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Toeplitz and Hankel determinants in random matrix theory

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Chapter 1

Introduction

In the present thesis, we provide new results for certain Toeplitz and Hankel determinants related to determinantal point processes which arise in random matrix theory. These determinants can be expressed in terms of orthogonal polynomials on the unit circle and on the real line. We use a Riemann-Hilbert characterisation of these orthogonal polynomials and apply the Deift/Zhou steepest descent method on it.

In sections 1.1-1.4, we first recall some definitions and known results concerning Toeplitz and Hankel determinants, random matrices, determinantal point processes and orthogonal polynomials. We also show how they are related to each other. In Section 1.5, we present in detail the results of the thesis. Section 1.6 is divided into two parts. First we define what a Riemann-Hilbert problem is and how the steepest descent method of Deift and Zhou works. The second part is devoted to Riemann surfaces.

1.1 Toeplitz and Hankel determinants

A Toeplitz matrix is a square matrix which has constant entries along each diagonal. More precisely, if n is the size of the Toeplitz matrix T_n, one has $T_n = (t_{j-k})_{j,k=1,\dots,n}$, where $t_{-n+1}, ..., t_{n-1}$ are some complex numbers. If f is a function defined on the unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ which possesses Fourier coefficients

$$f_k = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-ik\theta}d\theta, \qquad k \in \mathbb{Z}, \tag{1.1.1}$$

then f is called a symbol. The Toeplitz matrix $T_n(f)$ and its determinant $D_n(f)$ associated to the symbol f are given by

$$T_n(f) = (f_{j-k})_{j,k=1,\dots,n} = \begin{pmatrix} f_0 & f_{-1} & \cdots & f_{-n+1} \\ f_1 & f_0 & \cdots & f_{-n+2} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n-1} & f_{n-2} & \cdots & f_0 \end{pmatrix}, \tag{1.1.2}$$

and $D_n(f) = \det(T_n(f))$.

A Hankel matrix H_n of size n is also a square matrix, but which is constant along anti-diagonals, one has $H_n = (h_{j+k-2})_{j,k=1,\dots,n}$ with $h_0, ..., h_{2n-2}$ some complex numbers. If w is a function defined on $\mathbb{R}$ which possesses finite moments

$$w_k = \int_{\mathbb{R}} w(x) x^k dx, \qquad k \in \mathbb{N}, \tag{1.1.3}$$

then w is called a weight. The Hankel matrix $H_n(w)$ and its determinant $D_n(w)$ associated to the weight w are given by

$$H_n(w) = (w_{j+k-2})_{j,k=1,\dots,n} = \begin{pmatrix} w_0 & w_1 & \cdots & w_{n-1} \\ w_1 & w_2 & \cdots & w_n \\ \vdots & \vdots & \ddots & \vdots \\ w_{n-1} & w_n & \cdots & w_{2n-2} \end{pmatrix}, \tag{1.1.4}$$

and $D_n(w) = \det(H_n(w))$. In this thesis we will only use positive weights w.

Toeplitz matrices and determinants were introduced by Otto Toeplitz in 1907 [65, 66]. He was studying quadratic forms $\sum_{j,k=0}^{n-1} f_{j-k} x_j y_k$, associated to the matrix $T_n(f)$, in order to give concrete examples of Hilbert's theory of functional analysis. The first asymptotic results are due to Szegő with his weak limit theorem in 1915 [60], and 37 years later in 1952 with his strong limit theorem [61]. Since then, a lot of other asymptotic results have followed. See [25] for a historical review of Toeplitz matrices and determinants.

Toeplitz and Hankel determinants have applications in:

- statistical mechanics [33], in particular the Ising model [44].
- the emptiness formation probability for XY Spin Chain, see [39].
- the distribution of the length of the longest increasing subsequence of a random permutation [6].

- random matrix theory, which has in itself a lot of applications.

In this thesis we will focus on Toeplitz determinants $D_n(f)$ for which the symbol f has two jump discontinuities: more precisely, we take f of the form

$$f(e^{i\theta}) = e^{W(e^{i\theta})} \times \begin{cases} 1, & \text{for } -L \le \theta \le L, \\ s, & \text{for } L < \theta < 2\pi - L, \end{cases} \tag{1.1.5}$$

where $s \in [0,1]$ and $L \in (0,\pi)$ are two parameters and W is analytic in a neighbourhood of the unit circle. If $W = 0$, f is piecewise constant.

If $s = 0$ and L is fixed, i.e. independent of n, the symbol is supported on the single arc $\gamma = \{e^{i\theta} : -L \le \theta \le L\}$. If f is positive on γ and has the symmetry $f(e^{-i\theta}) = f(e^{i\theta})$, then large n asymptotics for $D_n(f)$ are well-known and are due to Widom [67],

$$\begin{aligned} \log D_n(f|_{s=0}) \quad = \quad & n^2 \log\sin\frac{L}{2} + n\widetilde{W}_0 - \frac{1}{4}\log n + \sum_{k=1}^{\infty} k\widetilde{W}_k\widetilde{W}_{-k} \\ & -\frac{1}{4}\log\cos\frac{L}{2} + \frac{1}{12}\log 2 + 3\zeta'(-1) + o(1), \qquad \text{as } n \to \infty, \end{aligned} \tag{1.1.6}$$

where $\widetilde{W}_k$ is the k-th Fourier coefficient of $\widetilde{W}$, defined by

$$\widetilde{W}(e^{i\theta}) = W(e^{2i\arcsin(\sin\frac{L}{2}\sin\frac{\theta}{2})}), \qquad \widetilde{W}_k = \frac{1}{2\pi}\int_0^{2\pi} \widetilde{W}(e^{i\theta})e^{-ik\theta}d\theta, \tag{1.1.7}$$

and where ζ is the Riemann ζ-function.

On the other hand, if $s > 0$ is fixed, the symbol f is supported on the whole unit circle and possesses two jump discontinuities at $e^{\pm iL}$. In this case f is a special case of a Fisher-Hartwig symbol whose asymptotics are also known [33, 68, 7, 8, 16, 32, 24, 26]

$$\begin{aligned} \log D_n(f|_{s>0}) \quad = \quad & n\left(1 - \frac{L}{\pi}\right)\log s + nW_0 + \frac{(\log s)^2}{2\pi^2}(\log n + \log(2\sin L)) \\ & -\frac{\log s}{\pi}\sum_{k=1}^{\infty}(W_k + W_{-k})\sin(kL) + \sum_{k=1}^{+\infty} kW_kW_{-k} \\ & +2\log\left(G\left(1 + \frac{\log s}{2\pi i}\right)G\left(1 - \frac{\log s}{2\pi i}\right)\right) + o(1), \end{aligned} \tag{1.1.8}$$

as $n \to \infty$, where G is Barnes' G-function, defined by

$$G(1+z) = (2\pi)^{z/2} \exp\left(-\frac{z + z^2(1+\epsilon)}{2}\right) \prod_{k=1}^{\infty} \left[\left(1+\frac{z}{k}\right)^k \exp\left(\frac{z^2}{2k} - z\right)\right],$$

and ϵ is the Euler-Mascheroni constant [53]. If we put $s = 1$ in (1.1.8), the asymptotics coincide with the result of Szegő.

If we let s tend to 0 in (1.1.5), the jump discontinuities of f turn into endpoints of the support of f, and f transforms from a Fisher-Hartwig symbol to an arc-supported symbol on $[-L, L]$. Nevertheless, letting $s \to 0$ in (1.1.8), we do not recover the Widom asymptotics (1.1.6). This means that a non-trivial critical transition in the asymptotic behavior for $D_n(f)$ takes place as $s \to 0$. We will analyse this transition in detail in Chapter 2 and Chapter 3.

1.2 Random matrices

Random matrices were introduced in 1928 by Wishart [70] in the study of normal multivariate populations. In the 1950s, this area received growing attention with the work of Wigner [69] in nuclear physics. Since then, random matrices found applications and connections in many areas, such as finance (see e.g. [5], Section 12.2). There are also some links with the Riemann ζ-function [45]. Some classical references in the field are Mehta's book [52], where a lot of applications and results are presented, and [2, 1, 5, 35, 62].

A random matrix ensemble is a set of matrices with a probability measure defined on it. Elements of random matrices are therefore random variables, possibly dependent. We will present here three of the most studied matrix ensembles:

- The Gaussian Unitary Ensemble (GUE) consists of the space of $n \times n$ complex Hermitian matrices with the distribution

$$P(M)dM = \frac{1}{\widetilde{Z}_n} e^{-n\mathrm{Tr}(M^2)} dM, \tag{1.2.1}$$

  where $\widetilde{Z}_n$ is the normalization constant and

$$dM = \prod_{i=1}^{n} dM_{ii} \prod_{1 \le i < j \le n} d\mathrm{Re}\, M_{ij} d\mathrm{Im}\, M_{ij} \tag{1.2.2}$$

is the Lebesgue measure. The distribution (1.2.1) implies that the entries of the upper triangular part of M are independent Gaussian variables, one has:

$$\begin{aligned} & M_{ii} \sim \mathcal{N}\left(0, \sigma^2 = \frac{1}{2n}\right) && 1 \leq i \leq n, \\ & \operatorname{Re} M_{ij},\ \operatorname{Im} M_{ij} \sim \mathcal{N}\left(0, \sigma^2 = \frac{1}{4n}\right) && 1 \leq i < j \leq n, \\ & M_{ji} = \overline{M_{ij}} && 1 \leq i \leq j \leq n, \end{aligned} \tag{1.2.3}$$

where σ^2 denotes the variance. The name GUE might appear misleading because this is a space of Hermitian matrices but is called a unitary ensemble. This is due to the fact that the distribution (1.2.1) is invariant under unitary conjugations, i.e. if U is an $n \times n$ unitary matrix, one has

$$P(UMU^*) = P(M) \quad \text{and} \quad d(UMU^*) = dM, \tag{1.2.4}$$

where U^* denotes the conjugate transpose of U. This implies the remarkable fact that the eigenvectors are independent of the eigenvalues. Integrating out the eigenvectors, and computing the Jacobian of this transformation, we are left with the joint probability distribution on $\mathbb{R}^n$ of the eigenvalues $x_1, ..., x_n$. One has

$$\frac{1}{n! Z_{n,\mathrm{GUE}}} \prod_{1 \leq i < j \leq n} (x_j - x_i)^2 \prod_{i=1}^{n} e^{-n x_i^2} dx_i, \tag{1.2.5}$$

where $Z_{n,\mathrm{GUE}}$ is the normalisation constant. From (1.2.5), we observe two different phenomena: it is unlikely to have (1) two eigenvalues equal or very close to each other and (2) an eigenvalue which is "large" in absolute value. These two opposite forces, i.e., the fact that the eigenvalues repel each other and that they are repulsed from $\pm\infty$, are of the same strength. More precisely, the limiting average counting measure of the eigenvalues has a continuous density which is given by the Wigner semi-circle law (see [23]):

$$\mathbb{E}\left[\frac{1}{n} \sum_{i=1}^{n} \delta_{x_i}\right] \to \frac{1}{\pi} \sqrt{2 - x^2} dx, \qquad \text{as } n \to \infty, \tag{1.2.6}$$

where $\to$ denotes the weak convergence. This limiting measure is supported on $[-\sqrt{2}, \sqrt{2}]$ and vanishes like a square root near the endpoints $\pm\sqrt{2}$. So the largest (resp. the smallest) eigenvalue is located around $\sqrt{2}$ (resp. $-\sqrt{2}$). These endpoints of the support are called soft edges because there can be eigenvalues outside of $[-\sqrt{2}, \sqrt{2}]$. The fluctuations of the rescaled extreme eigenvalues follow the Tracy-Widom distribution [63], one has

$$\begin{aligned} \lim_{n\to\infty} \mathbb{P}\left(\left(x_{\max} - \sqrt{2}\right)\sqrt{2}n^{2/3} \le x\right) &= \widetilde{F}_2(x), \\ \lim_{n\to\infty} \mathbb{P}\left(-\left(x_{\min} + \sqrt{2}\right)\sqrt{2}n^{2/3} \le x\right) &= \widetilde{F}_2(x), \end{aligned} \tag{1.2.7}$$

with

$$\widetilde{F}_2(x) = \exp\left(-\int_x^\infty (s-x)q^2(s)ds\right), \tag{1.2.8}$$

and q is the Hastings-McLeod [40] solution of the Painlevé II equation

$$q''(s) = sq(s) + 2q(s)^3 \tag{1.2.9}$$

satisfying the boundary condition $q(s) \sim \mathrm{Ai}(s)$ as $s \to \infty$, where Ai is the Airy function (see [53, Chapter 9] for the definition and properties of the Airy function).

- The Laguerre Unitary Ensemble (LUE) consists of the space of $n \times n$ complex positive definite Hermitian matrices endowed with the distribution

$$P(M)dM = \frac{1}{\widetilde{Z}_{n,\alpha}}(\det M)^\alpha e^{-n\mathrm{Tr}M}dM, \qquad \alpha > -1. \tag{1.2.10}$$

The eigenvalues $x_1, ..., x_n$ are thus positive and follow the joint probability distribution on $(\mathbb{R}^+)^n$ given by

$$\frac{1}{n!\widehat{Z}_{n,\alpha}} \prod_{1\le i<j\le n} (x_j - x_i)^2 \prod_{i=1}^n x_i^\alpha e^{-nx_i} dx_i. \tag{1.2.11}$$

If α is an integer, one can show that (1.2.11) is also the distribution of the eigenvalues of a matrix of the form X^*X, where X is an $(n+\alpha)\times n$ matrix whose entries X_{ij} are independent and identically distributed complex Gaussian variables

$$\mathrm{Re}\, X_{ij},\ \mathrm{Im}\, X_{ij} \sim \mathcal{N}\left(0, \sigma^2 = \frac{1}{2n}\right). \tag{1.2.12}$$

The limiting mean eigenvalue density function is given by the Marchenko-Pastur [51] law

$$\frac{1}{2\pi}\sqrt{\frac{4-x}{x}}, \tag{1.2.13}$$

supported on the interval $[0,4]$. This density vanishes like a square root near the soft edge 4. The largest eigenvalues are expected to be around 4, which is a soft edge. The rescaled fluctuations of the largest eigenvalue around 4 follow again the Tracy-Widom distribution $\widetilde{F}_2$ given by (1.2.8). The eigenvalues near 0 behave differently, since they are positive. On the level of the density, we observe that the Marchenko-Pastur law blows up like an inverse square root near 0. Zero is called a hard edge, and the fluctuations of the properly rescaled smallest eigenvalue follow another distribution $F_\alpha(x)$, also proved by Tracy and Widom [64]. We have

$$F_\alpha(x) = \lim_{n\to\infty} \mathbb{P}_n(4n^2 x_{\min} > x) = \exp\left(-\frac{1}{4}\int_0^x \log\left(\frac{x}{\xi}\right) q^2(\xi)d\xi\right) \tag{1.2.14}$$

where $q(\xi)$ is the solution of the Painlevé V equation

$$\xi(q^2-1)(\xi q')' = q(\xi q')^2 + \frac{1}{4}(\xi-\alpha^2)q + \frac{1}{4}\xi q^3(q^2-2), \tag{1.2.15}$$

with boundary condition $q(\xi) \sim J_\alpha(\sqrt{\xi})$ as $\xi \to 0_+$, and J_α is the Bessel function of the first kind of order α (see [53, Chapter 10] for definition and properties of the Bessel functions).

- The Circular Unitary Ensemble (CUE) consists of the space of $n\times n$ unitary matrices with the Haar measure on it. Eigenvalues of such matrices are of the form $e^{i\theta_1}, ..., e^{i\theta_n}$, with the eigenangles $\{\theta_i\}$ distributed on $[0,2\pi)^n$ as follows:

$$\frac{1}{(2\pi)^n n!}\prod_{1\le j<k\le n} |e^{i\theta_j} - e^{i\theta_k}|^2 \prod_{j=1}^n d\theta_j. \tag{1.2.16}$$

  The limiting mean eigenvalue density function is given by the uniform density $\frac{1}{2\pi}$. The support is the whole unit circle and has no edges.

An interesting quantity, often studied in random matrix theory, is the gap probability, i.e., the probability that all eigenvalues (or eigenangles in the case of the CUE) lie in a certain ensemble B ($B\subset\mathbb{R}$ for the GUE, $B\subset\mathbb{R}^+$ for the

LUE and $B \subset [0, 2\pi)$ for the CUE). Such probabilities in the case of the CUE can be expressed as the Toeplitz determinant associated to the symbol χ_B, the characteristic function of B. Indeed, starting from the definition of Toeplitz determinants, we have

$$\begin{aligned} D_n(\chi_B) &= \det\left(\frac{1}{2\pi}\int_{[0,2\pi)} \chi_B(\theta_j) e^{i(k-j)\theta_j} d\theta_j\right)_{1\le j,k\le n} \\ &= \frac{1}{(2\pi)^n}\int_{[0,2\pi)^n} \det\left(e^{i(k-1)\theta_j}\right)_{1\le j,k\le n} \prod_{j=1}^n e^{-i(j-1)\theta_j}\chi_B(\theta_j) d\theta_j. \end{aligned} \tag{1.2.17}$$

By Fubini's theorem, we can interchange order of integration and we obtain

$$D_n(\chi_B) = \frac{1}{(2\pi)^n}\int_{[0,2\pi)^n} \det\left(e^{i(k-1)\theta_{\sigma(j)}}\right)_{1\le j,k\le n} \prod_{j=1}^n e^{-i(j-1)\theta_{\sigma(j)}}\chi_B(\theta_{\sigma(j)}) d\theta_j, \tag{1.2.18}$$

where $\sigma \in S_n$, and S_n is the set of permutations of $\{1, ..., n\}$. Note that

$$\begin{aligned} \prod_{j=1}^n \chi_B(\theta_{\sigma(j)}) &= \prod_{j=1}^n \chi_B(\theta_j), \\ \det\left(e^{i(k-1)\theta_{\sigma(j)}}\right)_{1\le j,k\le n} &= \operatorname{sgn}(\sigma)\det\left(e^{i(k-1)\theta_j}\right)_{1\le j,k\le n}, \end{aligned} \tag{1.2.19}$$

where $\operatorname{sgn}(\sigma) \in \{-1, 1\}$ is the sign of σ. Summing over all the permutations of S_n, we get

$$\begin{aligned} n! D_n(\chi_B) = \frac{1}{(2\pi)^n} \times \\ \int_{B^n} \det\left(e^{i(k-1)\theta_j}\right)_{1\le j,k\le n} \left(\sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\prod_{j=1}^n e^{-i(j-1)\theta_{\sigma(j)}}\right) \prod_{j=1}^n d\theta_j. \end{aligned} \tag{1.2.20}$$

Using Vandermonde's formula we finally obtain

$$D_n(\chi_B) = \frac{1}{(2\pi)^n n!}\int_{B^n} \prod_{1\le j<k\le n} |e^{i\theta_j} - e^{i\theta_k}|^2 \prod_{j=1}^n d\theta_j. \tag{1.2.21}$$

The right-hand side of (1.2.21) is by definition equal to the gap probability $\mathbb{P}(\{\theta_1, ..., \theta_n\} \subset B)$. Similarly, we can prove that the gap probabilities in the GUE (1.2.5) and the LUE (1.2.11) can be expressed in terms of Hankel determinants.

More generally, there is an integral representation of Toeplitz and Hankel determinants. If f is a symbol defined on S^1 and w is a weight on $\mathbb{R}$, one has Heine's formulas

$$\begin{aligned} D_n(f) &= \frac{1}{(2\pi)^n n!}\int_{[0,2\pi)^n} \prod_{1\le j<k\le n} |e^{i\theta_j}-e^{i\theta_k}|^2 \prod_{j=1}^n f(e^{i\theta_j})d\theta_j, \\ D_n(w) &= \frac{1}{n!}\int_{\mathbb{R}^n} \prod_{1\le j<k\le n}(x_k-x_j)^2 \prod_{j=1}^n w(x_j)dx_j. \end{aligned} \tag{1.2.22}$$

1.3 Determinantal point processes and orthogonal polynomials

Let Λ be a complete separable metric space. In the present thesis we will use $\Lambda = \mathbb{R}$, $\mathbb{R}^+$ or $[0, 2\pi)$. We consider $\mathcal{N}(\Lambda)$ the space of all counting measures μ which are locally bounded on Λ, i.e., for which $\mu(B) \in \mathbb{N}$ for every bounded $B \subset \Lambda$. A point process on Λ is a probability measure on $\mathcal{N}(\Lambda)$ and is called finite if there exists $n \in \mathbb{N}$ such that this probability measure is zero on $\{\mu \in \mathcal{N}(\Lambda) : \mu(\Lambda) \ge n\}$. The reader can find general definitions and properties of point processes in [42, 59, 14].

The eigenvalue measures on $\mathbb{R}^n$ for the GUE and on $(\mathbb{R}^+)^n$ for the LUE, given by (1.2.5) and (1.2.11) respectively, define finite point processes on $\Lambda = \mathbb{R}$ and $\Lambda = \mathbb{R}^+$ by identifying an eigenvalue configuration $(x_1, ..., x_n)$ with the counting measure $\sum_{i=1}^n \delta_{x_i}$:

$$\Lambda^n \ni (x_1, ..., x_n) \mapsto \sum_{i=1}^n \delta_{x_i}. \tag{1.3.1}$$

Similarly, the measure (1.2.16) for the CUE defines a point process on $[0, 2\pi)$ (or on S^1).

Let $u_n(x_1, ..., x_n)$ be a probability density on Λ^n which is symmetric under permutations of the $\{x_j\}$. One can keep in mind the three examples of the density of the GUE or LUE eigenvalues, or of the CUE eigenangles. One defines the k-point correlation function ρ_k of the point process induced by u_n by

$$\rho_k(x_1, ..., x_k) = \frac{n!}{(n-k)!}\int_{\mathbb{R}^{n-k}} u_n(x_1, ..., x_n)dx_{k+1}...dx_n. \tag{1.3.2}$$

If $B \subset \Lambda$, one can show that

$$\begin{aligned}\int_B \rho_1(x_1)dx_1 &= \mathbb{E}\left(\#\{x_i \in B\}\right), \\ \int_B\int_B \rho_2(x_1,x_2)dx_1dx_2 &= \mathbb{E}\left(\#\{(i,j) : i \neq j \text{ and } x_i, x_j \in B\}\right),\end{aligned}$$

where the pairs (i,j) and (j,i) are both counted if $x_i, x_j \in B$.

In this context, a point process is called determinantal if there exists a function $K : \Lambda \times \Lambda \mapsto \mathbb{C}$ such that

$$\rho_k(x_1, ..., x_k) = \det(K(x_i, x_j))_{1 \leq i,j \leq k}, \tag{1.3.3}$$

for all $x_1, ..., x_k \in \Lambda$ and $k = 1, 2, ..., n$. K is called the correlation kernel. Since all correlation functions can be expressed in terms of K, the correlation kernel contains all statistical information about the point process. The three examples considered above are determinantal, and the associated correlation kernels can each of them be expressed in terms of orthogonal polynomials (OP) [23, 58].

Given a weight w on the real line, the associated family of monic orthogonal polynomials p_k of degree k is characterised by the relations

$$\int_{\mathbb{R}} p_k(x)p_l(x)w(x)dx = h_k\delta_{kl}, \qquad k,l = 0,1,2,... \tag{1.3.4}$$

If $w(x) = e^{-nx^2}$, the $p_k(x)$'s are the rescaled Hermite polynomials and if $w(x) = x^\alpha e^{-nx}\chi_{\mathbb{R}^+}(x)$, the $p_k(x)$'s are the rescaled Laguerre polynomials.

On the unit circle, given a symbol f, one has to consider two families of monic orthogonal polynomials $\phi_k(z)$, $\hat{\phi}_k(z)$ characterised by

$$\frac{1}{2\pi}\int_0^{2\pi} \phi_k(e^{i\theta})\hat{\phi}_l(e^{-i\theta})f(e^{i\theta})d\theta = h_k\delta_{kl}, \qquad k,l = 0,1,2,... \tag{1.3.5}$$

If f is real on S^1, $\hat{\phi}_n = \overline{\phi_n}$ and if f is symmetric on S^1, i.e. $f(e^{i\theta}) = f(e^{-i\theta})$, then $\hat{\phi}_n = \phi_n$. Measures (1.2.5), (1.2.11) and (1.2.16) all define a specific determinantal point process. For the GUE (resp. the LUE), the correlation kernel is given by

$$K_n(x,y) = \sqrt{w(x)w(y)}\sum_{i=0}^{n-1}\frac{p_i(x)p_i(y)}{h_i}, \tag{1.3.6}$$

where $w(x) = e^{-nx^2}$ and the p_k's are the rescaled Hermite polynomials (resp. $w(x) = x^\alpha e^{-nx}\chi_{\mathbb{R}^+}(x)$ and the $p_k(x)$'s are the rescaled Laguerre polynomials). Using the Christoffel-Darboux formula, we can express K_n only in terms of p_n and p_{n-1}

$$K_n(x,y) = \sqrt{w(x)w(y)}\frac{p_n(x)p_{n-1}(y) - p_{n-1}(x)p_n(y)}{h_{n-1}(x-y)}. \tag{1.3.7}$$

On the unit circle, we obtain a similar formula for the correlation kernel

$$K_n(e^{i\theta}, e^{i\mu}) = \frac{1}{2\pi h_n}\sqrt{f(e^{i\theta})f(e^{i\mu})}\frac{e^{in(\theta-\mu)}\phi_n(e^{i\mu})\hat{\phi}_n(e^{-i\theta}) - \hat{\phi}_n(e^{-i\mu})\phi_n(e^{i\theta})}{1-e^{i(\theta-\mu)}}. \tag{1.3.8}$$

In the case of the CUE, the orthogonal polynomial of degree n with respect to the constant symbol $f(e^{i\theta}) = 1$ on the unit circle is given by $\phi_n(z) = \hat{\phi}_n(z) = z^n$ and (1.3.8) is reduced to

$$K_n(e^{i\theta}, e^{i\mu}) = \frac{1}{2\pi}\frac{1-e^{in(\theta-\mu)}}{1-e^{i(\theta-\mu)}}. \tag{1.3.9}$$

Therefore if we want to analyse the limiting kernel as $n \to \infty$, one is led to find asymptotics for orthogonal polynomials of large degree. An important observation is that orthogonal polynomials can be characterised in terms of a Riemann-Hilbert problem (RH problem). We refer to Section 1.6 for more information.

1.4 Thinned CUE and conditional CUE

Thinning is a classical operation in the theory of point processes [22, 43, 56]. Given a locally finite configuration of points, it consists of removing points following a deterministic or probabilistic rule. Starting with a given point process, one can in this way create other, thinned, point processes.

Consider the operation of thinning the eigenvalues of an $n \times n$ CUE matrix, which consists of removing each eigenvalue $e^{i\theta_1}, ..., e^{i\theta_n}$ independently with probability $s = 1 - p \in (0,1)$. We call s the *removal probability* and p the *retention probability*. We denote $e^{i\phi_1}, ..., e^{i\phi_m}$, $0 \le m \le n$ for the retained eigenvalues, where the number of particles m is now itself a random variable following the binomial distribution $B(n,p)$. We write

$$\Theta = \{e^{i\theta_1}, \dots, e^{i\theta_n}\}, \qquad \Phi = \{e^{i\phi_1}, \dots, e^{i\phi_m}\} \tag{1.4.1}$$

for the complete and thinned spectra. On the formal level, one can define the *thinned CUE* by associating to each eigenvalue $e^{i\theta_k}$ an independent Bernoulli random variable taking the value 0 if the eigenvalue is removed and taking the value 1 if it is kept. The point process on S^1 defined by the thinned CUE is

$$(S^1)^n \times \{0,1\}^n \ni ((e^{i\theta_1}, ..., e^{i\theta_n}), (p_1, ..., p_n)) \mapsto \sum_{p_i=1} \delta_{e^{i\theta_i}} = \sum_{i=1}^{m} \delta_{e^{i\phi_i}}. \tag{1.4.2}$$

On the level of correlation kernels, random independent thinning of a general determinantal point process boils down to multiplying the correlation kernel with the retention probability p, as was observed in [50, Appendix A]. The correlation kernel for the particles $e^{i\phi_1}, ..., e^{i\phi_m}$ is in other words equal to $pK_n(e^{i\theta}, e^{i\mu})$, with K_n as in (1.3.9).

Consider the situation where we observe no particles outside a measurable subset γ of the unit circle, equivalently $\Phi \subset \gamma$. We now define a *conditional CUE* as the joint probability distribution of all n eigenvalues in the complete spectrum Θ, given the fact that $\Phi \subset \gamma$. In this conditional CUE, any eigenvalue configuration Θ can occur, but configurations with many eigenvalues on γ^c are less likely because all of them have to be removed by the thinning procedure. We note that, for finite n, the event $\Phi \subset \gamma$ has non-zero probability, so that the conditional probability is well-defined in the classical sense.

From the definition of conditional probability $\mathbb{P}(A|B) = \mathbb{P}(A \cap B)/\mathbb{P}(B)$, it is straightforward to see that the conditional joint probability distribution for the eigenvalues is given by

$$\frac{1}{Z_n} \prod_{1 \le j < k \le n} |e^{i\theta_j} - e^{i\theta_k}|^2 \prod_{j=1}^{n} f(e^{i\theta_j}) d\theta_j, \qquad \theta_1, \dots, \theta_n \in [0, 2\pi), \tag{1.4.3}$$

where $Z_n = Z_n(s, \gamma)$ is a normalization constant, and f is given by

$$f(e^{i\theta}) = \begin{cases} 1, & \text{for } e^{i\theta} \in \gamma, \\ s, & \text{for } e^{i\theta} \in \gamma^c. \end{cases} \tag{1.4.4}$$

If γ is an arc on S^1 of arclength $2L$ (which we can assume to be centered at 1 without loss of generality), then (1.4.4) is equivalent to (1.1.5) with $W = 0$.

The joint probability distribution (1.4.3) of the conditional CUE defines again a determinantal point process; the conditional eigenvalue correlation kernel can be expressed as in (1.3.8), with $\phi_n = \widehat{\phi}_n$ the orthogonal polynomials with respect to the symbol f given by (1.4.4).

1.5 New results in the thesis

Results in Chapter 2

We study large n asymptotics for Toeplitz determinants $D_n(f)$ associated to the symbol f given by (1.1.5) in the case where the parameter s is not fixed, but depends on n. In the case of f positive and symmetric on the arc $\gamma = \{e^{i\theta} : -L \leq \theta \leq L\}$, we recall that if $s = 0$, asymptotics for $D_n(f)$ are given by (1.1.6). Our main result in Chapter 2 is that asymptotics for $s = 0$ still hold in the case where $s \neq 0$, but $s = s(n) \to 0$ sufficiently rapidly as $n \to \infty$. More precisely, we have the following.

Theorem 1.5.1 *Let $L \in (0, \pi)$ and define*

$$x_c = -2 \log \tan \frac{L}{4}. \tag{1.5.1}$$

Let f be of the form (1.1.5) *with W analytic in a neighbourhood of the unit circle. As $n \to \infty$ and simultaneously $s \to 0$ in such a way that $0 \leq s \leq e^{-x_c n}$, we have*

$$\log D_n(f|_{s \leq e^{-x_c n}}) = \log D_n(f|_{s=0}) + o(1). \tag{1.5.2}$$

The error term $o(1)$ is uniform for $0 \leq s \leq e^{-x_c n}$ and $\epsilon \leq L \leq \pi - \epsilon$, $\epsilon > 0$, and can be specified as

$$o(1) = \mathcal{O}(n^{-1/2} e^{x_c n} s). \tag{1.5.3}$$

In addition, the result extends to the case where L approaches π at a sufficiently slow rate: (1.5.2) *holds for $\epsilon < L < \pi - \frac{M}{n}$ with M sufficiently large and $0 \leq s \leq e^{-x_c n}$, with the error term given by*

$$o(1) = \mathcal{O}((\pi - L)^{1/2} n^{-1/2} e^{n x_c} s). \tag{1.5.4}$$

Theorem 1.5.1 is illustrated in Figure 1.1.

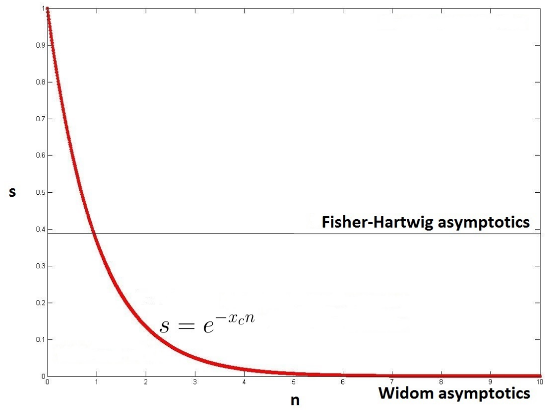

Figure 1.1: As $n \to \infty$, $\log D_n(f)$ follows Widom asymptotics for $s = 0$ and Fisher-Hartwig asymptotics for $s > 0$ fixed. Theorem 1.5.1 implies that the Widom asymptotics remain valid as $n \to \infty$ with $s = s(n)$ below the curve $s = e^{-x_c n}$. As $n \to \infty$ with $s = s(n)$ above the curve, a different type of asymptotic behaviour takes place, as shown in Chapter 3.

Results in Chapter 3

We show that if γ is a measurable subset of the unit circle, the Toeplitz determinant $D_n(f)$ with f given by (1.4.4) is equal to a gap probability in the thinned CUE, i.e. the probability that all the remaining eigenvalues belong to γ. With the notations introduced in Section 1.4, we have $D_n(f) = \mathbb{P}(\Phi \subset \gamma)$. Indeed, we can partition $[0, 2\pi)^n$ as

$$[0,2\pi)^n = \bigsqcup_{k=0}^{n} A_k, \qquad A_k = \{(\theta_1, ..., \theta_n) \in [0,2\pi)^n : \#(\Theta \cap \gamma^c) = k\}. \tag{1.5.5}$$

In this way A_k represents the event that exactly k eigenvalues belong to γ^c. A given eigenvalue configuration Θ leads to a thinned configuration Φ which lies entirely in γ with probability s^k, where $k = \#(\Theta \cap \gamma^c)$. Therefore,

$$\begin{aligned}\mathbb{P}(\Phi \subset \gamma) &= \sum_{k=0}^{n} s^k \mathbb{P}(A_k) = \frac{1}{(2\pi)^n n!} \sum_{k=0}^{n} s^k \int_{A_k} \prod_{1 \le j < k \le n} |e^{i\theta_j} - e^{i\theta_k}|^2 \prod_{j=1}^{n} d\theta_j \\ &= \frac{1}{(2\pi)^n n!} \int_{[0,2\pi]^n} \prod_{1 \le j < k \le n} |e^{i\theta_j} - e^{i\theta_k}|^2 \prod_{j=1}^{n} f(e^{i\theta_j}) d\theta_j = D_n(f),\end{aligned} \tag{1.5.6}$$

where we used the standard Heine integral representation (1.2.22) for Toeplitz determinants.

In Chapter 3, we generalize Theorem 1.5.1 and consider Toeplitz determinants $D_n(f)$ with f as in (1.1.5), in the case where $s(n) \to 0$ as $n \to \infty$ but at a slower rate. For technical reasons, we derive the leading term of $D_n(f)$ only in the case $W = 0$. More precisely, we have the following.

Theorem 1.5.2 *Let $D_n(f)$ be the Toeplitz determinant with f defined by*

$$f(e^{i\theta}) = \begin{cases} 1, & \text{for } -L \le \theta \le L, \\ s, & \text{for } L < \theta < 2\pi - L, \end{cases} \tag{1.5.7}$$

with $L \in (0,\pi)$ and $s \in (0,1)$. If $0 \le x \le x_c$ and $n \to \infty$, we have

$$\lim_{n\to\infty} \frac{1}{n^2} \log D_n(f|_{s=e^{-xn}}) = -\int_0^x \Omega_{\xi,L} d\xi. \tag{1.5.8}$$

Here $\Omega_{\xi,L}$ is given by

$$\Omega_{\xi,L} = \int_{\pi-T}^{\pi+T} d\mu_{\xi,L}(e^{i\theta}), \quad d\mu_{\xi,L}(e^{i\theta}) := \frac{1}{2\pi}\sqrt{\frac{\cos\theta + \cos T}{\cos\theta - \cos L}} d\theta, \tag{1.5.9}$$

where $0 \le T \le \pi - L$ is the unique solution of the equation

$$\frac{1}{\pi}\int_{[-L,L]\cup[\pi-T,\pi+T]} \log\left|\frac{1+e^{i\theta}}{1-e^{i\theta}}\right| \sqrt{\frac{\cos\theta + \cos T}{\cos\theta - \cos L}} d\theta = \xi. \tag{1.5.10}$$

In the conditional CUE, we express the expected number of eigenvalues lying on γ^c before the thinning procedure, and the variance of this quantity in terms of $D_n(f)$. For general removal probability $s \in (0,1)$, we have

$$\begin{aligned} \mathbb{E}\left(\#(\Theta \cap \gamma^c) | \Phi \subset \gamma\right) &= \sum_{k=0}^{n} k \mathbb{P}\left(\#(\Theta \cap \gamma^c) = k | \Phi \subset \gamma\right) \\ &= \sum_{k=0}^{n} k \frac{\mathbb{P}\left(\Phi \subset \gamma \text{ and } \#(\Theta \cap \gamma^c) = k\right)}{\mathbb{P}\left(\Phi \subset \gamma\right)}. \end{aligned} \tag{1.5.11}$$

Using (1.5.6), we obtain the identity

$$\mathbb{E}\left(\#(\Theta \cap \gamma^c) | \Phi \subset \gamma\right) = \sum_{k=0}^{n} k s^k \frac{\mathbb{P}\left(\#(\Theta \cap \gamma^c) = k\right)}{\mathbb{P}\left(\Phi \subset \gamma\right)} = s\frac{\partial}{\partial s} \log D_n(f). \tag{1.5.12}$$

Similarly, after a straightforward calculation,

$$\begin{aligned}\mathrm{Var}\left(\#(\Theta\cap\gamma^c)|\Phi\subset\gamma\right) &= \mathbb{E}\left([\#(\Theta\cap\gamma^c)]^2|\Phi\subset\gamma\right)-\mathbb{E}\left(\#(\Theta\cap\gamma^c)|\Phi\subset\gamma\right)^2\\ &= \sum_{k=0}^{n}k^2s^k\frac{\mathbb{P}\left(\#(\Theta\cap\gamma^c)=k\right)}{\mathbb{P}\left(\Phi\subset\gamma\right)}-\left(s\frac{\partial}{\partial s}\log D_n(f)\right)^2\\ &= s^2\frac{\partial^2}{\partial s^2}\log D_n(f)+s\frac{\partial}{\partial s}\log D_n(f).\end{aligned} \tag{1.5.13}$$

We denote

$$\mu_{n,s,L}(z):=\mathbb{E}\left(\frac{1}{n}\sum_{j=1}^{n}\delta(z-e^{i\theta_j})\right) \tag{1.5.14}$$

for the average counting measure of the eigenvalues in the conditional CUE, with density $\psi_{n,s,L}$, and we let $\nu_{n,s,L}$ be the zero average counting measure of the orthogonal polynomial ϕ_n.

The next theorem gives the leading order term for (1.5.12) and (1.5.13) as $n\to\infty$, as well as the large n behaviour of $\nu_{n,s,L}$ and $\mu_{n,s,L}$, in the case of $\gamma=\{e^{i\theta}:-L\le\theta\le L\}$ and in the regime where $s=e^{-xn}$, $x<x_c$.

Theorem 1.5.3 *Fix $0<L<\pi$. We take $s=e^{-xn}$ with $0<x<x_c=-2\log\tan\frac{L}{4}$.*

1. *The measures $\mu_{n,s,L}$ and $\nu_{n,s,L}$ converge weakly to the measure $\mu_{x,L}$ given by (1.5.9) as $n\to\infty$; moreover,*

$$\lim_{n\to\infty}\psi_{n,s,L}(e^{i\theta})=\frac{d\mu_{x,L}(e^{i\theta})}{d\theta}=\psi_{x,L}(e^{i\theta}), \tag{1.5.15}$$

uniformly for θ in compact subsets of $[-\pi,\pi]\setminus\{-\pi+T,-L,L,\pi-T\}$.

2. *We have*

$$\lim_{n\to\infty}\frac{1}{n}\mathbb{E}\left(\#(\Theta\cap\gamma^c)|\Phi\subset\gamma\right)=\int_{\pi-T}^{\pi+T}d\mu_{x,L}(e^{i\theta})d\theta=\Omega_{x,L}, \tag{1.5.16}$$

$$\lim_{n\to\infty}\frac{1}{n}\mathrm{Var}\left(\#(\Theta\cap\gamma^c)|\Phi\subset\gamma\right)=0. \tag{1.5.17}$$

Previous quantities like the gap probability as well as the expected number of eigenvalues lying in γ^c and its variance, concern the global behaviour of the thinned and conditional spectrum. The local behaviour is described in terms of the correlation kernel K_n. For the thinned CUE, as explained before, one has

$$K_n(e^{i\theta}, e^{i\mu}) = \frac{p}{2\pi} \frac{1 - e^{in(\theta-\mu)}}{1 - e^{i(\theta-\mu)}}, \tag{1.5.18}$$

where $p = 1 - s$ is the retention probability.

A direct computation shows that the rescaled limiting correlation kernel is the sine kernel, i.e. for any $\theta, u, v \in \mathbb{R}$, one has

$$\lim_{n\to\infty} \frac{1}{cn} K_n(e^{i(\theta+\frac{u}{cn})}, e^{i(\theta+\frac{v}{cn})}) = pe^{i\frac{u-v}{2c}} \frac{\sin(\pi(u-v))}{\pi(u-v)}, \tag{1.5.19}$$

where $c = \frac{1}{2\pi}$.

The rescaled limiting correlation kernel for the conditional CUE is far more involved. For finite n, we recall that we have (1.3.8) where $\phi_n = \widehat{\phi}_n$ is orthogonal with respect to the symbol f given in (1.4.4). In the case where $\gamma = \{e^{i\theta} : -L \leq \theta \leq L\}$ and $s = e^{-xn}$, $x < x_c$, we prove the following theorem.

Theorem 1.5.4 *We let $s = e^{-xn}$ with $x < x_c$, and K_n the correlation kernel of the conditional CUE defined in* (1.3.8) *with f given by* (1.5.7).

1. *If $e^{i\theta} \in \{e^{i\theta} : \theta \in (-L, L) \cup (\pi - T, \pi + T)\}$, $u, v \in \mathbb{R}$, we have the sine kernel limit*

$$\lim_{n\to\infty} \frac{1}{cn} K_n(e^{i(\theta+\frac{u}{cn})}, e^{i(\theta+\frac{v}{cn})}) = e^{i\frac{u-v}{2c}} \frac{\sin(\pi(u-v))}{\pi(u-v)}, \tag{1.5.20}$$

   with $c = \psi_{x,L}(e^{i\theta})$.

2. *At the edges of γ, we have the Bessel kernel limits*

$$\lim_{n\to\infty} \frac{1}{(cn)^2} K_n\left(e^{\pm i\left(L - \frac{u}{(cn)^2}\right)}, e^{\pm i\left(L - \frac{v}{(cn)^2}\right)}\right) = \frac{J_0(\sqrt{u})\sqrt{v}J_0'(\sqrt{v}) - J_0(\sqrt{v})\sqrt{u}J_0'(\sqrt{u})}{2(u-v)}, \tag{1.5.21}$$

   for $u, v > 0$, with $c = \sqrt{\frac{|z_0 - z_1||z_0 - \overline{z_1}|}{|z_0 - \overline{z_0}|}}$, $z_0 = e^{iL}$, and $z_1 = e^{i(\pi - T)}$.

3. At the soft edges, we have the Airy kernel limits

$$\lim_{n\to\infty} \frac{e^{-i\frac{n^{1/3}}{2c^{2/3}}(u-v)}}{(cn)^{2/3}} K_n\left(e^{\pm i\left(-\pi+T+\frac{u}{(cn)^{2/3}}\right)}, e^{\pm i\left(-\pi+T+\frac{v}{(cn)^{2/3}}\right)}\right) = \frac{\mathrm{Ai}\,(u)\mathrm{Ai}\,'(v) - \mathrm{Ai}\,'(u)\mathrm{Ai}\,(v)}{u-v}, \quad (1.5.22)$$

for $u, v \in \mathbb{R}$, with $c = \sqrt{\frac{|\overline{z_1}-z_1|}{4|\overline{z_1}-z_0||\overline{z_1}-\overline{z_0}|}}$.

Results in Chapter 4

In Chapter 4 we focus on the hard edge 0 of the LUE, where the asymptotic distribution of the smallest eigenvalue is given by the distribution $F_\alpha(x)$, see (1.2.14). Another distribution to consider is the gap probability

$$G_{n,\alpha}(r) = \mathbb{P}_{n,\alpha}\left(\lambda_{\text{smin}} - \lambda_{\text{min}} > r\right), \qquad r > 0, \quad (1.5.23)$$

between the smallest and second smallest (denoted by x_{smin}) eigenvalue of the LUE. Some results were obtained in [38], where it was shown that the density of $G_\alpha(x) = \lim_{n\to\infty} G_{n,\alpha}(\frac{x}{4n^2})$ exists and is characterised by the solution of a Painlevé III equation and its associated linear isomonodromic system. For results on the gap probability but at the soft edge, see [54] and [71].

The main focus in this chapter is the distribution of the ratio

$$Q_{n,\alpha}(r) = \mathbb{P}_{n,\alpha}\left(\frac{x_{\text{smin}}}{x_{\text{min}}} > r\right), \qquad r > 1, \quad (1.5.24)$$

between the smallest and second smallest eigenvalue of the LUE. Note that the ratio distribution $Q_{n,\alpha}$ cannot be straightforwardly related to the gap $G_{n,\alpha}$ and the distribution of the smallest eigenvalue, since the three variables x_{min}, $x_{\text{smin}} - x_{\text{min}}$ and $\frac{x_{\text{smin}}}{x_{\text{min}}}$ are not independent.

Our techniques differ from the techniques used in [38], [54] and [71]. We express the ratio probability in terms of a Hankel determinant and then apply well-known rigorous techniques from Riemann-Hilbert analysis to derive asymptotics. We obtain a description of the limit of $Q_{n,\alpha}(r)$ as $n \to \infty$ in terms of a solution to a special system of non-linear ODEs arising from a Lax pair associated to a model RH problem.

We begin the calculation of the gap probability between the smallest and second smallest eigenvalue for a general $\alpha > -1$ by writing the quantity $Q_{n,\alpha}(r)$

as an integral over a Hankel determinant. For $r > 1$, by definition of (1.2.11) we have

$$Q_{n,\alpha}(r) = n \int_0^\infty e^{-ny} y^\alpha$$
$$\times \left(\int_{yr}^\infty \dots \int_{yr}^\infty \frac{1}{n! \widehat{Z}_{n,\alpha}} \prod_{1 \le i < j \le n-1} (x_j - x_i)^2 \prod_{i=1}^{n-1} e^{-nx_i} x_i^\alpha (x_i - y)^2 dx_i \right) dy.$$

By changing variables $nx_i = (n-1)\tilde{x}_i$, we can then write

$$Q_{n,\alpha}(r) = \frac{1}{\widehat{Z}_{n,\alpha}} \left(\frac{n-1}{n} \right)^{(n-1)(n+1+\alpha)} \int_0^\infty y^\alpha e^{-ny} Z_{n-1,\alpha} \left(\frac{n}{n-1} y; r \right) dy, \tag{1.5.25}$$

where we defined $Z_{n,\alpha}(y;r)$ by

$$Z_{n,\alpha}(y;r) = \frac{1}{n!} \int_{yr}^\infty \dots \int_{yr}^\infty \prod_{1 \le i < j \le n} (x_j - x_i)^2 \prod_{i=1}^n (x_i - y)^2 x_i^\alpha e^{-nx_i} dx_i. \tag{1.5.26}$$

Note that $Z_{n,\alpha}(y;r)$ is the Hankel determinant with respect to the weight $w(x)$ defined by

$$w(x) = (x-y)^2 x^\alpha e^{-nx} \chi_{[yr,\infty)}(x), \tag{1.5.27}$$

and $\chi_{[yr,\infty)}(x)$ is the characteristic function of $[yr, \infty)$, i.e. we have

$$Z_{n,\alpha}(y;r) = \det \left(\int_{yr}^\infty x^{i+j} w(x) dx \right)_{i,j=0,\dots,n-1}. \tag{1.5.28}$$

Before stating our results, we present the following set of coupled non-linear ODEs, depending on $r > 1$,

$$\begin{aligned}
&\frac{1}{2}\left(-4r - \frac{2V''(x)}{V(x)} + \frac{V'(x)^2}{V(x)^2} \right) + \frac{\alpha^2}{2U_0(x)^4} + \frac{2U_0''(x)}{U_0(x)} = 0, \\
&\frac{1}{2}\left(-4r - \frac{2V''(x)}{V(x)} + \frac{V'(x)^2}{V(x)^2} + 4 \right) + \frac{2U_1''(x)}{U_1(x)} + \frac{2}{U_1(x)^4} = 0, \\
&V(x) = \frac{x}{2} - U_0(x)^2 - U_1(x)^2,
\end{aligned} \tag{1.5.29}$$

where U_0, U_1 and V are three analytic functions of x on $(0,\infty)$ depending on the parameter r with boundary conditions as $x \to \infty$

$$\begin{aligned} U_0(x) &= \sqrt{\frac{\alpha}{2\sqrt{r}}} + \mathcal{O}(x^{-1}), \\ U_1(x) &= \frac{1}{\sqrt[4]{r-1}} + \mathcal{O}(x^{-1}), \\ V(x) &= \frac{x}{2} - \frac{\alpha}{2\sqrt{r}} - \frac{1}{\sqrt{r-1}} + \mathcal{O}(x^{-1}). \end{aligned} \tag{1.5.30}$$

As $x \to 0_+$, we have

$$U_0(x) = \mathcal{O}(x^{-1/2}), \qquad U_1(x) = \mathcal{O}(x^{-1/2}) \quad \text{and} \quad V(x) = \mathcal{O}(x^{-1}). \tag{1.5.31}$$

Our main results concern asymptotics for the Hankel determinants $Z_{n,\alpha}(y;r)$ and the limiting distribution as $n \to \infty$ of the ratio $Q_{n,\alpha}(r)$.

Asymptotics for the Hankel determinant $Z_{n,\alpha}(y;r)$

Theorem 1.5.5 *For $\alpha \geq 0$, as $n \to \infty$ and $x = x(n)$ lies in compact subset of $[0,\infty)$ we have,*

$$\log Z_{n,\alpha}\left(\frac{x}{n^2};r\right) - \log(\widehat{Z}_{n,\alpha+2}) = I(x;r) + \mathcal{O}\left(n^{-1}\right), \tag{1.5.32}$$

where

$$\begin{aligned} I(x;r) &= \frac{1}{2}\int_0^x [v(4\xi;r) - v(0)]\,\frac{d\xi}{\xi} \\ &= 2\int_0^x \log\left(\frac{x}{\xi}\right) v'(4\xi;r)d\xi \end{aligned} \tag{1.5.33}$$

with $v(0) := \dfrac{1}{8}(4(\alpha+2)^2 - 1)$ *independent of r and*

$$v'(x^2;r) = -\frac{U_1(x)^2 + rV(x)}{x}. \tag{1.5.34}$$

Here U_1 and V are special solutions of the system (1.5.29)*, satisfying the boundary conditions as $x \to \infty$ and $x \to 0_+$ given by* (1.5.30) *and* (1.5.31)*. The function v satisfies*

$$\begin{aligned} v(x;r) &= -\frac{r}{2}x + (2\sqrt{r-1} + \alpha\sqrt{r})\sqrt{x} + \mathcal{O}(1), && \text{as } x \to \infty, \\ v(x;r) &= v(0) + \mathcal{O}(x), && \text{as } x \to 0_+. \end{aligned} \tag{1.5.35}$$

Limiting distribution of the ratio probability $Q_{n,\alpha}(r)$

Theorem 1.5.6 *For $\alpha \geq 0$, as $n \to \infty$, the limit $Q_\alpha(r) := \lim_{n\to\infty} Q_{n,\alpha}(r)$ exists and is given by,*

$$Q_\alpha(r) = \frac{1}{\Gamma(\alpha+1)\Gamma(\alpha+2)} \int_0^\infty x^\alpha e^{I(x;r)} dx \tag{1.5.36}$$

with $I(x;r)$ given in Theorem 1.5.5.

1.6 Tools and techniques

We will use RH problems and the Deift-Zhou steepest descent method in Chapter 2, Chapter 3 and Chapter 4, and Riemann surfaces in Chapter 3. For convenience we recall here some definitions and known results. For more general theory, the proofs, examples and details, we refer to lecture notes [48] and Deift's book [23] for RH problems and to both lecture notes [11] and [9] for Riemann surfaces.

Riemann-Hilbert problems

RH problems were introduced by Hilbert around 1900 in his list of 23 problems, in the purpose of proving existence of some linear differential equations. Now RH problems are a class of boundary value problems that arise in different areas of mathematics. Of particular interest for us are the links with orthogonal polynomials.

A RH problem is associated to a given contour Σ, which is a finite union of piecewise smooth oriented curves. By oriented we mean that each curve of Σ possesses a "+" side and a "-" side. Σ can contain a finite number of self-intersection points and edge points. We denote by $\widetilde{\Sigma}$ the contour Σ without those points. A RH problem associated to Σ consists in searching for a $k \times k$ matrix valued function, which we denote by $Y(z)$, satisfying the following conditions.

RH problem for Y

(a) each entry of Y is analytic on the complex plane except on the contour Σ.

(b) Y has continuous boundary values on both side of $\widetilde{\Sigma}$, denoted by Y_+ and Y_- and related to each other by

$$Y_+(z) = Y_-(z)J(z), \qquad \text{for } z \in \widetilde{\Sigma}, \tag{1.6.1}$$

where $J = J(z)$ is a $k \times k$ matrix valued function defined on $\widetilde{\Sigma}$, which is called the jump matrix. In most cases J is analytic on $\widetilde{\Sigma}$. Condition (b) is called the jump condition.

(c) the behaviour of Y has to be specified as $z \to \infty$ and as $z \to \Sigma \setminus \widetilde{\Sigma}$.

If $k = 1$ then Y is of size 1×1 and the problem becomes scalar. In the present thesis we will only use RH problems whose solutions are of size 2×2.

In random matrix theory, as seen in (1.3.7) and (1.3.8), correlation kernels are, in some cases, described in terms of orthogonal polynomials with respect either to a weight w on the real line or to a symbol f on the unit circle. Riemann-Hilbert problems for orthogonal polynomials were introduced by Fokas, Its and Kitaev [34]. It is possible to show that Y_1, Y_2, given by

$$Y_1(z) = \begin{pmatrix} \phi_n(z) & \displaystyle\int_{S^1} \frac{\phi_n(w)}{w-z} \frac{f(w)}{2\pi i w^n} dw \\ -h_{n-1}^{-1} z^{n-1} \widehat{\phi}_{n-1}(z^{-1}) & \displaystyle -h_{n-1}^{-1} \int_{S^1} \frac{\widehat{\phi}_{n-1}(w^{-1})}{w-z} \frac{f(w)}{2\pi i w} dw \end{pmatrix}, \tag{1.6.2}$$

where ϕ_n, $\widehat{\phi}_n$ are the monic orthogonal polynomials associated to the symbol f (1.1.5) and

$$Y_2(z) = \begin{pmatrix} p_n(z) & \displaystyle\frac{1}{2\pi i} \int_{\mathbb{R}} \frac{p_n(x) w(x)}{x-z} dx \\ \displaystyle -\frac{2\pi i}{h_{n-1}} p_{n-1}(z) & \displaystyle -\frac{1}{h_{n-1}} \int_{\mathbb{R}} \frac{p_{n-1}(x) w(x)}{x-z} dx \end{pmatrix}, \tag{1.6.3}$$

where p_n are orthogonal polynomials associated to the weight w (1.5.27), both satisfy a RH problem.

RH problem for Y_1

(a) $Y_1 : \mathbb{C} \setminus S^1 \to \mathbb{C}^{2\times 2}$ is analytic.

(b) Y_1 has the following jumps:

$$Y_{1+}(z) = Y_{1-}(z) \begin{pmatrix} 1 & z^{-n} f(z) \\ 0 & 1 \end{pmatrix}, \qquad \text{for } z \in S^1 \setminus \left\{ e^{iL}, e^{-iL} \right\},$$

where $Y_{1+}(z)$ $(Y_{1-}(z))$ denotes the limit as z is approached from the inside (outside) of the unit circle.

(c1) As $z \to \infty$, $Y_1(z) = \left(I + \mathcal{O}(z^{-1})\right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$.

(c2) As $z \to e^{\pm iL}$, $Y_1(z) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\log(z - e^{\pm iL})) \\ \mathcal{O}(1) & \mathcal{O}(\log(z - e^{\pm iL})) \end{pmatrix}$.

RH problem for Y_2

(a) $Y_2 : \mathbb{C} \setminus [yr, \infty) \to \mathbb{C}^{2\times 2}$ is analytic.

(b) Y_2 has the following jumps:

$$Y_{2+}(x) = Y_{2-}(x) \begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix}, \qquad x \in (yr, \infty), \tag{1.6.4}$$

where $Y_{2+}(z)$ $(Y_{2-}(z))$ denotes the limit as z is approached from above (below) of (yr, ∞).

(c1) As $z \to \infty$, $Y_2(z) = \left(I + \mathcal{O}(z^{-1})\right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$.

(c2) As $z \to yr$, $Y_2(z) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\log(yr - z)) \\ \mathcal{O}(1) & \mathcal{O}(\log(yr - z)) \end{pmatrix}$.

A standard argument shows that those RH problems possess a unique solution [23]. In general it is more complicated to prove the existence of a solution than the uniqueness. In this special case, the existence of Y_2 follows from the fact that w is positive, and therefore by Gram-Schmidt p_n exists for all n.

The RH problems for Y_1 and Y_2 depend on the parameter n. The well known Deift/Zhou steepest descent method [29, 30] allows to get asymptotics for Y_1 and Y_2 as $n \to \infty$. Basically, starting on a given RH problem for Y (one can think of $Y = Y_1$ or $Y = Y_2$ for example), the method consists of a series of invertible transformations, usually denoted by

$$Y \mapsto T \mapsto S \mapsto R.$$

The final transformation leads to a *small norm* RH problem for R, which will be of the following form.

RH problem for R

(a) $R : \mathbb{C} \setminus \Sigma_R \to \mathbb{C}^{2\times 2}$ is analytic.

(b) R satisfies the jump conditions

$$R_+(z) = R_-(z) J_R(z), \quad \text{for } z \in \widetilde{\Sigma}_R, \tag{1.6.5}$$

with $J_R(z) = I + \mathcal{O}(n^{-\lambda})$, $\lambda > 0$ uniformly for $z \in \widetilde{\Sigma}_R$ as $n \to \infty$. We will often have $\lambda = 1$.

(c) $R(z) = I + \mathcal{O}(z^{-1})$ as $z \to \infty$.

From the standard theory for small-norm RH problems [27, 28], it follows that R exists for n large and

$$R(z) = I + \mathcal{O}(n^{-\lambda}), \qquad n \to \infty, \tag{1.6.6}$$

uniformly for $z \in \mathbb{C} \setminus \Sigma_R$. Inverting the transformations $R \mapsto S \mapsto T \mapsto Y$ we get large n asymptotics for Y. Since the $(1,1)$ and $(2,1)$ entries of Y contain orthogonal polynomials of large degree, we can therefore obtain large n asymptotics for the corresponding correlation kernel K_n.

Riemann surfaces and elliptic curves

We recall here some definitions and results which we will make use of later. For more results and proofs, we refer to the lecture notes [11] and [9].

A Riemann surface X is a connected two-dimensional real manifold endowed with a complex atlas, i.e. for every $x \in X$, there exist a subset $U_x \subset X$ such that $x \in U_x$, and a homeomorphism z_x from U_x to an open subset of $\mathbb{C}$. Such a pair (U_x, z_x) is called a *local coordinate*, or *local chart*. We require in addition that for any two local coordinates (U_x, z_x), $(U_{x'}, z_{x'})$ with $U_x \cap U_{x'} \neq \emptyset$, the transition function

$$z_{x'} \circ z_x^{-1} : z_x(U_x \cap U_{x'}) \to z_{x'}(U_x \cap U_{x'}) \tag{1.6.7}$$

is holomorphic. X is a topological space with the topology inherited via z_x^{-1} from $\mathbb{C}$, i.e. $U \subset X$ is open if for all local coordinates z_x in the atlas, $z_x(U)$ is open in $\mathbb{C}$. In the present thesis we will only use compact Riemann surfaces, and therefore in the rest of this section we will assume that X is compact. It is known that any compact Riemann surface is homeomorphic to a sphere with handles in the Euclidian 3-space. The number $g \in \mathbb{N}$ of handles is called the *genus* of X. Riemann surfaces with different genera are not homeomorphic. X is homeomorphic to a sphere if $g = 0$ and to a torus if $g = 1$. A mapping $f : X \to \mathbb{C}$ is called a holomorphic (resp. meromorphic) function if for every local coordinate (U, z) on X, the mapping $f \circ z^{-1} : z(U) \to f(U)$ is holomorphic (resp. meromorphic).

The first homology group of X is a free Abelian group generated by exactly $2g$ generators. These generators can be chosen to be simple cycles and such that they form a canonical basis. We will denote them by $\{a_1, ..., a_g, b_1, ..., b_g\}$. The following theorem is known as Riemann's bilinear identity.

Theorem 1.6.1 (Riemann's bilinear identity) *Let ω and η be two closed differentials on X, then*

$$\int_X \omega \wedge \eta = \sum_{j=1}^{g} \left(\int_{a_j} \omega \int_{b_j} \eta - \int_{a_j} \eta \int_{b_j} \omega \right). \tag{1.6.8}$$

A differential ω on X is called holomorphic (resp. meromorphic) if in any local coordinate we have

$$\omega = h(z)dz, \tag{1.6.9}$$

where $h(z)$ is holomorphic (resp. meromorphic). The dimension of the vector space of holomorphic differentials on X is g. There is a unique basis $\omega_1, ..., \omega_g$ of this space normalized by

$$\int_{a_j} \omega_k = \delta_{jk}, \qquad 1 \le j, k \le g, \tag{1.6.10}$$

and we naturally define a $g \times g$ matrix B by $B_{jk} = \int_{b_j} \omega_k$. A divisor D is a finite formal sum of points with integer coefficients

$$D = \sum_{j=1}^{N} n_j P_j, \quad N \in \mathbb{N},\ n_j \in \mathbb{Z},\ P_j \in X. \tag{1.6.11}$$

The degree of D is $\deg D = \sum_{j=1}^{N} n_j$. Let f be a meromorphic function on X and $P_1, ..., P_M$ be its zeros with multiplicities $p_1, ..., p_m > 0$ and $Q_1, ..., Q_N$ be its poles with the multiplicities $q_1, ..., q_N > 0$. The divisor

$$D = p_1 P_1 + ... + p_M P_M - q_1 Q_1 - ... - q_N Q_N = (f) \tag{1.6.12}$$

is called the divisor of f and is denoted by (f). A divisor D is called principal if there exists a meromorphic function f on X with $(f) = D$. The Abel map is denoted by $\mathcal{A}$ and is defined by

$$\mathcal{A} : X \to \mathbb{C}^g/\Lambda, \qquad \mathcal{A}(P) = \int_{P_0}^{P} \omega, \tag{1.6.13}$$

where $\omega = (\omega_1, ..., \omega_g)$, $P_0 \in X$ and

$$\Lambda = \{N + BM : N, M \in \mathbb{Z}^g\}. \tag{1.6.14}$$

Theorem 1.6.2 (Abel) *The divisor D is principal if and only if* $\deg D = 0$ *and* $\mathcal{A}(D) \equiv 0 \mod \Lambda$*.*

Let D be a divisor on X. We will consider the two vector spaces

$$\begin{aligned} L(D) &= \{f \text{ meromorphic functions on } X : (f) \geq D \text{ or } f \equiv 0\}, \\ H(D) &= \{\Omega \text{ meromorphic differentials on } X : (\Omega) \geq D \text{ or } \Omega \equiv 0\}. \end{aligned}$$

Theorem 1.6.3 (Riemann-Roch) *If X is compact and D is a divisor on X, then*

$$\dim L(-D) = \deg D - g + 1 + \dim H(D). \tag{1.6.15}$$

A curve of the form $\{(w, z) \in \mathbb{C}^2 : w^2 = \prod_{j=1}^{N}(z - z_j)\}$, with $z_i \neq z_j$ if $i \neq j$, is called elliptic if $N = 3$ or $N = 4$ and hyperelliptic if $N \geq 5$. This is a special case of compact Riemann surface with genus $g = \frac{N-1}{2}$ if N is odd and $g = \frac{N-2}{2}$ if N is even. In this thesis we will only need elliptic curves with $N = 4$ (and therefore $g = 1$).

We will also use the following proposition.

Proposition 1.6.4 *If $g = 1$, then the Abel map* (1.6.13) *is a bijection.*

We will use the elliptic theta functions (see [53, Chapter 20] for definition and properties of these functions) on a Riemann surface to build approximations for certain orthogonal polynomials in Chapter 3.

Chapter 2

Asymptotics for Toeplitz determinants: perturbation of symbols with a gap

Abstract [1]

We study the determinants of Toeplitz matrices as the size of the matrices tends to infinity, in the particular case where the symbol has two jump discontinuities and tends to zero on an arc of the unit circle at a sufficiently fast rate. We generalize an asymptotic expansion by Widom [67], which was known for symbols supported on an arc. We highlight applications of our results in the Circular Unitary Ensemble and in the study of Fredholm determinants associated to the sine kernel.

2.1 Introduction

We consider Toeplitz determinants of the form

$$D_n(f) = \det(f_{j-k})_{j,k=0,\dots,n-1}, \qquad f_j = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-ij\theta}d\theta, \qquad (2.1.1)$$

[1] This chapter corresponds to the following paper [17]:

C. Charlier and T. Claeys, Asymptotics for Toeplitz determinants: perturbation of symbols with a gap, *J. Math. Phys.* **56** (2015), 022705.

where the symbol f is given by

$$f(e^{i\theta}) = e^{W(e^{i\theta})} \times \begin{cases} 1, & \text{for } -L \le \theta \le L, \\ s, & \text{for } L < \theta < 2\pi - L, \end{cases} \tag{2.1.2}$$

and f_k is the k-th Fourier coefficient of f. The symbol depends on the parameters $s \in [0,1]$ and $L \in (0,\pi)$, and on a function $W(z)$ which we assume to be analytic in a neighbourhood of the unit circle. In the simplest case where $W(z) = 0$, f is piecewise constant with jump discontinuities at $e^{\pm iL}$.

For $s \in [0,1]$ and $L \in (0,\pi)$ fixed, the large n asymptotic behavior for the Toeplitz determinants $D_n(f) = D_n(s,L,W)$ is well understood. For $s = 0$, the symbol is supported on the arc $\gamma = \{e^{i\theta} : -L \le \theta \le L\}$. If f is positive on γ and symmetric, $f(e^{i\theta}) = f(e^{-i\theta})$, we have the following asymptotics due to Widom [67],

$$\begin{aligned} \log D_n\,(s=0,L,W) &= n^2 \log\sin\frac{L}{2} + n\widetilde{W}_0 - \frac{1}{4}\log n + \sum_{k=1}^{\infty} k\widetilde{W}_k\widetilde{W}_{-k} \\ &\quad -\frac{1}{4}\log\cos\frac{L}{2} + \frac{1}{12}\log 2 + 3\zeta'(-1) + o(1), \end{aligned} \tag{2.1.3}$$

as $n \to \infty$, where $\widetilde{W}_k$ is the k-th Fourier coefficient of $\widetilde{W}$, defined by

$$\widetilde{W}(e^{i\theta}) = W(e^{2i\arcsin(\sin\frac{L}{2}\sin\frac{\theta}{2})}), \tag{2.1.4}$$

and where ζ is the Riemann ζ-function. For $W = 0$, it was shown in [46, 47] that this result holds not only for $0 < L < \pi$ fixed, but also if L approaches π slowly enough such that $n(\pi - L)$ is large. The error term in (2.1.3) then becomes $\mathcal{O}(n^{-1}(\pi - L)^{-1})$.

On the other end of the parameter range for s, we have $s = 1$: here the symbol is smooth and we have the Szegő asymptotics

$$\log D_n\,(s=1,L,W) = nW_0 + \sum_{k=1}^{\infty} kW_kW_{-k} + o(1), \qquad \text{as } n \to \infty. \tag{2.1.5}$$

For $0 < s < 1$, f has two jump discontinuities which are special cases of Fisher-Hartwig (FH) singularities. FH singularities are generally characterized by two parameters: α, which describes a root-type singularity, and β, which describes a jump discontinuity. Our symbol f has Fisher-Hartwig singularities at $e^{\pm iL}$ with parameters $\alpha = 0$ (which means that there are no root-type singularities) and

$\beta = \mp\frac{1}{2\pi i}\log s$. Asymptotics for Toeplitz determinants with Fisher-Hartwig singularities have been studied by many authors [33, 68, 7, 8, 16, 32, 24, 26]. Applied to our symbol f, the results from, e.g., [24] imply that, for $s \in (0,1)$ fixed, i.e. independent of n, we have

$$\begin{aligned}\log D_n(s,L,W) &= n\left(1-\frac{L}{\pi}\right)\log s + nW_0 + \frac{(\log s)^2}{2\pi^2}(\log n + \log(2\sin L)) \\ &-\frac{\log s}{\pi}\sum_{k=1}^{\infty}(W_k+W_{-k})\sin(kL) + \sum_{k=1}^{\infty}kW_kW_{-k} \\ &+2\log\left(G\left(1+\frac{\log s}{2\pi i}\right)G\left(1-\frac{\log s}{2\pi i}\right)\right) + o(1),\end{aligned} \tag{2.1.6}$$

as $n \to \infty$, where G is Barnes' G-function.

If we let s tend to 0, the jump discontinuities of f turn into endpoints of the support of f, and f transforms from a Fisher-Hartwig symbol to an arc-supported symbol on $[-L,L]$. Nevertheless, letting $s \to 0$ in (2.1.6), we do not recover the Widom asymptotics (2.1.3). This means that a non-trivial critical transition in the asymptotic behavior for $D_n(s,L,W)$ takes place as $s \to 0$. In this paper, we show that the Widom asymptotics (2.1.3) extend to the case where $s \neq 0$ but $s = s(n) \to 0$ sufficiently rapidly as $n \to \infty$.

Theorem 2.1.1 *Let $L \in (0,\pi)$ and define*

$$x_c = -2\log\tan\frac{L}{4}. \tag{2.1.7}$$

Let f be of the form (2.1.2) with W analytic in a neighbourhood of the unit circle. As $n \to \infty$ and simultaneously $s \to 0$ in such a way that $0 \leq s \leq e^{-x_c n}$, we have

$$\log D_n(s,L,W) = \log D_n(0,L,W) + o(1). \tag{2.1.8}$$

The error term $o(1)$ is uniform for $0 \leq s \leq e^{-x_c n}$ and $\epsilon \leq L \leq \pi-\epsilon$, $\epsilon > 0$, and can be specified as

$$o(1) = \mathcal{O}(n^{-1/2}e^{x_c n}s). \tag{2.1.9}$$

In addition, the result extends to the case where L approaches π at a sufficiently slow rate: (2.1.8) holds for $\epsilon < L < \pi - \frac{M}{n}$ with M sufficiently large and

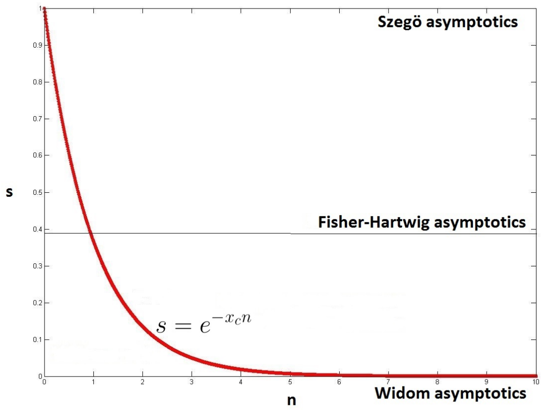

Figure 2.1: As $n \to \infty$, $\log D_n(s, L, W)$ follows Widom asymptotics for $s = 0$, Fisher-Hartwig asymptotics for $s \in (0, 1)$ fixed, and Szegő asymptotics for $s = 1$. Theorem 2.1.1 implies that the Widom asymptotics remain valid as $n \to \infty$ with $s = s(n)$ below the curve $s = e^{-x_c n}$. As $n \to \infty$ with $s = s(n)$ above the curve, a different type of asymptotic behavior takes place, as shown in Chapter 3.

$0 \le s \le e^{-x_c n}$, *with the error term given by*

$$o(1) = \mathcal{O}((\pi - L)^{1/2} n^{-1/2} e^{n x_c} s). \tag{2.1.10}$$

Remark 2.1.2 The bound $s \le e^{-x_c n}$ is sharp: as $n \to \infty$, $s \to 0$ with $s > e^{-x_c n}$, the asymptotic behavior for $\log D_n(s, L, W)$ is different. We refer to Chapter 3 for this case.

Remark 2.1.3 If f is positive on γ and even in θ, the asymptotics for $\log D_n(0, L, W)$ in (2.1.8) are given by (2.1.3). The perturbative result (2.1.8) is valid for general analytic W, even if the positivity and symmetry conditions needed for the Widom asymptotics do not hold. Note that the error term $o(1) = \mathcal{O}((\pi - L)^{1/2} n^{-1/2} e^{n x_c} s)$ in (2.1.8) improves as L approaches π. This is reasonable since the perturbation of the arc-supported symbol then takes place on a shrinking arc near -1. For our proof, it is however crucial that $n(\pi - L)$ is sufficiently large.

Application 1: the Circular Unitary Ensemble

Consider the Circular Unitary Ensemble (CUE) which is the set of unitary $n \times n$ matrices with the Haar measure. The eigenvalues $e^{i\theta_1}, \dots, e^{i\theta_n}$ in this ensemble have the joint probability distribution

$$\frac{1}{(2\pi)^n n!} \prod_{1 \le j < k \le n} |e^{i\theta_j} - e^{i\theta_k}|^2 \prod_{j=1}^{n} d\theta_j, \qquad \theta_1, \dots, \theta_n \in [0, 2\pi). \tag{2.1.11}$$

Define the random variable $X = X_L$ as the number of eigenvalues of a CUE matrix on the arc $\gamma = \{e^{i\theta} : -L \le \theta \le L\}$. The average value of X_L is given by

$$\mathbb{E}_n(X_L) = \frac{L}{\pi} n. \tag{2.1.12}$$

Define the moment generating function

$$F_n(\lambda) = \mathbb{E}_n(e^{\lambda X_L}). \tag{2.1.13}$$

By (2.1.11), we can write this as

$$\begin{aligned} F_n(\lambda) &= \frac{1}{(2\pi)^n n!} \int_{[0,2\pi]^n} \prod_{1 \le j < k \le n} |e^{i\theta_j} - e^{i\theta_k}|^2 e^{\lambda X_L} \prod_{j=1}^{n} d\theta_j & (2.1.14) \\ &= \frac{1}{(2\pi)^n n!} \int_{[0,2\pi]^n} \prod_{1 \le j < k \le n} |e^{i\theta_j} - e^{i\theta_k}|^2 \prod_{j=1}^{n} e^{\lambda \chi_\gamma(e^{i\theta_j})} d\theta_j, & (2.1.15) \end{aligned}$$

with χ_γ the characteristic function of the arc γ. But using the standard multiple integral representation for Toeplitz determinants, we obtain

$$F_n(\lambda) = e^{n\lambda} D_n(s = e^{-\lambda}, L, W = 0). \tag{2.1.16}$$

This is an identity for any $\lambda \ge 0$. The moment generating function $F_n(\lambda)$ at large positive values of λ contains statistical information about large deviations where X_L is much bigger than its average value. As an illustration, a rough estimate of the integral in (2.1.15) gives us the inequality

$$\mathrm{Prob}_n\left[X_L \ge p\right] \le e^{-p\lambda} F(\lambda) = e^{(n-p)\lambda} D_n(e^{-\lambda}, L, 0), \tag{2.1.17}$$

which holds for any $\lambda \ge 0$ and $p \in \{0, 1, \dots, n\}$. Theorem 2.1.1 gives asymptotics for $D_n(e^{-\lambda}, L, 0)$ for $\lambda \ge n x_c$. For large n, the minimum of the right hand side of (2.1.17) in the range $\lambda \geq nx_c$ is attained near

$$\lambda = nx_c = -2n \log \tan \frac{L}{4}. \tag{2.1.18}$$

Substituting this value for λ and the asymptotic behavior for $D_n(e^{-nx_c}, L, 0)$, we obtain, as $n \to \infty$,

$$\begin{aligned} \mathrm{Prob}_n\left[X_L \geq p\right] \quad &\leq \quad \left(\tan \frac{L}{4}\right)^{-2n^2+2pn} \left(\sin \frac{L}{2}\right)^{n^2} \\ &\times \quad n^{-1/4} \left(\cos \frac{L}{2}\right)^{-1/4} 2^{1/12} e^{3\zeta'(-1)}(1+o(1)), \end{aligned} \tag{2.1.19}$$

For typical values of $p \sim \frac{L}{\pi} n$, and even more generally for $p < (1-c)n$ with

$$c = \frac{\log \sin \frac{L}{2}}{2 \log \tan \frac{L}{4}}, \tag{2.1.20}$$

this estimate does not give any useful information since the right hand side is large for large n. We do get non-trivial information for large deviations from the typical value. If we let $p = n - \omega(n)$, we have

$$\begin{aligned} \mathrm{Prob_n}\left[X_L \geq p\right] \leq &\left(\tan \frac{L}{4}\right)^{-2n\omega(n)} \left(\sin \frac{L}{2}\right)^{n^2} \\ \times \quad & n^{-1/4} \left(\cos \frac{L}{2}\right)^{-1/4} 2^{1/12} e^{3\zeta'(-1)}(1+o(1)), \end{aligned} \tag{2.1.21}$$

as $n \to \infty$, and the right hand side decays whenever $\omega(n) < cn$. If $\omega(n) = o(n)$ as $n \to \infty$, we observe that the probability to have at least $n - \omega(n)$ eigenvalues on γ can be estimated by

$$\mathrm{Prob_n}\left[X_L \geq n - \omega(n)\right] = \mathcal{O}(n^{-1/4} e^{n^2 \phi(L) - 2n\omega(n) \log \tan \frac{L}{4}}), \qquad \text{as } n \to \infty, \tag{2.1.22}$$

where the rate function ϕ is given by $\phi(L) = \log \sin \frac{L}{2}$. Note that this is not a large deviation principle since we only have an asymptotic upper bound for the left hand side of (2.1.22). A more detailed analysis of the moment generating function $F(\lambda)$ may lead to better estimates, but it is not our aim to proceed in this direction.

Application 2: sine kernel determinants

In the double scaling limit where $n \to \infty$ and L approaches π at rate $\mathcal{O}(1/n)$, the Toeplitz determinant $D_n(s, L, 0)$ converges to the Fredholm determinant of the operator $1 - (1-s)K_y$, where K_y is the integral operator acting on $(-y, y)$ with kernel $\frac{\sin \pi(x-t)}{\pi(x-t)}$: we have [25, formulas (236)-(241)]

$$\lim_{n\to\infty} D_n(s, L = \pi(1 - \frac{2y}{n}), 0) = \det(1 - (1-s)K_y), \tag{2.1.23}$$

where the operator K_y is defined by

$$(K_y g)(x) = \int_{-y}^{y} \frac{\sin \pi(x-t)}{\pi(x-t)} g(t) dt. \tag{2.1.24}$$

Asymptotics for the right hand side of (2.1.23) have been studied by Dyson [31], see also [25] for a historical review; the double scaling limit where $y \to \infty$ and simultaneously $s \to 0$ appears to be subtle. As $y \to \infty$, $s \to 0$ in such a way that $s \leq e^{-2\pi y}$, with

$$c = \frac{1}{12} \log 2 + 3\zeta'(-1), \tag{2.1.25}$$

we have

$$\log \det(1 - (1-s)K_y) = -\frac{\pi^2 y^2}{2} - \frac{1}{4} \log \pi y + c + o(1). \tag{2.1.26}$$

Except for $s = 0$, (2.1.26) had not been proved rigorously until the recent paper [15], where double scaling asymptotics were obtained both for $s \leq e^{-2\pi y}$ and for $s > e^{-2\pi y}$. The value of the constant (2.1.25) was also proved in [15].

As a consequence of our Theorem 2.1.1, we re-derive (2.1.26) and (2.1.25). Indeed, by (2.1.3), (2.1.8), and (2.1.10), we have

$$\begin{aligned} \log D_n(s, \pi(1 - \frac{2y}{n}), 0) &= n^2 \log \cos \frac{\pi y}{n} - \frac{1}{4} \log n - \frac{1}{4} \log \sin \frac{\pi y}{n} + c \\ &\quad + \mathcal{O}(\frac{y^{1/2}}{n} e^{n x_c} s) + \mathcal{O}(y^{-1}), \end{aligned} \tag{2.1.27}$$

with the error term uniform for n, y large, $y^{1/2}/n$ small, and

$$s \leq e^{-x_c n} = e^{2n \log \tan \frac{L}{4}} = e^{2n \log \tan\left(\frac{\pi}{4} - \frac{\pi y}{2n}\right)}. \tag{2.1.28}$$

The right hand side of (2.1.28) can be written as $e^{-2\pi y}\left(1+\mathcal{O}(y^2n^{-1})\right)$ for n, y large and y^2/n small.
Now, let us choose an arbitrary small function $\epsilon(y) > 0$ and assume $s \leq (1-\epsilon(y))e^{-2\pi y}$. Then for n sufficiently large, i.e. for $n >> y^2/\epsilon(y)$, (2.1.28) holds, and letting $n \to \infty$ in (2.1.27), we obtain

$$\lim_{n\to\infty} \log D_n(s, \pi(1-\frac{2y}{n}), 0) = -\frac{\pi^2 y^2}{2} - \frac{1}{4}\log \pi y + c + \mathcal{O}(y^{-1}), \quad (2.1.29)$$

for large y. We thus re-derive (2.1.26) in the double scaling limit where $y \to \infty, s \to 0$ in such a way that $s \leq (1-\epsilon(y))e^{-2\pi y}$, and we confirm the value of the constant (2.1.25), using different methods than in [15].

In [15], the double scaling asymptotics $y \to \infty, s \to 0$ such that $s \geq e^{-2\pi y}$ have also been considered, and asymptotics for $\det(1-(1-s)K_y)$ in terms of elliptic θ-functions were obtained. This is consistent with Remark 2.1.2 and Chapter 3.

Outline

In Section 2.2, we will derive identities which express $\frac{d}{ds}\log D_n(s, L, W)$ in terms of orthogonal polynomials on the unit circle. In Section 2.3, we will study an equilibrium problem which is crucial to obtain asymptotics for the orthogonal polynomials. In Section 2.4, we will obtain asymptotics for the orthogonal polynomials as the degree tends to infinity, via the Riemann-Hilbert (RH) method. An important novel feature in the RH analysis is the construction of a *modified Bessel parametrix* compared to Bessel parametrices that appeared in the literature before. This modification is needed because $s \neq 0$. The asymptotics for the orthogonal polynomials will lead us towards large n asymptotics for $\frac{d}{ds}\log D_n(s, L, W)$. Integrating the differential identity will complete the proof of Theorem 2.1.1 in Section 2.5 for L fixed. In addition, we explain how the proof can be extended to the case where $L \to \pi$ at a sufficiently slow rate.

2.2 Differential identities for Toeplitz determinants

In this section, we derive an identity for $\frac{d}{ds}\log D_n(s, L, W)$ in terms of orthogonal polynomials on the unit circle. Define $\phi_j, \widehat{\phi}_j$, $j = 0, 1, 2, \ldots$ as the family of monic orthogonal polynomials on the unit circle characterized by

$$\frac{1}{2\pi}\int_0^{2\pi} \phi_k(e^{i\theta})\widehat{\phi}_m(e^{-i\theta})f(e^{i\theta})d\theta = h_k\delta_{km}, \qquad \forall k,m\in\mathbb{N}, \tag{2.2.1}$$

where the degree of $\phi_j, \widehat{\phi}_j$ is j and their norm $h_j \neq 0$.

For general weight functions $f(e^{i\theta})$, it is a standard fact that $\phi_n, \widehat{\phi}_n$ exist and are unique if $D_n(f), D_{n+1}(f) \neq 0$ and that they are given by the determinantal formulas [58]

$$\phi_n(z) = \frac{1}{D_n(f)}\begin{vmatrix} f_0 & f_{-1} & f_{-2} & \cdots & f_{-n} \\ f_1 & f_0 & f_{-1} & \cdots & f_{-n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_{n-1} & f_{n-2} & f_{n-3} & \cdots & f_{-1} \\ 1 & z & z^2 & \cdots & z^n \end{vmatrix}, \tag{2.2.2}$$

and

$$\widehat{\phi}_n(z) = \frac{1}{D_n(f)}\begin{vmatrix} f_0 & f_{-1} & f_{-2} & \cdots & f_{-n+1} & 1 \\ f_1 & f_0 & f_{-1} & \cdots & f_{-n+2} & z \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ f_n & f_{n-1} & f_{n-2} & \cdots & f_1 & z^n \end{vmatrix}, \tag{2.2.3}$$

where $D_n = D_n(f)$ is the Toeplitz determinant defined in (2.1.1), and where f_k is, as before, the k-th Fourier coefficient of f. If $f(e^{i\theta})$ is positive, $D_n(f) > 0$ for all $n \in \mathbb{N}$ and the OPs are well-defined. For complex f, existence and uniqueness of the orthogonal polynomials is not guaranteed for any n. For f given in (2.1.2), it will follow from our analysis that they are well-defined for n sufficiently large and $0 \leq s \leq e^{-x_c n}$. If $W = 0$, the symbol f is even in θ and positive, which implies that $\widehat{\phi}_k = \phi_k = \overline{\phi_k}$.

From (2.2.2), we see that the norm h_n of ϕ_n is equal to

$$h_n = h_n(s,L,W) = \frac{D_{n+1}(s,L,W)}{D_n(s,L,W)}.$$

We can thus express $D_n(s,L,W)$ in terms of the norms of the OPs if $D_0 \neq 0$, $D_1 \neq 0$, …, $D_n \neq 0$:

$$D_n(s,L,W) = \prod_{j=0}^{n-1} h_j. \tag{2.2.4}$$

To obtain an asymptotic formula for $D_n(s,L,W)$ from this formula, we would need information about all OPs of degree 0 up to $n-1$. To circumvent this problem, we derive differential identities for $\log D_n(s,L,W)$.

2.2.1 Differential identity for general W

For general sufficiently smooth symbols $f(z;s)$ depending on a parameter s, there exists an identity for the logarithmic derivative of $D_n(f)$ with respect to s: if $D_{n-1}, D_n, D_{n+1} \neq 0$, we have [26, Proposition 3.3]

$$\partial_s \log D_n(s,L,W) = \frac{1}{2\pi i}\int_{S^1} z^{-n}\left[Y^{-1}(z)Y'(z)\right]_{21}\partial_s f(z,s)dz,$$

where S^1 is the unit circle in the complex plane, $' = \frac{d}{dz}$, $\partial_s = \frac{d}{ds}$, and

$$Y(z) = \begin{pmatrix} \phi_n(z) & \displaystyle\int_{S^1}\frac{\phi_n(w)}{w-z}\frac{f(w)}{2\pi i w^n}dw \\ -h_{n-1}^{-1}z^{n-1}\widehat{\phi}_{n-1}(z^{-1}) & \displaystyle -h_{n-1}^{-1}\int_{S^1}\frac{\widehat{\phi}_{n-1}(w^{-1})}{w-z}\frac{f(w)}{2\pi i w}dw \end{pmatrix}, \tag{2.2.5}$$

For our symbol given by (2.1.2), this formula reduces to

$$\partial_s \log D_n(s,L,W) = \frac{1}{2\pi i}\int_{\gamma^c} z^{-n}\left[Y^{-1}(z)Y'(z)\right]_{21} e^{W(z)}dz, \tag{2.2.6}$$

where $\gamma^c = S^1 \setminus \gamma$.

2.2.2 Differential identity for $W=0$

If $W=0$, the integral at the right hand side of (2.2.6) can be simplified. Although (2.2.6) will be sufficient for the proof of Theorem 2.1.1, we will show here how to simplify the differential identity if $W=0$. This will allow us to give a more elegant proof of Theorem 2.1.1 in this case.

Proposition 2.2.1 *Let $D_n(s,L,0)$ be the Toeplitz determinant with symbol (2.1.2) in the case where $W(e^{i\theta})=0$. We have the following differential identity for $\log D_n$:*

$$\partial_s \log D_n(s,L,0) = n\frac{\partial_s h_n}{h_n} + \frac{2(1-s)}{\pi}\mathrm{Im}\left(h_n^{-1/2}\overline{\phi_n(e^{iL})}\partial_s\left(h_n^{-1/2}\phi_n(e^{iL})\right)\right). \tag{2.2.7}$$

Proof. Taking the logarithm of both sides in (2.2.4), and then differentiating with respect to s, we get

$$\partial_s \log D_n(s,L,0) = \sum_{j=0}^{n-1} \frac{\partial_s h_j}{h_j}.$$

On the other hand, by (2.2.1),

$$\frac{1}{2\pi}\int_0^{2\pi} \partial_s \left(h_j^{-1}\phi_j(e^{i\theta})\overline{\phi_j(e^{i\theta})} \right) f(e^{i\theta})d\theta = -\frac{\partial_s h_j}{h_j},$$

and this gives

$$\partial_s \log D_n(s,L,0) = -\frac{1}{2\pi}\int_0^{2\pi} \partial_s \left[\sum_{j=0}^{n-1} h_j^{-1}\phi_j(e^{i\theta})\overline{\phi_j(e^{i\theta})} \right] f(e^{i\theta})d\theta. \quad (2.2.8)$$

Here we can use the Christoffel-Darboux formula (see e.g. [58, 24] for a proof of it):

$$\begin{aligned} &\sum_{j=0}^{n-1} h_j^{-1}\phi_j(z)\overline{\phi_j}(z^{-1}) = \\ &\qquad - nh_n^{-1}\phi_n(z)\overline{\phi_n}(z^{-1}) + zh_n^{-1}\left(\overline{\phi_n}(z^{-1})\phi_n'(z) - \left(\overline{\phi_n}(z^{-1})\right)'\phi_n(z)\right). \end{aligned}$$

Substituting this into (2.2.8), we obtain

$$\begin{aligned} &\partial_s \log D_n(s,L,0) = -n\frac{\partial_s h_n}{h_n} \\ &- \frac{1}{2\pi}\int_0^{2\pi} \partial_s \left[e^{i\theta}h_n^{-1}\left(\overline{\phi_n}(e^{-i\theta})\phi_n'(e^{i\theta}) - \left(\overline{\phi_n}(z^{-1})\right)'\Big|_{z=e^{i\theta}}\phi_n(e^{i\theta})\right)\right] f(e^{i\theta})d\theta \\ &\qquad = -n\frac{\partial_s h_n}{h_n} + I_1 + I_2 + I_3 + I_4, \qquad (2.2.9) \end{aligned}$$

where

$$I_1 = -\frac{1}{2\pi}\int_0^{2\pi} e^{i\theta}\partial_s\left(h_n^{-1/2}\overline{\phi_n}(e^{-i\theta})\right)h_n^{-1/2}\phi_n'(e^{i\theta})f(e^{i\theta})d\theta, \tag{2.2.10}$$

$$I_2 = -\frac{1}{2\pi}\int_0^{2\pi} e^{i\theta}h_n^{-1/2}\overline{\phi_n}(e^{-i\theta})\partial_s\left(h_n^{-1/2}\phi_n'(e^{i\theta})\right)f(e^{i\theta})d\theta, \tag{2.2.11}$$

$$I_3 = \frac{1}{2\pi}\int_0^{2\pi} e^{i\theta}\partial_s\left(h_n^{-1/2}\left.\left(\overline{\phi_n}(z^{-1})\right)'\right|_{z=e^{i\theta}}\right)h_n^{-1/2}\phi_n(e^{i\theta})f(e^{i\theta})d\theta, \tag{2.2.12}$$

$$I_4 = \frac{1}{2\pi}\int_0^{2\pi} e^{i\theta}h_n^{-1/2}\left.\left(\overline{\phi_n}(z^{-1})\right)'\right|_{z=e^{i\theta}}\partial_s\left(h_n^{-1/2}\phi_n(e^{i\theta})\right)f(e^{i\theta})d\theta. \tag{2.2.13}$$

From the orthogonality relation (2.2.1), we easily get

$$I_2 = I_3 = \frac{n}{2}\frac{\partial_s h_n}{h_n}. \tag{2.2.14}$$

The computation of I_1 and I_4 is slightly more involved. Using (2.1.2), we have

$$\begin{aligned} I_1 = &-\frac{1}{2\pi i}\int_0^{2\pi}\partial_s\left(h_n^{-1/2}\overline{\phi_n}(e^{-i\theta})\right)\frac{d}{d\theta}\left(h_n^{-1/2}\phi_n(e^{i\theta})\right)d\theta \\ &+\frac{1-s}{2\pi i}\int_L^{2\pi-L}\partial_s\left(h_n^{-1/2}\overline{\phi_n}(e^{-i\theta})\right)\frac{d}{d\theta}\left(h_n^{-1/2}\phi_n(e^{i\theta})\right)d\theta. \end{aligned} \tag{2.2.15}$$

Integrating by parts and then using orthogonality, we obtain

$$\begin{aligned} I_1 &= \frac{1-s}{2\pi i}\left[\partial_s\left(h_n^{-1/2}\overline{\phi_n}(e^{-i\theta})\right)h_n^{-1/2}\phi_n(e^{i\theta})\right]_L^{2\pi-L} \\ &\quad+\frac{1}{2\pi i}\int_0^{2\pi}h_n^{-1/2}\phi_n(e^{i\theta})\partial_s\frac{d}{d\theta}\left(h_n^{-1/2}\overline{\phi_n}(e^{-i\theta})\right)f(e^{i\theta})d\theta \\ &= \frac{1-s}{2\pi i}\left[\partial_s\left(h_n^{-1/2}\overline{\phi_n}(e^{-i\theta})\right)h_n^{-1/2}\phi_n(e^{i\theta})\right]_L^{2\pi-L}+\frac{n}{2}\frac{\partial_s h_n}{h_n}. \end{aligned} \tag{2.2.16}$$

In the same way, we show that

$$\begin{aligned} I_4 &= \frac{1}{2\pi}\int_0^{2\pi}e^{i\theta}\left.\left(h_n^{-1/2}\overline{\phi_n}(z^{-1})\right)'\right|_{z=e^{i\theta}}\partial_s\left(h_n^{-1/2}\phi_n(e^{i\theta})\right)f(e^{i\theta})d\theta \\ &= -\frac{1-s}{2\pi i}\left[h_n^{-1/2}\overline{\phi_n}(e^{-i\theta})\partial_s\left(h_n^{-1/2}\phi_n(e^{i\theta})\right)\right]_L^{2\pi-L}+\frac{n}{2}\frac{\partial_s h_n}{h_n}. \end{aligned} \tag{2.2.17}$$

Summing up (2.2.14), (2.2.16), and (2.2.17), and using the fact that $\overline{\phi_n} = \phi_n$ if $W = 0$, we get the result. □

As a consequence of Proposition 2.2.1 and (2.2.5), we can express the right hand side of (2.2.7) in terms of $Y = Y^{(n)}$ given by (2.2.5):

Corollary 2.2.2 *We have*

$$\partial_s \log D_n(s, L, 0) = n\partial_s \log Y_{12}(0) + \frac{2(1-s)}{\pi} \mathrm{Im}\left(\frac{\overline{Y_{11}(e^{iL})}}{\sqrt{Y_{12}(0)}} \partial_s \left(\frac{Y_{11}(e^{iL})}{\sqrt{Y_{12}(0)}} \right) \right). \tag{2.2.18}$$

Proof. By (3.3.1), we have

$$Y_{12}(0) = h_n, \qquad Y_{11}(e^{iL}) = \phi_n(e^{iL}).$$

Substituting this into (2.2.7), we get (2.2.18). □

Integrating both sides from 0 to s_0 in (2.2.6) or (2.2.7), we obtain

$$\log D_n(s_0, L, W) - \log D_n(0, L, W) = \int_0^{s_0} [\partial_s \log D_n(s, L, W)]\, ds. \tag{2.2.19}$$

In order to obtain asymptotics for $\log D_n(s_0, L, W)$, we need large n asymptotics for the right-hand side of (2.2.6) and (2.2.7) uniformly for $0 \le s \le s_0$.

To obtain large n asymptotics for Y, we will use the following RH characterization [34]: $Y(z) = Y^{(n)}(z)$ is the unique 2×2 matrix-valued function which satisfies the following properties.

RH problem for Y

(a) $Y : \mathbb{C} \setminus S^1 \to \mathbb{C}^{2\times 2}$ is analytic.

(b) Y has the following jumps:

$$Y_+(z) = Y_-(z) \begin{pmatrix} 1 & z^{-n} f(z) \\ 0 & 1 \end{pmatrix}, \qquad \text{for } z \in S^1 \setminus \left\{ z_0 = e^{iL}, \overline{z_0} = e^{-iL} \right\},$$

where $Y_+(z)$ ($Y_-(z)$) denotes the limit as z is approached from the inside (outside) of the unit circle.

(c) $Y(z) = \left(I + \mathcal{O}(z^{-1})\right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$ as $z \to \infty$.

(d) As z tends to z_0 or z tends to $\overline{z_0}$, Y behaves as

$$Y(z) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\log|z-z_0|) \\ \mathcal{O}(1) & \mathcal{O}(\log|z-z_0|) \end{pmatrix} \qquad \text{as } z \to z_0,$$

$$Y(z) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\log|z-\overline{z_0}|) \\ \mathcal{O}(1) & \mathcal{O}(\log|z-\overline{z_0}|) \end{pmatrix} \qquad \text{as } z \to \overline{z_0}.$$

2.3 Equilibrium measures

It will be convenient to introduce a new parameter $x \in \mathbb{R}^+ \cup \{+\infty\}$ defined by

$$s = e^{-xn}, \tag{2.3.1}$$

such that f can be written as $f(e^{i\theta}) = e^{W(e^{i\theta})}e^{-nV(e^{i\theta})}$, where

$$V(e^{i\theta}) = \begin{cases} 0, & \text{for } e^{i\theta} \in \gamma, \\ x, & \text{for } e^{i\theta} \in S^1 \setminus \gamma. \end{cases} \tag{2.3.2}$$

An important ingredient for the large n analysis of the orthogonal polynomials and the RH problem for Y is the following minimization problem: find the measure $\mu = \mu^{(x)}$ which minimizes

$$\iint \log|z-s|^{-1} d\mu(z) d\mu(s) + \int V(z) d\mu(z) \tag{2.3.3}$$

among all Borel probability measures on the unit circle S^1.

This measure, which is unique and absolutely continuous with respect to the Lebesgue measure, is called the equilibrium measure, is denoted $d\mu^{(x)} = u^{(x)}(e^{i\theta})d\theta$, and its support is denoted $J^{(x)}$. The equilibrium measure $\mu = \mu^{(x)}$ and its support $J = J^{(x)}$ are uniquely determined by the following Euler-Lagrange variational conditions [6, 57]: there exists a real constant $\ell = \ell^{(x)}$ such that

$$2\int_{-\pi}^{\pi} \log|z-e^{i\theta}| d\mu(e^{i\theta}) - V(z) + \ell = 0, \qquad \text{for } z \in J, \tag{2.3.4}$$

$$2\int_{-\pi}^{\pi} \log|z-e^{i\theta}| d\mu(e^{i\theta}) - V(z) + \ell \leq 0, \qquad \text{for } z \in S^1 \setminus J. \tag{2.3.5}$$

The support and density of the equilibrium measure can be computed explicitly.

Proposition 2.3.1 *(a) For $x = +\infty$, the support of the equilibrium measure and its density are given by*

$$J = J^{(\infty)} = \gamma, \qquad u(e^{i\theta}) = u^{(\infty)}(e^{i\theta}) = \frac{1}{2\pi}\sqrt{\frac{\cos\theta + 1}{\cos\theta - \cos L}}. \tag{2.3.6}$$

The constant $\ell = \ell^{(\infty)}$ in the variational conditions (2.3.4)-(2.3.5) is given by

$$\ell^{(\infty)} = -2\log\sin\frac{L}{2}, \tag{2.3.7}$$

and the variational inequality (2.3.5) is strict for $z \in S^1 \setminus \gamma$.

(b) Let x_c be given by (2.1.7). For $x \geq x_c$, we have the same result as for $x = +\infty$:

$$J = \gamma\ , \qquad u(e^{i\theta}) = u^{(\infty)}(e^{i\theta}), \qquad \ell = \ell^{(\infty)}. \tag{2.3.8}$$

Moreover, the variational inequality (2.3.5) is strict for $z \in S^1 \setminus \gamma$ if $x \neq x_c$; if $x = x_c$, it is strict for $z \in S^1 \setminus (\gamma \cup \{-1\})$, and there is equality for $z = -1$.

(c) For $0 < x < x_c$, the support of μ consists of two disjoint arcs: we have

$$J = \{e^{i\theta} : \theta \in [-L, L] \cup [\pi - T, \pi + T]\}, \tag{2.3.9}$$

where $T = T(x) \in (0, \pi - L)$ is the unique solution of

$$2\int_{[-L,L]\cup[\pi-T,\pi+T]} \log\left|\frac{1+e^{i\theta}}{1-e^{i\theta}}\right| u(e^{i\theta})d\theta = x, \tag{2.3.10}$$

and the density is given by

$$u(e^{i\theta}) = \frac{1}{2\pi}\sqrt{\frac{\cos\theta + \cos T}{\cos\theta - \cos L}}. \tag{2.3.11}$$

The constant ℓ is given by

$$\ell = -\int_0^1 \frac{1}{s}\left(1 - \sqrt{\frac{s^2 + 2\cos(T)s + 1}{s^2 - 2\cos(L)s + 1}}\right) ds, \tag{2.3.12}$$

and the variational inequality (2.3.5) is strict for $z \in S^1 \setminus J$.

Remark 2.3.2 Although part (c) of the proposition is not needed for the proof of Theorem 2.1.1, we present it here for completeness and will be useful in Chapter 3, where we will discuss asymptotics for D_n if $x < x_c$.

Proof. Note first that the equation (2.3.10) has indeed a unique solution T for $x < x_c$, since the function

$$T \mapsto 2 \int_{[-L,L]\cup[\pi-T,\pi+T]} \log \left| \frac{1+e^{i\theta}}{1-e^{i\theta}} \right| u(e^{i\theta}) d\theta$$

decreases as a function of $T \in (0, \pi - L)$, and is bijective from $(0, \pi - L)$ to $(0, x_c)$.

The equilibrium measure μ is uniquely characterized by the conditions (2.3.4)-(2.3.5), which means that it is sufficient for us to show that the measure μ defined in cases (a), (b), and (c) satisfy these variational conditions. To do so, consider the function

$$f(z) = 2 \int \log |z - e^{i\theta}| d\mu^{(x)}(e^{i\theta}). \tag{2.3.13}$$

If we define $T > 0$ as the unique solution of (2.3.10) if $x < x_c$, and if we let $T = 0$ for $x \geq x_c$, we have

$$f(e^{i\alpha}) = \frac{1}{\pi} \int_{[-L,L]\cup[\pi-T,\pi+T]} \log |e^{i\alpha} - e^{i\theta}| \sqrt{\frac{\cos\theta + \cos T}{\cos\theta - \cos L}} d\theta. \tag{2.3.14}$$

The derivative $\frac{d}{d\alpha} f(e^{i\alpha})$ can be written as a contour integral

$$\frac{d}{d\alpha} f(e^{i\alpha}) = \frac{1}{2i} \frac{1}{2\pi i} \int_\Sigma \frac{1}{\xi} \frac{\xi + e^{i\alpha}}{\xi - e^{i\alpha}} \left(\frac{(\xi - z_1)(\xi - \overline{z_1})}{(\xi - z_0)(\xi - \overline{z_0})} \right)^{1/2} d\xi, \qquad z_1 = e^{i(\pi - T)}, \tag{2.3.15}$$

where the square root is analytic off J and tends to 1 as $\xi \to \infty$, and where the contour Σ consists of one (if $x \geq x_c$) or two (if $x < x_c$) counterclockwise oriented circles around the arc(s) of J. If $e^{i\alpha} \in S^1 \setminus J$, the contour has to be chosen sufficiently small such that $e^{i\alpha}$ lies in the exterior of Σ.

If $e^{i\alpha} \in J$, a residue calculation shows that $\frac{d}{d\alpha} f(e^{i\alpha}) = 0$. If $e^{i\alpha} \in S^1 \setminus J$ on the other hand, we have

$$\frac{d}{d\alpha} f(e^{i\alpha}) = \begin{cases} \sqrt{\frac{\cos T + \cos\alpha}{\cos L - \cos\alpha}}, & \text{if } L < \alpha < \pi - T, \\ -\sqrt{\frac{\cos T + \cos\alpha}{\cos L - \cos\alpha}}, & \text{if } \pi + T < \alpha < 2\pi - L. \end{cases} \tag{2.3.16}$$

It follows that $f(e^{i\alpha})$ is constant on $[-L, L] \cup [\pi - T, \pi + T]$, and that it achieves its maximum on $[\pi - T, \pi + T]$ (i.e. at π if $x \geq x_c$). We have

$$f(e^{i\alpha}) = f(1), \qquad \text{for } -L \leq \alpha \leq L, \tag{2.3.17}$$
$$f(e^{i\alpha}) = f(-1), \qquad \text{for } \pi - T \leq \alpha \leq \pi + T, \tag{2.3.18}$$
$$f(e^{i\alpha}) < f(-1), \qquad \text{for } \alpha \notin [\pi - T, \pi + T]. \tag{2.3.19}$$

If we show that

$$f(1) = -\ell,\ f(-1) < f(1) + x, \qquad \text{for } x > x_c, \tag{2.3.20}$$
$$f(1) = -\ell,\ f(-1) = f(1) + x, \qquad \text{for } x \leq x_c, \tag{2.3.21}$$

then (2.3.17)-(2.3.19) imply the Euler-Lagrange conditions (2.3.4)-(2.3.5).

For the case $x \geq x_c$, using (2.3.16), we obtain after a straightforward calculation,

$$f(-1) - f(1) = f(e^{i\pi}) - f(e^{iL}) = \int_L^\pi \sqrt{\frac{1 + \cos\theta}{\cos L - \cos\theta}} d\theta = -2 \log\tan\frac{L}{4} = x_c \leq x,$$

which proves the inequality in (2.3.20), and the fact that there is equality at -1 if $x = x_c$. Moreover, using another residue calculation, we get

$$f(1) = \int_0^1 f'(s) ds = \int_0^1 \frac{1}{s}\left(1 - \frac{1+s}{\sqrt{s^2 - 2\cos(L)s + 1}}\right) ds = 2\log\sin\frac{L}{2} = -\ell^{(\infty)}.$$

This proves the equality in (2.3.20). For $0 < x < x_c$, by (2.3.16),

$$f(-1) - f(1) = f(e^{i(\pi - T)}) - f(e^{iL}) = \int_L^{\pi - T} \sqrt{\frac{\cos T + \cos\theta}{\cos L - \cos\theta}} d\theta = x,$$

by definition of f and T, and

$$f(1) = \int_0^1 \frac{1}{s}\left(1 - \sqrt{\frac{s^2 + 2\cos(T)s + 1}{s^2 - 2\cos(L)s + 1}}\right) ds = -\ell.$$

This completes the proof. $\square$

2.4 Riemann-Hilbert analysis for $x \geq x_c$

We will now perform an asymptotic analysis of the RH problem for Y as $n \to \infty$ with $x \geq x_c$. Our analysis uses the Deift/Zhou steepest descent method [23, 27, 28] and shows many similarities with the analysis done in [46] for $s = 0$ (or $x = +\infty$). An important difference is that we need to modify the Bessel parametrices around the points $z_0, \overline{z_0}$, see Section 2.4.4 below.

2.4.1 First transformation $Y \mapsto T$

We define g by

$$g(z) = \int_{-L}^{L} \log(z - e^{i\theta}) d\mu^{(\infty)}(e^{i\theta}). \tag{2.4.1}$$

Below we write $\mu = \mu^{(\infty)}$, $u = u^{(\infty)}$, and $\ell = \ell^{(\infty)}$. We have that e^g is analytic for $z \in \mathbb{C} \setminus \gamma$, and by the Euler-Lagrange variational conditions (2.3.4)-(2.3.5), we also have

$$g_+(z) + g_-(z) - \log(z) - i\pi + \ell = 0, \qquad \text{for } z \in \gamma \setminus \{z_0, \overline{z_0}\}, \tag{2.4.2}$$

$$2g(z) - \log(z) - i\pi + \ell - x < 0, \qquad \text{for } z \in S^1 \setminus \gamma, \tag{2.4.3}$$

$$g_+'(z) - g_-'(z) = -\frac{2\pi}{z} u(z), \qquad \text{for } z \in \gamma \setminus \{z_0, \overline{z_0}\}. \tag{2.4.4}$$

Define

$$T(z) = e^{\frac{n\pi i}{2}\sigma_3} e^{\frac{n\ell}{2}\sigma_3} Y(z) e^{-ng(z)\sigma_3} e^{-\frac{n\ell}{2}\sigma_3} e^{-\frac{n\pi i}{2}\sigma_3}. \tag{2.4.5}$$

Then it is straightforward to check that T satisfies the following conditions.

RH problem for T

(a) $T : \mathbb{C} \setminus S^1 \to \mathbb{C}^{2\times 2}$ is analytic.

(b) T satisfies the jump relation

$$T_+(z) = T_-(z) J_T(z), \qquad \text{on } S^1 \setminus \{z_0, \overline{z_0}\}, \tag{2.4.6}$$

with

$$J_T(z) = \begin{pmatrix} e^{-n(g_+(z)-g_-(z))} & (-1)^n z^{-n} e^{-nV(z)} e^{n\ell} e^{n(g_+(z)+g_-(z))} e^{W(z)} \\ 0 & e^{n(g_+(z)-g_-(z))} \end{pmatrix}.$$

(c) $T(z) = I + \mathcal{O}(z^{-1})$ as $z \to \infty$.

(d) As $z \to z_0 = e^{iL}$ or $z \to \overline{z_0} = e^{-iL}$, we have

$$T(z) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\log|z-z_0|) \\ \mathcal{O}(1) & \mathcal{O}(\log|z-z_0|) \end{pmatrix}, \qquad \text{as } z \to z_0,$$

$$T(z) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\log|z-\overline{z_0}|) \\ \mathcal{O}(1) & \mathcal{O}(\log|z-\overline{z_0}|) \end{pmatrix}, \qquad \text{as } z \to \overline{z_0}.$$

We define

$$\phi(z) = 2g(z) - \log z - i\pi + \ell, \tag{2.4.7}$$

so that e^{ϕ} is analytic in a neighborhood of γ with the exception of γ itself. For $z \in \gamma \setminus \{z_0, \overline{z_0}\}$, we have

$$\begin{aligned} \phi_+(z) &= (g_+(z) + g_-(z) - \log z - i\pi + \ell) + (g_+(z) - g_-(z)) &(2.4.8)\\ &= g_+(z) - g_-(z), &(2.4.9) \end{aligned}$$

by (2.4.2). But $g_+(z_0) = g_-(z_0)$, and integrating (2.4.4) between z_0 and z, and then substituting it into (2.4.8), we obtain

$$\phi_+(z) = -2\pi \int_{z_0}^{z} \frac{u(\xi)}{\xi} d\xi. \tag{2.4.10}$$

Analytically continuing the left and right hand side, we obtain

$$\phi(z) = \int_{z_0}^{z} \frac{\xi + 1}{((\xi - z_0)(\xi - \overline{z_0}))^{1/2}} \frac{d\xi}{\xi}, \tag{2.4.11}$$

where the branch cut of the square root is chosen on γ. It is now convenient to express J_T in terms of ϕ:

$$J_T(z) = \begin{cases} \begin{pmatrix} e^{-n\phi_+(z)} & e^{W(z)} \\ 0 & e^{-n\phi_-(z)} \end{pmatrix}, & z \in \gamma \setminus \{z_0, \overline{z_0}\}, \\ \begin{pmatrix} 1 & e^{-nx} e^{n\phi(z)} e^{W(z)} \\ 0 & 1 \end{pmatrix}, & z \in S^1 \setminus \gamma. \end{cases}$$

2.4.2 Second transformation $T \mapsto S$

We can factorize J_T on γ as follows:

$$\begin{pmatrix} e^{-n\phi_+(z)} & e^{W(z)} \\ 0 & e^{-n\phi_-(z)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ e^{-n\phi_-(z)} e^{-W(z)} & 1 \end{pmatrix} \times \begin{pmatrix} 0 & e^{W(z)} \\ -e^{-W(z)} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{-n\phi_+(z)} e^{-W(z)} & 1 \end{pmatrix}. \tag{2.4.12}$$

Using this factorization, we can split the jump on γ into three different jumps on a lens-shaped oriented contour, see Figure 2.2. It is important that the jump contour lies in the region where W is analytic. Denote by γ_+ and γ_- the lenses around γ on the $|z| < 1$ side and the $|z| > 1$ side respectively. Define

$$S(z) = \begin{cases} T(z) \begin{pmatrix} 1 & 0 \\ -e^{-n\phi(z)} e^{-W(z)} & 1 \end{pmatrix}, & |z| < 1, z \text{ inside the lenses around } \gamma, \\ T(z) \begin{pmatrix} 1 & 0 \\ e^{-n\phi(z)} e^{-W(z)} & 1 \end{pmatrix}, & |z| > 1, z \text{ inside the lenses around } \gamma, \\ T(z), & z \text{ outside the lenses.} \end{cases} \tag{2.4.13}$$

Then S solves the following RH problem.

RH problem for S

(a) $S : \mathbb{C} \setminus (S^1 \cup \gamma_- \cup \gamma_+) \to \mathbb{C}^{2\times 2}$ is analytic.

(b) S satisfies the jump relations

$$S_+(z) = S_-(z) J_S(z), \qquad \text{for } z \in S^1 \cup \gamma_- \cup \gamma_+, \tag{2.4.14}$$

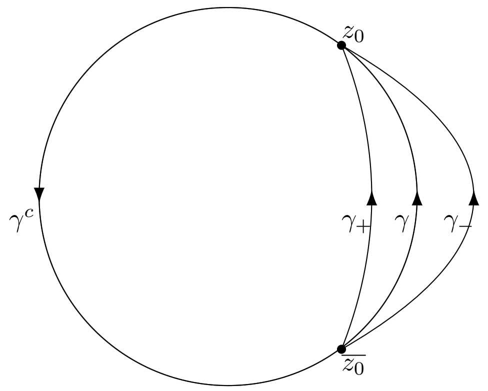

Figure 2.2: The jump contour for S.

where S_+ (S_-) denotes the boundary value from the left (right) of the contour, and where

$$J_S(z) = \begin{cases} \begin{pmatrix} 1 & e^{-nx}e^{n\phi(z)}e^{W(z)} \\ 0 & 1 \end{pmatrix}, & \text{for } z \in S^1 \setminus \gamma, \\ \begin{pmatrix} 0 & e^{W(z)} \\ -e^{-W(z)} & 0 \end{pmatrix}, & \text{for } z \in \gamma \setminus \{z_0, \overline{z_0}\}, \\ \begin{pmatrix} 1 & 0 \\ e^{-n\phi(z)}e^{-W(z)} & 1 \end{pmatrix}, & \text{for } z \in \gamma_+ \cup \gamma_-. \end{cases} \tag{2.4.15}$$

(c) $S(z) = I + \mathcal{O}(z^{-1})$ as $z \to \infty$.

(d) As $z \to z_0 = e^{iL}$ or $z \to \overline{z_0} = e^{-iL}$, we have

$$S(z) = \mathcal{O}(\log|z - z_0|), \qquad S(z) = \mathcal{O}(\log|z - \overline{z_0}|). \tag{2.4.16}$$

By (2.4.7) and (2.4.3), we observe that the jump matrices for S converge to the identity matrix on $S^1 \setminus \gamma$ as $n \to \infty$, except at -1 if $x = x_c$. On $\gamma_+ \cup \gamma_-$, one shows that $\operatorname{Re}\phi(z) > 0$ and consequently the jump matrices for S also converge to the identity matrix on $\gamma_+ \cup \gamma_-$. The convergence of the jump matrices is pointwise in z and breaks down as z approaches $z_0, \overline{z_0}$, and also as z approaches -1 if $x = x_c$. Therefore, we will need to construct approximations to S for large n in different regions of the complex plane: local parametrices will be constructed in small disks $D(z_0,r), D(\overline{z_0},r), D(-1,r)$ surrounding $z_0, \overline{z_0}, -1$, and a global parametrix will be constructed in $\mathbb{C}\setminus\overline{(D(z_0,r)\cup D(\overline{z_0},r)\cup D(-1,r))}$.

2.4.3 Global parametrix

Ignoring the exponentially small jumps for S and small neighborhoods of $z_0, \overline{z_0}$, -1, we are led to the following RH problem:

RH problem for $P^{(\infty)}$

(a) $P^{(\infty)}: \mathbb{C}\setminus\gamma \to \mathbb{C}^{2\times 2}$ is analytic.

(b) $P^{(\infty)}$ has the jump

$$P_+^{(\infty)}(z) = P_-^{(\infty)}(z)\begin{pmatrix} 0 & e^{W(z)} \\ -e^{-W(z)} & 0 \end{pmatrix}, \qquad z\in\gamma\setminus\{z_0,\overline{z_0}\}. \tag{2.4.17}$$

(c) $P^{(\infty)}(z) = I + \mathcal{O}(z^{-1})$ as $z\to\infty$.

(d) As $z\to z_0$ or $z\to\overline{z_0}$, we have

$$P^{(\infty)}(z) = \mathcal{O}(|z-z_0|^{-1/4}), \qquad P^{(\infty)}(z) = \mathcal{O}(|z-\overline{z_0}|^{-1/4}). \tag{2.4.18}$$

The solution of this RH problem is explicitly given by

$$P^{(\infty)}(z) = e^{h_\infty\sigma_3}\begin{pmatrix} \frac{1}{2}\left(\beta(z)+\beta^{-1}(z)\right) & -\frac{1}{2i}\left(\beta(z)-\beta^{-1}(z)\right) \\ \frac{1}{2i}\left(\beta(z)-\beta^{-1}(z)\right) & \frac{1}{2}\left(\beta(z)+\beta^{-1}(z)\right) \end{pmatrix} e^{-h(z)\sigma_3}, \tag{2.4.19}$$

where $\beta(z) = \left(\frac{z-\overline{z_0}}{z-z_0}\right)^{1/4}$ is analytic in $\mathbb{C}\setminus\gamma$ and $\beta(z)\to 1$ as $z\to\infty$, where $h_\infty = \lim_{z\to\infty} h(z)$, and

$$h(z) = \frac{((z-z_0)(z-\overline{z_0}))^{1/2}}{2\pi i}\int_\gamma \frac{W(\xi)}{((\xi-z_0)(\xi-\overline{z_0}))_+^{1/2}}\frac{1}{\xi-z}d\xi, \tag{2.4.20}$$

with $((z-z_0)(z-\overline{z_0}))^{1/2}$ analytic off γ. Note that h is analytic in $\mathbb{C}\setminus\gamma$, bounded near z_0, $\overline{z_0}$, and ∞, and that it satisfies the jump relation

$$h_+(z) + h_-(z) = W(z), \qquad z\in\gamma\setminus\{z_0,\overline{z_0}\}.$$

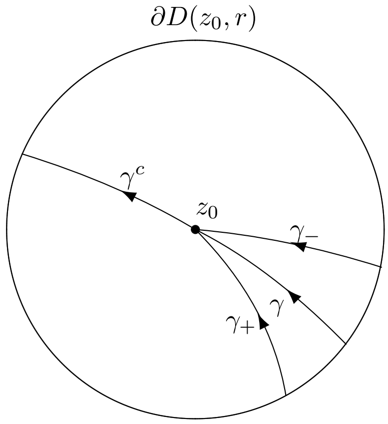

Figure 2.3: The jump contour for P.

Using these properties, it is straightforward to verify that $P^{(\infty)}$ solves the above RH problem.

2.4.4 Local parametrix near z_0

We want to construct a function P defined in $D(z_0, r)$ satisfying the following RH conditions.

RH problem for P

(a) $P : D(z_0, r) \setminus (S^1 \cup \gamma_+ \cup \gamma_-) \to \mathbb{C}^{2\times 2}$ is analytic.

(b) For z on the contour shown in Figure 2.3, P satisfies the jump conditions

$$\begin{aligned} P_+(z) &= P_-(z) \begin{pmatrix} 0 & e^{W(z)} \\ -e^{-W(z)} & 0 \end{pmatrix}, && \text{on } \gamma \setminus \{z_0, \overline{z_0}\}, \\ P_+(z) &= P_-(z) \begin{pmatrix} 1 & e^{n(\phi(z)-x)}e^{W(z)} \\ 0 & 1 \end{pmatrix}, && \text{on } \gamma^c, \\ P_+(z) &= P_-(z) \begin{pmatrix} 1 & 0 \\ e^{-n\phi(z)}e^{-W(z)} & 1 \end{pmatrix}, && \text{on } \gamma_- \cup \gamma_+. \end{aligned} \tag{2.4.21}$$

(c) For $z \in \partial D(z_0, r)$, we have

$$P(z) = \left(I + \mathcal{O}(n^{-1})\right) P^{(\infty)}(z), \qquad \text{as } n \to \infty. \tag{2.4.22}$$

(d) As z tend to z_0, the behaviour of P is

$$P(z) = \mathcal{O}(\log|z - z_0|). \tag{2.4.23}$$

Bessel model RH problem

We will construct P in terms of a model RH problem for which the solution is constructed using Bessel functions. Consider the following model RH problem:

RH problem for Ψ

(a) $\Psi : \mathbb{C} \setminus \Sigma_\Psi \to \mathbb{C}^{2\times 2}$ is analytic, where $\Sigma_\Psi = \mathbb{R}^- \cup \{xe^{\pm\frac{2\pi}{3}i} : x \in \mathbb{R}^+\}$, with the orientation from ∞ towards 0 for the three half-lines.

(b) Ψ satisfies the jump conditions

$$\begin{aligned}
\Psi_+(\zeta) &= \Psi_-(\zeta)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad && \zeta \in \mathbb{R}^-, \\
\Psi_+(\zeta) &= \Psi_-(\zeta)\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, && \zeta \in \{xe^{\frac{2\pi}{3}i} : x \in \mathbb{R}^+\}, \\
\Psi_+(\zeta) &= \Psi_-(\zeta)\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, && \zeta \in \{xe^{-\frac{2\pi}{3}i} : x \in \mathbb{R}^+\}.
\end{aligned}$$

(c) As $\zeta \to \infty$, $\zeta \notin \Sigma_\Psi$, we have

$$\Psi(\zeta) = \left(2\pi\zeta^{\frac{1}{2}}\right)^{-\frac{\sigma_3}{2}} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\left(I + \mathcal{O}(\zeta^{-\frac{1}{2}})\right) e^{2\zeta^{\frac{1}{2}}\sigma_3}. \tag{2.4.24}$$

(d) As ζ tend to 0, the behaviour of $\Psi(\zeta)$ is

$$\Psi(\zeta) = \mathcal{O}(\log|\zeta|). \tag{2.4.25}$$

This model RH problem is well-known and it can be solved explicitly using Bessel functions, see e.g. [49, 46]. The unique solution to this RH problem is given by

$$\Psi(\zeta) = \begin{cases} \begin{pmatrix} I_0(2\zeta^{\frac{1}{2}}) & \frac{i}{\pi} K_0(2\zeta^{\frac{1}{2}}) \\ 2\pi i \zeta^{\frac{1}{2}} I_0'(2\zeta^{\frac{1}{2}}) & -2\zeta^{\frac{1}{2}} K_0'(2\zeta^{\frac{1}{2}}) \end{pmatrix}, & |\arg\zeta| < \frac{2\pi}{3}, \\ \begin{pmatrix} \frac{1}{2} H_0^{(1)}(2(-\zeta)^{\frac{1}{2}}) & \frac{1}{2} H_0^{(2)}(2(-\zeta)^{\frac{1}{2}}) \\ \pi\zeta^{\frac{1}{2}} \left(H_0^{(1)}\right)'(2(-\zeta)^{\frac{1}{2}}) & \pi\zeta^{\frac{1}{2}} \left(H_0^{(2)}\right)'(2(-\zeta)^{\frac{1}{2}}) \end{pmatrix}, & \frac{2\pi}{3} < \arg\zeta < \pi, \\ \begin{pmatrix} \frac{1}{2} H_0^{(2)}(2(-\zeta)^{\frac{1}{2}}) & -\frac{1}{2} H_0^{(1)}(2(-\zeta)^{\frac{1}{2}}) \\ -\pi\zeta^{\frac{1}{2}} \left(H_0^{(2)}\right)'(2(-\zeta)^{\frac{1}{2}}) & \pi\zeta^{\frac{1}{2}} \left(H_0^{(1)}\right)'(2(-\zeta)^{\frac{1}{2}}) \end{pmatrix}, & -\pi < \arg\zeta < -\frac{2\pi}{3}, \end{cases} \tag{2.4.26}$$

where $H_0^{(1)}$ and $H_0^{(2)}$ are the Hankel functions of the first and second kind, and I_0 and K_0 are the modified Bessel functions of the first and second kind.

Modification of the model RH problem

We will use a conformal map which maps the curves $\gamma_-, \gamma, \gamma_+$ in the vicinity of z_0 to (part of) the jump contour for Ψ. This is similar to the construction in [46], which corresponds to the case $s = 0$. If $s = 0$, P has no jump on γ_c (see Figure 2.3), and the model RH problem fits perfectly to construct the local parametrix P. In our situation however, P does have a jump on γ_c, and for this reason we need to modify the model RH problem. A different but similar construction was done in [15].

Define $\widehat{\Psi}$ by

$$\widehat{\Psi}(\zeta) = (I + A(\zeta))\,\Psi(\zeta), \tag{2.4.27}$$

where A is given by

$$A(\zeta) = e^{-nx} F(\zeta) \begin{pmatrix} 0 & -\frac{1}{2\pi i}\log(-\zeta) \\ 0 & 0 \end{pmatrix} F^{-1}(\zeta), \tag{2.4.28}$$

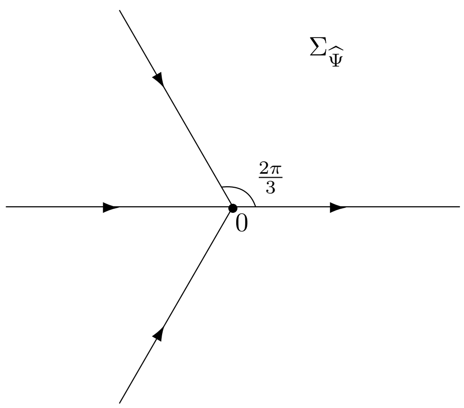

Figure 2.4: The jump contour for $\widehat{\Psi}$.

with F defined by

$$F(\zeta) = \begin{cases} \Psi(\zeta)\begin{pmatrix} 1 & -\frac{1}{2\pi i}\log\zeta \\ 0 & 1 \end{pmatrix} & |\arg\zeta| < \frac{2\pi}{3}, \\ \Psi(\zeta)\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -\frac{1}{2\pi i}\log\zeta \\ 0 & 1 \end{pmatrix} & \frac{2\pi}{3} < \arg\zeta < \pi, \\ \Psi(z)\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -\frac{1}{2\pi i}\log\zeta \\ 0 & 1 \end{pmatrix} & -\pi < \arg\zeta < -\frac{2\pi}{3}, \end{cases} \tag{2.4.29}$$

where in both (2.4.28) and (2.4.29) log has its branch cut on $\mathbb{R}^-$ with imaginary part between $-\pi$ and π. It is easy to check that F is an entire function. $\widehat{\Psi}$ is analytic in $\mathbb{C}\setminus\Sigma_{\widehat{\Psi}}$, with $\Sigma_{\widehat{\Psi}}$ as shown in Figure 2.4. On $\Sigma_{\widehat{\Psi}}$, it has the jump relations

$$\begin{aligned} \widehat{\Psi}_+(\zeta) &= \widehat{\Psi}_-(\zeta)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, && \text{on } \mathbb{R}^-, \\ \widehat{\Psi}_+(\zeta) &= \widehat{\Psi}_-(\zeta)\begin{pmatrix} 1 & e^{-nx} \\ 0 & 1 \end{pmatrix}, && \text{on } \mathbb{R}^+, \\ \widehat{\Psi}_+(\zeta) &= \widehat{\Psi}_-(\zeta)\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, && \text{on } \left\{xe^{\frac{2\pi}{3}i} : x\in\mathbb{R}^+\right\}, \\ \widehat{\Psi}_+(\zeta) &= \widehat{\Psi}_-(\zeta)\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, && \text{on } \left\{xe^{-\frac{2\pi}{3}i} : x\in\mathbb{R}^+\right\}. \end{aligned} \tag{2.4.30}$$

Construction of the local parametrix

We construct P as follows,

$$P(z) = E(z)\widehat{\Psi}(n^2\zeta(z))e^{-\frac{n}{2}\phi(z)\sigma_3}e^{-\frac{1}{2}W(z)\sigma_3}, \tag{2.4.31}$$

where E is an analytic function in $D(z_0, r)$, and where $\zeta(z) = \frac{1}{16}\phi(z)^2$. By (2.4.11), we have that $\zeta(z)$ is a conformal map near z_0, and that $\zeta(z_0) = 0$. Moreover, ζ maps $\gamma \cap D(z_0, r)$ to part of the real line. We now fix the lens-shaped contours γ_- and γ_+ by requiring that $\zeta(\gamma_- \cup \gamma_+) \subset \Sigma_{\widehat{\Psi}}$.

For any analytic function E, we have that P defined in (2.4.31) satisfies conditions (a), (b), and (d) of the RH problem for P. Indeed, P is analytic in $D(z_0, r) \setminus (\gamma \cup \gamma_+ \cup \gamma_- \cup \gamma^c)$ by construction, and by (2.4.31) and (2.4.30), it is straightforward to verify that the jump condition (2.4.21) holds. The logarithmic behavior (2.4.23) of P near z_0 follows from the logarithmic behavior (2.4.25) together with the definition (2.4.27)-(2.4.28) of $\widehat{\Psi}$. If we define $E(z)$ by

$$E(z) = P^{(\infty)}(z)e^{\frac{1}{2}W(z)\sigma_3}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\left(\frac{1}{2}n\pi\phi(z)\right)^{\frac{\sigma_3}{2}},$$

we have in addition the matching condition (2.4.22) for P. Using the jump relation for $P^{(\infty)}$, it is easily verified that E is analytic near z_0. This completes the construction of the local parametrix near z_0.

2.4.5 Local parametrix near $\overline{z_0}$

Once the local parametrix near z_0 constructed, the local parametrix near $\overline{z_0}$ is easy to construct. Define for $z \in D(\overline{z_0}, r)$, $P(z) = \overline{P(\overline{z})}$, where $\overline{P(\overline{z})}$ refers to the local parametrix constructed near z_0. Then, P satisfies the following RH conditions.

RH problem for P

(a) $P : D(\overline{z_0}, r) \setminus (S^1 \cup \gamma_+ \cup \gamma_-) \to \mathbb{C}^{2\times 2}$ is analytic.

(b) For $z \in D(\overline{z_0}, r)$ and z on the jump contour, P satisfies the jump conditions

$$\begin{aligned}
P_+(z) &= P_-(z) \begin{pmatrix} 0 & e^{W(z)} \\ -e^{-W(z)} & 0 \end{pmatrix}, && \text{on } \gamma \setminus \{z_0, \overline{z_0}\}, \\
P_+(z) &= P_-(z) \begin{pmatrix} 1 & e^{n(\phi(z)-x)} e^{W(z)} \\ 0 & 1 \end{pmatrix}, && \text{on } \gamma^c, \\
P_+(z) &= P_-(z) \begin{pmatrix} 1 & 0 \\ e^{-n\phi(z)} e^{-W(z)} & 1 \end{pmatrix}, && \text{on } \gamma_- \cup \gamma_+.
\end{aligned}$$

(c) For $z \in \partial D(\overline{z_0}, r)$, we have

$$P(z) = \left(I + \mathcal{O}(n^{-1})\right) P^{(\infty)}(z), \qquad \text{as } n \to \infty. \tag{2.4.32}$$

(d) As z tend to $\overline{z_0}$, the behaviour of P is

$$P(z) = \mathcal{O}(\log |z - \overline{z_0}|).$$

2.4.6 Local parametrix near -1

For $x > x_c + \delta$, the jump matrix for S converges exponentially fast to the identity matrix as $n \to \infty$ near -1. However, as x approaches x_c, the convergence becomes slower, and for $x = x_c$, the jump matrix for S does not converge to I any longer. Therefore, we need to construct a local parametrix near -1. This construction can be done for any $x \geq x_c$ but is only necessary when x is close to x_c. The local parametrix should satisfy the following conditions.

RH problem for P

(a) $P : D(-1, r) \setminus S^1 \to \mathbb{C}^{2\times 2}$ is analytic.

(b) P has the jump

$$P_+(z) = P_-(z) \begin{pmatrix} 1 & e^{n(\phi(z)-x)} e^{W(z)} \\ 0 & 1 \end{pmatrix}, \qquad z \in S^1 \cap D(-1, r). \tag{2.4.33}$$

(c) For $z \in \partial D(-1, r)$, we have

$$P(z) = (I + o(1)) P^{(\infty)}(z), \qquad \text{as } n \to \infty. \tag{2.4.34}$$

The solution of this RH problem is given by

$$P(z) = P^{(\infty)}(z) \begin{pmatrix} 1 & \tilde{h}(z) \\ 0 & 1 \end{pmatrix}, \tag{2.4.35}$$

with $\tilde{h}(z) = \dfrac{1}{2\pi i} \displaystyle\int_{S^1 \cap D(-1,r)} \frac{e^{n(\phi(s)-x)} e^{W(s)}}{s-z} ds$. Using the fact that $\phi(s) - x_c$ has a double zero at $s = 1$, it is straightforward to verify that

$$\tilde{h}(z) = \mathcal{O}(n^{-1/2} e^{n(x_c - x)}), \qquad \text{for } z \in \partial D(-1,r), \text{ as } n \to \infty, \tag{2.4.36}$$

and this implies the matching condition (2.4.34).

Note that we use the same notation P for the different local parametrices defined in $D(z_0, r)$, $D(\overline{z_0, r})$, and $D(-1, r)$.

2.4.7 Final transformation $S \mapsto R$

Define

$$R(z) = \begin{cases} S(z) P^{(\infty)}(z)^{-1}, & z \in \mathbb{C} \setminus (\overline{D(z_0,r)} \cup \overline{D(\overline{z_0},r)} \cup \overline{D(-1,r)}), \\ S(z) P(z)^{-1}, & z \in D(z_0,r) \cup D(\overline{z_0},r) \cup D(-1,r). \end{cases} \tag{2.4.37}$$

P was constructed in such a way that it has exactly the same jump relations as S in $D(z_0, r) \cup D(\overline{z_0}, r) \cup D(-1, r)$, and as a consequence R has no jumps at all inside those disks. Moreover, from the local behaviour of S and P near z_0, $\overline{z_0}$ and -1, it follows that R is analytic at these three points. We have the following RH problem for R:

RH problem for R

(a) $R : \mathbb{C} \setminus \Sigma_R \to \mathbb{C}^{2\times 2}$ is analytic, with Σ_R as in Figure 2.5.

(b) R satisfies the jump conditions:

$$\begin{aligned} R_+(z) &= R_-(z) \left(I + \mathcal{O}(n^{-1})\right), && \text{for } z \in \partial D(z_0, r) \cup \partial D(\overline{z_0}, r), \\ R_+(z) &= R_-(z) \left(I + \mathcal{O}(n^{-\frac{1}{2}} e^{n(x_c - x)})\right), && \text{for } z \in \partial D(-1, r), \\ R_+(z) &= R_-(z)(I + \mathcal{O}(e^{-cn})), c > 0 && \text{for } z \text{ elsewhere on } \Sigma_R. \end{aligned}$$

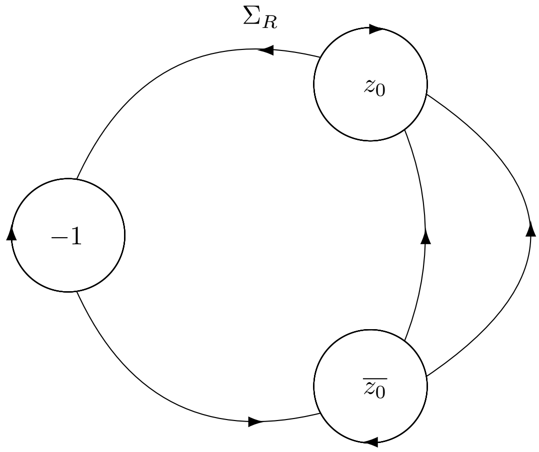

Figure 2.5: The jump contour for R.

(c) $R(z) = I + \mathcal{O}(z^{-1})$ as $z \to \infty$.

As $n \to \infty$ with $s \leq e^{-x_c n}$, it follows from the standard theory for small-norm RH problems that

$$R(z) = I + \mathcal{O}(n^{-\frac{1}{2}} e^{n(x_c - x)}) + \mathcal{O}(n^{-1}), \qquad R'(z) = \mathcal{O}(n^{-\frac{1}{2}} e^{n(x_c - x)}) + \mathcal{O}(n^{-1}), \tag{2.4.38}$$

uniformly for $z \in \mathbb{C} \setminus \Sigma_R$. By a more detailed analysis of Cauchy operators associated to the RH problem for R, one obtains in addition that

$$\partial_x R(z) = \mathcal{O}(n^{1/2} e^{n(x_c - x)}). \tag{2.4.39}$$

The latter estimate is not really needed for the proof of Theorem 2.1.1, but will be needed in our alternative proof if $W = 0$, see Section 2.5.2. Since $s = e^{-xn}$, it follows that

$$\partial_s R(z) = -\frac{e^{xn}}{n} \partial_x R(z) = \mathcal{O}(n^{-1/2} e^{n x_c}). \tag{2.4.40}$$

2.5 Asymptotics for $D_n(s, L, W)$

We will now complete the proof of Theorem 2.1.1 using the results from the previous sections. We first do this for general analytic W in the case where $\epsilon < L < \pi - \epsilon$ for some $\epsilon > 0$. Afterwards we will explain how the proof can be somewhat simplified if $W=0$, and we will show how it can be extended to the case where L approaches π at a sufficiently slow rate.

2.5.1 Proof of Theorem 2.1.1 for $\epsilon < L < \pi - \epsilon$

By inverting the transformations $Y \mapsto T \mapsto S \mapsto R$, we obtain an expression for R in terms of Y, involving the parametrices P and $P^{(\infty)}$. For $z \in \gamma^c \cap (D(z_0,r) \cup D(\overline{z_0},r) \cup D(-1,r))$, we have

$$Y_\pm(z) = e^{-\frac{n\pi i}{2}\sigma_3} e^{-\frac{n\ell}{2}\sigma_3} R(z) P_\pm(z) e^{ng(z)\sigma_3} e^{\frac{n\ell}{2}\sigma_3} e^{\frac{n\pi i}{2}\sigma_3}, \tag{2.5.1}$$

and for $z \in \gamma^c \setminus (D(z_0,r) \cup D(\overline{z_0},r) \cup D(-1,r))$,

$$Y_\pm(z) = e^{-\frac{n\pi i}{2}\sigma_3} e^{-\frac{n\ell}{2}\sigma_3} R_\pm(z) P^{(\infty)}(z) e^{ng(z)\sigma_3} e^{\frac{n\ell}{2}\sigma_3} e^{\frac{n\pi i}{2}\sigma_3}. \tag{2.5.2}$$

It follows that

$$\left[Y^{-1}(z)Y'(z)\right]_{21} = \begin{cases} (-1)^n e^{2ng(z)} e^{n\ell} A_1(z), & z \in \gamma^c \cap (D(z_0,r) \cup D(\overline{z_0},r) \cup D(-1,r)), \\ (-1)^n e^{2ng(z)} e^{n\ell} A_2(z), & z \in \gamma^c \setminus (D(z_0,r) \cup D(\overline{z_0},r) \cup D(-1,r)), \end{cases} \tag{2.5.3}$$

where

$$A_1(z) = \left[P^{-1}(z)R^{-1}(z)R'(z)P(z) + P^{-1}(z)P'(z)\right]_{21}, \tag{2.5.4}$$

$$A_2(z) = \left[{P^{(\infty)}}^{-1}(z)R^{-1}(z)R'(z)P^{(\infty)}(z) + {P^{(\infty)}}^{-1}(z){P^{(\infty)}}'(z)\right]_{21}. \tag{2.5.5}$$

Note that the boundary values of $A_1(z)$ and $A_2(z)$ as z is approached from the inside and the outside of the unit circle are the same. For $z \in D(-1,r)$, the local parametrix P is given by (2.4.35), and one verifies that the formulas for A_1 and A_2 coincide in this case.

Substituting (2.5.3) in the differential identity (2.2.6), we find

$$\partial_s \log D_n(s,L,W) = I_1 + I_2, \tag{2.5.6}$$

where

$$I_1 = \frac{1}{2\pi i}\int_{\gamma_1^c}(-1)^n z^{-n} e^{2ng(z)} e^{n\ell} A_1(z) e^{W(z)} dz,$$
$$I_2 = \frac{1}{2\pi i}\int_{\gamma_2^c}(-1)^n z^{-n} e^{2ng(z)} e^{n\ell} A_2(z) e^{W(z)} dz,$$

with $\gamma_1^c = \gamma^c \cap (D(z_0, r) \cup D(\overline{z_0}, r))$ and $\gamma_2^c = \gamma^c \setminus (D(z_0, r) \cup D(\overline{z_0}, r))$.

Now we need to know how I_1 and I_2 behave for large n and $s \leq e^{-x_c n}$. For A_2, we note that $P^{(\infty)}$ does not depend on n, and that R and R' are uniformly bounded by (2.4.38). This implies that $A_2(z)$ is uniformly bounded on γ_2^c for large n. For A_1, we need to take a closer look at the construction of P near z_0 and $\overline{z_0}$, but it is straightforward to show that $P(z) = \mathcal{O}(n)$, $P^{-1}(z) = \mathcal{O}(n)$, and $P'(z) = \mathcal{O}(n^2)$ for $z \in \gamma_1^c$. We get

$$|I_1| = \mathcal{O}\left(n^3 \int_{[L,L+r]\cup[2\pi-L-r,2\pi-L]} \left|e^{2ng(e^{i\alpha})+n\ell}\right| d\alpha\right),$$
$$|I_2| = \mathcal{O}\left(\int_{L+r}^{2\pi-L-r} \left|e^{2ng(e^{i\alpha})+n\ell}\right| d\alpha\right),$$

as $n \to \infty$ with $s \leq e^{-x_c n}$. The function $2\mathrm{Re}(g(z)) + \ell - x_c$ is always negative on γ^c except that it has a zero of order two at $z = -1$. Therefore we obtain after a straightforward analysis that

$$|I_1| = \mathcal{O}\left(e^{n(x_c - C)}\right), \qquad C > 0, \tag{2.5.7}$$
$$|I_2| = \mathcal{O}(n^{-1/2} e^{n x_c}), \tag{2.5.8}$$

as $n \to \infty$, $s \leq e^{-x_c n}$.

If we integrate (2.5.6) from 0 up to $s = e^{-xn} \leq e^{-x_c n}$, we finally obtain the desired estimate

$$\log D_n(s, L, W) - \log D_n(0, L, W) = \int_0^s (I_1 + I_2) ds' = \mathcal{O}(n^{-1/2} e^{-n(x - x_c)}).$$

2.5.2 Alternative proof of Theorem 2.1.1 if $W = 0$, $s = o(n^{-1/2} e^{-n x_c})$

In the case where $W = 0$, there is an alternative approach to prove Theorem 2.1.1: we can use the differential identity (2.2.18) instead of (2.2.6). This does not make the proof much shorter, but it has the advantage that no integrals have to be estimated, and that we only need information about Y at the points 0 and z_0, instead of on the entire curve γ_c. The objects in the differential identity can be computed more explicitly in this case by the following result.

Proposition 2.5.1 *Let $W = 0$. As $n \to \infty$ with $x > x_c$, we have*

$$Y_{12}(0) = e^{-n\ell}\left[\sin\frac{L}{2} + \mathcal{O}(n^{-1/2})\right], \tag{2.5.9}$$

$$\partial_s \log Y_{12}(0) = \mathcal{O}(n^{-1/2}e^{nx_c}), \tag{2.5.10}$$

$$Y_{11}(z_0) = e^{-\frac{n\ell}{2}}\mathcal{O}(n^{1/2}). \tag{2.5.11}$$

Proof. Using (2.5.2) and the expressions $g(0) = \pi i$ and $P_{12}^{(\infty)}(0) = \sin\frac{L}{2}$ (if $W = 0$), we get the result for $Y_{12}(0)$.

For $\partial_s \log Y_{12}(0)$, we have

$$\partial_s \log Y_{12}(0) = \frac{\partial_s\left(R_{11}(0)P_{12}^{(\infty)}(0) + R_{12}(0)P_{22}^{(\infty)}(0)\right)}{R_{11}(0)P_{12}^{(\infty)}(0) + R_{12}(0)P_{22}^{(\infty)}(0)}. \tag{2.5.12}$$

By (2.4.40), this yields (2.5.10).

For the rest of this proof, we assume that $|z| < 1$ and that z lies outside of the lenses and in $D(z_0, r)$. Then we have by (2.5.1),

$$Y_{11}(z) = e^{ng(z)}\left[R_{11}(z)P_{11}(z) + R_{12}(z)P_{21}(z)\right]. \tag{2.5.13}$$

By (2.4.1) and (2.4.2), we can show that

$$g(z_0) = -\frac{\ell}{2} + i\frac{L+\pi}{2}.$$

On the other hand, by (2.4.26), as $z \to z_0$ for fixed n, we have

$$\Psi_{11}(n^2\zeta(z)) = 1 + \mathcal{O}(z - z_0), \quad \Psi_{21}(n^2\zeta(z)) = \mathcal{O}(z - z_0).$$

This implies, by (2.4.27), that

$$P_{j1}(z_0) = E_{j1}(z_0)(1 + \mathcal{O}(e^{-nx}))e^{-\frac{n}{2}\phi(z_0)}, \qquad \text{as } n \to \infty.$$

Since $\phi(z_0) = 0$, we have $P_{j1}(z_0) = \mathcal{O}(\sqrt{n})$, $j = 1, 2$, and

$$Y_{11}(z_0) = e^{n\left(-\frac{\ell}{2}+i\frac{L+\pi}{2}\right)}\left(P_{11}(z_0) + \mathcal{O}(1)\right) = e^{-\frac{n\ell}{2}}\mathcal{O}(n^{1/2}),$$

as $n \to \infty$. $\square$

By Proposition 2.5.1 and (2.2.18), we have

$$n\partial_s \log Y_{12}(0) = \mathcal{O}(n^{1/2}e^{nx_c}), \tag{2.5.14}$$

$$\frac{2(1-s)}{\pi}\mathrm{Im}\left(\frac{\overline{Y_{11}(e^{iL})}}{\sqrt{Y_{12}(0)}}\partial_s\left(\frac{Y_{11}(e^{iL})}{\sqrt{Y_{12}(0)}}\right)\right) = \mathcal{O}(n), \tag{2.5.15}$$

as $n \to \infty$. Therefore, using (2.2.19), we get

$$\begin{aligned}\log D_n(s,L,0) &= \log D_n(0,L,0) + \int_0^s \mathcal{O}(n^{1/2}e^{nx_c})ds' \\ &= \log D_n(0,L,0) + \mathcal{O}(n^{1/2}e^{-n(x-x_c)}),\end{aligned}$$

and we rederive Theorem 2.1.1 with a slightly worse error term which is only small if $s = o(n^{-1/2}e^{-nx_c})$.

2.5.3 Extension to the case where L depends on n

The RH analysis done in the previous section is valid as $n \to \infty$ with $\epsilon < L < \pi - \epsilon$ and $\epsilon > 0$ independent of n. If $L = L(n)$ depends on n and approaches π as $n \to \infty$, the arc γ grows and the *gap* $(e^{iL}, e^{i(2\pi-L)})$ closes. Similarly to [46], we will show here that the RH analysis carried out before remains valid as long as $n(\pi - L)$ is large.

A first problem in the RH analysis is that we need to allow the radius $r = r(n)$ of the disks $D(z_0, r)$, $D(-1, r)$, and $D(\overline{z_0}, r)$ to depend on n (or on L) in order to prevent the disks to overlap. For instance, we can keep the disks separated from each other if we let $r(L) = \delta(\pi - L)$, with $\delta > 0$ a sufficiently small but fixed number. Then the local parametrices near $z_0, \overline{z_0}, -1$ can be constructed in exactly the same way as before. However, in order to make the asymptotic analysis work, it is crucial that the jump matrix for R tend to I uniformly as $n \to \infty$ on the n-dependent jump contour. Recall that the jump matrix for R is given by

$$J_R(z) = \begin{cases} P(z)P^{(\infty)}(z)^{-1}, & z \in \partial D(z_0,r) \cup \partial D(\overline{z_0},r) \cup \partial D(-1,r), \\ P^{(\infty)}(z)J_S(z)P^{(\infty)}(z)^{-1}, & z \in \Sigma_R \setminus (\partial D(z_0,r) \cup \partial D(\overline{z_0},r) \cup \partial D(-1,r)), \end{cases} \tag{2.5.16}$$

where J_S denotes the jump matrix for S given in (2.4.15).

The first thing to notice is that $h(z)$, defined in (2.4.20), and $\beta(z)$, defined right before (2.4.20), are uniformly bounded on the jump contour Σ_R. This implies that $P^{(\infty)}(z)$ is uniformly bounded for $z \in \Sigma_R$. Next, by (2.4.11), we have

$$|\phi(z)| \geq C\sin(\pi - L), \qquad z \in \Sigma_R, \tag{2.5.17}$$

for a constant $C > 0$ independent of z and L. By (2.4.15), this already implies that the jump matrix J_R is $I + \mathcal{O}(n^{-1}(\pi - L)^{-1})$ on Σ_R, except on the three circles around $z_0, \overline{z_0}$, and -1. On $\partial D(z_0, r)$ and $\partial D(\overline{z_0}, r)$, we have

$$\begin{aligned} J_R(z) &= P(z)P^{(\infty)}(z)^{-1} = P^{(\infty)}(z)\left(I + \mathcal{O}(n^{-1}\phi(z)^{-1})\right)P^{(\infty)}(z)^{-1} \\ &= I + \mathcal{O}(n^{-1}(\pi - L)^{-1}). \end{aligned} \tag{2.5.18}$$

Finally, on $\partial D(-1, r)$, by (2.4.35), we have

$$J_R(z) = P(z)P^{(\infty)}(z)^{-1} = I + \mathcal{O}(\tilde{h}(z)) = I + \mathcal{O}(n^{-1/2}e^{-n(x-x_c)}). \tag{2.5.19}$$

It follows from these considerations that

$$J_R(z) - I = \mathcal{O}(n^{-1}(\pi - L)^{-1}) + \mathcal{O}(n^{-1/2}e^{-n(x-x_c)}), \qquad \text{as } n \to \infty,$$

with L approaching π sufficiently slowly such that $n(\pi - L) \to +\infty$. By the small-norm theory for RH problems, it follows that

$$R(z) = I + \mathcal{O}\left(n^{-1}(\pi - L)^{-1}\right) + \mathcal{O}(n^{-1/2}e^{-n(x-x_c)}), \tag{2.5.20}$$

and

$$R'(z) = \mathcal{O}\left(n^{-1}(\pi - L)^{-1}\right) + \mathcal{O}(n^{-1/2}e^{-n(x-x_c)}), \tag{2.5.21}$$

as $n \to \infty$, $n(\pi - L) \to +\infty$. The proof of Theorem 2.1.1 can now be completed in the same way as before, where the estimate (2.5.7) remains valid, (2.5.8) becomes

$$|I_2| = \mathcal{O}((\pi - L)^{1/2}n^{-1/2}e^{nx_c}),$$

and we obtain the error term (2.1.10).

Chapter 3

Thinning and conditioning of the Circular Unitary Ensemble

Abstract[1]

We apply the operation of random independent thinning on the eigenvalues of $n \times n$ Haar distributed unitary random matrices. We study gap probabilities for the thinned eigenvalues, and we study the statistics of the eigenvalues of random unitary matrices which are conditioned such that there are no thinned eigenvalues on a given arc of the unit circle. Various probabilistic quantities can be expressed in terms of Toeplitz determinants and orthogonal polynomials on the unit circle, and we use these expressions to obtain asymptotics as $n \to \infty$.

3.1 Introduction

Randomly incomplete spectra of random matrices were introduced by Bohigas and Pato in [12, 13], motivated by problems in nuclear physics. Two natural questions arose in those works: (1) what can we say about the eigenvalues of the incomplete spectrum, and (2) to what extent does the incomplete spectrum give

[1]This chapter corresponds to the following paper [18]:

C. Charlier and T. Claeys, Thinning and conditioning of the Circular Unitary Ensemble, arXiv:1604.08399, submitted in *Annals of Probability*.

us information about the complete spectrum? We investigate these questions in detail for Haar distributed unitary matrices in the limit where the matrix size becomes large.

Thinning is a classical operation in the theory of point processes [22, 43, 56]. Given a locally finite configuration of points, it consists of removing points following a deterministic or probabilistic rule. Starting with a given point process, one can in this way create other, thinned, point processes. If one thins a determinantal point process (see e.g. [42, 59] for an overview of the theory of determinantal point processes) by independently removing each particle with a certain probability, it is remarkable, but straightforward to see, that the resulting process is still determinantal [50].

In this chapter, we start from the determinantal point process on the unit circle in the complex plane defined by the eigenvalues of a Haar distributed random unitary matrix, and we apply a random uniform independent thinning operation to it. This means that we remove each eigenvalue independently with a given probability which does not depend on the position of the eigenvalue. One can think of the remaining eigenvalues as observed particles and the removed eigenvalues as unobserved particles.

On one hand, we will study the asymptotic behaviour of gap probabilities for the thinned spectrum as the size of the matrices tends to infinity. On the other hand, we will study the asymptotic statistical behaviour of all eigenvalues, given some knowledge about the thinned (observed) eigenvalues. More precisely, we will assume that a certain arc of the unit circle is free of thinned particles, and we will investigate what this condition tells us about the other eigenvalues.

The eigenvalues of Haar distributed unitary matrices are of particular interest because of their remarkable connection to zeros with large imaginary part of the Riemann ζ-function on the critical line [45]. Spacings between thinned eigenvalues in this model were compared recently [37] with thinned data sets of zeros of the Riemann ζ-function, showing accurate agreement.

We begin with a more precise definition of the models which we study.

Thinned Circular Unitary Ensemble. An $n \times n$ matrix U from the Circular Unitary Ensemble (CUE) is a random Haar distributed unitary $n \times n$ matrix. We denote $e^{i\theta_1}, \dots, e^{i\theta_n}$ for the eigenvalues of U, with $\theta_j \in [0, 2\pi)$ for

$j = 1, \dots, n$. The joint probability distribution of the eigenvalues is given by

$$\frac{1}{(2\pi)^n n!} \prod_{1 \le j < k \le n} |e^{i\theta_j} - e^{i\theta_k}|^2 \prod_{j=1}^{n} d\theta_j, \qquad \theta_1, \dots, \theta_n \in [0, 2\pi). \tag{3.1.1}$$

This is a determinantal point process on the unit circle with correlation kernel [52]

$$K_n(e^{i\theta}, e^{i\mu}) = \frac{1}{2\pi} \sum_{k=0}^{n-1} e^{ik(\theta-\mu)}. \tag{3.1.2}$$

Consider the operation of thinning the eigenvalues of an $n \times n$ CUE matrix, which consists of removing each eigenvalue $e^{i\theta_1}, ..., e^{i\theta_n}$ independently with probability $s = 1 - p \in (0, 1)$. We call s the *removal probability* and p the *retention probability*. We denote $e^{i\phi_1}, ..., e^{i\phi_m}$, $0 \le m \le n$ for the retained eigenvalues, where the number of particles m is now itself a random variable following the binomial distribution $B(n, p)$. We write

$$\Theta = \{e^{i\theta_1}, \dots, e^{i\theta_n}\}, \qquad \Phi = \{e^{i\phi_1}, \dots, e^{i\phi_m}\}$$

for the complete and thinned spectra. On the formal level, one can define the thinned CUE by associating to each eigenvalue $e^{i\theta_k}$ an independent Bernoulli random variable taking the value 0 if the eigenvalue is removed and taking the value 1 if it is kept.

Thinned point processes interpolate in general between the original point process (for $p = 1$) and a Poisson process (as $p \to 0$, after a proper re-scaling) with uncorrelated points [43]. Since the eigenvalues of a CUE matrix are intrinsically repulsive by (3.1.1), in our situation the retention probability p can be thought of as a parameter measuring the repulsion between particles. The repulsion is strongest for $p = 1$, becomes weaker as p decreases, and disappears as $p \to 0$.

On the level of correlation kernels, random independent thinning of a general determinantal point process boils down to multiplying the correlation kernel with the retention probability p, as was observed in [50, Appendix A]. The correlation kernel for the particles $e^{i\phi_1}, ..., e^{i\phi_m}$ is in other words equal to $pK_n(e^{i\theta}, e^{i\mu})$, with K_n as in (3.1.2).

It is worth noting that the CUE kernel K_n, seen as the kernel of an integral operator acting on $L^2(S^1)$ with S^1 the unit circle, defines an orthogonal projection operator of rank n onto the space of polynomials of degree $\le n - 1$.

The integral operator associated to the kernel pK_n with $p \in (0,1)$ is still of rank n, but not a projection. By the general theory of determinantal point processes [59], this is consistent with the fact that the number of particles in the original process is equal to n, whereas it is a random number $\leq n$ in the thinned process.

Although eigenvalue correlations (and large n asymptotics thereof) in the thinned CUE are obtained directly from the non-thinned eigenvalue correlations, the question of computing large n asymptotics for gap probabilities and, related to that, eigenvalue spacings, is less trivial. For finite n, the probability $\mathbb{P}(\Phi \subset \gamma)$ that all thinned eigenvalues lie on a measurable part $\gamma \subset S^1$ is equal to the Fredholm determinant $\det\left(1 - p\, K_n|_{\gamma^c}\right)$, where $\gamma^c = S^1 \setminus \gamma$ and $pK_n|_{\gamma^c}$ is the integral operator with kernel pK_n acting on $L^2(\gamma^c)$. In the special case where γ^c is an arc of the unit circle with arclength $4\pi y/n$ with $y > 0$ fixed, the gap probability $\mathbb{P}(\Phi \subset \gamma)$ converges as $n \to \infty$ to the Fredholm determinant $\det\left(1 - p\, K^{\sin}|_{[-y,y]}\right)$ associated to the sine kernel operator with kernel $pK^{\sin}(u,v) = p\frac{\sin \pi(u-v)}{\pi(u-v)}$. Large gap asymptotics as $y \to \infty$ and $p \to 1$ for this determinant were studied in detail in [15].

For general measurable subsets γ of S^1, we can also express $\mathbb{P}(\Phi \subset \gamma)$ as a Toeplitz determinant in the following elementary way. We can partition $[0,2\pi)^n$ as

$$[0,2\pi)^n = \bigsqcup_{k=0}^{n} A_k, \qquad A_k = \{(\theta_1,...,\theta_n) \in [0,2\pi)^n : \#(\Theta \cap \gamma^c) = k\}. \quad (3.1.3)$$

In this way A_k represents the event that exactly k eigenvalues belong to γ^c. A given eigenvalue configuration Θ leads to a thinned configuration Φ which lies entirely in γ with probability s^k, where $k = \#(\Theta \cap \gamma^c)$. Therefore,

$$\mathbb{P}(\Phi \subset \gamma) = \sum_{k=0}^{n} s^k \mathbb{P}(A_k) = \frac{1}{(2\pi)^n n!} \sum_{k=0}^{n} s^k \int_{A_k} \prod_{1 \leq j < k \leq n} |e^{i\theta_j} - e^{i\theta_k}|^2 \prod_{j=1}^{n} d\theta_j. \quad (3.1.4)$$

Writing f for the piece-wise constant function

$$f(z) = \begin{cases} 1 & \text{for } z \in \gamma, \\ s & \text{for } z \in \gamma^c, \end{cases} \quad (3.1.5)$$

we can express the gap probability as a Toeplitz determinant with symbol f:

$$\begin{aligned}\mathbb{P}\left(\Phi \subset \gamma\right) &= \frac{1}{(2\pi)^n n!}\int_{[0,2\pi]^n} \prod_{1\le j<k\le n} |e^{i\theta_j} - e^{i\theta_k}|^2 \prod_{j=1}^{n} f(e^{i\theta_j})d\theta_j \\ &= \det\left(\frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-i(k-j)\theta}d\theta\right)_{j,k=1,\dots,n} =: D_n(f), \quad (3.1.6)\end{aligned}$$

where we used the standard Heine integral representation for Toeplitz determinants. If γ is an arc of arclength $2L$, f given by (3.1.5) has two jump discontinuities and is a special case of a symbol with two Fisher-Hartwig singularities. Throughout this chapter, we will write $D_n(s, L)$ for $D_n(f)$ in this case. The large n asymptotics of Toeplitz determinants for symbols with Fisher-Hartwig singularities have a long history, notable results in this context have been obtained e.g. in [7, 16, 24, 26, 32, 33, 68].

If $L \in (0, \pi)$ and $s \in (0, 1)$ are independent of n, large n asymptotics for $D_n(s, L)$ are known, and we have [24, 32]

$$\begin{aligned}\mathbb{P}\left(\Phi \subset \gamma\right) = D_n(s, L) &= s^{n(1-\frac{L}{\pi})} n^{\frac{(\log s)^2}{2\pi^2}} (2\sin L)^{\frac{(\log s)^2}{2\pi^2}} \\ \times\ G\left(1 + \frac{\log s}{2\pi i}\right)^2 G\left(1 - \frac{\log s}{2\pi i}\right)^2 &(1 + o(1)), \qquad \text{as } n \to \infty, \quad (3.1.7)\end{aligned}$$

where G is Barnes' G-function. This implies that the probability of having all thinned eigenvalues on γ decays exponentially as $n \to \infty$, which is not surprising since there are, before thinning, on average $\left(1 - \frac{L}{\pi}\right) n$ CUE eigenvalues outside γ. The factor $s^{n(1-\frac{L}{\pi})}$ corresponds to the probability that $\left(1 - \frac{L}{\pi}\right) n$ eigenvalues are removed. If we allow L and s to depend on n, several situations are possible. For instance, one observes that the decay in (3.1.7) becomes weaker if we let the removal probability $s \to 1$: the decay is destroyed when $1 - s = \mathcal{O}(1/n)$. Similarly, if we let, for fixed $s \in (0, 1)$, the half arclength $L \to \pi$, the decay becomes weaker. As $s \to 0$, the decay becomes faster instead. In Section 3.2, we will give a detailed overview of known asymptotic results for the Toeplitz determinants $D_n(s, L)$, depending on how L and s vary with n, and we will state a new result on the asymptotic behaviour of $D_n(s, L)$ if $s \to 0$ at an appropriate speed as $n \to \infty$.

Conditioning on a gap of the thinned spectrum. Let us now adopt an opposite point of view. Instead of studying statistics of the thinned spectrum, we assume that we have information about the incomplete thinned CUE spectrum, and we want to deduce statistical information about the complete non-thinned spectrum. In this context, it is natural to interpret the thinned eigenvalues as *observed particles*, and the others as *unobserved particles*. The general question which we want to investigate, is: *what do the observed particles tell us about the unobserved ones?*

We consider the situation where we observe no particles outside a measurable subset γ of the unit circle, equivalently $\Phi \subset \gamma$. We now define a conditional CUE as the joint probability distribution of all n eigenvalues in the complete spectrum Θ, given the fact that $\Phi \subset \gamma$. In this conditional CUE, any eigenvalue configuration Θ can occur, but configurations with many eigenvalues on γ^c are less likely because all of them have to be removed by the thinning procedure. We note that, for finite n, the event $\Phi \subset \gamma$ has non-zero probability, so the conditional probability is well-defined in the classical sense.

From the definition of conditional probability $\mathbb{P}(A|B) = \mathbb{P}(A \cap B)/\mathbb{P}(B)$, it is straightforward to see that the conditional joint probability distribution for the eigenvalues is given by

$$\frac{1}{Z_n} \prod_{1 \leq j < k \leq n} |e^{i\theta_j} - e^{i\theta_k}|^2 \prod_{j=1}^{n} f(e^{i\theta_j}) d\theta_j, \qquad \theta_1, \dots, \theta_n \in [0, 2\pi), \tag{3.1.8}$$

where $Z_n = Z_n(s, \gamma)$ is a normalization constant, and f is given by (3.1.5). We will refer to this probability measure as the *conditional CUE*. This is again a determinantal point process; the conditional eigenvalue correlation kernel can be expressed as

$$K_n(e^{i\theta}, e^{i\mu}) = \frac{1}{2\pi} \sum_{k=0}^{n-1} \frac{1}{h_k} \phi_k(e^{i\theta}) \overline{\phi_k(e^{i\mu})} \sqrt{f(e^{i\theta}) f(e^{i\mu})}, \tag{3.1.9}$$

where $\phi_k, k = 0, 1, \dots$ are the monic orthogonal polynomials on the unit circle with respect to the weight f, characterized by the orthogonality conditions

$$\frac{1}{2\pi} \int_0^{2\pi} \phi_k(e^{i\theta}) \overline{\phi_\ell(e^{i\theta})} f(e^{i\theta}) d\theta = h_k \delta_{k\ell}. \tag{3.1.10}$$

The polynomial ϕ_n is moreover the average characteristic polynomial of the conditional CUE. By the Christoffel-Darboux formula for orthogonal polynomials on the unit circle, see e.g. [58], we can express K_n in terms of ϕ_n only as follows:

$$K_n(e^{i\theta}, e^{i\mu}) = \frac{1}{2\pi h_n} \sqrt{f(e^{i\theta}) f(e^{i\mu})} \frac{\phi_n^*(e^{i\theta}) \overline{\phi_n^*(e^{i\mu})} - \phi_n(e^{i\theta}) \overline{\phi_n(e^{i\mu})}}{1 - e^{i(\theta - \mu)}}, \tag{3.1.11}$$

where $\phi_n^*(z) = z^n \overline{\phi_n}(z^{-1})$ is the reverse polynomial. If f has the symmetry $f(e^{-i\theta}) = f(e^{i\theta})$ then $\overline{\phi_n}(z) = \phi_n(z)$.

Conditional eigenvalue correlations and gap probabilities can be computed from the kernel K_n. The one-point function or mean eigenvalue density $\psi_{n,s,L}$ is given by

$$\begin{aligned}\psi_{n,s,L}(e^{i\theta}) &= \frac{1}{n} K_n(e^{i\theta}, e^{i\theta}) \\ &= \frac{1}{2\pi n h_n} f(e^{i\theta}) e^{i\theta} \left(\phi_n'(e^{i\theta}) \overline{\phi_n(e^{i\theta})} - \left(\phi_n^*\right)' (e^{i\theta}) \overline{\phi_n^*(e^{i\theta})} \right).\end{aligned} \tag{3.1.12}$$

Formulas (3.1.11) and (3.1.12) are particularly convenient for asymptotic analysis as $n \to \infty$, since they require only to know the large n behaviour of the average characteristic polynomial ϕ_n. Other interesting quantities are the conditional average and variance of the number of (unobserved) eigenvalues which lie on γ^c, given that no thinned (observed) eigenvalues lie on γ^c. It is clear that, for $s = 0$, this number is equal to 0 with probability 1 since all eigenvalues are observed. If $s = 1$, no particles are observed, and then one shows easily that the average is $\left(1 - \frac{L}{\pi}\right) n$ by rotational symmetry.

For a general removal probability $s \in (0, 1)$, we have

$$\begin{aligned}\mathbb{E}\left(\#(\Theta \cap \gamma^c) | \Phi \subset \gamma\right) &= \sum_{k=0}^{n} k \mathbb{P}\left(\#(\Theta \cap \gamma^c) = k | \Phi \subset \gamma\right) \\ &= \sum_{k=0}^{n} k \frac{\mathbb{P}\left(\Phi \subset \gamma \text{ and } \#(\Theta \cap \gamma^c) = k\right)}{\mathbb{P}\left(\Phi \subset \gamma\right)}.\end{aligned}$$

Using (3.1.4) and (3.1.6), we obtain the identity

$$\mathbb{E}\left(\#(\Theta \cap \gamma^c) | \Phi \subset \gamma\right) = \sum_{k=0}^{n} k s^k \frac{\mathbb{P}\left(\#(\Theta \cap \gamma^c) = k\right)}{\mathbb{P}\left(\Phi \subset \gamma\right)} = s \frac{\partial}{\partial s} \log D_n(f). \tag{3.1.13}$$

Similarly, after a straightforward calculation,

$$\begin{aligned}\mathrm{Var}\left(\#(\Theta \cap \gamma^c) | \Phi \subset \gamma\right) &= \mathbb{E}\left([\#(\Theta \cap \gamma^c)]^2 | \Phi \subset \gamma\right) - \mathbb{E}\left(\#(\Theta \cap \gamma^c) | \Phi \subset \gamma\right)^2 \\ &= \sum_{k=0}^{n} k^2 s^k \frac{\mathbb{P}\left(\#(\Theta \cap \gamma^c) = k\right)}{\mathbb{P}\left(\Phi \subset \gamma\right)} - \left(s \frac{\partial}{\partial s} \log D_n(f) \right)^2 \\ &= s^2 \frac{\partial^2}{\partial s^2} \log D_n(f) + s \frac{\partial}{\partial s} \log D_n(f).\end{aligned} \tag{3.1.14}$$

In the case where $s \in (0,1)$ is independent of n and where γ is an arc of arclength $2L$ independent of n, we obtain from (3.1.7) that

$$\mathbb{E}\left(\#(\Theta \cap \gamma^c) | \Phi \subset \gamma\right) = n\left(1 - \frac{L}{\pi}\right) + \frac{\log s}{\pi^2} \log n + \mathcal{O}(1), \tag{3.1.15}$$

$$\operatorname{Var}\left(\#(\Theta \cap \gamma^c) | \Phi \subset \gamma\right) = \frac{\log n}{\pi^2} + \mathcal{O}(1), \tag{3.1.16}$$

as $n \to \infty$, where the $\mathcal{O}(1)$ term can be expressed in terms of Barnes' G function. Remarkably, the leading order terms of the average and number variance are independent of s, which suggests that the conditioning does not have a major effect on the CUE eigenvalues, in other words the fact that no particles are observed outside γ does not give us much additional information on all particles. As we will see below, this changes drastically if we allow L and s to depend on n.

Remark 3.1.1 It is a priori not rigorously justified to take logarithmic derivatives of the asymptotic expansion (3.1.7) in order to obtain (3.1.15) and (3.1.16). However, this can be justified in two ways. First, the asymptotics for the Toeplitz determinants $D_n(s, L)$ in [26] were obtained precisely by integrating asymptotics for $\frac{\partial}{\partial s} \log D_n(s, L)$, see e.g. [26, Section 3]. Secondly, the asymptotics (3.1.15) are valid for s in a complex neighbourhood of $(0, 1)$ with a $o(1)$ error term which is analytic in s and uniform in compact subsets of this neighbourhood. From Cauchy's integral formula, it then follows that s-derivatives of the $o(1)$ term are also $o(1)$. This allows one to justify (3.1.15) and (3.1.16).

We will study the large n asymptotics for various quantities in the conditional CUE:

1. the zero counting measure $\nu_{n,s,L}$ of the conditional average characteristic polynomial ϕ_n,

2. the conditional mean eigenvalue density

$$\frac{d\mu_{n,s,L}(e^{i\theta})}{d\theta} = \psi_{n,s,L}(e^{i\theta}) = \frac{1}{n} K_n(e^{i\theta}, e^{i\theta}),$$

3. the conditional eigenvalue correlation kernels, suitably scaled near different types of points on the unit circle,

4. the conditional average and variance of the number $\#(\Theta \cap \gamma^c)$, i.e. the number of particles on γ^c.

Similarly to the gap probabilities of the thinned spectrum, the above quantities exhibit different types of asymptotic behaviour, depending on how L and s behave as $n \to \infty$. We will give an overview of known results in Section 3.2, and we will in addition derive new results if $s \to 0$ and $n \to \infty$ simultaneously at an appropriate speed.

3.2 Statement of results

In what follows, we take γ to be an arc of the unit circle with arclength $2L \in (0, 2\pi)$. By rotational invariance, we can assume without loss of generality that γ is an arc between $z_0 = e^{iL}$ and $\overline{z_0} = e^{-iL}$, such that $1 \in \gamma$. Note that in this case $\overline{\phi_n} = \phi_n$.

3.2.1 Large n asymptotics for gap probabilities in the thinned CUE

We describe the asymptotics of the quantity $\mathbb{P}(\Phi \subset \gamma) = D_n(s, L)$ in 4 different regimes depending on the behaviour of s and L. Recall that

$$D_n(s, L) = \det(f_{k-j})_{j,k=1,\dots,n}, \tag{3.2.1}$$

where the f_k's are the Fourier coefficients of f given in (3.1.5). They can be computed explicitly, we have

$$f_k = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{-ik\theta} d\theta = \begin{cases} \frac{L}{\pi} + \frac{\pi - L}{\pi} s, & \text{if } k = 0, \\ \frac{(1-s)\sin(kL)}{\pi k}, & \text{if } k \neq 0. \end{cases} \tag{3.2.2}$$

Case I: s and L fixed. The case where $L \in (0, \pi)$ and $s \in (0, 1)$ are independent of n was already discussed in the introduction: we have (3.1.7), which implies that $\mathbb{P}(\Phi \subset \gamma)$ decays exponentially fast as $n \to \infty$. The precise rate of decay depends on the values of s and L. The decay is slower for larger s and larger L.

Case II: s fixed, $L \to \pi$. If $s \in (0, 1)$ is independent of n and $L \to \pi$, we can use large n asymptotics for $D_n(s, L)$ obtained in [21, Theorem 1.5] (our situation corresponds to $\alpha_1 = \alpha_2 = 0$, $\beta_1 = -\beta_2 = \frac{\log s}{2\pi i}$, and $t = \pi - L$ in the language of [21]). This leads to

$$P(\Phi \subset \gamma) = \exp\left(\int_0^{-2in(\pi-L)} \frac{\sigma(\xi;s)}{\xi} d\xi\right)(1+o(1)), \tag{3.2.3}$$

as $n \to \infty$ and simultaneously $L \to \pi_-$. Here, $\sigma(\xi;s)$ is a solution of the σ-form of the Painlevé V equation

$$\xi^2(\sigma'')^2 = (\sigma - \xi\sigma' + 2(\sigma')^2)^2 - 4(\sigma')^4, \tag{3.2.4}$$

where $'$ denotes a derivative with respect to ξ, and we have the asymptotic behaviour (depending on s)

$$\begin{aligned} \sigma(\xi;s) &= \mathcal{O}(\xi\log\xi), & \xi \to i0_-, \\ \sigma(\xi;s) &= -\frac{\log s}{2\pi i}\xi + \frac{(\log s)^2}{2\pi^2} + \mathcal{O}(\xi^{-1}), & \xi \to -i\infty. \end{aligned} \tag{3.2.5}$$

In the special case $L = \pi\left(1 - \frac{4y}{n}\right)$ with $y > 0$, this can also be derived from the classical results by Jimbo, Miwa, Môri, and Sato on the sine kernel Fredholm determinant [41], recall the discussion above (3.1.3). With this scaling of L, $\mathbb{P}(\Phi \subset \gamma)$ does not decay as $n \to \infty$ but tends to a constant depending on the values of y and s. Asymptotics of the integral at the right hand side of (3.2.3) as $n(\pi - L) \to \infty$ are described in [21, Formula (1.26)].

Case III: $s \to 0$ sufficiently fast and L fixed or $L \to \pi$ sufficiently slow. If $L \in (0,\pi)$ is fixed and $s \to 0$ sufficiently fast such that

$$\frac{1}{n}\log s \leq -x_c := 2\log\tan\frac{L}{4}, \tag{3.2.6}$$

we can use the results from Theorem 2.1.1 to obtain

$$\mathbb{P}(\Phi \subset \gamma) = \left(\sin\frac{L}{2}\right)^{n^2} n^{-1/4}\left(\cos\frac{L}{2}\right)^{-1/4} 2^{\frac{1}{12}} e^{3\zeta'(-1)}\left(1+o(1)\right), \tag{3.2.7}$$

as $n \to \infty$, where ζ is the Riemann ζ-function. This implies that the gap probability decays faster than exponentially in this case. The estimate (3.2.7) is the same as in the case $s = 0$, in which the asymptotics for the Toeplitz determinants were obtained in [67]. This estimate remains valid if the arclength L tends to π as $n \to \infty$, as long as $n(\pi - L) \to \infty$, see Chapter 2.

In this case, a very small fraction of eigenvalues are removed for n large. In fact, the expected number of removed eigenvalues tends to 0 exponentially fast as $n \to \infty$, so typically we observe all eigenvalues, and it is not surprising that the gap probability behaves like in the case $s = 0$. However, this is no longer true if $s \to 0$ at a slower rate.

Case IV: $s \to 0$ at a slower rate, L fixed. In the case where $s \to 0$ in such a way that $\log s = -xn$ with $0 < x < x_c$, asymptotics for $D_n(s, L)$ are not available in the literature. We will study this case in detail and prove the following result.

Theorem 3.2.1 *Let $D_n(s, L)$ be the Toeplitz determinant defined in* (3.2.1)-(3.2.2)*. If $0 \leq x \leq x_c$ with x_c as in* (3.2.6)*, we have*

$$\lim_{n\to\infty} \frac{1}{n^2} \log D_n(e^{-xn}, L) = -\int_0^x \Omega_{\xi,L} d\xi. \tag{3.2.8}$$

Here $\Omega_{\xi,L}$ is given by

$$\Omega_{\xi,L} = \frac{1}{2\pi} \int_{\pi-T}^{\pi+T} \sqrt{\frac{\cos\theta + \cos T}{\cos\theta - \cos L}} d\theta, \tag{3.2.9}$$

where $0 \leq T \leq \pi - L$ is the unique solution of the equation

$$\frac{1}{\pi} \int_{[-L,L]\cup[\pi-T,\pi+T]} \log\left|\frac{1+e^{i\theta}}{1-e^{i\theta}}\right| \sqrt{\frac{\cos\theta + \cos T}{\cos\theta - \cos L}} d\theta = \xi. \tag{3.2.10}$$

Remark 3.2.2 The fact that equation (3.2.10) has a unique solution was shown in Proposition 2.3.1.

Remark 3.2.3 The decay of $\mathbb{P}(\Phi \subset \gamma) = D_n(e^{-xn}, L)$ as $n \to \infty$ is superexponential. Setting $x = x_c$ in (3.2.8), we recover

$$P(\Phi \subset \gamma) = D_n(e^{-xn}, L) = \left(\sin\frac{L}{2}\right)^{n^2} o(\exp(n^2))$$

as in (3.2.7), provided that the identity

$$\int_0^{x_c} \Omega_{\xi,L} d\xi = -\log\sin\frac{L}{2} \tag{3.2.11}$$

holds. We will give an independent proof of this identity in Appendix A. As $x \to 0$, by (3.2.8), the superexponential decay disappears, which makes the connection with the behaviour in case I. Further terms in the asymptotic expansion of $D_n(e^{-xn}, L)$ are expected to contain quantities related to elliptic θ-functions, but are hard to compute explicitly.

Remark 3.2.4 The quantities $\Omega_{x,L}$ and T are closely related to an equilibrium problem. Define $\mu_{x,L}$ as the measure supported on the two disjoint arcs $\theta \in [-L,L] \cup [\pi - T, \pi + T]$ with density

$$\frac{d\mu_{x,L}(e^{i\theta})}{d\theta} = \frac{1}{2\pi}\sqrt{\frac{\cos\theta + \cos T}{\cos\theta - \cos L}} =: \psi_{x,L}(e^{i\theta}), \tag{3.2.12}$$

see Figure 3.1. It was shown in Proposition 2.3.1 that $\mu_{x,L}$ is the unique equilibrium measure minimizing the logarithmic energy

$$\iint \log|z-u|^{-1} d\mu(z) d\mu(u) + \int V(z) d\mu(z), \tag{3.2.13}$$

among all Borel probability measures μ on the unit circle, where

$$V(e^{i\theta}) = \begin{cases} 0, & \text{for } e^{i\theta} \in \gamma, \\ x, & \text{for } e^{i\theta} \in S^1 \setminus \gamma. \end{cases} \tag{3.2.14}$$

It is characterized by the Euler-Lagrange variational conditions

$$\begin{aligned} &2\int_{-\pi}^{\pi} \log|z - e^{i\theta}| d\mu_{x,L}(e^{i\theta}) - V(z) + \ell_{x,L} = 0, && \text{for } z \in \operatorname{supp} \mu_{x,L}, \\ &2\int_{-\pi}^{\pi} \log|z - e^{i\theta}| d\mu_{x,L}(e^{i\theta}) - V(z) + \ell_{x,L} < 0, && \text{for } z \in S^1 \setminus \operatorname{supp} \mu_{x,L}, \end{aligned} \tag{3.2.15}$$

where

$$\ell_{x,L} = -\int_0^1 \frac{1}{u}\left(1 - \sqrt{\frac{u^2 + 2u\cos T + 1}{u^2 - 2u\cos L + 1}}\right) du. \tag{3.2.16}$$

We have $\Omega_{x,L} = \int_{\pi-T}^{\pi+T} d\mu_{x,L}(e^{i\theta})$. For $x \geq x_c$, the measure $\mu_{x,L} = \mu_{\infty,L}$ minimizing (3.2.13) does not depend on the value of x and is supported on the single arc $\theta \in [-L, L]$.

3.2.2 Large n asymptotics in the conditional CUE

We now take a look at the large n behaviour of the zero counting measure of the average characteristic polynomial ϕ_n, the limiting mean eigenvalue distribution, scaling limits of the eigenvalue correlation kernels, and the average and variance of the number of particles on γ^c, in the conditional CUE. All these objects can be expressed in terms of the average characteristic polynomial ϕ_n and

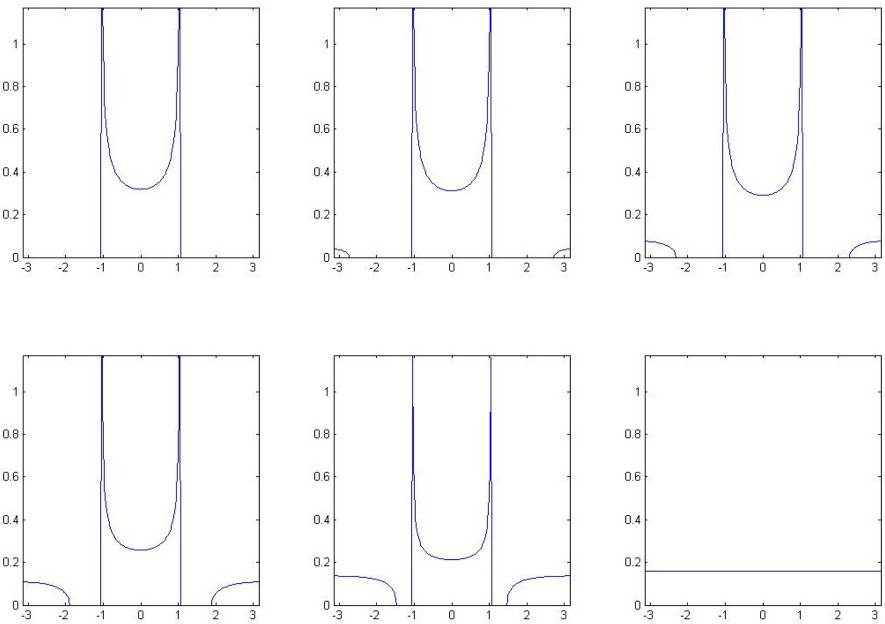

Figure 3.1: Density of the equilibrium measure $\mu_{x,L}$ as a function of $e^{i\theta}$ with $\theta \in [-\pi, \pi]$, for $L = \pi/3$ and for different values of x: the upper left picture corresponds to $x \geq x_c$, then x decreases from left to right and from top to bottom, and the bottom right picture corresponds to $x = 0$.

the Toeplitz determinant $D_n(s, L)$, as explained in Section 3.1 (recall formulas (3.1.11)–(3.1.14)). We denote

$$\mu_{n,s,L}(z) := \mathbb{E}\left(\frac{1}{n}\sum_{j=1}^{n}\delta(z - e^{i\theta_j})\right) \tag{3.2.17}$$

for the average counting measure of the eigenvalues in the conditional CUE, with density $\psi_{n,s,L}$ given by (3.1.12), and we let $\nu_{n,s,L}$ be the zero counting measure of the average characteristic polynomial ϕ_n.

We again distinguish between the 4 different regimes identified before. In Cases I-III, large n asymptotics can be derived from results for the Toeplitz determinants and average characteristic polynomials that are (explicitly or implicitly) available in the literature. In Case IV, large n asymptotics for ϕ_n and D_n are not available in the literature. We will perform a detailed asymptotic analysis using Riemann-Hilbert (RH) methods in Section 3.3, and this will enable us to obtain asymptotic results in the conditional CUE.

Case I: s and L fixed. An asymptotic analysis of ϕ_n in this case was done in [24]. Using the results obtained in that paper, we will show in Section 3.4 that $\nu_{n,s,L}$ and $\mu_{n,s,L}$, for $0 < s < 1$, converge weakly to the uniform measure on the unit circle; on the level of densities, we have

$$\lim_{n\to\infty} \psi_{n,s,L}(e^{i\theta}) = \frac{1}{2\pi}, \tag{3.2.18}$$

uniformly for θ in compact subsets of $[-\pi, \pi] \setminus \{-L, L\}$. Note that for $s = 1$, we have $\phi_n(z) = z^n$ and then $\nu_{n,1,L}$ does not converge to the uniform measure on the circle, since it is a Dirac measure at 0.

For the eigenvalue correlation kernel K_n, one expects from the analysis in [24] to have the scaling limit

$$\lim_{n\to\infty} \frac{1}{cn} K_n\left(e^{i\left(\theta+\frac{u}{cn}\right)}, e^{i\left(\theta+\frac{v}{cn}\right)}\right) = e^{i\frac{u-v}{2c}} \frac{\sin\pi(u-v)}{\pi(u-v)}, \tag{3.2.19}$$

for any $u, v \in \mathbb{R}$ and $\theta \in [0, 2\pi] \setminus \{L, 2\pi - L\}$, and where $c = \frac{1}{2\pi}$. Near $e^{\pm iL}$, the asymptotics for ϕ_n are described in terms of confluent hypergeometric functions, and instead of the sine kernel, one expects a limiting kernel built out of confluent hypergeometric functions. These scaling limits can be derived from the asymptotics for ϕ_n, using similar computations as the ones in Sections 3.4.4–3.4.6 (where we prove scaling limits in Case IV). The asymptotics for the expected number of eigenvalues on γ^c were already given in (3.1.15), and for the number variance in (3.1.16).

The above results can heuristically be explained as follows: the expected fraction of CUE eigenvalues on γ^c is equal to $1 - \frac{L}{\pi}$. From the theory of large deviations, see [2], it follows that the probability of having a smaller fraction of eigenvalues on γ^c decays exponentially fast in n^2 as $n \to \infty$. On the other hand, the probability that all eigenvalues on γ^c are removed by the thinning procedure is only exponentially small in n. In other words, if there are no observed particles on γ^c, that does not say much about the complete spectrum, it just means that the thinning procedure has accidentally removed all the eigenvalues on γ^c. It is expected that the eigenvalues uniformly fill out the arc γ^c as $n \to \infty$, just like in the CUE without conditioning. The conditioning only has an effect on the local behaviour of eigenvalues near $e^{\pm iL}$: near those points, eigenvalues are more likely to lie on γ than outside.

Case II: s fixed, $L \to \pi$. Here, asymptotics for ϕ_n were obtained in [21]. We will show in Section 3.4, relying on the analysis in [21], that $\mu_{n,s,L}$ and $\nu_{n,s,L}$

converge weakly to the uniform measure on the unit circle. For the correlation kernel, we expect convergence to the sine kernel as in (3.2.19), except when $\theta = \pi$, in which case one expects a limiting kernel built out of functions related to the Painlevé V equation, see [19] for a similar situation on the real line.

Taking logarithmic derivatives of (3.2.3) with respect to s, we obtain by (3.1.13)–(3.1.14) that the average number of particles on γ^c and the number variance are given by

$$\mathbb{E}\left(\#(\Theta\cap\gamma^c)|\Phi\subset\gamma\right) = s\int_0^{-2in(\pi-L)} \frac{\partial_s\sigma(\xi;s)}{\xi}d\xi + o(1), \tag{3.2.20}$$

$$\mathrm{Var}\left(\#(\Theta\cap\gamma^c)|\Phi\subset\gamma\right) = \int_0^{-2in(\pi-L)} \frac{s^2\partial_s^2\sigma(\xi;s)+s\partial_s\sigma(\xi;s)}{\xi}d\xi + o(1), \tag{3.2.21}$$

as $n\to\infty$, where $\sigma(\xi;s)$ is the Painlevé V transcendent defined in (3.2.4)–(3.2.5). This can be justified rigorously as described in Remark 3.1.1.

The case where $L = \pi\left(1-\frac{4y}{n}\right)$ with $y>0$ independent of n is of particular interest because the probability of having no eigenvalues on γ^c does not decay as $n\to\infty$, and this means that we condition on an event which is not unlikely to occur for large n. The average and variance of the number of particles on γ^c are described in terms of a Painlevé V transcendent which depends on the removal probability s. Here, the fact that we do not observe particles on γ^c does have implications for the unobserved particles.

Case III: $s\to 0$ sufficiently fast and L fixed or $L\to\pi$ sufficiently slow. As $n\to\infty$ and at the same time s goes to 0 in such a way that $s\leq\left(\tan\frac{L}{4}\right)^{2n}$, asymptotics for ϕ_n were obtained in Chapter 2. Both $\mu_{n,s,L}$ and $\nu_{n,s,L}$ converge weakly to the measure

$$d\mu_{\infty,L}(e^{i\theta}) := \frac{1}{2\pi}\sqrt{\frac{\cos\theta+1}{\cos\theta-\cos L}}d\theta, \qquad \theta\in[-L,L], \tag{3.2.22}$$

see Section 3.4. Moreover,

$$\lim_{n\to\infty}\psi_{n,s,L}(e^{i\theta}) = \frac{1}{2\pi}\sqrt{\frac{\cos\theta+1}{\cos\theta-\cos L}}\chi_{[-L,L]}(\theta), \tag{3.2.23}$$

uniformly for θ in compact subsets of $[-\pi,\pi]\setminus\{-L,L\}$. For the eigenvalue correlation kernel, we expect the sine kernel scaling limit (3.2.19) with $c = \frac{1}{2\pi}\sqrt{\frac{\cos\theta+1}{\cos\theta-\cos L}}$ for $\theta\in(-L,L)$. Near the edges of γ, we obtain the Bessel kernel,

$$\lim_{n\to\infty}\frac{1}{(cn)^2}K_n\left(e^{\pm i\left(L-\frac{u}{(cn)^2}\right)},e^{\pm i\left(L-\frac{v}{(cn)^2}\right)}\right)$$
$$=\frac{J_0(\sqrt{u})\sqrt{v}J_0'(\sqrt{v})-J_0(\sqrt{v})\sqrt{u}J_0'(\sqrt{u})}{2(u-v)},\quad(3.2.24)$$

for $u,v>0$, where $c=\sqrt{\cot\frac{L}{2}}$. This is remarkable because the Bessel kernel usually appears near hard edges of the spectrum. In our situation, for finite n and $s\in(0,1)$, it can happen that there are eigenvalues on γ^c, so we do not have a hard edge. Taking the logarithmic s-derivative of (3.2.7), we obtain by (3.1.13)–(3.1.14),

$$\mathbb{E}\left(\#(\Theta\cap\gamma^c)|\Phi\subset\gamma\right)=o(s),\quad(3.2.25)$$
$$\operatorname{Var}\left(\#(\Theta\cap\gamma^c)|\Phi\subset\gamma\right)=o(s),\quad(3.2.26)$$

as $n\to\infty$. In other words, the expected number of eigenvalues on γ^c is exponentially small, as well as the number variance.

In this case, observing no particles on γ^c implies that it is unlikely to have unobserved particles there too. This is natural because typically no or very few particles are removed by the thinning procedure. Then, because of the repulsion between the eigenvalues, there tend to be more eigenvalues near the edges $e^{\pm iL}$ of γ, which heuristically explains the blow-up of the limiting mean eigenvalue density. Asymptotically for large n, the eigenvalues in the conditional CUE behave locally near $e^{\pm iL}$ like near a hard edge.

Case IV: $s\to 0$ at a slower rate, L fixed. This is the intermediate regime between Case I and Case III. In contrast to Case I, not observing particles outside γ does have an effect on the large n macroscopic behaviour of the eigenvalues, but it is not as drastic as in Case III. Typically, for large n, there will be a non-zero fraction of eigenvalues outside γ, but this fraction is smaller than in Case I. We prove the following results.

Theorem 3.2.5 *Fix $0<L<\pi$. We take $s=e^{-xn}$ with $0<x<x_c=-2\log\tan\frac{L}{4}$.*

1. *The measures $\mu_{n,s,L}$ and $\nu_{n,s,L}$ converge weakly to the measure $\mu_{x,L}$ given by* (3.2.12) *as $n\to\infty$; moreover,*

$$\lim_{n\to\infty}\psi_{n,s,L}(e^{i\theta})=\frac{d\mu_{x,L}(e^{i\theta})}{d\theta},\quad(3.2.27)$$

uniformly for θ in compact subsets of $[-\pi,\pi]\setminus\{-\pi+T,-L,L,\pi-T\}$.

2. *We have*

$$\lim_{n\to\infty}\frac{1}{n}\mathbb{E}\left(\#(\Theta\cap\gamma^c)|\Phi\subset\gamma\right)=\int_{\pi-T}^{\pi+T}d\mu_{x,L}(e^{i\theta})d\theta=\Omega_{x,L}, \tag{3.2.28}$$

$$\lim_{n\to\infty}\frac{1}{n}\mathrm{Var}\left(\#(\Theta\cap\gamma^c)|\Phi\subset\gamma\right)=0. \tag{3.2.29}$$

Remark 3.2.6 We will also provide a result on the location of the zeros of the orthogonal polynomials: we will show that ϕ_n has, for n sufficiently large, all its zeros either close to the unit circle or to a point w_n on the real line, see Proposition 3.4.1 below. A similar but stronger result is also shown in Case I and Case III, see Proposition 3.4.3.

Remark 3.2.7 The eigenvalues outside γ are expected on an arc containing -1, and it is unlikely to have eigenvalues close to the edges of γ^c. This is at first sight surprising, but can be explained (a posteriori) by the repulsion between the eigenvalues. As $x\to x_c$, $T\to 0$ and the average of the number of particles on γ^c becomes smaller; as $x\to 0$, $\mu_{x,L}$ converges weakly to the uniform measure on the unit circle, and we recover the leading order of the corresponding result in Case I. It is worth noting that the variance is small compared to the average, this means that we can accurately guess the fraction of eigenvalues lying on γ^c for large n.

Next, we consider scaling limits of the eigenvalue correlation kernels.

Theorem 3.2.8 *We let $s=e^{-xn}$ with $x<x_c$.*

1. *If $e^{i\theta}$ lies in the interior of the support of $\mu_{x,L}$, we have the sine kernel limit* (3.2.19) *with $c=\psi_{x,L}(e^{i\theta})$.*

2. *At the edges of γ, we have the Bessel kernel limits* (3.2.24) *for $u,v>0$, with $c=\sqrt{\frac{|z_0-z_1||z_0-\overline{z_1}|}{|z_0-\overline{z_0}|}}$, $z_0=e^{iL}$, and $z_1=e^{i(\pi-T)}$.*

3. *At the soft edges, we have the Airy kernel limits*

$$\lim_{n\to\infty}\frac{e^{-i\frac{n^{1/3}}{2c^{2/3}}(u-v)}}{(cn)^{2/3}}K_n\left(e^{\pm i\left(-\pi+T+\frac{u}{(cn)^{2/3}}\right)},e^{\pm i\left(-\pi+T+\frac{v}{(cn)^{2/3}}\right)}\right)$$
$$=\frac{\mathrm{Ai}\,(u)\mathrm{Ai}\,'(v)-\mathrm{Ai}\,'(u)\mathrm{Ai}\,(v)}{u-v}, \tag{3.2.30}$$

for $u, v \in \mathbb{R}$, *with* $c = \sqrt{\frac{|\overline{z_1}-z_1|}{4|\overline{z_1}-z_0||\overline{z_1}-\overline{z_0}|}}$.

Remark 3.2.9 As expected, we obtain the sine kernel in the interior of the support of $\mu_{x,L}$. Near the edges, we have two different types of behaviour: near $\theta = \pm L$, scaling limits lead to the Bessel kernel, as if there were hard edges; near $\theta = \pi - T$ and $\theta = \pi + T$, we have soft edges with square root vanishing of the limiting mean eigenvalue density, and obtain the usual Airy kernel.

Outline

In Section 3.3, we will use the Deift/Zhou steepest descent method applied to the RH problem for orthogonal polynomials on the unit circle to obtain large n asymptotics for ϕ_n in Case IV. The main ingredients of the RH analysis will be a g-function related to the equilibrium measure $\mu_{x,L}$, the construction of standard Airy parametrices near $e^{\pm i(\pi-T)}$, and the construction of Bessel parametrices near $e^{\pm iL}$. Because those points are not hard edges, we will need to use a non-standard Bessel parametrix similar to the one constructed in Chapter 2 and in [15]. The RH analysis yields asymptotics for ϕ_n, which we describe in detail in Section 3.3.8.

Next, in Section 3.4, we will use the asymptotics for ϕ_n to obtain asymptotics for the zero counting measure $\nu_{n,s,L}$, the mean eigenvalue distribution $\mu_{n,s,L}$, and the eigenvalue correlation kernel. In this way, we will be able to prove Theorem 3.2.1, Theorem 3.2.5, and Theorem 3.2.8, as well as the corresponding results in Cases I-III.

In Appendix A, we prove (3.2.11), and in Appendix B, we extend the asymptotic analysis from Section 3.3 to the case of small x.

3.3 Riemann-Hilbert analysis for $0 < x < x_c$

The goal of this section is to obtain large n asymptotics for $\phi_n(z)$ for z anywhere in the complex plane.

3.3.1 RH problem for orthogonal polynomials on the unit circle

Consider the matrix Y defined by

$$Y(z) = \begin{pmatrix} \phi_n(z) & \displaystyle\int_{S^1} \frac{\phi_n(w)}{w-z} \frac{f(w)}{2\pi i w^n} dw \\ -h_{n-1}^{-1} z^{n-1} \phi_{n-1}(z^{-1}) & -h_{n-1}^{-1} \displaystyle\int_{S^1} \frac{\phi_{n-1}(w^{-1})}{w-z} \frac{f(w)}{2\pi i w} dw \end{pmatrix}, \quad (3.3.1)$$

where ϕ_n is the monic orthogonal polynomial with respect to the weight f, given by (3.1.5) and depending on s and L, and where h_n is the norming constant, defined in (3.1.10).

The first column of Y contains the orthogonal polynomials of degree n and $n-1$, while the second column contains their Cauchy transforms with respect to the weight function f. Y depends on n and also on $s = e^{-xn}$ and L through the weight function f. It is well known that $Y(z)$ is the unique 2×2 matrix-valued function which satisfies the following RH problem [34]:

RH problem for Y

(a) $Y : \mathbb{C} \setminus S^1 \to \mathbb{C}^{2\times 2}$ is analytic.

(b) Y has the following jumps:

$$Y_+(z) = Y_-(z) \begin{pmatrix} 1 & z^{-n} f(z) \\ 0 & 1 \end{pmatrix}, \qquad \text{for } z \in S^1 \setminus \left\{e^{iL}, e^{-iL}\right\}.$$

(c) $Y(z) = \left(I + \mathcal{O}(z^{-1})\right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$ as $z \to \infty$.

(d) As z tends to $e^{\pm iL}$, the behaviour of Y is

$$Y(z) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\log|z - e^{\pm iL}|) \\ \mathcal{O}(1) & \mathcal{O}(\log|z - e^{\pm iL}|) \end{pmatrix}.$$

We want to obtain large n asymptotics for the matrix Y, uniformly for x in compact subsets of $(0, x_c)$. Such asymptotics can be obtained using the Deift/Zhou steepest descent method [30] applied to the RH problem for Y. Following the general procedure developed in [27, 28] and applied first to orthogonal polynomials on the unit circle in [6], we will apply a series of transformations to the RH problem for Y, with the goal of obtaining, in the end, a RH problem for which we can easily compute asymptotics of the solution. The first transformations are similar to those in Section 2.4.

3.3.2 First transformation $Y \mapsto T$

We define g, depending on x and L, by

$$g(z) = \int_{[-L,L]\cup[\pi-T,\pi+T]} \log(z - e^{i\theta}) d\mu(e^{i\theta}), \tag{3.3.2}$$

where we write μ for $\mu_{x,L}$, omitting the subscripts x, L, where $\mu_{x,L}$ is the equilibrium measure with density (3.2.12) and satisfying the variational conditions (3.2.15). In (3.3.2), for each θ, the branch is chosen such that $\log(z - e^{i\theta})$ is analytic in $\mathbb{C} \setminus ((-\infty, -1] \cup \{e^{it} : -\pi \leq t \leq \theta\})$ and $\log(x - e^{i\theta}) \sim \log x$ as $x \in \mathbb{R}^+$, $x \to \infty$.

The following transformation has the effect of normalizing the RH problem at infinity. We define

$$T(z) = e^{-\frac{n\pi i}{2}\sigma_3} e^{\frac{n\ell}{2}\sigma_3} Y(z) e^{-ng(z)\sigma_3} e^{-\frac{n\ell}{2}\sigma_3} e^{\frac{n\pi i}{2}\sigma_3}, \tag{3.3.3}$$

with $\ell = \ell_{x,L}$ given by (3.2.16), and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then T solves the following RH problem:

RH problem for T

(a) $T : \mathbb{C} \setminus S^1 \to \mathbb{C}^{2\times 2}$ is analytic.

(b) T satisfies the jump relation

$$T_+(z) = T_-(z) J_T(z), \qquad \text{on } S^1 \setminus \left\{e^{iL}, e^{-iL}\right\},$$

with

$$J_T(z) = \begin{pmatrix} e^{-n(g_+(z)-g_-(z))} & (-1)^n z^{-n} e^{-nV(z)} e^{n\ell} e^{n(g_+(z)+g_-(z))} \\ 0 & e^{n(g_+(z)-g_-(z))} \end{pmatrix}.$$

(c) $T(z) = I + O(z^{-1})$ as $z \to \infty$.

(d) As $z \to e^{\pm iL}$, we have

$$T(z) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\log|z - e^{\pm iL}|) \\ \mathcal{O}(1) & \mathcal{O}(\log|z - e^{\pm iL}|) \end{pmatrix}.$$

Writing

$$\Sigma_1 = \{e^{i\theta} : L < \theta < \pi - T\}, \tag{3.3.4}$$

$$\Sigma_2 = \{e^{i\theta} : -\pi + T < \theta < -L\}, \tag{3.3.5}$$

$$\widetilde{\gamma} = \{e^{i\theta} : \pi - T < \theta < \pi + T\}, \tag{3.3.6}$$

we can use (3.2.15) and the definition of g (3.3.2) to write the jump matrices for T in the following form,

$$J_T(z) = \begin{cases} \begin{pmatrix} e^{-2n\pi i \int_{\arg z}^{\pi} d\mu(e^{i\theta})} & 1 \\ 0 & e^{2n\pi i \int_{\arg z}^{\pi} d\mu(e^{i\theta})} \end{pmatrix}, & z \in \gamma \cup \widetilde{\gamma}, \\ \begin{pmatrix} e^{-n\pi i\Omega} & e^{n\left[2\int_{-\pi}^{\pi} \log|z-e^{i\theta}| d\mu(e^{i\theta}) - x + \ell\right]} \\ 0 & e^{n\pi i\Omega} \end{pmatrix}, & z \in \Sigma_1, \\ \begin{pmatrix} e^{n\pi i\Omega} & e^{n\left[2\int_{-\pi}^{\pi} \log|z-e^{i\theta}| d\mu(e^{i\theta}) - x + \ell\right]} \\ 0 & e^{-n\pi i\Omega} \end{pmatrix}, & z \in \Sigma_2, \end{cases} \tag{3.3.7}$$

where $\Omega = \int_{\pi-T}^{\pi+T} d\mu(e^{i\theta})$, as in (3.2.9). We write $z_0 = e^{iL}$, $z_1 = e^{i(\pi-T)}$ and define

$$\phi(z) = \int_{z_0}^{z} \left(\frac{(\xi - z_1)(\xi - \overline{z_1})}{(\xi - z_0)(\xi - \overline{z_0})}\right)^{1/2} \frac{d\xi}{\xi}, \tag{3.3.8}$$

and

$$\widetilde{\phi}(z) = \int_{\overline{z_1}}^{z} \left(\frac{(\xi - z_1)(\xi - \overline{z_1})}{(\xi - z_0)(\xi - \overline{z_0})}\right)^{1/2} \frac{d\xi}{\xi}, \tag{3.3.9}$$

where we define the square roots with branch cuts on $\gamma \cup \widetilde{\gamma}$ and such that they tend to 1 as $\xi \to \infty$. For ϕ, we take the path of integration such that it does not cross $\overline{\gamma \cup \widetilde{\gamma} \cup \Sigma_2}$, and for $\widetilde{\phi}$, we take it such that it does not cross $\overline{\gamma \cup \widetilde{\gamma} \cup \Sigma_1}$. Then, it is straightforward to check that e^{ϕ} is single-valued and analytic on $\mathbb{C} \setminus \left(\overline{\gamma \cup \widetilde{\gamma} \cup \Sigma_2} \cup \{0\}\right)$, and that $e^{\widetilde{\phi}}$ is single-valued and analytic on $\mathbb{C} \setminus \left(\overline{\gamma \cup \widetilde{\gamma} \cup \Sigma_1} \cup \{0\}\right)$. We can rewrite the jump matrix J_T in terms of ϕ

and $\widetilde{\phi}$, in the same manner as in Section 2.4.1:

$$J_T(z) = \begin{cases} \begin{pmatrix} e^{n(\phi_-(z)-\pi i\Omega)} & 1 \\ 0 & e^{-n(\phi_-(z)-\pi i\Omega)} \end{pmatrix}, & z \in \gamma, \\ \begin{pmatrix} e^{n(\widetilde{\phi}_-(z)+\pi i\Omega)} & 1 \\ 0 & e^{-n(\widetilde{\phi}_-(z)+\pi i\Omega)} \end{pmatrix}, & z \in \widetilde{\gamma}, \\ \begin{pmatrix} e^{-n\pi i\Omega} & e^{n(\phi(z)-x)} \\ 0 & e^{n\pi i\Omega} \end{pmatrix}, & z \in \Sigma_1, \\ \begin{pmatrix} e^{n\pi i\Omega} & e^{n\widetilde{\phi}(z)} \\ 0 & e^{-n\pi i\Omega} \end{pmatrix}, & z \in \Sigma_2. \end{cases} \tag{3.3.10}$$

Now we are able to define the next transformation: the opening of the lenses.

3.3.3 Second transformation $T \mapsto S$

We can factorize J_T on γ as follows:

$$\begin{pmatrix} e^{n(\phi_-(z)-\pi i\Omega)} & 1 \\ 0 & e^{-n(\phi_-(z)-\pi i\Omega)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ e^{-n(\phi_-(z)-\pi i\Omega)} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{n(\phi_-(z)-\pi i\Omega)} & 1 \end{pmatrix}.$$

There is a similar factorization of J_T on $\widetilde{\gamma}$. Using this factorization, we can split the jump on γ into three different jumps on a lens-shaped contour, see Figure 3.2. Denote by γ_+ and γ_- the lenses around γ on the $|z| < 1$ side and the $|z| > 1$ side respectively and similarly by $\widetilde{\gamma}_+$ and $\widetilde{\gamma}_-$ the lenses around $\widetilde{\gamma}$,

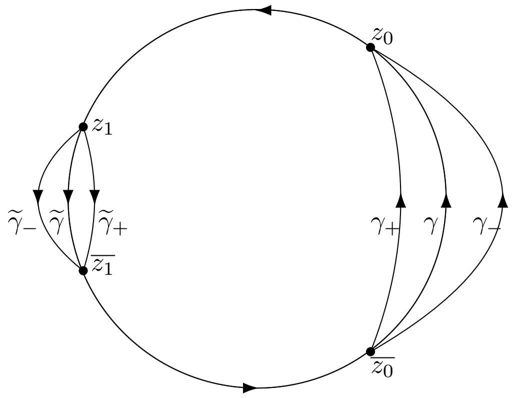

Figure 3.2: The jump contour for S.

see Figure 3.2. Define

$$S(z) = T(z) \times \begin{cases} \begin{pmatrix} 1 & 0 \\ -e^{-n\phi(z)}e^{-n\pi i\Omega} & 1 \end{pmatrix}, & |z| < 1, z \text{ inside the lenses around } \gamma, \\ \begin{pmatrix} 1 & 0 \\ e^{-n\phi(z)}e^{n\pi i\Omega} & 1 \end{pmatrix}, & |z| > 1, z \text{ inside the lenses around } \gamma, \\ \begin{pmatrix} 1 & 0 \\ -e^{-n\widetilde{\phi}(z)}e^{n\pi i\Omega} & 1 \end{pmatrix}, & |z| < 1, z \text{ inside the lenses around } \widetilde{\gamma}, \\ \begin{pmatrix} 1 & 0 \\ e^{-n\widetilde{\phi}(z)}e^{-n\pi i\Omega} & 1 \end{pmatrix}, & |z| > 1, z \text{ inside the lenses around } \widetilde{\gamma}, \\ I, & z \text{ outside the lenses.} \end{cases} \tag{3.3.11}$$

Then S solves the following RH problem.

RH problem for S

(a) $S : \mathbb{C} \setminus (S^1 \cup \gamma_+ \cup \gamma_- \cup \widetilde{\gamma}_+ \cup \widetilde{\gamma}_-) \to \mathbb{C}^{2\times 2}$ is analytic.

(b) S satisfies the jump relations

$$\begin{aligned}
S_+(z) &= S_-(z)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, && \text{for } z \in \gamma \cup \widetilde{\gamma}, \\
S_+(z) &= S_-(z)\begin{pmatrix} e^{-n\pi i\Omega} & e^{n(\phi(z)-x)} \\ 0 & e^{n\pi i\Omega} \end{pmatrix}, && \text{for } z \in \Sigma_1, \\
S_+(z) &= S_-(z)\begin{pmatrix} e^{n\pi i\Omega} & e^{n\widetilde{\phi}(z)} \\ 0 & e^{-n\pi i\Omega} \end{pmatrix}, && \text{for } z \in \Sigma_2, \\
S_+(z) &= S_-(z)\begin{pmatrix} 1 & 0 \\ e^{-n\phi(z)}e^{-n\pi i\Omega} & 1 \end{pmatrix}, && \text{for } z \in \gamma_+, \\
S_+(z) &= S_-(z)\begin{pmatrix} 1 & 0 \\ e^{-n\phi(z)}e^{n\pi i\Omega} & 1 \end{pmatrix}, && \text{for } z \in \gamma_-, \\
S_+(z) &= S_-(z)\begin{pmatrix} 1 & 0 \\ e^{-n\widetilde{\phi}(z)}e^{n\pi i\Omega} & 1 \end{pmatrix}, && \text{for } z \in \widetilde{\gamma}_+, \\
S_+(z) &= S_-(z)\begin{pmatrix} 1 & 0 \\ e^{-n\widetilde{\phi}(z)}e^{-n\pi i\Omega} & 1 \end{pmatrix}, && \text{for } z \in \widetilde{\gamma}_-.
\end{aligned} \tag{3.3.12}$$

(c) $S(z) = I + O(z^{-1})$ as $z \to \infty$.

(d) As $z \to e^{\pm iL}$, we have

$$S(z) = \mathcal{O}(\log|z - e^{\pm iL}|).$$

On $\gamma_+ \cup \gamma_-$ (resp. $\widetilde{\gamma}_+ \cup \widetilde{\gamma}_-$), one shows as in Section 2.4.2 that $\operatorname{Re}\phi(z) > 0$ (resp. $\operatorname{Re}\widetilde{\phi}(z) > 0$) and consequently the jump matrices for S converge to the identity matrix on $\gamma_+ \cup \gamma_- \cup \widetilde{\gamma}_+ \cup \widetilde{\gamma}_-$. On Σ_1, we have that Re $(\phi(z) - x) < 0$, so that the jump matrix converges to a diagonal matrix. On Σ_2 finally, we similarly have $\operatorname{Re}\widetilde{\phi}(z) < 0$, and the jump matrix converges to a diagonal matrix here as well. The convergence of the jump matrices is point-wise in z and breaks down as z approaches $e^{\pm iL}$ and $e^{i(\pi \pm T)}$.

Therefore, we will construct approximations to S for large n in different regions of the complex plane: local parametrices will be constructed in small disks $D(e^{\pm iL}, r)$ and $D(e^{i(\pi\pm T)}, r)$ surrounding the edge points of the support of μ, and a global parametrix elsewhere.

3.3.4 Global parametrix

We will construct the solution to the following RH problem, which is obtained from the RH problem for S by ignoring the exponentially small jumps. The global parametrix will give us a good approximation to S for z away from the endpoints of γ and $\widetilde{\gamma}$.

RH problem for $P^{(\infty)}$

(a) $P^{(\infty)} : \mathbb{C} \setminus S^1 \to \mathbb{C}^{2\times 2}$ is analytic.

(b) $P^{(\infty)}$ has the following jumps:

$$\begin{aligned} P_+^{(\infty)}(z) &= P_-^{(\infty)}(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, && \text{for } z \in \gamma \cup \widetilde{\gamma}, \\ P_+^{(\infty)}(z) &= P_-^{(\infty)}(z) e^{-n\pi i \Omega \sigma_3}, && \text{for } z \in \Sigma_1, \\ P_+^{(\infty)}(z) &= P_-^{(\infty)}(z) e^{n\pi i \Omega \sigma_3}, && \text{for } z \in \Sigma_2. \end{aligned} \tag{3.3.13}$$

(c) $P^{(\infty)}(z) = I + \mathcal{O}(z^{-1})$ as $z \to \infty$.

(d) As $z \to \zeta$ with $\zeta = e^{\pm iL}$ or $\zeta = e^{i(\pi \pm T)}$, we have

$$P^{(\infty)}(z) = \mathcal{O}(|z-\zeta|^{-1/4}).$$

This problem can be explicitly solved in terms of the elliptic theta-function of the third kind. This is typical for situations where the support of the equilibrium measure consists of two disjoint intervals or arcs. Similar RH problems have been solved several times with jumps on the real line, see e.g. [27, 28, 10]. We follow a similar procedure here, adapted to the unit circle.

First we apply the following transformation:

$$Q^{(\infty)}(z) = e^{-n\pi i \frac{\Omega}{2}\sigma_3} P^{(\infty)}(z) \begin{cases} e^{-n\pi i \frac{\Omega}{2}\sigma_3}, & |z| < 1, \\ e^{n\pi i \frac{\Omega}{2}\sigma_3}, & |z| > 1. \end{cases} \tag{3.3.14}$$

It is easy to verify that $Q^{(\infty)}$ satisfies the following RH problem.

RH problem for $Q^{(\infty)}$

(a) $Q^{(\infty)} : \mathbb{C} \setminus (\operatorname{supp} \mu \cup \Sigma_1) \to \mathbb{C}^{2\times 2}$ is analytic.

(b) $Q^{(\infty)}$ has the following jumps:

$$\begin{aligned} Q_+^{(\infty)}(z) &= Q_-^{(\infty)}(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, && \text{for } z \in \gamma \cup \widetilde{\gamma}, \\ Q_+^{(\infty)}(z) &= Q_-^{(\infty)}(z) e^{-2n\pi i \Omega \sigma_3}, && \text{for } z \in \Sigma_1. \end{aligned} \tag{3.3.15}$$

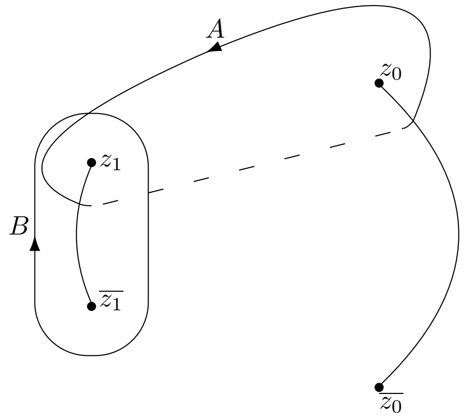

Figure 3.3: The cycles A and B.

(c) $Q^{(\infty)}(z) = I + O(z^{-1})$ as $z \to \infty$.

(d) As $z \to \zeta$ with $\zeta = e^{\pm iL}$ or $\zeta = e^{i(\pi \pm T)}$, we have

$$Q^{(\infty)}(z) = \mathcal{O}(|z-\zeta|^{-1/4}).$$

Riemann surface and elliptic theta function

To construct Q, we need to introduce quantities related to an elliptic theta-function on a Riemann surface. We consider the elliptic curve

$$X = \{(z,w) : w^2 = R(z)\}, \qquad R(z) = (z-\overline{z_0})(z-z_0)(z-z_1)(z-\overline{z_1}),$$

of genus one. We represent X as the two-sheeted Riemann surface associated to $\sqrt{R(z)}$. We let $\sqrt{R(z)} \sim z^2$ as $z \to \infty$ on the first sheet and $\sqrt{R(z)} \sim -z^2$ as $z \to \infty$ on the second sheet.

We define cycles A and B on X as in Figure 3.3: B encircles the arc $\widetilde{\gamma}$ in the clockwise sense on the first sheet, and A encircles the arc Σ_1. The solid part in Figure 3.3 lies on the first sheet, the dashed part on the second sheet. A and B form a canonical homology basis for X.

Note that by an explicit calculation we have

$$\begin{aligned}
\int_A \frac{1}{\sqrt{R(z)}}dz &= 2\int_{z_0}^{z_1} \frac{dz}{iz\sqrt{|R(z)|}} = 2\int_L^{\pi-T} \frac{d\theta}{\sqrt{|R(e^{i\theta})|}} \in \mathbb{R}^+, \\
\int_B \frac{1}{\sqrt{R(z)}}dz &= 2\int_{z_1}^{\overline{z_1}} \frac{dz}{z\sqrt{|R(z)|}} = 2i\int_{\pi-T}^{\pi+T} \frac{d\theta}{\sqrt{|R(e^{i\theta})|}} \in i\mathbb{R}^+.
\end{aligned}$$

We define the one-form

$$\omega = \frac{c_0 dz}{\sqrt{R(z)}}, \qquad c_0 = \frac{1}{\int_A \frac{dz}{\sqrt{R(z)}}} \in \mathbb{R}^+, \tag{3.3.16}$$

such that ω is the unique holomorphic one-form on X which satisfies $\int_A \omega = 1$. We also let

$$\tau = \int_B \omega = \frac{\int_B \frac{dz}{\sqrt{R(z)}}}{\int_A \frac{dz}{\sqrt{R(z)}}} \in i\mathbb{R}^+. \tag{3.3.17}$$

The associated theta-function of the third kind $\theta(z;\tau) = \theta(z)$ is given by

$$\theta(z) = \sum_{m=-\infty}^{+\infty} e^{2\pi i m z} e^{\pi i m^2 \tau}. \tag{3.3.18}$$

It is an entire and even function satisfying

$$\theta(z+1) = \theta(z) \qquad \text{and} \qquad \theta(z+\tau) = e^{-2\pi i z} e^{-\pi i \tau} \theta(z) \qquad \text{for all } z \in \mathbb{C}. \tag{3.3.19}$$

Finally, define

$$u(z) = \int_{\overline{z_1}}^{z} \omega, \tag{3.3.20}$$

for z on the Riemann surface, where the path of integration lies in $\mathbb{C}\setminus(\overline{\gamma \cup \widetilde{\gamma} \cup \Sigma_1})$ and on the same sheet as z. Since $\oint_{C(0,R)} \omega = 0$ for all $R > 1$, with $C(0,R)$ a circle of radius R around 0, on each sheet, u can be seen as a single-valued and analytic function of $z \in \mathbb{C} \setminus (\overline{\gamma \cup \widetilde{\gamma} \cup \Sigma_1})$. For z on the first sheet, it satisfies the relations

$$\begin{aligned}
&\text{for } z \in \Sigma_1, \quad u_+(z) - u_-(z) = -\int_B \omega = -\tau, \\
&\text{for } z \in \widetilde{\gamma}, \quad\; u_+(z) + u_-(z) = 0, \\
&\text{for } z \in \gamma, \quad\; u_+(z) + u_-(z) = -\int_A \omega = -1.
\end{aligned} \tag{3.3.21}$$

Construction of $Q^{(\infty)}$

Now we define

$$f_1(z;c,d)=\frac{\theta(u(z)+d+c)}{\theta(u(z)+d)},\qquad f_2(z;c,d)=\frac{\theta(-u(z)+d+c)}{\theta(-u(z)+d)},\tag{3.3.22}$$

where we take z on the first sheet. These functions are meromorphic on $\mathbb{C}\setminus(\overline{\gamma\cup\widetilde{\gamma}\cup\Sigma_1})$ with possible poles at the zeros of $\theta(u(z)+d)$ and $\theta(-u(z)+d)$. Furthermore, by the periodicity properties of the θ-function, we have the following relations:

$$\begin{aligned}
&\text{for } z\in\Sigma_1, && f_1(z;c,d)_+=e^{2\pi ic}f_1(z;c,d)_-,\\
& && f_2(z;c,d)_+=e^{-2\pi ic}f_2(z;c,d)_-,\\
&\text{for } z\in\gamma\cup\tilde{\gamma}, && f_1(z;c,d)_+=f_2(z;c,d)_-,\\
& && f_2(z;c,d)_+=f_1(z;c,d)_-.
\end{aligned}\tag{3.3.23}$$

This implies that

$$F(z;c,d_1,d_2)=\begin{pmatrix} f_1(z;c,d_1) & f_2(z;c,d_1)\\ f_1(z;c,d_2) & f_2(z;c,d_2)\end{pmatrix}$$

satisfies the jump relations

$$F_+(z)=F_-(z)e^{2\pi ic\sigma_3},\qquad \text{for } z\in\Sigma_1,\tag{3.3.24}$$

$$F_+(z)=F_-(z)\begin{pmatrix}0&1\\1&0\end{pmatrix},\qquad \text{for } z\in\gamma\cup\widetilde{\gamma}.\tag{3.3.25}$$

On the other hand, the function

$$N(z)=\begin{pmatrix}\frac{1}{2}(\beta(z)+\beta^{-1}(z)) & \frac{1}{-2i}(\beta(z)-\beta^{-1}(z))\\ \frac{1}{2i}(\beta(z)-\beta^{-1}(z)) & \frac{1}{2}(\beta(z)+\beta^{-1}(z))\end{pmatrix},\tag{3.3.26}$$

where

$$\beta(z)=\left(\frac{z-\overline{z_0}}{z-z_0}\frac{z-z_1}{z-\overline{z_1}}\right)^{1/4}\tag{3.3.27}$$

with branch cut on $\gamma\cup\widetilde{\gamma}$ and such that $\beta(z)\to 1$ as $z\to\infty$, satisfies the following RH problem, see e.g. [10] for similar situations:

RH problem for N

(a) $N : \mathbb{C} \setminus \operatorname{supp}\mu \to \mathbb{C}^{2\times 2}$ is analytic.

(b) N has the following jump:

$$N_+(z) = N_-(z)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \text{for } z \text{ on } \gamma \cup \widetilde{\gamma}.$$

(c) $N(z) = I + O(z^{-1})$ as $z \to \infty$.

(d) As $z \to \zeta$, $\zeta = z_0, \overline{z_0}, z_1$ or $\overline{z_1}$, we have

$$N(z) = \mathcal{O}(|z-\zeta|^{-1/4}).$$

Now, we take $c = -n\Omega$ and $d_1 = -d_2 = d$, and define $Q^{(\infty)}$ by

$$Q^{(\infty)}(z) = \begin{pmatrix} \frac{\theta(u_\infty+d-n\Omega)}{\theta(u_\infty+d)} & 0 \\ 0 & \frac{\theta(u_\infty+d+n\Omega)}{\theta(u_\infty+d)} \end{pmatrix}^{-1} \times \begin{pmatrix} N_{11}(z)F_{11}(z;-n\Omega,d,-d) & N_{12}(z)F_{12}(z;-n\Omega,d,-d) \\ N_{21}(z)F_{21}(z;-n\Omega,d,-d) & N_{22}(z)F_{22}(z;-n\Omega,d,-d) \end{pmatrix}, \tag{3.3.28}$$

where we define $u_\infty = \lim_{z\to\infty} u(z)$. Combining the jump relations for F and N, it is straightforward to verify that $Q^{(\infty)}$ satisfies the RH problem for $Q^{(\infty)}$, except for the possible problem that it may have poles at the zeros of the functions $\theta(u(z)-d)$ and $\theta(u(z)+d)$. We will exploit the freedom we have to choose the value of d to ensure that the zeros in the denominator are cancelled out by zeros in the numerator, so that $Q^{(\infty)}$ is analytic in $\mathbb{C} \setminus (\operatorname{supp}\mu \cup \Sigma_1)$. Note also that $\theta(z) \neq 0$ at $z = u_\infty + d$, $z = u_\infty + d - n\Omega$ and $z = u_\infty + d - n\Omega$ since these three points are purely real and that $\theta(z)$ doesn't vanish on $\mathbb{R}$. This will be a consequence of Proposition 3.3.1, see below, and the fact that $\Omega \in \mathbb{R}$.

For $w \in \mathbb{C} \setminus \operatorname{supp}\mu$, we denote by $w^{(1)}$ the representation of w on the first sheet of X, and $w^{(2)}$ for the one on the second sheet. We define

$$z_\star = \frac{z_0\overline{z_1} - \overline{z_0}z_1}{(z_0 - \overline{z_0}) - (z_1 - \overline{z_1})}. \tag{3.3.29}$$

This is the projection of z_1 along the vector $z_1 - z_0$ on the real axis $\mathbb{R}$. So if $0 < \sin T < \sin L$, $z_\star \in]-\infty, -1[$, if $\sin T = \sin L$, $z_\star = \infty$, and if $\sin L < \sin T$, $z_\star \in]1, +\infty[$.

Proposition 3.3.1 *Let* $d = -K + \int_{\overline{z_1}}^{z_\star^{(1)}} \omega$, *where* $K = \frac{1}{2} + \frac{\tau}{2}$. *Then* $Q^{(\infty)}$ *defined by equation* (3.3.28) *is the solution of the RH problem for* $Q^{(\infty)}$. *Moreover,* $u_\infty + d \equiv 0 \mod 1$.

Proof. First of all, we identify the zeros of the functions $\beta(z) \pm \beta^{-1}(z)$ on the Riemann surface. One of those functions vanishes at a point z on the Riemann surface if and only if

$$\beta(z)^2 \pm 1 = 0 \Leftrightarrow \beta^4(z) = 1 \Leftrightarrow z = z_\star. \tag{3.3.30}$$

From the definition of β on the first sheet, we have that $\beta(z_\star^{(1)}) > 0$. So $\beta(z) - \beta^{-1}(z)$ vanishes only at $z_\star^{(1)}$ and $\beta(z) + \beta^{-1}(z)$ does not vanish on the first sheet. Since X is of genus one, we know that $u(z)$ is a bijection from the Riemann surface X to the Jacobi variety $\mathbb{C}/\Lambda$, $\Lambda = \{n + \tau m, n, m \in \mathbb{Z}\}$. Therefore, since $\theta(K) = 0$,

$$\theta(u(z) - d) = \theta\left(\int_{\overline{z_1}}^{z} \omega - \int_{\overline{z_1}}^{z_\star^{(1)}} \omega + K\right)$$

vanishes only at $z = z_\star^{(1)}$. A similar argument shows that

$$\theta(u(z) + d) = \theta\left(\int_{\overline{z_1}}^{z} \omega - \int_{\overline{z_1}}^{z_\star^{(2)}} \omega - K\right)$$

vanishes only at $z = z_\star^{(2)}$. By (3.3.28), it follows that $Q^{(\infty)}$ has no poles in $\mathbb{C} \setminus (\operatorname{supp} \mu \cup \Sigma_1)$. Hence it solves the RH problem for $Q^{(\infty)}$.

Now, note that $\beta^2(z) - 1 = \left(\frac{(z-\overline{z_0})(z-z_1)}{(z-z_0)(z-\overline{z_1})}\right)^{1/2} - 1$ is meromorphic on X, has simple zeros at $z_\star^{(1)}$ and $\infty^{(1)}$, and simple poles at z_0 and $\overline{z_1}$. By the Abel theorem, we have

$$\int_{\overline{z_1}}^{\infty^{(1)}} \omega + \int_{z_0}^{z_\star^{(1)}} \omega \equiv 0 \mod \Lambda. \tag{3.3.31}$$

By the choice of cycles and the definition of ω, there exist $n, m \in \mathbb{Z}$ such that

$$\int_{z_\star^{(1)}}^{-1_-^{(1)}} \omega + \int_{\infty^{(1)}}^{-1_-^{(1)}} \omega = \frac{1}{2} + n + m\tau, \tag{3.3.32}$$

where $-1_-^{(1)}$ means that we start the integration path from -1 on the side $|z| > 1$ of the first sheet. For $z < -1$ on the first sheet, $\omega > 0$ so that $m = 0$ if

$z_\star < -1$ or if $z_\star = \infty$. From the definitions of u_∞, d, and K, and by (3.3.31) and the choice of cycles, it is then straightforward to see that

$$u_\infty + d = \int_{\overline{z_1}}^{z_*^{(1)}} \omega + \int_{z_*^{(1)}}^{z_0} \omega - n - \frac{1}{2} - \frac{\tau}{2} \equiv 0 \mod 1.$$

If $z_\star > 1$, this can be shown in a similar way. □

Summarizing, by (3.3.14) and (3.3.28), we have the following expression for $P^{(\infty)}$:

$$P^{(\infty)}(z) = e^{n\pi i \frac{\Omega}{2}\sigma_3} \begin{pmatrix} \frac{1}{2}(\beta(z)+\beta^{-1}(z))\Theta_{11}(z) & \frac{1}{-2i}(\beta(z)-\beta^{-1}(z))\Theta_{12}(z) \\ \frac{1}{2i}(\beta(z)-\beta^{-1}(z))\Theta_{21}(z) & \frac{1}{2}(\beta(z)+\beta^{-1}(z))\Theta_{22}(z) \end{pmatrix} \times \begin{cases} e^{n\pi i \frac{\Omega}{2}\sigma_3}, & |z|<1, \\ e^{-n\pi i \frac{\Omega}{2}\sigma_3}, & |z|>1, \end{cases} \tag{3.3.33}$$

with

$$\Theta_{11}(z) = \frac{\theta(0)}{\theta(n\Omega)} \frac{\theta(u(z)+d-n\Omega)}{\theta(u(z)+d)}, \qquad \Theta_{12}(z) = \frac{\theta(0)}{\theta(n\Omega)} \frac{\theta(u(z)-d+n\Omega)}{\theta(u(z)-d)},$$
$$\Theta_{21}(z) = \frac{\theta(0)}{\theta(n\Omega)} \frac{\theta(u(z)-d-n\Omega)}{\theta(u(z)-d)}, \qquad \Theta_{22}(z) = \frac{\theta(0)}{\theta(n\Omega)} \frac{\theta(u(z)+d+n\Omega)}{\theta(u(z)+d)}. \tag{3.3.34}$$

In the above formulas involving $u(z)$, z is taken on the first sheet.

3.3.5 Local parametrices near $z_0 = e^{iL}$ and $\overline{z_0} = e^{-iL}$

Near z_0 and $\overline{z_0}$, we want to construct a function which has exactly the same jump conditions than S, and which matches with the global parametrix on the boundaries of small fixed disks of radius $r > 0$, $D(z_0, r)$ or $D(\overline{z_0}, r)$, around z_0 and $\overline{z_0}$. More precisely, near z_0, we want to have the following conditions.

RH problem for P

(a) $P : D(z_0, r) \setminus (S^1 \cup \gamma_+ \cup \gamma_-) \to \mathbb{C}^{2\times 2}$ is analytic.

(b) For $z \in D(z_0,r)\cap(S^1\cup\gamma_+\cup\gamma_-)$, P satisfies the jump conditions

$$\begin{aligned}
P_+(z) &= P_-(z)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, && \text{on } \gamma,\\
P_+(z) &= P_-(z)\begin{pmatrix} e^{-n\pi i\Omega} & e^{n(\phi(z)-x)} \\ 0 & e^{n\pi i\Omega} \end{pmatrix}, && \text{on } \Sigma_1,\\
P_+(z) &= P_-(z)\begin{pmatrix} 1 & 0 \\ e^{-n\phi(z)}e^{-n\pi i\Omega} & 1 \end{pmatrix}, && \text{on } \gamma_+,\\
P_+(z) &= P_-(z)\begin{pmatrix} 1 & 0 \\ e^{-n\phi(z)}e^{n\pi i\Omega} & 1 \end{pmatrix}, && \text{on } \gamma_-.
\end{aligned} \tag{3.3.35}$$

(c) For $z \in \partial D(z_0,r)$, we have

$$P(z) = \left(I+\mathcal{O}(n^{-1})\right)P^{(\infty)}(z), \qquad \text{as } n\to\infty. \tag{3.3.36}$$

(d) As z tends to z_0, the behaviour of P is

$$P(z) = \mathcal{O}(\log|z-z_0|). \tag{3.3.37}$$

We can solve this RH problem in the same way as done in Chapter 2 in terms of Bessel functions. The construction in Chapter 2 is an adaptation of the standard Bessel parametrix construction from [49].

Define Ψ by

$$\Psi(\zeta) = \begin{cases}
\begin{pmatrix} I_0(2\zeta^{\frac12}) & \frac{i}{\pi}K_0(2\zeta^{\frac12}) \\ 2\pi i\zeta^{\frac12}I_0'(2\zeta^{\frac12}) & -2\zeta^{\frac12}K_0'(2\zeta^{\frac12}) \end{pmatrix}, & |\arg\zeta| < \frac{2\pi}{3},\\
\begin{pmatrix} \frac12 H_0^{(1)}(2(-\zeta)^{\frac12}) & \frac12 H_0^{(2)}(2(-\zeta)^{\frac12}) \\ \pi\zeta^{\frac12}\left(H_0^{(1)}\right)'(2(-\zeta)^{\frac12}) & \pi\zeta^{\frac12}\left(H_0^{(2)}\right)'(2(-\zeta)^{\frac12}) \end{pmatrix}, & \frac{2\pi}{3} < \arg\zeta < \pi,\\
\begin{pmatrix} \frac12 H_0^{(2)}(2(-\zeta)^{\frac12}) & -\frac12 H_0^{(1)}(2(-\zeta)^{\frac12}) \\ -\pi\zeta^{\frac12}\left(H_0^{(2)}\right)'(2(-\zeta)^{\frac12}) & \pi\zeta^{\frac12}\left(H_0^{(1)}\right)'(2(-\zeta)^{\frac12}) \end{pmatrix}, & -\pi < \arg\zeta < -\frac{2\pi}{3},
\end{cases} \tag{3.3.38}$$

where $H_0^{(1)}$ and $H_0^{(2)}$ are the Hankel functions of the first and second kind, and I_0 and K_0 are the modified Bessel functions of the first and second kind. Next, let

$$\widehat{\Psi}(\zeta) = (I+A(\zeta))\,\Psi(\zeta), \tag{3.3.39}$$

where $A(\zeta)$ is a nilpotent matrix which is needed to take care of the jump on Σ_1,

$$A(\zeta) = e^{-nx} F(\zeta) \begin{pmatrix} 0 & -\frac{1}{2\pi i}\log(-\zeta) \\ 0 & 0 \end{pmatrix} F^{-1}(\zeta), \tag{3.3.40}$$

with the branch cut of $\log(-\zeta)$ on $\mathbb{R}^+$, and F is the entire function given by

$$F(\zeta) = \begin{cases} \Psi(\zeta) \begin{pmatrix} 1 & -\frac{1}{2\pi i}\log\zeta \\ 0 & 1 \end{pmatrix} & |\arg\zeta| < \frac{2\pi}{3}, \\ \Psi(\zeta) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2\pi i}\log\zeta \\ 0 & 1 \end{pmatrix} & \frac{2\pi}{3} < \arg\zeta < \pi, \\ \Psi(z) \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2\pi i}\log\zeta \\ 0 & 1 \end{pmatrix} & -\pi < \arg\zeta < -\frac{2\pi}{3}. \end{cases} \tag{3.3.41}$$

It was shown in Chapter 2 that $\widehat{\Psi}$ is the solution of the following RH problem.

RH problem for $\widehat{\Psi}$

(a) $\widehat{\Psi} : \mathbb{C} \setminus \Sigma_{\widehat{\Psi}} \to \mathbb{C}^{2\times 2}$ is analytic, where $\Sigma_{\widehat{\Psi}}$ is shown in Figure 3.4.

(b) $\widehat{\Psi}$ satisfies the jump conditions

$$\begin{aligned} \widehat{\Psi}_+(\zeta) &= \widehat{\Psi}_-(\zeta) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, && \text{on } \mathbb{R}^-, \\ \widehat{\Psi}_+(\zeta) &= \widehat{\Psi}_-(\zeta) \begin{pmatrix} 1 & e^{-nx} \\ 0 & 1 \end{pmatrix}, && \text{on } \mathbb{R}^+, \\ \widehat{\Psi}_+(\zeta) &= \widehat{\Psi}_-(\zeta) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, && \text{on } e^{\frac{2\pi}{3}i}\mathbb{R}^+, \\ \widehat{\Psi}_+(\zeta) &= \widehat{\Psi}_-(\zeta) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, && \text{on } e^{-\frac{2\pi}{3}i}\mathbb{R}^+. \end{aligned} \tag{3.3.42}$$

(c) As $\zeta \to \infty$, $\zeta \notin \Sigma_\Psi$, we have

$$\widehat{\Psi}(\zeta) = (I + A(\zeta)) \left(2\pi\zeta^{\frac{1}{2}}\right)^{-\frac{\sigma_3}{2}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \left(I + \mathcal{O}(\zeta^{-\frac{1}{2}})\right) e^{2\zeta^{\frac{1}{2}}\sigma_3}. \tag{3.3.43}$$

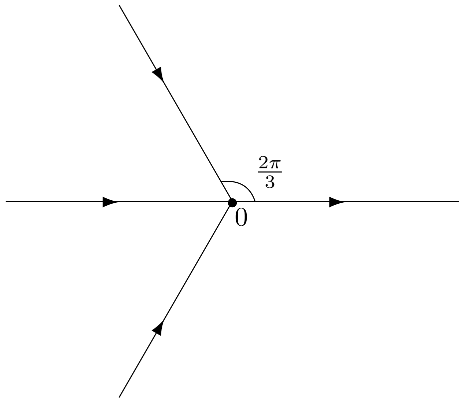

Figure 3.4: The jump contour for $\widehat{\Psi}$ and Φ.

(d) As ζ tends to 0, the behaviour of $\widehat{\Psi}(\zeta)$ is

$$\widehat{\Psi}(\zeta) = \mathcal{O}(\log|\zeta|). \tag{3.3.44}$$

Then, we define $\zeta : D(z_0, r) \to \mathbb{C}$ such that $\zeta(z) = \frac{1}{16}\phi(z)^2$. One can check that ζ is a conformal map which maps z_0 to 0. We can now fix the lenses γ_+ and γ_- such that ζ maps them to the half-lines $e^{\frac{2\pi}{3}i}\mathbb{R}^+$ and $e^{-\frac{2\pi}{3}i}\mathbb{R}^+$.

We can now construct the local parametrix as follows,

$$P(z) = E(z)\widehat{\Psi}(n^2\zeta(z))e^{-\frac{n}{2}\phi(z)\sigma_3} \times \left\{ \begin{array}{ll} e^{-n\pi i\frac{\Omega}{2}\sigma_3}, & |z| < 1, \\ e^{n\pi i\frac{\Omega}{2}\sigma_3}, & |z| > 1, \end{array} \right\} \tag{3.3.45}$$

where

$$E(z) = P^{(\infty)}(z) \times \left\{ \begin{array}{ll} e^{n\pi i\frac{\Omega}{2}\sigma_3}, & |z| < 1, \\ e^{-n\pi i\frac{\Omega}{2}\sigma_3}, & |z| > 1, \end{array} \right\} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \left(\frac{1}{2}n\pi\phi(z)\right)^{\sigma_3/2}, \tag{3.3.46}$$

where the principal branch is chosen for $(\cdot)^{1/2}$, in such a way that E is analytic in $D(z_0, r)$. Then, P solves the RH problem for P.

The local parametrix near $\overline{z_0}$ can simply be constructed by symmetry. For $z \in D(\overline{z_0}, r)$, we have

$$P(z) = \overline{P(\overline{z})}.$$

3.3.6 Local parametrices near $z_1 = e^{i(\pi - T)}$ and $\overline{z_1} = e^{i(\pi+T)}$

Near z_1 and $\overline{z_1}$, we again want to construct a function P which has the same jump relations than S, and which matches with the global parametrix.

In $D(\overline{z_1}, r)$, we need the following RH conditions.

RH problem for P

(a) $P : D(\overline{z_1}, r) \setminus (S^1 \cup \widetilde{\gamma}_+ \cup \widetilde{\gamma}_-) \to \mathbb{C}^{2\times 2}$ is analytic.

(b) For $z \in D(\overline{z_1}, r) \cap (S^1 \cup \widetilde{\gamma}_+ \cup \widetilde{\gamma}_-)$, P satisfies the jump conditions

$$\begin{aligned}
P_+(z) &= P_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, && \text{on } \widetilde{\gamma}, \\
P_+(z) &= P_-(z) \begin{pmatrix} e^{n\pi i\Omega} & e^{n\widetilde{\phi}(z)} \\ 0 & e^{-n\pi i\Omega} \end{pmatrix}, && \text{on } \Sigma_2, \\
P_+(z) &= P_-(z) \begin{pmatrix} 1 & 0 \\ e^{-n\widetilde{\phi}(z)} e^{n\pi i\Omega} & 1 \end{pmatrix}, && \text{on } \widetilde{\gamma}_+, \\
P_+(z) &= P_-(z) \begin{pmatrix} 1 & 0 \\ e^{-n\widetilde{\phi}(z)} e^{-n\pi i\Omega} & 1 \end{pmatrix}, && \text{on } \widetilde{\gamma}_-.
\end{aligned} \tag{3.3.47}$$

(c) For $z \in \partial D(\overline{z_1}, r)$, we have

$$P(z) = \left(I + \mathcal{O}(n^{-1})\right) P^{(\infty)}(z), \qquad \text{as } n \to \infty. \tag{3.3.48}$$

(d) As z tends to $\overline{z_1}$, the behaviour of P is

$$P(z) = \mathcal{O}(1). \tag{3.3.49}$$

We will construct the solution of this RH problem in terms of the Airy function. For this, we use the following well-known model RH problem, see e.g. [27, 28, 10].

Airy model RH problem

(a) $\Phi : \mathbb{C} \setminus \Sigma_\Phi \to \mathbb{C}^{2\times 2}$ is analytic, with Σ_Φ as in Figure 3.4.

(b) For $z \in \Sigma_\Phi$, Φ satisfies the jump conditions

$$\begin{aligned} \Phi_+(z) &= \Phi_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, && \text{on } \mathbb{R}^-, \\ \Phi_+(z) &= \Phi_-(z) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, && \text{on } \mathbb{R}^+, \\ \Phi_+(z) &= \Phi_-(z) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, && \text{on } e^{\frac{2\pi}{3}i}\mathbb{R}^+, \\ \Phi_+(z) &= \Phi_-(z) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, && \text{on } e^{-\frac{2\pi}{3}i}\mathbb{R}^+. \end{aligned} \tag{3.3.50}$$

(c) As $z \to \infty$, we have

$$\Phi(z) = \frac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}\sigma_3} \begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix} \left(I + \mathcal{O}(z^{-3/2}) \right) e^{-\frac{2}{3}z^{3/2}\sigma_3}. \tag{3.3.51}$$

(d) As z tends to 0, the behaviour of Φ is

$$\Phi(z) = \mathcal{O}(1). \tag{3.3.52}$$

The unique solution of this RH problem is given by

$$\Phi(z) = \begin{cases} \begin{pmatrix} y_0(z) & -y_2(z) \\ y_0'(z) & -y_2'(z) \end{pmatrix}, & \text{for } 0 < \arg z < \frac{2\pi}{3}, \\ \begin{pmatrix} -y_1(z) & -y_2(z) \\ -y_1'(z) & -y_2'(z) \end{pmatrix}, & \text{for } \frac{2\pi}{3} < \arg z < \pi, \\ \begin{pmatrix} -y_2(z) & y_1(z) \\ -y_2'(z) & y_1'(z) \end{pmatrix}, & \text{for } -\pi < \arg z < -\frac{2\pi}{3}, \\ \begin{pmatrix} y_0(z) & y_1(z) \\ y_0'(z) & y_1'(z) \end{pmatrix}, & \text{for } -\frac{2\pi}{3} < \arg z < 0, \end{cases} \tag{3.3.53}$$

where $y_0(z) = \mathrm{Ai}(z)$, $y_1(z) = e^{\frac{2\pi}{3}i}\mathrm{Ai}(e^{\frac{2\pi}{3}i}z)$ and $y_2(z) = e^{-\frac{2\pi}{3}i}\mathrm{Ai}(e^{-\frac{2\pi}{3}i}z)$.

Construction of the local parametrix

We construct P using the solution Φ to the Airy model RH problem. We take P of the form

$$P(z) = E(z)\Phi(n^{2/3}\widetilde{\zeta}(z))e^{-\frac{n}{2}\widetilde{\phi}(z)\sigma_3} \times \begin{cases} e^{n\pi i\frac{\Omega}{2}\sigma_3}, & |z| < 1 \\ e^{-n\pi i\frac{\Omega}{2}\sigma_3}, & |z| > 1 \end{cases}. \tag{3.3.54}$$

Here $\widetilde{\zeta}$ is a conformal map near $\overline{z_1}$, defined by $\widetilde{\zeta}(z) = \left(-\frac{3}{4}\widetilde{\phi}(z)\right)^{\frac{2}{3}}$. Then, we have $\widetilde{\zeta}(\widetilde{\gamma} \cap D(\overline{z_1}, r)) \subset \mathbb{R}^-$ and $\widetilde{\zeta}(\Sigma_2 \cap D(\overline{z_1}, r)) \subset \mathbb{R}^+$. We had some freedom in the choice of the lenses, we use it now, by choosing $\widetilde{\gamma}_+$ and $\widetilde{\gamma}_-$ such that

$$\widetilde{\zeta}(\widetilde{\gamma}_+ \cap D(\overline{z_1}, r)) \subset e^{\frac{2\pi}{3}i}\mathbb{R}^+,$$
$$\widetilde{\zeta}(\widetilde{\gamma}_- \cap D(\overline{z_1}, r)) \subset e^{-\frac{2\pi}{3}i}\mathbb{R}^+.$$

In this way, if E is analytic in $D(\overline{z_1}, r)$, P has its jumps precisely on the jump contour for S. Using (3.3.54), one verifies moreover that P has exactly the same jumps than S inside $D(\overline{z_1}, r)$.

In order to have the matching condition (3.3.48), we now define E as

$$E(z) = P^{(\infty)}(z) \times \left\{ \begin{array}{ll} e^{-n\pi i\frac{\Omega}{2}\sigma_3}, & |z| < 1 \\ e^{n\pi i\frac{\Omega}{2}\sigma_3}, & |z| > 1 \end{array} \right\} \times \left[\frac{1}{2\sqrt{\pi}} n^{-\frac{1}{6}\sigma_3} \widetilde{\zeta}(z)^{-\frac{1}{4}\sigma_3} \begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix} \right]^{-1}, \tag{3.3.55}$$

with $\widetilde{\zeta}(z)^{-1/4\sigma_3}$ analytic except for $\widetilde{\zeta}(z) \in \mathbb{R}^-$, or $z \in \widetilde{\gamma}$. Using the jump relations satisfied by $P^{(\infty)}$, it is straightforward to check that E is analytic in $D(\overline{z_1}, r))$. In this way, P satisfies the RH conditions needed near $\overline{z_1}$. In $D(z_1, r)$, the local parametrix is directly constructed as $P(z) = \overline{P(\overline{z})}$.

3.3.7 Final transformation $S \mapsto R$

Define

$$R(z) = \begin{cases} S(z)P(z)^{-1}, & z \in D(z_0, r) \cup D(\overline{z_0}, r) \cup D(z_1, r) \cup D(\overline{z_1}, r), \\ S(z)P^{(\infty)}(z)^{-1}, & \text{elsewhere.} \end{cases} \tag{3.3.56}$$

P was constructed in such a way that it has exactly the same jump relations as S in the four small disks, and as a consequence R has no jumps at all inside those disks. Moreover, from the local behaviour of S and P near z_0, $\overline{z_0}$, z_1, $\overline{z_1}$, it follows that R is analytic at these four points.

If we orient the circles in the clockwise sense, the jump matrices for R are given by

$$R_-(z)^{-1}R_+(z) = P(z)P^{(\infty)}(z)^{-1}, \qquad \text{for } z \text{ on the four circles,} \tag{3.3.57}$$

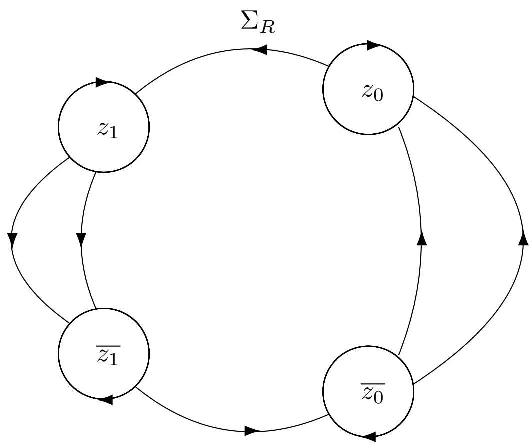

Figure 3.5: The jump contour for R.

and by

$$R_-(z)^{-1}R_+(z) = P^{(\infty)}(z)S_-(z)^{-1}S_+(z)P^{(\infty)}(z)^{-1}, \tag{3.3.58}$$

for $z \in \Sigma_S \setminus \operatorname{supp}\mu$ and z outside the four disks, where Σ_S is the jump contour for S. Notice that R is analytic on the part of $\gamma \cup \widetilde{\gamma}$ outside the four disks. By the fact that the jump matrices for S converge to the identity matrix rapidly outside the four disks, by the uniform boundedness of $P^{(\infty)}$ away from the four edge points, and by the matching conditions (3.3.36) and (3.3.48), the jump matrices for R become small as $n \to \infty$.

In conclusion, we have the following RH problem for R:

RH problem for R

(a) $R : \mathbb{C} \setminus \Sigma_R \to \mathbb{C}^{2\times 2}$ is analytic, where Σ_R consists of the four circles and the part of $\Sigma_S \setminus \operatorname{supp}\mu$ outside the four disks, as in Figure 3.5.

(b) As $n \to \infty$ with $0 < x < x_c$, R satisfies the jump conditions

$$\begin{aligned} R_+(z) &= R_-(z)\left(I + \mathcal{O}(n^{-1})\right), && \text{for } z \text{ on the four circles,} \\ R_+(z) &= R_-(z)(I + \mathcal{O}(e^{-cn})), && \text{for } z \text{ elsewhere on } \Sigma_R, \end{aligned} \tag{3.3.59}$$

with $c > 0$ a constant independent of n.

(c) $R(z) = I + \mathcal{O}(z^{-1})$ as $z \to \infty$.

From the standard theory for small-norm RH problems [27, 28], it follows that

$$R(z) = I + \mathcal{O}(n^{-1}), \qquad R'(z) = \mathcal{O}(n^{-1}), \qquad n \to \infty, \tag{3.3.60}$$

uniformly for $z \in \mathbb{C} \setminus \Sigma_R$. The asymptotics (3.3.60) hold uniformly as long as x lies in a compact subset of $(0, x_c)$, or if s is in a region of the type $e^{-(x_c-\delta)n} \leq s \leq e^{-\delta n}$, with $\delta > 0$ a fixed constant.

3.3.8 Asymptotics for the orthogonal polynomials

We can now use the RH analysis to obtain large n asymptotics for the orthogonal polynomials $\phi_n = Y_{11}$, valid for $\delta < x < x_c - \delta$, for any $\delta > 0$.

The outer region

For any $\epsilon > 0$, if $|z| < 1 - \epsilon$ or $|z| > 1 + \epsilon$, we can take the lenses sufficiently close to the unit circle and the disks for the local parametrices sufficiently small, such that z lies in the region outside the lenses and outside the disks.

Inverting the transformations $Y \mapsto T \mapsto S \mapsto R$, we can express Y in terms of R, and obtain the identity

$$Y(z) = e^{\frac{n\pi i}{2}\sigma_3} e^{-\frac{n\ell}{2}\sigma_3} R(z) P^{(\infty)}(z) e^{-\frac{n\pi i}{2}\sigma_3} e^{\frac{n\ell}{2}\sigma_3} e^{ng(z)\sigma_3}. \tag{3.3.61}$$

By (3.3.33), this leads to

$$\phi_n(z) = e^{ng(z)} \left(P_{11}^{(\infty)}(z) R_{11}(z) + P_{21}^{(\infty)}(z) R_{12}(z) \right). \tag{3.3.62}$$

Since $P^{(\infty)}(z)$ is uniformly bounded in n for z in this region and by (3.3.60), we get

$$\phi_n(z) = e^{ng(z)} \left(P_{11}^{(\infty)}(z) + \mathcal{O}(n^{-1}) \right), \qquad \text{as } n \to \infty. \tag{3.3.63}$$

For the norming constants h_n, we have $h_n = Y_{12}(0)$ by (3.3.1). Using the fact that $g(0) = \pi i$ and the identity $\beta(0) - \beta^{-1}(0) = -2i \sin\left(\frac{L+T}{2}\right)$, we obtain

$$\begin{aligned} h_n = Y_{12}(0) &= e^{-n\ell} P_{12}^{(\infty)}(0) \left(1 + \mathcal{O}(n^{-1})\right) \\ &= e^{-n\ell} \sin\left(\frac{L+T}{2}\right) \Theta_{12}(0) \left(1 + \mathcal{O}(n^{-1})\right), \end{aligned} \tag{3.3.64}$$

as $n \to \infty$, where we note that $\Theta_{12}(0)$ is real since $u(0) - d \in \mathbb{R}$ and satisfies

$$\frac{\theta(1/2)}{\theta(0)} \leq \Theta_{12}(0) \leq \frac{\theta(0)^2}{\theta(1/2)^2}. \tag{3.3.65}$$

Inside the lenses away from edges

In this subsection, we will only consider the cases where z is outside the disks near the edge points. For z on the unit circle, or close to it, we need to use the formula for S valid inside the lens-shaped regions, see (3.3.11). This leads to different asymptotic expressions for ϕ_n.

Inside the lenses around γ, for $|z| > 1$, we have

$$\begin{aligned} \phi_n(z) &= e^{ng(z)} \sum_{j=1,2} R_{1j}(z) \left(P_{j1}^{(\infty)}(z) - e^{-n\phi(z)} e^{n\pi i\Omega} P_{j2}^{(\infty)}(z) \right) \\ &= e^{ng(z)} \left(P_{11}^{(\infty)}(z) - e^{-n\phi(z)} e^{n\pi i\Omega} P_{12}^{(\infty)}(z) + \mathcal{O}(n^{-1}) \right), \end{aligned} \tag{3.3.66}$$

as $n \to \infty$, since $\mathrm{Re}\,(\phi(z)) > 0$ for z inside the lenses and $P^{(\infty)} = \mathcal{O}(1)$ in this region.

Similarly to (3.3.66), we obtain asymptotics for ϕ_n in the other regions. For $|z| < 1$, inside the lenses around γ, we have

$$\phi_n(z) = e^{ng(z)} \left(P_{11}^{(\infty)}(z) + e^{-n\phi(z)} e^{-n\pi i\Omega} P_{12}^{(\infty)}(z) + \mathcal{O}(n^{-1}) \right) \qquad \text{as } n \to +\infty. \tag{3.3.67}$$

For z inside the lenses around $\widetilde{\gamma}$, we obtain similar expressions:

$$\phi_n(z) = e^{ng(z)} \left(P_{11}^{(\infty)}(z) - e^{-n\widetilde{\phi}(z)} e^{-n\pi i\Omega} P_{12}^{(\infty)}(z) + \mathcal{O}(n^{-1}) \right) \qquad \text{as } n \to +\infty \tag{3.3.68}$$

for $|z| > 1$ and

$$\phi_n(z) = e^{ng(z)} \left(P_{11}^{(\infty)}(z) + e^{-n\widetilde{\phi}(z)} e^{n\pi i\Omega} P_{12}^{(\infty)}(z) + \mathcal{O}(n^{-1}) \right) \qquad \text{as } n \to +\infty \tag{3.3.69}$$

for $|z| < 1$.

Near the hard edges

Inside the disk $D(z_0, r)$, for $|z| > 1$ and z inside the lenses, inverting the transformations in the RH analysis, we obtain the identity

$$Y(z) = e^{\frac{n\pi i}{2}\sigma_3} e^{-\frac{n\ell}{2}\sigma_3} R(z) P(z) \begin{pmatrix} 1 & 0 \\ -e^{-n\phi(z)} e^{n\pi i \Omega} & 1 \end{pmatrix} e^{ng(z)\sigma_3} e^{\frac{n\ell}{2}\sigma_3} e^{-\frac{n\pi i}{2}\sigma_3}. \tag{3.3.70}$$

Taking the $1,1$ entry, we have

$$\phi_n(z) = e^{ng(z)} \Big[\Big(P_{11}(z) - e^{-n\phi(z)} e^{n\pi i\Omega} P_{12}(z) \Big) R_{11}(z) + \Big(P_{21}(z) - e^{-n\phi(z)} e^{n\pi i\Omega} P_{22}(z) \Big) R_{12}(z) \Big], \tag{3.3.71}$$

where P is the local Bessel parametrix (3.3.45). Similar expressions hold near $\overline{z_0}$; they can easily be obtained from the equation $\phi_n(\overline{z_0}) = \overline{\phi_n(z_0)}$.

Near the soft edges

Near $\overline{z_1}$, we can express ϕ_n asymptotically in terms of the Airy parametrix. For $|z| > 1$, z inside the lenses around $\widetilde{\gamma}$, and $z \in D(\overline{z_1}, r)$, we obtain after a similar calculation than in Section 3.3.8 that

$$\phi_n(z) = e^{ng(z)} \Big[\Big(P_{11}(z) - e^{-n\widetilde{\phi}(z)} e^{-n\pi i\Omega} P_{12}(z) \Big) R_{11}(z) + \Big(P_{21}(z) - e^{-n\widetilde{\phi}(z)} e^{-n\pi i\Omega} P_{22}(z) \Big) R_{12}(z) \Big], \tag{3.3.72}$$

where P is the local Airy parametrix (3.3.54).

3.4 Proofs of main results

In this section, we use the asymptotics obtained for the orthogonal polynomials ϕ_n to prove our main results, Theorem 3.2.1, Theorem 3.2.5, and Theorem 3.2.8, corresponding to Case IV where $s = e^{-xn}$ with x in compact subsets of $(0, x_c)$. Afterwards we will also indicate how one can prove the corresponding results in Cases I-III, without giving full details.

3.4.1 Zeros of ϕ_n

If we denote by $z_j^{(n)}$, $j = 1, ..., n$ the zeros of the polynomial $\phi_n(z)$ and by $\nu_{n,s,L} = \frac{1}{n}\sum_{j=1}^n \delta_{z_j^{(n)}}$ the normalized zero counting measure of ϕ_n, we have by definition that

$$\phi_n(z) = \prod_{j=1}^n (z - z_j^{(n)}) = e^{\sum_{j=1}^n \log(z - z_j^{(n)})} = e^{n\int \log(z-w)d\nu_{n,s,L}(w)}.$$

We know from the general theory for orthogonal polynomials on the unit circle that all the zeros lie strictly inside the unit circle. It follows from Helly's theorem that there exists a subsequence $(\nu_{n_k,s,L})_k$ which converges weakly to a probability measure $\nu_{s,L}$ supported on a subset of $|z| \leq 1$. From (3.3.63), it follows that

$$\lim_{n\to\infty} \frac{1}{n}\mathrm{Re}(\log \phi_n(z)) = \int \log|z - e^{i\theta}| d\mu_{x,L}(e^{i\theta}), \qquad \forall z \in \mathbb{C} \setminus S^1.$$

In particular we have for any converging subsequence $\nu_{n_k,s,L}$ that

$$\lim_{k\to\infty} \int \log|z - w| d\nu_{n_k,s,L}(w) = \int \log|z - e^{i\theta}| d\mu_{x,L}(e^{i\theta}),$$

for all z off the unit circle. By the unicity theorem [57, Theorem II.2.1], this implies that $\nu_{n_k,s,L}$ converges weakly to $\mu_{x,L}$. Hence the entire sequence $\nu_{n,s,L}$ converges weakly to $\mu_{x,L}$. This proves a first part of Theorem 3.2.5.

We can also prove a result about the location of the zeros of ϕ_n: for n large enough, there cannot be zeros which are not close to the unit circle or to a point w_n on the real line, as stated in the following proposition.

Proposition 3.4.1 *Let $0 < L < \pi$ and $s = e^{-xn}$ with $0 < x < x_c$. There exists a real sequence $(w_n)_{n\in\mathbb{N}}$ such that for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, $\phi_n(z)$ has no zeros in $\{z \in \mathbb{C} : |z| \leq 1 - \epsilon, |z - w_n| \geq \epsilon\}$.*

Proof. We can use (3.3.63) for $|z| \leq 1 - \epsilon$ and n sufficiently large, which implies that $\phi_n(z)$ has, for sufficiently large n, no zeros if $|P_{11}^{(\infty)}(z)| > c > 0$ with c independent of n. Since $\beta(z) + \beta^{-1}(z)$ and $\theta(u(z) + d)$ both are bounded away from zero in this region, $P_{11}^{(\infty)}(z)$ vanishes only at the zeros of $\theta(u(z) + d - n\Omega)$. Since the Riemann surface X is of genus one, $u(z)$ is a bijection from X to the Jacobi variety $\mathbb{C}/\Lambda$, $\Lambda = \{n + \tau m, n, m \in \mathbb{Z}\}$, implying that $\theta(u(z) + d - n\Omega)$ vanishes in at most one point which we denote by w_n. For any n, the point w_n is necessarily real because of the symmetry $P^{(\infty)}(z) = \overline{P^{(\infty)}(\overline{z})}$. This relation follows from the fact that $\overline{P^{(\infty)}(\overline{z})}$ satisfies the RH conditions for $P^{(\infty)}$, and the uniqueness of the RH solution. □

3.4.2 Limit of the mean eigenvalue density

We first prove that the mean eigenvalue density $\psi_{n,s,L}(e^{i\theta}) = \frac{1}{n}K_n(e^{i\theta}, e^{i\theta})$ converges to 0 for $e^{i\theta}$ on $S^1 \setminus \operatorname{supp}\mu_{x,L}$.

For z in the exterior region of Σ_R, we have (3.3.63). If we let z approach a point $e^{i\theta} \in S^1 \setminus \operatorname{supp}\mu_{x,L} = \Sigma_1 \cup \Sigma_2$, we obtain

$$\phi_n(e^{i\theta}) = e^{ng_-(e^{i\theta})} P^{(\infty)}_{11,-}(e^{i\theta}) \left(1 + \mathcal{O}(n^{-1})\right), \qquad n \to \infty,$$

where we note that $P^{(\infty)}_{11,-}(e^{i\theta})$ is bounded away from 0. A direct calculation shows that

$$\phi_n'(e^{i\theta})\overline{\phi_n(e^{i\theta})} - (\phi_n^*)'(e^{i\theta})\overline{\phi_n^*(e^{i\theta})} = e^{2n\mathrm{Re}(g_-(e^{i\theta}))}\mathcal{O}(n), \qquad n \to \infty.$$

By equations (2.3.16) and (2.3.21), it follows that

$$2\mathrm{Re}(g_-(e^{i\theta})) = 2\int_{-\pi}^{\pi} \log|e^{i\theta} - e^{i\alpha}| d\mu(e^{i\alpha}) < -\ell + x$$

for $e^{i\theta}$ in $\Sigma_1 \cup \Sigma_2$. Combining this with (3.1.12) and (3.3.64), we obtain

$$\psi_{n,s,L}(e^{i\theta}) = \exp\left(\left(2\int_{-\pi}^{\pi} \log|e^{i\theta} - e^{i\alpha}| d\mu(e^{i\alpha}) + \ell - x\right) n\right) \mathcal{O}(1), \quad n \to \infty. \tag{3.4.1}$$

This shows that $\psi_{n,s,L}(e^{i\theta})$ tends to 0 uniformly for $e^{i\theta}$ in any compact subset of $\Sigma_1 \cup \Sigma_2$.

Next, we show that $\psi_{n,s,L}(e^{i\theta}) \to \psi_{x,L}(e^{i\theta})$ uniformly for $e^{i\theta}$ in compact subsets of γ, where $\psi_{x,L}$ is given by (3.2.12). Taking the limit as $z \to e^{i\theta}$ in (3.3.66), we get

$$\phi_n(e^{i\theta}) = e^{ng_-(e^{i\theta})} \left(Q(e^{i\theta}) + \mathcal{O}(n^{-1})\right), \qquad n \to \infty, \tag{3.4.2}$$

where $Q(e^{i\theta}) = P^{(\infty)}_{11,-}(e^{i\theta}) - e^{-n\phi_-(e^{i\theta})} e^{n\pi i\Omega} P^{(\infty)}_{12,-}(e^{i\theta})$.

Now, (3.1.12) can be rewritten as

$$\psi_{n,s,L}(e^{i\theta}) = \frac{f(e^{i\theta})}{2\pi n h_n}\left[-n|\phi_n(e^{i\theta})|^2 + 2\mathrm{Re}\left(e^{i\theta}\phi_n'(e^{i\theta})\overline{\phi_n(e^{i\theta})}\right)\right]. \quad (3.4.3)$$

Substituting (3.3.64) and (3.4.2) in (3.4.3), taking into account that $\mathrm{Re}(g_-(e^{i\theta})) = -\frac{\ell}{2}$ and $f(e^{i\theta}) = 1$ for $e^{i\theta}\in\gamma$, we obtain

$$\begin{aligned}\psi_{n,s,L}(e^{i\theta}) = \frac{1}{2\pi P_{12}^{(\infty)}(0)} &\left[|Q(e^{i\theta})|^2\left(2\mathrm{Re}(e^{i\theta}g_-'(e^{i\theta})) - 1\right)\right. \\ &\left. + 2\mathrm{Re}\left(e^{i\theta}\phi_-'(e^{i\theta})e^{-n\phi_-(e^{i\theta})}e^{n\pi i\Omega}P_{12,-}^{(\infty)}(e^{i\theta})\overline{Q(e^{i\theta})}\right) + \mathcal{O}(n^{-1})\right],\end{aligned}$$

as $n\to\infty$.

Equation (3.3.8) together with the properties

$$\begin{aligned} g_+(z) + g_-(z) &= \log z + i\pi - \ell, && \text{for } z\in\gamma, \\ g_+(z) - g_-(z) &= 2\pi i\int_{\arg z}^{\pi} d\mu(e^{i\theta}), && \text{for } z\in S^1, \end{aligned} \quad (3.4.4)$$

where $-\pi < \arg z \le \pi$, implies

$$2\mathrm{Re}(e^{i\theta}g_-'(e^{i\theta})) - 1 = e^{i\theta}\phi_-'(e^{i\theta}) = 2\pi\psi_{x,L}(e^{i\theta}).$$

We thus have, as $n\to\infty$,

$$\psi_{n,s,L}(e^{i\theta}) = \psi_{x,L}(e^{i\theta})\left(\frac{|P_{11,-}^{(\infty)}(e^{i\theta})|^2 - |P_{12,-}^{(\infty)}(e^{i\theta})|^2}{P_{12}^{(\infty)}(0)} + \mathcal{O}(n^{-1})\right). \quad (3.4.5)$$

The convergence of $\psi_{n,s,L}$ to $\psi_{x,L}$ then follows from the following identity.

Proposition 3.4.2 *For all $z\in\gamma\cup\widetilde{\gamma}$, we have*

$$|P_{11,-}^{(\infty)}(e^{i\theta})|^2 - |P_{12,-}^{(\infty)}(e^{i\theta})|^2 = P_{12}^{(\infty)}(0). \quad (3.4.6)$$

Proof. We recall the relation $P^{(\infty)}(z) = \overline{P^{(\infty)}(\overline{z})}$ (see the proof of Proposition 3.4.1). This allows us to rewrite the left hand side of (3.4.6) as

$$P_{11,-}^{(\infty)}(e^{i\theta})P_{11,-}^{(\infty)}(e^{-i\theta}) - P_{12,-}^{(\infty)}(e^{i\theta})P_{12,-}^{(\infty)}(e^{-i\theta}). \quad (3.4.7)$$

By another argument of uniqueness of the solution of the RH problem for $P^{(\infty)}$, we have the relation

$$P^{(\infty)}(z;\Omega) = \sigma_1 P^{(\infty)}(0;-\Omega)^{-1} P^{(\infty)}(z^{-1};-\Omega)\sigma_1, \tag{3.4.8}$$

where we have now explicitly written the dependence on the parameter Ω of $P^{(\infty)}$, and $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Noting from (3.3.33) that $P_{21}^{(\infty)}(0;-\Omega) = -P_{12}^{(\infty)}(0;\Omega)$ and using (3.4.8), we obtain

$$P_{12}^{(\infty)}(0;\Omega) = P_{22}^{(\infty)}(z^{-1};-\Omega)P_{12}^{(\infty)}(z;\Omega) - P_{21}^{(\infty)}(z^{-1};-\Omega)P_{11}^{(\infty)}(z;\Omega).$$

From (3.3.33), for $|z| \neq 1$,

$$P_{22}^{(\infty)}(z;-\Omega) = P_{11}^{(\infty)}(z;\Omega) \qquad \text{and} \qquad P_{21}^{(\infty)}(z;-\Omega) = -P_{12}^{(\infty)}(z;\Omega). \tag{3.4.9}$$

We are then led to an expression involving only $P^{(\infty)}(z;\Omega)$ (so we omit again the dependence in Ω):

$$P_{12}^{(\infty)}(0) = P_{11}^{(\infty)}(z^{-1})P_{12}^{(\infty)}(z) + P_{12}^{(\infty)}(z^{-1})P_{11}^{(\infty)}(z), \qquad \text{for } |z| \neq 1.$$

Taking the limit of the above expression as $z \to e^{i\theta} \in \gamma \cup \widetilde{\gamma}$, $|z| > 1$ and noting that

$$\begin{aligned} \lim_{z \to e_-^{i\theta}} P_{11}^{(\infty)}(z^{-1}) &= P_{11,+}^{(\infty)}(e^{-i\theta}) = -P_{12,-}^{(\infty)}(e^{-i\theta}), \\ \lim_{z \to e_-^{i\theta}} P_{12}^{(\infty)}(z^{-1})_- &= P_{12,+}^{(\infty)}(e^{-i\theta}) = P_{11,-}^{(\infty)}(e^{-i\theta}), \end{aligned}$$

we obtain that $P_{12}^{(\infty)}(0)$ is equal to (3.4.7), which proves the proposition. □

The convergence of $\psi_{n,s,L}(e^{i\theta})$ to $\psi_{x,L}(e^{i\theta})$ for $e^{i\theta}$ on $\widetilde{\gamma}$ is obtained in a similar way.

To complete the proof of Part 1 of Theorem 3.2.5, we still need to show weak convergence of $\mu_{n,s,L}$ to $\mu_{x,L}$. To that end, let h be a continuous function on the unit circle. For any $\epsilon > 0$, we can take a compact subset A of the unit circle which does not contain the points $z_0, \overline{z_0}, z_1, \overline{z_1}$ and which is such that

$$\mu_{x,L}(A) > 1 - \frac{\epsilon}{3\max_{z\in S^1}|h(z)|}.$$

Because $\psi_{n,s,L}$ converges uniformly to $\psi_{x,L}$ for $e^{i\theta} \in A$, we then have

$$\left| \int_{e^{i\theta} \in A} h(e^{i\theta}) \left(\psi_{n,s,L}(e^{i\theta}) - \psi_{x,L}(e^{i\theta}) \right) d\theta \right| < \epsilon/3, \tag{3.4.10}$$

for n sufficiently large. We also have

$$\left| \int_{e^{i\theta} \in S^1 \setminus A} h(e^{i\theta}) \psi_{x,L}(e^{i\theta}) d\theta \right| \leq \max_{z \in S^1} |h(z)| (1 - \mu_{x,L}(A)) < \epsilon/3. \tag{3.4.11}$$

Since $\mu_{n,s,L}(A)$ converges to $\mu_{x,L}(A)$ as $n \to \infty$, we also have

$$\left| \int_{e^{i\theta} \in S^1 \setminus A} h(e^{i\theta}) \psi_{n,s,L}(e^{i\theta}) d\theta \right| \leq \max_{z \in S^1} |h(z)| (1 - \mu_{n,s,L}(A)) < \epsilon/3, \tag{3.4.12}$$

for sufficiently large n. Combining (3.4.10)–(3.4.12), we obtain

$$\left| \int_0^{2\pi} h(e^{i\theta}) \left(\psi_{n,s,L}(e^{i\theta}) - \psi_{x,L}(e^{i\theta}) \right) d\theta \right| < \epsilon, \tag{3.4.13}$$

which proves the weak convergence of $\mu_{n,s,L}$ to $\mu_{x,L}$.

The asymptotics (3.2.28) for the average number of eigenvalues are obtained in a similar way: we have

$$\mathbb{E} \left(\#(\Theta \cap \gamma^c) | \Phi \subset \gamma \right) = n \int_{e^{i\theta} \in \gamma^c} \psi_{n,s,L}(e^{i\theta}) d\theta.$$

Repeating the same argument as above, but now with h the characteristic function of γ^c, we obtain (3.2.28).

For the variance, we use (3.1.14), and obtain (3.2.29) after differentiating (3.2.28) with respect to $s = e^{-xn}$. This can be justified using a standard argument in RH analysis: first, it follows from the RH analysis that (3.2.28) holds not only for real $0 < x < x_c$, but also uniformly for x in a small complex neighbourhood of $(0, x_c)$. Next, using Cauchy's integral formula

$$\partial_x \left(\partial_x \log D_n(e^{-nx}, L) \right) = \frac{1}{2\pi i} \int \frac{\partial_{x'} \log D_n(e^{-nx'}, L)}{(x - x')^2} dx',$$

where the integral is over a small circle in the x-plane. Using (3.2.28) and the fact that the error term is uniform on the little circle, we obtain (3.2.29).

3.4.3 Proof of Theorem 3.2.1

Since

$$\frac{1}{n}\, s\partial_s \log D_n(s,L)\big|_{s=e^{-xn}} = \frac{1}{n}\mathbb{E}\left(\#(\Theta\cap\gamma^c)|\Phi\subset\gamma\right) \leq 1,$$

we can integrate (3.2.28) with respect to x and use Lebesgue's dominated convergence theorem. From the point-wise convergence (3.2.28) for $0 < x < x_c$, we obtain

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n^2}\int_0^x \partial_\xi \log D_n(e^{-\xi n},L)d\xi &= -\lim_{n\to\infty}\frac{1}{n}\int_0^x s\partial_s \log D_n(s,L)\big|_{s=e^{-\xi n}}\, d\xi \\ &= -\int_0^x \Omega(\xi,L)d\xi,\end{aligned}$$

for $x\in[0,x_c]$.

3.4.4 Bessel kernel limit

Let u, v be two positive real numbers and c a positive constant. To simplify the expressions we will use the following notations:

$$u_n = L - \frac{u}{(cn)^2}, \qquad v_n = L - \frac{v}{(cn)^2}. \tag{3.4.14}$$

Using (3.3.64) in (3.1.11), we obtain as $n\to\infty$,

$$\begin{aligned}\frac{1}{(cn)^2}K_n(e^{iu_n},e^{iv_n}) = \frac{e^{in(u_n-v_n)}\phi_n(e^{-iu_n})\phi_n(e^{iv_n}) - \phi_n(e^{iu_n})\phi_n(e^{-iv_n}))}{2\pi e^{-n\ell}P_{12}^{(\infty)}(0)(u-v)} \\ \times\ -i(1+\mathcal{O}(n^{-1})).\end{aligned} \tag{3.4.15}$$

We can also use the asymptotic expression (3.3.71) for $\phi_n(e^{iu_n})$. Noting that

$$n^2\zeta(e^{iu_n}) = -n^2\zeta'(z_0)e^{iL}\frac{iu}{(cn)^2}\left(1+\mathcal{O}\left(\frac{u}{n^2}\right)\right),$$

and that

$$\zeta'(z_0) = \frac{1}{4}e^{-iL}e^{-i\frac{\pi}{2}}\frac{|z_0-z_1||z_0-\overline{z_1}|}{|z_0-\overline{z_0}|},$$

we obtain

$$n^2\zeta(e^{iu_n}) = -\frac{u}{4}\left(1+\mathcal{O}\left(\frac{u}{n^2}\right)\right), \qquad n^2\zeta(e^{iv_n}) = -\frac{v}{4}\left(1+\mathcal{O}\left(\frac{v}{n^2}\right)\right), \tag{3.4.16}$$

as $n \to \infty$, if we choose $c = \sqrt{\frac{|z_0-z_1||z_0-\overline{z_1}|}{|z_0-\overline{z_0}|}}$. This implies by (3.3.45) and (3.3.46) that $P_-(e^{iu_n}) = \mathcal{O}(n^{1/2})$ as $n \to \infty$ uniformly for u in a compact subset of $(0,\infty)$, and similarly for v_n. By (3.3.60) and (3.3.71), we thus have

$$\phi_n(e^{iu_n}) = e^{ng_-(e^{iu_n})}\left(P_{11,-}(e^{iu_n}) - e^{-n\phi_-(e^{iu_n})}e^{n\pi i\Omega}P_{12,-}(e^{iu_n}) + \mathcal{O}(n^{-1/2})\right), \tag{3.4.17}$$

as $n \to \infty$.

The definition of $\widehat{\Psi}$ and one of the connection formulas for the Bessel functions (see [53]),

$$H_0^{(1)}(z) + H_0^{(2)}(z) = 2J_0(z),$$

imply by (3.3.45) that, for $\delta < x < x_c - \delta$,

$$P_{11,-}(e^{iu_n}) - e^{-n\phi_-(e^{iu_n})}e^{n\pi i\Omega}P_{12,-}(e^{iu_n}) = e^{-\frac{n}{2}\phi_-(e^{iu_n})}e^{n\pi i\frac{\Omega}{2}}F(e^{iu_n}) + \mathcal{O}(e^{-nx}),$$

as $n \to \infty$, where

$$F(z) = E_{11}(z)J_0(2\sqrt{-n^2\zeta(z)}) + 2i\pi\sqrt{-n^2\zeta(z)}E_{12}(z)J_0'(2\sqrt{-n^2\zeta(z)}).$$

Using (3.3.8) and (3.4.4), we also have the following relation,

$$g_-(e^{i\theta}) - \frac{1}{2}\phi_-(e^{i\theta}) + \pi i\frac{\Omega}{2} = -\frac{\ell}{2} + \frac{i(\theta+\pi)}{2}, \qquad \text{for } -L \le \theta \le L, \tag{3.4.18}$$

which allows us to rewrite (3.4.17) in a more explicit form in terms of the Bessel function of the first kind:

$$\phi_n(e^{iu_n}) = e^{-\frac{n\ell}{2}}e^{\frac{ni}{2}(u_n+\pi)}\left(F(e^{iu_n}) + \mathcal{O}(n^{-1/2})\right), \qquad n \to \infty. \tag{3.4.19}$$

Substituting (3.4.19) in (3.4.15), we get

$$\frac{1}{(cn)^2}K_n(e^{iu_n}, e^{iv_n}) = \frac{-1}{\pi P_{12}^{(\infty)}(0)(u-v)}\mathrm{Im}\left(F(e^{iu_n})\overline{F(e^{iv_n})}\right) + \mathcal{O}(n^{-1/2}), \tag{3.4.20}$$

as $n \to \infty$. We now compute $F(e^{iu_n})\overline{F(e^{iv_n})}$: we have

$$F(e^{iu_n})\overline{F(e^{iv_n})} \quad = \quad F_1 + F_2 + F_3 + F_4,$$

where

$$
\begin{aligned}
F_1 &= E_{11}(e^{iu_n})\overline{E_{11}(e^{iv_n})}J_0(2\sqrt{-n^2\zeta(e^{iu_n})})J_0(2\sqrt{-n^2\zeta(e^{iv_n})}),\\
F_2 &= 4\pi^2 n^2\sqrt{\zeta(e^{iu_n})\zeta(e^{iv_n})}E_{12}(e^{iu_n})\overline{E_{12}(e^{iv_n})}\\
&\qquad\qquad \times J_0'(2\sqrt{-n^2\zeta(e^{iu_n})})J_0'(2\sqrt{-n^2\zeta(e^{iv_n})}),\\
F_3 &= -2i\pi\sqrt{-n^2\zeta(e^{iv_n})}E_{11}(e^{iu_n})\overline{E_{12}(e^{iv_n})}\\
&\qquad\qquad \times J_0(2\sqrt{-n^2\zeta(e^{iu_n})})J_0'(2\sqrt{-n^2\zeta(e^{iv_n})}),\\
F_4 &= 2i\pi\sqrt{-n^2\zeta(e^{iu_n})}E_{12}(e^{iu_n})\overline{E_{11}(e^{iv_n})}\\
&\qquad\qquad \times J_0(2\sqrt{-n^2\zeta(e^{iv_n})})J_0'(2\sqrt{-n^2\zeta(e^{iu_n})}).
\end{aligned}
$$

Here, we have used the fact that $J_0(x)$ is real for $x > 0$. It is straightforward to see that the imaginary parts of F_1 and F_2 tend to 0 as $n \to +\infty$. More precisely, one has

$$
\operatorname{Im}(F_j) = \mathcal{O}\left(\frac{|u|+|v|}{n}\right), \qquad j = 1, 2, \qquad n \to \infty. \tag{3.4.21}
$$

The computation for F_3 and F_4 is slightly more involved. Using (3.3.46) and Proposition 3.4.2, we have

$$
\operatorname{Im}(F_3) = -\frac{\pi}{2}J_0(\sqrt{u})\sqrt{v}J_0'(\sqrt{v})P_{12}^{(\infty)}(0)\left(1+\mathcal{O}(n^{-1})\right) \tag{3.4.22}
$$

and

$$
\operatorname{Im}(F_4) = \frac{\pi}{2}J_0(\sqrt{v})\sqrt{u}J_0'(\sqrt{u})P_{12}^{(\infty)}(0)\left(1+\mathcal{O}(n^{-1})\right), \tag{3.4.23}
$$

as $n \to \infty$. Note that it follows from Proposition 3.4.2 that $P_{12}^{(\infty)}(0)$ is real. Putting everything together in (3.4.20), one gets

$$
\frac{1}{(cn)^2}K_n(e^{iu_n}, e^{iv_n}) = \frac{J_0(\sqrt{u})\sqrt{v}J_0'(\sqrt{v}) - J_0(\sqrt{v})\sqrt{u}J_0'(\sqrt{u})}{2(u-v)} + \mathcal{O}(n^{-1}), \tag{3.4.24}
$$

as $n \to \infty$, uniformly for u, v in compact subsets of $(0, \infty)$.

3.4.5 Airy kernel limit

For convenience, we make use of the following notations:

$$u_n = -\pi + T + \frac{u}{(cn)^{2/3}}, \qquad v_n = -\pi + T + \frac{v}{(cn)^{2/3}}.$$

Even if (3.2.30) is true for any u, v real, we will only provide explicit computation for $u, v < 0$. Other cases can be dealt with in a similar manner (and are in fact easier). Using (3.3.64) in (3.1.11), we obtain

$$\begin{aligned}\frac{1}{(cn)^{2/3}} K_n(e^{iu_n}, e^{iv_n}) = & \frac{e^{in(u_n - v_n)}\phi_n(e^{-iu_n})\phi_n(e^{iv_n}) - \phi_n(e^{iu_n})\phi_n(e^{-iv_n})}{2\pi e^{-n\ell} P_{12}^{(\infty)}(0)(u-v)} \\ & \times\ ie^{-xn}(1 + \mathcal{O}(n^{-2/3})), \qquad n \to \infty. \quad (3.4.25)\end{aligned}$$

If we choose $c = \sqrt{\frac{|\overline{z_1} - z_1|}{4|\overline{z_1} - z_0||\overline{z_1} - \overline{z_0}|}}$, we have $n^{2/3}\zeta(e^{iu_n}) = u(1 + \mathcal{O}(\frac{u}{(cn)^{2/3}}))$ as $n \to \infty$. With this choice of c, it is straightforward to see from (3.3.54) and (3.3.55) that $P_-(e^{iu_n}) = \mathcal{O}(n^{1/6})$. Taking the limit as $z \to e^{iu_n}$ in (3.3.72), we obtain the large n asymptotics

$$\begin{aligned}\phi_n(e^{iu_n}) &= e^{ng_-(e^{iu_n})}\Big(P_{11,-}(e^{iu_n}) - e^{-n\widetilde{\phi}_-(e^{iu_n})}e^{-n\pi i\Omega}P_{12,-}(e^{iu_n}) + \mathcal{O}(n^{-5/6})\Big), \\ &= e^{n(g_-(e^{iu_n}) - \frac{1}{2}\widetilde{\phi}_-(e^{iu_n}) - \frac{1}{2}\pi i\Omega)}\left(K(e^{iu_n}) + \mathcal{O}(n^{-5/6})\right), \\ &= e^{n\frac{-\ell+x}{2}}e^{ni\frac{u_n - \pi}{2}}\left(K(e^{iu_n}) + \mathcal{O}(n^{-5/6})\right), \qquad (3.4.26)\end{aligned}$$

where K is defined by

$$K(z) = E_{11}(z)\mathrm{Ai}(n^{2/3}\tilde{\zeta}(z)) + E_{12}(z)\mathrm{Ai}'(n^{2/3}\tilde{\zeta}(z)).$$

Note that we have used the analogue of (3.4.18) for $e^{i\theta} \in \widetilde{\gamma}$ with $-\pi < \theta \leq -\pi + T$, namely

$$g_-(e^{i\theta}) - \frac{1}{2}\widetilde{\phi}_-(e^{i\theta}) - \pi i\frac{\Omega}{2} = -\frac{\ell}{2} + \frac{x}{2} + \frac{i(\theta - \pi)}{2}, \quad -\pi < \theta \leq -\pi + T, \ (3.4.27)$$

and the Airy function relation $y_0(z) + y_1(z) + y_2(z) = 0$ (see [53]) which implies by (3.3.53) that, for $-\pi < \arg z < -\frac{2\pi}{3}$,

$$\begin{aligned}\Phi_{11}(z) - \Phi_{12}(z) &= \mathrm{Ai}(z), \\ \Phi_{21}(z) - \Phi_{22}(z) &= \mathrm{Ai}'(z).\end{aligned}$$

Inserting (3.4.26) in (3.4.25), we have as $n \to \infty$,

$$\frac{1}{(cn)^{2/3}} K_n(e^{iu_n}, e^{iv_n}) = \frac{e^{i\frac{n}{2}(u_n - v_n)}}{\pi P_{12}^{(\infty)}(0)(u-v)} \mathrm{Im}(K(e^{iu_n})\overline{K(e^{iv_n})}) + \mathcal{O}(n^{-2/3}).$$

Using the fact that $\mathrm{Ai}(x) \in \mathbb{R}$ for x real, we have

$$K(e^{iu_n})\overline{K(e^{iv_n})} = K_1 + K_2 + K_3 + K_4, \tag{3.4.28}$$

with

$$\begin{aligned}
K_1 &= E_{11}(e^{iu_n})\overline{E_{11}(e^{iv_n})}\mathrm{Ai}(n^{2/3}\widetilde{\zeta}(e^{iu_n}))\mathrm{Ai}(n^{2/3}\widetilde{\zeta}(e^{iv_n})),\\
K_2 &= E_{12}(e^{iu_n})\overline{E_{12}(e^{iv_n})}\mathrm{Ai}'(n^{2/3}\widetilde{\zeta}(e^{iu_n}))\mathrm{Ai}'(n^{2/3}\widetilde{\zeta}(e^{iv_n})),\\
K_3 &= E_{11}(e^{iu_n})\overline{E_{12}(e^{iv_n})}\mathrm{Ai}(n^{2/3}\widetilde{\zeta}(e^{iu_n}))\mathrm{Ai}'(n^{2/3}\widetilde{\zeta}(e^{iv_n})),\\
K_4 &= E_{12}(e^{iu_n})\overline{E_{11}(e^{iv_n})}\mathrm{Ai}'(n^{2/3}\widetilde{\zeta}(e^{iu_n}))\mathrm{Ai}(n^{2/3}\widetilde{\zeta}(e^{iv_n})).
\end{aligned}$$

Analogously to (3.4.21), (3.4.22) and (3.4.23), we have

$$\mathrm{Im}(K_j) = \mathcal{O}\left(\frac{|u|+|v|}{n^{1/3}}\right), \qquad j = 1, 2, \qquad n \to \infty, \tag{3.4.29}$$

and

$$\begin{aligned}
\mathrm{Im}(K(e^{iu_n})\overline{K(e^{iv_n})}) &= (\mathrm{Ai}(u)\mathrm{Ai}'(v) - \mathrm{Ai}'(u)\mathrm{Ai}(v))\mathrm{Im}(E_{11}(\overline{z_1})\overline{E_{12}(\overline{z_1})})\\
&\quad + \mathcal{O}((u-v)n^{-1/3}),
\end{aligned}$$

as $n \to \infty$. Using the explicit expression for E given by (3.3.55) and Proposition 3.4.2, one gets

$$\mathrm{Im}(E_{11}(\overline{z_1})\overline{E_{12}(\overline{z_1})}) = \pi P_{12}^{(\infty)}(0), \tag{3.4.30}$$

and we obtain the Airy kernel limit (3.2.30).

3.4.6 Sine kernel limit

For the sine kernel, we use the notations

$$u_n = w + \frac{u}{cn} \quad \text{and} \quad v_n = w + \frac{v}{cn},$$

and in this section we will complete the computation only for $w \in (-L, L)$; the case $w \in (\pi - T, \pi + T)$ is similar. In this case, the expression for the kernel takes the asymptotic form as $n \to \infty$,

$$\frac{1}{cn} K_n(e^{iu_n}, e^{iv_n}) = \frac{i(e^{in(u_n - v_n)}\phi_n(e^{-iu_n})\phi_n(e^{iv_n}) - \phi_n(e^{iu_n})\phi_n(e^{-iv_n}))}{2\pi e^{-n\ell} P_{12}^{(\infty)}(0)(u-v)} \times \ (1 + \mathcal{O}(n^{-1})). \quad (3.4.31)$$

Starting from (3.3.66) and taking the limit as $z \to e^{i\theta} \in \gamma$, one has

$$\phi_n(e^{i\theta}) = e^{ng_-(e^{i\theta})} \left(\widetilde{K}(e^{i\theta}) + \mathcal{O}(n^{-1}) \right), \quad (3.4.32)$$

where $\widetilde{K}(e^{i\theta}) = P_{11,-}^{(\infty)}(e^{i\theta}) - e^{-n\phi_-(e^{i\theta})} e^{n\pi i \Omega} P_{12,-}^{(\infty)}(e^{i\theta})$. Using (3.4.4) we immediately obtain for $e^{i\theta} \in \gamma$ that

$$e^{ng_-(e^{iu_n})} = e^{-\frac{n\ell}{2}} e^{\frac{ni}{2}(u_n + \pi)} \exp\left(-n\pi i \int_{u_n}^{\pi} d\mu(e^{i\alpha}) \right). \quad (3.4.33)$$

Substituting (3.4.33) and (3.4.32) in (3.4.31), we have

$$\frac{1}{cn} K_n(e^{iu_n}, e^{iv_n}) = \frac{e^{\frac{ni}{2}(u_n - v_n)}}{\pi P_{12}^{(\infty)}(0)(u-v)} \mathrm{Im}\left(e^{\alpha_n} \widetilde{K}(e^{iu_n}) \overline{\widetilde{K}(e^{iv_n})} \right) + \mathcal{O}(n^{-1}), \quad (3.4.34)$$

as $n \to \infty$, where $\alpha_n = \dfrac{n}{2}(\phi_-(e^{iu_n}) - \phi_-(e^{iv_n})) = n\pi i \displaystyle\int_{v_n}^{u_n} d\mu(e^{i\theta})$. An explicit computation shows that

$$\begin{aligned} \widetilde{K}(e^{iu_n})\overline{\widetilde{K}(e^{iv_n})} = e^{-\alpha_n} \Big(& e^{\alpha_n} P_{11,-}^{(\infty)}(e^{iu_n}) \overline{P_{11,-}^{(\infty)}(e^{iv_n})} + e^{-\alpha_n} P_{12,-}^{(\infty)}(e^{iu_n}) \overline{P_{12,-}^{(\infty)}(e^{iv_n})} \\ & - e^{\beta_n} e^{-n\pi i\Omega} P_{11,-}^{(\infty)}(e^{iu_n}) \overline{P_{12,-}^{(\infty)}(e^{iv_n})} - e^{-\beta_n} e^{n\pi i\Omega} P_{12,-}^{(\infty)}(e^{iu_n}) \overline{P_{11,-}^{(\infty)}(e^{iv_n})} \Big), \end{aligned}$$

where $\beta_n = \frac{n}{2}\left(\phi_-(e^{iu_n}) + \phi_-(e^{iv_n}) \right)$. If we choose $c = \psi_{x,L}(e^{iw})$, then $e^{\alpha_n} = e^{i\pi(u-v)}(1 + \mathcal{O}(n^{-1}))$ as $n \to \infty$ and we obtain finally

$$\begin{aligned} \mathrm{Im}\left(e^{\alpha_n} \widetilde{K}(e^{iu_n}) \overline{\widetilde{K}(e^{iv_n})} \right) &= \mathrm{Im}\left(e^{\pi i(u-v)} |P_{11,-}^{(\infty)}(e^{iw})|^2 + e^{-\pi i(u-v)} |P_{12,-}^{(\infty)}(e^{iw})|^2 \right) \\ &\quad + \mathcal{O}((u-v)n^{-1}), \\ &= \sin(\pi(u-v)) P_{12}^{(\infty)}(0) + \mathcal{O}((u-v)n^{-1}), \end{aligned}$$

as $n \to \infty$, where we have again used Proposition 3.4.2. Using this in (3.4.34), we obtain

$$\lim_{n\to\infty} \frac{1}{cn} K_n\left(e^{i\left(w + \frac{u}{cn}\right)}, e^{i\left(w + \frac{v}{cn}\right)} \right) = e^{\frac{i(u-v)}{2c}} \frac{\sin \pi(u-v)}{\pi(u-v)}, \quad (3.4.35)$$

which corresponds to (3.2.19) with $c = \psi_{x,L}(e^{iw})$.

3.4.7 Cases I-III

Zeros of ϕ_n

In Case III, it was shown in Chapter 2 that for any $\epsilon > 0$ fixed, if $|z| \geq 1+\epsilon$ or $|z| \leq 1-\epsilon$ we have

$$\phi_n(z) = e^{n\int_\gamma \log(z-e^{i\theta})d\mu_{\infty,L}(e^{i\theta})}\frac{1}{2}\left(\left(\frac{z-\overline{z_0}}{z-z_0}\right)^{1/4}+\left(\frac{z-\overline{z_0}}{z-z_0}\right)^{-1/4}\right) \times \left(1+\mathcal{O}(n^{-1})\right),$$

as $n\to\infty$, where the branch cuts are chosen on γ. In a similar way as in Section 3.4.1, one has

$$\lim_{n\to\infty}\frac{1}{n}\mathrm{Re}(\log\phi_n(z)) = \int_{\mathrm{supp}(\mu_{\infty,L})}\log|z-e^{i\theta}|d\mu_{\infty,L}(e^{i\theta}), \qquad |z|\neq 1, \quad (3.4.36)$$

and one shows exactly as in Section 3.4.1 that $\nu_{n,s,L}$ converges weakly to $\mu_{\infty,L}$. In Case I, one can use [24, formulas (4.1), (4.7), (4.56)] to prove that

$$\lim_{n\to\infty}\frac{1}{n}\mathrm{Re}(\log\phi_n(z)) = \begin{cases}\log|z|, & |z|>1\\ 0, & |z|<1\end{cases} = \frac{1}{2\pi}\int_0^{2\pi}\log|z-e^{i\theta}|d\theta. \quad (3.4.37)$$

The same holds in Case II by results from [21]. This implies, by similar arguments as in Section 3.4.1, that $\nu_{n,s,L}$ converges weakly to the uniform measure on the circle.

Similar to Case IV, we have the following stronger result in Case I and in Case II, illustrated in Figure 3.6.

Proposition 3.4.3 *1. Fix $0<s<1$ and $0<L<\pi$. For any $\epsilon>0$, there exists $n_0\in\mathbb{N}$ such that for $n\geq n_0$, $\phi_n(z)$ has at most one zero in $\{z\in\mathbb{C}: |z|\leq 1-\epsilon\}$.*

2. Fix $0<L<\pi$ and let $s=e^{-xn}$, $x\geq x_c$. For any $\epsilon>0$, there exists $n_0\in\mathbb{N}$ such that for $n\geq n_0$, $\phi_n(z)$ has no zeros in $\{z\in\mathbb{C}: |z|\leq 1-\epsilon\}$.

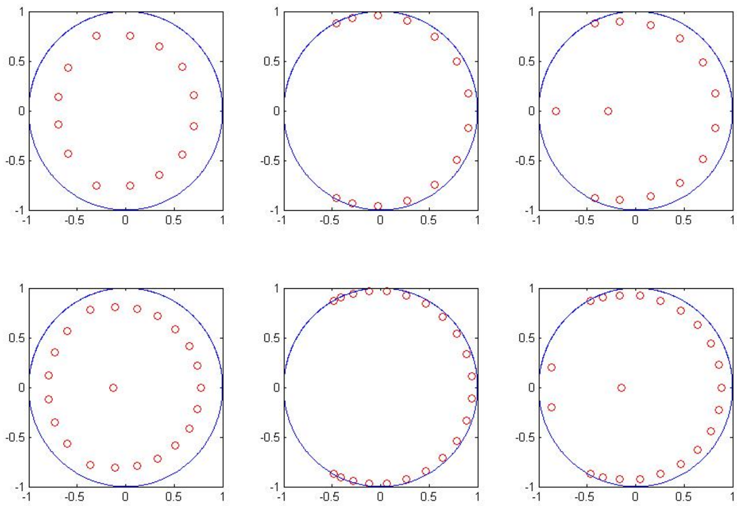

Figure 3.6: Zeros of $\phi_{14}(z)$ on the first row and of $\phi_{22}(z)$ on the second one, with $L = \frac{2\pi}{3}$. The first column corresponds to Case I with $s = 1/2$, the second column to Case III with $x = 2x_c$ and the third column to Case IV with $x = x_c/2$.

Proof. 1. In the first case, we have for $|z| \leq 1 - \epsilon$ by [24, formulas (4.1), (4.3), (4.7), (4.8), (4.54), (4.56), (4.71)] that

$$\phi_n(z) = -\exp\left(-\frac{\log s}{2\pi i}\int_L^{2\pi - L}\frac{1}{w-z}dw\right)\left(\frac{c}{z-z_0}+\frac{\overline{c}}{z-\overline{z_0}}\right) \times \frac{1}{n}\left(1+\mathcal{O}(n^{-1})\right),$$

as $n \to \infty$, where

$$c = z_0^{n+1}\beta_0\frac{\Gamma(\beta_0)}{\Gamma(-\beta_0)}|z_0-\overline{z_0}|^{-2\beta_0}n^{-2\beta_0}, \qquad \beta_0 = \frac{-1}{2\pi i}\log s.$$

Since this expression has just one root for sufficiently large n, the conclusion follows.

2. Here, we can use the asymptotics analysis done in Chapter 2 to obtain, as $n \to \infty$ for $|z| \leq 1 - \epsilon$,

$$\phi_n(z) = e^{n \int \log(z - e^{i\theta}) d\mu_{\infty,L}(e^{i\theta})} \frac{1}{2} \left(\left(\frac{z - \overline{z_0}}{z - z_0} \right)^{1/4} + \left(\frac{z - \overline{z_0}}{z - z_0} \right)^{-1/4} \right) \times \left(1 + \mathcal{O}(n^{-1})\right),$$

where the branch cut lies on γ. We easily verify that this expression vanishes nowhere for sufficiently large n.

$\square$

Limiting mean eigenvalue density and correlation kernel

Asymptotics for the mean eigenvalue density and for the correlation kernel can be obtained in Cases I-III using similar straightforward but lengthy calculations as in Section 3.4.4, Section 3.4.5, and Section 3.4.6, based on the asymptotic results in the papers [24], [21], and Chapter 2. In this way, one can prove for example formulas (3.2.18), (3.2.19), (3.2.22), and (3.2.23). For the sake of brevity, we do not present details here.

A Proof of identity (3.2.11)

Recall the definition of the equilibrium measure $\mu_{x,L}$ in (3.2.12), and the related quantities $\Omega_{x,L} = \int_{\pi-T}^{\pi+T} d\mu(e^{i\theta})$ given in (3.2.9) and $\ell_{x,L}$ given in (3.2.16). We will use the identity $\ell_{\infty,L} = -2 \log \sin \frac{L}{2}$ (see Chapter 2 for a proof of it).

Proposition A.1 *Let $0 < L < \pi$ and $0 < x < x_c$. There holds a relation between $\Omega_{x,L}$, the Euler-Lagrange constant $\ell_{x,L}$, and x:*

$$\Omega_{x,L} = \frac{\partial \ell_{x,L}}{\partial x} + x \frac{\partial \Omega_{x,L}}{\partial x} = -\frac{\partial}{\partial x} \left[-\frac{\ell_{\infty,L}}{2} \left(\frac{x}{x_c} \right)^2 - \int_{\frac{x}{x_c}}^{1} \zeta \ell_{\frac{x}{\zeta},L} d\zeta \right]. \quad \text{(A.1)}$$

Proof. We consider the following meromorphic differential of the third kind on the Riemann surface X,

$$\widehat{\omega} = \sqrt{\frac{(z - z_1)(z - \overline{z_1})}{(z - z_0)(z - \overline{z_0})}} \frac{dz}{iz}. \quad \text{(A.2)}$$

By the Riemann bilinear identity we have

$$\int_X \omega \wedge \widehat{\omega} = A_1\widehat{B}_1 - \widehat{A}_1 B_1, \tag{A.3}$$

where ω is given by (3.3.16), and

$$A_1 = \int_A \omega = 1, \qquad B_1 = \int_B \omega = \tau,$$
$$\widehat{A}_1 = \int_A \widehat{\omega} = -2ix, \qquad \widehat{B}_1 = \int_B \widehat{\omega} = -4\pi\Omega_{x,L}.$$

As $z \to 0$ on the first (respectively, second) sheet of X, we have

$$\widehat{\omega} = \pm\frac{i}{z}(1+\mathcal{O}(z))dz, \qquad \omega = \mp c_0(1+\mathcal{O}(z))dz. \tag{A.4}$$

In terms of the local coordinate $\xi = z^{-1}$, as $\xi \to 0$ on the first (respectively, second) sheet of X, we have

$$\widehat{\omega} = \pm\frac{i}{\xi}(1+\mathcal{O}(\xi))d\xi, \qquad \omega = \mp c_0(1+\mathcal{O}(\xi))d\xi. \tag{A.5}$$

The left hand side of (A.3) can now be computed using Stokes' theorem and the residue theorem. After a long calculation, we get $\int_X \omega \wedge \widehat{\omega} = -8\pi \int_{1_-^{(1)}}^{\infty^{(1)}} \omega$, where the notation $1_-^{(1)}$ means that we start the integration path from 1 on the first sheet at the side $|z| > 1$. We obtain

$$\Omega_{x,L} = 2\int_{1_-^{(1)}}^{\infty^{(1)}} \omega + \frac{i\tau}{2\pi}x. \tag{A.6}$$

Now a direct computation shows that

$$\frac{\partial \Omega_{x,L}}{\partial x} = \frac{\partial \Omega_{x,L}}{\partial T}\frac{\partial T}{\partial x} = \frac{i\tau}{2\pi} \quad \text{and} \quad \frac{\partial \ell_{x,L}}{\partial x} = \frac{\partial \ell_{x,L}}{\partial T}\frac{\partial T}{\partial x} = 2\int_{1_-^{(1)}}^{\infty^{(1)}} \omega. \tag{A.7}$$

This proves the first equality in (A.1). Now, note that

$$\begin{aligned} -\frac{\partial}{\partial x}\left[-\frac{\ell_{\infty,L}}{2}\left(\frac{x}{x_c}\right)^2 - \int_{\frac{x}{x_c}}^1 \zeta \ell_{\frac{x}{\zeta},L} d\zeta\right] &= \int_{\frac{x}{x_c}}^1 \left.\frac{\partial \ell_{u,L}}{\partial u}\right|_{u=\frac{x}{\zeta}} d\zeta \\ &= x\int_x^{x_c} \frac{\Omega_{u,L} - u\frac{\partial \Omega_{u,L}}{\partial u}}{u^2} du, \end{aligned}$$

where we proceeded by substitution and used the first equality in (A.1). Integrating by parts, we obtain

$$\int_x^{x_c} \frac{\Omega_{u,L}}{u^2} du = \frac{\Omega_{x,L}}{x} + \int_x^{x_c} \frac{1}{u} \frac{\partial \Omega_{u,L}}{\partial u} du.$$

This proves the second equality. □

Corollary A.2

$$-\int_0^{x_c} \Omega_{x,L} dx = \frac{-\ell_{\infty,L}}{2} = \log \sin \frac{L}{2} \tag{A.8}$$

Proof. It suffices to integrate (A.1) from 0 to x_c and to note that $\ell_{0,L} = 0$. □

B Extension of the RH analysis as $x \to 0$: shrinking Airy and Bessel disks

In this section we extend the RH analysis, up to now justified for x in compact subsets of $(0, x_c)$, to the case of small x. This is the intermediate regime between Case IV and Case I. We first have a closer look at the behaviour of $T = T(x)$ as $x \to 0$.

Proposition B.1 *As $\alpha := \pi - T - L \to 0_+$, we have*

$$\begin{aligned} x &= \frac{\pi}{2}\alpha + \mathcal{O}(\alpha^2), \\ \tau &= -\frac{2i}{\pi} \log(\alpha) + \mathcal{O}(1). \end{aligned} \tag{B.1}$$

Proof. It was shown in the proof of Proposition 2.3.1 that

$$x = \int_L^{L+\alpha} \sqrt{\frac{\cos\theta - \cos(L+\alpha)}{\cos L - \cos\theta}} d\theta. \tag{B.2}$$

The equation for x in (B.1) is therefore straightforward, since for any $a < b$,

$$\int_a^b \sqrt{\frac{y-a}{b-y}} dy = \frac{\pi}{2}(b-a). \tag{B.3}$$

To prove the second one, one starts from (3.3.17), which gives

$$\tau = i\frac{\displaystyle\int_{L+\alpha}^{2\pi-(L+\alpha)} \frac{d\theta}{\sqrt{\cos L - \cos\theta}\sqrt{\cos(L+\alpha)-\cos\theta}}}{\displaystyle\int_{L}^{L+\alpha} \frac{d\theta}{\sqrt{\cos L - \cos\theta}\sqrt{\cos\theta-\cos(L+\alpha)}}}. \tag{B.4}$$

Since for any $a < b < c$,

$$\int_a^b \frac{dy}{\sqrt{b-y}\sqrt{y-a}} = \pi$$

and

$$\int_a^b \frac{dy}{\sqrt{b-y}\sqrt{c-y}} = -\log(c-b) + 2\log(\sqrt{b-a}+\sqrt{c-a}),$$

the denominator in (B.4) is $\frac{\pi}{\sin L} + \mathcal{O}(\alpha)$ and the numerator is $\frac{-2}{\sin L}\log(\alpha) + \mathcal{O}(1)$ as $\alpha \to 0_+$, and the result follows. $\square$

If we choose the radii of the disks of the four local parametrices to be, say, $r = \frac{x}{2\pi}$, it follows from Proposition B.1 that the disks do not intersect if x is sufficiently small.

We now estimate the jump matrices for R as $x \to 0$.

Proposition B.2 *We let $r = \frac{x}{2\pi}$, and we let R be as constructed in Section 3.3.7. As $n \to \infty$ and $x \to 0$ in such a way that $xn \to \infty$, we have*

$$R_+(z) = R_-(z)\left(I + \mathcal{O}(x^{-1}n^{-1})\right), \quad \text{for } z \text{ on } \partial D(\zeta, r),\ \zeta = z_0, \overline{z_0}, z_1, \overline{z_1}, \tag{B.5}$$

$$R_+(z) = R_-(z)(I + \mathcal{O}(e^{-cxn})), \quad \text{for } z \text{ elsewhere on } \Sigma_R, \tag{B.6}$$

and $c > 0$ is a constant independent of x and n.

Proof. By (3.3.57) and (3.3.45)–(3.3.54), the jump matrices for R are

$$\begin{aligned} &I + P^{(\infty)}(z)\mathcal{O}\left(\frac{1}{n\phi(z)}\right)P^{(\infty)}(z)^{-1}, && \text{for } z \in \partial D(z_0, r), \\ &I + P^{(\infty)}(z)\mathcal{O}\left(\frac{1}{n\widetilde{\phi}(z)}\right)P^{(\infty)}(z)^{-1}, && \text{for } z \in \partial D(\overline{z_1}, r), \end{aligned} \tag{B.7}$$

and similarly on $\partial D(\overline{z_0}, r)$ and $\partial D(z_1, r)$. Uniformly for $z \in \mathcal{D}_x := \bigcup_{\zeta = z_0, \overline{z_0}, z_1, \overline{z_1}} \partial D(\zeta, r)$, we have from (3.3.8) and (3.3.9), as $x \to 0$, that there exists $\epsilon > 0$ such that $|\frac{\phi(z)}{r}| \geq \epsilon > 0$ and $|\frac{\widetilde{\phi}(z)}{r}| \geq \epsilon > 0$. To obtain (B.5) from (B.7), it is now sufficient to prove that $P^{(\infty)}(z)$ is uniformly bounded for n large, x small and for $z \in \Sigma_R$.

For z outside the four disks, since the zero of $\theta(u(z) - d)$ is cancelled by the one of $\beta(z) - \beta^{-1}(z)$, it is straightforward to see that $P^{(\infty)}(z) = \mathcal{O}(1)$ outside of a neighbourhood of the four edge points, as $n \to \infty$ and $x \to 0$. On $\partial D(z_0, r)$, from (3.3.27) and (B.1), we have

$$|\beta(z)|^4 \leq 2 \frac{\frac{2}{\pi} x + r}{r} = 10 \qquad \text{and} \qquad |\beta(z)^{-1}|^4 \leq 2 \frac{r}{\frac{2}{\pi} x - r} = \frac{2}{3}. \tag{B.8}$$

A similar computation shows that β and β^{-1} are also bounded on the other disks.

By (3.3.33), we still need to show that

$$\sup_{z \in \mathcal{D}_x} \Theta_{kl}(z), \qquad \text{with } 1 \leq k, l \leq 2, \tag{B.9}$$

are uniformly bounded for n large and x small. From Proposition B.1, we have that $|\tau| \to \infty$ as $x \to 0$. It is clear from (3.3.18) that

$$\lim_{\tau \to i\infty} \sup_{y \in \mathbb{R}} |\theta(y; \tau) - 1| = 0, \tag{B.10}$$

and it follows that $\frac{\theta(0;\tau)}{\theta(n\Omega;\tau)} \to 1$ as $n \to \infty$, $x \to 0$.

A next important observation is that there exist $x_0 > 0$ and $0 < \epsilon < \frac{1}{4}$ such that for all $x \leq x_0$ and $z \in \mathcal{D}_x$ we have

$$\begin{aligned} \operatorname{Im}(u(z) \pm d) \in \left[-\left(\frac{1}{4} + \epsilon \right) |\tau|, -\left(\frac{1}{4} - \epsilon \right) |\tau| \right] \cup \left[\left(\frac{1}{4} - \epsilon \right) |\tau|, \left(\frac{1}{4} + \epsilon \right) |\tau| \right] \\ \cup \left[\left(\frac{3}{4} - \epsilon \right) |\tau|, \left(\frac{3}{4} + \epsilon \right) |\tau| \right]. \end{aligned} \tag{B.11}$$

From the periodicity properties (3.3.19) of θ, we have also that for any $y \in \mathbb{R}$,

$$\frac{|\theta(u(z) - d + y)|}{|\theta(u(z) - d)|} = \frac{|\theta(u(z) - d + y - \tau)|}{|\theta(u(z) - d - \tau)|}. \tag{B.12}$$

Combining these two facts with the following estimate for any $0 \leq \delta < 1$,

$$\sup_{-\delta\frac{|\tau|}{2}\leq \mathrm{Im}\,(z)\leq \delta\frac{|\tau|}{2}} |\theta(z;\tau)-1| \leq 2e^{-\pi|\tau|(1-\delta)}+\frac{2e^{-2\pi|\tau|}}{1-e^{-\pi|\tau|}} \to 0, \qquad \text{as } \tau\to i\infty,$$

we obtain that (B.9) is bounded, which leads the first estimate in (B.5). By (3.3.58), we similarly obtain (B.6). □

If we choose x such that $x \to 0$ and $xn \to \infty$ as $n \to \infty$, we thus have that R exists for sufficiently large n and satisfies

$$R(z) = I + \mathcal{O}(x^{-1}n^{-1}), \qquad R'(z) = \mathcal{O}(x^{-1}n^{-1}). \tag{B.13}$$

This allows one to generalize Theorem 3.2.1 and Theorem 3.2.5 to the case where $x \to 0$ and $n \to \infty$ in such a way that $xn \to \infty$. From these asymptotics for R, we can obtain asymptotics for the orthogonal polynomials ϕ_n, for the Toeplitz determinants, the mean eigenvalue density, and the eigenvalue correlation kernel. In particular, as $x \to 0$ and $n \to \infty$ in such a way that $xn \to \infty$, we have that the zero counting measure $\nu_{n,s,L}$ of ϕ_n and the mean eigenvalue distribution $\mu_{n,s,L}$ converge weakly to the uniform measure on the unit circle. For the eigenvalue correlation kernel K_n, we have convergence to the sine kernel everywhere except near the points $e^{\pm iL}$.

Chapter 4

The ratio probability of the smallest eigenvalues in the Laguerre Unitary Ensemble

Abstract [1]

In the $n \times n$ Laguerre Unitary Ensemble, we study the distribution of the ratio probability between the smallest and second smallest eigenvalue. The limiting distribution as $n \to \infty$ is found as an integral over $(0, \infty)$ containing a function $v(x)$. This function is related to solutions of a system of ODEs associated to a Riemann-Hilbert problem. We also compute asymptotic behaviours of $v(x)$ as $x \to \infty$ and $x \to 0_+$.

4.1 Introduction and main results

The Laguerre Unitary Ensemble (LUE) consists of the space of $n \times n$ complex positive definite Hermitian matrices endowed with the distribution

$$\frac{1}{\widetilde{Z}_{n,\alpha}} (\det M)^{\alpha} e^{-n\mathrm{TrM}} dM, \qquad \alpha > -1, \tag{4.1.1}$$

[1]This chapter is joint work with Max Atkin and Stefan Zohren.

where $\widetilde{Z}_{n,\alpha}$ is the normalisation constant and dM is the Lebesgue measure

$$dM = \prod_{i=1}^{n} dM_{ii} \prod_{1\leq i<j\leq n} d\mathrm{Re}M_{ij} d\mathrm{Im}M_{ij}. \tag{4.1.2}$$

The probability measure (4.1.1) is invariant under unitary conjugation and induces a joint probability distribution on the eigenvalues $\lambda_1, ..., \lambda_n$ on $(\mathbb{R}^+)^n$ given by

$$\frac{1}{n!\widehat{Z}_{n,\alpha}} \Delta_n(\lambda)^2 \prod_{i=1}^{n} e^{-n\lambda_i} \lambda_i^{\alpha} \chi_{\mathbb{R}^+}(\lambda_i) d\lambda_i, \tag{4.1.3}$$

where $\Delta_n(\lambda)$ denote the Vandermonde determinant

$$\Delta_n(\lambda) \equiv \Delta_n(\{\lambda_i\}) := \prod_{1\leq i<j\leq n} (\lambda_j - \lambda_i). \tag{4.1.4}$$

The normalisation constant $\widehat{Z}_{n,\alpha}$, also known as the partition function, is given by

$$\widehat{Z}_{n,\alpha} = \frac{1}{n!} \int_0^{\infty} ... \int_0^{\infty} \Delta_n(\lambda)^2 \prod_{i=1}^{n} e^{-n\lambda_i} \lambda_i^{\alpha} d\lambda_i. \tag{4.1.5}$$

The theory of random matrices has enjoyed a growing interest and study for a number of decades in part due to the surprising number of connections between it and a priori unrelated fields. Probably the earliest appearance of random matrix theory dates back to Wishart in 1928 in the context of multivariate data analysis [70]. He was studying matrices of the form X^*X, where X is an $m \times n$ matrix ($m \geq n$) whose entries X_{ij} are independent and identically distributed complex Gaussian variables

$$\mathrm{Re}\, X_{ij}, \mathrm{Im}\, X_{ij} \sim \mathcal{N}\left(0, \sigma^2 = \frac{1}{2n}\right), \tag{4.1.6}$$

and X^* is the conjugate transpose of X. Such positive semi-definite matrices, called Wishart matrices, possess eigenvalues which are also distributed according to (4.1.3), where $\alpha = m - n$ is an integer. Since then, the LUE has been studied a lot, and has found applications in different areas, for example in finance (see e.g. [5], Section 12.2).

Some known results

The limiting mean eigenvalue density function is given by the Marchenko-Pastur law

$$\frac{1}{2\pi}\sqrt{\frac{4-x}{x}}, \tag{4.1.7}$$

supported on the interval $[0, 4]$. The most well-known extreme value statistics in the field of random matrix theory is the Tracy-Widom distribution [63] which describes the properly rescaled fluctuations of the largest (or smallest) eigenvalue of a matrix from the Gaussian Unitary Ensemble (GUE). This distribution is given by $\widetilde{F}_2(x) = \exp\left(-\int_x^\infty (s-x)q^2(s)ds\right)$ and q is the Hastings-McLeod [40] solution of the Painlevé II equation $q''(s) = sq(s) + 2q(s)^3$ satisfying the boundary condition $q(s) \sim \mathrm{Ai}(s)$ as $s \to \infty$, where Ai is the Airy function. For the case of the LUE, the distribution of the rescaled fluctuations of the largest eigenvalue at the soft edge 4 is also given by the Tracy-Widom distribution $\widetilde{F}_2$.

In this chapter we focus on the hard edge 0, where the distribution of the smallest eigenvalue (denoted by $\lambda_{\min}$) is given by another distribution $F_\alpha(x)$ in terms of a transcendent of the Painlevé V equation, also proved by Tracy and Widom [64]. More precisely, one has

$$F_\alpha(x) = \lim_{n\to\infty} \mathbb{P}_n(4n^2\lambda_{\min} > x) = \exp\left(-\frac{1}{4}\int_0^x \log\left(\frac{x}{\xi}\right) q^2(\xi)d\xi\right) \tag{4.1.8}$$

where $q(\xi)$ is the solution of the Painlevé V equation

$$\xi(q^2-1)(\xi q')' = q(\xi q')^2 + \frac{1}{4}(\xi - \alpha^2)q + \frac{1}{4}\xi q^3(q^2-2), \tag{4.1.9}$$

with boundary condition $q(\xi) \sim J_\alpha(\sqrt{\xi})$ for $\xi \to 0_+$, and J_α is the Bessel function of the first kind of order α.

Another quantity of interest is the gap probability

$$G_{n,\alpha}(r) = \mathbb{P}_{n,\alpha}\left(\lambda_{\mathrm{smin}} - \lambda_{\min} > r\right), \qquad r > 0, \tag{4.1.10}$$

between the smallest and second smallest (denoted by λ_{smin}) eigenvalue of the LUE. Some results were obtained in [38], where it was shown that the density of $G_\alpha(x) = \lim_{n\to\infty} G_{n,\alpha}(\frac{x}{4n^2})$ exists and is characterised by the solution of a Painlevé III equation and its associated linear isomonodromic system. For results on the gap probability but at the soft edge, see [54] and [71]. In [55], using heuristics arguments and numerical simulations, the authors generalized the results obtained in [54] to a more general context.

The main focus of this chapter is the distribution of the ratio

$$Q_{n,\alpha}(r) = \mathbb{P}_{n,\alpha}\left(\frac{\lambda_{\text{smin}}}{\lambda_{\min}} > r\right), \qquad r > 1, \tag{4.1.11}$$

between the smallest and second smallest eigenvalue of the LUE. Note that the ratio distribution $Q_{n,\alpha}$ cannot be straightforwardly related to the gap $G_{n,\alpha}$ and the distribution of the smallest eigenvalue, since the three variables $\lambda_{\min}$, $\lambda_{\text{smin}} - \lambda_{\min}$ and $\frac{\lambda_{\text{smin}}}{\lambda_{\min}}$ are not independent.

Our techniques differ from the techniques used in [38], [54] and [71]. We express the ratio probability in terms of a Hankel determinant and then apply well-known rigorous techniques from Riemann-Hilbert analysis to derive asymptotics. We obtain a description of this quantity in terms of a solution to a special system of non-linear ODEs arising from a Lax pair associated to a Riemann-Hilbert problem.

Statement of results

We begin the calculation of the gap probability between the smallest and second smallest eigenvalue for general $\alpha > -1$ by writing the quantity $Q_{n,\alpha}(r)$ as an integral over a Hankel determinant. For $r > 1$, by definition of (4.1.3) we have

$$Q_{n,\alpha}(r) =$$
$$n\int_0^\infty e^{-ny}y^\alpha\left(\int_{yr}^\infty \dots \int_{yr}^\infty \frac{1}{n!\widehat{Z}_{n,\alpha}}\Delta_{n-1}^2(\lambda)\prod_{i=1}^{n-1} e^{-n\lambda_i}\lambda_i^\alpha(\lambda_i - y)^2 d\lambda_i\right)dy.$$

By changing variables $n\lambda_i = (n-1)\tilde{\lambda}_i$, we can then write

$$Q_{n,\alpha}(r) = \frac{1}{\widehat{Z}_{n,\alpha}}\left(\frac{n-1}{n}\right)^{(n-1)(n+1+\alpha)}\int_0^\infty y^\alpha e^{-ny} Z_{n-1,\alpha}\left(\frac{n}{n-1}y; r\right)dy, \tag{4.1.12}$$

where we defined $Z_{n,\alpha}(y;r)$ by

$$Z_{n,\alpha}(y;r) = \frac{1}{n!}\int_{yr}^\infty \dots \int_{yr}^\infty \Delta_n(\lambda)^2\prod_{i=1}^n(\lambda_i - y)^2\lambda_i^\alpha e^{-n\lambda_i}d\lambda_i. \tag{4.1.13}$$

Note that $Z_{n,\alpha}(y;r)$ is the Hankel determinant with respect to the weight $w(x)$ defined by

$$w(x) = (x-y)^2 x^\alpha e^{-nx} \chi_{[yr,\infty)}(x), \tag{4.1.14}$$

and $\chi_{[yr,\infty)}(x)$ is the characteristic function of $[yr,\infty)$, i.e. we have

$$Z_{n,\alpha}(y;r) = \det\left(\int_{yr}^{\infty} x^{i+j} w(x) dx\right)_{i,j=0,\dots,n-1}. \tag{4.1.15}$$

Before stating our results, we associate to each $r>1$ the following set of coupled non-linear ODEs,

$$\begin{aligned}
&\frac{1}{2}\left(-4r - \frac{2V''(x)}{V(x)} + \frac{V'(x)^2}{V(x)^2}\right) + \frac{\alpha^2}{2U_0(x)^4} + \frac{2U_0''(x)}{U_0(x)} = 0,\\
&\frac{1}{2}\left(-4r - \frac{2V''(x)}{V(x)} + \frac{V'(x)^2}{V(x)^2} + 4\right) + \frac{2U_1''(x)}{U_1(x)} + \frac{2}{U_1(x)^4} = 0,\\
&V(x) = \frac{x}{2} - U_0(x)^2 - U_1(x)^2,
\end{aligned} \tag{4.1.16}$$

where U_0, U_1 and V are three analytic functions of x on $(0,\infty)$ depending on the parameter r with boundary conditions as $x\to\infty$

$$\begin{aligned}
U_0(x) &= \sqrt{\frac{\alpha}{2\sqrt{r}}} + \mathcal{O}(x^{-1}),\\
U_1(x) &= \frac{1}{\sqrt[4]{r-1}} + \mathcal{O}(x^{-1}),\\
V(x) &= \frac{x}{2} - \frac{\alpha}{2\sqrt{r}} - \frac{1}{\sqrt{r-1}} + \mathcal{O}(x^{-1}).
\end{aligned} \tag{4.1.17}$$

As $x\to 0_+$, we have

$$U_0(x) = \mathcal{O}(x^{-1/2}), \qquad U_1(x) = \mathcal{O}(x^{-1/2}) \quad \text{and} \quad V(x) = \mathcal{O}(x^{-1}). \tag{4.1.18}$$

Our main results concern asymptotics for the Hankel determinants $Z_{n,\alpha}(y;r)$ and the limiting distribution as $n\to\infty$ of the ratio probability $Q_{n,\alpha}(r)$.

Asymptotics for the Hankel determinant $Z_{n,\alpha}(y;r)$

Theorem 4.1.1 *For $\alpha \geq 0$, as $n\to\infty$ and $x = x(n)$ lies in compact subset of $[0,\infty)$ we have,*

$$\log Z_{n,\alpha}\left(\frac{x}{n^2};r\right) - \log(\widehat{Z}_{n,\alpha+2}) = I(x;r) + \mathcal{O}\left(n^{-1}\right), \tag{4.1.19}$$

where

$$\begin{aligned} I(x;r) &= \frac{1}{2}\int_0^x [v(4\xi;r)-v(0)]\,\frac{d\xi}{\xi} \\ &= 2\int_0^x \log\left(\frac{x}{\xi}\right) v'(4\xi;r)d\xi \end{aligned} \tag{4.1.20}$$

with $v(0) := \frac{1}{8}(4(\alpha+2)^2-1)$ *independent of* r *and*

$$v'(x^2;r) = -\frac{U_1(x)^2 + rV(x)}{x}. \tag{4.1.21}$$

Here U_1 *and* V *are special solutions of the system* (4.1.16)*, satisfying the boundary conditions as* $x\to\infty$ *and* $x\to 0_+$ *given by* (4.1.17) *and* (4.1.18)*. The function* v *satisfies*

$$\begin{aligned} v(x;r) &= -\frac{r}{2}x + (2\sqrt{r-1}+\alpha\sqrt{r})\sqrt{x} + \mathcal{O}(1), && \text{as } x\to\infty, \\ v(x;r) &= v(0) + \mathcal{O}(x), && \text{as } x\to 0_+. \end{aligned} \tag{4.1.22}$$

Limiting distribution of the ratio probability $Q_{n,\alpha}(r)$

Theorem 4.1.2 *For* $\alpha \geq 0$*, as* $n\to\infty$*, the limit* $Q_\alpha(r) := \lim_{n\to\infty} Q_{n,\alpha}(r)$ *exists and is given by,*

$$Q_\alpha(r) = \frac{1}{\Gamma(\alpha+1)\Gamma(\alpha+2)}\int_0^\infty x^\alpha e^{I(x;r)}dx \tag{4.1.23}$$

with $I(x;r)$ *given in Theorem 4.1.1.*

Remark 4.1.3 The reason we could only treat the case $\alpha \geq 0$ and not $\alpha > -1$ is technical and will be explained later in Section 4.3.

Remark 4.1.4 In [36], the authors obtain an exact identity for the density $p_2^{(n)}(\lambda_{\min}, \lambda_{\text{smin}})$ of the first two smallest eigenvalues, for finite n and in the special case where α is an integer. Therefore, it could be interesting to compare the result we obtain with their result by the relation

$$Q_{n,\alpha}(r) = \int_0^\infty \left(\int_{ry}^\infty p_2^{(n)}(y,y_2)dy_2\right) dy. \tag{4.1.24}$$

Outline

In Section 4.2, we introduce a family of monic orthogonal polynomials in terms of which the Hankel determinant $Z_{n,\alpha}(y;r)$ can be expressed. We also use the Riemann-Hilbert problem for orthogonal polynomials introduced by Fokas, Its and Kitaev [34] and derive a differential identity in y for $Z_{n,\alpha}(y;r)$.

We apply Deift/Zhou steepest descent method [29] on this Riemann-Hilbert problem in Section 4.6 to obtain large n asymptotics of $Z_{n,\alpha}(y;r)$ uniformly in y small enough.

In the analysis we will need a non standard model Riemann-Hilbert problem, which we introduce in Section 4.3. We derive a system of ODEs using a Lax pair in Section 4.4 and show asymptotic properties of solutions of these ODEs for $x \to \infty$ and $x \to 0$ in Section 4.5.

Finally we give a proof of Theorem 4.1.1 and Theorem 4.1.2 in Section 4.7 and 4.8 respectively, by integrating the differential identity and (4.1.12).

4.2 Differential identity for Hankel determinants

In this section we relate the Hankel determinant $Z_{n,\alpha}(y;r)$ to a Riemann-Hilbert (RH) problem by making use of the orthogonal polynomials. We consider a family of monic orthogonal polynomials p_j of degree j characterised by the relations

$$\int_{yr}^{\infty} p_j(x)p_m(x)w(x)dx = h_j\delta_{jm}, \tag{4.2.1}$$

where the weight w is defined in (4.1.14) and h_j is the squared norm of p_j, given by

$$h_j = \frac{Z_{j+1,\alpha}(y;r)}{Z_{j,\alpha}(y;r)}, \qquad Z_{0,\alpha}(y;r) := 1. \tag{4.2.2}$$

It will prove useful for the later analysis to consider $Y(z) = Y_n(z;y,r)$ the matrix valued function defined by,

$$Y(z) = \begin{pmatrix} p_n(z) & q_n(z) \\ -\frac{2\pi i}{h_{n-1}}p_{n-1}(z) & -\frac{2\pi i}{h_{n-1}}q_{n-1}(z) \end{pmatrix}, \tag{4.2.3}$$

where q_j is the Cauchy transform of p_j defined by

$$q_j(z) = \frac{1}{2\pi i}\int_{yr}^{\infty} \frac{p_j(x)w(x)}{x-z}dx. \tag{4.2.4}$$

The function Y can be characterised as the unique function satisfying a set of conditions [34], known as the RH problem for Y, which are as follows,

RH problem for Y

(a) $Y : \mathbb{C}\setminus[yr,\infty) \to \mathbb{C}^{2\times 2}$ is analytic.

(b) The limits of Y as z approaches (yr,∞) from above and below exist, are continuous on (yr,∞) and are denoted by Y_+ and Y_- respectively. Furthermore they are related by

$$Y_+(x) = Y_-(x)\begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix}, \qquad x \in (yr,\infty). \tag{4.2.5}$$

(c) $Y(z) = (I+\mathcal{O}(z^{-1}))z^{n\sigma_3}$ as $z\to\infty$.

(d) $Y(z) = Y_0(z)\begin{pmatrix} 1 & -\frac{w(z)}{2\pi i}\log(yr-z) \\ 0 & 1 \end{pmatrix}$ as $z \to yr$, where Y_0 is analytic in a neighbourhood of yr and the principal branch of the logarithm is taken.

Having introduced the above objects we are now in a position to state the following lemma which is central for the asymptotic analysis of $Z_{n,\alpha}(y;r)$.

Lemma 4.2.1 *The following identity holds,*

$$\partial_y \log Z_{n,\alpha}(y;r) = \frac{n^2+(\alpha+2)n}{y} + \frac{n}{2y}\mathrm{Res}\left(z\mathrm{Tr}\left(Y^{-1}(z;y,r)Y'(z;y,r)\sigma_3\right);\infty\right). \tag{4.2.6}$$

Remark 4.2.2 Since Y is not analytic in a neighbourhood of ∞, by "Res" we mean

$$\mathrm{Res}\left(z\mathrm{Tr}\left(Y^{-1}(z)Y'(z)\sigma_3\right);\infty\right) = \lim_{R\to\infty} -\frac{1}{2\pi i}\int_{C(0,R)} z\mathrm{Tr}\left(Y^{-1}(z)Y'(z)\sigma_3\right)dz, \tag{4.2.7}$$

where $C(0,R)$ is the circle oriented counter-clockwise, centred at 0 and of radius R.

Proof. We begin by making the substitution $x_i = y\xi_i$, which gives,

$$Z_{n,\alpha}(y;r) = y^{n^2+(\alpha+2)n}\widetilde{Z}_{n,\alpha}(y;r), \tag{4.2.8}$$

where

$$\widetilde{Z}_{n,\alpha}(y;r) := \int_{[r,\infty)} \cdots \int_{[r,\infty)} \Delta_n(\xi)^2 \prod_{j=1}^{n} (\xi_j - 1)^2 \xi_j^{\alpha} e^{-ny\xi_j} d\xi_j. \tag{4.2.9}$$

The above quantity may be computed by introducing monic orthogonal polynomials $\widetilde{p}_j$ satisfying

$$\int_r^{\infty} \widetilde{p}_\ell(x)\widetilde{p}_m(x)\widetilde{w}(x)dx = \widetilde{h}_\ell \delta_{\ell m}, \qquad \widetilde{w}(x) = (x-1)^2 x^{\alpha} e^{-nyx}. \tag{4.2.10}$$

Analogously to (4.2.2), we have

$$\tilde{h}_j = \frac{\widetilde{Z}_{j+1,\alpha}(y;r)}{\widetilde{Z}_{j,\alpha}(y;r)}, \qquad \widetilde{Z}_{0,\alpha}(y;r) := 1. \tag{4.2.11}$$

Recalling the orthogonality conditions for p_n given in (4.2.1), we easily obtain

$$\widetilde{p}_n(x) = y^{-n}p_n(yx), \qquad \widetilde{h}_n = y^{-(2n+\alpha+3)}h_n. \tag{4.2.12}$$

A similar calculation for the Cauchy transform q_n appearing in (4.2.3) shows that

$$\widetilde{q}_n(z) = \frac{1}{2\pi i}\int_r^{\infty} \frac{\widetilde{p}_n(x)\widetilde{w}(x)}{x-z}dx = y^{-(n+\alpha+2)}q_n(yz). \tag{4.2.13}$$

Summarising, if we define

$$\widetilde{Y}(z) := \begin{pmatrix} \widetilde{p}_n(z) & \widetilde{q}_n(z) \\ -\frac{2\pi i}{\widetilde{h}_{n-1}}\widetilde{p}_{n-1}(z) & -\frac{2\pi i}{\widetilde{h}_{n-1}}\widetilde{q}_{n-1}(z) \end{pmatrix}, \tag{4.2.14}$$

we obtain the relationship

$$\widetilde{Y}(z) = y^{-n\sigma_3} y^{-\frac{\alpha+2}{2}\sigma_3} Y(yz) y^{\frac{\alpha+2}{2}\sigma_3}. \tag{4.2.15}$$

We now use the well known relation, which can be straightforwardly deduced from (4.2.11),

$$\widetilde{Z}_{n,\alpha}(y;r) = \prod_{i=0}^{n-1} \tilde{h}_i, \tag{4.2.16}$$

from which it follows that,

$$\partial_y \log \widetilde{Z}_{n,\alpha}(y;r) = \sum_{i=0}^{n-1} \frac{\partial_y \tilde{h}_i}{\tilde{h}_i}. \tag{4.2.17}$$

Now note that,

$$\partial_y \tilde{h}_i = \int_r^\infty \tilde{p}_i(x)^2 \partial_y \tilde{w}(x) dx, \tag{4.2.18}$$

where in the above line we have used the fact that $\tilde{p}$ and $\partial_y \tilde{p}$ are orthogonal. Combining the above expressions then yields,

$$\partial_y \tilde{h}_i = -n \int_r^\infty x \tilde{p}_i(x)^2 \tilde{w}(x) dx. \tag{4.2.19}$$

Using the above in (4.2.17) we have,

$$\partial_y \log \widetilde{Z}_{n,\alpha}(y;r) = -n \int_r^\infty x \tilde{w}(x) \sum_{i=0}^{n-1} \frac{\tilde{p}_i(x)^2}{\tilde{h}_i} dx. \tag{4.2.20}$$

The summation can now be removed by use of the Christoffel-Darboux formula in the form,

$$\tilde{w}(x) \sum_{i=0}^{n-1} \frac{\tilde{p}_i(x)^2}{\tilde{h}_i} = -\frac{1}{4\pi i} \left(\mathrm{Tr}\left(\tilde{Y}_+^{-1}(x) \tilde{Y}_+'(x) \sigma_3\right) - \mathrm{Tr}\left(\tilde{Y}_-^{-1}(x) \tilde{Y}_-'(x) \sigma_3\right)\right), \tag{4.2.21}$$

where $\tilde{Y}_\pm$ correspond to the limiting values of $\tilde{Y}$ above and below (r, ∞). We now obtain,

$$\partial_y \log \widetilde{Z}_{n,\alpha}(y;r) = \frac{n}{4\pi i} \int_r^\infty x \left(\mathrm{Tr}\left(\tilde{Y}_+^{-1}(x) \tilde{Y}_+'(x) \sigma_3\right) - \mathrm{Tr}\left(\tilde{Y}_-^{-1}(x) \tilde{Y}_-'(x) \sigma_3\right)\right) dx. \tag{4.2.22}$$

Combining the above with the fact that

$$\mathrm{Tr}(\widetilde{Y}^{-1}(z;y,r)\widetilde{Y}'(z;y,r)\sigma_3) = y\mathrm{Tr}(Y^{-1}(yz;y,r)Y'(yz;y,r)\sigma_3) \tag{4.2.23}$$

gives

$$\partial_y \log \widetilde{Z}_{n,\alpha}(y;r) = \frac{n}{4\pi i y} \int_{yr}^\infty x \left(\mathrm{Tr}\left(Y_+^{-1}(x) Y_+'(x) \sigma_3\right) - \mathrm{Tr}\left(Y_-^{-1}(x) Y_-'(x) \sigma_3\right)\right) dx.$$

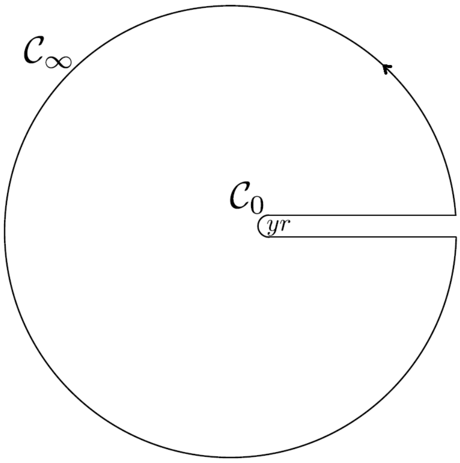

Figure 4.1: The contour used in establishing a differential identity for the partition function $Z(y, r, t)$.

$$(4.2.24)$$

Consider the integral of $z\mathrm{Tr}\left(Y^{-1}(z; y, r)Y'(z; y, r)\sigma_3\right)$ over the contour $\mathcal{C}$ shown in Figure 4.1. As $\mathcal{C}$ does not enclose any singularities this integral is zero and therefore we have,

$$\int_{yr}^{\infty} x\left(\mathrm{Tr}\left(Y_+^{-1}(x)Y_+'(x)\sigma_3\right) - \mathrm{Tr}\left(Y_-^{-1}(x)Y_-'(x)\sigma_3\right)\right) dx$$
$$= -\int_{\mathcal{C}_0+\mathcal{C}_\infty} z\mathrm{Tr}\left(Y^{-1}(z)Y'(z)\sigma_3\right) dz$$

Using the property (d) in the RH problem for Y we see that there is no contribution from the integration along $\mathcal{C}_0$ and we are therefore left with only the contribution coming from $\mathcal{C}_\infty$ which yields,

$$\partial_y \log \widetilde{Z}_{n,\alpha}(y; r) = \frac{n}{2y}\mathrm{Res}\left(z\mathrm{Tr}\left(Y^{-1}(z; y, r)Y'(z; y, r)\sigma_3\right); \infty\right), \quad (4.2.25)$$

The result follows from (4.2.8). □

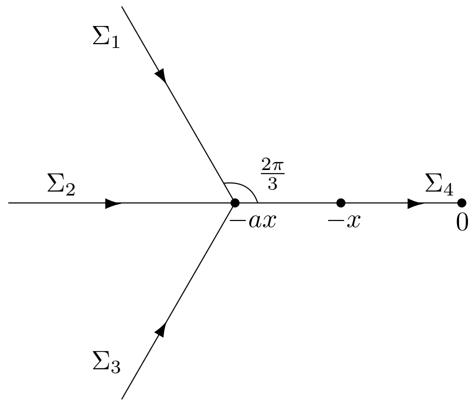

Figure 4.2: The jump contours appearing in the Riemann-Hilbert problem for Φ. The arcs Σ_j, $j = 1, 2, 3, 4$ depend on x and a. For convenience, they are chosen such that they do not contain $-ax$, $-x$ and 0.

4.3 A Riemann-Hilbert problem related to the system of ODEs

We first introduce some notations for the sake of convenience. We define the piecewise constant function

$$\theta(z) = \begin{cases} +1 & \text{if } \operatorname{Im} z > 0, \\ -1 & \text{if } \operatorname{Im} z < 0, \end{cases} \tag{4.3.1}$$

and for $x \geq 0$, we define

$$H_x(z) = \begin{cases} I, & \text{for } -\frac{2\pi}{3} < \arg(z + ax) < \frac{2\pi}{3}, \\ \begin{pmatrix} 1 & 0 \\ -e^{\pi i\alpha} & 1 \end{pmatrix}, & \text{for } \frac{2\pi}{3} < \arg(z + ax) < \pi, \\ \begin{pmatrix} 1 & 0 \\ e^{-\pi i\alpha} & 1 \end{pmatrix}, & \text{for } -\pi < \arg(z + ax) < -\frac{2\pi}{3}. \end{cases} \tag{4.3.2}$$

In the course of computing the asymptotics for the above Hankel determinants $Z_{n,\alpha}(y; r)$ one is led to consider a model RH problem, which we will denote by Φ. The matrix-valued function Φ is defined as the unique solution of the following RH problem, depending on parameters $x > 0$ and $a > 1$. As we will see later, x is related to y and a is related to r.

RH for $\Phi(z) = \Phi(z; x, a)$

(a) $\Phi : \mathbb{C} \setminus \Sigma_{x,a} \to \mathbb{C}^{2\times 2}$ analytic, with $\Sigma_{x,a} = \cup_{i=1}^4 \Sigma_i \cup \{-ax, -x, 0\}$ as illustrated in Figure 4.2.

(b) Φ has continuous boundary values $\Phi_\pm(z)$ as $z \in \Sigma_{x,a} \setminus \{-ax, -x, 0\}$ is approached from the left (+) or right (−) side of $\Sigma_{x,a}$, and they are related by

$$\Phi_+(z) = \Phi_-(z) \begin{pmatrix} 1 & 0 \\ e^{\pi i\alpha} & 1 \end{pmatrix}, \qquad z \in \Sigma_1, \tag{4.3.3}$$

$$\Phi_+(z) = \Phi_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad z \in \Sigma_2, \tag{4.3.4}$$

$$\Phi_+(z) = \Phi_-(z) \begin{pmatrix} 1 & 0 \\ e^{-\pi i\alpha} & 1 \end{pmatrix}, \qquad z \in \Sigma_3, \tag{4.3.5}$$

$$\Phi_+(z) = \Phi_-(z) e^{\pi i\alpha\sigma_3}, \qquad z \in \Sigma_4. \tag{4.3.6}$$

(c) As $z \to \infty$, there exist functions $p(x), q(x), v(x), q_2(x), v_2(x), r_2(x), p_2(x)$ depending also on a, such that Φ has the asymptotic behaviour

$$\Phi(z) = \left(I + \frac{1}{z}\Phi_1(x) + \frac{1}{z^2}\Phi_2(x) + \mathcal{O}(z^{-3}) \right) z^{-\frac{1}{4}\sigma_3} N e^{z^{\frac{1}{2}}\sigma_3}, \tag{4.3.7}$$

where

$$\Phi_1(x) = \begin{pmatrix} q(x) & iv(x) \\ ip(x) & -q(x) \end{pmatrix}, \qquad \Phi_2(x) = \begin{pmatrix} q_2(x) & iv_2(x) \\ ip_2(x) & r_2(x) \end{pmatrix},$$

$N = \frac{1}{\sqrt{2}}(I + i\sigma_1)$, with $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(d) As $z \to -ax$, Φ has the asymptotic behaviour

$$\Phi(z) = \mathcal{O}(1) \begin{pmatrix} 1 & \frac{1}{2\pi i}\log(z + ax) \\ 0 & 1 \end{pmatrix} e^{\frac{\pi i\alpha}{2}\theta(z)\sigma_3} H_x(z). \tag{4.3.8}$$

As $z \to -x$, Φ has the asymptotic behaviour

$$\Phi(z) = \mathcal{O}(1) e^{\frac{\pi i\alpha}{2}\theta(z)\sigma_3} (z + x)^{\sigma_3}. \tag{4.3.9}$$

As $z \to 0$, Φ has the asymptotic behaviour

$$\Phi(z) = \mathcal{O}(1) z^{\frac{\alpha\sigma_3}{2}}. \tag{4.3.10}$$

Remark 4.3.1 It can be verified that the $\mathcal{O}(1)$ terms in asymptotics (4.3.8), (4.3.9) and (4.3.10) are analytic functions.

Remark 4.3.2 The existence of Φ has been proved for $\alpha \geq 0$ in [3] (our situation corresponds to $I = (-\infty, -ax)$, $B = \{-x, 0\}$, $\hat{\alpha}_{-x} = 1$, $\hat{\alpha}_0 = \frac{\alpha}{2}$ and $\tau_{\infty,0} = -\frac{1}{2}$ in the language of [3]).

4.4 The Lax pair and the system of ODEs

We begin by making the transformation,

$$\widetilde{\Phi}(z; x, a) := \begin{pmatrix} 1 & 0 \\ -x^{-1} v(x^2) & 1 \end{pmatrix} x^{\frac{\sigma_3}{2}} e^{\frac{1}{4}\pi i \sigma_3} \Phi(x^2 z; x^2, a). \tag{4.4.1}$$

The jump contour for $\widetilde{\Phi}(z)$ is $\Sigma_{1,a}$ and is independent of x, while its asymptotic expansion as $z \to \infty$ is given by

$$\widetilde{\Phi}(z) = \begin{pmatrix} 1 & 0 \\ -\frac{v(x^2)}{x} & 1 \end{pmatrix} \left(I + \frac{1}{z}\widetilde{\Phi}_1(x) + \frac{1}{z^2}\widetilde{\Phi}_2(x) + \mathcal{O}(z^{-3}) \right) e^{\frac{\pi i}{4}\sigma_3} z^{-\frac{1}{4}\sigma_3} N e^{x z^{\frac{1}{2}} \sigma_3}, \tag{4.4.2}$$

where

$$\widetilde{\Phi}_1(x) = \begin{pmatrix} \frac{q(x^2)}{x^2} & -\frac{v(x^2)}{x} \\ \frac{p(x^2)}{x^3} & -\frac{q(x^2)}{x^2} \end{pmatrix} \quad \text{and} \quad \widetilde{\Phi}_2(x) = \begin{pmatrix} \frac{q_2(x^2)}{x^4} & -\frac{v_2(x^2)}{x^3} \\ \frac{p_2(x^2)}{x^5} & \frac{r_2(x^2)}{x^4} \end{pmatrix}. \tag{4.4.3}$$

By standard argument in RH analysis, $\tilde{\Phi}$ is analytic in $x \in (0, \infty)$. The Lax pair $(A, B) = (A(z; x, a), B(z; x, a))$ is defined by,

$$\partial_z \widetilde{\Phi}(z) = A(z) \widetilde{\Phi}(z), \tag{4.4.4}$$
$$\partial_x \widetilde{\Phi}(z) = B(z) \widetilde{\Phi}(z). \tag{4.4.5}$$

Since the jump matrices for $\widetilde{\Phi}(z)$ are independent of z and x, A and B are analytic on $\mathbb{C} \setminus \{-a, -1, 0\}$. Using the asymptotic behaviour at $z = 0$, $z = -1$, $z = -a$ and $z = \infty$, it is easy to show that $A(z)$ is meromorphic on $\mathbb{C}$ with single pole at $-a$, -1 and 0 while $B(z)$ is an entire function. One has

$$A(z) = A_{\infty,0}(x) + A_{0,1}(x)z^{-1} + A_{1,1}(x)(z+1)^{-1} + A_{a,1}(x)(z+a)^{-1}, \tag{4.4.6}$$

$$B(z) = \begin{pmatrix} 0 & 1 \\ z - u(x) & 0 \end{pmatrix}, \tag{4.4.7}$$

where $u = 2\partial_x(v(x^2)x^{-1})$, $A_{0,1}(x)$, $A_{1,1}(x)$, $A_{a,1}(x)$ are analytic matrices in $x \in (0,\infty)$ and $A_{\infty,0}(x) = \begin{pmatrix} 0 & 0 \\ \frac{x}{2} & 0 \end{pmatrix}$.

We now turn to the compatibility condition,

$$\partial_z \partial_x \widetilde{\Phi} = \partial_x \partial_z \widetilde{\Phi}, \tag{4.4.8}$$

which upon rewriting the derivatives in terms of the Lax matrices becomes,

$$\partial_x A - \partial_z B + AB - BA = 0 \tag{4.4.9}$$

If we parameterise A as

$$A(z;x,a) = \begin{pmatrix} d(z;x,a) & b(z;x,a) \\ c(z;x,a) & -d(z;x,a) \end{pmatrix}, \tag{4.4.10}$$

then (4.4.9) is equivalent to three coupled ODEs,

$$d(z) = -\frac{1}{2}\partial_x b(z), \tag{4.4.11}$$

$$c(z) = (z-u)b(z) - \frac{1}{2}\partial_x^2 b(z), \tag{4.4.12}$$

$$\partial_x c(z) = 1 + 2(z - u(x))d(z). \tag{4.4.13}$$

The first two equations provide d and c in terms of b. Taking the determinant yields,

$$\det A = -\frac{3}{4}(b')^2 - (z-u(x))b^2 + \frac{1}{4}(b^2)'', \tag{4.4.14}$$

where primes denote derivatives with respect to x. From (4.4.6) we have that $b(z)$ is of the form,

$$b(z) = b_0 z^{-1} + b_1(z+1)^{-1} + b_2(z+a)^{-1}, \tag{4.4.15}$$

where b_0, b_1 and b_2 depend on x and a. Equation (4.4.15) together with (4.4.14), allow us to compute the asymptotics of $\det A$ at $z=0$, $z=-1$, $z=-a$ and $z=\infty$ in terms of b_0, b_1 and b_2.

Alternatively, we may compute $\det A$ at these four points using the asymptotic expansion of $\widetilde{\Phi}$ there. Equating these asymptotics with those expressed in terms of b_0 and b_1 we reach the equations,

$$\frac{\alpha^2}{4}+b_0(x)^2u(x)-\frac{1}{4}b_0'(x)^2+\frac{1}{2}b_0(x)b_0''(x)=0 \tag{4.4.16}$$
$$b_1(x)^2(u(x)+1)-\frac{1}{4}b_1'(x)^2+\frac{1}{2}b_1(x)b_1''(x)+1=0 \tag{4.4.17}$$
$$b_2(x)^2(a+u(x))-\frac{1}{4}b_2'(x)^2+\frac{1}{2}b_2(x)b_2''(x)=0 \tag{4.4.18}$$
$$\frac{x^2}{4}-\left(b_0(x)+b_1(x)+b_2(x)\right)^2=0 \tag{4.4.19}$$

These equations can be simplified by changing variables,

$$b_0(x)=U_0(x)^2, \tag{4.4.20}$$
$$b_1(x)=U_1(x)^2, \tag{4.4.21}$$
$$b_2(x)=V(x), \tag{4.4.22}$$

after which we obtain the first two equations in (4.1.16). On the other hand, by expanding the expression $A_{12}(z)=b(z)$ in a Laurent series about $z=\infty$, we get these following identities:

$$\begin{array}{ccccccc} b_0(x) & + & b_1(x) & + & b_2(x) & = & \dfrac{x}{2}, \\ & & b_1(x) & + & ab_2(x) & = & -xv'(x^2), \\ & & 2b_1(x) & + & 2a^2b_2(x) & = & \dfrac{F(x)}{x^3}, \end{array} \tag{4.4.23}$$

where

$$F(x)=2q(x^2)^2+q_2(x^2)-r_2(x^2)-v_2(x^2)(2v(x^2)-3)-(p(x^2)+q(x^2))v(x^2). \tag{4.4.24}$$

The first two equations in (4.4.23) can be rewritten in terms of U_0, U_1 and V which correspond to (4.1.21) and the third equation in (4.1.16).

If we follow the same idea for B using the fact that this is an entire matrix function, we can express all the quantities involved in (4.4.24) in terms of v and

v_2:

$$\begin{aligned} q(x) &= \frac{1}{2}(2xv'(x)+v^2(x)-v(x)), \\ p(x) &= -v_2(x)+v(x)q(x)+2xq'(x)-2q(x), \\ q_2(x)-r_2(x) &= v(x)v_2(x)-3v_2(x)+2xv_2'(x). \end{aligned} \tag{4.4.25}$$

4.5 Further properties of the special solutions

Our main goal in this section is to get asymptotics for $U_0(x)$, $U_1(x)$ and $V(x)$ as $x \to \infty$ and $x \to 0_+$.

Note that (4.4.23) and (4.4.25) allow us to express $b_0(x)$, $b_1(x)$ and $b_2(x)$ (and therefore $U_0(x)$, $U_1(x)$ and $V(x)$) in terms of $v(x)$, $v'(x)$, $v''(x)$, $v_2(x)$, $v_2'(x)$.

4.5.1 Asymptotic analysis when $x \to \infty$

Re-scaling of the model problem

In order to have a jump contour independent of x, we make the transformation $C(z;x,a) = x^{\frac{1}{4}\sigma_3}\Phi(zx;x,a)$. C is the solution of the following RH problem:

RH Problem for U

(a) $C : \mathbb{C} \setminus \Sigma_{1,a} \to \mathbb{C}^{2\times 2}$ is analytic.

(b) C has the following jumps

$$C_+(z) = C_-(z)\begin{pmatrix} 1 & 0 \\ e^{\pi i\alpha} & 1 \end{pmatrix}, \qquad z \in \Sigma_1, \tag{4.5.1}$$

$$C_+(z) = C_-(z)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad z \in \Sigma_2, \tag{4.5.2}$$

$$C_+(z) = C_-(z)\begin{pmatrix} 1 & 0 \\ e^{-\pi i\alpha} & 1 \end{pmatrix}, \qquad z \in \Sigma_3, \tag{4.5.3}$$

$$C_+(z) = C_-(z)e^{\pi i\alpha\sigma_3}, \qquad z \in \Sigma_4. \tag{4.5.4}$$

(c) As $z \to \infty$,

$$C(z) = \left(I + \frac{1}{z}C_1(x) + \frac{1}{z^2}C_2(x) + \mathcal{O}(z^{-3})\right) z^{-\frac{1}{4}\sigma_3} N e^{\sqrt{x}z^{\frac{1}{2}}\sigma_3}, \quad (4.5.5)$$

where

$$C_1(x) = \begin{pmatrix} \frac{q(x)}{x} & \frac{iv(x)}{x^{1/2}} \\ \frac{ip(x)}{x^{3/2}} & -\frac{q(x)}{x} \end{pmatrix} \quad \text{and} \quad C_2(x) = \begin{pmatrix} \frac{q_2(x)}{x^2} & \frac{iv_2(x)}{x^{3/2}} \\ \frac{ip_2(x)}{x^{5/2}} & \frac{r_2(x)}{x^2} \end{pmatrix}. \quad (4.5.6)$$

(d) As $z \to -a$, C has the asymptotic behaviour

$$C(z) = \mathcal{O}(1) \begin{pmatrix} 1 & \frac{1}{2\pi i}\log(z+a) \\ 0 & 1 \end{pmatrix} e^{\frac{\pi i \alpha}{2}\theta(z)\sigma_3} H_1(z). \quad (4.5.7)$$

As $z \to -1$, C has the asymptotic behaviour

$$C(z) = \mathcal{O}(1) e^{\frac{\pi i \alpha}{2}\theta(z)\sigma_3}(z+1)^{\sigma_3}. \quad (4.5.8)$$

As $z \to 0$, C has the asymptotic behaviour

$$C(z) = \mathcal{O}(1) z^{\frac{\alpha\sigma_3}{2}}. \quad (4.5.9)$$

Normalisation at ∞ of the RH problem

We define the g-function by

$$g(z) = \sqrt{z+a} \quad (4.5.10)$$

where the principal branch is taken for the square root. As $z \to \infty$, g has the asymptotic behaviour

$$g(z) = z^{1/2} + g_1 z^{-1/2} + g_2 z^{-3/2} + \mathcal{O}(z^{-5/2}), \qquad g_1 = \frac{a}{2}, \quad g_2 = -\frac{a^2}{8}. \quad (4.5.11)$$

Now we define

$$W(z) = \begin{pmatrix} 1 & 0 \\ i\sqrt{x}g_1 & 1 \end{pmatrix} C(z) e^{-\sqrt{x}g(z)\sigma_3}. \quad (4.5.12)$$

RH Problem for W

(a) $W : \mathbb{C} \setminus \Sigma_{1,a} \to \mathbb{C}^{2\times 2}$ is analytic.

(b) W has the following jumps:

$$W_+(z) = W_-(z)\begin{pmatrix} 1 & 0 \\ e^{\pi i\alpha}e^{-2\sqrt{x}g(z)} & 1 \end{pmatrix}, \qquad z \in \Sigma_1, \tag{4.5.13}$$

$$W_+(z) = W_-(z)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad z \in \Sigma_2, \tag{4.5.14}$$

$$W_+(z) = W_-(z)\begin{pmatrix} 1 & 0 \\ e^{-\pi i\alpha}e^{-2\sqrt{x}g(z)} & 1 \end{pmatrix}, \qquad z \in \Sigma_3, \tag{4.5.15}$$

$$W_+(z) = W_-(z)e^{\pi i\alpha\sigma_3}, \qquad z \in \Sigma_4. \tag{4.5.16}$$

(c) As $z \to \infty$,

$$W(z) = \left(I + \frac{1}{z}W_1(x) + \frac{1}{z^2}W_2(x) + \mathcal{O}(z^{-3})\right) z^{-\frac{1}{4}\sigma_3}N, \tag{4.5.17}$$

where

$$(W_1(x))_{12} = \frac{iv(x)}{\sqrt{x}} + ig_1\sqrt{x} \tag{4.5.18}$$

and

$$(W_2(x))_{12} = \frac{i}{6x^{3/2}}(6g_1xq(x) + g_1^3x^3 + 6g_2x^2 + 3g_1^2x^2v(x) + 6v_2(x)). \tag{4.5.19}$$

(d) As $z \to -a$,

$$W(z) = \mathcal{O}(1)\begin{pmatrix} 1 & \frac{\log(z+a)}{2\pi i} \\ 0 & 1 \end{pmatrix} e^{\frac{\pi i\alpha}{2}\theta(z)\sigma_3}H_1(z)e^{-\sqrt{x}g(z)\sigma_3}. \tag{4.5.20}$$

As $z \to -1$,

$$W(z) = \mathcal{O}(1)(z+1)^{\sigma_3}. \tag{4.5.21}$$

As $z \to 0$,

$$W(z) = \mathcal{O}(1)z^{\frac{\alpha\sigma_3}{2}}. \tag{4.5.22}$$

On $\Sigma_1 \cup \Sigma_3$, we observe that $\mathrm{Re}(g(z)) > 0$ and therefore these jumps are exponentially small when $x \to \infty$. Nevertheless, this convergence is not uniform as z approaches $-a$ and therefore we will have to construct a global parametrix which will be a good approximation of W as z stays away from a neighbourhood of $-a$, and a local parametrix around $-a$.

Global parametrix

Ignoring exponentially small jumps and a small neighbourhood of $-a$, we are led to the following RH problem.

RH Problem for $P^{(\infty)}$

(a) $P^{(\infty)} : \mathbb{C} \setminus \mathbb{R}^-$ is analytic.

(b) $P^{(\infty)}$ has the following jumps on $\mathbb{R}^-$:

$$P_+^{(\infty)}(z) = P_-^{(\infty)}(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad z \in (-\infty, -a), \tag{4.5.23}$$

$$P_+^{(\infty)}(z) = P_-^{(\infty)}(z) e^{\pi i \alpha \sigma_3}, \qquad z \in (-a, 0) \setminus \{-1\}, \tag{4.5.24}$$

(c) As $z \to \infty$,

$$P^{(\infty)}(z) = \left(I + \frac{1}{z} P_1^{(\infty)} + \frac{1}{z^2} P_2^{(\infty)} + \mathcal{O}(z^{-3}) \right) z^{-\frac{1}{4}\sigma_3} N. \tag{4.5.25}$$

(d) As $z \to -1$,

$$P^{(\infty)}(z) = \mathcal{O}(1)(z+1)^{\sigma_3}. \tag{4.5.26}$$

As $z \to 0$,

$$P^{(\infty)}(z) = \mathcal{O}(1) z^{\frac{\alpha \sigma_3}{2}}. \tag{4.5.27}$$

We can check that the solution of this RH problem is explicitly given by

$$\begin{aligned} P^{(\infty)}(z) = &\begin{pmatrix} 1 & 0 \\ i(\alpha\sqrt{a} + 2\sqrt{a-1}) & 1 \end{pmatrix} (z+a)^{-\frac{\sigma_3}{4}} N \\ &\times \left(\frac{\sqrt{z+a} + \sqrt{a}}{\sqrt{z+a} - \sqrt{a}} \right)^{-\frac{\alpha}{2}\sigma_3} \left(\frac{\sqrt{z+a} + \sqrt{a-1}}{\sqrt{z+a} - \sqrt{a-1}} \right)^{-\sigma_3}. \end{aligned} \tag{4.5.28}$$

Note that

$$
\begin{aligned}
(P_1^{(\infty)})_{12} &= i(\alpha\sqrt{a} + 2\sqrt{a-1}),\\
(P_2^{(\infty)})_{12} &= \frac{i}{12}\Big(\alpha a^{3/2}(2\alpha^2+19) + 6a\sqrt{a-1}(2\alpha^2+1) - 24\alpha\sqrt{a} - 24\sqrt{a-1}\Big).
\end{aligned}
\tag{4.5.29}
$$

Local parametrix near $-a$

We want to construct a function $P^{(-a)}$ defined in a fixed open disk D_{-a} around $-a$ which satisfies the following RH conditions.

RH Problem for $P^{(-a)}$

(a) $P^{(-a)} : D_{-a} \setminus \Sigma_{1,a}$ is analytic.

(b) $P^{(-a)}$ has the following jumps:

$$
P_+^{(-a)}(z) = P_-^{(-a)}(z)\begin{pmatrix} 1 & 0 \\ e^{\pi i\alpha}e^{-2\sqrt{x}g(z)} & 1 \end{pmatrix}, \qquad z \in \Sigma_1 \cap D_{-a}, \tag{4.5.30}
$$

$$
P_+^{(-a)}(z) = P_-^{(-a)}(z)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad z \in \Sigma_2 \cap D_{-a}, \tag{4.5.31}
$$

$$
P_+^{(-a)}(z) = P_-^{(-a)}(z)\begin{pmatrix} 1 & 0 \\ e^{-\pi i\alpha}e^{-2\sqrt{x}g(z)} & 1 \end{pmatrix}, \qquad z \in \Sigma_3 \cap D_{-a}, \tag{4.5.32}
$$

$$
P_+^{(-a)}(z) = P_-^{(-a)}(z)e^{\pi i\alpha\sigma_3}, \qquad z \in \Sigma_4 \cap D_{-a}. \tag{4.5.33}
$$

(c) As $x \to \infty$,

$$
P^{(-a)}(z) = (I + \mathcal{O}(x^{-1/2}))P^{(\infty)}(z) \tag{4.5.34}
$$

uniformly for $z \in \partial D_{-a}$.

(d) As $z \to -a$,

$$
P^{(-a)}(z) = \mathcal{O}(1)\begin{pmatrix} 1 & \frac{\log(z+a)}{2\pi i} \\ 0 & 1 \end{pmatrix} e^{\frac{\pi i\alpha}{2}\theta(z)\sigma_3}H_1(z)e^{-\sqrt{x}g(z)\sigma_3}. \tag{4.5.35}
$$

The solution of this RH problem can constructed in terms of the Bessel RH problem of order 0. Because we will need later of the Bessel RH problem for general α, we present it here.

The Bessel model RH problem of order α

(a) $\Upsilon = \Upsilon^{(\alpha)} : \mathbb{C} \setminus \Sigma_{0,0}$ is analytic.

(b) Υ has the following jumps on $\Sigma_{0,0} \setminus \{0\}$:

$$\Upsilon_+(z) = \Upsilon_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad z \in \Sigma_2, \tag{4.5.36}$$

$$\Upsilon_+(z) = \Upsilon_-(z) \begin{pmatrix} 1 & 0 \\ e^{\pi i \alpha} & 1 \end{pmatrix}, \qquad z \in \Sigma_1, \tag{4.5.37}$$

$$\Upsilon_+(z) = \Upsilon_-(z) \begin{pmatrix} 1 & 0 \\ e^{-\pi i \alpha} & 1 \end{pmatrix}, \qquad z \in \Sigma_3, \tag{4.5.38}$$

(c) As $z \to \infty$,

$$\Upsilon(z) = \left(I + \frac{1}{z}\Upsilon_1 + \frac{1}{z^2}\Upsilon_2 + \mathcal{O}(z^{-3}) \right) z^{-\frac{1}{4}\sigma_3} N e^{z^{1/2}\sigma_3}. \tag{4.5.39}$$

(d) As $z \to 0$,

(i) If $\alpha < 0$, $\Upsilon(z) = \mathcal{O}\begin{pmatrix} |z|^{\frac{\alpha}{2}} & |z|^{\frac{\alpha}{2}} \\ |z|^{\frac{\alpha}{2}} & |z|^{\frac{\alpha}{2}} \end{pmatrix}$.

(ii) If $\alpha = 0$, $\Upsilon(z) = \mathcal{O}\begin{pmatrix} \log|z| & \log|z| \\ \log|z| & \log|z| \end{pmatrix}$.

(iii) If $\alpha > 0$,

$$\Upsilon(z) = \begin{cases} \mathcal{O}\begin{pmatrix} |z|^{\frac{\alpha}{2}} & |z|^{-\frac{\alpha}{2}} \\ |z|^{\frac{\alpha}{2}} & |z|^{-\frac{\alpha}{2}} \end{pmatrix}, & \text{for } -\frac{2\pi}{3} < \arg(z) < \frac{2\pi}{3}, \\ \mathcal{O}\begin{pmatrix} |z|^{-\frac{\alpha}{2}} & |z|^{-\frac{\alpha}{2}} \\ |z|^{-\frac{\alpha}{2}} & |z|^{-\frac{\alpha}{2}} \end{pmatrix}, & \text{for } \arg(z) \in (-\pi, -\frac{2\pi}{3}) \cup (\frac{2\pi}{3}, \pi). \end{cases} \tag{4.5.40}$$

The solution of this RH problem is explicitly given in terms of the Bessel functions (see [49] or [4]), one has

$$\Upsilon^{(\alpha)}(z) = \begin{pmatrix} 1 & 0 \\ \frac{i}{8}(4\alpha^2+3) & 1 \end{pmatrix} \pi^{\sigma_3/2} \begin{pmatrix} I_\alpha(z^{1/2}) & \frac{i}{\pi}K_\alpha(z^{1/2}) \\ \pi i z^{1/2} I'_\alpha(z^{1/2}) & -z^{1/2}K'_\alpha(z^{1/2}) \end{pmatrix} H_0(z), \tag{4.5.41}$$

and

$$\begin{aligned} (\Upsilon_1)_{12} &= \frac{i}{8}(2\alpha-1)(2\alpha+1), \\ (\Upsilon_2)_{12} &= \frac{i}{3072}(2\alpha-5)(2\alpha-3)(2\alpha-1)(2\alpha+1)(2\alpha+3)(2\alpha+5). \end{aligned} \tag{4.5.42}$$

Construction of the local parametrix near $-a$

We can therefore construct the local parametrix using the model RH problem $\Upsilon^{(0)}$, one has

$$P^{(-a)}(z) = E(z)\Upsilon^{(0)}(xf(z))e^{\frac{\pi i\alpha}{2}\theta(z)\sigma_3}e^{-\sqrt{x}g(z)\sigma_3}, \tag{4.5.43}$$

where

$$f(z) = g(z)^2 = z + a \quad \text{and} \quad E(z) = P^{(\infty)}(z)e^{-\frac{\pi i\alpha}{2}\theta(z)\sigma_3}N^{-1}f(z)^{\frac{\sigma_3}{4}}x^{\frac{1}{4}\sigma_3}. \tag{4.5.44}$$

A direct check shows that E is analytic in D_{-a}.

Small norm RH problem

Define

$$R(z) = \begin{cases} W(z)P^{(\infty)}(z)^{-1}, & \text{for } z \in \mathbb{C}\setminus(\overline{D_{-a}}\cup\Sigma_1\cup\Sigma_3), \\ W(z)P^{(-a)}(z)^{-1}, & \text{for } z \in D_{-a}. \end{cases} \tag{4.5.45}$$

On ∂D_{-a} by (4.5.34), we have $R_-(z)^{-1}R_+(z) = I+\mathcal{O}(x^{-1/2})$ and by (4.5.13) and (4.5.15), on $(\Sigma_1\cup\Sigma_3)\setminus D_{-a}$, $R_-(z)^{-1}R_+(z) = I+\mathcal{O}(e^{-c\sqrt{x}})$ where $c>0$ is a constant. From (4.5.17) and (4.5.25), as $z\to\infty$ one has $R(z) = I+\mathcal{O}(z^{-1})$.

By the small norm theory for RH problems, it follows that $R(z) = I + \mathcal{O}(x^{-1/2})$ uniformly in $z \in \mathbb{C} \setminus ((\partial D_{-a} \cup \Sigma_1 \cup \Sigma_3) \setminus D_{-a})$ and as $z \to \infty$,

$$R(z) = I + \frac{R_1(x)}{z} + \frac{R_2(x)}{z^2} + \mathcal{O}(z^{-3}). \tag{4.5.46}$$

We have in particular that $R_1 = \mathcal{O}(x^{-1/2})$, $R_2 = \mathcal{O}(x^{-1/2})$. One can also prove by a more detailed analysis that

$$\partial_x R(z) = \mathcal{O}(x^{-3/2}), \qquad \partial_x^2 R(z) = \mathcal{O}(x^{-5/2}). \tag{4.5.47}$$

For $z \in \mathbb{C} \setminus (\overline{D_{-a}} \cup \Sigma_1 \cup \Sigma_3)$, we have $W(z) = R(z)P^{(\infty)}(z)$ and therefore

$$W_1(x) = R_1(x) + P_1^{(\infty)} = P_1^{(\infty)} + \mathcal{O}(x^{-1/2}), \tag{4.5.48}$$

$$W_2(x) = R_2(x) + R_1(x)P_1^{(\infty)} + P_2^{(\infty)} = P_2^{(\infty)} + \mathcal{O}(x^{-1/2}). \tag{4.5.49}$$

Using (4.5.18), (4.5.19), (4.5.48), (4.5.49), (4.5.47) and the system (4.4.23) we have finally that

$$\begin{aligned}
v(x) &= -\frac{a}{2}x + (2\sqrt{a-1} + \alpha\sqrt{a})\sqrt{x} + \mathcal{O}(1), \\
b_0(x) &= \frac{\alpha}{2\sqrt{a}} + \mathcal{O}(x^{-1}), \\
b_1(x) &= \frac{1}{\sqrt{a-1}} + \mathcal{O}(x^{-1}), \\
b_2(x) &= \frac{x}{2} - \frac{\alpha}{2\sqrt{a}} - \frac{1}{\sqrt{a-1}} + \mathcal{O}(x^{-1}).
\end{aligned} \tag{4.5.50}$$

If we express (4.5.50) in terms of U_0, U_1 and V, we get (4.1.17) with $a = r$.

4.5.2 Asymptotic analysis when $x \to 0$

In order to have a jump contour which doesn't depend on x, we make the following transformation on Φ:

$$\widetilde{W}(z) = \Phi(z)H_x(z)^{-1}H_0(z). \tag{4.5.51}$$

It is easy to verify that $\widetilde{W}$ satisfies the following RH Problem.

RH Problem for $\widetilde{W}$

(a) $\widetilde{W} : \mathbb{C} \setminus \Sigma_{0,0} \to \mathbb{C}^{2\times 2}$ is analytic.

(b) $\widetilde{W}$ has the following jumps

$$\widetilde{W}_+(z) = \widetilde{W}_-(z) \begin{pmatrix} 1 & 0 \\ e^{\pi i\alpha} & 1 \end{pmatrix}, \qquad \arg(z) = \frac{2\pi}{3}, \quad (4.5.52)$$

$$\widetilde{W}_+(z) = \widetilde{W}_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad z \in (-\infty, -ax), \quad (4.5.53)$$

$$\widetilde{W}_+(z) = \widetilde{W}_-(z) \begin{pmatrix} 1 & 0 \\ e^{-\pi i\alpha} & 1 \end{pmatrix}, \qquad \arg(z) = -\frac{2\pi}{3}, \quad (4.5.54)$$

$$\widetilde{W}_+(z) = \widetilde{W}_-(z) \begin{pmatrix} e^{\pi i\alpha} & 0 \\ -2 & e^{-\pi i\alpha} \end{pmatrix}, \quad z \in (-ax, 0) \setminus \{-x\}. \quad (4.5.55)$$

(c) As $z \to \infty$,

$$\widetilde{W}(z) = \left(I + \frac{1}{z}\Phi_1(x) + \frac{1}{z^2}\Phi_2(x) + \mathcal{O}(z^{-3}) \right) z^{-\frac{1}{4}\sigma_3} N e^{z^{\frac{1}{2}}\sigma_3}. \quad (4.5.56)$$

(d) As $z \to -ax$, $\widetilde{W}$ has the asymptotic behaviour

$$\widetilde{W}(z) = \mathcal{O}(1) \begin{pmatrix} 1 & \frac{\log(z+ax)}{2\pi i} \\ 0 & 1 \end{pmatrix} e^{\frac{\pi i\alpha}{2}\theta(z)\sigma_3} H_0(z). \quad (4.5.57)$$

As $z \to -x$,

$$\widetilde{W}(z) = \mathcal{O}(1) e^{\frac{\pi i\alpha}{2}\theta(z)\sigma_3} (z+x)^{\sigma_3} H_0(z). \quad (4.5.58)$$

As $z \to 0$,

$$\widetilde{W}(z) = \mathcal{O}(1) z^{\frac{\alpha\sigma_3}{2}} H_0(z). \quad (4.5.59)$$

In equations (4.5.73), (4.5.74) and (4.5.75), the $\mathcal{O}(1)$ are analytic functions in a neighbourhood of their respective point.

Global parametrix

As $x \to 0$, the length of $(-ax, 0)$ tends to 0 and the two poles at 0 and $-x$ are merging together. Therefore, for z outside of a neighbourhood of 0, we expect that the Bessel parametrix of order $\alpha + 2$ will be a good approximation of $\widetilde{W}$.

Of course this is not true in a small neighbourhood of 0 and we will construct a local parametrix around the origin.

RH Problem for $P^{(\infty)}$

(a) $P^{(\infty)}:\mathbb{C}\setminus\Sigma_{0,0}$ is analytic.

(b) $P^{(\infty)}$ has the following jumps on $\Sigma_{0,0}$:

$$P_+^{(\infty)}(z) = P_-^{(\infty)}(z)\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \qquad z\in\Sigma_2, \tag{4.5.60}$$

$$P_+^{(\infty)}(z) = P_-^{(\infty)}(z)\begin{pmatrix}1 & 0\\ e^{\pi i\alpha} & 1\end{pmatrix}, \qquad z\in\Sigma_1, \tag{4.5.61}$$

$$P_+^{(\infty)}(z) = P_-^{(\infty)}(z)\begin{pmatrix}1 & 0\\ e^{-\pi i\alpha} & 1\end{pmatrix}, \qquad z\in\Sigma_3, \tag{4.5.62}$$

(c) As $z\to\infty$,

$$P^{(\infty)}(z) = \left(I+\mathcal{O}(z^{-1})\right) z^{-\frac{1}{4}\sigma_3} N e^{z^{1/2}\sigma_3}. \tag{4.5.63}$$

(d) As $z\to 0$,

$$P^{(\infty)}(z) = \begin{cases} \mathcal{O}\begin{pmatrix}|z|^{\frac{\alpha+2}{2}} & |z|^{-\frac{\alpha+2}{2}}\\ |z|^{\frac{\alpha+2}{2}} & |z|^{-\frac{\alpha+2}{2}}\end{pmatrix}, & \text{for } -\frac{2\pi}{3}<\arg(z)<\frac{2\pi}{3},\\ \mathcal{O}\begin{pmatrix}|z|^{-\frac{\alpha+2}{2}} & |z|^{-\frac{\alpha+2}{2}}\\ |z|^{-\frac{\alpha+2}{2}} & |z|^{-\frac{\alpha+2}{2}}\end{pmatrix}, & \text{for } \arg(z)\in(-\pi,-\frac{2\pi}{3})\cup(\frac{2\pi}{3},\pi).\end{cases} \tag{4.5.64}$$

The only solution of this RH problem is

$$P^{(\infty)}(z) = \Upsilon_{\alpha+2}(z). \tag{4.5.65}$$

Note that if we don't specify condition (d) in the RH problem for $P^{(\infty)}$, the solution is not unique. From a mathematical point of view, we remark that Υ_α or $\Upsilon_{\alpha+4}$ for example could have also been a suitable choice for $P^{(\infty)}$, but $\Upsilon_{\alpha+2}$ is the only one which allows us to create a local parametrix around 0 respecting the matching condition (4.5.72).

In the construction of the local parametrix for $\widetilde{W}$ near 0, we will need a more explicit knowledge of the behaviour of $P^{(\infty)}$ at the origin. It can be verified (see [4]) that

$$P^{(\infty)}(z) = P_0^{(\infty)}(z) z^{\frac{\alpha+2}{2}\sigma_3} \begin{pmatrix} 1 & h(z) \\ 0 & 1 \end{pmatrix} H_0(z), \tag{4.5.66}$$

where $P_0^{(\infty)}(z) = P_{0,\alpha}^{(\infty)}(z)$ is an entire function in z for every α while

$$h(z) = \begin{cases} \frac{1}{2i\sin(\pi\alpha)}, & \alpha \notin \mathbb{Z}, \\ \frac{i}{\pi}(-1)^{\alpha+1}\log\frac{\sqrt{z}}{2}, & \alpha \in \mathbb{Z}. \end{cases} \tag{4.5.67}$$

Local parametrix near 0

We want to construct a function $P^{(0)}$ defined in a fixed open disk D_0 around 0 which satisfies exactly the same RH conditions than $\widetilde{W}$ on D_0 and matches with $P^{(\infty)}$ on ∂D_0.

RH Problem for $P^{(0)}$

(a) $P^{(0)} : D_0 \setminus \Sigma_{0,0}$ is analytic.

(b) $P^{(0)}$ has the following jumps

$$P_+^{(0)}(z) = P_-^{(0)}(z) \begin{pmatrix} 1 & 0 \\ e^{\pi i\alpha} & 1 \end{pmatrix}, \qquad z \in \left\{\arg(z) = \frac{2\pi}{3}\right\} \cap D_0, \tag{4.5.68}$$

$$P_+^{(0)}(z) = P_-^{(0)}(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad z \in (-\infty, -ax) \cap D_0, \tag{4.5.69}$$

$$P_+^{(0)}(z) = P_-^{(0)}(z) \begin{pmatrix} 1 & 0 \\ e^{-\pi i\alpha} & 1 \end{pmatrix}, \qquad z \in \left\{\arg(z) = -\frac{2\pi}{3}\right\} \cap D_0, \tag{4.5.70}$$

$$P_+^{(0)}(z) = P_-^{(0)}(z) \begin{pmatrix} e^{\pi i\alpha} & 0 \\ -2 & e^{-\pi i\alpha} \end{pmatrix}, \qquad z \in (-ax, 0) \setminus \{-x\}. \tag{4.5.71}$$

(c) As $x \to 0$,

$$P^{(0)}(z) = (I + \mathcal{O}(x)) P^{(\infty)}(z) \tag{4.5.72}$$

uniformly for $z \in \partial D_0$.

(d) As $z \to -ax$, $P^{(0)}$ has the asymptotic behaviour

$$P^{(0)}(z) = \mathcal{O}(1) \begin{pmatrix} 1 & \frac{\log(z+ax)}{2\pi i} \\ 0 & 1 \end{pmatrix} e^{\frac{\pi i \alpha}{2}\theta(z)\sigma_3} H_0(z). \tag{4.5.73}$$

As $z \to -x$,

$$P^{(0)}(z) = \mathcal{O}(1) e^{\frac{\pi i \alpha}{2}\theta(z)\sigma_3} (z+x)^{\sigma_3} H_0(z). \tag{4.5.74}$$

As $z \to 0$,

$$P^{(0)}(z) = \mathcal{O}(1) z^{\frac{\alpha\sigma_3}{2}} H_0(z). \tag{4.5.75}$$

We can check that the solution of this RH problem is explicitly given by:

$$P^{(0)}(z) = P_0^{(\infty)}(z) \left(\frac{z+x}{z}\right)^{\sigma_3} \begin{pmatrix} 1 & f(z;x) \\ 0 & 1 \end{pmatrix} z^{\frac{\alpha+2}{2}\sigma_3} \begin{pmatrix} 1 & h(z) \\ 0 & 1 \end{pmatrix} H_0(z) \tag{4.5.76}$$

where

$$f(z;x) = \frac{-z^2}{2\pi i} \int_{-ax}^{0} \frac{|s|^\alpha}{s-z} ds. \tag{4.5.77}$$

Small norm RH problem

Define

$$R(z) = \begin{cases} \widetilde{W}(z) P^{(\infty)}(z)^{-1}, & \text{for } z \in \mathbb{C} \setminus \overline{D_0}, \\ \widetilde{W}(z) P^{(0)}(z)^{-1}, & \text{for } z \in D_0. \end{cases} \tag{4.5.78}$$

The only jumps of R are on ∂D and we have $R_-(z)^{-1} R_+(z) = I + \mathcal{O}(x)$ as $x \to 0$. From (4.5.63) and (4.5.56), as $z \to \infty$ we have $R(z) = I + \mathcal{O}(z^{-1})$.

By the small norm theory for RH problems, as $x \to 0$ we have that R exists and $R(z) = I + \mathcal{O}(x)$ uniformly in $z \in \mathbb{C} \setminus \partial D_0$ and as $z \to \infty$,

$$R(z) = I + \frac{R_1(x)}{z} + \frac{R_2(x)}{z^2} + \mathcal{O}(z^{-3}). \tag{4.5.79}$$

We have in particular that $R_1(x) = \mathcal{O}(x)$ and $R_2(x) = \mathcal{O}(x)$.

For $z \in \mathbb{C} \setminus \overline{D_0}$, we have $\widetilde{W}(z) = R(z)P^{(\infty)}(z)$ and therefore, in the same way we proceeded in Section 4.5.1, we obtain as $x \to 0_+$

$$v(x) = v(0) + \mathcal{O}(x), \quad b_0(x) = \mathcal{O}(x^{-1}), \quad b_1(x) = \mathcal{O}(x^{-1}) \quad \text{and} \quad b_2(x) = \mathcal{O}(x^{-1}), \tag{4.5.80}$$

with $v(0) := \frac{1}{8}(4(\alpha+2)^2 - 1)$.

We need explicit expression for $R_1(x)$ and $R_2(x)$ to get more precise asymptotics for $b_0(x)$, $b_1(x)$ and $b_2(x)$ as $x \to 0$. If we express (4.5.80) in terms of U_0, U_1 and V, we get (4.1.18).

4.6 Steepest descent analysis of Y

Define the spectral curve as

$$y(z) = \sqrt{\frac{z-4}{z}}. \tag{4.6.1}$$

together with the density,

$$\rho(x) = \frac{1}{2\pi i} y(x)_+ = \frac{1}{2\pi}\sqrt{\frac{4-x}{x}}, \tag{4.6.2}$$

supported on $\overline{\mathcal{S}}$, where $\mathcal{S} := (0,4)$. We now introduce the measure μ with density $\frac{d\mu(x)}{dx} = \rho(x)$.

This measure μ is the unique measure which minimizes

$$\iint_{\mathbb{R}^+ \times \mathbb{R}^+} \log \frac{1}{|x-y|} d\tilde{\mu}(x) d\tilde{\mu}(y) + \int_{\mathbb{R}^+} y \, d\tilde{\mu}(y), \tag{4.6.3}$$

among all Borel probability measures $\tilde{\mu}$ on $\mathbb{R}^+$, and is therefore called the equilibrium measure.

The equilibrium measure μ satisfies the following identities, known as the Euler-Lagrange variational conditions:

$$2\int_{\mathcal{S}} \log|x-y| \rho(y) dy - x = \ell, \quad x \in \overline{\mathcal{S}}, \tag{4.6.4}$$

$$2\int_{\mathcal{S}} \log|x-y| \rho(y) dy - x < \ell, \quad x \in \mathbb{R}^+ \setminus \overline{\mathcal{S}}, \tag{4.6.5}$$

where ℓ is a constant.

We define the g-function by

$$g(z) = \int_{\mathcal{S}} \log(z - x) d\mu(x), \tag{4.6.6}$$

where the principal branch of the logarithm is taken, meaning g is analytic on $\mathbb{C} \setminus (-\infty, 4]$. The g-function has a number of properties we will make use of, in particular

$$g_+(x) + g_-(x) - x - \ell = 0, \qquad x \in \mathcal{S}, \tag{4.6.7}$$

$$2g(x) - x - \ell < 0, \qquad x \in \mathbb{R}^+ \setminus \overline{\mathcal{S}}, \tag{4.6.8}$$

$$g_+(x) - g_-(x) = 2\pi i \int_x^4 \rho(s) ds, \qquad x \in \mathcal{S}, \tag{4.6.9}$$

$$g_+(x) - g_-(x) = 2\pi i, \qquad x \in (-\infty, 0). \tag{4.6.10}$$

where ℓ is a constant. Let us also define,

$$\xi(z) = -\frac{1}{2} \int_4^z y(s) ds \qquad \text{for } z \in \mathbb{C} \setminus (-\infty, 4], \tag{4.6.11}$$

where the integration path does not cross $(-\infty, 4]$. By (4.6.9) we have,

$$g_+(x) - g_-(x) = 2\xi_+(x), \qquad x \in \mathcal{S}, \tag{4.6.12}$$

and using (4.6.7), we obtain the identity

$$2\xi(z) = 2g(z) - z - \ell, \qquad z \in \mathbb{C} \setminus (-\infty, 4]. \tag{4.6.13}$$

The jumps of ξ follow from those of g,

$$\xi_+(x) + \xi_-(x) = 0, \qquad x \in \mathcal{S}, \tag{4.6.14}$$

$$2\xi(x) < 0, \qquad x \in \mathbb{R}^+ \setminus \overline{\mathcal{S}}, \tag{4.6.15}$$

$$\xi_+(x) - \xi_-(x) = 2\pi i \int_x^4 \rho(x) dx, \qquad x \in \mathcal{S}, \tag{4.6.16}$$

$$\xi_+(x) - \xi_-(x) = 2\pi i, \qquad x \in (-\infty, 0). \tag{4.6.17}$$

(4.6.18)

4.6.1 Transformation to constant jumps

Defining $\Psi(z) = Y(z) w(z)^{\frac{\sigma_3}{2}}$ we have,

RH Problem for Ψ

(a) $\Psi : \mathbb{C} \setminus [yr, \infty) \setminus (-\infty, 0] \setminus \{y\} \to \mathbb{C}^{2\times 2}$ is analytic.

(b) Let $j_\Psi(z) := \Psi_-(z)^{-1}\Psi_+(z)$. Then,

$$j_\Psi(z) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad z \in (yr, \infty). \tag{4.6.19}$$

$$j_\Psi(z) = e^{\pi i \alpha \sigma_3}, \qquad z \in (-\infty, 0). \tag{4.6.20}$$

(c) $\Psi(z) = (I + \mathcal{O}(z^{-1}))z^{(n+\alpha/2+1)\sigma_3}e^{-\frac{nz}{2}\sigma_3}$ as $z \to \infty$.

(d) As $z \to 0$,

$$\Psi(z) = \mathcal{O}(1)z^{\frac{\alpha}{2}\sigma_3}. \tag{4.6.21}$$

As $z \to y$,

$$\Psi(z) = \mathcal{O}(1)(z - y)^{\sigma_3}. \tag{4.6.22}$$

As $z \to yr$,

$$\Psi(z) = \Psi_{y,0}(z) \begin{pmatrix} 1 & -\frac{\log(yr-z)}{2\pi i} \\ 0 & 1 \end{pmatrix}. \tag{4.6.23}$$

Here $\Psi_{y,0}$ is analytic in z in a neighbourhood of yr.

4.6.2 Opening of the lenses

We now perform the standard step of opening the lens. The difference here is that, since the jumps on the lens contours are constant, the lens contours are unconstrained. In a subsequent transformation we will use the g-function to normalise at infinity, at which point the lens contours will be required to stay within a region in which they converge to the identity as $n \to \infty$.

Let D_0 and D_4 denote small and fixed open discs centered at 0 and 4 respectively and $U = D_0 \cup D_4$. We require the discs to have a radius greater than yr. Let us define $\gamma = (yr, 4)$. We now make the transformation $S(z) :=$

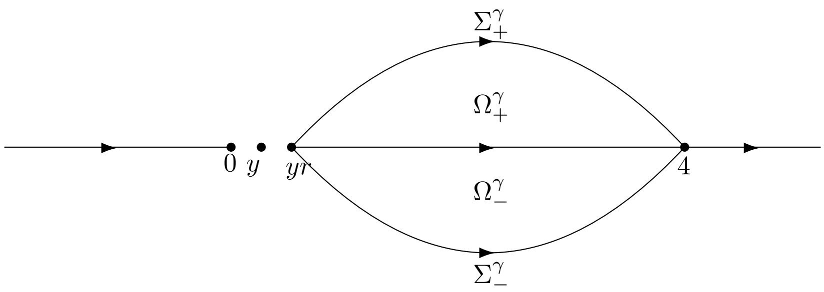

Figure 4.3: Jump contours for the S Riemann-Hilbert problem. The lens contours are labelled Σ_+^γ and Σ_-^γ while the upper and lower lens regions are labelled Ω_+^γ and Ω_-^γ.

$\Psi(z)K(z)$ where K is a piecewise function designed to open the lens,

$$K(z) := \begin{cases} I, & \text{for } z \in \mathbb{C} \setminus \Omega_+^\gamma \setminus \Omega_-^\gamma, \\ \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, & \text{for } z \in \Omega_+^\gamma, \\ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & \text{for } z \in \Omega_-^\gamma. \end{cases} \tag{4.6.24}$$

The regions $\Omega_\pm^\gamma$ are shown in Figure 4.3. They consist of contours in the upper and lower half planes joining yr to 4.

The function S satisfies the following RH problem.

RH Problem for S

(a) $S : \mathbb{C}\backslash[yr,\infty)\backslash(-\infty,0]\backslash\{y\}\backslash\Sigma^\gamma \to \mathbb{C}^{2\times 2}$ is analytic, where $\Sigma^\gamma = \Sigma_-^\gamma \cup \Sigma_+^\gamma$, see Figure 4.3.

(b) Let $j_S(z) := S_-(z)^{-1}S_+(z)$. We have,

$$j_S(z) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad z \in \gamma, \tag{4.6.25}$$

$$j_S(z) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad z \in (4,\infty), \tag{4.6.26}$$

$$j_S(z) = e^{\pi i \alpha \sigma_3}, \qquad z \in (-\infty, 0), \tag{4.6.27}$$

$$j_S(z) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad z \in \Sigma^\gamma. \tag{4.6.28}$$

(c) $S(z) = (I + \mathcal{O}(z^{-1}))z^{(n+\alpha/2+1)\sigma_3}e^{-\frac{nz}{2}\sigma_3}$ as $z \to \infty$.

(d) Near the endpoints, the behaviour of S takes the form

$$\begin{aligned} S(z) &= \mathcal{O}(1)z^{\frac{\alpha}{2}\sigma_3}, && \text{as } z \to 0, \\ S(z) &= \mathcal{O}(1)(z-y)^{\sigma_3}, && \text{as } z \to y, \\ S(z) &= \mathcal{O}(1)\begin{pmatrix} 1 & -\frac{\log(yr-z)}{2\pi i} \\ 0 & 1 \end{pmatrix} K(z), && \text{as } z \to yr, \\ S(z) &= \mathcal{O}(1), && \text{as } z \to 4. \end{aligned} \tag{4.6.29}$$

4.6.3 Normalisation at infinity

The next transformation takes the form,

$$T(z) = e^{-\frac{n\ell\sigma_3}{2}}S(z) \times \begin{cases} e^{-n\xi(z)\sigma_3}, & \text{if } z \in \mathbb{C} \setminus \overline{U}, \\ I, & \text{if } z \in U. \end{cases} \tag{4.6.30}$$

The above transformation has the effect of normalising the problem at infinity.

RH Problem for T

(a) $T : \mathbb{C} \setminus [yr,\infty) \setminus (-\infty, 0] \setminus \{y\} \setminus \Sigma^\gamma \setminus \partial U \to \mathbb{C}^{2\times 2}$ is analytic.

(b) Let $j_T(z) := T_-(z)^{-1}T_+(z)$. Then,

$$
\begin{aligned}
j_T(z) &= j_S(z), && z \in ((yr,\infty)\cup(-\infty,0)\cup\Sigma^\gamma)\cap U,\\
j_T(z) &= \begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}, && z\in\gamma\setminus\overline{U},\\
j_T(z) &= \begin{pmatrix} 1 & e^{2n\xi(z)}\\ 0 & 1\end{pmatrix}, && z\in(yr,\infty)\setminus\overline{\gamma}\setminus\overline{U},\\
j_T(z) &= \begin{pmatrix} 1 & 0\\ e^{-2n\xi(z)} & 1\end{pmatrix}, && z\in\Sigma^\gamma\setminus\overline{U},\\
j_T(z) &= e^{\pi i\alpha\sigma_3}, && z\in(-\infty,0),\\
j_T(z) &= e^{-n\xi(z)\sigma_3}, && z\in\partial U,
\end{aligned}
$$

where the orientation of ∂U is clockwise.

(c) As $z\to\infty$,

$$T(z) = (1+\mathcal{O}(z^{-1}))z^{\frac{\alpha+2}{2}\sigma_3}. \tag{4.6.31}$$

(d) Near the endpoints, the behaviour of T takes the form

$$
\begin{aligned}
T(z) &= \mathcal{O}(1)z^{\frac{\alpha}{2}\sigma_3}, && \text{as } z\to 0,\\
T(z) &= \mathcal{O}(1)(z-y)^{\sigma_3}, && \text{as } z\to y,\\
T(z) &= \mathcal{O}(1)\begin{pmatrix} 1 & -\frac{\log(yr-z)}{2\pi i}\\ 0 & 1\end{pmatrix}K(z), && \text{as } z\to yr,\\
T(z) &= \mathcal{O}(1), && \text{as } z\to 4.
\end{aligned} \tag{4.6.32}
$$

4.6.4 Global parametrix

Finally we define the *global parametrix* as a function $P^{(\infty)}$ satisfying the following RH problem,

RH problem for $P^{(\infty)}$

(a) $P^{(\infty)}:\mathbb{C}\setminus(-\infty,4]\to\mathbb{C}^{2\times 2}$ is analytic.

(b) $P^{(\infty)}$ has the jump relations

$$P_+^{(\infty)}(z) = P_-^{(\infty)}(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad z \in \mathcal{S} \tag{4.6.33}$$

$$P_+^{(\infty)}(z) = P_-^{(\infty)}(z) e^{\pi i \alpha \sigma_3}, \qquad z \in (-\infty, 0). \tag{4.6.34}$$

(c) As $z \to \infty$,

$$P^{(\infty)}(z) = (I + \mathcal{O}(z^{-1})) z^{\frac{\alpha+2}{2}\sigma_3}. \tag{4.6.35}$$

(d) As $z \to \hat{z} \in \partial\mathcal{S} = \{0, 4\}$, we have

$$P^{(\infty)}(z) = \mathcal{O}((z - \hat{z})^{-\frac{1}{4}}). \tag{4.6.36}$$

The solution is explicitly given by

$$P^{(\infty)}(z) = N^{-1} \left(\frac{z-4}{z} \right)^{-\frac{\sigma_3}{4}} N \varphi \left(\frac{z}{2} - 1 \right)^{\frac{\alpha+2}{2}\sigma_3}, \tag{4.6.37}$$

where $\varphi(z) = z + \sqrt{z^2 - 1}$ is analytic in $\mathbb{C} \setminus [-1, 1]$.

We now need to defined *local parametricies* valid in the fixed open discs D_0 and D_4 around 0 and 4.

4.6.5 Local parametrix near 4

RH problem for $P^{(4)}$

(a) $P^{(4)} : D_4 \setminus \Sigma^\gamma \setminus \mathbb{R} \to \mathbb{C}^{2\times 2}$ is analytic.

(b) $P^{(4)}$ has the jump relations

$$\begin{aligned} P_+^{(4)}(z) &= P_-^{(4)}(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, && \text{on } (-\infty, 4) \cap D_4, \\ P_+^{(4)}(z) &= P_-^{(4)}(z) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, && \text{on } (4, \infty) \cap D_4, \\ P_+^{(4)}(z) &= P_-^{(4)}(z) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, && \text{on } \Sigma^\gamma \cap D_4. \end{aligned} \tag{4.6.38}$$

(c) As $z \to 4$, $P^{(4)}(z) = \mathcal{O}(1)$.

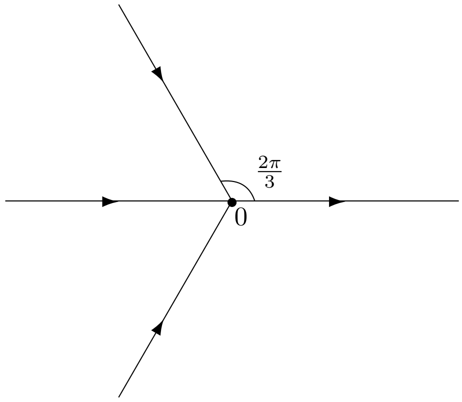

Figure 4.4: The jump contour Σ_A for $\tilde{\Upsilon}$.

(d) As $n \to \infty$, $P^{(4)}(z) = (I + \mathcal{O}(n^{-1})P^{(\infty)}(z)e^{n\xi(z)\sigma_3}$ uniformly for $z \in \partial D_4$.

The solution $P^{(4)}$ can be constructed in term of the well-known Airy parametrix, the construction of which is standard and now briefly review. We first introduce the matrix-valued function

$$\tilde{\Upsilon}(z) = M_A \times \begin{cases} \begin{pmatrix} \mathrm{Ai}(z) & \mathrm{Ai}(\omega^2 z) \\ \mathrm{Ai}'(z) & \omega^2 \mathrm{Ai}'(\omega^2 z) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3}, & 0 < \arg z < \frac{2\pi}{3}, \\ \begin{pmatrix} \mathrm{Ai}(z) & \mathrm{Ai}(\omega^2 z) \\ \mathrm{Ai}'(z) & \omega^2 \mathrm{Ai}'(\omega^2 z) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, & \frac{2\pi}{3} < \arg z < \pi, \\ \begin{pmatrix} \mathrm{Ai}(z) & -\omega^2 \mathrm{Ai}(\omega z) \\ \mathrm{Ai}'(z) & -\mathrm{Ai}'(\omega z) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & -\pi < \arg z < -\frac{2\pi}{3}, \\ \begin{pmatrix} \mathrm{Ai}(z) & -\omega^2 \mathrm{Ai}(\omega z) \\ \mathrm{Ai}'(z) & -\mathrm{Ai}'(\omega z) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3}, & -\frac{2\pi}{3} < \arg z < 0, \end{cases} \tag{4.6.39}$$

with $\omega = e^{\frac{2\pi i}{3}}$, Ai the Airy function and

$$M_A := \sqrt{2\pi} e^{\frac{\pi i}{6}} \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}. \tag{4.6.40}$$

The function $\tilde{\Upsilon}$ solves the following RH problem, see [20].

Airy model RH problem

(a) $\tilde{\Upsilon}(z) : \mathbb{C} \setminus \Sigma_A \to \mathbb{C}^{2\times 2}$ is analytic where Σ_A is shown in Figure 4.4.

(b) $\tilde{\Upsilon}$ has the jump relations

$$\begin{aligned}
\tilde{\Upsilon}_+(z) &= \tilde{\Upsilon}_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, && \text{on } \mathbb{R}^-, \\
\tilde{\Upsilon}_+(z) &= \tilde{\Upsilon}_-(z) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, && \text{on } \mathbb{R}^+, \\
\tilde{\Upsilon}_+(z) &= \tilde{\Upsilon}_-(z) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, && \text{on } e^{\frac{2\pi}{3}i}\mathbb{R}^+, \\
\tilde{\Upsilon}_+(z) &= \tilde{\Upsilon}_-(z) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, && \text{on } e^{-\frac{2\pi}{3}i}\mathbb{R}^+.
\end{aligned} \tag{4.6.41}$$

(c) As $z \to \infty$, we have

$$\tilde{\Upsilon}(z) = z^{-\frac{\sigma_3}{4}} N \left(I + \frac{1}{z^{3/2}} \tilde{\Upsilon}_1 + \mathcal{O}(z^{-3}) \right) e^{-\frac{2}{3} z^{3/2} \sigma_3}, \tag{4.6.42}$$

with

$$\tilde{\Upsilon}_1 = \frac{1}{8} \begin{pmatrix} \frac{1}{6} & i \\ i & -\frac{1}{6} \end{pmatrix}. \tag{4.6.43}$$

(d) As $z \to 0$, $\tilde{\Upsilon}(z) = \mathcal{O}(1)$.

The local parametrix inside D_4 can then be written as

$$P^{(4)}(z) = \tilde{E}(z) \tilde{\Upsilon}(n^{2/3} \tilde{f}(z)), \tag{4.6.44}$$

where $\tilde{f}(z)$ is the conformal map at 4 given by

$$\tilde{f}(z) := \left(-\frac{3}{2} \xi(z) \right)^{2/3} \tag{4.6.45}$$

and $\tilde{E}$ is defined in D_4 by

$$\tilde{E}(z) = P^{(\infty)}(z) N^{-1} \tilde{f}(z)^{\frac{\sigma_3}{4}} n^{\frac{\sigma_3}{6}}, \tag{4.6.46}$$

where the principal branch is taken for $(\cdot)^{\frac{1}{4}}$. It can be easily verified that $\tilde{E}$ is analytic in D_4.

4.6.6 Local parametrix near 0

Inside D_0 we require a local parametrix satisfying the following RH problem.

RH problem for $P^{(0)}$

(a) $P^{(0)} : D_0 \setminus [yr, \infty) \setminus (-\infty, 0] \setminus \{y\} \setminus \Sigma^\gamma \to \mathbb{C}^{2\times 2}$ is analytic.

(b) $P^{(0)}$ has the jump relations

$$\begin{aligned} P_+^{(0)}(z) &= P_-^{(0)}(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, && \text{on } (yr, \infty) \cap D_0, \\ P_+^{(0)}(z) &= P_-^{(0)}(z) e^{\pi i \alpha \sigma_3}, && \text{on } (-\infty, 0) \cap D_0, \\ P_+^{(0)}(z) &= P_-^{(0)}(z) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, && \text{on } \Sigma^\gamma \cap D_0. \end{aligned} \tag{4.6.47}$$

(c) As $z \to 0$, $P^{(0)}(z) = \mathcal{O}(1) z^{\frac{\alpha}{2}\sigma_3}$.
As $z \to y$, $P^{(0)}(z) = \mathcal{O}(1)(z-y)^{\sigma_3}$.
As $z \to yr$, $P^{(0)}(z) = \mathcal{O}(1) \begin{pmatrix} 1 & -\frac{\log(yr-z)}{2\pi i} \\ 0 & 1 \end{pmatrix} K(z)$.

(d) As $n \to \infty$, $P^{(0)}(z) = (I + \mathcal{O}(n^{-1})) P^{(\infty)}(z) e^{n\xi(z)\sigma_3}$ uniformly for $z \in \partial D_0$.

The explicit construction of $P^{(0)}$ in terms of the model RH problem Φ introduced in Section 4.3 is given in the following Lemma.

Lemma 4.6.1 *As $n \to \infty$ and simultaneously $y \to 0$ in such a way that $n^2 y \to x \in [0, \infty)$, the function*

$$P^{(0)}(z) = E(z)\sigma_3 \Phi(-n^2 f(z), n^2 f(y), f(yr)/f(y)) \sigma_3 e^{\frac{\pi i \alpha}{2}\theta(z)\sigma_3} \tag{4.6.48}$$

satisfies the RH problem for $P^{(0)}$, where E is the analytic function in D_0 given by

$$E(z) = (-1)^n P^{(\infty)}(z) e^{-\frac{\pi i \alpha}{2}\theta(z)\sigma_3} \sigma_3 N^{-1} \sigma_3 (-f(z))^{\frac{\sigma_3}{4}} n^{\frac{\sigma_3}{2}}, \tag{4.6.49}$$

f is the conformal map

$$f(z) = -(\xi(z) - \xi(0))^2 \tag{4.6.50}$$

satisfying $f'(0) = 4$ and $\Phi(z; x, a)$ is the model RH problem introduced in Section 4.3.

Proof. The asymptotics (4.3.7) is valid as long as $x = n^2 f(y)$ lies in a compact subset of $(0, +\infty)$. By developing equation (4.5.78), we have that (4.3.7) still holds for x small, i.e. (4.3.7) remain valid for x in a compact subset of $[0, \infty)$. The rest of the proof consists to check that

$$P^{(0)}(z)e^{-n\xi(z)\sigma_3}P^{(\infty)}(z)^{-1} = I + \mathcal{O}(n^{-1}), \tag{4.6.51}$$

uniformly for $z \in D_0$ as $n \to \infty$ and that f is conformal and E is analytic. □

Finally, we now define $R(z)$ as,

$$R(z) = \begin{cases} T(z)P^{(\infty)}(z)^{-1}, & \text{for } z \in \mathbb{C} \setminus \overline{D_0 \cup D_4}, \\ T(z)P^{(0)}(z)^{-1}, & \text{for } z \in D_0, \\ T(z)P^{(4)}(z)^{-1}, & \text{for } z \in D_4. \end{cases} \tag{4.6.52}$$

Using the above definition we can derive the following RH problem for R.

RH problem for R

(a) $R : \mathbb{C} \setminus \Sigma_R \to \mathbb{C}^{2\times 2}$ is analytic, where $\Sigma_R = ((4, \infty) \cup \Sigma^\gamma \cup \partial D_0 \cup \partial D_4) \setminus (D_0 \cup D_4)$.

(b) Let $j_R(z) := R_-(z)^{-1}R_+(z)$. We have

$$j_R(z) = P^{(\infty)}(z)\begin{pmatrix} 1 & 0 \\ e^{-2n\xi(z)} & 1 \end{pmatrix}P^{(\infty)}(z)^{-1}, \qquad z \in \Sigma^\gamma \setminus \overline{D_0 \cup D_4}, \tag{4.6.53}$$

$$j_R(z) = P^{(\infty)}(z)\begin{pmatrix} 1 & e^{2n\xi(z)} \\ 0 & 1 \end{pmatrix}P^{(\infty)}(z)^{-1}, \qquad z \in (4, \infty) \setminus \overline{D_4}, \tag{4.6.54}$$

$$j_R(z) = P^{(4)}(z)\begin{pmatrix} e^{-n\xi(z)} & 0 \\ 0 & e^{n\xi(z)} \end{pmatrix}P^{(\infty)}(z)^{-1}, \qquad z \in \partial D_4, \tag{4.6.55}$$

$$j_R(z) = P^{(0)}(z)\begin{pmatrix} e^{-n\xi(z)} & 0 \\ 0 & e^{n\xi(z)} \end{pmatrix}P^{(\infty)}(z)^{-1}, \qquad z \in \partial D_0. \tag{4.6.56}$$

$$\tag{4.6.57}$$

(c) As $z \to \infty$, $R(z) = I + \frac{R_1}{z} + \mathcal{O}(z^{-2})$.

(d) As $z \to b \in \{0, y, yr, 4\}$, $R(z) = \mathcal{O}(1)$.

From (4.6.37), (4.6.44) and (4.6.48), as $n \to \infty$ we have

$$\begin{aligned} j_R(z) &= I + \mathcal{O}(n^{-1}), \qquad \text{uniformly for } z \in \partial D_0 \cup \partial D_4, \\ j_R(z) &= I + \mathcal{O}(e^{-cn}), \qquad \text{uniformly for } z \in \Sigma_R \setminus (\partial D_0 \cup \partial D_4), \end{aligned} \tag{4.6.58}$$

where $c > 0$ is a constant independent of n. It follows from standard theory for small norm RH problem that R exists for n sufficiently large and satisfies

$$R(z) = I + \mathcal{O}(n^{-1}), \qquad \text{uniformly for } z \text{ in compact subsets of } \mathbb{C} \setminus \Sigma_R. \tag{4.6.59}$$

4.6.7 Computation of R_1

The quantity R_1 can be computed via a perturbative calculation. From (4.6.58) we can write for $z \in \Sigma_R$,

$$j_R(z) = I + \frac{1}{n} J_1(z) + \mathcal{O}(n^{-2}), \qquad n \to \infty, \tag{4.6.60}$$

where the matrix $J_1(z)$ is non-zero only on ∂U. By developing (4.6.59), the large n behaviour of R has the form

$$R(z) = I + R^{(1)}(z) n^{-1} + \mathcal{O}(n^{-2}), \qquad z \in \mathbb{C} \setminus \Sigma_R. \tag{4.6.61}$$

The quantity $R^{(1)}(z)$ may be expressed in terms of $J_1(z)$ by substituting (4.6.60) into the jump relation $R_+(z) = R_-(z) J_R(z)$, from which we obtain the following RH problem for $R^{(1)}$.

RH problem for $R^{(1)}$

(a) $R^{(1)} : \mathbb{C} \setminus U \to \mathbb{C}^{2\times 2}$ is analytic,

(b) $R^{(1)}_+(z) = R^{(1)}_-(z) + J_1(z)$ for $z \in \partial D_0 \cup \partial D_4$,

(c) $R^{(1)}(z) \to 0$ as $z \to \infty$.

The above RH problem can be solved explicitly in terms of a Cauchy transform,

$$R^{(1)}(z) = \frac{1}{2\pi i} \int_{\partial D_0 \cup \partial D_4} \frac{J_1(\xi)}{\xi - z} d\xi, \tag{4.6.62}$$

where the integral is taken entry-wise. To obtain an expression for the leading term of R_1 we merely need to compute $\lim_{z\to\infty} \frac{R^{(1)}(z)z}{n}$, one has

$$R_1 = -\frac{1}{2\pi i n}\int_{\partial D_0 \cup \partial D_4} J_1(\xi)d\xi + \mathcal{O}(n^{-2}), \qquad n \to \infty. \tag{4.6.63}$$

Explicit computations using property (d) from (4.6.6) and (4.3.7) gives for $z \in \partial D_0$,

$$J_1(z) = \frac{v(n^2 f(y); f(ry)/f(y))}{2\sqrt{-f(z)}} P^{(\infty)}(z) \begin{pmatrix} -1 & -ie^{-i\pi\alpha\theta(z)} \\ -ie^{i\pi\alpha\theta(z)} & 1 \end{pmatrix} P^{(\infty)}(z)^{-1}, \tag{4.6.64}$$

The function $v(x;a)$ is the special function appearing in the model problem Φ. $J_1(z)$ on ∂D_4 is given by,

$$J_1(z) = \frac{1}{8\tilde{f}(z)^{3/2}} P^{(\infty)}(z) \begin{pmatrix} \frac{1}{6} & i \\ i & -\frac{1}{6} \end{pmatrix} P^{(\infty)}(z)^{-1} \tag{4.6.65}$$

Putting this together and using the above in (4.6.63) gives,

$$n\mathrm{Tr}(R_1\sigma_3) = v(n^2 f(y); f(ry)/f(y)) - v(0) + \mathcal{O}(ny) \qquad \text{as } n \to \infty, \tag{4.6.66}$$

where $v(0) = \frac{1}{8}(4(\alpha+2)^2 - 1)$.

4.7 Proof of Theorem 4.1.1

In order to utilise Lemma 4.2.1 we require large n asymptotics for the quantity Y uniform in z. The outcome of the steepest descent analysis is that for z bounded away from $\{0, 4\}$, as $n \to \infty$ and $y \to 0$ in such a way that $n^2 y$ is a compact subset of $[0, \infty)$ we have,

$$Y(z; y, r) = e^{\frac{n\ell\sigma_3}{2}} R(z; y, r) P^{(\infty)}(z) e^{n\xi(z)\sigma_3} w(z)^{-\frac{\sigma_3}{2}}. \tag{4.7.1}$$

Using the above expression for Y we first compute,

$$\begin{aligned} \mathrm{Tr}(Y^{-1}(z)\partial_z Y(z)\sigma_3) \quad &= \quad -\partial_z \log w(z) + 2n\partial_z \xi(z) \\ &\quad +\mathrm{Tr}(P^{(\infty)}(z)^{-1}\partial_z P^{(\infty)}(z)\sigma_3) \\ &\quad +\mathrm{Tr}(P^{(\infty)}(z)^{-1}R(z)^{-1}\partial_z R(z) P^{(\infty)}(z)\sigma_3), \end{aligned}$$

into which the explicit expression for the functions on the right hand side must be substituted.

We now consider of the above expression as $z \to \infty$. Using (4.6.37) one can show,

$$\mathrm{Tr}(P^{(\infty)}(z)^{-1}\partial_z P^{(\infty)}(z)\sigma_3) = \frac{\alpha+2}{z} + \frac{2\alpha+4}{z^2} + \mathcal{O}(z^{-3}) \tag{4.7.2}$$

as $z \to \infty$. Finally, we have also,

$$-\partial_z \log w(z) + 2n\partial_z \xi(z) = 2n(\frac{1}{z} + \frac{1}{z^2}) - \frac{\alpha+2}{z} - \frac{2y}{z^2} + \mathcal{O}(z^{-3}) \tag{4.7.3}$$

as $z \to \infty$. Using the above expressions in Lemma 4.2.1 gives,

$$\partial_y \log Z_{n,\alpha}(y;r) = n + \frac{n}{2y}\mathrm{Tr}(R_1(y,r)\sigma_3). \tag{4.7.4}$$

By using (4.6.66) in (4.7.4) and rewriting it in terms of $x := n^2 y$, we have as $n \to \infty$ and x in compact subsets of $[0,\infty)$ that

$$\begin{aligned} \partial_x \log Z_{n,\alpha}\left(\frac{x}{n^2};r\right) &= \frac{1}{2x}\left(v\left(n^2 f(xn^{-2}); \frac{f(rxn^{-2})}{f(xn^{-2})}\right) - v(0)\right) + \mathcal{O}(n^{-1}), \\ &= \frac{1}{2x}\left(v(4x;r) - v(0)\right) + \mathcal{O}(n^{-1}), \end{aligned} \tag{4.7.5}$$

where the $\mathcal{O}(n^{-1})$ term is uniform in x in compact subsets of $[0,\infty)$. Integrating this from 0 up to a constant x and noting that

$$Z_{n,\alpha}(0;r) = \widehat{Z}_{n,\alpha+2}, \tag{4.7.6}$$

we have

$$\log\left(\frac{Z_{n,\alpha}(xn^{-2};r)}{\widehat{Z}_{n,\alpha+2}}\right) = \frac{1}{2}\int_0^x \left[v(4\zeta;r) - v(0)\right]\frac{d\zeta}{\zeta} + \mathcal{O}(xn^{-1}). \tag{4.7.7}$$

This can be rewritten as

$$Z_{n,\alpha}(xn^{-2};r) = \widehat{Z}_{n,\alpha+2}e^{I(x;r)}\left(1 + \mathcal{O}(xn^{-1})\right), \tag{4.7.8}$$

where the notation $I(x;r)$ has been introduced in (4.1.20). We remind the reader that the above expression is valid when x is in compact subsets of $[0,+\infty)$. This completes the proof of Theorem 4.1.1.

4.8 Proof of Theorem 4.1.2

From (4.1.12) with the change of variable $y = xn^{-2}$, we obtain

$$Q_{n,\alpha}(r) = \frac{\widehat{Z}_{n-1,\alpha+2}}{\widehat{Z}_{n,\alpha}} \left(\frac{n-1}{n}\right)^{(n-1)(n+1+\alpha)} n^{-2-\alpha} \times \int_0^\infty x^\alpha e^{-\frac{x}{n}} \frac{1}{\widehat{Z}_{n-1,\alpha+2}} Z_{n-1,\alpha}\left(\frac{n}{n-1}\frac{x}{n^2}; r\right) dx. \quad (4.8.1)$$

It is known that (see [52])

$$\widehat{Z}_{n,\alpha} = \frac{1}{n!} n^{-n^2-\alpha n} \prod_{j=1}^{n} j!\Gamma(j+\alpha), \quad (4.8.2)$$

and therefore we have

$$\begin{aligned} \frac{\widehat{Z}_{n-1,\alpha+2}}{\widehat{Z}_{n,\alpha}} &= \frac{(n-1)^{\alpha+1}\Gamma(n+\alpha+1)}{(n-1)!\Gamma(\alpha+1)\Gamma(\alpha+2)} \left(\frac{n}{n-1}\right)^{n^2+\alpha n}, \\ &= \frac{n^{2\alpha+2}}{\Gamma(\alpha+1)\Gamma(\alpha+2)} \left(\frac{n}{n-1}\right)^{n^2+\alpha n} \left(1+\mathcal{O}\left(n^{-1}\right)\right), \quad \text{as } n\to\infty. \end{aligned} \quad (4.8.3)$$

We can thus simplify expression (4.8.1) as

$$Q_{n,\alpha}(r) = \frac{1+\mathcal{O}(n^{-1})}{\Gamma(\alpha+1)\Gamma(\alpha+2)}(I_1+I_2), \quad (4.8.4)$$

where

$$I_1 = \int_0^M x^\alpha e^{-\frac{x}{n}} e^{I(x,r)} dx, \qquad I_2 = \int_M^\infty x^\alpha e^{-\frac{x}{n}} \frac{1}{Z_{n-1}^{\alpha+2}} Z_{n-1}\left(\frac{n}{n-1}\frac{x}{n^2}; r\right) dx, \quad (4.8.5)$$

and $M > 0$ is a constant.

Furthermore, we can derive the following inequality

$$\begin{aligned}
\frac{Z_{n-1,\alpha}(\frac{x}{n^2};r)}{\widehat{Z}_{n-1,\alpha+2}} &= \frac{1}{\widehat{Z}_{n-1,\alpha+2}}\int_{\frac{xr}{n^2}}^{\infty}\cdots\int_{\frac{xr}{n^2}}^{\infty}\Delta_{n-1}(\xi)^2\prod_{i=1}^{n-1}\left(\xi_i-\frac{x}{n^2}\right)^2\xi_i^{\alpha}e^{-n\xi_i}d\xi_i \\
&\le \frac{1}{\widehat{Z}_{n-1,\alpha+2}}\int_{\frac{xr}{n^2}}^{\infty}\cdots\int_{\frac{xr}{n^2}}^{\infty}\Delta_{n-1}(\xi)^2\prod_{i=1}^{n-1}\xi_i^{\alpha+2}e^{-n\xi_i}d\xi_i \\
&= \mathbb{P}_{n-1}^{\alpha+2}\left(\lambda_{\min}>\frac{xr}{n^2}\right)\le e^{-Cxr}
\end{aligned} \tag{4.8.6}$$

where $C>0$ is a constant and x and n are sufficiently large. We thus have

$$I_2 \le \int_M^{\infty} x^{\alpha}e^{-\frac{x}{n}}e^{-Cxr}dx\le e^{-\frac{C}{2}Mr}, \tag{4.8.7}$$

if M is chosen big enough. Therefore we obtain,

$$\lim_{n\to\infty}Q_n^{\alpha}(r)=\frac{1}{\Gamma(\alpha+1)\Gamma(\alpha+2)}\int_0^M x^{\alpha}e^{I(x;r)}dx+\mathcal{O}(e^{-\frac{C}{2}Mr}). \tag{4.8.8}$$

Letting $M\to\infty$ in (4.8.8) finishes the proof of Theorem 4.1.2.

Conclusion

We proved two new results concerning asymptotics for Toeplitz determinants associated to a symbol which tends to zero on an arc of the unit circle at a certain rate:

- we generalized the asymptotic expansion proved by Widom [67] for this symbol when the rate is sufficiently fast (Theorem 1.5.1),
- we found the leading order term of these asymptotics in the slower regime (Theorem 1.5.2), which interpolates between asymptotics of Toeplitz determinants associated to a symbol with Fisher-Hartwig singularities [24] and a symbol supported on an arc [67].

We proved a new result about Hankel determinants:

- we found asymptotics for Hankel determinants associated to the weight $w(x) = (x-y)^2 x^\alpha e^{-nx} \chi_{[yr,\infty)}(x)$ (Theorem 1.5.5), which correspond to the Laguerre weight of order $\alpha + 2$ if $y = 0$.

In random matrix theory, we proved the following results:

- in the thinned CUE, we found asymptotics for gap probabilities, when the thinning parameter decays sufficiently fast (see (3.2.7)), and in the slower regime (Theorem 1.5.2)
- in the conditional CUE, we obtained asymptotics for the expected number of eigenvalues lying on γ^c, and the variance of this quantity, both when the thinning parameter decays sufficiently fast (see (3.2.25) and (3.2.26)) and in the slower regime (Theorem 1.5.3).
- in the LUE, we obtained the limiting ratio probability between the smallest and second smallest eigenvalue in terms of a function related to a Lax pair of a model RH problem (Theorem 1.5.6).

We also proved a result for orthogonal polynomials:

- we obtained asymptotics for orthogonal polynomials $\phi_n(z)$ associated to the symbol f given in (1.1.5) in different region of the complex plane, in the double scaling limit $n \to \infty$ and $s \to 0$. Its zero average counting measure $\nu_{n,L,s}$ converges weakly to the uniform measure on the unit circle if $s \in (0,1)$ is fixed. If $s = s(n) \to 0$ sufficiently rapidly, $\nu_{n,L,s}$ converges weakly to the measure $\mu_{\infty,L}$ which is independent of s. If $s = s(n) \to 0$ in the slower regime, $\nu_{n,L,s}$ converges weakly to the measure $\mu_{x,L}$ which depends on s (Theorem 1.5.3 and Figure 3.6).

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