

A topological characterization of the Moufang property for compact polygons

Nicolas Radu*

UCLouvain, 1348 Louvain-la-Neuve, Belgium

July 15, 2016

Abstract

We prove a purely topological characterization of the Moufang property for disconnected compact polygons in terms of convergence groups. As a consequence, we recover the fact that a locally finite thick affine building of rank 3 is a Bruhat–Tits building if and only if its automorphism group is strongly transitive. We also study automorphism groups of general compact polygons without any homogeneity assumption. A compactness criterion for sets of automorphisms is established, generalizing the theorem by Burns and Spatzier that the full automorphism group, endowed with the compact-open topology, is a locally compact group.

Contents

1	Introduction	2
2	Preliminaries	5
2.1	Compact spherical buildings	5
2.2	Strong transitivity and the Moufang property	7
3	A compactness criterion for subsets of $\text{Auttop}(\Delta)$	7
3.1	Relative compactness in $\text{Auttop}(\Delta)$	8
3.2	Sequences collapsed by a sequence of automorphisms	9
3.3	Proof of the compactness criterion	12
4	Topological characterization of the Moufang property	18
4.1	Convergence groups	18
4.2	Group of projectivities	19
4.3	Conventions for the rest of the section	22
4.4	Automorphisms fixing an apartment	23
4.5	Proof of the characterization of the Moufang property	26

2010 Mathematics Subject Classification: 20E42 51E24 (Primary), 20F65 22D05 (Secondary).

*F.R.S.-FNRS Research Fellow.

1 Introduction

Compact buildings form a natural generalization of compact projective planes. They were introduced by Burns and Spatzier in [BS87a], and used in their proof of the Rank Rigidity Theorem for Riemannian manifolds of non-positive sectional curvature, see [BS87b]. A prominent example of a compact building is provided by the spherical building associated with a semisimple algebraic group over a non-discrete locally compact field. Since the origin, the problem of characterizing those buildings associated with semisimple algebraic groups among all compact buildings has attracted much attention; in some sense, this is the topic of the encyclopaedic book [SBG⁺95], mainly devoted to the study of *connected* compact projective planes. The relevance of that problem is also well illustrated by the work of Burns and Spatzier: a key result from [BS87a] that is used in the proof of Rank Rigidity in [BS87b] asserts that a thick irreducible *connected* compact spherical building of rank ≥ 2 is the spherical building associated with a simple Lie group if and only if its automorphism group is strongly transitive (see Subsection 2.2 for the definition of strong transitivity). This result has been improved by Grundhöfer, Knarr and Kramer who showed in [GKK95] and [GKK00] that the same classification is true for connected buildings admitting a chamber transitive automorphism group. The starting point of the present article is the question whether such a statement could hold beyond the connected case. A precise formulation can be stated as follows.

Conjecture. *Let Δ be an infinite thick irreducible compact spherical building of rank at least 2. Then Δ is the spherical building associated with a semisimple algebraic group over a non-discrete locally compact field if and only if Δ is strongly transitive.*

Here again, one could even replace strong transitivity by chamber transitivity in this conjecture. Historically, a similar problem had been suggested by J. Tits in the 1970's for finite buildings. More precisely, Tits conjectured that a finite thick irreducible building of rank ≥ 2 should be the building associated with a finite simple group of Lie type if and only if the building has a strongly transitive automorphism group, see [Tit74, Conjecture 11.5.1]. Tits' hope was that this could be helpful to the classification of the finite simple groups, which was an ongoing project at the time. Ironically, the latter classification was achieved first, and then used by Buekenhout and Van Maldeghem in their proof that Tits' conjecture is indeed accurate, see [BVM94].

As mentioned above, the conjecture is certainly true for connected buildings by the work of Burns and Spatzier; we may therefore assume that Δ is totally disconnected, since any disconnected compact building is so. Moreover, Grundhöfer, Kramer, Van Maldeghem and Weiss have proved that, under the same hypotheses as the conjecture, the building Δ is the spherical building associated with a semisimple algebraic group over a non-discrete locally compact field if and only if Δ is Moufang: this follows from [GKVMW12, Theorem 1.1]. As is well known, Tits proved that every thick irreducible spherical building of rank ≥ 3 is Moufang (see [Tit74, Addenda]), so that the conjecture above can be reduced to the following one, which appears as Question 1.4 in [GKVMW12].

Conjecture (Reformulation). *Let Δ be an infinite thick irreducible compact totally disconnected spherical building of rank 2. If Δ is strongly transitive, then Δ is Moufang.*

In the rest of this paper, we shall therefore focus on compact spherical buildings of rank 2, also called **compact polygons**. More precisely, a compact polygon of diameter m is called a **compact m -gon**. It is irreducible if and only if $m \geq 3$.

Our first main result may be viewed as a first step towards the conjecture above.

Theorem A. *Let Δ be an infinite thick compact totally disconnected m -gon with $m \geq 3$ and let G be a closed subgroup of the group $\text{Autop}(\Delta)$ of topological automorphisms which is strongly transitive on Δ . The following assertions are equivalent.*

- (i) Δ is Moufang.
- (ii) For each panel π of Δ , the closure in $\text{Homeo}(\text{Cham}(\pi))$ of the group of projectivities $\Pi(\pi)$ acts as a convergence group on $\text{Cham}(\pi)$.
- (iii) For each panel π of Δ , the closure of the natural image of $\text{Stab}_G(\pi)$ in $\text{Homeo}(\text{Cham}(\pi))$ acts as a convergence group on $\text{Cham}(\pi)$.

The notion of a convergence group is recalled in Subsection 4.1 while the group of projectivities $\Pi(\pi)$ is defined in Subsection 4.2. Conditions (i) and (ii) are both independent of the group G . In fact, Condition (ii) in Theorem A implies in particular that the closure in $\text{Homeo}(\text{Cham}(\pi))$ of the group of projectivities $\Pi(\pi)$ is a locally compact group (see Lemma 4.1 below). In the case of *connected* compact polygons, the latter property in turn is known to be equivalent to the Moufang condition, without assuming any extra homogeneity assumption on Δ a priori (see [L 81, Theorem 5.1] or [SBG⁺95, Theorem 66.1]). If one believes that the connected case is representative of the general case, one should therefore not expect Condition (ii) from Theorem A to be satisfied by all infinite compact totally disconnected polygons.

We point out the following corollary of Theorem A.

Corollary B (Caprace–Monod). *Let X be a locally finite thick irreducible affine building of rank 3. If X is strongly transitive, then the building at infinity X_∞ is Moufang. In particular X is the Bruhat–Tits building associated with a semisimple algebraic group over a non-Archimedean local field.*

The special case of Corollary B where X is of type $\tilde{A}_2$ was obtained by Van Maldeghem and Van Steen in [VMVS98, Main Theorem], while the general case has been obtained by Caprace and Monod in [CM15, Corollary E] using CAT(0) geometry. Our approach to Theorem A is different, but was partly inspired by theirs: instead of the CAT(0) Levi decomposition used in their work, we use a purely algebraic counterpart of that result, due to Baumgartner and Willis [BW04], and valid in the realm of totally disconnected locally compact groups.

The basic result that makes the structure theory of locally compact groups available in the study of compact buildings is the theorem, due to Burns–Spatzier and valid for all thick irreducible compact spherical building Δ of rank ≥ 2 , asserting that the topological automorphism group $\text{Autop}(\Delta)$ is a locally compact group, see [BS87a, Theorem 2.1]. Equivalently, there exists an identity neighbourhood in the group $\text{Autop}(\Delta)$ endowed with the compact-open topology that is compact. Our second main result is a compactness criterion for more general subsets of $\text{Autop}(\Delta)$ in the rank 2 case. Notice that we do not make any transitivity assumption on the automorphism group.

Theorem C. *Let Δ be a thick compact m -gon with $m \geq 3$ and C, C' be opposite chambers of Δ . Denote by D_0 and D_1 the two chambers adjacent to (but different from) C in the apartment containing C and C' and by v_0 (resp. v_1) the common vertex of C and D_0 (resp.*

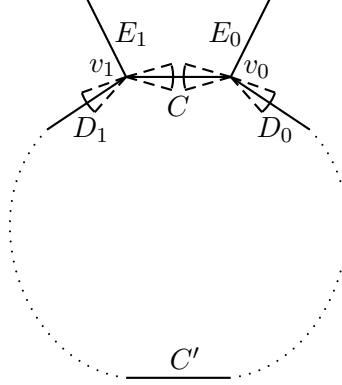


Figure 1: Illustration of Theorem C.

D_1). Let also E_0 (resp. E_1) be a chamber having vertex v_0 (resp. v_1) but different from C and D_0 (resp. D_1). Let finally U and U' be closed subsets of $\text{Cham } \Delta$ such that every chamber of U is opposite every chamber of U' . Then for all $\varepsilon > 0$, the (possibly empty) set

$$J_\varepsilon(C, C', U, U', E_0, E_1) := \left\{ \varphi \in \text{Auttop}(\Delta) \left| \begin{array}{l} \varphi(C) \in U, \varphi(C') \in U', \\ \rho(\varphi(C), \varphi(B_{v_i}(D_i, \varepsilon))) \geq \varepsilon \quad \forall i \in \{0, 1\}, \\ \rho(\varphi(D_i), \varphi(B_{v_i}(C, \varepsilon))) \geq \varepsilon \quad \forall i \in \{0, 1\}, \\ \rho(\varphi(C), \varphi(E_i)) \geq \varepsilon \quad \forall i \in \{0, 1\}, \\ \rho(\varphi(D_i), \varphi(E_i)) \geq \varepsilon \quad \forall i \in \{0, 1\} \end{array} \right. \right\}$$

is compact, where ρ is some fixed metric associated to the topology on $\text{Cham } \Delta$ and $B_v(C, r)$ denotes the open ball in $\text{Cham}(v)$ centered at C and of radius r (with respect to the metric ρ).

Remark that the conclusion of Theorem C is also valid for irreducible compact spherical buildings of higher rank, but the proof that we are aware of in that case is very indirect: one deduces it from the fact that those buildings are Moufang, hence are buildings at infinity of Bruhat-Tits buildings, via the works of Tits [Tit74, Addenda] and Grundhöfer–Kramer–Van Maldeghem–Weiss [GKVMW12, Theorem 1.1].

A particular case of this criterion is the following result, which can be seen as a topological version of a well-known theorem [Tit74, Theorem 4.1.1] of Tits. Indeed, the latter basically states that $L_0(C, C')$ (as defined below) is reduced to the identity in every thick (not necessarily topological) spherical building.

Corollary D. *Let Δ be a thick compact m -gon with $m \geq 3$ and C, C' be opposite chambers of Δ . There exists $\varepsilon > 0$ such that the set*

$$L_\varepsilon(C, C') := \{ \varphi \in \text{Auttop}(\Delta) \mid \rho(X, \varphi(X)) \leq \varepsilon \quad \forall X \in E_1(C) \cup \{C'\} \}$$

is compact, where ρ is some fixed metric associated to the topology on $\text{Cham } \Delta$ and $E_1(C)$ denotes the set of chambers adjacent to C .

We remark that the local compactness theorem of Burns and Spatzier in the rank 2 case can now be recovered as an immediate consequence of Corollary D. Actually, our proof of Theorem C was inspired by their work; note however that our approach is uniform in the sense that it does not require to distinguish the cases where the diameter of Δ is even or odd. In the particular case of compact projective planes (i.e. for $m = 3$), this result was also proved by Grundhöfer in a very short and elegant way, see [Gru86, Theorem 1].

Corollary E (Burns–Spatzier). *Let Δ be a thick compact m -gon with $m \geq 3$. Then $\text{Auttop}(\Delta)$ is locally compact.*

One could ask if the definition of the set $J_\varepsilon(C, C', U, U', E_0, E_1)$ whose compactness is ensured by Theorem C is optimal: is there any natural bigger subset of $\text{Auttop}(\Delta)$ whose compactness could be proved as well for all compact polygons? In that direction, we suggest the following question; we feel that the answer is likely to be positive.

Question. *Under the same hypotheses and with the same notation as in Theorem C, is the following set compact?*

$$\tilde{J}_\varepsilon(C, C', U, U', E_0, E_1) := \left\{ \varphi \in \text{Auttop}(\Delta) \left| \begin{array}{l} \varphi(C) \in U, \varphi(C') \in U', \\ \rho(\varphi(C), \varphi(E_i)) \geq \varepsilon \quad \forall i \in \{0, 1\}, \\ \rho(\varphi(D_i), \varphi(E_i)) \geq \varepsilon \quad \forall i \in \{0, 1\} \end{array} \right. \right\}.$$

Actually, this question is a natural generalization of the well-known conjecture that in any compact projective plane, the stabilizer of four points, none three of which are collinear, is compact (see [SBG⁺95, Section 44]). It should also be emphasized that this question is strongly related to the conjecture mentioned at the beginning of this introduction. Indeed, in order to prove the conjecture, it only remains to show that Condition (ii) or (iii) in Theorem A is satisfied by any strongly transitive compact polygon. A positive answer to the above question would ensure that a slightly weaker condition than (iii) holds for all compact polygons; in view of Theorem A, this would be an important step toward the proof of the conjecture.

The proof of Theorem C is given in Section 3, while Theorem A is proved in Section 4 (see Corollary 4.6, Proposition 4.9 and Theorem 4.19). Some of the preparatory results of Section 3 are also used in Section 4 to prove the implication (iii) $\Rightarrow$ (i) of Theorem A.

Acknowledgement

The results of this paper are the outcomes of my Master’s thesis, which was supervised by Pierre-Emmanuel Caprace. I am very grateful to him for suggesting the topic and for his support and guidance throughout the preparation of this work. I also thank Linus Kramer for several comments on the history of compact polygons.

2 Preliminaries

In this section, we give the definition and first properties of compact spherical buildings and recall what the strong transitivity and the Moufang property exactly are.

2.1 Compact spherical buildings

Let Δ be a spherical building of rank k viewed as a simplicial complex, and denote by Δ_r the set of simplices of dimension $r - 1$ in Δ (for $r \in \{1, \dots, k\}$). We fix an ordering of the k types of vertices in Δ . Then Δ is called a **compact spherical building** if the set $\text{Vert } \Delta := \Delta_1$ carries a compact topology such that the set $\text{Cham } \Delta := \Delta_k$ is closed in $(\text{Vert } \Delta)^k$ (where a chamber of Δ is viewed as a k -tuple of vertices, ordered according to their type). For each $r \in \{1, \dots, k\}$, Δ_r is given the induced topology from the product topology on $(\text{Vert } \Delta)^r$.

We also say that Δ is connected or totally disconnected when $\text{Cham } \Delta$ has the said property. When Δ is thick and has no factor of rank 1, one can actually show that the topology on $\text{Vert } \Delta$ is metrizable (see [GKVMW12, Proposition 6.14]). We will therefore always suppose that the topology comes from a metric. The **topological automorphism group** $\text{Autop}(\Delta)$ of Δ is the group of all automorphisms of Δ whose restriction to each Δ_r are homeomorphisms. The group $\text{Autop}(\Delta)$ is endowed with the topology induced from the compact-open topology of $C(\text{Cham } \Delta, \text{Cham } \Delta)$ (where $C(X, Y)$ denotes the set of continuous maps from X to Y). Since $\text{Cham } \Delta$ is compact and metrizable, it also actually is the topology of uniform convergence.

We can now give the first properties of compact buildings. The following results all come from article [BS87a] of Burns and Spatzier and are almost direct consequences of the definition of compact buildings, except Lemma 2.8 whose proof is more technical.

Proposition 2.1. *Let Δ be a compact spherical building of rank k . The set*

$$\{(R, R') \in \Delta_r^2 \mid R \text{ is opposite } R'\}$$

is open in Δ_r^2 for each $1 \leq r \leq k$.

Proof. See [BS87a, Proposition 1.9]. □

Proposition 2.2. *Let C be a chamber of a compact spherical building Δ and R be a face of a chamber opposite C . If $R_n \rightarrow R$ and $C_n \rightarrow C$, then $\text{proj}_{R_n} C_n \rightarrow \text{proj}_R C$.*

Proof. See [BS87a, Proposition 1.10]. □

Corollary 2.3. *Let R and R' be two opposite residues of a compact spherical building Δ . The projection $\text{proj}_R |_{R'}: \text{Cham } R' \rightarrow \text{Cham } R$ is a homeomorphism.*

Proof. This follows from Proposition 2.2. □

In a polygon (i.e. a spherical building of rank 2), we will say that a gallery $(C_1, \dots, C_n)$ **stammers** if $C_i = C_{i-1}$ for some $i \in \{2, \dots, n\}$. The two following results are true in any usual (non-topological) polygon. Recall that an m -gon is a polygon of diameter m .

Lemma 2.4. *Let Δ be an m -gon. A gallery $(C_1, \dots, C_n)$ in Δ with $n \leq m + 1$ is minimal if and only if it does not stammer. In particular, there is no non-stammering closed gallery in Δ having less than $2m$ chambers.*

Proof. This is a direct consequence of [BS87a, Lemma 0.4]. □

Corollary 2.5. *Let Δ be a polygon and x, y be distinct non-opposite vertices of Δ . There is a unique minimal gallery with initial vertex x and final vertex y , which we denote by $[x, y]$.*

Proof. The existence of two minimal galleries from x to y would contradict Lemma 2.4. □

When Δ is a polygon, we denote by $D: \text{Vert } \Delta \times \text{Vert } \Delta \rightarrow \mathbf{N}$ the graph distance in the 1-dimensional simplicial complex Δ .

Lemma 2.6. *Let Δ be a compact polygon and x, y be distinct non-opposite vertices of Δ . If $x_n \rightarrow x$, $y_n \rightarrow y$ and $D(x_n, y_n) = D(x, y)$ for all $n \in \mathbf{N}$, then $[x_n, y_n] \rightarrow [x, y]$.*

Proof. See [BS87a, Lemma 1.12]. □

We say that two vertices v and v' in an m -gon are **almost opposite** if $D(v, v') = m - 1$ (note that $D(v, v') = m$ if and only if v and v' are opposite). We then have a result similar to Proposition 2.1 for almost opposite vertices.

Proposition 2.7. *Let Δ be a compact polygon. The set*

$$\{(v, v') \in (\text{Vert } \Delta)^2 \mid v \text{ is almost opposite } v'\}$$

is open in $(\text{Vert } \Delta)^2$.

Proof. See [BS87a, Lemma 1.13]. □

The following result will finally be helpful in the proofs of Theorems A and C.

Lemma 2.8. *Let Δ be an infinite thick compact m -gon with $m \geq 3$. For each $v \in \text{Vert } \Delta$, no chamber of $\text{Cham}(v)$ is isolated in $\text{Cham}(v)$. In particular, $\text{Cham}(v)$ is infinite.*

Proof. See [BS87a, Lemma 1.14]. □

2.2 Strong transitivity and the Moufang property

Let Δ be a spherical building and $\text{Aut}(\Delta)$ be the automorphism group of Δ . A subgroup G of $\text{Aut}(\Delta)$ is said to be **strongly transitive** if G is transitive on the set of all pairs (A, C) where A is an apartment of Δ and C is a chamber of A . If Δ is a usual (non-topological) building, we say that Δ is **strongly transitive** when $\text{Aut}(\Delta)$ is strongly transitive. If Δ is a compact building, this terminology is rather used when $\text{Auttop}(\Delta)$ is strongly transitive.

The following fact is almost immediate but worth mentioning.

Lemma 2.9. *Let Δ be a spherical building and π, π' be opposite panels of Δ . Let also G be a strongly transitive subgroup of $\text{Aut}(\Delta)$. Then the stabilizer $\text{Stab}_G(\pi, \pi')$ of π and π' in G acts 2-transitively on $\text{Cham}(\pi)$.*

Proof. Let $C_1 \neq D_1$ and $C_2 \neq D_2$ be chambers in $\text{Cham}(\pi)$. We want to find $g \in \text{Stab}_G(\pi, \pi')$ such that $g(C_1) = C_2$ and $g(D_1) = D_2$. Let A_1 (resp. A_2) be the apartment of Δ containing C_1 (resp. C_2), D_1 (resp. D_2) and π' . By strong transitivity of G , there is $g \in G$ such that $g(A_1) = A_2$ and $g(C_1) = C_2$. Clearly, we also have $g(\pi) = \pi$, $g(\pi') = \pi'$ and $g(D_1) = D_2$. □

The Moufang property is another transitivity hypothesis on the automorphism group of a building. We say that a thick irreducible spherical building Δ is **Moufang** (or satisfies the **Moufang property**) if for each root α of Δ , the **root group**

$$U_\alpha := \{g \in \text{Aut}(\Delta) \mid g \text{ fixes every chamber having a panel in } \alpha \setminus \partial\alpha\}$$

acts transitively on the set of apartments of Δ containing α . A natural wish would be to replace $\text{Aut}(\Delta)$ with $\text{Auttop}(\Delta)$ in the case where Δ is a compact building, but this is actually not necessary. Indeed, one can show that every element of a root group is automatically continuous, hence contained in $\text{Auttop}(\Delta)$ (see [GKVMW12, Corollary 6.17]).

3 A compactness criterion for subsets of $\text{Auttop}(\Delta)$

This section is dedicated to the proof of Theorem C which gives, in a compact polygon Δ , a sufficient condition on subsets of $\text{Auttop}(\Delta)$ for being compact.

3.1 Relative compactness in $\text{Auttop}(\Delta)$

To prove Theorem C, we need some characterization of relative compactness in $\text{Auttop}(\Delta)$. The tool that will help us is the Arzela-Ascoli theorem. For the reader's convenience, we recall its statement below. First recall that when X is a topological space and Y is a metric space, a subset $\mathcal{F}$ of $C(X, Y)$ is said to be **equicontinuous at** $x_0 \in X$ if, for all $\varepsilon > 0$, there exists a neighbourhood U of x_0 such that

$$f(U) \subseteq B(f(x_0), \varepsilon) \quad \text{for all } f \in \mathcal{F},$$

where $B(x, r)$ denotes the open ball of radius r centered at x . The set $\mathcal{F}$ is then called **equicontinuous on** X if it is equicontinuous at each point of X .

Theorem (Arzela-Ascoli). *Let X be a topological space and Y be a metric space. Let $\mathcal{F}$ be a subset of $C(X, Y)$. If $\mathcal{F}$ is equicontinuous on X and if $\{f(x) \mid f \in \mathcal{F}\}$ is relatively compact in Y for all $x \in X$, then $\mathcal{F}$ is relatively compact in $C(X, Y)$ equipped with the compact-open topology. The converse holds if X is locally compact.*

Proof. See [Mun00, Theorem 47.1]. □

We can now already give some characterization of relative compactness in $\text{Auttop}(\Delta)$.

Lemma 3.1. *Let Δ be a compact spherical building. A set $S \subseteq \text{Auttop}(\Delta)$ is relatively compact in $\text{Auttop}(\Delta)$ if and only if S and S^{-1} are equicontinuous on $\text{Cham } \Delta$.*

Proof. First assume that S is relatively compact in $\text{Auttop}(\Delta)$. The set S is thus also relatively compact in $C(\text{Cham } \Delta, \text{Cham } \Delta)$, hence S is equicontinuous on $\text{Cham } \Delta$ by Arzela-Ascoli. Since $\text{Auttop}(\Delta)$ is a topological group, the inverse function of $\text{Auttop}(\Delta)$ is a homeomorphism and S^{-1} is in turn relatively compact and equicontinuous on $\text{Cham } \Delta$.

Conversely, assume that S and S^{-1} are equicontinuous. We want to show that S is relatively compact in $\text{Auttop}(\Delta)$. We already know by Arzela-Ascoli that S is relatively compact in $C(\text{Cham } \Delta, \text{Cham } \Delta)$. A way to conclude is therefore to prove that the closure of S in $C(\text{Cham } \Delta, \text{Cham } \Delta)$ is contained in $\text{Auttop}(\Delta)$. We thus consider a sequence $\varphi_n \rightarrow \varphi$ with $\{\varphi_n\} \subseteq \text{Auttop}(\Delta)$ and $\varphi \in C(\text{Cham } \Delta, \text{Cham } \Delta)$ and prove that $\varphi \in \text{Auttop}(\Delta)$. As a direct consequence of the fact that the adjacency relation is closed in a compact building, φ is a (continuous) building morphism. To complete the proof, it suffices to find another such morphism ψ so that $\varphi\psi = \psi\varphi = \text{id}_\Delta$. Since S^{-1} is equicontinuous and thus relatively compact in $C(\text{Cham } \Delta, \text{Cham } \Delta)$, it follows that some subsequence of (φ_n^{-1}) converges to an element of $C(\text{Cham } \Delta, \text{Cham } \Delta)$. Taking this limit for ψ , we obtain $\varphi\psi = \psi\varphi = \text{id}_\Delta$. □

To prove that a given subset of $\text{Auttop}(\Delta)$ is relatively compact, we will generally proceed by contradiction. In this context, the next proposition will be helpful. We introduce a convenient definition to state it.

Definition 3.2. Let Δ be a compact polygon. Two convergent sequences $a_n \rightarrow a$ and $b_n \rightarrow b$ in $\text{Vert } \Delta$ are said to be **collapsed by** a sequence $\{\varphi_n\} \subseteq \text{Auttop}(\Delta)$ if $a \neq b$ and if $\varphi_n a_n \rightarrow x$ and $\varphi_n b_n \rightarrow x$ for some $x \in \text{Vert } \Delta$.

Proposition 3.3. *Let Δ be a compact polygon and $S \subseteq \text{Auttop}(\Delta)$. If S is not relatively compact in $\text{Auttop}(\Delta)$, then there exist two sequences $a_n \rightarrow a$ and $b_n \rightarrow b$ in $\text{Vert } \Delta$ which are collapsed by some sequence (φ_n) in S or in S^{-1} .*

Proof. Let S be a subset of $\text{Auttop}(\Delta)$ which is not relatively compact. By Lemma 3.1, S or S^{-1} is not equicontinuous on $\text{Cham}\Delta$ and therefore also not equicontinuous on $\text{Vert}\Delta$. Suppose S is not equicontinuous. Then there exist some $x \in \text{Vert}\Delta$ and $\varepsilon > 0$ such that for every neighbourhood U of x , there exists $\psi \in S$ satisfying $\psi(U) \not\subseteq B(\psi x, \varepsilon)$. In other words, for every $n \in \mathbf{N}^*$ there exist $\psi_n \in S$ and $y_n \in B(x, \frac{1}{n})$ such that $\psi_n y_n \notin B(\psi_n x, \varepsilon)$. Clearly $y_n \rightarrow x$ and we can assume by passing to a subsequence that $\psi_n x \rightarrow a$ and $\psi_n y_n \rightarrow b$ for some $a \neq b$. Hence, the sequences $a_n = \psi_n x \rightarrow a$ and $b_n = \psi_n y_n \rightarrow b$ are collapsed by $\{\varphi_n = \psi_n^{-1}\} \subseteq S^{-1}$. The case where S^{-1} is not equicontinuous is identical but gives two sequences which are collapsed by a sequence in $(S^{-1})^{-1} = S$. $\square$

3.2 Sequences collapsed by a sequence of automorphisms

In this section, we consider a compact polygon Δ . Proposition 3.3 states that one can deduce from non-relative compactness of a subset S of $\text{Auttop}(\Delta)$ that there exist two sequences $a_n \rightarrow a$ and $b_n \rightarrow b$ of vertices of Δ collapsed by a sequence in S or in S^{-1} . The goal of this subsection is to show that we can actually obtain additional constraints on these two sequences. The following result of Burns and Spatzier already goes in this direction by asserting that we can assume $D(a_n, b_n) = D(a, b) = 2$.

Proposition 3.4. *Let Δ be a thick compact polygon. If there exist two sequences $a_n \rightarrow a$ and $b_n \rightarrow b$ in $\text{Vert}\Delta$ collapsed by a sequence $\{\varphi_n\} \subseteq \text{Auttop}(\Delta)$, then there exist two sequences $a'_n \rightarrow a'$ and $b'_n \rightarrow b'$ collapsed by (φ_n) and such that $D(a'_n, b'_n) = D(a', b') = 2$ for all $n \in \mathbf{N}$.*

Proof. See [BS87a, Discussion between Assertions 2.3 and 2.4]. $\square$

Now suppose that we are given a converging sequence of apartments $A_n \rightarrow A$, i.e. $2m$ converging sequences $v_i^{(n)} \rightarrow v_i$ ($i \in \{0, \dots, 2m-1\}$) where $v_0^{(n)}, \dots, v_{2m-1}^{(n)}$ (resp. $v_0, \dots, v_{2m-1}$) are the vertices of an apartment A_n (resp. A). In this context, we can generally even assume that the middle vertex c_n of a_n and b_n (which are such that $D(a_n, b_n) = 2$) is a vertex of A_n . For this reason we introduce the following definition.

Definition 3.5. Let Δ be a compact polygon. Two convergent sequences $a_n \rightarrow a$ and $b_n \rightarrow b$ in $\text{Vert}\Delta$ are said to be **centered at** $c_n \rightarrow c$ if $D(a_n, c_n) = D(c_n, b_n) = 1$ for all $n \in \mathbf{N}$. This also implies that $D(a, c) = D(c, b) = 1$.

Before proving the announced result, we show the next lemma, which provides a way to go from a sequence of vertices to a sequence of opposite vertices.

Lemma 3.6. *Let Δ be a compact polygon and let $a_n \rightarrow a$ and $b_n \rightarrow b$ be two sequences in $\text{Vert}\Delta$ centered at $c_n \rightarrow c$ and collapsed by a sequence $\{\varphi_n\} \subseteq \text{Auttop}(\Delta)$. Let also $c'_n \rightarrow c'$ be a sequence in $\text{Vert}\Delta$ such that c'_n (resp. c') is opposite c_n (resp. c) for all $n \in \mathbf{N}$. Suppose that $\varphi_n c_n \rightarrow \tilde{c}$ and $\varphi_n c'_n \rightarrow \tilde{c}'$ for some opposite vertices $\tilde{c}$ and $\tilde{c}'$. Denote by C_n (resp. D_n) the chamber whose vertices are c_n and a_n (resp. b_n) and by a'_n (resp. b'_n) the vertex of $\text{proj}_{c'_n}(C_n)$ (resp. $\text{proj}_{c'_n}(D_n)$) different from c'_n . Then the sequences (a'_n) and (b'_n) converge, are centered at (c'_n) and are collapsed by (φ_n) .*

Proof. By Proposition 2.2, the sequence (a'_n) converges to a' , the vertex of $\text{proj}_{c'}(C)$ different from c' where C is the chamber having vertices c and a . The sequence (b'_n) converges to b' defined in the same way and $a' \neq b'$ since $a \neq b$. The fact that these two sequences are collapsed by (φ_n) directly comes from the observation that the two sequences

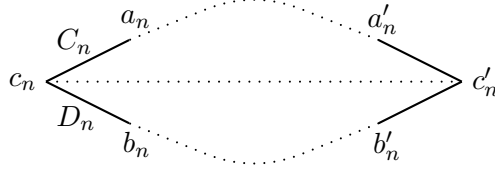


Figure 2: Illustration of Lemma 3.6.

$\left(\varphi_n \text{proj}_{c'_n}(C_n)\right) = \left(\text{proj}_{\varphi_n c'_n}(\varphi_n C_n)\right)$ and $\left(\varphi_n \text{proj}_{c'_n}(D_n)\right) = \left(\text{proj}_{\varphi_n c'_n}(\varphi_n D_n)\right)$ have the same limit by Proposition 2.2. $\square$

We draw attention to the fact that it is essential to have $\tilde{c}$ and $\tilde{c}'$ opposite each other to apply the previous lemma.

Proposition 3.7. *Let Δ be a thick compact polygon and $A_n \rightarrow A$ be a converging sequence of apartments of Δ . If there exist two sequences $a_n \rightarrow a$ and $b_n \rightarrow b$ in $\text{Vert } \Delta$ collapsed by a sequence $\{\varphi_n\} \subseteq \text{Auttop}(\Delta)$ and if $\varphi_n A_n \rightarrow \tilde{A}$ for some apartment $\tilde{A}$ of Δ , then there exist two sequences $a'_n \rightarrow a'$ and $b'_n \rightarrow b'$ centered at $c'_n \rightarrow c'$ and collapsed by some subsequence of (φ_n) , with $c'_n \in A_n$ for all $n \in \mathbf{N}$.*

Proof. By Proposition 3.4, we can assume that the two sequences $a_n \rightarrow a$ and $b_n \rightarrow b$ are centered at some sequence of vertices $c_n \rightarrow c$. Let d be a vertex of A opposite c (to find such a vertex, one can take a gallery from c to any vertex of A and extend this one with chambers of A until an opposite vertex is reached) and let d_n be the vertex of A_n (for $n \in \mathbf{N}$) such that $d_n \rightarrow d$. Denote also by $\tilde{d}$ the limit of $(\varphi_n d_n)$. By Proposition 2.1, c_n is opposite d_n for sufficiently large n . We can also assume by passing to a subsequence that $(\varphi_n c_n)$ converges to some $\tilde{c}$. If $\tilde{c}$ is opposite $\tilde{d}$, then we can immediately complete the proof using Lemma 3.6.

If, on the contrary, $\tilde{c}$ is not opposite $\tilde{d}$, then we cannot proceed in this way. In this case, pick a sequence $e_n \rightarrow e$ with e_n (resp. e) adjacent to d_n (resp. d) but not contained in A_n (resp. A). It is easy to show that such a sequence exists, using for example Proposition 2.2. Since e is almost opposite c , e_n is almost opposite c_n for sufficiently large n (Proposition 2.7). For these n , there is a gallery from d_n to c_n of length $m = \text{diam } \Delta$ passing through e_n . Let $d_n = e_0^{(n)}, e_1^{(n)} = e_n, \dots, e_{m-1}^{(n)}, e_m^{(n)} = c_n$ be the vertices of this gallery, as illustrated in Figure 3. After passing to subsequences, we can suppose that $e_i^{(n)} \rightarrow e_i$ and $\varphi_n e_i^{(n)} \rightarrow \tilde{e}_i$ for each $i \in \{0, \dots, m\}$. In particular, $\tilde{e}_0 = \tilde{d}$ and $\tilde{e}_m = \tilde{c}$. Since $\tilde{c}$ is not opposite $\tilde{d}$, the gallery of vertices $\tilde{d} = \tilde{e}_0, \tilde{e}_1, \dots, \tilde{e}_m = \tilde{c}$ must stammer. This means that there exists $k \in \{0, \dots, m-2\}$ such that $\tilde{e}_k = \tilde{e}_{k+2}$ (if there is more than one such k , then we choose the smallest one). In

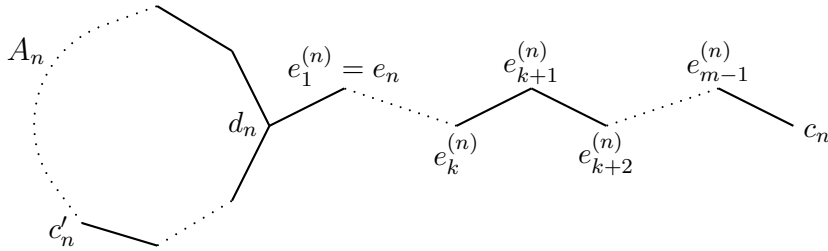


Figure 3: Illustration of Proposition 3.7.

other words, since $e_k \neq e_{k+2}$, the two sequences $e_k^{(n)} \rightarrow e_k$ and $e_{k+2}^{(n)} \rightarrow e_{k+2}$ are centered at $e_{k+1}^{(n)} \rightarrow e_{k+1}$ and collapsed by (φ_n) . Now take c' a vertex of A such that $D(c', d) = m - k - 1$ and c'_n the vertex of A_n (for $n \in \mathbf{N}$) such that $c'_n \rightarrow c'$. Thanks to this choice, e_{k+1} is opposite c' and $\tilde{e}_{k+1}$ is opposite the limit $\tilde{c}'$ of $(\varphi_n c'_n)$ (in the particular case where $\tilde{e}_1 \in \tilde{A}$, we choose c' so that the minimal gallery from $\tilde{d}$ to $\tilde{c}'$ does not contain $\tilde{e}_1$). We can therefore apply Lemma 3.6 to the sequences $e_k^{(n)} \rightarrow e_k$ and $e_{k+2}^{(n)} \rightarrow e_{k+2}$ centered at $e_{k+1}^{(n)} \rightarrow e_{k+1}$ and the opposite sequence $c'_n \rightarrow c'$. $\square$

The following result eventually shows that we can even suppose that one of the two sequences which are collapsed is contained in the sequence of apartments $A_n \rightarrow A$. The proof only works when the diameter m of Δ is at least 3, i.e. when Δ is irreducible.

Proposition 3.8. *Let Δ be a thick compact m -gon with $m \geq 3$ and $A_n \rightarrow A$ be a converging sequence of apartments of Δ . If there exist two sequences $a_n \rightarrow a$ and $b_n \rightarrow b$ in $\text{Vert } \Delta$ collapsed by a sequence $\{\varphi_n\} \subseteq \text{Autop}(\Delta)$ and if $\varphi_n A_n \rightarrow \tilde{A}$ for some apartment $\tilde{A}$ of Δ , then there exist two sequences $a'_n \rightarrow a'$ and $b'_n \rightarrow b'$ centered at $c'_n \rightarrow c'$ and collapsed by some subsequence of (φ_n) , with $c'_n \in A_n$ and $a'_n \in A_n$ for all $n \in \mathbf{N}$.*

Proof. By Proposition 3.7, we can assume that the sequences $a_n \rightarrow a$ and $b_n \rightarrow b$ are centered at $v_n \rightarrow v$ with $v_n \in A_n$ (and thus $v \in A$). Let $\tilde{x}$ be the common limit of $(\varphi_n a_n)$ and $(\varphi_n b_n)$.

If $\tilde{x}$ is a vertex of $\tilde{A}$, then consider $s_n \rightarrow s$ the sequence of vertices with $s_n \in A_n$ and $s \in A$ such that $\varphi_n s_n \rightarrow \tilde{x}$. Since $a \neq b$, we can assume without loss of generality that $a \neq s$. The sequences $a_n \rightarrow a$ and $s_n \rightarrow s$ are then collapsed by (φ_n) , which ends the proof.

Suppose now that $\tilde{x} \notin \tilde{A}$. Let v'_n (resp. v') be the vertex of A_n (resp. A) opposite v_n (resp. v), $\tilde{v}'$ be the limit of $(\varphi_n v'_n)$ and $w_n \rightarrow w$ be a sequence of vertices where w_n (resp. w) is a vertex of A_n (resp. A) and is adjacent to v'_n (resp. v'). Consider also a sequence $u_n \rightarrow u$ with u_n (resp. u) adjacent to w_n (resp. w) but outside A_n (resp. A). Passing to a subsequence if necessary, we can assume that $\varphi_n u_n \rightarrow \tilde{u}$ for some vertex $\tilde{u}$ (adjacent to the

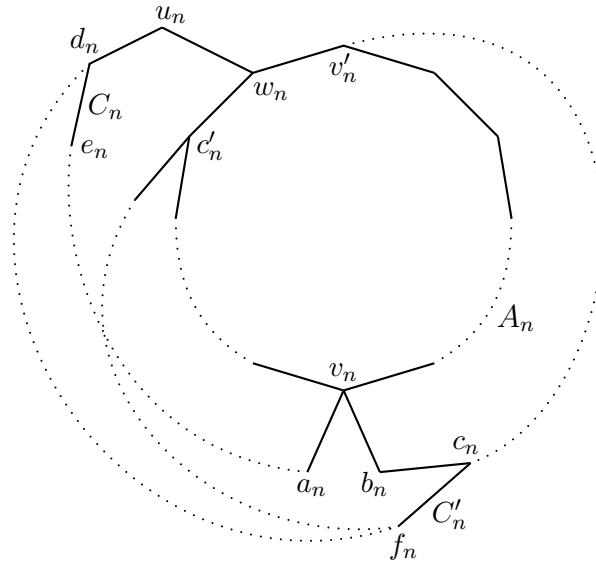


Figure 4: Illustration of Proposition 3.8.

limit of $(\varphi_n w_n)$). Once again, if $\tilde{u} \in \tilde{A}$ then the proof is completed. We therefore assume that $\tilde{u} \notin \tilde{A}$. For sufficiently large n , a_n is almost opposite u_n and b_n is almost opposite v'_n (see Proposition 2.7). We can thus draw $[a_n, u_n]$ and $[b_n, v'_n]$ as in Figure 4. Thanks to Lemma 2.6, $\varphi_n[a_n, u_n] \rightarrow [\tilde{x}, \tilde{u}]$ and $\varphi_n[b_n, v'_n] \rightarrow [\tilde{x}, \tilde{v}']$. The fact that $\tilde{u} \notin \tilde{A}$ ensures that the two galleries $[\tilde{x}, \tilde{u}]$ and $[\tilde{x}, \tilde{v}']$ have no shared chambers. Let c_n be the vertex of $[b_n, v'_n]$ adjacent to b_n , d_n be the vertex of $[a_n, u_n]$ adjacent to u_n , e_n be the vertex of $[d_n, a_n]$ adjacent to d_n , and C_n be the chamber whose vertices are d_n and e_n . The vertices e_n and c_n are opposite for large n . Now, let C'_n be the projection of C_n on c_n and f_n be the vertex of C'_n different from c_n . Since d_n and c_n are almost opposite as well as the limits of $(\varphi_n d_n)$ and $(\varphi_n c_n)$, Lemma 2.6 gives $\varphi_n f_n \rightarrow \tilde{x}$ and the two sequences (b_n) and (f_n) centered at (c_n) are collapsed by (φ_n) .

Let finally c' be a vertex of A at distance 2 from v' such that c' is opposite the limit of (c_n) and let c'_n be the vertex of A_n (for $n \in \mathbf{N}$) such that $c'_n \rightarrow c'$. As the limits of $(\varphi_n c_n)$ and $(\varphi_n c'_n)$ are opposite, we can apply Lemma 3.6 to get two sequences of vertices centered at $c'_n \rightarrow c'$, collapsed by (φ_n) and so that one of them is contained in $A_n \rightarrow A$. $\square$

3.3 Proof of the compactness criterion

The following lemma is almost evident but will play an important role in the next results. As in the statement of Theorem C, we write $B_v(C, r)$ for the open ball in $\text{Cham}(v)$ centered at C and of radius r .

Lemma 3.9. *Let Δ be a compact polygon and $v^{(n)} \rightarrow v$ and $v'^{(n)} \rightarrow v'$ be two sequences in $\text{Vert } \Delta$ such that $v^{(n)}$ (resp. v) is opposite $v'^{(n)}$ (resp. v') for all $n \in \mathbf{N}$. Let also $X^{(n)} \rightarrow X$ be a sequence in $\text{Cham } \Delta$ such that $X^{(n)}$ (resp. X) has vertex $v^{(n)}$ (resp. v) for all $n \in \mathbf{N}$. For each $\eta > 0$, there exists $\eta' > 0$ such that*

$$B_{v'^{(n)}}(\text{proj}_{v'^{(n)}}(X^{(n)}), \eta') \subseteq \text{proj}_{v'^{(n)}}(B_{v^{(n)}}(X^{(n)}, \eta)) \text{ for all } n \in \mathbf{N}.$$

Proof. Fix $\eta > 0$ and suppose by contradiction that η' does not exist. This means that we can find, after passage to a subsequence, a sequence $(Y^{(n)})$ with $Y^{(n)} \in B_{v'^{(n)}}(\text{proj}_{v'^{(n)}}(X^{(n)}), \frac{1}{n})$ but $Y^{(n)} \notin \text{proj}_{v'^{(n)}}(B_{v^{(n)}}(X^{(n)}, \eta))$ for all $n \in \mathbf{N}^*$. By Proposition 2.2, $Y^{(n)} \rightarrow \text{proj}_{v'}(X)$ and thus $\text{proj}_{v^{(n)}}(Y^{(n)}) \rightarrow \text{proj}_v(\text{proj}_{v'}(X)) = X$. This contradicts the fact that $\text{proj}_{v^{(n)}}(Y^{(n)}) \notin B_{v^{(n)}}(X^{(n)}, \eta)$ for all $n \in \mathbf{N}^*$. $\square$

The next definition will turn out to be convenient.

Definition 3.10. Let $X^{(n)} \rightarrow X$ and $Y^{(n)} \rightarrow Y$ be two sequences in $\text{Cham } \Delta$ with $X^{(n)}$ (resp. X) adjacent to $Y^{(n)}$ (resp. Y) in vertex $v^{(n)}$ (resp. v) for all $n \in \mathbf{N}$. Let (φ_n) be a sequence in $\text{Auttop}(\Delta)$ such that $\varphi_n Y^{(n)} \rightarrow \tilde{Y}$ for some $\tilde{Y} \in \text{Cham } \Delta$. For $\eta > 0$, we say that (X, Y) is η -**correct** (the sequences $X_n \rightarrow X$, $Y_n \rightarrow Y$ and (φ_n) being clear from the context) if for each sequence $(T^{(n)})$ with $T^{(n)} \in B_{v^{(n)}}(X^{(n)}, \eta)$ the sequence $(\varphi_n T^{(n)})$ does not accumulate at $\tilde{Y}$. We say that (X, Y) is **correct** if it is η -correct for some $\eta > 0$.

The following lemma is the key result for the proof of Theorem C. Indeed, it makes it possible to find some kind of uniform convergence from just a simple convergence. Figure 5 should make the statement easier to digest.

Lemma 3.11. *Let Δ be a compact polygon and $A^{(n)} \rightarrow A$ be a converging sequence of apartments of Δ . Let $v_0^{(n)}, \dots, v_{2m-1}^{(n)}$ (resp. $v_0, \dots, v_{2m-1}$) be the vertices of $A^{(n)}$ (resp. A) such*

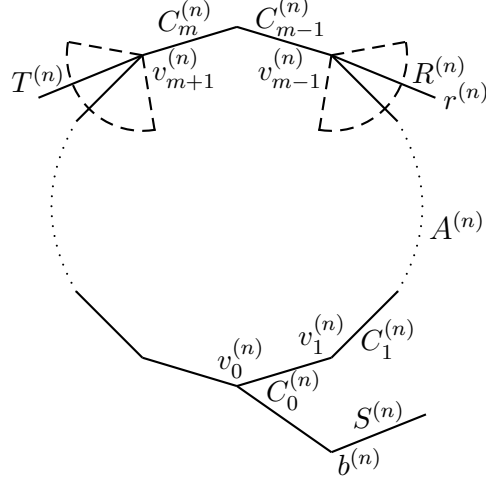


Figure 5: Illustration of Lemma 3.11.

that $v_i^{(n)} \rightarrow v_i$ for $i \in \{0, \dots, 2m-1\}$ and let $C_i^{(n)}$ (resp. C_i) be the chamber having vertices $v_i^{(n)}$ (resp. v_i) and $v_{i+1}^{(n)}$ (resp. v_{i+1}) for $i \in \{0, \dots, 2m-1\}$, where $v_{2m}^{(n)} := v_0^{(n)}$ and $v_{2m} := v_0$. Let also $b^{(n)} \rightarrow b \notin A$ be a sequence in $\text{Vert } \Delta$ such that $b^{(n)}$ is adjacent to $v_0^{(n)}$ for all $n \in \mathbf{N}$. Fix $\eta > 0$. Then there exists $\eta' > 0$ such that if (φ_n) is a sequence in $\text{Auttop}(\Delta)$ satisfying

- (a) $\varphi_n v_i^{(n)} \rightarrow \tilde{v}_i$ for each $i \in \{0, \dots, 2m-1\}$ and the set of vertices $\tilde{v}_0, \dots, \tilde{v}_{2m-1}$ form an apartment $\tilde{A}$ (whose chambers are denoted by $\tilde{C}_0, \dots, \tilde{C}_{2m-1}$ with $\varphi_n C_i^{(n)} \rightarrow \tilde{C}_i$),
- (b) (C_{m+1}, C_m) is η -correct (with respect to (φ_n)),
- (c) $\varphi_n b^{(n)} \rightarrow \tilde{v}_1$;

then for each sequence $(R^{(n)})$ with $R^{(n)} \in B_{v_{m-1}^{(n)}}(C_{m-2}^{(n)}, \eta')$ for all $n \in \mathbf{N}$, $\varphi_n R^{(n)} \rightarrow \tilde{C}_{m-2}$.

Proof. Since $b \notin A$, we can assume that $b^{(n)} \notin A^{(n)}$ for all $n \in \mathbf{N}$. We consider the balls $B_{v_{m+1}^{(n)}}(C_{m+1}^{(n)}, \eta)$ and apply Lemma 3.9 twice: first with the sequences $v_{m+1}^{(n)} \rightarrow v_{m+1}$ and $b^{(n)} \rightarrow b$ and then with the sequences $b^{(n)} \rightarrow b$ and $v_{m-1}^{(n)} \rightarrow v_{m-1}$. This gives $\eta' > 0$ such that

$$B_{v_{m-1}^{(n)}}(C_{m-2}^{(n)}, \eta') \subseteq \text{proj}_{v_{m-1}^{(n)}}(\text{proj}_{b^{(n)}}(B_{v_{m+1}^{(n)}}(C_{m+1}^{(n)}, \eta))) \text{ for all } n \in \mathbf{N}. \quad (*)$$

We now prove that η' satisfies the statement.

Consider a sequence $(R^{(n)})$ with $R^{(n)} \in B_{v_{m-1}^{(n)}}(C_{m-2}^{(n)}, \eta')$ and suppose for a contradiction that there exists a sequence (φ_n) satisfying the conditions of the statement but such that $(\varphi_n R^{(n)})$ does not converge to $\tilde{C}_{m-2}$. Passing to a subsequence, we can assume that $\varphi_n R^{(n)} \rightarrow \tilde{R} \neq \tilde{C}_{m-2}$ and $R^{(n)} \rightarrow R$. Now take $S^{(n)} = \text{proj}_{b^{(n)}}(R^{(n)})$ and $S = \text{proj}_b(R)$. By Proposition 2.2, $S^{(n)} \rightarrow S$. Finally, define $T^{(n)} = \text{proj}_{v_{m+1}^{(n)}}(S^{(n)})$ and $T = \text{proj}_{v_{m+1}}(S)$.

Once again, $T^{(n)} \rightarrow T$. By $(*)$, $T^{(n)} \in B_{v_{m+1}^{(n)}}(C_{m+1}^{(n)}, \eta)$ for all $n \in \mathbf{N}$. We now show that the convergence of $(\varphi_n R^{(n)})$ to $\tilde{R} \neq \tilde{C}_{m-2}$ necessarily implies $\varphi_n T^{(n)} \rightarrow \tilde{C}_m$, which is impossible since (C_{m+1}, C_m) is η -correct.

Denote by $r^{(n)}$ the vertex of $R^{(n)}$ different from $v_{m-1}^{(n)}$ and by $\tilde{r}$ the vertex of $\tilde{R}$ different from $\tilde{v}_{m-1}$. Since $b^{(n)}$ and $r^{(n)}$ are almost opposite (for large enough n) as well as $\tilde{v}_1$ and $\tilde{r}$, $\varphi_n[b^{(n)}, r^{(n)}] \rightarrow [\tilde{v}_1, \tilde{r}]$ (Lemma 2.6) and in particular $\varphi_n S^{(n)} \rightarrow \tilde{C}_1$. As v_1 is opposite v_{m+1} , we therefore get by Proposition 2.2:

$$\varphi_n T^{(n)} = \varphi_n \left(\text{proj}_{v_{m+1}^{(n)}}(S^{(n)}) \right) = \text{proj}_{\varphi_n v_{m+1}^{(n)}}(\varphi_n S^{(n)}) \rightarrow \text{proj}_{\tilde{v}_{m+1}}(\tilde{C}_1) = \tilde{C}_m. \quad \square$$

We are now able to prove Theorem C.

Proof of Theorem C. If Δ is finite, then $\text{Auttop}(\Delta)$ and $J_\varepsilon := J_\varepsilon(C, C', U, U', E_0, E_1)$ are finite and the latter is thus compact. We will therefore assume that Δ is infinite. By Lemma 2.8, this implies that $\text{Cham}(v)$ is infinite for each vertex v of Δ .

Since J_ε is closed in $\text{Auttop}(\Delta)$, it is compact if and only if it is relatively compact in $\text{Auttop}(\Delta)$. Suppose for a contradiction that it is not relatively compact. By Proposition 3.3, there exist two sequences $a^{(n)} \rightarrow a$ and $b^{(n)} \rightarrow b$ collapsed by a sequence (φ_n) in J_ε or in J_ε^{-1} . Observe both cases:

- (1) If $\{\varphi_n\} \subseteq J_\varepsilon$, then we define $C^{(n)} = C$, $C'^{(n)} = C'$, $D_i^{(n)} = D_i$ and $E_i^{(n)} = E_i$ for all $n \in \mathbf{N}$ and $i \in \{0, 1\}$. Let also $A^{(n)} = A$ be the apartment containing C and C' (for all $n \in \mathbf{N}$). Passing to a subsequence, we can assume that $\varphi_n C^{(n)} \rightarrow \tilde{C}$ and $\varphi_n C'^{(n)} \rightarrow \tilde{C}'$. By definition of J_ε , $\tilde{C}$ and $\tilde{C}'$ are opposite, which means that $\varphi_n A^{(n)} \rightarrow \tilde{A}$, the apartment containing $\tilde{C}$ and $\tilde{C}'$. The definition of J_ε is also such that (C, D_i) and (D_i, C) are ε -correct for $i \in \{0, 1\}$. Finally, we can assume that $\varphi_n E_i^{(n)} \rightarrow \tilde{E}_i$ and $\tilde{E}_i \notin \tilde{A}$ for $i \in \{0, 1\}$.
- (2) If $\{\varphi_n\} \subseteq J_\varepsilon^{-1}$, then we define $F^{(n)} = \varphi_n^{-1} C$, $F'^{(n)} = \varphi_n^{-1} C'$, $G_i^{(n)} = \varphi_n^{-1} D_i$ and $H_i^{(n)} = \varphi_n^{-1} E_i$ for all $n \in \mathbf{N}$ and $i \in \{0, 1\}$. Passing to a subsequence, we can assume that $F^{(n)} \rightarrow F$, $F'^{(n)} \rightarrow F'$, $G_i^{(n)} \rightarrow G_i$ and $H_i^{(n)} \rightarrow H_i$. Note that F is opposite F' since $\{\varphi_n^{-1}\} \subseteq J_\varepsilon$, which allows us to denote by $A^{(n)}$ (resp. A) the apartment containing $F^{(n)}$ (resp. F) and $F'^{(n)}$ (resp. F') for all $n \in \mathbf{N}$. By definition of J_ε , $H_i \notin A$. Moreover, $\varphi_n F^{(n)} = C \rightarrow C =: \tilde{F}$, $\varphi_n F'^{(n)} = C' \rightarrow C' =: \tilde{F}'$ and $\varphi_n H_i^{(n)} = E_i \rightarrow E_i =: \tilde{H}_i$. Hence, $\varphi_n A^{(n)} \rightarrow \tilde{A}$, the apartment containing $\tilde{F}$ and $\tilde{F}'$. It is easy to see that (F, G_i) and (G_i, F) are ε -correct (with respect to (φ_n)) and clearly $\tilde{H}_i \notin \tilde{A}$ for $i \in \{0, 1\}$.

The situations in (1) and (2) are exactly the same, replacing letters F , G and H in (2) by letters C , D and E respectively. For this reason, we from now on assume that we are in case (1) and only use the objects $C^{(n)} \rightarrow C$, $C'^{(n)} \rightarrow C'$, $D_i^{(n)} \rightarrow D_i$ and $E_i^{(n)} \rightarrow E_i$ satisfying the conditions stated above. Let $v_0, \dots, v_{2m-1}$ be the vertices of A in the natural order so that E_0 has vertex v_0 and E_1 has vertex v_1 . Once again, $v_0^{(n)}, \dots, v_{2m-1}^{(n)}$ denotes the corresponding vertices in $A^{(n)}$ and $\tilde{v}_0, \dots, \tilde{v}_{2m-1}$ the corresponding vertices in $\tilde{A}$.

Claim 1. (X, Y) is correct for any two adjacent chambers X and Y in A .

Proof of the claim: In the proof of this claim, we say that v_i is correct if (X, Y) and (Y, X) are correct where X and Y are the two chambers of A having vertex v_i . We already know that v_0 and v_1 are correct, and we want to prove that every vertex of A is correct.

We actually show that if v_i is correct, then v_{-i} and v_{2-i} are also correct (the indices being considered modulo $2m$). To do so, draw the minimal gallery from E_0 (resp. $E_0^{(n)}$) to v_m

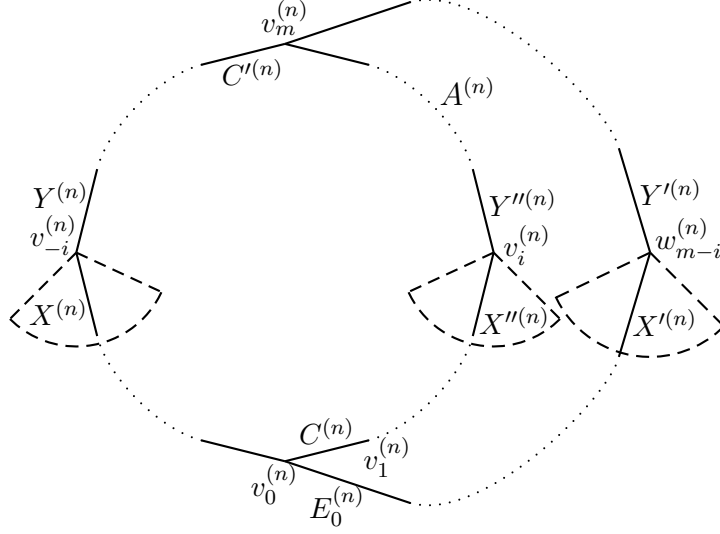


Figure 6: Illustration of Claim 1.

(resp. $v_m^{(n)}$) and denote by $w_1, \dots, w_{m-1}$ (resp. $w_1^{(n)}, \dots, w_{m-1}^{(n)}$) its vertices (see Figure 6). By Lemma 2.6, $w_i^{(n)} \rightarrow w_i$ for each $i \in \{1, \dots, m-1\}$. Now assume that v_i is correct for some $i \in \{1, \dots, m-1\}$. To show that v_{-i} is also correct, consider X a chamber of A having vertex v_{-i} and Y the other chamber of this apartment having vertex v_{-i} . Project these two chambers on w_{m-i} by defining $X' = \text{proj}_{w_{m-i}}(X)$ and $Y' = \text{proj}_{w_{m-i}}(Y)$, and then on v_i by defining $X'' = \text{proj}_{v_i}(X')$ and $Y'' = \text{proj}_{v_i}(Y')$. The chambers X'' and Y'' obviously are the chambers of A having vertex v_i . Define $X^{(n)}, Y^{(n)}, X'^{(n)}, Y'^{(n)}, X''^{(n)}$ and $Y''^{(n)}$ in the natural way. Since v_i is correct, there exists $\eta > 0$ such that (X'', Y'') is η -correct. We can assume that $w_{m-i}^{(n)}$ is opposite $v_i^{(n)}$ and $v_{-i}^{(n)}$ for all $n \in \mathbf{N}$ since $E_0 \notin A$. This allows us to go from v_i to v_{-i} via w_{m-i} using Lemma 3.9 twice. This gives $\eta' > 0$ such that

$$B_{v_{-i}^{(n)}}(X^{(n)}, \eta') \subseteq \text{proj}_{v_{-i}^{(n)}}(\text{proj}_{w_{m-i}^{(n)}}(B_{v_i^{(n)}}(X''^{(n)}, \eta))) \text{ for all } n \in \mathbf{N}. \quad (**)$$

We now show that (X, Y) is η' -correct. Suppose for a contradiction that there exists $(T^{(n)})$ with $T^{(n)} \in B_{v_{-i}^{(n)}}(X^{(n)}, \eta')$ such that $(\varphi_n T^{(n)})$ accumulates at $\tilde{Y}$, the limit of $(\varphi_n Y^{(n)})$. Considering $T'^{(n)} = \text{proj}_{w_{m-i}^{(n)}}(T^{(n)})$ and $T''^{(n)} = \text{proj}_{v_i^{(n)}}(T'^{(n)})$, Proposition 2.2 directly shows that $(\varphi_n T''^{(n)})$ accumulates at $\tilde{Y}''$, the limit of $(\varphi_n Y''^{(n)})$. This is a contradiction since (X'', Y'') is η -correct and $T''^{(n)} \in B_{v_i^{(n)}}(X''^{(n)}, \eta)$ for all $n \in \mathbf{N}$ by (**). Note that we assumed $i \in \{1, \dots, m-1\}$ but the reasoning is obviously the same for $i \in \{m+1, \dots, 2m-1\}$.

The same construction with E_1 instead of E_0 enables us to go from v_i to v_{2-i} . We directly conclude that all vertices are correct since v_0 and v_1 are correct and $v_{i+2} = v_{2-(-i)}$ is correct as soon as v_i is. The claim stands proven. $\blacksquare$

From now on, we denote by C_i the chamber having vertices v_i and v_{i+1} (indices being taken modulo $2m$).

Claim 2. *After possible passage to a subsequence, there exists for each $i \in \{0, \dots, 2m-1\}$ a sequence $b_i^{(n)} \rightarrow b_i \notin A$ with $b_i^{(n)}$ adjacent to $v_i^{(n)}$ for all $n \in \mathbf{N}$ and $\varphi_n b_i^{(n)} \rightarrow \tilde{v}_{i+1}$. Similarly, there exists a sequence $c_i^{(n)} \rightarrow c_i \notin A$ with $c_i^{(n)}$ adjacent to $v_i^{(n)}$ and $\varphi_n c_i^{(n)} \rightarrow \tilde{v}_{i-1}$.*

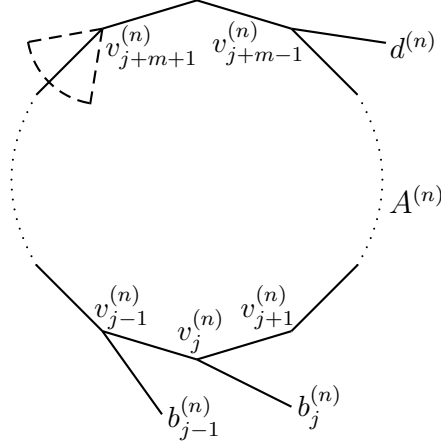


Figure 7: Illustration of Claim 2.

Proof of the claim: Recall that there are two sequences $a^{(n)} \rightarrow a$ and $b^{(n)} \rightarrow b$ collapsed by (φ_n) . All the hypotheses of Proposition 3.8 being met, we can assume that $a^{(n)} \rightarrow a$ and $b^{(n)} \rightarrow b$ are centered at some $v_j^{(n)} \rightarrow v_j$ and that $a^{(n)} \in A^{(n)}$ for all $n \in \mathbf{N}$. Without loss of generality, we may assume that $a^{(n)} = v_{j+1}^{(n)}$. Clearly $b \neq v_{j+1}$, and $b \neq v_{j-1}$ since (C_{j-1}, C_j) is correct. The sequence $b_j^{(n)} \rightarrow b_j$ defined by $b_j^{(n)} = b^{(n)}$ satisfies the statement of the claim for $i = j$.

We now prove that one can construct $b_{j-1}^{(n)} \rightarrow b_{j-1}$ from $b_j^{(n)} \rightarrow b_j$. Since (C_{j+m+1}, C_{j+m}) is correct, there is by Lemma 3.11 a sequence $d^{(n)} \rightarrow d \notin A$ with $d^{(n)}$ adjacent to $v_{j+m-1}^{(n)}$ such that $\varphi_n d^{(n)} \rightarrow \tilde{v}_{j+m-2}$. After projection on $v_{j-1}^{(n)} \rightarrow v_{j-1}$ (see Lemma 3.6), there is a sequence $b_{j-1}^{(n)} \rightarrow b_{j-1} \notin A$ with $b_{j-1}^{(n)}$ adjacent to $v_{j-1}^{(n)}$ such that $\varphi_n b_{j-1}^{(n)} \rightarrow \tilde{v}_j$. By repeating this process, we get the sequences $b_i^{(n)} \rightarrow b_i$ satisfying the statement for each $i \in \{0, \dots, 2m-1\}$. Using Lemma 3.6, there obviously also is for each vertex v_i a sequence $c_i^{(n)} \rightarrow c_i \notin A$ with $c_i^{(n)}$ adjacent to $v_i^{(n)}$ such that $\varphi_n c_i^{(n)} \rightarrow \tilde{v}_{i-1}$. ■

Let us now focus on $\text{Cham}(v_0)$. In the rest of this proof, we associate to each $X \in \text{Cham}(v_0)$ a sequence $X^{(n)} \rightarrow X$ with $X^{(n)} \in \text{Cham}(v_0^{(n)})$. Such a sequence exists, take for instance $X^{(n)} = \text{proj}_{v_0^{(n)}}(\text{proj}_{v_m}(X))$ (v_m being opposite $v_0^{(n)}$ for sufficiently large n).

Claim 3. *For each $X \in \text{Cham}(v_0)$, there exists $\eta_X > 0$ such that if $\varphi_{k(n)} X^{(k(n))} \rightarrow \tilde{X}$ for some subsequence $(\varphi_{k(n)})$ of (φ_n) and some chamber $\tilde{X}$, then $\varphi_{k(n)} R^{(k(n))} \rightarrow \tilde{X}$ for each sequence $(R^{(n)})$ with $R^{(n)} \in B_{v_0^{(n)}}(X^{(n)}, \eta_X)$ for all $n \in \mathbf{N}$.*

Proof of the claim: Let $X \in \text{Cham}(v_0)$ and first assume that $X \neq C, D_0$. Consider the sequence of apartments $A_1^{(n)} \rightarrow A_1$ where $A_1^{(n)}$ is the apartment containing $C^{(n)}$, $X^{(n)}$ and $v_m^{(n)}$, and the sequence $A_2^{(n)} \rightarrow A_2$ where $A_2^{(n)}$ is the apartment containing $D_0^{(n)}$, $X^{(n)}$ and $v_m^{(n)}$ ($X^{(n)}$ is different from $C^{(n)}$ and $D_0^{(n)}$ for sufficiently large n). The idea is to apply Lemma 3.11 to these two sequences of apartments simultaneously. We do it for $A_1^{(n)} \rightarrow A_1$. Denote by $w_{m+1}^{(n)}, \dots, w_{2m-1}^{(n)}$ the new vertices of $A_1^{(n)}$ as in Figure 8. The sequence $c_1^{(n)} \rightarrow c_1 \notin A$ is such

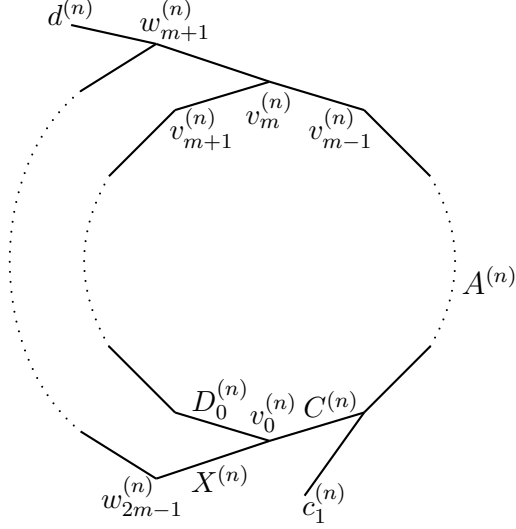


Figure 8: Illustration of Claim 3.

that $\varphi_n c_1^{(n)} \rightarrow \tilde{v}_0$ and, after projecting on $w_{m+1}^{(n)} \rightarrow w_{m+1}$ (see Lemma 3.6), there also is a sequence $d^{(n)} \rightarrow d \notin A_1$ with $d^{(n)}$ adjacent to $w_{m+1}^{(n)}$. Moreover, by Claim 1 there exists $\eta_1 > 0$ such that (C_2, C_1) is η_1 -correct. We can therefore apply Lemma 3.11 which provides $\eta'_1 > 0$ satisfying the properties stated in this same lemma. In the same way with $A_2^{(n)} \rightarrow A_2$, there is $\eta'_2 > 0$ with similar properties.

We now prove that $\eta_X = \min(\eta'_1, \eta'_2)$ is adequate. Suppose that $\varphi_{k(n)} X^{(k(n))} \rightarrow \tilde{X}$ for some subsequence $(\varphi_{k(n)})$ of (φ_n) and some chamber $\tilde{X}$. If $\tilde{X} \neq C$, then the sequence of apartments $\varphi_{k(n)} A_1^{(k(n))}$ converges to the apartment $\tilde{A}_1$ containing $\tilde{C}$, $\tilde{X}$ and $\tilde{v}_m$. Also, we have in this case $\varphi_{k(n)} d^{(k(n))} \rightarrow \tilde{w}_{m+2}$ (the limit of $(\varphi_{k(n)} w_{m+2}^{(k(n))})$) because $\varphi_n c_1^{(n)} \rightarrow \tilde{v}_0$. This means that $(\varphi_{k(n)})$ satisfies the three conditions (a), (b) and (c) of Lemma 3.11 for η_1 and thus $\varphi_{k(n)} R^{(k(n))} \rightarrow \tilde{X}$ as soon as $(R^{(n)})$ is a sequence with $R^{(n)} \in B_{v_0^{(n)}}(X^{(n)}, \eta'_1)$ for all $n \in \mathbf{N}$. On the other hand, if $\tilde{X} = C$ then $\tilde{X} \neq D_0$ and we can do the same trick with $A_2^{(n)} \rightarrow A_2$ and η'_2 . In either case, we have shown that $\eta_X = \min(\eta'_1, \eta'_2)$ is adequate.

Now if $X = C$, the convergence $\varphi_{k(n)} X^{(k(n))} \rightarrow \tilde{D}_0$ cannot happen because (C, D_0) is correct. We can therefore in this case do the same reasoning with only $A_2^{(n)} \rightarrow A_2$ and take $\eta_X = \eta'_2$. Similarly, for $X = D_0$ we only consider $A_1^{(n)} \rightarrow A_1$ and take $\eta_X = \eta'_1$. $\blacksquare$

To complete the proof, we show that the situation described in Claim 3 is actually impossible. Indeed, by compactness of $\text{Cham}(v_0)$ we get

$$\text{Cham}(v_0) = \bigcup_{j=1}^r B_{v_0}(X_j, \eta_{X_j})$$

for some $X_1, \dots, X_r \in \text{Cham}(v_0)$. Passing to a subsequence, we can assume that $\varphi_n X_j^{(n)} \rightarrow \tilde{X}_j$ for each $j \in \{1, \dots, r\}$. Now consider some $S \in \text{Cham}(\tilde{v}_0) \setminus \{\tilde{X}_1, \dots, \tilde{X}_r\}$ (which is non-empty since each panel is infinite). By a construction similar to the construction of $X^{(n)}$ from X , we can build a sequence $S^{(n)} \rightarrow S$ with $S^{(n)} \in \text{Cham}(\varphi_n v_0^{(n)})$ for all $n \in \mathbf{N}$. Let

$R^{(n)} = \varphi_n^{-1}S^{(n)}$; we can assume that $R^{(n)} \rightarrow R \in \text{Cham}(v_0)$. There exists $j \in \{1, \dots, r\}$ such that $R \in B_{v_0}(X_j, \eta_{X_j})$ and this means that $R^{(n)} \in B_{v_0^{(n)}}(X_j^{(n)}, \eta_{X_j})$ for sufficiently large n . Hence, $(\varphi_n R^{(n)})$ should converge to $\tilde{X}_j$, which is impossible as $\varphi_n R^{(n)} = S^{(n)} \rightarrow S \neq \tilde{X}_j$. $\square$

Proof of Corollary D. Define D_0, D_1, v_0 and v_1 as in Theorem C and take $\varepsilon > 0$ sufficiently small so that

$$(A) \quad \overline{B(C, \varepsilon)} \times \overline{B(C', \varepsilon)} \subseteq \{(X, X') \in (\text{Cham } \Delta)^2 \mid X \text{ is opposite } X'\},$$

$$(B) \quad \rho(C, D_i) \geq 4\varepsilon \text{ for } i \in \{0, 1\},$$

$$(C) \quad \text{there exists } E_i \in \text{Cham}(v_i) \text{ such that } \rho(C, E_i) \geq 3\varepsilon \text{ and } \rho(D_i, E_i) \geq 3\varepsilon \text{ for } i \in \{0, 1\}.$$

It is obvious that there exists $\varepsilon > 0$ satisfying (B) and (C), and Proposition 2.1 tells us that we can choose it so that it also satisfies (A). It is then immediate to see that $L_\varepsilon(C, C')$ is a closed subset of $J_\varepsilon(C, C', \overline{B(C, \varepsilon)}, \overline{B(C', \varepsilon)}, E_0, E_1)$, the latter being compact by Theorem C. $\square$

Proof of Corollary E. Consider two opposite chambers C and C' of Δ . The set $L_\varepsilon(C, C')$ is a neighbourhood of the identity in $\text{Autop}(\Delta)$ and is compact for some $\varepsilon > 0$ by Corollary D. $\square$

4 Topological characterization of the Moufang property

In this last section, our ultimate goal is to prove Theorem A. The implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are explained in the first two subsections while the most difficult part (iii) $\Rightarrow$ (i) is the subject of the rest of the section.

4.1 Convergence groups

If M is a topological space and G is a topological group acting continuously on M , then the action of G on M is **proper** if for every compact set $K \subseteq M$, the set $\{g \in G \mid g(K) \cap K \neq \emptyset\}$ is compact. When M is compact, we also say that the action of G on M is **n -proper** if the componentwise action of G on the space

$$M^{(n)} = \{(x_1, \dots, x_n) \in M^n \mid x_i \neq x_j \ \forall i \neq j\}$$

is proper, where $M^{(n)}$ is equipped with the induced topology from the product topology on M^n . We finally say that G acts as a **convergence group** on M if G acts 3-properly on M .

We will be particularly interested in subgroups of the homeomorphism group of a compact metric space M that act as a convergence group on M .

Lemma 4.1. *Let (M, d) be a compact metric space and G be a subgroup of $\text{Homeo}(M)$ (equipped with the compact-open topology) acting as a convergence group on M . Then G is closed in $\text{Homeo}(M)$ and G is locally compact.*

Proof. If M has less than 3 elements, then $\text{Homeo}(M)$ is finite and hence G is locally compact and closed in $\text{Homeo}(M)$. Now suppose that $|M| \geq 3$.

We first show that G is closed in $\text{Homeo}(M)$. Let $g_n \rightarrow g$ be a sequence with $g_n \in G$ for all $n \in \mathbf{N}$ and $g \in \text{Homeo}(M)$. We want to prove that $g \in G$. Take x_1, x_2, x_3 three distinct elements of M and consider $\varepsilon > 0$ such that $d(g(x_i), g(x_j)) > 2\varepsilon$ for each $i \neq j$. Define

$$K = \{(x_1, x_2, x_3)\} \cup (\overline{B(g(x_1), \varepsilon)} \times \overline{B(g(x_2), \varepsilon)} \times \overline{B(g(x_3), \varepsilon)}) \subseteq M^{(3)}.$$

This is clearly a compact subset of $M^{(3)}$, and $g_n \in \{h \in G \mid h(K) \cap K \neq \emptyset\}$ for sufficiently large n . This set being compact by hypothesis, it is closed in G and $g \in G$.

We now prove that G is locally compact. Once again, take x_1, x_2, x_3 three distinct elements of M . Choose $\varepsilon > 0$ such that $d(x_i, x_j) > 3\varepsilon$ for each $i \neq j$ and define

$$K = \{(y_1, y_2, y_3) \in M^3 \mid d(y_i, y_j) \geq \varepsilon \ \forall i \neq j\} \subseteq M^{(3)}.$$

This is a compact subset of $M^{(3)}$ and thus $V := \{h \in G \mid h(K) \cap K \neq \emptyset\}$ is compact. But V contains the ball

$$B(\text{id}, \varepsilon) := \{h \in G \mid d(h(x), x) < \varepsilon \ \forall x \in M\}.$$

Indeed, if $h \in B(\text{id}, \varepsilon)$, then $(h(x_1), h(x_2), h(x_3)) \in h(K) \cap K$. Therefore, V is a compact neighbourhood of the identity of G . $\square$

The notion of convergence group appears in the theory of trees.

Lemma 4.2. *Let T be a locally finite tree. The automorphism group $\text{Aut}(T)$ of T with the topology of pointwise convergence acts as a convergence group on the space of ends T_∞ of T .*

Proof. The action of $\text{Aut}(T)$ on the set of vertices of T (endowed with the discrete topology) is proper, since T is locally finite. The result then follows from the fact that one can canonically associate a vertex of T to each element of $T_\infty^{(3)}$. $\square$

The reason why we are interested in convergence groups is the existence of the following reciprocal result, due to Carrière and Dreesen.

Theorem 4.3 (Carrière–Dreesen). *Let G be a σ -compact, locally compact, non-compact group acting continuously and transitively as a convergence group on an infinite compact totally disconnected space M . Then G acts continuously, properly and faithfully on some locally finite tree T and the spaces M and T_∞ are equivariantly homeomorphic.*

Proof. This is a particular case of [CD14, Theorem D]. $\square$

4.2 Group of projectivities

If π and π' are two opposite panels of a compact spherical building Δ , then the projection $\text{proj}_\pi|_{\pi'}: \text{Cham}(\pi') \rightarrow \text{Cham}(\pi)$ is a homeomorphism (see Corollary 2.3). A map of the form

$$[\pi_0, \pi_1, \dots, \pi_n] := \text{proj}_{\pi_n} \circ \text{proj}_{\pi_{n-1}} \circ \dots \circ \text{proj}_{\pi_1}|_{\pi_0}: \text{Cham}(\pi_0) \rightarrow \text{Cham}(\pi_n)$$

where $\pi_0, \pi_1, \dots, \pi_n$ is a sequence of panels with π_i opposite π_{i-1} for each $i \in \{1, \dots, n\}$ is called a **projectivity** from π_0 to π_n . The **group of projectivities** $\Pi(\pi)$ associated to a panel π is the group of all projectivities from π to π (seen as a subgroup of $\text{Homeo}(\text{Cham}(\pi))$, so that two projectivities acting in the same way on $\text{Cham}(\pi)$ are considered equal).

Now let X be a locally finite thick affine building of rank at least 3 and of irreducible type. It is well known (see [GKVMW12, Proposition 6.4]) that the spherical building X_∞ at infinity of X can be given a structure of totally disconnected compact building. Also, one can construct for each panel π of X_∞ a locally finite thick tree $T(\pi)$, often called *panel-tree*, whose space of ends $T_\infty(\pi)$ (we should actually write $T(\pi)_\infty$) is in bijective correspondence with $\text{Cham}(\pi)$ (see [Ron89, Lemma 10.4] or [Wei09, Proposition 11.22]). This bijection $\psi: T_\infty(\pi) \rightarrow \text{Cham}(\pi)$ is a homeomorphism if we equip $T_\infty(\pi)$ with the usual topology. There also is a natural action of the group of projectivities $\Pi(\pi)$ on $T(\pi)$ and ψ is equivariant under $\Pi(\pi)$. These remarks are summarized in the following proposition.

Proposition 4.4. *Let X be a locally finite thick irreducible affine building of rank at least 3. For each panel π of X_∞ , there exist a locally finite thick tree $T(\pi)$ such that the group of projectivities $\Pi(\pi)$ acts on $T(\pi)$ and a homeomorphism $\psi : T_\infty(\pi) \rightarrow \text{Cham}(\pi)$ equivariant under $\Pi(\pi)$.*

Proof. See the results mentioned in the previous discussion. $\square$

We can now prove the implication (i) $\Rightarrow$ (ii) in Theorem A.

Proposition 4.5. *Let X be a locally finite thick irreducible affine building of rank at least 3. For each panel π of X_∞ , the closure of the group of projectivities $\Pi(\pi)$ in $\text{Homeo}(\text{Cham}(\pi))$ acts as a convergence group on $\text{Cham}(\pi)$.*

Proof. In view of Proposition 4.4, given a panel π of X_∞ there is a tree $T(\pi)$ such that $\Pi(\pi)$ acts on $T(\pi)$ and the spaces $T_\infty(\pi)$ and $\text{Cham}(\pi)$ are equivariantly homeomorphic. We can therefore see $\Pi(\pi)$ as a subgroup of $\text{Aut}(T(\pi))$, and the closure $\overline{\Pi}(\pi)$ of $\Pi(\pi)$ in $\text{Homeo}(T_\infty(\pi))$ remains in $\text{Aut}(T(\pi))$ since $\text{Aut}(T(\pi))$ is closed in $\text{Homeo}(T_\infty(\pi))$. Lemma 4.2 then states that $\text{Aut}(T(\pi))$ acts as a convergence group on $T_\infty(\pi)$. Since the restriction to a closed subgroup of an action as a convergence group is itself an action as a convergence group, this is also the case of $\overline{\Pi}(\pi)$. Recalling that $T_\infty(\pi)$ and $\text{Cham}(\pi)$ are equivariantly homeomorphic, we see that $\overline{\Pi}(\pi)$ acts as a convergence group on $\text{Cham}(\pi)$. $\square$

Corollary 4.6 (Theorem A, (i) $\Rightarrow$ (ii)). *Let Δ be an infinite thick compact totally disconnected m -gon with $m \geq 3$. If Δ is Moufang, then for each panel π of Δ the closure in $\text{Homeo}(\text{Cham}(\pi))$ of the group of projectivities $\Pi(\pi)$ acts as a convergence group on $\text{Cham}(\pi)$.*

Proof. By [GKVMW12, Theorem 1.1], Δ is the building at infinity of some locally finite thick irreducible affine building X of rank 3, i.e. $\Delta \cong X_\infty$. The result then follows from Proposition 4.5. $\square$

To prove the implication (ii) $\Rightarrow$ (iii) in Theorem A, we need some basic observations about groups acting on a locally finite thick tree T so that the induced action on T_∞ is 2-transitive. We use a result of Tits to prove the following lemma, but the reader should be aware that it can also be shown without using such a deep result.

Lemma 4.7. *Let T be a locally finite thick tree and let $G \leq \text{Aut}(T)$. If G acts 2-transitively on T_∞ , then there exists a hyperbolic element in G , i.e. some $g \in G$ that stabilizes a bi-infinite geodesic ℓ of T and acts non-trivially on ℓ by translation.*

Proof. Assume for a contradiction that G does not contain any hyperbolic element. Then by [Tit70, Proposition 3.4], G is contained in the stabilizer of a vertex, an edge or an end of T . However, none of these three situations is possible since G acts 2-transitively on T_∞ . $\square$

Lemma 4.8. *Let T be a locally finite thick tree and let $G \leq \text{Aut}(T)$. Consider an infinite ray $r = (v_i)_{i \geq 1}$ in T and let v_0 and v'_0 be two vertices of T adjacent to v_1 and different from v_2 . If G acts 2-transitively on T_∞ , then there exists $g \in G$ such that $g(r) = r$ and $g(v_0) = v'_0$.*

Proof. Consider a bi-infinite geodesic ℓ of T containing r and v_0 . By Lemma 4.7, there exists a hyperbolic element in G , and by 2-transitivity of G there even is an element $t \in G$ stabilizing ℓ and acting on it non-trivially by translation. Replacing t by t^{-1} if necessary, we can assume that $t(v_0) = v_m$ for some $m > 0$. Now let ℓ' be a bi-infinite geodesic containing r and v'_0 . By 2-transitivity of G , we can conjugate t by an adequate element of G to find $t' \in G$ stabilizing ℓ' and such that $t'(v'_0) = v_m$. Then $t'^{-1}t \in G$ fixes r and sends v_0 on v'_0 . $\square$

The implication (ii) $\Rightarrow$ (iii) in Theorem A can now be shown.

Proposition 4.9 (Theorem A, (ii) $\Rightarrow$ (iii)). *Let Δ be an infinite thick compact totally disconnected m -gon with $m \geq 3$ and let π be a panel of Δ . Suppose that the closure of the group of projectivities $\Pi(\pi)$ in $\text{Homeo}(\text{Cham}(\pi))$ acts as a convergence group on $\text{Cham}(\pi)$. Then for all $G \leq \text{Auttop}(\Delta)$, the closure of the natural image of $\text{Stab}_G(\pi)$ in $\text{Homeo}(\text{Cham}(\pi))$ acts as a convergence group on $\text{Cham}(\pi)$.*

Proof. By [Kna88, Lemma 1.2], the action of $\Pi(\pi)$ on $\text{Cham}(\pi)$ is 2-transitive. Now suppose that the closure $\overline{\Pi}(\pi)$ of $\Pi(\pi)$ in $\text{Homeo}(\text{Cham}(\pi))$ acts as a convergence group on $\text{Cham}(\pi)$.

Claim 1. *The group $\overline{\Pi}(\pi)$ acts on a locally finite thick tree $T(\pi)$ such that $T_\infty(\pi)$ and $\text{Cham}(\pi)$ are equivariantly homeomorphic.*

Proof of the claim: We want to apply Theorem 4.3 and thus check its hypotheses. The group $\overline{\Pi}(\pi)$ is locally compact (Lemma 4.1), σ -compact (it is second-countable by [Are46, Theorem 5] and a locally compact second-countable space is always σ -compact), $\text{Cham}(\pi)$ is infinite (Lemma 2.8), compact and totally disconnected, and the action of $\overline{\Pi}(\pi)$ on $\text{Cham}(\pi)$ is continuous and transitive (even 2-transitive). It only remains to show that $\overline{\Pi}(\pi)$ is non-compact. By contradiction, suppose that $\Pi(\pi)$ is relatively compact in $\text{Homeo}(\text{Cham}(\pi))$. It is therefore also relatively compact in $C(\text{Cham}(\pi), \text{Cham}(\pi))$ and, by the Arzela-Ascoli theorem, $\Pi(\pi)$ is equicontinuous on $\text{Cham}(\pi)$. This is a contradiction with the fact that the action of $\Pi(\pi)$ on $\text{Cham}(\pi)$ is 2-transitive. By Theorem 4.3, we therefore have a locally finite tree $T(\pi)$ as wanted. This tree may be not thick, but we can actually assume it is. Indeed, the vertices of degree 1 can clearly be deleted, as well as the vertices of degree 2 by joining their two neighbours. Note that there is no infinite ray in $T(\pi)$ only containing vertices of degree 2 (which would prevent the deletion of these vertices) since this would imply the existence of an isolated chamber in $\text{Cham}(\pi)$, which is impossible in view of Lemma 2.8. ■

The next claim shows that the structure of $T(\pi)$ is encoded in the group structure of $\overline{\Pi}(\pi)$. For any vertex v of $T(\pi)$, we write S_v for the stabilizer of v in $\overline{\Pi}(\pi)$.

Claim 2. *The groups S_v are pairwise distinct and are exactly the maximal compact subgroups K of $\overline{\Pi}(\pi)$ such that $[K : (K \cap H)] \geq 3$ for any other maximal compact subgroup H of $\overline{\Pi}(\pi)$. Moreover, two distinct vertices v and v' of $T(\pi)$ are adjacent if and only if $[S_v : (S_v \cap S_{v'})] \leq [S_v : (S_v \cap S_w)]$ for any vertex w different from v .*

Proof of the claim: In view of Lemma 4.8, the groups S_v are pairwise distinct. Moreover, they are maximal compact subgroups of $\overline{\Pi}(\pi)$ and the only other maximal compact subgroups are the stabilizers of those edges e for which there is an element of $\overline{\Pi}(\pi)$ inverting e (see [FTN91, Theorem 5.2]). If S_e denotes the stabilizer in $\overline{\Pi}(\pi)$ of such an edge $e = [v_1, v_2]$, then $[S_e : (S_e \cap S_{v_1})] = 2$. On the other hand, if v is a vertex of $T(\pi)$ then by Lemma 4.8 and since $T(\pi)$ is thick we have $[S_v : (S_v \cap S_{v'})] \geq 3$ for v' a vertex different from v and $[S_v : (S_v \cap S_e)] \geq 3$ for e an edge of $T(\pi)$. The first part of the claim is therefore proven.

Now fix a vertex v of T . By Lemma 4.8, for any other vertex w the equality

$$[S_v : (S_v \cap S_w)] = \deg(v) \cdot \prod_{i=1}^{n-1} (\deg(v_i) - 1)$$

holds, where $v, v_1, \dots, v_{n-1}, w$ is the path in $T(\pi)$ joining v and w (this formula can be obtained by induction on n). The second part of the claim follows from this observation. ■

Claim 3. *The stabilizer $\text{Stab}_{\text{Auttop}(\Delta)}(\pi)$ of π in $\text{Auttop}(\Delta)$ acts on $T(\pi)$ and the homeomorphism between $T_\infty(\pi)$ and $\text{Cham}(\pi)$ is equivariant under $\text{Stab}_{\text{Auttop}(\Delta)}(\pi)$.*

Proof of the claim: First remark that $\text{Stab}_{\text{Auttop}(\Delta)}(\pi)$ normalizes $\Pi(\pi)$ in $\text{Homeo}(\text{Cham}(\pi))$. Indeed, if $[\pi, \pi_1, \dots, \pi_{n-1}, \pi]$ is a projectivity and $g \in \text{Stab}_{\text{Auttop}(\Delta)}(\pi)$, one can check that

$$g \circ [\pi, \pi_1, \dots, \pi_{n-1}, \pi] \circ g^{-1} = [\pi, g(\pi_1), \dots, g(\pi_{n-1}), \pi].$$

In view of Claim 2, the action by conjugation of $\text{Stab}_{\text{Auttop}(\Delta)}(\pi)$ on $\Pi(\pi)$ induces an action of $\text{Stab}_{\text{Auttop}(\Delta)}(\pi)$ on $T(\pi)$. It remains to show that the homeomorphism $\psi : T_\infty(\pi) \rightarrow \text{Cham}(\pi)$ given by Claim 1 is equivariant under $\text{Stab}_{\text{Auttop}(\Delta)}(\pi)$. Consider $g \in \text{Stab}_{\text{Auttop}(\Delta)}(\pi)$, $C \in \text{Cham}(\pi)$ and $(v_i)_{i \geq 1}$ a ray in $T(\pi)$ representing the end corresponding to C . We want to show that the ray $(g(v_i))_{i \geq 1}$ represents the end corresponding to $g(C)$. To do so, remark that

$$\bigcup_{k=1}^{\infty} \bigcap_{i=k}^{\infty} S_{v_i}$$

fixes C but does not fix any chamber different from C (this follows from Lemma 4.8). Now we also have

$$g \left(\bigcup_{k=1}^{\infty} \bigcap_{i=k}^{\infty} S_{v_i} \right) g^{-1} = \bigcup_{k=1}^{\infty} \bigcap_{i=k}^{\infty} S_{g(v_i)} =: R$$

by definition of the action of $\text{Stab}_{\text{Auttop}(\Delta)}(\pi)$ on $T(\pi)$. But R fixes $g(C)$ and also fixes the chamber corresponding to the end represented by the ray $(g(v_i))_{i \geq 1}$. Since R can only fix one chamber, the claim stands proven. $\blacksquare$

In view of Claim 3 and since $\text{Aut}(T(\pi))$ is closed in $\text{Homeo}(T_\infty(\pi))$, for any $G \leq \text{Auttop}(\Delta)$ the closure of the image of $\text{Stab}_G(\pi)$ in $\text{Homeo}(\text{Cham}(\pi))$ can be seen as a (closed) subgroup of $\text{Aut}(T(\pi))$. The conclusion thus follows from Lemma 4.2, as the restriction to a closed subgroup of an action as a convergence group is itself an action as a convergence group. $\square$

4.3 Conventions for the rest of the section

The rest of this section is dedicated to the proof of the implication (iii) $\Rightarrow$ (i) in Theorem A. Unless otherwise stated, we will from now on study an infinite thick compact totally disconnected m -gon Δ with $m \geq 3$. We will also assume that we are in presence of a strongly transitive closed subgroup G of $\text{Auttop}(\Delta)$ such that Δ satisfies the *property* $\mathcal{CG}_G$:

Definition 4.10. Let Δ be a compact polygon and let G be a subgroup of $\text{Auttop}(\Delta)$. We will say that Δ has the **property** $\mathcal{CG}_G$ if for each vertex (i.e. panel) v of Δ , the closure of the image of $\text{Stab}_G(v)$ in $\text{Homeo}(\text{Cham}(v))$ acts as a convergence group on $\text{Cham}(v)$.

Our ultimate goal is to prove that, under these conditions, Δ has the Moufang property. We can already directly use Theorem 4.3 to construct a tree associated to each vertex of Δ .

Proposition 4.11. *Let Δ be an infinite thick compact totally disconnected m -gon with $m \geq 3$ and let G be a strongly transitive subgroup of $\text{Auttop}(\Delta)$ satisfying property $\mathcal{CG}_G$. Then for each vertex v of Δ there exists a locally finite thick tree $T(v)$ such that $\text{Stab}_G(v)$ acts on $T(v)$ and the spaces $T_\infty(v)$ and $\text{Cham}(v)$ are equivariantly homeomorphic.*

Proof. The property $\mathcal{CG}_G$ tells us that the group $\overline{i(\text{Stab}_G(v))}$ where i is the natural map from $\text{Stab}_G(v)$ to $\text{Homeo}(\text{Cham}(v))$ acts as a convergence group on $\text{Cham}(v)$. All the hypotheses of Theorem 4.3 are then satisfied. Indeed, $\overline{i(\text{Stab}_G(v))}$ is locally compact (Lemma 4.1) and σ -compact (it is second-countable by [Are46, Theorem 5] and a locally compact second-countable space is always σ -compact), $\text{Cham}(v)$ is infinite (Lemma 2.8), compact and totally disconnected, and the action of $\overline{i(\text{Stab}_G(v))}$ on $\text{Cham}(v)$ is continuous and transitive (even 2-transitive, see Lemma 2.9). It only remains to show that $\overline{i(\text{Stab}_G(v))}$ is non-compact. Suppose for a contradiction that $i(\text{Stab}_G(v))$ is relatively compact in $\text{Homeo}(\text{Cham}(v))$. It is therefore also relatively compact in $C(\text{Cham}(v), \text{Cham}(v))$ and, by the Arzela-Ascoli theorem, $i(\text{Stab}_G(v))$ is equicontinuous on $\text{Cham}(v)$. This is a contradiction with the fact that the action of $i(\text{Stab}_G(v))$ on $\text{Cham}(v)$ is 2-transitive.

Hence, there is a locally finite tree $T(v)$ such that $\overline{i(\text{Stab}_G(v))}$ acts on $T(v)$ and so that $T_\infty(v)$ and $\text{Cham}(v)$ are equivariantly homeomorphic. The group $\text{Stab}_G(v)$ obviously acts on $T(v)$ too, and the homeomorphism remains equivariant under $\text{Stab}_G(v)$. We can also assume that $T(v)$ is thick since the vertices of degree ≤ 2 can be deleted (for those of degree 2, the two neighbours have to be joined by an edge). Note that there is no infinite ray only containing vertices of degree 2 since it would contradict Lemma 2.8. $\square$

4.4 Automorphisms fixing an apartment

In this subsection, we focus on the fixator $H = \text{Fix}_G(A)$ of an apartment A of Δ (where Δ and G are as described in Subsection 4.3). By Proposition 4.11, for each vertex v of A there exists a tree $T(v)$ whose space of ends corresponds to $\text{Cham}(v)$. The group H acts simultaneously on all these trees. Moreover, for each vertex v of A , H fixes the two chambers of A having vertex v , which means that it fixes the two corresponding ends of $T(v)$. In this tree, H thus stabilizes the bi-infinite geodesic between these two ends, which we denote by ℓ_v . We therefore have for each vertex v of A an induced action of H on ℓ_v .

Our goal is now to show that for every vertex v of A , there exists some element of H fixing ℓ_v and $\ell_{v'}$ pointwise where v' is the vertex of A opposite v but acting non-trivially on every other ℓ_w . We will actively use this result in the last subsection to prove that Δ is Moufang.

As a first step in this direction, we show that for each vertex v of A , there is an element of H that acts non-trivially on ℓ_v .

Lemma 4.12. *Let Δ be an infinite thick compact totally disconnected m -gon with $m \geq 3$ and let G be a strongly transitive subgroup of $\text{Auttop}(\Delta)$ satisfying property $\mathcal{CG}_G$. Let A be an apartment of Δ , v be a vertex of A and $H = \text{Fix}_G(A)$. There is an element of H that acts non-trivially on ℓ_v .*

Proof. Let v' be the vertex of A opposite v and let $L = \text{Stab}_G(v, v')$. Since G is strongly transitive, the natural image $\tilde{L}$ of L in $\text{Aut}(T(v))$ acts 2-transitively on $T_\infty(v)$ (see Lemma 2.9). By Lemma 4.7, there is a hyperbolic element in $\tilde{L}$ and hence an element that stabilizes ℓ_v but does not fix it pointwise. It corresponds to an element of H that does not fix ℓ_v pointwise. $\square$

To go further in the discussion, we denote by $v_0, \dots, v_{2m-1}$ the vertices of A in the natural order and distinguish three types of possible actions of an element $t \in H$ around a vertex v_i :

- (a) If t fixes ℓ_{v_i} pointwise, we will say that t acts **trivially around** v_i .
- (b) If t translates ℓ_{v_i} toward the end representing the chamber of A having vertex v_{i+1} , we will say that t acts **positively around** v_i .

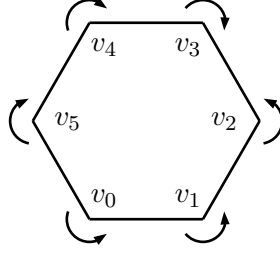


Figure 9: Representation of an action.

- (c) If t translates ℓ_{v_i} toward the end representing the chamber of A having vertex v_{i-1} , we will say that t acts **negatively around** v_i .

An example of such an action of an element $t \in H$ on A is shown in Figure 9. An arrow in the anticlockwise direction means that the action is positive while an arrow in the clockwise direction means that it is negative. In this figure, the action is positive around v_0, v_1 and v_2 and negative around v_3, v_4 and v_5 .

The three types of action can easily be spotted by observing how t^n acts on Δ when n goes to infinity. Indeed, denoting by C_+ the chamber having vertices v_i and v_{i+1} , by C_- the chamber having vertices v_i and v_{i-1} , and by C any chamber having vertex v_i different from C_+ and C_- , we get the following characterizations.

- (a) t acts trivially around $v_i \Leftrightarrow t^n(C)$ does not accumulate at C_+ nor at C_- .
- (b) t acts positively around $v_i \Leftrightarrow t^n(C) \rightarrow C_+$.
- (c) t acts negatively around $v_i \Leftrightarrow t^n(C) \rightarrow C_-$.

These characterizations already allow us to prove the following lemma giving a relationship between the action of an element t around v_i and its action around v_{i+m} , the opposite vertex in A . The action represented in Figure 9 is coherent with this information.

Lemma 4.13. *The action of an element $t \in H$ around a vertex v_i of A is opposite to its action around v_{i+m} : t either acts trivially around the two vertices, or it acts positively around one and negatively around the other.*

Proof. Let $C \notin A$ be a chamber having vertex v_i . Then $\text{proj}_{v_{i+m}}(C)$ is a chamber having vertex v_{i+m} and $t^n(\text{proj}_{v_{i+m}}(C)) = \text{proj}_{v_{i+m}}(t^n(C))$. By Proposition 2.2, the action of t around v_i is opposite to its action around v_{i+m} . $\square$

We can now prove that H always contains some particular elements.

Proposition 4.14. *Let Δ be an infinite thick compact totally disconnected m -gon with $m \geq 3$ and let G be a strongly transitive subgroup of $\text{Auttop}(\Delta)$ satisfying property $\mathcal{CG}_G$. Let A be an apartment of Δ , $v_0, \dots, v_{2m-1}$ be the vertices of A and $H = \text{Fix}_G(A)$. For each $i \in \{0, \dots, 2m-1\}$, there is an element of H that acts positively around all vertices of A closer to v_i than to v_{i+1} and negatively around all vertices of A closer to v_{i+1} than to v_i (indices being taken modulo $2m$).*

Proof. We divide the proof into three claims.

Claim 1. *There exists an element of H that acts positively around v_s and negatively around v_{s+1} for some $s \in \{0, \dots, 2m-1\}$.*

Proof of the claim: In view of Lemma 4.12, there is an element of H acting positively around at least one vertex of A . We consider $t \in H$ an element acting positively around a maximal number of consecutive vertices of A , and denote by $v_r, v_{r+1}, \dots, v_s$ these vertices. Note that t does not act positively around all vertices of A in view of Lemma 4.13. Hence, t acts negatively or trivially around v_{s+1} . Suppose that t acts trivially around v_{s+1} . By Lemma 4.12, there is $t' \in H$ acting non-trivially around v_{s+1} . Replacing t' by t'^{-1} if necessary, we can assume that t' acts positively around v_{s+1} . Then for sufficiently large $n \in \mathbf{N}$, $t^n t' \in H$ acts positively around $v_r, \dots, v_s, v_{s+1}$ which contradicts the maximality of t . This means that t acts negatively around v_{s+1} , and the claim stands proven. ■

Claim 2. *For each $i \in \{0, \dots, 2m-1\}$, there exists an element of H that acts positively around v_i and negatively around v_{i+1} .*

Proof of the claim: By Claim 1, there is $t \in H$ acting positively around v_s and negatively around v_{s+1} for some $s \in \{0, \dots, 2m-1\}$. Now take $i \in \{0, \dots, 2m-1\}$. Since G is strongly transitive, there is an element $g \in H$ sending the chamber with vertices v_s and v_{s+1} to the chamber having vertices v_i and v_{i+1} . It can be directly checked that $gtg^{-1} \in H$ acts positively around v_i and negatively around v_{i+1} . ■

Claim 3. *Let $s \in \{0, \dots, 2m-1\}$ and let $t \in H$ be an element acting positively around v_s and negatively around v_{s+1} . Then t acts positively around all vertices of A closer to v_s than to v_{s+1} and negatively around all vertices of A closer to v_{s+1} than to v_s .*

Proof of the claim: By Lemma 4.13, t acts negatively around v_{s+m} and positively around v_{s+m+1} (see Figure 10). We now want to show that t acts negatively around each vertex between v_{s+1} and v_{s+m} and positively around all other vertices. We therefore consider some vertex v_i with $s+1 < i < s+m$ and show that the action around v_i is negative (and it

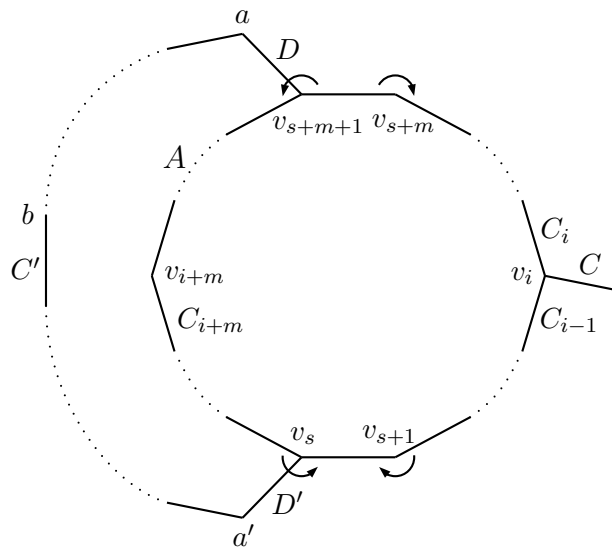


Figure 10: Illustration of Claim 3.

will follow that the action around v_{i+m} is positive). Denoting by C_j the chamber having vertices v_j and v_{j+1} for $j \in \{0, \dots, 2m-1\}$, it suffices to find some chamber $C \notin A$ having vertex v_i and such that $t^n(C) \rightarrow C_{i-1}$. To get one, consider a chamber $D \notin A$ having vertex v_{s+m+1} and a chamber $D' \notin A$ having vertex v_s . Let also a (resp. a') be the vertex of D (resp. D') different from v_{s+m+1} (resp. v_s). The vertices a and a' are almost opposite while $t^n(a) \rightarrow v_{s+m+2}$ and $t^n(a') \rightarrow v_{s+1}$ with v_{s+m+2} and v_{s+1} almost opposite. Hence, $t^n([a, a']) \rightarrow [v_{s+m+2}, v_{s+1}]$ (Lemma 2.6). Now consider b the first vertex of $[a, a']$ opposite v_i and C' the chamber just after b in this new gallery from v_{s+m+1} to v_s (see Figure 10). Clearly $t^n(b) \rightarrow v_{i+m}$ and $t^n(C') \rightarrow C_{i+m}$. Define finally $C = \text{proj}_{v_i}(C')$ to get $t^n(C) \rightarrow C_{i-1}$ as wanted (see Proposition 2.2). Note that $C \neq C_{i-1}$ since the vertex of C' different from b is opposite v_{i-1} . $\blacksquare$

The proposition follows from Claim 2 and Claim 3. $\square$

The desired result can finally be shown.

Proposition 4.15. *Let Δ be an infinite thick compact totally disconnected m -gon with $m \geq 3$ and let G be a strongly transitive subgroup of $\text{Auttop}(\Delta)$ satisfying property $\mathcal{CG}_G$. Let A be an apartment of Δ , $v_0, \dots, v_{2m-1}$ be the vertices of A and $H = \text{Fix}_G(A)$. For each $i \in \{0, \dots, m-1\}$, there is an element of H that acts trivially around v_i and v_{i+m} but non-trivially around all other vertices of A .*

Proof. Fix $i \in \{0, \dots, m-1\}$. By Proposition 4.14, there exists $t_1 \in H$ acting positively around all vertices of A closer to v_{i-1} than to v_i and negatively around all vertices of A closer to v_i than to v_{i-1} . There also exists $t_2 \in H$ acting positively around all vertices of A closer to v_i than to v_{i+1} and negatively around all vertices of A closer to v_{i+1} than to v_i . Denote by $T_1 \in \mathbf{N}^*$ (resp. $T_2 \in \mathbf{N}^*$) the length of the translation of ℓ_{v_i} induced by t_1 (resp. t_2). Now consider the element $t = t_1^{T_2} t_2^{T_1} \in H$. Clearly, t acts trivially around v_i (and hence around v_{i+m}) and non-trivially around all other vertices of A . $\square$

4.5 Proof of the characterization of the Moufang property

The proof of implication (iii) $\Rightarrow$ (i) in Theorem A will use the theory of contraction groups. If G is a topological group and $t \in G$, the **contraction group** associated to t is the subgroup of G defined by

$$\text{Con}_G(t) = \{g \in G \mid t^n g t^{-n} \rightarrow e\}$$

where e is the identity element of G . The **parabolic subgroup** associated to t is the subgroup of G defined by

$$\text{Par}_G(t) = \{g \in G \mid \{t^n g t^{-n}\}_{n \in \mathbf{N}} \text{ is relatively compact in } G\}.$$

The following lemma is immediate.

Lemma 4.16. *Let $f : G \rightarrow H$ be a continuous homomorphism of topological groups and let $t \in G$. We have the following inclusions.*

$$f(\text{Con}_G(t)) \subseteq \text{Con}_H(f(t)) \quad \text{and} \quad f(\text{Par}_G(t)) \subseteq \text{Par}_H(f(t))$$

Proof. Using the continuity of f , we directly get $f(\text{Con}_G(t)) \subseteq \text{Con}_H(f(t))$. Similarly, the inclusion $f(\text{Par}_G(t)) \subseteq \text{Par}_H(f(t))$ follows from the fact that a continuous image of a compact set is compact. $\square$

In the case where G is the automorphism group of a tree, we can compute the contraction groups and the parabolic subgroups associated to some particular elements.

Lemma 4.17. *Let T be a locally finite tree and let t be an element of $\text{Aut}(T)$ stabilizing a bi-infinite geodesic (a, b) in T (where $a, b \in T_\infty$).*

- (i) *If t fixes (a, b) pointwise, then $\text{Con}_{\text{Aut}(T)}(t) = \{e\}$.*
- (ii) *If t translates (a, b) toward b , then $\text{Par}_{\text{Aut}(T)}(t) = \text{Fix}_{\text{Aut}(T)}(a)$.*

Proof. Point (i) directly comes from the fact that $\{t^n\}_{n \in \mathbf{N}}$ is relatively compact in $\text{Aut}(T)$ when t fixes (a, b) pointwise. The proof of (ii) is given in [CDM13, Lemma 2.3]. $\square$

Finally, the following theorem is due to Baumgartner and Willis.

Theorem 4.18 (Baumgartner–Willis). *Let G be a totally disconnected locally compact group and let $t \in G$. Then*

$$\text{Par}_G(t) = \text{Con}_G(t) \cdot (\text{Par}_G(t) \cap \text{Par}_G(t^{-1})).$$

Proof. See [BW04, Corollary 3.17]. $\square$

We are now able to prove the implication (iii) $\Rightarrow$ (i) in Theorem A.

Theorem 4.19 (Theorem A, (iii) $\Rightarrow$ (i)). *Let Δ be an infinite thick compact totally disconnected m -gon with $m \geq 3$ and let G be a closed strongly transitive subgroup of $\text{Autop}(\Delta)$ satisfying property $\mathcal{CG}_G$. Then Δ is Moufang.*

Proof. Fix some root α of Δ and denote by $v_0, v_1, \dots, v_m$ its vertices in the natural order. We want to prove that the root group

$$U_\alpha = \{g \in \text{Aut}(\Delta) \mid g \text{ fixes every chamber having a panel in } \alpha \setminus \partial\alpha\}$$

acts transitively on the set of apartments containing α . This is equivalent to saying that U_α acts transitively on $\text{Cham}(v_0) \setminus \{C\}$ where C is the chamber having vertices v_0 and v_1 . First remark that $V_0 := \text{Fix}_G(\alpha)$ acts transitively on $\text{Cham}(v_0) \setminus \{C\}$ since G is strongly transitive (see Lemma 2.9). We will therefore proceed by induction, by defining successively

$$\begin{aligned} V_1 &= \text{Ker}(V_0 \curvearrowright \text{Cham}(v_1)), \\ V_2 &= \text{Ker}(V_1 \curvearrowright \text{Cham}(v_2)), \\ &\vdots \\ V_{m-1} &= \text{Ker}(V_{m-2} \curvearrowright \text{Cham}(v_{m-1})). \end{aligned}$$

In this way, V_1 is the fixator in G of α and every chamber having vertex v_1 , V_2 is the fixator in G of α and every chamber having vertex v_1 or v_2 , and so on until V_{m-1} which is exactly $U_\alpha \cap G$. The action of V_0 on $\text{Cham}(v_0) \setminus \{C\}$ is transitive and we would like to prove that the action of V_{m-1} on this set remains transitive. It thus suffices to show that if the action of V_i is transitive, then the action of V_{i+1} is transitive too.

Suppose that the action of V_i on $\text{Cham}(v_0) \setminus \{C\}$ is transitive for some $i \in \{0, \dots, m-2\}$. Consider A any apartment containing α and $H = \text{Fix}_G(A)$. By Proposition 4.15, there exists $t \in H$ fixing $\ell_{v_{i+1}}$ pointwise but not ℓ_{v_j} for $j \in \{0, \dots, i\}$. Let a be the end of $T(v_0)$

corresponding to the chamber C and b be the end of $T(v_0)$ corresponding to the other chamber of A having vertex v_0 . Replacing t by t^{-1} if necessary, we can assume that t acts negatively around v_0 , i.e. that it translates $\ell_{v_0} = (a, b)$ toward b . Now consider W_i the semidirect product in G of V_i and $\langle t \rangle$ (this is well defined as H normalizes V_i). To prove that the action of V_{i+1} on $\text{Cham}(v_0) \setminus \{C\}$ remains transitive, we show that $\text{Con}_{W_i}(t)$ is included in V_{i+1} and that its action on the set remains transitive.

Claim 1. *The equality $\text{Par}_{W_i}(t) = W_i$ holds.*

Proof of the claim: We prove the claim by contradiction. Suppose that there is some element $g \in W_i$ such that $\{t^n g t^{-n}\}_{n \geq 0}$ is not relatively compact in W_i . Since W_i is closed in $\text{Auttop}(\Delta)$, it is not relatively compact in $\text{Auttop}(\Delta)$ either. By Proposition 3.3 with $S = \{t^n g t^{-n}\}_{n \geq 0}$, there exist two sequences in $\text{Vert } \Delta$ which are collapsed by a subsequence of $(t^n g t^{-n})$ or $(t^n g^{-1} t^{-n})$. Replacing g by g^{-1} if necessary, we can assume that it is a subsequence of $(t^n g t^{-n})$. We would then like to use Proposition 3.7 with the sequence of apartments $A \rightarrow A$. Let $v_{m+1}, \dots, v_{2m-1}$ be the vertices of A outside α (in the natural order) and D the chamber having vertices v_0 and v_{2m-1} . Passing to a subsequence, we can assume that $t^n g t^{-n} D \rightarrow \tilde{D}$ for some chamber $\tilde{D}$. If $\tilde{D} \neq C$, then $t^n g t^{-n} A \rightarrow \tilde{A}$ where $\tilde{A}$ is the unique apartment containing α and $\tilde{D}$ and Proposition 3.7 gives two sequences centered at $v_k \rightarrow v_k$ for some $k \in \{0, \dots, 2m-1\}$ and collapsed by a subsequence of $(t^n g t^{-n})$. By Lemma 3.6, we can assume that $k \in \{0, \dots, m-1\}$. On the other hand, if $\tilde{D} = C$, then $v_1 \rightarrow v_1$ and $v_{2m-1} \rightarrow v_{2m-1}$ are sequences centered at $v_0 \rightarrow v_0$ and collapsed by $(t^n g t^{-n})$.

In either case, there are two sequences centered at $v_k \rightarrow v_k$ for some $k \in \{0, \dots, m-1\}$ and collapsed by a subsequence of $(t^n g t^{-n})$. We introduce the notation (for $j \in \{0, \dots, m\}$)

$$\varphi_j : W_i \rightarrow \text{Aut}(T(v_j))$$

for the natural map from $W_i \subseteq \text{Stab}_{\text{Auttop}(\Delta)}(v_j)$ to $\text{Aut}(T(v_j))$. For each $j \in \{1, \dots, m-1\}$, since $t^n g t^{-n}$ translates ℓ_{v_j} exactly as g does for all $n \in \mathbb{N}$, the set $\{\varphi_j(t^n g t^{-n})\}_{n \geq 0}$ is relatively compact in $\text{Aut}(T(v_j))$. It is also relatively compact in $\text{Homeo}(T_\infty(v_j))$ and hence equicontinuous on $T_\infty(v_j)$ by the Arzela-Ascoli theorem. It is therefore impossible to have two sequences centered at $v_j \rightarrow v_j$ and collapsed by a subsequence of $(t^n g t^{-n})$ for $j \in \{1, \dots, m-1\}$. But it is not possible either to have them centered at $v_0 \rightarrow v_0$. Indeed, the existence of two such sequences would imply that $\varphi_0(g) \notin \text{Par}_{\text{Aut}(T(v_0))}(\varphi_0(t)) = \text{Fix}_{\text{Aut}(T(v_0))}(a)$ (see Lemma 4.17 (ii)) which is impossible since $g \in W_i$ fixes a . The claim is therefore proven. ■

Claim 2. *$\text{Con}_{W_i}(t)$ acts transitively on $\text{Cham}(v_0) \setminus \{C\}$.*

Proof of the claim: Applying Theorem 4.18 to $t \in W_i$ (note that W_i is locally compact as a closed subgroup of $\text{Auttop}(\Delta)$ which is locally compact (Corollary E) and totally disconnected since it acts continuously and faithfully on the totally disconnected space $\text{Cham } \Delta$), we get

$$\text{Par}_{W_i}(t) = \text{Con}_{W_i}(t) \cdot (\text{Par}_{W_i}(t) \cap \text{Par}_{W_i}(t^{-1})).$$

Now by Lemma 4.16,

$$\varphi_0(\text{Par}_{W_i}(t)) \subseteq \varphi_0(\text{Con}_{W_i}(t)) \cdot (\text{Par}_{\text{Aut}(T(v_0))}(\varphi_0(t)) \cap \text{Par}_{\text{Aut}(T(v_0))}(\varphi_0(t^{-1}))).$$

Claim 1 together with Lemma 4.17 (ii) finally gives

$$\varphi_0(W_i) \subseteq \varphi_0(\text{Con}_{W_i}(t)) \cdot \text{Fix}_{\text{Aut}(T(v_0))}(a, b).$$

But V_i acts transitively on $\text{Cham}(v_0) \setminus \{C\}$ and so does W_i , which means that $\varphi_0(W_i)(b) = T_\infty(v_0) \setminus \{a\}$. Hence,

$$T_\infty(v_0) \setminus \{a\} = \varphi_0(W_i)(b) \subseteq \varphi_0(\text{Con}_{W_i}(t))(b)$$

and $\text{Con}_{W_i}(t)$ also acts transitively on $\text{Cham}(v_0) \setminus \{C\}$. ■

Claim 3. *The inclusion $\text{Con}_{W_i}(t) \subseteq V_{i+1}$ holds.*

Proof of the claim: It is not even clear for the moment that $\text{Con}_{W_i}(t)$ is included in V_i . If $i = 0$, then $t \in H \subseteq V_0$ and $W_0 = V_0$ so this is true. If $i > 0$, then $t \notin V_i$ (that is actually why we introduced the semidirect product W_i) but we can prove that $\text{Con}_{W_i}(t) \subseteq V_i$. Indeed, suppose by contradiction that there is an element of $\text{Con}_{W_i}(t)$ that is not in V_i , i.e. of the form xt^k with $x \in V_i$ and $k \in \mathbf{Z}_0$. This means that $t^n xt^{k-n} \rightarrow e$ in W_i when n goes to infinity. But t does not fix ℓ_{v_i} pointwise while x does as it acts trivially on $\text{Cham}(v_i)$. Hence, $t^n xt^{k-n}$ does not fix ℓ_{v_i} pointwise for each $n \in \mathbf{N}$ and $t^n xt^{k-n} \not\rightarrow e$. This proves $\text{Con}_{W_i}(t) \subseteq V_i$. Finally, since

$$\varphi_{i+1}(\text{Con}_{W_i}(t)) \subseteq \text{Con}_{\text{Aut}(T(v_{i+1}))}(\varphi_{i+1}(t)) = \{e\}$$

by Lemma 4.17 (i), we get $\text{Con}_{W_i}(t) \subseteq V_{i+1}$. ■

As explained above, the conclusion follows from Claim 2 and Claim 3. □

We finally give the proof of Corollary B.

Proof of Corollary B. Let X be a locally finite thick affine building of rank 3 and of irreducible type which is strongly transitive. It is clear that $\text{Aut}(X)$ is a closed subgroup of $\text{Auttop}(X_\infty)$ and, by Proposition 4.5 and Proposition 4.9, X_∞ has the property $\mathcal{CG}_{\text{Aut}(X)}$. Moreover, $\text{Aut}(X)$ is strongly transitive on X and hence on X_∞ . By Theorem 4.19, X_∞ is Moufang. In other words, X is a Bruhat-Tits building. Those buildings have been classified by Bruhat and Tits; this classification is the subject of an important part of the book [Wei09]. □

References

- [Are46] Richard F. Arens, *A topology for spaces of transformations*, Ann. of Math. (2) **47** (1946), no. 3, 480–495.
- [BW04] Udo Baumgartner and George A. Willis, *Contraction groups and scales of automorphisms of totally disconnected locally compact groups*, Israel J. Math. **142** (2004), 221–248.
- [BVM94] Francis Buekenhout and Hendrik Van Maldeghem, *Finite distance-transitive generalized polygons*, Geom. Dedicata **52** (1994), no. 1, 41–51.
- [BS87a] Keith Burns and Ralf Spatzier, *On topological Tits building and their classification*, Publ. Math. Inst. Hautes Études Sci. **65** (1987), 5–34.
- [BS87b] Keith Burns and Ralf Spatzier, *Manifolds of nonpositive curvature and their buildings*, Publ. Math. Inst. Hautes Études Sci. **65** (1987), 35–59.
- [CDM13] Pierre-Emmanuel Caprace and Tom De Medts, *Trees, contraction groups, and Moufang sets*, Duke Math. J **162** (2013), no. 13, 2413–2449.
- [CM15] Pierre-Emmanuel Caprace and Nicolas Monod, *An indiscrete Bieberbach theorem: from amenable CAT(0) groups to Tits buildings*, J. Éc. polytech. Math. **2** (2015), 333–383.
- [CD14] Mathieu Carette and Denis Dreesen, *Locally compact convergence groups and n -transitive actions*, Math. Z. **278** (2014), no. 3–4, 795–827.

- [FTN91] Alessandro Figà-Talamanca and Claudio Nebbia, *Harmonic analysis and representation theory for groups acting on homogeneous trees*, London Math. Soc. Lecture Note Ser., vol. 162, Cambridge University Press, Cambridge, 1991.
- [Gru86] Theo Grundhöfer, *Automorphism groups of compact projective planes*, Geom. Dedicata **21** (1986), no. 3, 291–298.
- [GKK95] Theo Grundhöfer, Norbert Knarr, and Linus Kramer, *Flag-homogeneous compact connected polygons*, Geom. Dedicata **55** (1995), no. 1, 95–114.
- [GKK00] Theo Grundhöfer, Norbert Knarr, and Linus Kramer, *Flag-homogeneous compact connected polygons II*, Geom. Dedicata **83** (2000), no. 1–3, 1–29.
- [GKVMW12] Theo Grundhöfer, Linus Kramer, Hendrik Van Maldeghem, and Richard M. Weiss, *Compact totally disconnected Moufang buildings*, Tohoku Math. J. (2) **64** (2012), no. 3, 333–360.
- [Kna88] Norbert Knarr, *Projectivities of generalized polygons*, Ars Combin. **25** (1988), no. B, 265–275.
- [Löw81] Rainer Löwen, *Projectivities and the geometric structure of topological planes*, Geometry - von Staudt's point of view, 1981, pp. 339–372.
- [Mun00] James R. Munkres, *Topology*, 2nd ed., Prentice Hall, 2000.
- [Ron89] Mark Ronan, *Lectures on buildings*, Perspectives in Mathematics, vol. 7, Academic Press, Inc., Boston, 1989.
- [SBG⁺95] Helmut Salzmann, Dieter Betten, Theo Grundhöfer, Hermann Hähl, Rainer Löwen, and Markus Stroppel, *Compact projective planes: with an introduction to octonion geometry*, de Gruyter Exp. Math., vol. 21, Walter de Gruyter & Co., New York, 1995.
- [Tit70] Jacques Tits, *Sur le groupe des automorphismes d'un arbre*, Essays on topology and related topics (Mémoires dédiés à Georges de Rham), 1970, pp. 188–211 (French).
- [Tit74] Jacques Tits, *Buildings of spherical type and finite BN-pairs*, Lecture Notes in Math., vol. 386, Springer-Verlag, Berlin, 1974.
- [VMVS98] Hendrik Van Maldeghem and Kristel Van Steen, *Characterizations by automorphism groups of some rank 3 buildings: I. Some properties of half strongly-transitive triangle buildings*, Geom. Dedicata **73** (1998), no. 2, 119–142.
- [Wei09] Richard M. Weiss, *The structure of affine buildings*, Ann. of Math. Stud., vol. 168, Princeton University Press, Princeton, 2009.