. (39) in the wavenumber  $k$  is implicit and hidden by our choice of the peculiar time  $\eta_b = -1/k$  at which all the Hubble flow functions are evaluated. In order to make contact with the cosmological observations, one needs to recast this expression at a fixed wavenumber, say  $k_*$ , chosen by the observer. For CMB measurements, the typical wavenumber would be  $k_* = 0.05 \text{ Mpc}^{-1}$  [97] and this implies that the Hubble flow functions should be expanded around another time uniquely fixed by  $k_*$ . Let us choose a time, say  $\eta_*$ , close to the Hubble crossing and verifying

$$k_* \eta_* = -1. \tag{41}$$

Using the Hubble flow expansions of Eq. (12), we can expand all parameters  $\epsilon_{ib}$  in terms of  $\epsilon_{i*} = \epsilon_i(\eta_*)$  and powers of

$$\ln\left(\frac{\eta_b}{\eta_*}\right) = -\ln\left(\frac{k}{k_*}\right). \tag{42}$$

Let us stress that the normalization  $H_b^2$  appearing in Eq. (38) has to be expanded around the new time  $\eta_*$ . This can be straightforwardly done from Eq. (8) by remarking that

$$\frac{d \ln H}{d \ln |\eta|} = -\epsilon_1 \eta \mathcal{H}. \tag{43}$$

After some algebra, one finally obtains one of the main results of this work, namely, the tensor mode power spectrum, at N3LO, fully expanded around the pivot wavenumber  $k_*$ . It reads



$$\begin{aligned}
\mathcal{P}_h(k) = & \frac{2H_*^2}{\pi^2} \left\{ 1 - 2(C+1)\epsilon_{1*} + \frac{1}{2}(\pi^2 + 4C^2 + 4C - 6)\epsilon_{1*}^2 + \frac{1}{12}(\pi^2 - 12C^2 - 24C - 24)\epsilon_{1*}\epsilon_{2*} \right. \\
& - \frac{1}{3}[4C^3 + 3(\pi^2 - 8)C + 14\zeta(3) - 16]\epsilon_{1*}^3 + \frac{1}{12}[24C^3 + 13\pi^2 + 2(5\pi^2 - 36)C + 36C^2 - 96]\epsilon_{1*}^2\epsilon_{2*} \\
& - \frac{1}{12}[4C^3 - \pi^2 - (\pi^2 - 24)C + 12C^2 + 8\zeta(3) + 8](\epsilon_{1*}\epsilon_{2*}^2 + \epsilon_{1*}\epsilon_{2*}\epsilon_{3*}) \\
& + \left[ -2\epsilon_{1*} + 2(2C+1)\epsilon_{1*}^2 - 2(C+1)\epsilon_{1*}\epsilon_{2*} - (\pi^2 + 4C^2 - 8)\epsilon_{1*}^3 + \frac{1}{6}(5\pi^2 + 36C^2 + 36C - 36)\epsilon_{1*}^2\epsilon_{2*} \right. \\
& + \frac{1}{12}(\pi^2 - 12C^2 - 24C - 24)(\epsilon_{1*}\epsilon_{2*}^2 + \epsilon_{1*}\epsilon_{2*}\epsilon_{3*}) \left. \right] \ln\left(\frac{k}{k_*}\right) \\
& + \left[ 2\epsilon_{1*}^2 - \epsilon_{1*}\epsilon_{2*} - 4C\epsilon_{1*}^3 + 3(2C+1)\epsilon_{1*}^2\epsilon_{2*} - (C+1)(\epsilon_{1*}\epsilon_{2*}^2 + \epsilon_{1*}\epsilon_{2*}\epsilon_{3*}) \right] \ln^2\left(\frac{k}{k_*}\right) \\
& + \frac{1}{3} \left( -4\epsilon_{1*}^3 + 6\epsilon_{1*}^2\epsilon_{2*} - \epsilon_{1*}\epsilon_{2*}^2 - \epsilon_{1*}\epsilon_{2*}\epsilon_{3*} \right) \ln^3\left(\frac{k}{k_*}\right) \left. \right\}. \tag{44}
\end{aligned}$$

This formula corrects the one derived independently in an unpublished work [98] (master thesis).

### III. CURVATURE PERTURBATIONS

The dominant anisotropies in the universe are sourced by scalar perturbations and, in this section, we derive the primordial power spectrum of the comoving curvature perturbation  $\zeta(\eta, k)$

$$\mathcal{P}_\zeta(k) \equiv \frac{k^3}{2\pi^2} |\zeta|^2. \tag{45}$$

As a warm-up, we start with an inflationary era driven by a scalar field having a minimal kinetic term before turning to the more exotic situation of K-inflation.

#### A. Minimal kinetic term

##### 1. Equation of motion

The evolution of the curvature perturbations during inflation can be recast as a parametric oscillator by introducing the Mukhanov-Sasaki variable

$$v(\eta, k) \equiv a(\eta)\sqrt{2\epsilon_1(\eta)}\zeta(\eta, k). \tag{46}$$

In Fourier space, it satisfies [93]

$$\frac{d^2v}{d\eta^2} + \left( k^2 - \frac{1}{z} \frac{d^2z}{d\eta^2} \right) v = 0, \tag{47}$$

where we have defined

$$z(\eta) \equiv a(\eta)\sqrt{\epsilon_1(\eta)}. \tag{48}$$

Since this equation is formally identical to the one of the tensor modes, see Eq. (2), one could repeat the calculations done in Section II. We provide a summary of these calculations in Appendix B 2, while here, we give the result using the mapping method discussed in Ref. [57].

##### 2. Mapping method

This method consists in introducing a generalized conformal Hubble parameter  $\tilde{\mathcal{H}}$  and a generalized e-fold number  $\tilde{N}$  such that

$$\tilde{\mathcal{H}} \equiv \frac{z'}{z} = \frac{d\tilde{N}}{d\eta}. \tag{49}$$

In other word, the function  $z(\eta)$  is viewed as a generalized scale factor. Defining  $\tilde{H} \equiv \tilde{\mathcal{H}}/z$ , one can define a hierarchy of generalized flow functions

$$\alpha_{i+1}(\tilde{N}) \equiv \frac{d \ln |\alpha_i|}{d\tilde{N}}, \quad \alpha_1(\tilde{N}) \equiv -\frac{d \ln \tilde{H}}{d\tilde{N}}. \tag{50}$$

Let us notice that, although not explicitly needed, the generalized cosmic time is then defined as  $d\tilde{t} = z d\eta$ . Because Eqs. (2) and (47) are formally identical and have the same initial conditions (a Bunch-Davies vacuum), up to an overall normalization accounting for the different number of polarization degrees of freedom, the scalar power spectrum must be given by Eq. (44) up to the replacement rule

$$\epsilon_{i*} \rightarrow \alpha_{i*}, \quad H_* \rightarrow \tilde{H}_*. \tag{51}$$

The only work to do is expressing the generalized parameters in terms of the standard Hubble flow functions. From Eqs. (49) and (50), one has the exact relations

$$\tilde{H} = \frac{H}{\sqrt{\epsilon_1}} \left( 1 + \frac{\epsilon_2}{2} \right), \quad \tilde{N} = N + \frac{1}{2} \ln \epsilon_1, \tag{52}$$

from which we get [57]

$$\alpha_1 = \frac{4\epsilon_1 + \epsilon_2(2 + 2\epsilon_1 + \epsilon_2 - 2\epsilon_3)}{(\epsilon_2 + 2)^2}, \quad (53)$$

$$\alpha_{i+1} = \left(1 + \frac{\epsilon_2}{2}\right) \frac{d \ln |\alpha_i|}{dN}.$$

Using Eq. (1) in this last equation allows us to determine, by recurrence, all the needed function  $\alpha_1$ ,  $\alpha_2$  and  $\alpha_3$  in

terms of  $\epsilon_i$  (see Ref. [57]).

### 3. Power spectrum

Plugging Eqs. (52) and (53) in the tensor power spectrum of Eq. (44), dropping all terms of order equal or higher than  $\mathcal{O}(\epsilon^4)$ , and dividing by 16 to account for the scalar versus tensor degrees of freedom, we finally obtain

$$\begin{aligned} \mathcal{P}_\zeta(k) = & \frac{H_*^2}{8\pi^2 \epsilon_{1*}} \left\{ 1 - 2(C+1)\epsilon_{1*} - C\epsilon_{2*} + \left(\frac{\pi^2}{2} + 2C^2 + 2C - 3\right)\epsilon_{1*}^2 + \left(\frac{7\pi^2}{12} + C^2 - C - 6\right)\epsilon_{1*}\epsilon_{2*} \right. \\ & + \frac{1}{8}(\pi^2 + 4C^2 - 8)\epsilon_{2*}^2 + \frac{1}{24}(\pi^2 - 12C^2)\epsilon_{2*}\epsilon_{3*} - \frac{1}{24}[4C^3 + 3(\pi^2 - 8)C + 14\zeta(3) - 16](8\epsilon_{1*}^3 + \epsilon_{2*}^3) \\ & + \frac{1}{12}[13\pi^2 - 8(\pi^2 - 9)C + 36C^2 - 84\zeta(3)]\epsilon_{1*}^2\epsilon_{2*} - \frac{1}{24}[8C^3 - 15\pi^2 + 6(\pi^2 - 4)C - 12C^2 + 100\zeta(3) + 16]\epsilon_{1*}\epsilon_{2*}^2 \\ & + \frac{1}{24}[\pi^2 C - 4C^3 - 8\zeta(3) + 16](\epsilon_{2*}\epsilon_{3*}^2 + \epsilon_{2*}\epsilon_{3*}\epsilon_{4*}) + \frac{1}{24}[12C^3 + (5\pi^2 - 48)C]\epsilon_{2*}^2\epsilon_{3*} \\ & + \frac{1}{12}[8C^3 + \pi^2 + 6(\pi^2 - 12)C - 12C^2 - 8\zeta(3) - 8]\epsilon_{1*}\epsilon_{2*}\epsilon_{3*} \\ & + \left[ -2\epsilon_{1*} - \epsilon_{2*} + 2(2C+1)\epsilon_{1*}^2 + (2C-1)\epsilon_{1*}\epsilon_{2*} + C\epsilon_{2*}^2 - C\epsilon_{2*}\epsilon_{3*} - \frac{1}{8}(\pi^2 + 4C^2 - 8)(8\epsilon_{1*}^3 + \epsilon_{2*}^3) \right. \\ & - \frac{2}{3}(\pi^2 - 9C - 9)\epsilon_{1*}^2\epsilon_{2*} - \frac{1}{4}(\pi^2 + 4C^2 - 4C - 4)\epsilon_{1*}\epsilon_{2*}^2 + \frac{1}{2}(\pi^2 + 4C^2 - 4C - 12)\epsilon_{1*}\epsilon_{2*}\epsilon_{3*} \\ & + \frac{1}{24}(\pi^2 - 12C^2)(\epsilon_{2*}\epsilon_{3*}^2 + \epsilon_{2*}\epsilon_{3*}\epsilon_{4*}) + \frac{1}{24}(5\pi^2 + 36C^2 - 48)\epsilon_{2*}^2\epsilon_{3*} \left. \right] \ln\left(\frac{k}{k_*}\right) \\ & + \frac{1}{2} \left[ 4\epsilon_{1*}^2 + 2\epsilon_{1*}\epsilon_{2*} + \epsilon_{2*}^2 - \epsilon_{2*}\epsilon_{3*} + 6\epsilon_{1*}^2\epsilon_{2*} - (2C-1)(\epsilon_{1*}\epsilon_{2*}^2 - 2\epsilon_{1*}\epsilon_{2*}\epsilon_{3*}) \right. \\ & - C(8\epsilon_{1*}^3 + \epsilon_{2*}^3 - 3\epsilon_{2*}^2\epsilon_{3*} + \epsilon_{2*}\epsilon_{3*}^2 + \epsilon_{2*}\epsilon_{3*}\epsilon_{4*}) \left. \right] \ln^2\left(\frac{k}{k_*}\right) \\ & + \frac{1}{6}(-8\epsilon_{1*}^3 - 2\epsilon_{1*}\epsilon_{2*}^2 + 4\epsilon_{1*}\epsilon_{2*}\epsilon_{3*} - \epsilon_{2*}^3 + 3\epsilon_{2*}^2\epsilon_{3*} - \epsilon_{2*}\epsilon_{3*}^2 - \epsilon_{2*}\epsilon_{3*}\epsilon_{4*}) \ln^3\left(\frac{k}{k_*}\right) \left. \right\}. \end{aligned} \quad (54)$$

As announced in the introduction, let us notice the appearance of  $\epsilon_{4*}$  and an explicit dependence into the third power of  $\ln(k/k_*)$ . Both Eq. (44), and Eq. (54) determine the tensor and scalar perturbations generated during single field slow-roll inflation and they are the main results of this article.

They are, however, not applicable to the more exotic single field models with varying speed of sound  $c_s$ . We now turn to this case.

### B. K-Inflation

These models are characterized by a non-minimal kinetic term for the scalar field  $\phi$  while gravity is still given by General Relativity [99–101]. Denoting the minimal ki-

netic term by

$$X = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi, \quad (55)$$

the Lagrangian density governing the field evolution introduces a new function  $P(X, \phi)$ . For a minimal kinetic term, it would be  $P = X - V(\phi)$ ,  $V(\phi)$  being the field's potential, but, here, we consider it to be a free function. However, not all choices for  $P(X, \phi)$  are acceptable, and, for the equations of motion to remain hyperbolic, one should have [102]

$$\frac{\partial P}{\partial X} > 0, \quad \frac{\partial P}{\partial X} + 2X \frac{\partial^2 P}{\partial X^2} > 0. \quad (56)$$

Let us notice that these conditions are automatically satisfied when the non-minimal kinetic term arises as low-dimensional projection of string theory models, such as in the Dirac–Born–Infeld inflation [103, 104].



### 1. Equation of motion

At the perturbative level, K-inflation exhibits a speed of sound  $c_s$  which does no longer match the speed of light, one has

$$c_s^2 \equiv \frac{\frac{\partial P}{\partial X}}{\frac{\partial P}{\partial X} + 2X \frac{\partial^2 P}{\partial X^2}}, \quad (57)$$

and this is, in general, a function of the time. As a result, the sonic radius does no longer match the Hubble radius, and we introduce a new hierarchy of sound flow functions

$$\delta_{i+1} \equiv \frac{d \ln |\delta_i|}{dN}, \quad \delta_1 \equiv -\frac{d \ln c_s}{dN}. \quad (58)$$

In the following, we assume that all  $\delta_i = \mathcal{O}(\epsilon)$  as to allow for consistent expansions in all the Hubble and sound flow functions.

The evolution of the comoving curvature perturbation is governed by an equation formally identical to Eq. (47), namely [47]

$$\frac{d^2 v}{d\tau^2} + \left( k^2 - \frac{1}{z} \frac{d^2 z}{d\tau^2} \right) v = 0, \quad (59)$$

where we have defined a new time  $\tau$

$$\tau(\eta) = - \int_{\eta}^{0^-} c_s(\eta) d\eta, \quad (60)$$

and the new function

$$z(\tau) \equiv a(\tau) \sqrt{\frac{\epsilon_1(\tau)}{c_s(\tau)}}. \quad (61)$$

The Mukhanov-Sasaki variable remains formally unchanged and reads

$$v(\tau, k) \equiv \sqrt{2} \zeta(\tau, k) z(\tau). \quad (62)$$

### 2. Mapping from the tensor modes

We can again use the mapping method summarized in Section III A 2, constructed on the function  $z(\tau)$  defined in Eq. (61). The generalized conformal Hubble parameter and e-fold number now read

$$\tilde{H} = \frac{H}{\sqrt{\epsilon_1 c_s}} \left( 1 + \frac{\epsilon_2 + \delta_1}{2} \right), \quad \tilde{N} = N + \frac{\ln \epsilon_1 - \ln c_s}{2}, \quad (63)$$

from which we get the exact relations

$$\alpha_1 = \frac{(2\epsilon_1 + \epsilon_2 - \delta_1)(\epsilon_2 + \delta_1 + 2) - 2(\delta_1 \delta_2 + \epsilon_2 \epsilon_3)}{(\epsilon_2 + \delta_1 + 2)^2},$$

$$\alpha_{i+1} = \left( 1 + \frac{\epsilon_2 + \delta_1}{2} \right) \frac{d \ln |\alpha_i|}{dN}. \quad (64)$$

Formally replacing  $H_* \rightarrow \tilde{H}$  and  $\epsilon_{i*} \rightarrow \alpha_i$  into Eq. (44), using the above relations and discarding all terms equal of higher than  $\mathcal{O}(\epsilon^4)$ , one gets the wanted expression for  $\mathcal{P}_\zeta(k)$ , fully expanded around a pivot wavenumber, say  $k_\triangleright$ . This expression can be found in the Appendix B 2, see Eq. (B14), and we do not repeat it here. The important point is to notice that the above relations imply that we have also mapped  $\eta$  to  $\tau$  and the actual value of  $\tau_\triangleright$  now verifies

$$k_\triangleright \tau(\eta_\triangleright) = -1. \quad (65)$$

It is different from the pivot definition of the tensor modes ( $k_* \eta_* = -1$ ). This is problematic because having different pivots for the scalar and tensor spectra forbids to extract various observable quantities such as the tensor-to-scalar ratio [51].

### 3. Change of pivot

One possibility is to change the scalar mode pivot from  $k_\triangleright \tau(\eta_\triangleright) = -1$  to the one associated with the tensor modes  $k_* \eta_* = -1$ , and we discuss this possibility later on. However, because the primordial gravitational waves have not yet been detected, let us first follow what is usually done and define the pivot using a characteristic time associated with the scalar perturbations. For instance, let us define  $\eta_\circ$  when a scalar mode of peculiar wavenumber  $k_\circ$  crossed the sonic radius and became frozen:

$$k_\circ \eta_\circ c_s(\eta_\circ) = -1. \quad (66)$$

In order to find the transformation from  $k_\triangleright \tau(\eta_\triangleright) = -1$  to Eq. (66), the explicit dependence of  $\tau(\eta)$  needs to be determined. In analogy with the expansion presented in Section II A, this one can be perturbatively obtained by successively integrating Eq. (60) by parts over the scale factor  $a$ . One gets

$$\begin{aligned} -\tau &= \int_a^{+\infty} \frac{c_s(a)}{a^2 H} da \\ &= \frac{c_s}{aH} + \int_a^{+\infty} \left[ \frac{1}{aH} \frac{dc_s}{da} + \frac{c_s}{a} \frac{d(H^{-1})}{da} \right] da. \end{aligned} \quad (67)$$

From the definition of the sound flow functions in Eq. (58), one has

$$\frac{dc_s}{da} = -\frac{c_s}{a} \delta_1, \quad \frac{d\delta_i}{da} = \frac{\delta_i \delta_{i+1}}{a}. \quad (68)$$

Replacing all derivative terms in Eq. (67) by the sound and Hubble flow functions, and successively integrating by parts, one obtains

$$\begin{aligned} -\tau &= \frac{c_s}{\tilde{H}} \left( 1 + \epsilon_1 - \delta_1 + \epsilon_1^2 + \epsilon_1 \epsilon_2 - 2\epsilon_1 \delta_1 + \delta_1^2 - \delta_1 \delta_2 \right. \\ &\quad + \epsilon_1^3 + 3\epsilon_1^2 \epsilon_2 + \epsilon_1 \epsilon_2^2 + \epsilon_1 \epsilon_2 \epsilon_3 - 3\epsilon_1^2 \delta_1 - 3\epsilon_1 \epsilon_2 \delta_1 \\ &\quad + 3\delta_1^2 \epsilon_1 - 3\epsilon_1 \delta_1 \delta_2 - \delta_1^3 + 3\delta_1^2 \delta_2 - \delta_1 \delta_2^2 - \delta_1 \delta_2 \delta_3 \Big) \\ &\quad + \mathcal{O}(\epsilon^4). \end{aligned} \quad (69)$$

Combined with the expansion of the conformal Hubble parameter  $\mathcal{H}$  given in Eq. (8), we finally get

$$\begin{aligned} \frac{\eta c_s}{\tau} &= 1 + \delta_1 + \epsilon_1 \delta_1 + \delta_1 \delta_2 + 2\epsilon_1 \epsilon_2 \delta_1 + \epsilon_1^2 \delta_1 + 2\epsilon_1 \delta_1 \delta_2 \\ &\quad + \delta_1 \delta_2 \delta_3 + \delta_1 \delta_2^2 - \delta_1^2 \delta_2 + \mathcal{O}(\epsilon^4). \end{aligned} \quad (70)$$

Performing the change of pivot can be made in two strictly equivalent ways. Either implicitly assuming that  $k_{\triangleright} = k_{\circ}$  and re-expressing all the flow functions  $\epsilon_{i\triangleright}$  and  $\delta_{i\triangleright}$  in terms of  $\epsilon_{i\circ}$  and  $\delta_{i\circ}$  by their expansion in  $\ln(\eta_{\triangleright}/\eta_{\circ})$ , or, implicitly assuming that  $\eta_{\triangleright} = \eta_{\circ}$  and remarking that Eq. (65) divided by Eq. (66) gives

$$\ln(k_{\triangleright}) = \ln(k_{\circ}) + \ln \left[ \frac{\eta_{\circ} c_s(\eta_{\circ})}{\tau(\eta_{\triangleright})} \right]. \quad (71)$$

Choosing the latter way, plugging  $\eta_{\circ} = \eta_{\triangleright}$  into the previous equation, we get, with the help of Eq. (70), the simple relation

$$\begin{aligned} \ln \left( \frac{k}{k_{\triangleright}} \right) &= \ln \left( \frac{k}{k_{\circ}} \right) - \delta_{1\circ} - \delta_{1\circ} \epsilon_{1\circ} - \delta_{1\circ} \delta_{2\circ} + \frac{1}{2} \delta_{1\circ}^2 \\ &\quad - 2\delta_{1\circ} \epsilon_{1\circ} \epsilon_{2\circ} - \delta_{1\circ} \epsilon_{1\circ}^2 - 2\delta_{1\circ} \delta_{2\circ} \epsilon_{1\circ} + \delta_{1\circ}^2 \epsilon_{1\circ} \\ &\quad - \delta_{1\circ} \delta_{2\circ} \delta_{3\circ} - \delta_{1\circ} \delta_{2\circ}^2 + 2\delta_{1\circ}^2 \delta_{2\circ} - \frac{1}{3} \delta_{1\circ}^3 + \mathcal{O}(\epsilon^4). \end{aligned} \quad (72)$$

#### 4. Power spectrum

Using this formula in Eq. (B14), one gets the power spectrum of the curvature perturbation for K-inflation, at N3LO, fully expanded around the pivot wavenumber  $k_{\circ}$ ,

$$\mathcal{P}_{\zeta}(k) = \frac{H_{\circ}^2}{8\pi^2 \epsilon_{1\circ} c_{s\circ}} \left[ b_{0\circ}^{(s)} + b_{1\circ}^{(s)} \ln \left( \frac{k}{k_{\circ}} \right) + b_{2\circ}^{(s)} \ln^2 \left( \frac{k}{k_{\circ}} \right) + b_{3\circ}^{(s)} \ln^3 \left( \frac{k}{k_{\circ}} \right) \right]. \quad (73)$$

The first term encodes the corrections to the amplitude and reads

$$\begin{aligned} b_{0\circ}^{(s)} &= 1 - 2(C+1)\epsilon_{1\circ} - C\epsilon_{2\circ} + (C+2)\delta_{1\circ} + \frac{1}{2}(\pi^2 + 4C^2 + 4C - 6)\epsilon_{1\circ}^2 + \frac{1}{12}(7\pi^2 + 12C^2 - 12C - 72)\epsilon_{1\circ}\epsilon_{2\circ} \\ &\quad - \frac{1}{2}(\pi^2 + 4C^2 + 6C - 8)\epsilon_{1\circ}\delta_{1\circ} + \frac{1}{8}(\pi^2 + 4C^2 - 8)\epsilon_{2\circ}^2 + \frac{1}{24}(\pi^2 - 12C^2)\epsilon_{2\circ}\epsilon_{3\circ} - \frac{1}{4}(\pi^2 + 4C^2 + 4C - 12)\epsilon_{2\circ}\delta_{1\circ} \\ &\quad + \frac{1}{8}(\pi^2 + 4C^2 + 8C - 8)\delta_{1\circ}^2 - \frac{1}{24}(\pi^2 - 12C^2 - 48C - 48)\delta_{1\circ}\delta_{2\circ} \\ &\quad - \frac{1}{24}[4C^3 + (3\pi^2 - 8)C + 14\zeta(3) - 16](8\epsilon_{1\circ}^3 + \epsilon_{2\circ}^3) \\ &\quad + \frac{1}{12}[13\pi^2 - 8(\pi^2 - 9)C + 36C^2 - 84\zeta(3)]\epsilon_{1\circ}^2\epsilon_{2\circ} + \frac{1}{8}[4C^3 + (3\pi^2 - 32)C + 14\zeta(3) - 16](4\epsilon_{1\circ}^2\delta_{1\circ} + \epsilon_{2\circ}^2\delta_{1\circ}) \\ &\quad - \frac{1}{24}[8C^3 - 15\pi^2 + 6(\pi^2 - 4)C - 12C^2 + 100\zeta(3) + 16]\epsilon_{1\circ}\epsilon_{2\circ}^2 \\ &\quad + \frac{1}{12}[8C^3 + \pi^2 + 6(\pi^2 - 12)C - 12C^2 - 8\zeta(3) - 8]\epsilon_{1\circ}\epsilon_{2\circ}\epsilon_{3\circ} \\ &\quad + \frac{1}{24}[24C^3 - 13\pi^2 + 2(13\pi^2 - 132)C - 36C^2 + 168\zeta(3) - 48]\epsilon_{1\circ}\epsilon_{2\circ}\delta_{1\circ} \\ &\quad - \frac{1}{8}[4C^3 + (3\pi^2 - 40)C + 14\zeta(3) - 20](2\epsilon_{1\circ}\delta_{1\circ}^2 + \epsilon_{2\circ}\delta_{1\circ}^2) \\ &\quad - \frac{1}{24}[12C^3 + 13\pi^2 + (5\pi^2 - 24)C + 36C^2 - 120](2\epsilon_{1\circ}\delta_{1\circ}\delta_{2\circ} + \epsilon_{2\circ}\delta_{1\circ}\delta_{2\circ}) + \frac{1}{24}[12C^3 + (5\pi^2 - 48)C]\epsilon_{2\circ}^2\epsilon_{3\circ} \\ &\quad + \frac{1}{24}[\pi^2 C - 4C^3 - 8\zeta(3) + 16](\epsilon_{2\circ}\epsilon_{3\circ}^2 + \epsilon_{2\circ}\epsilon_{3\circ}\epsilon_{4\circ}) - \frac{1}{24}[12C^3 + (5\pi^2 - 72)C]\epsilon_{2\circ}\epsilon_{3\circ}\delta_{1\circ} \\ &\quad + \frac{1}{24}[4C^3 + 3(\pi^2 - 16)C + 14\zeta(3) - 28]\delta_{1\circ}^3 + \frac{1}{24}[12C^3 + 13\pi^2 + (5\pi^2 - 48)C + 36C^2 - 168]\delta_{1\circ}^2\delta_{2\circ} \\ &\quad + \frac{1}{24}[4C^3 - 2\pi^2 - (\pi^2 - 48)C + 24C^2 + 8\zeta(3) + 32](\delta_{1\circ}\delta_{2\circ}^2 + \delta_{1\circ}\delta_{2\circ}\delta_{3\circ}). \end{aligned} \quad (74)$$

The second term gives the deviations from scale invariance and reads

$$\begin{aligned}
b_{1\circ}^{(S)} = & -2\epsilon_{1\circ} - \epsilon_{2\circ} + \delta_{1\circ} + (2C+1)(2\epsilon_{1\circ}^2 - \epsilon_{2\circ}\delta_{1\circ}) + (2C-1)\epsilon_{1\circ}\epsilon_{2\circ} - (4C+3)\epsilon_{1\circ}\delta_{1\circ} + C(\epsilon_{2\circ}^2 - \epsilon_{2\circ}\epsilon_{3\circ}) + (C+1)\delta_{1\circ}^2 \\
& + (C+2)\delta_{1\circ}\delta_{2\circ} - \frac{1}{8}(\pi^2 + 4C^2 - 8)(8\epsilon_{1\circ}^3 + \epsilon_{2\circ}^3) - \frac{2}{3}(\pi^2 - 9C - 9)\epsilon_{1\circ}^2\epsilon_{2\circ} + \frac{1}{8}(3\pi^2 + 12C^2 - 32)(4\epsilon_{1\circ}^2\delta_{1\circ} + \epsilon_{2\circ}^2\delta_{1\circ}) \\
& - \frac{1}{4}(\pi^2 + 4C^2 - 4C - 4)\epsilon_{1\circ}\epsilon_{2\circ}^2 + \frac{1}{2}(\pi^2 + 4C^2 - 4C - 12)\epsilon_{1\circ}\epsilon_{2\circ}\epsilon_{3\circ} + \frac{1}{12}(13\pi^2 + 36C^2 - 36C - 132)\epsilon_{1\circ}\epsilon_{2\circ}\delta_{1\circ} \\
& - \frac{1}{8}(3\pi^2 + 12C^2 - 40)(2\epsilon_{1\circ}\delta_{1\circ}^2 + \epsilon_{2\circ}\delta_{1\circ}^2) - \frac{1}{24}(5\pi^2 + 36C^2 + 72C - 24)(2\epsilon_{1\circ}\delta_{1\circ}\delta_{2\circ} + \epsilon_{2\circ}\delta_{1\circ}\delta_{2\circ}) \\
& + \frac{1}{24}(5\pi^2 + 36C^2 - 48)\epsilon_{2\circ}^2\epsilon_{3\circ} + \frac{1}{24}(\pi^2 - 12C^2)(\epsilon_{2\circ}\epsilon_{3\circ}^2 + \epsilon_{2\circ}\epsilon_{3\circ}\epsilon_{4\circ}) - \frac{1}{24}(5\pi^2 + 36C^2 - 72)\epsilon_{2\circ}\epsilon_{3\circ}\delta_{1\circ} \\
& + \frac{1}{8}(\pi^2 + 4C^2 - 16)\delta_{1\circ}^3 + \frac{1}{24}(5\pi^2 + 36C^2 + 72C - 48)\delta_{1\circ}^2\delta_{2\circ} - \frac{1}{24}(\pi^2 - 12C^2 - 48C - 48)(\delta_{1\circ}\delta_{2\circ}^2 + \delta_{1\circ}\delta_{2\circ}\delta_{3\circ}).
\end{aligned} \tag{75}$$

The higher order terms encode deviations from a pure power law and are

$$\begin{aligned}
b_{2\circ}^{(S)} = & 2\epsilon_{1\circ}^2 + \epsilon_{1\circ}\epsilon_{2\circ} - 2\epsilon_{1\circ}\delta_{1\circ} + \frac{1}{2}\epsilon_{2\circ}^2 - \frac{1}{2}\epsilon_{2\circ}\epsilon_{3\circ} - \epsilon_{2\circ}\delta_{1\circ} + \frac{1}{2}\delta_{1\circ}^2 + \frac{1}{2}\delta_{1\circ}\delta_{2\circ} - 4C\epsilon_{1\circ}^3 + 3\epsilon_{1\circ}^2\epsilon_{2\circ} + 6C\epsilon_{1\circ}^2\delta_{1\circ} \\
& + \frac{1}{2}(2C-1)(2\epsilon_{1\circ}\epsilon_{2\circ}\epsilon_{3\circ} - \epsilon_{1\circ}\epsilon_{2\circ}^2 + 3\epsilon_{1\circ}\epsilon_{2\circ}\delta_{1\circ}) - 3C\epsilon_{1\circ}\delta_{1\circ}^2 - 3(C+1)\epsilon_{1\circ}\delta_{1\circ}\delta_{2\circ} - \frac{C}{2}(\epsilon_{2\circ}^3 + \epsilon_{2\circ}\epsilon_{3\circ}^2 + \epsilon_{2\circ}\epsilon_{3\circ}\epsilon_{4\circ} - \delta_{1\circ}^3) \\
& + \frac{3}{2}C(\epsilon_{2\circ}^2\epsilon_{3\circ} + \epsilon_{2\circ}^2\delta_{1\circ} - \epsilon_{2\circ}\epsilon_{3\circ}\delta_{1\circ} - \epsilon_{2\circ}\delta_{1\circ}^2) - \frac{3}{2}(C+1)(\epsilon_{2\circ}\delta_{1\circ}\delta_{2\circ} - \delta_{1\circ}^2\delta_{2\circ}) + \frac{1}{2}(C+2)(\delta_{1\circ}\delta_{2\circ}^2 + \delta_{1\circ}\delta_{2\circ}\delta_{3\circ}),
\end{aligned} \tag{76}$$

and

$$\begin{aligned}
b_{3\circ}^{(S)} = & -\frac{4}{3}\epsilon_{1\circ}^3 + 2\epsilon_{1\circ}^2\delta_{1\circ} - \frac{1}{3}\epsilon_{1\circ}\epsilon_{2\circ}^2 + \frac{2}{3}\epsilon_{1\circ}\epsilon_{2\circ}\epsilon_{3\circ} + \epsilon_{1\circ}\epsilon_{2\circ}\delta_{1\circ} - \epsilon_{1\circ}\delta_{1\circ}^2 - \epsilon_{1\circ}\delta_{1\circ}\delta_{2\circ} - \frac{1}{6}\epsilon_{2\circ}^3 + \frac{1}{2}\epsilon_{2\circ}^2\epsilon_{3\circ} + \frac{1}{2}\epsilon_{2\circ}^2\delta_{1\circ} \\
& - \frac{1}{6}\epsilon_{2\circ}\epsilon_{3\circ}^2 - \frac{1}{6}\epsilon_{2\circ}\epsilon_{3\circ}\epsilon_{4\circ} - \frac{1}{2}\epsilon_{2\circ}\epsilon_{3\circ}\delta_{1\circ} - \frac{1}{2}\epsilon_{2\circ}\delta_{1\circ}^2 - \frac{1}{2}\epsilon_{2\circ}\delta_{1\circ}\delta_{2\circ} + \frac{1}{6}\delta_{1\circ}^3 + \frac{1}{2}\delta_{1\circ}^2\delta_{2\circ} + \frac{1}{6}\delta_{1\circ}\delta_{2\circ}^2 + \frac{1}{6}\delta_{1\circ}\delta_{2\circ}\delta_{3\circ}.
\end{aligned} \tag{77}$$

## 5. Tensor modes

During K-inflation, the tensor modes evolution remains unaffected by any modification in the scalar sector, and, as such, their primordial power spectrum is still given by Eq. (44). However, we still need to re-express Eq. (44) in terms of the Hubble flow functions evaluated at the same time as the ones appearing in the scalar power spectrum, namely at  $\eta_\circ$ . As discussed in Section III B 3, the easiest way to implement the change of pivot is to implicitly identify  $\eta_*$  and  $\eta_\circ$  while letting the pivot wavenumber running. From Eqs. (41) and (66), we have

$$\ln(k_*) = \ln(k_\circ) + \ln\left(\frac{\eta_\circ c_{s\circ}}{\eta_*}\right), \tag{78}$$

such that the tensor power spectrum  $\mathcal{P}_h(k)$ , at N3LO, fully expanded around the pivot  $\eta_\circ$  is given by Eq. (44) with the following simple replacement rule

$$k_* \rightarrow k_\circ c_{s\circ}, \quad \epsilon_{i*} \rightarrow \epsilon_{i\circ}. \tag{79}$$

Because this transformation is trivial, and for the sake of clarity, we do not rewrite the full expression of  $\mathcal{P}_h(k)$  at  $\eta_\circ$ . Let us notice that, if one expands all the logarithms  $\ln[k/(k_\circ c_{s\circ})]$ , the only formal difference with respect to Eq. (44) are terms which are powers of  $\ln(c_s)$ . Up to

N2LO, these terms match the ones previously derived in Ref. [57].

## 6. Other possible pivot

Another way to get consistent formulas for both the scalar and tensor mode power spectra for K-inflation would be to shift the scalar mode pivot from  $\eta_\circ$  to  $\eta_*$ . Using Eq. (78) again, we have now the replacement rule

$$k_\circ \rightarrow \frac{k_*}{c_{s*}}, \quad \epsilon_{i\circ} \rightarrow \epsilon_{i*}, \quad \delta_{i\circ} \rightarrow \delta_{i*}, \tag{80}$$

to be applied to Eqs. (73) to (77) while the expression for the tensor modes, Eq. (44), is left unchanged.

## IV. CONCLUSION

The main results of this work are the explicit expressions of the tensor and scalar primordial power spectra generated during single field inflation with a minimal kinetic term, see Eqs. (44) and (54), and, with a non-minimal kinetic term, see Eq. (73). Let us notice that, as expected, forcing  $c_{s\circ} = 1$  and  $\delta_{i\circ} = 0$  into Eq. (73) gives back Eq. (54). From a data analysis point of view, these

expressions are the ones that should be used when comparing the observable predictions of slow-roll inflation to any cosmological data. As motivated in the introduction, one should expect the next generation of cosmological observations to provide information on  $\epsilon_{3\circ}$ . This slow-roll parameter appears at N2LO and the expressions at N3LO derived in this paper therefore encode the dominant theoretical uncertainties. In other words, in order to search for a non-vanishing running of the spectral index in the data, one should perform a data analysis involving  $\epsilon_{4\circ}$  and marginalize over it to get a robust posterior probability distribution on the slow-roll parameter  $\epsilon_{3\circ}$ .

Another possible interest of the formulas derived here, and more particularly the ones in Appendix C, are for extrapolations of the power spectra. For instance, if one needs to estimate the amplitude of the curvature perturbations, or gravitational waves, at wavenumbers significantly different than  $k_*$  (or  $k_\circ$ ), all higher order terms may play a significant role.

Finally, a last but not least interest of having at our disposal highly accurate formulas for the semi-classical slow-roll predictions is to allow for searching in the data unexpected deviations. As mentioned in the introduction, for standard slow-roll, quantum backreaction is expected to be negligible, genuinely not showing up before the sixth order in the slow-roll parameters. However, the presence in the potential of any flat regions, or inflection points, between the time at which observable modes left the Hubble radius and the end of inflation are expected to boost quantum backreaction [92]. As such, it could induce small deviations to the semi-classical slow-roll predictions, the precise form of which would still need to be determined.

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## Appendix A: Generating functionals

### 1. The $F_n(x)$ hierarchy

To determine the asymptotic expansion of the functions  $F_n$  defined in Eq. (27), let us define the generating functional

$$f(\nu, x) \equiv \sum_n \frac{\nu^n}{n!} F_n(x), \quad (\text{A1})$$

so that the functions  $F_n$  satisfy

$$F_n(x) = \left. \frac{\partial^n f(\nu, x)}{\partial \nu^n} \right|_{\nu=0}. \quad (\text{A2})$$

Hence, the functional  $f(\nu, x)$  completely determines the behavior of the functions  $F_n$ .

While it is difficult to determine the functions  $F_n(x)$  for any  $n$ , it is easy to find the form of the functional  $f(\nu, x)$ . Indeed

$$f(\nu, x) = \int_x^\infty \left( \sum_n \frac{\nu^n \ln^n u}{n!} \right) \frac{e^{2iu}}{u} du \quad (\text{A3})$$

$$= \int_x^\infty u^{\nu-1} e^{2iu} du = x^\nu E_{1-\nu}(-2ix) \quad (\text{A4})$$

in which  $E_{1-\nu}$  is the generalized exponential integral.

In this paper, we only need the asymptotic expansion of  $F_n(x)$  in the limit  $x \rightarrow 0$ . Therefore, we only need an asymptotic form of  $f(\nu, x)$  in this limit

$$f(\nu, x) \underset{x \rightarrow 0}{\sim} -\frac{x^\nu}{\nu} + 2^{-\nu} e^{i\pi\nu/2} \Gamma(\nu). \quad (\text{A5})$$

From this expression, we obtain systematically the asymptotic behavior of all the functions  $F_n$

$$F_0(x) = -\ln x - B + \mathcal{O}(x), \quad (\text{A6})$$

$$F_1(x) = -\frac{1}{2} \ln^2 x + \frac{B^2}{2} + \frac{\pi^2}{12} + \mathcal{O}(x), \quad (\text{A7})$$

$$F_2(x) = -\frac{1}{3} \ln^3 x - \frac{B^3}{3} - \frac{\pi^2}{6} B - \frac{2}{3} \zeta(3) + \mathcal{O}(x), \quad (\text{A8})$$

where we have used the complex constant

$$B \equiv \gamma_E + \ln(2) - \frac{i\pi}{2}, \quad (\text{A9})$$

and with  $\gamma_E$  the Euler-Mascheroni constant.

### 2. The $F_{0^n}$ hierarchy

Let us introduce the hierarchy of integrals  $I_n$  defined by

$$I_{n+1} = \int_x^{+\infty} \frac{e^{+2iy}}{y} \overline{I_n}(y) dy, \quad I_0 = 1. \quad (\text{A10})$$

From this definition, we see that

$$I_{2n}(x) = \overline{F_{0^{2n}}}(x), \quad I_{2n+1}(x) = F_{0^{2n+1}}(x), \quad (\text{A11})$$

while the  $I_n(x)$  are solutions of an infinite coupled system of complex differential equations

$$\frac{dI_{n+1}}{dx} = -\frac{e^{+2ix}}{x} \overline{I_n}(x). \quad (\text{A12})$$

A generating functional  $h(\nu, x)$  can be constructed as

$$h(\nu, x) \equiv \sum_{k=0}^{+\infty} I_k(x) \nu^k, \quad (\text{A13})$$

from which one has

$$I_n(x) = \frac{1}{n!} \left. \frac{\partial^n h(\nu, x)}{\partial \nu^n} \right|_{\nu=0}. \quad (\text{A14})$$

From Eq. (A10), the generating functional verifies the complex differential equation

$$\frac{\partial h}{\partial x} + \frac{\nu e^{2ix}}{x} \bar{h}(x) = 0. \quad (\text{A15})$$

Separating the real and imaginary parts with  $h = a(x) + ib(x)$ , one obtains the differential system

$$\begin{aligned} \frac{da}{dx} &= -\frac{\nu}{x} \cos(2x)a - \frac{\nu}{x} \sin(2x)b, \\ \frac{db}{dx} &= -\frac{\nu}{x} \sin(2x)a + \frac{\nu}{x} \cos(2x)b. \end{aligned} \quad (\text{A16})$$

Defining

$$X(x) \equiv \begin{bmatrix} a(x) \\ b(x) \end{bmatrix}, \quad A(x) \equiv \begin{bmatrix} \cos(2x) & \sin(2x) \\ \sin(2x) & -\cos(2x) \end{bmatrix}, \quad (\text{A17})$$

it can be recast into the matrix form

$$\frac{dX}{dx} = -\frac{\nu}{x} AX. \quad (\text{A18})$$

The matrix  $A$  can be diagonalized by a rotation as

$$A = P \Lambda P^{-1}, \quad (\text{A19})$$

where

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad P = \begin{bmatrix} \cos(x) & -\sin(x) \\ \sin(x) & \cos(x) \end{bmatrix}. \quad (\text{A20})$$

We can now perform the change of variable

$$Z(x) = \begin{bmatrix} z_1(x) \\ z_2(x) \end{bmatrix} \equiv P^{-1} X, \quad (\text{A21})$$

in terms of which Eq. (A18) greatly simplifies into

$$\frac{dZ}{dx} = -\left( \frac{\nu}{x} \Lambda + P^{-1} \frac{dP}{dx} \right) Z = \begin{pmatrix} -\frac{\nu}{x} & 1 \\ -1 & \frac{\nu}{x} \end{pmatrix} Z. \quad (\text{A22})$$

This system has no longer any oscillatory terms and can be decoupled by differentiation. One gets

$$\frac{d^2 z_1}{dx^2} = -\left[ 1 - \frac{\nu(\nu+1)}{x^2} \right] z_1, \quad z_2 = \frac{dz_1}{dx} + \frac{\nu}{x} z_1. \quad (\text{A23})$$

The first of these equations is, once more, a Riccati-Bessel differential equation which admits the exact solutions

$$z_1(x) = C_1(\nu) x j_\nu(x) + C_2(\nu) x y_\nu(x), \quad (\text{A24})$$

where  $j_\nu$  and  $y_\nu$  are the spherical Bessel functions of first and second kind [95]. The yet undetermined functions  $C_1(\nu)$  and  $C_2(\nu)$  are the integration constants with respect to  $x$ . Plugging Eq. (A24) into Eq. (A23), and using the standard recurrence relations on the spherical Bessel functions, one obtains

$$z_2(x) = C_1(\nu) x j_{\nu-1}(x) + C_2(\nu) x y_{\nu-1}(x). \quad (\text{A25})$$

From the definition of the  $I_n(x)$  in Eq. (A10), assuming a smooth convergence at infinity and no branch cut, we have, for all  $\nu$

$$\lim_{x \rightarrow \infty} h(\nu, x) = 1. \quad (\text{A26})$$

This condition fixes the integration constants  $C_1(\nu)$  and  $C_2(\nu)$ , which, by using the asymptotic forms of the spherical Bessel functions, read

$$C_1(\nu) = -\sin\left(\frac{\pi\nu}{2}\right), \quad C_2(\nu) = -\cos\left(\frac{\pi\nu}{2}\right). \quad (\text{A27})$$

From Eqs. (A21), (A24) and (A25), one finally obtains the exact expression for the generating functional

$$\begin{aligned} h(\nu, x) &= -x e^{ix} \left\{ \sin\left(\frac{\pi\nu}{2}\right) [j_\nu(x) + i j_{\nu-1}(x)] \right. \\ &\quad \left. + \cos\left(\frac{\pi\nu}{2}\right) [y_\nu(x) + i y_{\nu-1}(x)] \right\}. \end{aligned} \quad (\text{A28})$$

The exact expression of all the integrals  $F_{0^n}(x)$  can now be obtained from Eqs. (A11), (A14) and (A28), i.e., by differentiation of the spherical Bessel functions with respect to  $\nu$ . Let us mention that some of these integrals have been discussed in Ref. [105].

For our purpose, the  $F_{0^n}(x)$  need to be evaluated only in the super-Hubble limit  $x \rightarrow 0$ , and, in practice, we only need an asymptotic form of  $h(\nu, x)$  in the same limit. From Eq. (A28), using the expansion of the spherical Bessel functions near the origin [95], we obtain

$$\begin{aligned} h(\nu, x) &\underset{x \rightarrow 0}{\sim} \frac{2^\nu}{x^\nu \sqrt{\pi}} \cos\left(\frac{\pi\nu}{2}\right) \Gamma\left(\nu + \frac{1}{2}\right) \\ &\quad + \frac{x^\nu \sqrt{\pi}}{2^\nu \cos(\pi\nu)} \frac{i \sin\left(\frac{\pi\nu}{2}\right)}{\Gamma\left(\nu + \frac{1}{2}\right)}. \end{aligned} \quad (\text{A29})$$

From Eq. (A14), we have near the origin

$$F_0(x) = -B - \ln(x) + \mathcal{O}(x), \quad (\text{A30})$$

where  $B$  is given in Eq. (A9). This expression matches the one derived by the other method of Appendix A1. The second derivative of  $h(\nu, x)$  with respect to  $\nu$  gives

$$F_{00}(x) = \frac{\pi^2}{4} + \frac{B^2}{2} + B \ln(x) + \frac{1}{2} \ln^2(x) + \mathcal{O}(x), \quad (\text{A31})$$

while the third derivative allows us to determine

$$F_{000}(x) = -\frac{7}{3}\zeta(3) - \frac{\pi^2}{4}B - \frac{1}{6}B^3 - \left(\frac{\pi^2}{4} + \frac{B^2}{2}\right)\ln(x) - \frac{B}{2}\ln^2(x) - \frac{1}{6}\ln^3(x) + \mathcal{O}(x). \quad (\text{A32})$$

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## Appendix B: Direct calculation

### 1. General solution

As an intermediate step, using Eqs. (21), (24), (25) and (36), we give the super-Hubble limit for the numerator. At N3LO in slow-roll, one gets

$$\begin{aligned} |x\hat{\mu}|^2 = & 1 + \frac{2}{3}g_{1b}[2 + \text{Re}(F_0)] + \frac{2}{27}g_{1b}^2[4 + 3|F_0|^2 + 11\text{Re}(F_0)] + \frac{2}{9}g_{2b}[8 + 7\text{Re}(F_0) + 3\text{Re}(F_1) + 6\ln x] \\ & + \frac{2}{243}g_{1b}^3[-8 + 14\text{Re}(F_0) + 30|F_0|^2 + 9\text{Re}(\overline{F_{00}}F_0) + 9\text{Re}(F_{000})] \\ & + \frac{4}{81}g_{1b}g_{2b}[-4 + 21|F_0|^2 + 9\text{Re}(\overline{F_1}F_0) + 40\text{Re}(F_0) + 15\text{Re}(F_1) + 12\ln x + 18\text{Re}(F_0)\ln x] \\ & + \frac{2}{27}g_{3b}[52 + 50\text{Re}(F_0) + 42\text{Re}(F_1) + 9\text{Re}(F_2) + 48\ln x + 18\ln^2 x] + \mathcal{O}(\epsilon^4, x), \end{aligned} \quad (\text{B1})$$

where the  $x$ -dependency of the integrals has been omitted. Let us notice we have removed the integrals  $F_{01}(x)$  and  $F_{10}(x)$  from this expression since they only appear as the peculiar combination

$$\text{Re}\{[F_{10}(x) + \overline{F_{01}}(x)]\} = \text{Re}\{[\overline{F_0}(x)F_1(x)]\}. \quad (\text{B2})$$

This equality stems from an integration by parts of  $F_{01}(x)$ , given in Eq. (28), which gives the relation

$$F_{01}(x) + \overline{F_{10}}(x) = \overline{F_0}(x)F_1(x). \quad (\text{B3})$$

Note that we have also simplified Eq. (B1) using a similar equality

$$F_{00} + \overline{F_{00}} = F_0\overline{F_0}. \quad (\text{B4})$$

### 2. Application to scalars

In this section, we derive the scalar power spectrum at N3LO in the pivot of Eq. (65). For completeness, we use the method of the Green's function all the way through and this allows us to cross-check the mapping technique described in Sections III A 2 and III B 2. A fearless reader may indeed notice that the two methods yield identical results, as expected.

From the definition of the new time  $\tau$  in Eq. (60), one finds

$$\frac{d}{d\tau} = \frac{1}{c_s} \frac{d}{d\eta} = \frac{\mathcal{H}}{c_s} \frac{d}{dN}, \quad (\text{B5})$$

where Eq. (10) has been used to express the derivative with respect to  $\eta$ .

To use the Green's function method, we need to recast the equation of motion (59) into the form of Eq. (14). We do so by expressing  $z^{-1} d^2z/d\tau^2$  in terms of Hubble flow and sound flow functions

$$\frac{1}{z} \frac{d^2z}{d\tau^2} = \left(\frac{\mathcal{H}}{c_s}\right)^2 \left(2 - \epsilon_1 + \frac{3}{2}\epsilon_2 + \frac{5}{2}\delta_1 - \frac{1}{2}\epsilon_1\epsilon_2 - \frac{1}{2}\epsilon_1\delta_1 + \frac{1}{4}\epsilon_2^2 + \frac{1}{2}\epsilon_2\epsilon_3 + \epsilon_2\delta_1 + \frac{3}{4}\delta_1^2 + \frac{1}{2}\delta_1\delta_2\right). \quad (\text{B6})$$

We recall that  $z$  is defined in Eq. (61). Note that the equation above is exact.



From Eqs. (1) and (69), one has

$$\begin{aligned} \frac{d\epsilon_i}{d\ln\tau} = \frac{\tau\mathcal{H}}{c_s}\epsilon_i\epsilon_{i+1} = -\epsilon_i\epsilon_{i+1} & \left( 1 + \epsilon_1 - \delta_1 + \epsilon_1^2 + \epsilon_1\epsilon_2 - 2\epsilon_1\delta_1 + \delta_1^2 - \delta_1\delta_2 + \epsilon_1^3 + 3\epsilon_1^2\epsilon_2 + \epsilon_1\epsilon_2^2 \right. \\ & \left. + \epsilon_1\epsilon_2\epsilon_3 - 3\epsilon_1^2\delta_1 - 3\epsilon_1\epsilon_2\delta_1 + 3\delta_1^2\epsilon_1 - 3\epsilon_1\delta_1\delta_2 - \delta_1^3 + 3\delta_1^2\delta_2 - \delta_1\delta_2^2 - \delta_1\delta_2\delta_3 \right) + \mathcal{O}(\epsilon^6), \end{aligned} \quad (\text{B7})$$

and similarly for the sound flow parameters

$$\frac{d\delta_i}{d\ln\tau} = \frac{\tau\mathcal{H}}{c_s}\delta_i\delta_{i+1}. \quad (\text{B8})$$

Extending Eq. (B6) around the pivot time  $\tau_\flat$ , i.e.,

$$\tau_\flat = -\frac{1}{k_\flat}, \quad (\text{B9})$$

one can identify the coefficients  $g_\flat$  appearing in a decomposition identical to the one of Eq. (14), but, now, in terms of the variable  $x \equiv -k\tau$ . The leading term reads

$$\begin{aligned} g_{1\flat} = & 3\epsilon_{1\flat} + \frac{3}{2}\epsilon_{2\flat} - \frac{3}{2}\delta_{1\flat} + 4\epsilon_{1\flat}^2 + \frac{13}{2}\epsilon_{1\flat}\epsilon_{2\flat} - \frac{11}{2}\epsilon_{1\flat}\delta_{1\flat} + \frac{1}{2}\epsilon_{2\flat}\epsilon_{3\flat} + \frac{1}{4}\epsilon_{2\flat}^2 - 2\epsilon_{2\flat}\delta_{1\flat} + \frac{7}{4}\delta_{1\flat}^2 - \frac{7}{2}\delta_{1\flat}\delta_{2\flat} + 5\epsilon_{1\flat}^3 \\ & + \frac{35}{2}\epsilon_{1\flat}^2\epsilon_{2\flat} - \frac{23}{2}\epsilon_{1\flat}^2\delta_{1\flat} + \frac{15}{2}\epsilon_{1\flat}\epsilon_{2\flat}^2 + 5\epsilon_{1\flat}\epsilon_{2\flat}\epsilon_{3\flat} - 17\epsilon_{1\flat}\epsilon_{2\flat}\delta_{1\flat} + \frac{17}{2}\epsilon_{1\flat}\delta_{1\flat}^2 - 13\epsilon_{1\flat}\delta_{1\flat}\delta_{2\flat} - \epsilon_{2\flat}\epsilon_{3\flat}\delta_{1\flat} - \frac{1}{2}\epsilon_{2\flat}^2\delta_{1\flat} \\ & + \frac{5}{2}\epsilon_{2\flat}\delta_{1\flat}^2 - 3\epsilon_{2\flat}\delta_{1\flat}\delta_{2\flat} - 2\delta_{1\flat}^3 + 10\delta_{1\flat}^2\delta_{2\flat} - 4\delta_{1\flat}\delta_{2\flat}^2 - 4\delta_{1\flat}\delta_{2\flat}\delta_{3\flat} \\ & + \frac{1}{2} \left( 6\epsilon_{1\flat}\epsilon_{2\flat} + 3\epsilon_{2\flat}\epsilon_{3\flat} - 3\delta_{1\flat}\delta_{2\flat} + 22\epsilon_{1\flat}^2\epsilon_{2\flat} + 13\epsilon_{1\flat}\epsilon_{2\flat}^2 + 16\epsilon_{1\flat}\epsilon_{2\flat}\epsilon_{3\flat} - 17\epsilon_{1\flat}\epsilon_{2\flat}\delta_{1\flat} - 14\epsilon_{1\flat}\delta_{1\flat}\delta_{2\flat} + \epsilon_{2\flat}^2\epsilon_{3\flat} + \epsilon_{2\flat}\epsilon_{3\flat}^2 \right. \\ & \left. + \epsilon_{2\flat}\epsilon_{3\flat}\epsilon_{4\flat} - 7\epsilon_{2\flat}\epsilon_{3\flat}\delta_{1\flat} - 4\epsilon_{2\flat}\delta_{1\flat}\delta_{2\flat} + 10\delta_{1\flat}^2\delta_{2\flat} - 7\delta_{1\flat}\delta_{2\flat}^2 - 7\delta_{1\flat}\delta_{2\flat}\delta_{3\flat} \right) \ln\left(\frac{k}{k_\flat}\right) \\ & + \frac{3}{4} \left( 2\epsilon_{1\flat}\epsilon_{2\flat}^2 + 2\epsilon_{1\flat}\epsilon_{2\flat}\epsilon_{3\flat} + \epsilon_{2\flat}\epsilon_{3\flat}^2 + \epsilon_{2\flat}\epsilon_{3\flat}\epsilon_{4\flat} - \delta_{1\flat}\delta_{2\flat}^2 - \delta_{1\flat}\delta_{2\flat}\delta_{3\flat} \right) \ln^2\left(\frac{k}{k_\flat}\right) + \mathcal{O}(\epsilon^4), \end{aligned} \quad (\text{B10})$$

then, for the second order, one obtains

$$\begin{aligned} g_{2\flat} = & -3\epsilon_{1\flat}\epsilon_{2\flat} - \frac{3}{2}\epsilon_{2\flat}\epsilon_{3\flat} + \frac{3}{2}\delta_{1\flat}\delta_{2\flat} - 11\epsilon_{1\flat}^2\epsilon_{2\flat} - \frac{13}{2}\epsilon_{1\flat}\epsilon_{2\flat}^2 - 8\epsilon_{1\flat}\epsilon_{2\flat}\epsilon_{3\flat} + \frac{17}{2}\epsilon_{1\flat}\epsilon_{2\flat}\delta_{1\flat} + 7\epsilon_{1\flat}\delta_{1\flat}\delta_{2\flat} - \frac{1}{2}\epsilon_{2\flat}^2\epsilon_{3\flat} \\ & - \frac{1}{2}\epsilon_{2\flat}\epsilon_{3\flat}^2 - \frac{1}{2}\epsilon_{2\flat}\epsilon_{3\flat}\epsilon_{4\flat} + \frac{7}{2}\epsilon_{2\flat}\epsilon_{3\flat}\delta_{1\flat} + 2\epsilon_{2\flat}\delta_{1\flat}\delta_{2\flat} - 5\delta_{1\flat}^2\delta_{2\flat} + \frac{7}{2}\delta_{1\flat}\delta_{2\flat}^2 + \frac{7}{2}\delta_{1\flat}\delta_{2\flat}\delta_{3\flat} \\ & + \frac{3}{2}(-2\epsilon_{1\flat}\epsilon_{2\flat}^2 - 2\epsilon_{1\flat}\epsilon_{2\flat}\epsilon_{3\flat} - \epsilon_{2\flat}\epsilon_{3\flat}^2 - \epsilon_{2\flat}\epsilon_{3\flat}\epsilon_{4\flat} + \delta_{1\flat}\delta_{2\flat}^2 + \delta_{1\flat}\delta_{2\flat}\delta_{3\flat}) \ln\left(\frac{k}{k_\flat}\right) + \mathcal{O}(\epsilon^4), \end{aligned} \quad (\text{B11})$$

and, finally,

$$g_{3\flat} = \frac{3}{2}\epsilon_{1\flat}\epsilon_{2\flat}^2 + \frac{3}{2}\epsilon_{1\flat}\epsilon_{2\flat}\epsilon_{3\flat} + \frac{3}{4}\epsilon_{2\flat}\epsilon_{3\flat}^2 + \frac{3}{4}\epsilon_{2\flat}\epsilon_{3\flat}\epsilon_{4\flat} - \frac{3}{4}\delta_{1\flat}\delta_{2\flat}^2 - \frac{3}{4}\delta_{1\flat}\delta_{2\flat}\delta_{3\flat} + \mathcal{O}(\epsilon^4). \quad (\text{B12})$$

Notice that, contrary to Eq. (17), the coefficients  $g_\flat$  depend explicitly on  $k$ . This is due to our choice of the particular time  $\tau_\flat$  at which we have performed the expansion, which is directly defined from the pivot scale  $k_\flat$ . On the contrary, the dependence on  $k$  is implicit in Eq. (17) due to the choice of the  $k$ -dependent pivot time of Eq. (15), namely,

$$\eta_\flat = -\frac{1}{k}. \quad (\text{B13})$$

This implies that all the  $\epsilon_\flat$  are implicit functions of  $k$ . Replacing the coefficients  $g_\flat$  in Eq. (B1) by their expansion given in Eqs. (B10) to (B12), after some algebra, one finds the scalar power spectrum at the pivot scale  $k_\flat$

$$\mathcal{P}_\zeta(k) = \frac{H_\flat^2}{8\pi^2\epsilon_{1\flat}c_{s\flat}} \left[ b_{0\flat}^{(s)} + b_{1\flat}^{(s)} \ln\left(\frac{k}{k_\flat}\right) + b_{2\flat}^{(s)} \ln^2\left(\frac{k}{k_\flat}\right) + b_{3\flat}^{(s)} \ln^3\left(\frac{k}{k_\flat}\right) \right]. \quad (\text{B14})$$

The corrections to the amplitude read

$$\begin{aligned}
b_{0\triangleright}^{(S)} = & 1 - 2(C+1)\epsilon_{1\triangleright} - C\epsilon_{2\triangleright} + (C+2)\delta_{1\triangleright} + \frac{1}{2}(\pi^2 + 4C^2 + 4C - 6)\epsilon_{1\triangleright}^2 + \frac{1}{12}(7\pi^2 + 12C^2 - 12C - 72)\epsilon_{1\triangleright}\epsilon_{2\triangleright} \\
& - \frac{1}{2}(\pi^2 + 4C^2 + 6C - 4)\epsilon_{1\triangleright}\delta_{1\triangleright} + \frac{1}{8}(\pi^2 + 4C^2 - 8)\epsilon_{2\triangleright}^2 + \frac{1}{24}(\pi^2 - 12C^2)\epsilon_{2\triangleright}\epsilon_{3\triangleright} - \frac{1}{4}(\pi^2 + 4C^2 + 4C - 8)\epsilon_{2\triangleright}\delta_{1\triangleright} \\
& + \frac{1}{8}(\pi^2 + 4C^2 + 8C)\delta_{1\triangleright}^2 - \frac{1}{24}(\pi^2 - 12C^2 - 48C - 48)\delta_{1\triangleright}\delta_{2\triangleright} \\
& + \frac{1}{24}[4C^3 + 3(\pi^2 - 8)C + 14\zeta(3) - 16](-8\epsilon_{1\triangleright}^3 + 12\epsilon_{1\triangleright}^2\delta_{1\triangleright} - 6\epsilon_{1\triangleright}\delta_{1\triangleright}^2 - \epsilon_{2\triangleright}^3 + 3\epsilon_{2\triangleright}^2\delta_{1\triangleright} - 3\epsilon_{2\triangleright}\delta_{1\triangleright}^2 + \delta_{1\triangleright}^3) \\
& + \frac{1}{12}[13\pi^2 - 8(\pi^2 - 9)C + 36C^2 - 84\zeta(3)]\epsilon_{1\triangleright}^2\epsilon_{2\triangleright} \\
& - \frac{1}{24}[8C^3 - 15\pi^2 + 6(\pi^2 - 4)C - 12C^2 + 100\zeta(3) + 16]\epsilon_{1\triangleright}\epsilon_{2\triangleright}^2 \\
& + \frac{1}{12}[8C^3 + \pi^2 + 6(\pi^2 - 12)C - 12C^2 - 8\zeta(3) - 8]\epsilon_{1\triangleright}\epsilon_{2\triangleright}\epsilon_{3\triangleright} \\
& + \frac{1}{24}[24C^3 - 13\pi^2 + 2(13\pi^2 - 108)C - 36C^2 + 168\zeta(3) - 96]\epsilon_{1\triangleright}\epsilon_{2\triangleright}\delta_{1\triangleright} \\
& + \frac{1}{24}[12C^3 + 13\pi^2 + (5\pi^2 - 24)C + 36C^2 - 96](-2\epsilon_{1\triangleright}\delta_{1\triangleright}\delta_{2\triangleright} - \epsilon_{2\triangleright}\delta_{1\triangleright}\delta_{2\triangleright} + \delta_{1\triangleright}^2\delta_{2\triangleright}) \\
& + \frac{1}{24}[12C^3 + (5\pi^2 - 48)C](\epsilon_{2\triangleright}^2\epsilon_{3\triangleright} - \epsilon_{2\triangleright}\epsilon_{3\triangleright}\delta_{1\triangleright}) + \frac{1}{24}[\pi^2 C - 4C^3 - 8\zeta(3) + 16](\epsilon_{2\triangleright}\epsilon_{3\triangleright}^2 + \epsilon_{2\triangleright}\epsilon_{3\triangleright}\epsilon_{4\triangleright}) \\
& + \frac{1}{24}[4C^3 - 2\pi^2 - (\pi^2 - 48)C + 24C^2 + 8\zeta(3) + 32](\delta_{1\triangleright}\delta_{2\triangleright}^2 + \delta_{1\triangleright}\delta_{2\triangleright}\delta_{3\triangleright}).
\end{aligned} \tag{B15}$$

Deviations from scale invariance are encoded in

$$\begin{aligned}
b_{1\triangleright}^{(S)} = & -2\epsilon_{1\triangleright} - \epsilon_{2\triangleright} + \delta_{1\triangleright} + (2C+1)(2\epsilon_{1\triangleright}^2 - \epsilon_{2\triangleright}\delta_{1\triangleright}) + (2C-1)\epsilon_{1\triangleright}\epsilon_{2\triangleright} - (4C+3)\epsilon_{1\triangleright}\delta_{1\triangleright} + C(\epsilon_{2\triangleright}^2 - \epsilon_{2\triangleright}\epsilon_{3\triangleright}) + (C+1)\delta_{1\triangleright}^2 \\
& + (C+2)\delta_{1\triangleright}\delta_{2\triangleright} - \frac{1}{8}(\pi^2 + 4C^2 - 8)(8\epsilon_{1\triangleright}^3 - 12\epsilon_{1\triangleright}^2\delta_{1\triangleright} + 6\epsilon_{1\triangleright}\delta_{1\triangleright}^2 + \epsilon_{2\triangleright}^3 - 3\epsilon_{2\triangleright}^2\delta_{1\triangleright} + 3\epsilon_{2\triangleright}\delta_{1\triangleright}^2 - \delta_{1\triangleright}^3) \\
& - \frac{2}{3}(\pi^2 - 9C - 9)\epsilon_{1\triangleright}^2\epsilon_{2\triangleright} - \frac{1}{4}(\pi^2 + 4C^2 - 4C - 4)\epsilon_{1\triangleright}\epsilon_{2\triangleright}^2 + \frac{1}{2}(\pi^2 + 4C^2 - 4C - 12)\epsilon_{1\triangleright}\epsilon_{2\triangleright}\epsilon_{3\triangleright} \\
& + \frac{1}{12}(13\pi^2 + 36C^2 - 36C - 108)\epsilon_{1\triangleright}\epsilon_{2\triangleright}\delta_{1\triangleright} - \frac{1}{24}(5\pi^2 + 36C^2 + 72C - 24)(2\epsilon_{1\triangleright}\delta_{1\triangleright}\delta_{2\triangleright} + \epsilon_{2\triangleright}\delta_{1\triangleright}\delta_{2\triangleright} - \delta_{1\triangleright}^2\delta_{2\triangleright}) \\
& + \frac{1}{24}(5\pi^2 + 36C^2 - 48)(\epsilon_{2\triangleright}^2\epsilon_{3\triangleright} - \epsilon_{2\triangleright}\epsilon_{3\triangleright}\delta_{1\triangleright}) + \frac{1}{24}(\pi^2 - 12C^2)(\epsilon_{2\triangleright}\epsilon_{3\triangleright}^2 + \epsilon_{2\triangleright}\epsilon_{3\triangleright}\epsilon_{4\triangleright}) \\
& - \frac{1}{24}(\pi^2 - 12C^2 - 48C - 48)(\delta_{1\triangleright}\delta_{2\triangleright}^2 + \delta_{1\triangleright}\delta_{2\triangleright}\delta_{3\triangleright}).
\end{aligned} \tag{B16}$$

while the deviations from a pure power law spectrum are given by

$$\begin{aligned}
b_{2\triangleright}^{(S)} = & 2\epsilon_{1\triangleright}^2 + \epsilon_{1\triangleright}\epsilon_{2\triangleright} - 2\epsilon_{1\triangleright}\delta_{1\triangleright} + \frac{1}{2}\epsilon_{2\triangleright}^2 - \frac{1}{2}\epsilon_{2\triangleright}\epsilon_{3\triangleright} - \epsilon_{2\triangleright}\delta_{1\triangleright} + \frac{1}{2}\delta_{1\triangleright}^2 + \frac{1}{2}\delta_{1\triangleright}\delta_{2\triangleright} - 4C\epsilon_{1\triangleright}^3 + 3\epsilon_{1\triangleright}^2\epsilon_{2\triangleright} + 6C\epsilon_{1\triangleright}^2\delta_{1\triangleright} \\
& - \frac{1}{2}(2C-1)(\epsilon_{1\triangleright}\epsilon_{2\triangleright}^2 - 2\epsilon_{1\triangleright}\epsilon_{2\triangleright}\epsilon_{3\triangleright} - 3\epsilon_{1\triangleright}\epsilon_{2\triangleright}\delta_{1\triangleright}) - 3C\epsilon_{1\triangleright}\delta_{1\triangleright}^2 - \frac{3}{2}(C+1)(2\epsilon_{1\triangleright}\delta_{1\triangleright}\delta_{2\triangleright} + \epsilon_{2\triangleright}\delta_{1\triangleright}\delta_{2\triangleright} - \delta_{1\triangleright}^2\delta_{2\triangleright}) \\
& + \frac{1}{2}C(-\epsilon_{2\triangleright}^3 + 3\epsilon_{2\triangleright}^2\epsilon_{3\triangleright} + 3\epsilon_{2\triangleright}^2\delta_{1\triangleright} - \epsilon_{2\triangleright}\epsilon_{3\triangleright}^2 - 3\epsilon_{2\triangleright}\epsilon_{3\triangleright}\delta_{1\triangleright} - \epsilon_{2\triangleright}\epsilon_{3\triangleright}\epsilon_{4\triangleright} - 3\epsilon_{2\triangleright}\delta_{1\triangleright}^2 + \delta_{1\triangleright}^3) + \frac{1}{2}(C+2)(\delta_{1\triangleright}\delta_{2\triangleright}^2 + \delta_{1\triangleright}\delta_{2\triangleright}\delta_{3\triangleright}),
\end{aligned} \tag{B17}$$

and

$$\begin{aligned}
b_{3\triangleright}^{(S)} = & -\frac{4}{3}\epsilon_{1\triangleright}^3 + 2\epsilon_{1\triangleright}^2\delta_{1\triangleright} - \frac{1}{3}\epsilon_{1\triangleright}\epsilon_{2\triangleright}^2 + \frac{2}{3}\epsilon_{1\triangleright}\epsilon_{2\triangleright}\epsilon_{3\triangleright} - \epsilon_{1\triangleright}\delta_{1\triangleright}^2 - \epsilon_{1\triangleright}\delta_{1\triangleright}\delta_{2\triangleright} + \epsilon_{1\triangleright}\epsilon_{2\triangleright}\delta_{1\triangleright} - \frac{1}{6}\epsilon_{2\triangleright}^3 + \frac{1}{2}\epsilon_{2\triangleright}^2\epsilon_{3\triangleright} - \frac{1}{2}\epsilon_{2\triangleright}\delta_{1\triangleright}\delta_{2\triangleright} \\
& + \frac{1}{2}\epsilon_{2\triangleright}^2\delta_{1\triangleright} - \frac{1}{6}\epsilon_{2\triangleright}\epsilon_{3\triangleright}^2 - \frac{1}{6}\epsilon_{2\triangleright}\epsilon_{3\triangleright}\epsilon_{4\triangleright} - \frac{1}{2}\epsilon_{2\triangleright}\epsilon_{3\triangleright}\delta_{1\triangleright} - \frac{1}{2}\epsilon_{2\triangleright}\delta_{1\triangleright}^2 + \frac{1}{6}\delta_{1\triangleright}^3 + \frac{1}{2}\delta_{1\triangleright}^2\delta_{2\triangleright} + \frac{1}{6}\delta_{1\triangleright}\delta_{2\triangleright}^2 + \frac{1}{6}\delta_{1\triangleright}\delta_{2\triangleright}\delta_{3\triangleright}.
\end{aligned} \tag{B18}$$

### Appendix C: Power-law quantities at N4LO

Having at our disposal the expression of the scalar and tensor power spectra at third order allows us to derive the power law parameters at N4LO by using the invariance of the results with respect to the choice of the pivot wavenumber  $k_\diamond$ . The method is detailed in Ref. [35] and, here, we apply it using our N3LO results.

#### 1. Scalar power-law parameters

The scalar spectral index  $n_s$  is defined by

$$n_s - 1 \equiv \left. \frac{d \ln [\mathcal{P}_\zeta(k)]}{d \ln k} \right|_{k=k_\diamond}, \quad (\text{C1})$$

and, to determine its value at N4LO, one can use the invariance of the power spectrum with respect to the choice of the pivot, i.e.,

$$\frac{d \ln \mathcal{P}_\zeta}{d \ln k_\diamond} = 0. \quad (\text{C2})$$

Let us consider the logarithmic expansion

$$\ln [\mathcal{P}_\zeta(k)] = \ln P_\diamond + \sum_{n=0}^{+\infty} \frac{c_{n_\diamond}^{(s)}}{n!} \ln^n \left( \frac{k}{k_\diamond} \right), \quad (\text{C3})$$

where the coefficients  $c_{n_\diamond}^{(s)}$  can be straightforwardly obtained from the  $b_{n_\diamond}^{(s)}$ , i.e., at N3LO in the slow-roll parameters. In this expression  $P_\diamond$  stands for the overall amplitude

$$P_\diamond \equiv \frac{H_\diamond^2}{8\pi^2 \epsilon_{1_\diamond} c_{s_\diamond}}. \quad (\text{C4})$$

Plugging Eq. (C3) into Eq. (C2) one gets the recurrence relations

$$c_{1_\diamond}^{(s)} = \frac{dc_{0_\diamond}^{(s)}}{d \ln k_\diamond} + \frac{d \ln P_\diamond}{d \ln k_\diamond}, \quad c_{n+1_\diamond}^{(s)} = \frac{dc_{n_\diamond}^{(s)}}{d \ln k_\diamond}. \quad (\text{C5})$$

Using Eq. (66) to trade the derivative with respect to  $k_\diamond$  by a derivative with respect to  $\eta_\diamond$ , one obtains, for the spectral index,

$$n_s - 1 = c_{1_\diamond}^{(s)} = \frac{1}{\eta_\diamond \mathcal{H}_\diamond \delta_{1_\diamond} - 1} \left[ \frac{dc_{0_\diamond}^{(s)}}{d \ln |\eta_\diamond|} + \frac{d \ln P_\diamond}{d \ln |\eta_\diamond|} \right]. \quad (\text{C6})$$

From the expression of  $\eta \mathcal{H}$  given in Eq. (8), and the one of  $b_{0_\diamond}^{(s)}$  given in Eq. (74), after some basic but rather long algebra, one gets

$$n_s = 1 + n_s^{(1)} + n_s^{(2)} + n_s^{(3)} + n_s^{(4)} + \mathcal{O}(\epsilon^5) \quad (\text{C7})$$

where

$$\begin{aligned} n_s^{(1)} &= -2\epsilon_{1_\diamond} - \epsilon_{2_\diamond} + \delta_{1_\diamond}, \\ n_s^{(2)} &= -2\epsilon_{1_\diamond}^2 - (2C + 3)\epsilon_{1_\diamond}\epsilon_{2_\diamond} - C\epsilon_{2_\diamond}\epsilon_{3_\diamond} + 3\epsilon_{1_\diamond}\delta_{1_\diamond} + \epsilon_{2_\diamond}\delta_{1_\diamond} - \delta_{1_\diamond}^2 + (C + 2)\delta_{1_\diamond}\delta_{2_\diamond}, \end{aligned} \quad (\text{C8})$$

and

$$\begin{aligned} n_s^{(3)} &= -2\epsilon_{1_\diamond}^3 + (\pi^2 - 6C - 15)\epsilon_{1_\diamond}^2\epsilon_{2_\diamond} + \frac{1}{12}(7\pi^2 - 12C^2 - 36C - 84)\epsilon_{1_\diamond}\epsilon_{2_\diamond}^2 + \left(\frac{7}{12}\pi^2 - C^2 - 4C - 6\right)\epsilon_{1_\diamond}\epsilon_{2_\diamond}\epsilon_{3_\diamond} \\ &+ \frac{1}{24}(\pi^2 - 12C^2)\epsilon_{2_\diamond}\epsilon_{3_\diamond}^2 + \left(\frac{\pi^2}{4} - 2\right)\epsilon_{2_\diamond}^2\epsilon_{3_\diamond} + \frac{1}{24}(\pi^2 - 12C^2)\epsilon_{2_\diamond}\epsilon_{3_\diamond}\epsilon_{4_\diamond} - 4\epsilon_{1_\diamond}\delta_{1_\diamond}^2 - \left(\frac{\pi^2}{2} - 4C - 10\right)\epsilon_{1_\diamond}\delta_{1_\diamond}\delta_{2_\diamond} \\ &- \left(\frac{\pi^2}{4} - C - 3\right)\epsilon_{2_\diamond}\delta_{1_\diamond}\delta_{2_\diamond} - \epsilon_{2_\diamond}\delta_{1_\diamond}^2 + 5\epsilon_{1_\diamond}^2\delta_{1_\diamond} - \left(\frac{\pi^2}{2} - 5C - 13\right)\epsilon_{1_\diamond}\epsilon_{2_\diamond}\delta_{1_\diamond} - \left(\frac{\pi^2}{4} - 2C - 3\right)\epsilon_{2_\diamond}\epsilon_{3_\diamond}\delta_{1_\diamond} \\ &+ \delta_{1_\diamond}^3 + \frac{1}{4}(\pi^2 - 12C - 32)\delta_{1_\diamond}^2\delta_{2_\diamond} - \frac{1}{24}(\pi^2 - 12C^2 - 48C - 48)(\delta_{1_\diamond}\delta_{2_\diamond}^2 + \delta_{1_\diamond}\delta_{2_\diamond}\delta_{3_\diamond}). \end{aligned} \quad (\text{C9})$$

The Eqs. (C8) and (C9) match the ones previously derived in Ref. [57], obtained using the same method but starting from the N2LO corrections to the amplitude. Our N3LO result allows us to determine the next term, it reads

$$\begin{aligned}
n_s^{(4)} = & -2\epsilon_{1\circ}^4 - \frac{1}{24} [8C^3 - 21\pi^2 - 14(\pi^2 - 12)C + 36C^2 + 100\zeta(3) + 88] \epsilon_{1\circ}\epsilon_{2\circ}^3 + \left[ \frac{1}{2}(\pi^2 - 8)C \right] \epsilon_{2\circ}^2\epsilon_{3\circ}^2 \\
& + \left[ \frac{61}{12}\pi^2 + (2\pi^2 - 35)C - 7C^2 - 14\zeta(3) - 37 \right] \epsilon_{1\circ}^2\epsilon_{2\circ}^2 + [4\pi^2 - 12C - 14\zeta(3) - 29] \epsilon_{1\circ}^3\epsilon_{2\circ} + \left[ 2 - \frac{7}{4}\zeta(3) \right] \epsilon_{2\circ}^3\epsilon_{3\circ} \\
& + \left[ \frac{17}{6}\pi^2 + (\pi^2 - 19)C - 4C^2 - 7\zeta(3) - 20 \right] \epsilon_{1\circ}^2\epsilon_{2\circ}\epsilon_{3\circ} - \left[ C^3 - \frac{13}{6}\pi^2 - \frac{7}{4}(\pi^2 - 12)C + 5C^2 + 9\zeta(3) + 9 \right] \epsilon_{1\circ}\epsilon_{2\circ}^2\epsilon_{3\circ} \\
& + \left[ -\frac{1}{3}C^3 + \frac{5}{24}\pi^2 + \frac{1}{12}(7\pi^2 - 72)C - \frac{5}{2}C^2 - \frac{2}{3}\zeta(3) - \frac{2}{3} \right] (\epsilon_{1\circ}\epsilon_{2\circ}\epsilon_{3\circ}^2 + \epsilon_{1\circ}\epsilon_{2\circ}\epsilon_{3\circ}\epsilon_{4\circ}) + \frac{1}{4}(\pi^2 - 8)C\epsilon_{2\circ}^2\epsilon_{3\circ}\epsilon_{4\circ} \\
& + \left[ \frac{\pi^2}{8}C - \frac{1}{2}C^3 - \zeta(3) + 2 \right] \epsilon_{2\circ}\epsilon_{3\circ}^2\epsilon_{4\circ} + \frac{1}{24} [\pi^2C - 4C^3 - 8\zeta(3) + 16] (\epsilon_{2\circ}\epsilon_{3\circ}^3 + \epsilon_{2\circ}\epsilon_{3\circ}\epsilon_{4\circ}^2 + \epsilon_{2\circ}\epsilon_{3\circ}\epsilon_{4\circ}\epsilon_{5\circ}) + 5\epsilon_{1\circ}\delta_{1\circ}^3 \\
& + \left[ \frac{13}{4}\pi^2 - 15C - 7\zeta(3) - 44 \right] \epsilon_{1\circ}\delta_{1\circ}^2\delta_{2\circ} + \left[ -\frac{29}{24}\pi^2 - \frac{1}{2}(\pi^2 - 24)C + \frac{5}{2}C^2 + 16 \right] (\epsilon_{1\circ}\delta_{1\circ}\delta_{2\circ}^2 + \epsilon_{1\circ}\delta_{1\circ}\delta_{2\circ}\delta_{3\circ}) \\
& + \epsilon_{2\circ}\delta_{1\circ}^3 + \left[ \frac{5}{4}\pi^2 - 3C - \frac{7}{2}\zeta(3) - 10 \right] \epsilon_{2\circ}\delta_{1\circ}^2\delta_{2\circ} + \left[ -\frac{13}{24}\pi^2 - \frac{1}{4}(\pi^2 - 12)C + \frac{1}{2}C^2 + 5 \right] (\epsilon_{2\circ}\delta_{1\circ}\delta_{2\circ}^2 + \epsilon_{2\circ}\delta_{1\circ}\delta_{2\circ}\delta_{3\circ}) \\
& - 9\epsilon_{1\circ}^2\delta_{1\circ}^2 - \left[ \frac{5}{2}\pi^2 - 9C - 7\zeta(3) - 24 \right] \epsilon_{1\circ}^2\delta_{1\circ}\delta_{2\circ} - \left[ \frac{\pi^2}{4} - \frac{7}{4}\zeta(3) \right] \epsilon_{2\circ}^2\delta_{1\circ}\delta_{2\circ} + \left[ \frac{7}{4}\pi^2 - 9C - \frac{7}{2}\zeta(3) - 28 \right] \epsilon_{1\circ}\epsilon_{2\circ}\delta_{1\circ}^2 \\
& - \left[ \frac{29}{8}\pi^2 + (\pi^2 - 25)C - \frac{9}{2}C^2 - 7\zeta(3) - 35 \right] \epsilon_{1\circ}\epsilon_{2\circ}\delta_{1\circ}\delta_{2\circ} - \left[ \frac{5}{8}\pi^2 + \frac{1}{2}(\pi^2 - 12)C - \frac{3}{2}C^2 - 5 \right] \epsilon_{2\circ}\epsilon_{3\circ}\delta_{1\circ}\delta_{2\circ} \\
& + \left[ \frac{3}{4}\pi^2 - 3C - \frac{7}{4}\zeta(3) - \frac{13}{2} \right] \epsilon_{2\circ}\epsilon_{3\circ}\delta_{1\circ}^2 + 7\epsilon_{1\circ}^3\delta_{1\circ} - \left[ \frac{67}{24}\pi^2 + \frac{1}{2}(\pi^2 - 32)C - \frac{7}{2}C^2 - 7\zeta(3) - 25 \right] \epsilon_{1\circ}\epsilon_{2\circ}^2\delta_{1\circ} \\
& - \left[ \frac{\pi^2}{8} + \frac{1}{4}(\pi^2 - 12)C - \frac{3}{2}C^2 \right] \epsilon_{2\circ}\epsilon_{3\circ}^2\delta_{1\circ} - \left[ \frac{11}{2}\pi^2 - 21C - 14\zeta(3) - 60 \right] \epsilon_{2\circ}\epsilon_{1\circ}^2\delta_{1\circ} \\
& - \left[ \frac{3}{4}\pi^2 - \frac{7}{2}\zeta(3) - 2 \right] \epsilon_{2\circ}^2\epsilon_{3\circ}\delta_{1\circ} - \left[ \frac{73}{24}\pi^2 + \frac{1}{2}(\pi^2 - 38)C - \frac{7}{2}C^2 - 7\zeta(3) - 26 \right] \epsilon_{1\circ}\epsilon_{2\circ}\epsilon_{3\circ}\delta_{1\circ} \\
& - \left[ \frac{\pi^2}{8} + \frac{1}{4}(\pi^2 - 12)C - \frac{3}{2}C^2 \right] \epsilon_{2\circ}\epsilon_{3\circ}\epsilon_{4\circ}\delta_{1\circ} - \delta_{1\circ}^4 - \frac{1}{4} [4\pi^2 - 24C - 7\zeta(3) - 74] \delta_{1\circ}^3\delta_{2\circ} \\
& + \frac{1}{24} [31\pi^2 + 12(\pi^2 - 36)C - 84C^2 - 576] \delta_{1\circ}^2\delta_{2\circ}^2 \\
& + \frac{1}{24} [4C^3 - 2\pi^2 - (\pi^2 - 48)C + 24C^2 + 8\zeta(3) + 32] (\delta_{1\circ}\delta_{2\circ}^3 + \delta_{1\circ}\delta_{2\circ}\delta_{3\circ}^2 + \delta_{1\circ}\delta_{2\circ}\delta_{3\circ}\delta_{4\circ}) \\
& + \left[ \frac{2}{3}\pi^2 + \frac{1}{4}(\pi^2 - 40)C - 2C^2 - 13 \right] \delta_{1\circ}^2\delta_{2\circ}\delta_{3\circ} + \left[ \frac{1}{2}C^3 - \frac{1}{4}\pi^2 - \frac{1}{8}(\pi^2 - 48)C + 3C^2 + \zeta(3) + 4 \right] \delta_{1\circ}\delta_{2\circ}^2\delta_{3\circ}.
\end{aligned} \tag{C10}$$

Using the same technique, we can derive the running of the spectral index

$$\alpha_s \equiv \left. \frac{d^2 \ln [\mathcal{P}_\zeta(k)]}{d(\ln k)^2} \right|_{k=k_\circ} = \alpha_s^{(2)} + \alpha_s^{(3)} + \alpha_s^{(4)} + \mathcal{O}(\epsilon^5). \tag{C11}$$

The leading-order term is second order in the slow-roll parameters and reads

$$\alpha_s^{(2)} = -2\epsilon_{1\circ}\epsilon_{2\circ} - \epsilon_{2\circ}\epsilon_{3\circ} + \delta_{1\circ}\delta_{2\circ}, \tag{C12}$$

while the third order term is given by

$$\begin{aligned}
\alpha_s^{(3)} = & -6\epsilon_{1\circ}^2\epsilon_{2\circ} - (2C + 3)\epsilon_{1\circ}\epsilon_{2\circ}^2 - C(\epsilon_{2\circ}\epsilon_{3\circ}^2 + \epsilon_{2\circ}\epsilon_{3\circ}\epsilon_{4\circ}) - (2C + 4)\epsilon_{1\circ}\epsilon_{2\circ}\epsilon_{3\circ} + 5\epsilon_{1\circ}\epsilon_{2\circ}\delta_{1\circ} + 2\epsilon_{2\circ}\epsilon_{3\circ}\delta_{1\circ} \\
& + 4\epsilon_{1\circ}\delta_{1\circ}\delta_{2\circ} + \epsilon_{2\circ}\delta_{1\circ}\delta_{2\circ} - 3\delta_{1\circ}^2\delta_{2\circ} + (C + 2)(\delta_{1\circ}\delta_{2\circ}^2 + \delta_{1\circ}\delta_{2\circ}\delta_{3\circ}).
\end{aligned} \tag{C13}$$

Both  $\alpha_s^{(2)}$  and  $\alpha_s^{(3)}$  match the expressions previously derived in Ref. [57]. The new term is the fourth order and reads

$$\begin{aligned}
\alpha_s^{(4)} = & -12\epsilon_{1\circ}^3\epsilon_{2\circ} + (2\pi^2 - 14C - 35)\epsilon_{1\circ}^2\epsilon_{2\circ}^2 + (\pi^2 - 8C - 19)\epsilon_{1\circ}^2\epsilon_{2\circ}\epsilon_{3\circ} + \frac{1}{12}(7\pi^2 - 12C^2 - 36C - 84)\epsilon_{1\circ}\epsilon_{2\circ}^3 \\
& + \left(\frac{7}{4}\pi^2 - 3C^2 - 10C - 21\right)\epsilon_{1\circ}\epsilon_{2\circ}^2\epsilon_{3\circ} + \left(\frac{\pi^2}{8} - \frac{3}{2}C^2\right)\epsilon_{2\circ}\epsilon_{3\circ}^2\epsilon_{4\circ} + \left(\frac{\pi^2}{4} - 2\right)(2\epsilon_{2\circ}^2\epsilon_{3\circ}^2 + \epsilon_{2\circ}^2\epsilon_{3\circ}\epsilon_{4\circ}) \\
& + \left(\frac{7}{12}\pi^2 - C^2 - 5C - 6\right)(\epsilon_{1\circ}\epsilon_{2\circ}\epsilon_{3\circ}^2 + \epsilon_{1\circ}\epsilon_{2\circ}\epsilon_{3\circ}\epsilon_{4\circ}) + \frac{1}{24}(\pi^2 - 12C^2)(\epsilon_{2\circ}\epsilon_{3\circ}^3 + \epsilon_{2\circ}\epsilon_{3\circ}\epsilon_{4\circ}^2 + \epsilon_{2\circ}\epsilon_{3\circ}\epsilon_{4\circ}\epsilon_{5\circ}) \\
& - 15\epsilon_{1\circ}\delta_{1\circ}^2\delta_{2\circ} - \left(\frac{\pi^2}{2} - 5C - 12\right)(\epsilon_{1\circ}\delta_{1\circ}\delta_{2\circ}^2 + \epsilon_{1\circ}\delta_{1\circ}\delta_{2\circ}\delta_{3\circ}) - \left(\frac{\pi^2}{4} - C - 3\right)(\epsilon_{2\circ}\delta_{1\circ}\delta_{2\circ}^2 + \epsilon_{2\circ}\delta_{1\circ}\delta_{2\circ}\delta_{3\circ}) \\
& - 3\epsilon_{2\circ}\delta_{1\circ}^2\delta_{2\circ} + 9\epsilon_{1\circ}^2\delta_{1\circ}\delta_{2\circ} - 9\epsilon_{1\circ}\epsilon_{2\circ}\delta_{1\circ}^2 - 3\epsilon_{1\circ}\epsilon_{3\circ}\delta_{1\circ}^2 - (\pi^2 - 9C - 25)\epsilon_{1\circ}\epsilon_{2\circ}\delta_{1\circ}\delta_{2\circ} - \left(\frac{\pi^2}{2} - 3C - 6\right)\epsilon_{2\circ}\epsilon_{3\circ}\delta_{1\circ}\delta_{2\circ} \\
& + 21\epsilon_{1\circ}^2\epsilon_{2\circ}\delta_{1\circ} - \left(\frac{\pi^2}{2} - 7C - 16\right)\epsilon_{1\circ}\epsilon_{2\circ}^2\delta_{1\circ} - \left(\frac{\pi^2}{4} - 3C - 3\right)\epsilon_{2\circ}\epsilon_{3\circ}^2\delta_{1\circ} - \left(\frac{\pi^2}{2} - 7C - 19\right)\epsilon_{1\circ}\epsilon_{2\circ}\epsilon_{3\circ}\delta_{1\circ} \\
& - \left(\frac{\pi^2}{4} - 3C - 3\right)\epsilon_{2\circ}\epsilon_{3\circ}\epsilon_{4\circ}\delta_{1\circ} + 6\delta_{1\circ}^3\delta_{2\circ} + \frac{1}{2}(\pi^2 - 14C - 36)\delta_{1\circ}^2\delta_{2\circ}^2 + \left(\frac{\pi^2}{4} - 4C - 10\right)\delta_{1\circ}^2\delta_{2\circ}\delta_{3\circ} \\
& - \left(\frac{\pi^2}{8} - \frac{3}{2}C^2 - 6C - 6\right)\delta_{1\circ}\delta_{2\circ}^2\delta_{3\circ} - \frac{1}{24}(\pi^2 - 12C^2 - 48C - 48)(\delta_{1\circ}\delta_{2\circ}^3 + \delta_{1\circ}\delta_{2\circ}\delta_{3\circ}^2 + \delta_{1\circ}\delta_{2\circ}\delta_{3\circ}\delta_{4\circ}).
\end{aligned} \tag{C14}$$

Next is the running of the running of the scalar spectral index

$$\beta_s \equiv \frac{d^3 \ln [\mathcal{P}_\zeta(k)]}{d(\ln k)^3} \Big|_{k=k_\circ} = \beta_s^{(3)} + \beta_s^{(4)} + \mathcal{O}(\epsilon^5), \tag{C15}$$

with

$$\beta_s^{(3)} = -2\epsilon_{1\circ}\epsilon_{2\circ}^2 - 2\epsilon_{1\circ}\epsilon_{2\circ}\epsilon_{3\circ} - \epsilon_{2\circ}\epsilon_{3\circ}^2 - \epsilon_{2\circ}\epsilon_{3\circ}\epsilon_{4\circ} + \delta_{1\circ}\delta_{2\circ}^2 + \delta_{1\circ}\delta_{2\circ}\delta_{3\circ}, \tag{C16}$$

and

$$\begin{aligned}
\beta_s^{(4)} = & -14\epsilon_{1\circ}^2\epsilon_{2\circ}^2 - 8\epsilon_{1\circ}^2\epsilon_{2\circ}\epsilon_{3\circ} - (2C + 3)\epsilon_{1\circ}\epsilon_{2\circ}^3 - C(\epsilon_{2\circ}\epsilon_{3\circ}^3 + 3\epsilon_{2\circ}\epsilon_{3\circ}^2\epsilon_{4\circ} + \epsilon_{2\circ}\epsilon_{3\circ}\epsilon_{4\circ}^2 + \epsilon_{2\circ}\epsilon_{3\circ}\epsilon_{4\circ}\epsilon_{5\circ}) \\
& - (2C + 5)(\epsilon_{1\circ}\epsilon_{2\circ}\epsilon_{3\circ}^2 + \epsilon_{1\circ}\epsilon_{2\circ}\epsilon_{3\circ}\epsilon_{4\circ}) - (6C + 10)\epsilon_{1\circ}\epsilon_{2\circ}^2\epsilon_{3\circ} + 5\epsilon_{1\circ}\delta_{1\circ}\delta_{2\circ}^2 + 5\epsilon_{1\circ}\delta_{1\circ}\delta_{2\circ}\delta_{3\circ} + \epsilon_{2\circ}\delta_{1\circ}\delta_{2\circ}^2 + \epsilon_{2\circ}\delta_{1\circ}\delta_{2\circ}\delta_{3\circ} \\
& + 9\epsilon_{1\circ}\epsilon_{2\circ}\delta_{1\circ}\delta_{2\circ} + 3\epsilon_{2\circ}\epsilon_{3\circ}\delta_{1\circ}\delta_{2\circ} + 3\epsilon_{2\circ}\epsilon_{3\circ}^2\delta_{1\circ} + 7\epsilon_{1\circ}\epsilon_{2\circ}^2\delta_{1\circ} + 7\epsilon_{1\circ}\epsilon_{2\circ}\epsilon_{3\circ}\delta_{1\circ} + 3\epsilon_{2\circ}\epsilon_{3\circ}\epsilon_{4\circ}\delta_{1\circ} - 7\delta_{1\circ}^2\delta_{2\circ}^2 - 4\delta_{1\circ}^2\delta_{2\circ}\delta_{3\circ} \\
& + (C + 2)(\delta_{1\circ}\delta_{2\circ}^3 + \delta_{1\circ}\delta_{2\circ}\delta_{3\circ}^2\delta_{1\circ}\delta_{2\circ}\delta_{3\circ}\delta_{4\circ} + 3\delta_{1\circ}\delta_{2\circ}^2\delta_{3\circ}).
\end{aligned} \tag{C17}$$

Finally, we also get the leading-order term of the running of the running of the running of the scalar spectral index

$$\gamma_s \equiv \frac{d^4 \ln [\mathcal{P}_\zeta(k)]}{d(\ln k)^4} \Big|_{k=k_\circ} = \gamma_s^{(4)} + \mathcal{O}(\epsilon^5), \tag{C18}$$

where

$$\begin{aligned}
\gamma_s^{(4)} = & -2\epsilon_{1\circ}\epsilon_{2\circ}^3 - 6\epsilon_{1\circ}\epsilon_{2\circ}^2\epsilon_{3\circ} - 2\epsilon_{1\circ}\epsilon_{2\circ}\epsilon_{3\circ}^2 - \epsilon_{2\circ}\epsilon_{3\circ}^3 - \epsilon_{2\circ}\epsilon_{3\circ}\epsilon_{4\circ}^2 - \epsilon_{2\circ}\epsilon_{3\circ}\epsilon_{4\circ}\epsilon_{5\circ} - 2\epsilon_{1\circ}\epsilon_{2\circ}\epsilon_{3\circ}\epsilon_{4\circ} - 3\epsilon_{2\circ}\epsilon_{3\circ}^2\epsilon_{4\circ} \\
& + \delta_{1\circ}\delta_{2\circ}^3 + 3\delta_{1\circ}\delta_{2\circ}^2\delta_{3\circ} + \delta_{1\circ}\delta_{2\circ}\delta_{3\circ}^2 + \delta_{1\circ}\delta_{2\circ}\delta_{3\circ}\delta_{4\circ}.
\end{aligned} \tag{C19}$$

## 2. Tensor power-law parameters

The equivalent power-law quantities can be derived in the exact same way for the tensor modes. From Eq. (44), implementing the change of pivot of Eq. (79), the tensor spectral index is defined by

$$n_T \equiv \frac{d \ln [\mathcal{P}_h(k)]}{d \ln k} \Big|_{k=k_\circ} = n_T^{(1)} + n_T^{(2)} + n_T^{(3)} + n_T^{(4)} + \mathcal{O}(\epsilon^5). \tag{C20}$$

We have

$$\begin{aligned}
n_{\text{T}}^{(1)} &= -2\epsilon_{1\circ}, \\
n_{\text{T}}^{(2)} &= -2\epsilon_{1\circ}^2 - 2(C + 1 - \ln c_{\text{s}\circ}) \epsilon_{1\circ} \epsilon_{2\circ}, \\
n_{\text{T}}^{(3)} &= -2\epsilon_{1\circ}^3 + (\pi^2 - 6C - 14 + 6\ln^2 c_{\text{s}\circ}) \epsilon_{1\circ}^2 \epsilon_{2\circ} + \left[ \frac{\pi^2}{12} - C^2 + 2(C + 1) \ln c_{\text{s}\circ} - \ln^2 c_{\text{s}\circ} - 2C - 2 \right] (\epsilon_{1\circ} \epsilon_{2\circ}^2 + \epsilon_{1\circ} \epsilon_{2\circ} \epsilon_{3\circ}),
\end{aligned} \tag{C21}$$

and the N4LO correction

$$\begin{aligned}
n_{\text{T}}^{(4)} &= -2\epsilon_{1\circ}^4 + [4\pi^2 - 12C + 12\ln c_{\text{s}\circ} - 14\zeta(3) - 28] \epsilon_{1\circ}^3 \epsilon_{2\circ} \\
&+ \left[ \frac{31}{12} \pi^2 + 2(\pi^2 - 16)C - 7C^2 - 2(\pi^2 - 7C - 16) \ln c_{\text{s}\circ} - 7\ln^2 c_{\text{s}\circ} - 30 \right] \epsilon_{1\circ}^2 \epsilon_{2\circ}^2 \\
&+ \left[ -\frac{C^3}{3} - (C + 1) \ln^2 c_{\text{s}\circ} + \frac{1}{3} \ln^3 c_{\text{s}\circ} + \frac{\pi^2}{12} + \frac{1}{12} (\pi^2 - 24)C - C^2 - \frac{1}{12} (\pi^2 - 12C^2 - 24C - 24) \ln c_{\text{s}\circ} \right. \\
&\quad \left. - \frac{2}{3} \zeta(3) - \frac{2}{3} \right] (\epsilon_{1\circ} \epsilon_{2\circ}^3 + 3\epsilon_{1\circ} \epsilon_{2\circ}^2 \epsilon_{3\circ} + \epsilon_{1\circ} \epsilon_{2\circ} \epsilon_{3\circ}^2 + \epsilon_{1\circ} \epsilon_{2\circ} \epsilon_{3\circ} \epsilon_{4\circ}) \\
&+ \left[ \frac{4}{3} \pi^2 + (\pi^2 - 16)C - 4C^2 - (\pi^2 - 8C - 16) \ln c_{\text{s}\circ} - 4\ln^2 c_{\text{s}\circ} - 16 \right] \epsilon_{1\circ}^2 \epsilon_{2\circ} \epsilon_{3\circ}.
\end{aligned} \tag{C22}$$

The running of the tensor spectral index is defined by

$$\alpha_{\text{T}} \equiv \left. \frac{d^2 \ln [\mathcal{P}_h(k)]}{d(\ln k)^2} \right|_{k=k_\circ} = \alpha_{\text{T}}^{(2)} + \alpha_{\text{T}}^{(3)} + \alpha_{\text{T}}^{(4)} + \mathcal{O}(\epsilon^5), \tag{C23}$$

and we find

$$\begin{aligned}
\alpha_{\text{T}}^{(2)} &= -2\epsilon_{1\circ} \epsilon_{2\circ}, \\
\alpha_{\text{T}}^{(3)} &= -6\epsilon_{1\circ}^2 \epsilon_{2\circ} - 2(C + 1 - \ln c_{\text{s}\circ}) (\epsilon_{1\circ} \epsilon_{2\circ}^2 + \epsilon_{1\circ} \epsilon_{2\circ} \epsilon_{3\circ}),
\end{aligned} \tag{C24}$$

and

$$\begin{aligned}
\alpha_{\text{T}}^{(4)} &= -12\epsilon_{1\circ}^3 \epsilon_{2\circ} + (2\pi^2 - 14C + 14\ln c_{\text{s}\circ} - 32) \epsilon_{1\circ}^2 \epsilon_{2\circ}^2 + (\pi^2 - 8C + 8\ln c_{\text{s}\circ} - 16) \epsilon_{1\circ}^2 \epsilon_{2\circ} \epsilon_{3\circ} \\
&+ \left[ \frac{\pi^2}{12} - C^2 + 2(C + 1) \ln c_{\text{s}\circ} - \ln^2 c_{\text{s}\circ} - 2C - 2 \right] (\epsilon_{1\circ} \epsilon_{2\circ}^3 + 3\epsilon_{1\circ} \epsilon_{2\circ}^2 \epsilon_{3\circ} + \epsilon_{1\circ} \epsilon_{2\circ} \epsilon_{3\circ}^2 + \epsilon_{1\circ} \epsilon_{2\circ} \epsilon_{3\circ} \epsilon_{4\circ}).
\end{aligned} \tag{C25}$$

The running of the running of the tensor spectral index is

$$\beta_{\text{T}} \equiv \left. \frac{d^3 \ln [\mathcal{P}_h(k)]}{d(\ln k)^3} \right|_{k=k_\circ} = \beta_{\text{T}}^{(3)} + \beta_{\text{T}}^{(4)} + \mathcal{O}(\epsilon^5), \tag{C26}$$

with

$$\begin{aligned}
\beta_{\text{T}}^{(3)} &= -2\epsilon_{1\circ} \epsilon_{2\circ}^2 - 2\epsilon_{1\circ} \epsilon_{2\circ} \epsilon_{3\circ}, \\
\beta_{\text{T}}^{(4)} &= -14\epsilon_{1\circ}^2 \epsilon_{2\circ}^2 - 8\epsilon_{1\circ}^2 \epsilon_{2\circ} \epsilon_{3\circ} - 2(C + 1 - \ln c_{\text{s}\circ}) (\epsilon_{1\circ} \epsilon_{2\circ}^3 + 3\epsilon_{1\circ} \epsilon_{2\circ}^2 \epsilon_{3\circ} + \epsilon_{1\circ} \epsilon_{2\circ} \epsilon_{3\circ}^2 + \epsilon_{1\circ} \epsilon_{2\circ} \epsilon_{3\circ} \epsilon_{4\circ}).
\end{aligned} \tag{C27}$$

Finally, the running of the running of the running of the tensor spectral index is defined by

$$\gamma_{\text{T}} \equiv \left. \frac{d^4 \ln [\mathcal{P}_h(k)]}{d(\ln k)^4} \right|_{k=k_\circ} = \gamma_{\text{T}}^{(4)} + \mathcal{O}(\epsilon^5), \tag{C28}$$

and the N4LO correction reads

$$\gamma_{\text{T}}^{(4)} = -2 (\epsilon_{1\circ} \epsilon_{2\circ}^3 + 3\epsilon_{1\circ} \epsilon_{2\circ}^2 \epsilon_{3\circ} + \epsilon_{1\circ} \epsilon_{2\circ} \epsilon_{3\circ}^2 + \epsilon_{1\circ} \epsilon_{2\circ} \epsilon_{3\circ} \epsilon_{4\circ}). \tag{C29}$$



### 3. Tensor-to-scalar ratio

Our calculation allows up to derive an expression for the tensor-to-scalar ratio  $r$  at fourth order in the slow-roll parameters. From the definition

$$r \equiv \frac{\mathcal{P}_h(k_\circ)}{\mathcal{P}_\zeta(k_\circ)}, \quad (\text{C30})$$

using Eqs. (44), (73) and (79), we obtain

$$r = 16\epsilon_{1\circ}c_{\text{s}\circ} [1 + r^{(1)} + r^{(2)} + r^{(3)}] + \mathcal{O}(\epsilon^5), \quad (\text{C31})$$

and we recover the already known terms, namely

$$r^{(1)} = C\epsilon_{2\circ} - (C+2)\delta_{1\circ} + 2\epsilon_{1\circ}\ln c_{\text{s}\circ}, \quad (\text{C32})$$

and

$$\begin{aligned} r^{(2)} = & -\frac{1}{8}(\pi^2 - 4C^2 - 8)\epsilon_{2\circ}^2 - \left[\frac{\pi^2}{2} - 2(2C+1)\ln c_{\text{s}\circ} + \ln^2 c_{\text{s}\circ} - C - 4\right]\epsilon_{1\circ}\epsilon_{2\circ} - \frac{1}{24}(\pi^2 - 12C^2)\epsilon_{2\circ}\epsilon_{3\circ} \\ & + 2\ln c_{\text{s}\circ}(\ln c_{\text{s}\circ} + 1)\epsilon_{1\circ}^2 + \left[\frac{\pi^2}{2} - 2(C+2)\ln c_{\text{s}\circ} - 3C - 8\right]\epsilon_{1\circ}\delta_{1\circ} + \left(\frac{\pi^2}{4} - C^2 - 3C - 3\right)\epsilon_{2\circ}\delta_{1\circ} \\ & - \frac{1}{8}(\pi^2 - 4C^2 - 24C - 40)\delta_{1\circ}^2 + \frac{1}{24}(\pi^2 - 12C^2 - 48C - 48)\delta_{1\circ}\delta_{2\circ}. \end{aligned} \quad (\text{C33})$$

The new correction reads

$$\begin{aligned} r^{(3)} = & \frac{1}{24}[4C^3 - 3(\pi^2 - 8)C + 14\zeta(3) - 16]\epsilon_{2\circ}^3 - \frac{1}{24}[\pi^2 C - 4C^3 - 8\zeta(3) + 16](\epsilon_{2\circ}\epsilon_{3\circ}^2 + \epsilon_{2\circ}\epsilon_{3\circ}\epsilon_{4\circ}) \\ & + [(6C+1)\ln^2 c_{\text{s}\circ} - 2\ln^3 c_{\text{s}\circ} - \pi^2 - 2(\pi^2 - 5C - 11)\ln c_{\text{s}\circ} + C + 7\zeta(3)]\epsilon_{1\circ}^2\epsilon_{2\circ} \\ & + \left[\frac{1}{2}C^3 - \frac{1}{24}(7\pi^2 - 48)C\right]\epsilon_{2\circ}^2\epsilon_{3\circ} + \left(\frac{4}{3}\ln^3 c_{\text{s}\circ} + 4\ln^2 c_{\text{s}\circ} + 2\ln c_{\text{s}\circ}\right)\epsilon_{1\circ}^3 \\ & - \left[(C+1)\ln^2 c_{\text{s}\circ} - \frac{1}{3}\ln^3 c_{\text{s}\circ} + \frac{\pi^2}{12} + \frac{C}{2}(\pi^2 - 8) - C^2 + \frac{1}{6}(\pi^2 - 12C^2 - 12C - 12)\ln c_{\text{s}\circ}\right]\epsilon_{1\circ}\epsilon_{2\circ}\epsilon_{3\circ} \\ & + \left[\frac{\ln^3 c_{\text{s}\circ}}{3} - (2C+1)\ln^2 c_{\text{s}\circ} - \frac{19}{24}\pi^2 - (\pi^2 - 9)C + \frac{3}{2}C^2 - \frac{1}{3}(\pi^2 - 12C^2 - 12C - 12)\ln c_{\text{s}\circ} + \frac{7}{2}\zeta(3) + 2\right]\epsilon_{1\circ}\epsilon_{2\circ}^2 \\ & + \left[-\frac{9}{4}\pi^2 - \frac{1}{2}(\pi^2 - 36)C + 3C^2 - \frac{1}{4}(\pi^2 - 4C^2 - 24C - 40)\ln c_{\text{s}\circ} + \frac{7}{2}\zeta(3) + 29\right]\epsilon_{1\circ}\delta_{1\circ}^2 \\ & + \left[\frac{1}{2}C^3 - \pi^2 - \frac{1}{8}(3\pi^2 - 88)C + 4C^2 + \frac{7}{4}\zeta(3) + \frac{19}{2}\right]\epsilon_{2\circ}\delta_{1\circ}^2 \\ & + \left[\frac{7}{6}\pi^2 + \frac{1}{2}(\pi^2 - 20)C - 2C^2 + \frac{1}{12}(\pi^2 - 12C^2 - 48C - 48)\ln c_{\text{s}\circ} - 14\right]\epsilon_{1\circ}\delta_{1\circ}\delta_{2\circ} \\ & + \left[-\frac{1}{2}C^3 + \frac{13}{24}\pi^2 + \frac{1}{24}(7\pi^2 - 120)C - \frac{5}{2}C^2 - 5\right]\epsilon_{2\circ}\delta_{1\circ}\delta_{2\circ} \\ & + [2\pi^2 - 2(C+2)\ln^2 c_{\text{s}\circ} + (\pi^2 - 8C - 20)\ln c_{\text{s}\circ} - 5C - 7\zeta(3) - 14]\epsilon_{1\circ}^2\delta_{1\circ} \\ & + \left[\frac{\pi^2}{2} - \frac{1}{2}C^3 + \frac{1}{8}(3\pi^2 - 32)C - 2C^2 - \frac{7}{4}\zeta(3) - 2\right]\epsilon_{2\circ}^2\delta_{1\circ} \\ & + \left[(C+2)\ln^2 c_{\text{s}\circ} + \frac{77}{24}\pi^2 + \frac{3}{2}(\pi^2 - 18)C - \frac{13}{2}C^2 + \frac{1}{2}(\pi^2 - 8C^2 - 24C - 20)\ln c_{\text{s}\circ} - 7\zeta(3) - 24\right]\epsilon_{1\circ}\epsilon_{2\circ}\delta_{1\circ} \\ & + \left[\frac{\pi^2}{6} - \frac{1}{2}C^3 + \frac{1}{24}(7\pi^2 - 72)C - 2C^2\right]\epsilon_{2\circ}\epsilon_{3\circ}\delta_{1\circ} - \frac{1}{24}[4C^3 - 12\pi^2 - 3(\pi^2 - 64)C + 48C^2 + 14\zeta(3) + 260]\delta_{1\circ}^3 \\ & + \frac{1}{24}[12C^3 - 17\pi^2 - 7(\pi^2 - 48)C + 108C^2 + 360]\delta_{1\circ}^2\delta_{2\circ} \\ & - \frac{1}{24}[4C^3 - 2\pi^2 - (\pi^2 - 48)C + 24C^2 + 8\zeta(3) + 32](\delta_{1\circ}\delta_{2\circ}^2 + \delta_{1\circ}\delta_{2\circ}\delta_{3\circ}). \end{aligned} \quad (\text{C34})$$

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