

Consistency of universally nonminimally coupled $f(R, T, R_{\mu\nu}T^{\mu\nu})$ theoriesIsmael Ayuso,¹ Jose Beltrán Jiménez,^{2,3} and Álvaro de la Cruz-Dombriz¹¹*Departamento de Física Teórica I, Ciudad Universitaria, Universidad Complutense de Madrid, E-28040 Madrid, Spain*²*Centre for Cosmology, Particle Physics and Phenomenology, Institute of Mathematics and Physics, Louvain University, 2 Chemin du Cyclotron, 1348 Louvain-la-Neuve, Belgium*³*CPT, Aix Marseille Université, UMR 7332, 13288 Marseille, France*

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We discuss the consistency of a recently proposed class of theories described by an arbitrary function of the Ricci scalar, the trace of the energy-momentum tensor, and the contraction of the Ricci tensor with the energy-momentum tensor. We briefly discuss the limitations of including the energy-momentum tensor in the action, as it is a nonfundamental quantity but a quantity that should be derived from the action. The fact that theories containing nonlinear contractions of the Ricci tensor usually lead to the presence of pathologies associated with higher-order equations of motion will be shown to constrain the stability of this class of theories. We provide a general framework and show that the conformal and nonminimal couplings to the matter fields usually lead to higher-order equations of motion. To illustrate such limitations, we explicitly study the cases of a canonical scalar field, a K -essence field, and a massive vector field, whereas for the scalar field cases, it is possible to find healthy theories, and for the vector field case, the presence of instabilities is unavoidable.

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I. INTRODUCTION

A key question in gravitational physics and possible extensions of Einsteinian gravity resides in the coupling of gravity and matter fields. Even assuming the correctness of the equivalence principle—and the subsequent minimal coupling between matter and geometry as dictated by general relativity—strongly supported by astrophysical and laboratory tests, violations of the minimal coupling may still be allowed in scales and at times at which experiments have not been performed yet. In the existing literature, there is a whole host of proposals for nonminimal couplings such as those provided by scalar-tensor theories [1], vector-tensor theories [2], and different couplings between matter and geometry [3], among others. A recently proposed departing point consists of suggesting the coupling of a function of the Ricci scalar to the matter Lagrangian, a proposal that has produced numerous studies for gravitational and cosmological issues on the subject (cf. Refs. [4,5] and references therein). In this work, we shall consider a class of extended gravity theories in which the gravitational action is given by a general function $f(R, T, R_{\mu\nu}T^{\mu\nu})$, where R and T denote the Ricci scalar and the trace of the energy-momentum tensor, respectively, and $R_{\mu\nu}T^{\mu\nu}$ holds for the contraction of the Ricci and energy-momentum tensors. We shall herein address the theoretical consistency of this class of theories with special emphasis on couplings to either scalar or vector fields. Before proceeding, let us point out that, as it is well known, there are important criteria to be fulfilled by any extended theories beyond Einsteinian gravity that would like to be

claimed as a well-founded theory able to describe the gravitational interaction. These criteria aim to guarantee the absence of instabilities such as the appearance of ghostlike modes and the exponential growth of perturbations around well-established spacetime backgrounds, among others. For instance, an undesirable instability is the so-called Dolgov–Kawasaki instability, which appears when at least one extra degree of freedom of the theory behaves as a ghost and therefore this mode would act to destabilize the theory with no stable ground state. The avoidance of the Dolgov–Kawasaki instability has been developed to constrain the extensively studied $f(R)$ gravity with minimal [6–9] and nonminimal couplings of the curvature with matter [10]. Authors in Refs. [11] and [12] have recently addressed this instability issue for $f(R, T, R_{\mu\nu}T^{\mu\nu})$ theories.

Another important requirement usually demanded by extended theories consists of the avoidance of the Ostrogradski instability, i.e., the fact that a linear instability appears in Hamiltonians associated with Lagrangians that depend upon more than one time derivative nondegenerately [13,14]. Consequently, this type of Hamiltonians turns out not to be bounded from below, and well-defined vacuum states are absent. Theories with higher-order equations of motion can, however, be sensible, provided that they are regarded within the framework of effective field theories. In such a framework, the operators leading to the higher-order equations of motion are simply the first terms of some expansion of which the adequate resummation might give rise to well-behaved theories (this is the situation for instance when one integrates heavy degrees of

freedom out). Another possibility to make sense of theories with higher-order equations of motion is to remove the undesired unstable degrees of freedom from the physical spectrum of the theory or, at the classical level, constraining the physically allowed set of boundary conditions. However, one needs to make sure that such a procedure does not get spoiled by either time evolution or coupling to other fields. This approach was followed in Ref. [15] for the case of the degenerate Pais–Uhlenbeck field.

A widely accepted way of circumventing the Ostrogradski instability when considering scalar-tensor theories of gravitation consists of requiring the Euler–Lagrange equations to be second order even if higher-order derivatives are present in the action. Following this line of reasoning, Horndeski’s theorem [16] provides the most general Lagrangian density for a scalar-tensor theory that provides second-order Euler–Lagrange equations, for instance with Galileon theories [17] a remarkable example. Nonetheless, recent proposals have ensured second-order equations of motion and hence the absence of Ostrogradski ghost degrees of freedom for theories that do not fall under the form of Horndeski-like theories. Among others, let us mention healthy theories beyond Horndeski [18], the introduction of derivative couplings through a disformal metric between the scalar and the matter degrees of freedom [19], multiscalar field theories [20], nonlocal gravity theories [21], and nonlinear combinations of purely kinetic gravity terms [22].

With sole dependency on R , the theories under consideration of course correspond to the extremely popular $f(R)$ gravity modified theories, which might be thought of as the only local, metric-based, and generally coordinate invariant and stable modifications of gravity [14,21]. Both viability and stability conditions for $f(R)$ theories have been widely studied and guarantee the attractive character, the aforementioned avoidance of Dolgov–Kawasaki instability, the agreement with solar system tests, and evolution of geodesics [23].

With regard to Lagrangians with both R and T dependence, these kinds of modified gravity were originally introduced in Ref. [24] and later on considered in Ref. [25], and some cosmological aspects have been already explored, such as the reconstruction of cosmological solutions [26], in particular late-time acceleration ones [27]. Also the energy conditions have been analyzed in Ref. [28]. The thermodynamics of Friedmann–Lemaître–Robertson–Walker (FLRW) spacetimes has been studied in Ref. [29]. More recently, the possibility of irreversible matter creation processes and the possibility of the occurrence of future singularities were addressed in Refs. [30] and [31], respectively. Theories with nonstandard couplings between the geometry and the matter Lagrangian (see Ref. [32]) usually suffer from not conserving the energy-momentum tensor, which implies a stringent shortcoming for their viability. For $f(R, T)$ theories, this issue

was studied in Ref. [33], showing that gravitational Lagrangians of the form $f_1(R) + f_2(T)$ can always be constructed to be consistent with the energy-momentum tensor standard conservation, at least for a single perfect fluid for adequate choices of functions $f_{1,2}$. Despite this partial success, the growth rate in $f(R, T)$ theories was shown [33] to be highly compromised by the existence of oscillations in the density contrast evolution, the occurrence of singularities, and the fast growth of the density contrast in the studied models.

Theories including also terms of the form $R_{\mu\nu}T^{\mu\nu}$ have attracted some attention in recent years. Authors in Ref. [11] claimed some motivation for these theories arguing that the Hořava-like gravity power-counting renormalizable covariant gravity might represent the simplest power-law version of $f(R, T, R_{\mu\nu}T^{\mu\nu})$ theories [34]. One could then expect that such theories provide some insight between the usual approach in extended theories of gravity and the Hořava–Lifshitz theory. Several aspects of these theories are already available in the literature. For instance, the energy conditions for these theories were originally addressed in Ref. [35], in which the authors used models presented in Refs. [11] and [12] finding constraints on the model parameters from the Raychaudhuri equation. Finally, some efforts have been made to try to establish the thermodynamics for black holes embedded in a FLRW spacetime developing the Friedmann equations for spatially curved space-time and showing that for those theories these equations can be transformed into the form of the Clausius relation [36].

The paper is structured as follows. In Sec. II, we present some generalities of the theories under consideration, paying special attention to the theoretical limitations imposed by the fact of considering the energy-momentum tensor at the level of the gravitational action. There we shall also include the multiscalar representation that will allow us to identify the potential instabilities in a transparent way. Therein, we shall specify the analysis for the restricted cases of $f(R)$ and $f(R, T)$ theories. In Sec. III, we shall focus on the case of the canonical scalar field, the appearance of instabilities for such a choice, and the appropriate gravitational Lagrangians capable of preserving second-order field equations either in the original formulation or in the multiscalar representation. The same kind of studies as in the previous section shall be performed in Secs. IV and V for K -essence theories and vector fields, respectively. Section VI is then devoted to illustrating the shortcomings of classes of models for the gravitational Lagrangian and constraints to be imposed on the parameters in order to guarantee viability. Finally, we shall end with in Sec. VII with the main conclusions.

Throughout this paper, Greek indices run from 0 to 3, the symbol ∇ denotes the standard covariant derivative, and the signature $+, -, -, -$ is used. The Riemann tensor definition is $R^\mu{}_{\nu\alpha\beta} = \partial_\alpha \Gamma^\mu_{\nu\beta} - \partial_\beta \Gamma^\mu_{\nu\alpha} + \Gamma^\mu_{\sigma\alpha} \Gamma^\sigma_{\nu\beta} - \Gamma^\mu_{\sigma\beta} \Gamma^\sigma_{\nu\alpha}$.

II. GENERALITIES AND MULTISCALAR REPRESENTATION

The theories that we shall consider throughout this work are based on an action of the form

$$S = \int d^4x \sqrt{-g} [f(R, T, R_{\mu\nu} T^{\mu\nu}) + \mathcal{L}_m(g_{\mu\nu}, \Psi)], \quad (1)$$

where f is an arbitrary function of its arguments, $R_{\mu\nu}$ is the Ricci tensor corresponding to the Levi-Civita connection of the spacetime metric $g_{\mu\nu}$, $R \equiv g^{\mu\nu} R_{\mu\nu}$ holds for the scalar curvature, and $T^{\mu\nu}$ is the energy-momentum tensor of the matter fields Ψ described by the Lagrangian $\mathcal{L}_m$ and defined as

$$T^{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g} \mathcal{L}_m)}{\delta g_{\mu\nu}}. \quad (2)$$

Finally, $T \equiv g_{\mu\nu} T^{\mu\nu}$ holds for the trace of the energy-momentum tensor. In addition, we will further assume that the matter fields in $\mathcal{L}_m$ are minimally coupled to gravity so that all the nonminimal couplings will come from $f(R, T, R_{\mu\nu} T^{\mu\nu})$. Since the appearance of the energy-momentum tensor in the action might be worrisome, a few words about the construction of the theories as given in (1) are in order here. This digression lies in the fact that the standard lore consists of defining the energy-momentum tensor as the variation of the action itself so this procedure would lead to an endless loop. One could try to construct the theory by a perturbative expansion analogous to the Gupta problem for gravity; i.e., one could start with a linear coupling of the Ricci tensor to the energy-momentum tensor of the matter fields. This will modify the energy-momentum tensor of the matter field, so the coupling will acquire a correction. This process will in general generate an infinite series that should be resummed in order to obtain the full theory. We will not pursue this approach here but will regard (1) as a purely procedural way of defining the theory. In fact, theories described by (1) can be consistent without the infinite contributions from the nonminimal coupling to the energy-momentum tensor provided that we assume the contributions entering as arguments of the function f solely correspond to the *energy-momentum* tensor of the matter Lagrangian alone defined through (2). It remains doubtful that one can dub the energy-momentum tensor to such an object within the context of these universally nonminimally coupled theories. We could mention at least two reasons for this skepticism.

First of all, that object is obviously not conserved and does not correspond to the Noether current associated with infinitesimal translations. Second, with all fields being nonminimally coupled to gravity, the difference of $T^{\mu\nu}$ and the canonical energy-momentum tensor is not simply

a total divergence.¹ For this particular study, we shall consider the action (1) together with (2) as an operational approach to describe the theory. Thus, keeping in mind the aforementioned objections, $T^{\mu\nu}$ will be denoted in the following as energy-momentum tensor.

An even more worrisome aspect of theories described by (1) is the coupling of the Ricci tensor to the energy-momentum tensor: since $R_{\mu\nu}$ contains second derivatives of the metric tensor, and $T^{\mu\nu}$ will typically have first derivatives of the matter fields, the equations of motion are expected to be higher than second order, and the Ostrogradski instability [13] is likely to be present. It is worth stressing that it is precisely the coupling to the Ricci tensor that will generically render the theory unstable. As it is well known, despite containing second derivatives in the Lagrangian, $f(R)$ theories avoid the Ostrogradski instability because they are constructed out of the Ricci scalar solely [14]. However, for arbitrary functions explicitly containing the Ricci tensor, e.g., $R_{\mu\nu} R^{\mu\nu}$, a ghost associated with the higher-order derivatives arises. In fact, in the following three sections, Secs. III, IV, and V, we shall study this feature for cases in which the matter fields are described either by scalar or by vector fields. We shall thus show that ghost modes are generally present in these theories due to the coupling $R_{\mu\nu} T^{\mu\nu}$ and that its avoidance considerably restricts the allowed form for the function f as will then be illustrated in Sec. VI.

A. Multiscalar representation

Before considering the presence of the Ostrogradski instabilities in these theories, we will present a general framework in which this instability (and its origin) can be more easily identified. To that end, we will rewrite the theory presented in (1) using the multiscalar-tensor representation. This way we intend to illustrate the generality with which the Ostrogradski instability will show up within these theories due to the aforementioned coupling $R_{\mu\nu} T^{\mu\nu}$. Let's start by rewriting action (1) as

$$S = \int d^4x \sqrt{-g} \left[f(\chi_1, \chi_2, \chi_3) + \sum_{i=1}^3 f_{\chi_i} (P_i - \chi_i) + \mathcal{L}_m \right], \quad (3)$$

where $\chi_{i=1,2,3}$ are auxiliary fields, $P_1 \equiv R$, $P_2 \equiv T$, $P_3 \equiv R_{\mu\nu} T^{\mu\nu}$, and $f_{\chi_j} \equiv \partial f / \partial \chi_j$, $j = 1, 2, 3$. The corresponding field equations for those auxiliary fields are given by

$$\frac{\partial^2 f}{\partial \chi_i \partial \chi_j} (P_i - \chi_i) = 0. \quad (4)$$

¹This is actually a feature present in all nonminimally coupled theories. These nonminimal couplings can be understood as arising from couplings of the graviton to surface terms [37].

Thus, provided that $\det \frac{\partial^2 f}{\partial \chi_i \partial \chi_j} \neq 0$, the only solution of (4) turns out to be $\chi_1 = R$, $\chi_2 = T$, $\chi_3 = R_{\mu\nu} T^{\mu\nu}$, and consequently the action (3) is dynamically equivalent to the original action (1). Let us stress that a necessary condition for the transformation to the multiscalar-tensor representation (3) to be valid lies in the nondegeneracy, i.e., non-vanishing determinant, of the matrix $\frac{\partial^2 f}{\partial \chi_i \partial \chi_j}$. This element will play an important role later on. At this stage, we will introduce a field redefinition as follows: $\varphi_i = -f_{\chi_i}$. Assuming that this redefinition is invertible, so that χ_i can be expressed as a function of $\{\varphi_1, \varphi_2, \varphi_3\}$, the action (3) can be written as

$$S = \int d^4x \sqrt{-g} [\mathcal{U}(\varphi_1, \varphi_2, \varphi_3) - \varphi_1 R - \varphi_2 T - \varphi_3 R_{\mu\nu} T^{\mu\nu} + \mathcal{L}_m], \quad (5)$$

where we have introduced the definition $\mathcal{U}(\varphi_1, \varphi_2, \varphi_3) \equiv f(\varphi_1, \varphi_2, \varphi_3) + \sum_{i=1}^3 \varphi_i \chi_i(\varphi_1, \varphi_2, \varphi_3)$.

Now, we can follow the usual approach to disentangle the nonminimal coupling $\varphi_1 R$ by means of a conformal transformation of the form $g_{\mu\nu} = e^{2\Omega} \tilde{g}_{\mu\nu}$, with $\Omega = \log \frac{1}{\sqrt{16\pi G \varphi_1}}$, so the action (5) becomes²

$$S = \int d^4x \sqrt{-\tilde{g}} \left\{ e^{4\Omega} \mathcal{U}(\Omega, \varphi_2, \varphi_3) - \frac{1}{16\pi G} (\tilde{R} - 6\tilde{g}^{\alpha\beta} \partial_\alpha \Omega \partial_\beta \Omega) - \varphi_2 \tilde{T} - e^{-2\Omega} \varphi_3 [\tilde{R}_{\mu\nu} - 2\tilde{\nabla}_\mu \tilde{\nabla}_\nu \Omega + 2\tilde{\nabla}_\mu \Omega \tilde{\nabla}_\nu \Omega - (2\tilde{g}^{\alpha\beta} \partial_\alpha \Omega \partial_\beta \Omega + \tilde{\square} \Omega) \tilde{g}_{\mu\nu}] \tilde{T}^{\mu\nu} + e^{4\Omega} \mathcal{L}_m(e^{2\Omega} \tilde{g}_{\mu\nu}, \Psi) \right\}, \quad (6)$$

where we have dropped a total divergence and used the transformation properties of the Ricci tensor and scalar curvature under conformal rescaling given by

$$R_{\mu\nu} = \tilde{R}_{\mu\nu} - 2\tilde{\nabla}_\mu \tilde{\nabla}_\nu \Omega + 2\tilde{\nabla}_\mu \Omega \tilde{\nabla}_\nu \Omega - (2\tilde{g}^{\alpha\beta} \partial_\alpha \Omega \partial_\beta \Omega + \tilde{\square} \Omega) \tilde{g}_{\mu\nu}, \quad (7)$$

$$R = e^{-2\Omega} (\tilde{R} - 6\tilde{g}^{\alpha\beta} \partial_\alpha \Omega \partial_\beta \Omega - 6\tilde{\square} \Omega). \quad (8)$$

²Notice that the scalar Ω is dimensionless. To restore its natural dimension and have a canonically normalized scalar field, it should be rescaled as $\Omega \rightarrow \sqrt{\frac{8\pi G}{3}} \Omega$.

We have also defined the energy-momentum tensor in the conformally transformed frame as³

$$\tilde{T}_{\mu\nu} = e^{2\Omega} T_{\mu\nu} \quad (9)$$

and its trace with respect to $\tilde{g}_{\mu\nu}$ as $\tilde{T} \equiv \tilde{g}^{\mu\nu} \tilde{T}_{\mu\nu} = e^{4\Omega} T$. After the conformal transformation, we see that the degree of freedom contained in φ_1 has been transferred to the conformal mode Ω . We will see later that, in some cases, the conformal transformation needs to be more general (depending also on the matter field and φ_3).

At this stage, let us further clarify the procedure sketched above when specified by several paradigmatic and simpler scenarios of extended gravity theories, such as Lagrangians of the forms $f(R)$ and $f(R, T)$ [immediately afterward, we will go back to the general $f(R, T, R_{\mu\nu} T^{\mu\nu})$ case under study in this paper]:

- (i) $f(R)$ case: In the extensively studied scenario of pure $f(R)$ fourth-order gravity theories [38], the previous derivation is nothing but the usual approach that clearly shows how the conformal degree of freedom behaves as a standard scalar field coupled to matter. Thus, these theories avoid the Ostrogradski instability. The nondegeneracy condition reduces in that case to $f_{RR} \neq 0$. Whenever the latter condition does not hold, it simply means that the theory is linear in R ; i.e., we are dealing with the usual Einstein–Hilbert action.
- (ii) $f(R, T)$ case: For Lagrangians with an arbitrary function depending only upon both R and T , i.e., for the so-called $f(R, T)$ theories, the auxiliary field φ_2 in (6) can be integrated out by using its own equation of motion, which is given by

$$\frac{\partial \mathcal{U}}{\partial \varphi_2} - T = 0 \Rightarrow \varphi_2 = \varphi_2(\varphi_1, T). \quad (10)$$

If the previous equation is rewritten after having performed the conformal transformation, we would obtain $\varphi_2 = \varphi_2(\Omega, \tilde{T})$ instead. To proceed with our analysis, we need to assume that this algebraic equation is in fact solvable with respect to φ_2 . In some cases, this does not need to be the case, and in addition, when solving Eq. (10), one might find several branches corresponding to different solutions of this equation. Moreover, there is a special case when the function $\mathcal{U}$ is linear in φ_2 since, for such case, φ_2 acts as a Lagrange multiplier imposing a constraint equation rather than being an auxiliary field. These cases are usually related to those models

³Notice that this is not the energy-momentum tensor one would obtain from (2) in the Einstein frame, but it is defined as $\tilde{T}_{\mu\nu} = -\frac{2}{\sqrt{-\tilde{g}}} \frac{\delta(\sqrt{-\tilde{g}} \mathcal{L}_m)}{\delta \tilde{g}_{\mu\nu}}$.

for which the auxiliary field χ_2 cannot be introduced. Leaving pathological cases aside and assuming the solvability of the above equation (10), the action (6) for $f(R, T)$ theories after integrating out the auxiliary field φ_2 reads

$$\begin{aligned} S &= \int d^4x \sqrt{-\tilde{g}} \left[e^{4\Omega} \mathcal{U}(\Omega, \tilde{T}) + e^{4\Omega} \mathcal{L}_m(e^{2\Omega} \tilde{g}_{\mu\nu}, \Psi) \right. \\ &\quad \left. - \frac{1}{16\pi G} (\tilde{R} - 6\tilde{g}^{\alpha\beta} \partial_\alpha \Omega \partial_\beta \Omega) - \varphi_2(\Omega, \tilde{T}) \tilde{T} \right] \\ &= \int d^4x \sqrt{-\tilde{g}} \left[-\frac{1}{16\pi G} (\tilde{R} - 6\tilde{g}^{\alpha\beta} \partial_\alpha \Omega \partial_\beta \Omega) \right. \\ &\quad \left. + \mathcal{P}(\Omega, \Psi) \right], \end{aligned} \quad (11)$$

where we have defined

$$\begin{aligned} \mathcal{P}(\Omega, \Psi) &\equiv e^{4\Omega} \mathcal{U}(\Omega, \tilde{T}) - \varphi_2(\Omega, \tilde{T}) \tilde{T} \\ &\quad + e^{4\Omega} \mathcal{L}_m(e^{2\Omega} \tilde{g}_{\mu\nu}, \Psi). \end{aligned} \quad (12)$$

This function comprises all the matter sector terms including couplings to the conformal mode Ω . Notice that the conformal mode appears with no derivatives and only the matter fields Ψ will enter with derivatives in $\mathcal{P}$. In the standard case, the energy-momentum tensor will depend upon both the matter fields and their first derivatives.⁴ Hence, the above action with these kinds of functions $f(R, T)$ will generically avoid the Ostrogradski instability except for some pathological cases such as scenarios in which, for instance, expression (4) is not invertible or equation (10) cannot be solved for φ_2 and therefore this construction fails. The expression (11) also tells us that, for the case of scalar fields, the resulting Lagrangian term $\mathcal{P}$ in (11) will resemble that of K -essence models, and the stability conditions can then be obtained in an analogous manner to $f(R, \mathcal{L}_m)$ theories as is done in Ref. [4]. It is important to keep in mind that, even though these theories can generically avoid the Ostrogradski instability, they can still have instabilities of a different nature around specific backgrounds.

- (iii) General $f(R, T, R_{\mu\nu} T^{\mu\nu})$ case: Let us now return to the general case of universally nonminimally coupled Lagrangians of the form $f(R, T, R_{\mu\nu} T^{\mu\nu})$ as given in (6). In this case, we can see the appearance of two types of problematic terms. They can be more easily identified by rewriting the action (6) as

$$\begin{aligned} S &= \int d^4x \sqrt{-\tilde{g}} \left\{ \hat{\mathcal{U}}(\Omega, \tilde{T}, \varphi_3) \right. \\ &\quad - \frac{1}{16\pi G} (\tilde{R} - 6\tilde{g}^{\alpha\beta} \partial_\alpha \Omega \partial_\beta \Omega) \\ &\quad - e^{-2\Omega} \varphi_3 [\tilde{R}_{\mu\nu} \tilde{T}^{\mu\nu} - (2\tilde{T}^{\mu\nu} + \tilde{T} \tilde{g}^{\mu\nu}) \tilde{\nabla}_\mu \tilde{\nabla}_\nu \Omega \\ &\quad + 2(\tilde{T}^{\mu\nu} - \tilde{T} \tilde{g}^{\mu\nu}) \tilde{\nabla}_\mu \Omega \tilde{\nabla}_\nu \Omega] \\ &\quad \left. + e^{4\Omega} \mathcal{L}_m(e^{2\Omega} \tilde{g}_{\mu\nu}, \Psi) \right\}, \end{aligned} \quad (13)$$

where, in the very same manner as for the $f(R, T)$ case, we have integrated out the auxiliary field φ_2 by using its own equation of motion and we have rearranged terms including φ_2 in $\hat{\mathcal{U}}(\Omega, \tilde{T}, \varphi_3) \equiv e^{4\Omega} \mathcal{U} - \varphi_2(\Omega, \tilde{T}, \varphi_3) \tilde{T}$. With the action expressed in this form, we can clearly identify therein two potential stability problems that we describe in the following:

- (1) On the one hand, there are terms with second-order derivatives of the conformal mode of the form $\mathcal{K}^{\mu\nu} \tilde{\nabla}_\mu \tilde{\nabla}_\nu \Omega$ where $\mathcal{K}^{\mu\nu}$ contains first derivatives of the matter fields. Therefore, this term will lead to higher-order equations of motion and, thus, the propagation of additional degrees of freedom which will correspond to unstable Ostrogradski modes. As we will discuss below in more detail, there are cases in which the structure of $\mathcal{K}^{\mu\nu}$ makes it possible to avoid higher-order equations of motion. This opens the possibility of having consistent theories of the discussed type, but the universal validity of theories with nonminimal couplings must be abandoned, as we will see later. For instance, we shall show how standard vector field theories do lead to Ostrogradski modes.
- (2) On the other hand, there is also a nonminimal coupling of the Ricci tensor to the energy-momentum tensor which might be the origin of additional instability problems. In particular:
 - (a) For a fixed curved background, this coupling will modify the kinetic term of the matter field and could turn it into a ghost (and/or other type of instabilities) due to the non-definite signature of the Ricci tensor.
 - (b) For dynamical gravitational fields, these nonminimal couplings will generally introduce additional propagating degrees of freedom associated with higher-order equations of motion with the corresponding Ostrogradski instability. Again in this case, for specific types of matter and particular choices of the function f , these nonminimal couplings might actually be stable.

⁴This is so because we are considering matter fields of which the Lagrangian only depends upon first-order derivatives.

The two points above are general features of these theories, but there can be ways of avoiding the Ostrogradski instabilities. In the general case, we also have the auxiliary field φ_3 that can be integrated out after having used its equation of motion, given by

$$e^{4\Omega} \frac{\partial \mathcal{U}(\Omega, \varphi_2, \varphi_3)}{\partial \varphi_3} - e^{-2\Omega} [\tilde{R}_{\mu\nu} - 2\tilde{\nabla}_\mu \tilde{\nabla}_\nu \Omega + 2\tilde{\nabla}_\mu \Omega \tilde{\nabla}_\nu \Omega - (2\tilde{g}^{\alpha\beta} \partial_\alpha \Omega \partial_\beta \Omega + \tilde{\square} \Omega) \tilde{g}_{\mu\nu}] \tilde{T}^{\mu\nu} = 0. \quad (14)$$

Here, one should bear in mind the same subtleties as discussed after Eq. (10) for the case of φ_2 . Assuming again that the above equation can be algebraically solved for φ_3 , the solution can be plugged back into the action (13) to remove the dependence on φ_3 . To be more precise, when both φ_2 and φ_3 are present, the condition for the solvability of φ_2 and φ_3 in terms of their own equations of motion is given by the nondegeneracy of the matrix $\partial^2 \mathcal{U} / \partial \varphi_i \partial \varphi_j$ with $i, j = 2, 3$. In other words, this will be the condition for such fields to be actual auxiliary fields and not Lagrange multipliers. This condition is actually linked to the condition for the validity of the Legendre transformation of the original action because one can easily show that

$$\frac{\partial^2 \mathcal{U}}{\partial \varphi_i \partial \varphi_j} = - \left(\frac{\partial^2 f}{\partial \chi_i \partial \chi_j} \right)^{-1} \quad (15)$$

again, for $i, j = 2, 3$. The fact that φ_3 is an auxiliary field will be a crucial obstruction for the stability of the theory as we shall exemplify below in the subsequent sections. In fact, this shortcoming will motivate the use of gravitational Lagrangians that are linear in $R_{\mu\nu} T^{\mu\nu}$ so that the system (4) is degenerate and φ_3 cannot be defined as an auxiliary field. This will make contact with Horndeski-type interactions because, provided that φ_3 is not really an auxiliary field that needs to be integrated out, we see from (14) that the potentially dangerous term for the conformal factor is linear in the second derivatives of the conformal mode Ω ; i.e., it is of the form $\mathcal{K}^{\mu\nu} \tilde{\nabla}_\mu \tilde{\nabla}_\nu \Omega$ where $\mathcal{K}^{\mu\nu}$ only depends on derivatives of the matter fields. Thus, it is a general result that, provided $\mathcal{K}^{\mu\nu}$ does not contain time derivatives,⁵ the field equations are actually second order. Consequently, the Ostrogradski instability is avoided. For this purpose, it is crucial that φ_3 is not an auxiliary field since, otherwise, even if the structure of $\mathcal{K}^{\mu\nu} \tilde{\nabla}_\mu \tilde{\nabla}_\nu \Omega$ is correct, after integrating out

⁵Actually, if $\mathcal{K}^{\mu\nu}$ only depends on Ω , even if it contains time derivatives, the equations of motion are second order. See the Appendix for more details.

φ_3 , the avoidance of the Ostrogradski instability will be compromised.

Upcoming sections shall in fact be devoted to better explaining and illustrating the aforementioned general statements when applied to specific choices of the matter Lagrangian, in particular to scalar and vector fields.

III. CANONICAL SCALAR FIELD

To clarify the general discussion performed in the previous section, we shall focus here on a setup where the matter sector is given by a scalar field. Thence, we start by considering the simplest case of a canonical scalar field, and we will make contact with Horndeski-type interactions to guarantee the avoidance of the Ostrogradski instability. Thus, the matter Lagrangian will be given by

$$\mathcal{L}_m = \mathcal{L}_\phi = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi), \quad (16)$$

with $V(\phi)$ the potential of the scalar field. The corresponding energy-momentum tensor for the scalar field is

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \mathcal{L}_\phi, \quad (17)$$

and the trace of the energy-momentum tensor and the contraction $R_{\mu\nu} T^{\mu\nu}$ provide

$$T = -(\partial\phi)^2 + 4V(\phi), \quad R_{\mu\nu} T^{\mu\nu} = G_{\mu\nu} \partial^\mu \phi \partial^\nu \phi + RV(\phi), \quad (18)$$

with the usual definition for the Einstein tensor $G_{\mu\nu} \equiv R_{\mu\nu} - 1/2 g_{\mu\nu} R$ and where we have used the notation $(\partial\phi)^2 \equiv \partial_\mu \phi \partial^\mu \phi$. Once the arguments of the gravitational Lagrangian have been expressed in terms of the scalar field, action (1) will take the form

$$S = \int d^4x \sqrt{-g} [f(R, -(\partial\phi)^2 + 4V(\phi), G^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + RV(\phi)) + \mathcal{L}_\phi(g_{\mu\nu}, \phi)] \quad (19)$$

so that we obtain a nonminimally coupled theory with derivative couplings of the scalar field to the Ricci curvature. As it is well known, this type of couplings leads to higher-order equations of motion, and consequently they generally suffer from the Ostrogradski instability [13]. This instability is actually present in general theories containing arbitrary contractions of the Riemann tensor, the $f(R)$ theories being an exceptional case in which the presence of constraints removes the unstable dynamical degree of freedom⁶ as explained in the previous section.

⁶This fact is a consequence of the degeneracy of the transformation to canonical variables, which is a crucial step in the Ostrogradski construction [13].

The usual approach to avoid this instability *a priori* consists of building actions leading to second-order equations of motion. For instance, in a purely gravitational context, such actions are given by the Lovelock invariants, which in four dimensions reduce to a cosmological constant, the Ricci scalar, and the Gauss–Bonnet term, the latter being a topological invariant. In this realm, in the context of scalar-tensor theories, the analogous Lagrangians with second-order equations of motion were obtained by Horndeski [16]. The appropriate gravitational Lagrangian can be written as a sum of the four terms

$$\mathcal{L}_2 = K(\phi, X), \quad (20)$$

$$\mathcal{L}_3 = G_3(\phi, X)\square\phi, \quad (21)$$

$$\begin{aligned} \mathcal{L}_4 = & G_4(\phi, X)R \\ & - G_{4,X}(\phi, X)[(\square\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2], \end{aligned} \quad (22)$$

$$\begin{aligned} \mathcal{L}_5 = & G_5(\phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi + \frac{1}{6}G_{5,X}(\phi, X)[(\square\phi)^3 \\ & - 3(\square\phi)(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3], \end{aligned} \quad (23)$$

where $X \equiv \frac{1}{2}\partial_\mu\phi\partial^\mu\phi$, K , and $G_{3,4,5}$ are arbitrary functions of ϕ and X and the subindex X refers to the derivative with respect to X . Therefore, to avoid the Ostrogradski instability for the theories considered in (1), it is sufficient to guarantee that the action lies within the aforementioned Horndeski theories either for the original form of the action or in the multiscalar representation. Notice that in the multiscalar representation we will eventually have two scalar fields (the matter scalar field plus the conformal mode), so the considered actions will actually be more general than the above Horndeski terms. We clearly see that this requirement will extremely constrain the permitted form of the gravitational Lagrangians $f(R, T, R_{\mu\nu}T^{\mu\nu})$. We should stress here that, while the Horndeski terms are the most general ones explicitly leading to second-order equations of motion, they are not the most general theories that propagate one spin-2 plus one spin-1 fields. A more general class of theories has been shown to propagate exactly the same degrees of freedom as the Horndeski terms even though the equations are of a higher order [18]. The reason can be traced to the existence of *hidden* constraints that reduce the required number of boundary conditions. Some terms within that class of theories can actually be related to some Horndeski terms by means of a general disformal transformation [19].

A. Preservation of second-order field equations

We will first look for conditions on the function f so that the theory contains no higher than second-order equations of motion in its original form. This will guarantee that neither the gravitational sector nor the matter sector will

propagate more degrees of freedom than it corresponds to a massless graviton plus one scalar field (or, in other words, the conformal mode is not excited). In the next subsection, we will drop this condition to let the conformal mode propagate as well, and we will find conditions for the absence of the Ostrogradski instabilities also in that case.

From the form of the Horndeski Lagrangians, one can *a priori* infer that curvature-scalar field couplings need to be linear in the curvature according to $\mathcal{L}_{4,5}$ to maintain the second-order nature of the field equations. This linearity in the curvature implies a first stringent constraint on the function f , which consequently needs to be of the form

$$f(R, T, R_{\mu\nu}T^{\mu\nu}) = f_1(T)R + f_2(T)R_{\mu\nu}T^{\mu\nu} + f_3(T), \quad (24)$$

for arbitrary functions $f_{1,2,3}(T)$. Of course this is not the most general case free from the Ostrogradski instability because it is well known that higher-order terms in derivatives can lead to stable field equations provided they correspond to the special class of degenerate theories. For instance, we could have added an arbitrary function of the Ricci scalar in (24) or considered more general scalar-tensor interactions as commented above, without introducing this instability. Let us also remember that $f_3(T)$ is in fact a function of X and ϕ and therefore lies in $\mathcal{L}_2$.

For our scalar field scenario with the result in (18) and the form for the function f given in (24), the corresponding action becomes

$$\begin{aligned} S = & \int d^4x \sqrt{-g} [f_1(T) + f_2(T)V]R \\ & + f_2(T)G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + f_3(T) + \mathcal{L}_\phi], \end{aligned} \quad (25)$$

with $f_{1,2,3}$ functions only of $T = -2X + 4V(\phi)$. Now, we want to obtain further constraints on $f_{1,2}$ so that the action can be mapped into Horndeski-like terms. The absence of terms of the type $(\nabla\nabla\phi)^3$ in (25) suggests that f_2 should be a function of ϕ only and not of X . However, f_2 is a function of the energy-momentum tensor trace $T = -2X + 4V(\phi)$ so that it also explicitly depends upon X . Consequently, the only possibility left is that f_2 is simply a constant. Thus, the only remaining arbitrary function would be $f_1(T)$. Nonetheless, this function is not arbitrary either since, if (25) is required to be Horndeski-like, the only way the first term in (25) can be mapped to G_4 is for $f_1 + f_2V$ being solely a function of ϕ , which, analogously to the reasoning used for f_2 , leads one to conclude that f_1 also needs to be a constant. Therefore, the requirement explicitly guaranteeing second-order field equations in $f(R, T, R_{\mu\nu}T^{\mu\nu})$ theories with a standard scalar field as the matter sector leads to actions of the form

$$S = \int d^4x \sqrt{-g} \left[(c_1 + c_2 V(\phi)) R + \frac{1}{2} (g^{\mu\nu} + c_2 G^{\mu\nu}) \partial_\mu \phi \partial_\nu \phi - V(\phi) + f_3 (-2X + 4V) \right] \quad (26)$$

with $c_{1,2}$ some constants,⁷ One can now immediately identify the different terms in the above action with the corresponding Horndeski Lagrangians with

$$\begin{aligned} G_2 &= \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) + f_3 (2X + 4V(\phi)), \\ G_3 &= 0, \\ G_4 &= c_1 + c_2 V(\phi), \\ G_5 &= -c_2 \phi, \end{aligned} \quad (27)$$

where we have used that, via integration by parts, $G^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi \rightarrow -\phi G^{\alpha\beta} \nabla_\alpha \nabla_\beta \phi$. For a scalar field without potential, i.e., with a shift symmetry $\phi \rightarrow \phi + c$ with c a constant, the first term in (26) simply renders the Einstein–Hilbert term (and we should identify $c_1 \equiv -(16\pi G)^{-1}$), whereas the last term gives a contribution in the form of a K -essence term. If we further set $f_3 = 0$, then we end up with a nonminimally derivatively coupled scalar field of which the nonminimal coupling is to the Einstein tensor. This simplified case was explored in Ref. [39] as a model of inflation, and in Ref. [40] black-hole solutions were obtained. The nonminimal derivative coupling to the Einstein tensor also arises in the covariantization of the decoupling limit of massive gravity [41]. Thus, these models can be regarded as specific cases of the general $f(R, T, R_{\mu\nu} T^{\mu\nu})$ theories where the function is subject to be simply $f = R_{\mu\nu} T^{\mu\nu}$ or, in other words, the derivative coupling to the Einstein tensor can be alternatively seen as a coupling of the Ricci tensor to the energy-momentum tensor of the scalar field.

B. Multiscalar-tensor representation analysis

Let us now consider more general actions by using the multiscalar representation for the canonical scalar field studied above; i.e., we will now let the conformal mode be excited. This approach will enable us to detect and identify the instabilities that $f(R, T, R_{\mu\nu} T^{\mu\nu})$ Lagrangians may suffer for more general models with field equations beyond second order. Equivalently, this procedure will allow us to find general actions with higher-order field equations that are actually *healthy* in a similar manner to $f(R)$ theories. Before proceeding, a subtlety should be remarked upon at

this stage: as we can see from (18), the coupling $R_{\mu\nu} T^{\mu\nu}$ will also generate a coupling between the scalar field potential and the Ricci scalar $V(\phi)R$. Therefore, after performing the conformal transformation, the definition of the conformal mode needs to be modified to $\Omega \equiv \log \frac{1}{\sqrt{16\pi G(\varphi_1 + V(\phi)\varphi_3)}}$. After taking this into account, the action becomes⁸

$$S = \int d^4x \sqrt{-\tilde{g}} \left\{ \hat{\mathcal{U}}(\Omega, \tilde{T}, \varphi_3) - \frac{1}{16\pi G} [\tilde{R} - 6(\partial\Omega)^2] - \varphi_3 [\tilde{G}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + 2(\partial_\mu \Omega \partial^\mu \phi)^2 + (\partial\Omega)^2 (\partial\phi)^2 + 2(\tilde{g}^{\mu\nu} (\partial\phi)^2 - \partial^\mu \phi \partial^\nu \phi) \tilde{\nabla}_\mu \tilde{\nabla}_\nu \Omega] \right\}. \quad (28)$$

In this case, the last expression explicitly shows why the fact that φ_3 is an auxiliary field will be problematic. Although the terms in brackets in the second line of the above expression provide the *appropriate* structure in order to guarantee second-order equations of motion, one will in general generate potentially dangerous terms again after integrating φ_3 out. For instance, for a theory with second derivatives of a scalar field (the conformal mode in our case), one typically needs the second derivatives to appear linearly, as it happens in (28). However, after integrating φ_3 out, such second derivatives will enter nonlinearly in the action, thus spoiling the required structure. A loophole in this conclusion occurs for a Lagrangian f with a linear dependence on the argument $R_{\mu\nu} T^{\mu\nu}$. In such a case, the auxiliary field φ_3 cannot be defined, and the resulting action would read

$$S = \int d^4x \sqrt{-\tilde{g}} \left\{ \hat{\mathcal{U}}(\Omega, \tilde{T}, \varphi_3) - \frac{1}{16\pi G} [\tilde{R} - 6(\partial\Omega)^2] - \alpha [\tilde{G}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + 2(\partial_\mu \Omega \partial^\mu \phi)^2 + (\partial\Omega)^2 (\partial\phi)^2 + 2(\tilde{g}^{\mu\nu} (\partial\phi)^2 - \partial^\mu \phi \partial^\nu \phi) \tilde{\nabla}_\mu \tilde{\nabla}_\nu \Omega] \right\} \quad (29)$$

with α a constant parameter. The potentially dangerous terms are in the second line of the above expression. The nonminimal coupling of the field ϕ is to the Einstein tensor and multiplied by a function of only Ω and not its derivatives. Therefore, this coupling will be of the Horndeski form. The other term that can potentially lead to higher-order equations of motion is the coupling of ϕ to the second derivatives of Ω . However, this is also a safe interaction because the tensor structure of $\mathcal{K}^{\mu\nu} \equiv \tilde{g}^{\mu\nu} (\partial\phi)^2 - \partial^\mu \phi \partial^\nu \phi$ is such that $\mathcal{K}^{00}$ does not contain any time derivatives and this guarantees that the field

⁷The curly brackets after f_3 must be understood as the argument of f_3 since $T = -2X + 4V$ in this case according to (18).

⁸In the expression (28), it must be understood that indices are now raised and lowered with the metric $\tilde{g}_{\mu\nu}$. For instance, $(\partial\Omega)^2 \equiv \tilde{g}^{\alpha\beta} \partial_\alpha \Omega \partial_\beta \Omega$.

equations will be second order (see the Appendix for further details).

IV. K -ESSENCE THEORIES

After studying in detail the case of a canonical scalar field, let us move on to a more general case in which the action for the scalar field is given by a K -essence model, i.e., the matter Lagrangian now reads

$$\mathcal{L}_m = \mathcal{L}_K = K(\phi, X), \quad (30)$$

where $K(\phi, X)$ is an arbitrary function of its arguments. The previous case for a canonical scalar field corresponds to the particular case $K(\phi, X) = X - V(\phi)$. We will proceed in a similar manner as the preceding section, and, in addition, to obtain constraints on the gravitational action f , we will obtain conditions on K . However, here we will aim to show that the freedom in the choice for the function $K(\phi, X)$ allows us more general functions $f(R, T, R_{\mu\nu}T^{\mu\nu})$. The relevant quantities in this case are given by

$$T_{\mu\nu} = K_{,X}\partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}K, \quad (31)$$

$$T = 2K_{,X}X - 4K, \quad (32)$$

$$R_{\mu\nu}T^{\mu\nu} = K_{,X}G^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + R(K_{,X}X - K), \quad (33)$$

where we can start to see the role that the form of the K -function and its dependence on X might play in the coupling terms. In particular, we see that the coupling $R_{\mu\nu}T^{\mu\nu}$ generates a derivative interaction with the Ricci scalar in addition to the nonminimal coupling to the Einstein tensor that we obtained in the canonical scalar field case. In fact, this might be the origin of pathologies for general K -essence models. The general action will take the form

$$S = \int d^4x \sqrt{-g} \{ f(R, 2K_{,X}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 4K(X, \phi), 2K_{,X}G^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + R[K_{,X}X - K(\phi, X)]) + K(\phi, X) \}. \quad (34)$$

Analogously to the scheme in the previous section, we will look for conditions to be imposed on the function f so that the theory does not lead to higher than second-order equations of motion in its original form. Again, for the action to be of the Horndeski type, the nonminimal coupling must be linear in the curvature, and therefore the function f should also be linear in R and $R_{\mu\nu}T^{\mu\nu}$. After imposing such restrictions, the action (34) simply reads

$$\begin{aligned} S &= \int d^4x \sqrt{-g} [f_1(T)R + f_2(T) + f_3(T)R_{\mu\nu}T^{\mu\nu} \\ &\quad + K(\phi, X)] \\ &= \int d^4x \sqrt{-g} \{ [f_1(T) + f_3(T)(K_{,X}X - K)]R \\ &\quad + f_2(T) + f_3(T)K_{,X}G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + K(\phi, X) \}, \end{aligned} \quad (35)$$

with T given by (32). A reasoning similar to the one below (25) allows us to establish that $f_3K_{,X}$ should be a function of ϕ only and not of X , i.e.,

$$\partial_X(f_3K_{,X}) = 0, \quad (36)$$

so that $f_3K_{,X} = g_1(\phi)$. Again, proceeding analogously to the reasoning below (25), we can also conclude that $f_1(T) + f_3(T)(K_{,X}X - K)$ should also be a function of only ϕ . Now, if we combine these two conditions, we finally obtain that the equation

$$\partial_X[f_1(T) + g_1(\phi)X - f_3K] = 0 \quad (37)$$

must be filled, which, by using that $\partial_X f_i = 2f'_i(K_{,XX}X - K_{,X})$ for $i = 1, 3$, can be expressed as

$$2(f'_1 - f'_3K)XK_{,XX} - 2(f'_1 + f_3)K_{,X} + f'_3(K^2)_{,X} = -g_1, \quad (38)$$

where primes denote a derivative with respect to the argument (T in this case). We see that for the canonical scalar field, for which $K = X - V(\phi)$, the above conditions imply that f_1 and f_3 can only depend on ϕ so that, being functions of T , this is only possible if they are indeed constant functions, in agreement with our previous result. In the present case, however, the freedom in the choice of the function $K(\phi, X)$ allows for more general cases, provided Eq. (36) together with (38). For instance, a straightforward generalization of the canonical scalar field is to impose $K_{,XX} = 0$ or, more explicitly, a model with

$$K(\phi, X) = h(\phi)X - V(\phi), \quad (39)$$

where $h(\phi)$ and $V(\phi)$ are arbitrary functions of the scalar field. According to (36), this condition also implies that f_3 must be a constant, and, additionally, Eq. (38) further imposes that f_1 also needs to be a constant. Notice that for this particular case we have that $K_{,X}X - K = V(\phi)$, and therefore all the nonminimal derivative couplings in the action are through the functions $f_1(T)$ and $f_3(T)$, as in the canonical scalar field case. Thus, our action for (39) with f_1 and f_3 constants will adopt the form

$$S = \int d^4x \sqrt{-g} \{ \hat{g}(\phi)R + \hat{h}(\phi)G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \hat{K}(\phi, X) \}, \quad (40)$$

where $\hat{g}(\phi) = f_1 + f_3 g(\phi)$, $\hat{h}(\phi) = f_3 h(\phi)$ and $\hat{K}(\phi, X) \equiv f_2(T) + K(\phi, X)$. One can now easily verify that this action is of the Horndeski type with

$$\begin{aligned} G_2 &= \hat{K}(\phi, X), \\ G_3 &= 0, \\ G_4 &= \hat{g}(\phi), \\ G_5 &= -\hat{h}(\phi)\phi. \end{aligned} \quad (41)$$

We will not explore more general models here, but we will stress that for any choice of nonminimal couplings, i.e., given $f_1(T)$ and $f_3(T)$, the conditions expressed in (36) and (38) will impose tight constraints on the possible form of $K(\phi, X)$. Analogously, inverting the argument, given a function $K(\phi, X)$, said conditions will strongly limit the allowed functions $f_1(T)$ and $f_3(T)$. As a final comment, let us remember that our aim here was to obtain theories with second-order equations of motion, so our conditions will be sufficient to avoid the Ostrogradski instability, but models free from such instabilities might also exist. Finally, one could also allow for models where the conformal mode can be excited in a healthy way within the context of K -essence models by going to the multiscalar representation of the theory. In such a formulation, analogous equations might be obtained relating the possible forms of the functions f and K to avoid the Ostrogradski instability for the conformal mode.

V. VECTOR FIELDS

In the preceding sections, we have considered a scalar field as our matter Lagrangian. Let us now consider the case of a massive vector field of which the Lagrangian is given by

$$\mathcal{L}_m = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M^2A^2, \quad (42)$$

with $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ and M the mass of the field. Consequently, the energy-momentum tensor reads

$$T_{\mu\nu} = -F_{\mu\alpha}F_\nu{}^\alpha + \frac{1}{4}g_{\mu\nu}F^2 - \frac{M^2}{2}g_{\mu\nu}A^2 + M^2A_\mu A_\nu, \quad (43)$$

of which the trace and contraction with the Ricci scalar become, respectively,

$$T = -M^2A^2, \quad (44)$$

$$\begin{aligned} R_{\mu\nu}T^{\mu\nu} &= \frac{1}{4}(RF_{\mu\nu}F^{\mu\nu} - 4R_{\mu\nu}F^{\mu\alpha}F^\nu{}_\alpha) + \frac{1}{4}RF^2 \\ &\quad + M^2G_{\mu\nu}A^\mu A^\nu. \end{aligned} \quad (45)$$

As expected, the trace T vanishes for the case of a massless vector field as it corresponds to a conformally invariant theory. Along the very same line of reasoning as discussed in the previous section for the scalar field case, these theories will typically contain the Ostrogradski instability arising from the nonminimal coupling of the vector field with the scalar curvature.

In principle, one could also attempt to obtain the corresponding Hordenski vector-tensor interaction. In this case, however, the allowed interaction between the Faraday tensor $F_{\mu\nu}$ and the Ricci curvature are much more restricted, and in fact there is a unique coupling between them that leads to second-order equations of motion. Such a coupling was already computed by Horndeski [42], and it is given by

$$\begin{aligned} L^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta} &= RF_{\mu\nu}F^{\mu\nu} - 4R_{\mu\nu}F^{\mu\sigma}F^\nu{}_\sigma \\ &\quad + R_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta}, \end{aligned} \quad (46)$$

where $L^{\alpha\beta\gamma\delta} = -\frac{1}{2}\epsilon^{\alpha\beta\mu\nu}\epsilon^{\gamma\delta\rho\sigma}R_{\mu\nu\rho\sigma}$ stands for the double dual Riemann tensor. The cosmology of this type of interaction was studied in Ref. [43], and its stability was thoroughly analyzed in Ref. [44]. A careful comparison between (45) and (46) shows that only the first two terms in the right-hand side of (46) appear in (45), but the last one in (46) is absent. Thus, one is led to infer that for both massive and massless vector fields actions of the form (1) cannot lead to second-order equations of motion. It should be noticed that kinetic interactions involving second derivatives of the vector field as those considered in Ref. [45] cannot appear because of the gauge invariance of the kinetic term in the Proca action.

Let us also show for completeness the multiscalar representation for the vector field just studied. After performing the redefinitions introduced in Sec. II, the action (6) becomes

$$\begin{aligned} S &= \int d^4x \sqrt{-\tilde{g}} \left\{ e^{4\Omega} \tilde{\mathcal{L}} - \frac{1}{16\pi G} [\tilde{R} - 6(\partial\Omega)^2] - \varphi_3 M^2 \tilde{G}^{\mu\nu} A_\mu A_\nu - \varphi_3 e^{-2\Omega} \left(\frac{1}{4} \tilde{F}^2 \tilde{R} - \tilde{R}_{\mu\nu} \tilde{F}^{\mu\alpha} \tilde{F}^\nu{}_\alpha \right) \right. \\ &\quad + 2\varphi_3 \left[e^{-2\Omega} \left(\frac{1}{4} \tilde{F}^2 \tilde{g}^{\mu\nu} - \tilde{F}^{\mu\alpha} \tilde{F}^\nu{}_\alpha \right) - M^2 (\tilde{A}^2 \tilde{g}^{\mu\nu} - \tilde{A}^\mu \tilde{A}^\nu) \right] \tilde{\nabla}_\mu \tilde{\nabla}_\nu \Omega \\ &\quad \left. - 2\varphi_3 \left[e^{-2\Omega} \left(\frac{1}{4} \tilde{F}^2 \tilde{g}^{\mu\nu} - \tilde{F}^{\mu\alpha} \tilde{F}^\nu{}_\alpha \right) + M^2 (\tilde{A}^2 \tilde{g}^{\mu\nu} + \tilde{A}^\mu \tilde{A}^\nu) \right] \tilde{\nabla}_\mu \Omega \tilde{\nabla}_\nu \Omega + e^{4\Omega} \mathcal{L}_m(e^{2\Omega} \tilde{g}_{\mu\nu}, \tilde{A}) \right\}, \end{aligned} \quad (47)$$

where $\tilde{F}^2 \equiv \tilde{F}^{\mu\nu} F_{\mu\nu}$, $\tilde{F}^{\mu\nu} \equiv \tilde{g}^{\mu\alpha} \tilde{g}^{\nu\beta} F_{\alpha\beta}$, and $\tilde{F}^\mu{}_\nu \equiv \tilde{g}^{\mu\alpha} F_{\alpha\nu}$. Here again, one can identify the instability problems discussed above. Again, as it happened for the scalar field case, the fact that φ_3 is an auxiliary field will typically lead to undesired terms even if the structure of the couplings of the vector field, curvature, and the conformal mode are appropriate. However, for theories linear in $R_{\mu\nu} T^{\mu\nu}$, φ_3 appearing in (47) will not be an auxiliary field but a simple constant parameter. We will assume this in the following discussion.

The first problem with the vector field could have arisen from the direct coupling of A_μ to the scalar curvature since such couplings usually introduce additional modes associated with an extra propagating degree of freedom for the vector field. However, for the simple case of a Proca field, the direct coupling to the curvature happens through the Einstein tensor, which guarantees the absence of such an extra mode. This can easily be seen by resorting to the Stueckelberg trick, i.e., if we consider the purely longitudinal mode $A_\mu = \partial_\mu \theta$. Such a mode will couple to the Einstein tensor precisely in the form required to be of the Horndeski form, and therefore it will not introduce any additional modes. This kind of interaction was studied in Ref. [46] as a nonminimal coupling to the electromagnetic field that could serve as a mechanism to generate magnetic fields from neutral but rotating bodies. It also arises in a natural manner within the context of Weyl geometries [47].

Another problem with the above action is the derivative nonminimal coupling, i.e., the coupling between $F_{\mu\nu}$ and the curvature. Again, this is not of the Horndeski vector-tensor type of interaction, and thus it will lead to higher-order equations of motion with the Ostrogradski instability. This would be enough to prove the instability of these theories in curved backgrounds. However, also the coupling of the conformal mode to $F_{\mu\nu}$ is pathological. Such a coupling possesses a structure of the form $e^{2\Omega}(F^{\mu\alpha} F^\nu{}_\alpha - \frac{1}{4} F^2 \tilde{g}^{\mu\nu}) \tilde{\nabla}_\mu \tilde{\nabla}_\nu \Omega$. If we focus on second time derivatives, we find $(1/2 F_{0i} F^{0i} - 1/4 F_{ij} F^{ij}) \tilde{\nabla}^0 \tilde{\nabla}_0 \Omega$ that will lead to higher-order equations of motion due to the presence of the $F_{0i} F^{0i}$ term.

We can conclude that gravitational theories described by $f(R, T, R_{\mu\nu} T^{\mu\nu})$ lead to Ostrogradski instabilities in a very general manner when coupled to vector fields. We therefore find it reasonable to abandon the universality of the nonminimal couplings of these theories and consider them only for specific forms of the matter sector, e.g., a certain class of scalar fields, for which the instabilities can be avoided.

VI. PARTICULAR MODELS

In this section, we shall illustrate our general discussions with specific examples of the gravitational theories under study. We will choose the models so we can explicitly see some of the points raised above. For simplicity, in the

following, we will consider a canonical scalar field as corresponding to the matter Lagrangian.

A. Model I: $f(R, T, R_{\mu\nu} T^{\mu\nu}) = \alpha R^n + \beta(R_{\mu\nu} T^{\mu\nu})^m$

We start by considering a superposition of two power laws so that the action takes the form

$$S_I = \int d^4x \sqrt{-g} \left[\alpha R^n + \beta (G^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + R V(\phi))^m - \frac{1}{2} (\partial\phi)^2 - V(\phi) \right]. \quad (48)$$

The definitions for the auxiliary fields $\varphi_i = -\frac{\partial f}{\partial \chi_i}$ in this case yield $\varphi_1 = -n\alpha\chi_1^{n-1}$ and $\varphi_3 = -m\beta\chi_3^{m-1}$. We see that the condition for the invertibility of these relations is that m and n are different from 1, i.e., whenever the dependence on R and $R_{\mu\nu} T^{\mu\nu}$ is not linear. In the case of $n = m = 1$, the theory is stable because we simply have the Einstein–Hilbert term plus a canonical scalar field with a nonminimal derivative coupling to the Einstein tensor, which, as discussed several times throughout this work, corresponds to a healthy coupling. Moreover, even for $n \neq 1$, the theory will be free of Ostrogradski instabilities. To see this more clearly, it is convenient to go to the multiscalar representation. After the appropriate conformal transformation, our action for arbitrary n and $m = 1$ reads

$$S_I = \int d^4x \sqrt{-\tilde{g}} \left\{ e^{4\Omega} \mathcal{U}(\Omega) - \frac{1}{16\pi G} [\tilde{R} - 6(\partial\Omega)^2] - \beta [\tilde{G}_{\mu\nu} \partial^\mu \phi \partial^\nu \phi + 2(\partial_\alpha \Omega \partial^\alpha \Omega)^2 + (\partial\Omega)^2 (\partial\phi)^2 + 2(\tilde{g}^{\mu\nu} (\partial\phi)^2 - (\partial^\mu \phi \partial^\nu \phi)) \tilde{\nabla}_\mu \tilde{\nabla}_\nu \Omega] + e^{4\Omega} \mathcal{L}_m(\phi, e^{2\Omega} \tilde{g}_{\mu\nu}) \right\}. \quad (49)$$

Here again, we see that the conformal mode couples to the scalar field in the appropriate way not to lead to higher-order equations of motion, and the derivative nonminimal coupling of the scalar field ϕ belongs to the Horndeski type. Therefore, for this case, the theory avoids the Ostrogradski instability. However, for arbitrary m , the coupling $R_{\mu\nu} T^{\mu\nu}$ will spoil the nice structure of the above equation, and therefore the Ostrogradski instability will reappear. A particular case with $n = 1$ and arbitrary m is a special case that we treat in more detail in the next subsection.

B. Model II: $f(R, T, R_{\mu\nu} T^{\mu\nu}) = -\frac{R}{16\pi G} + \beta(R_{\mu\nu} T^{\mu\nu})^m$

This is a particular case of the class of models above with $n = 1$. This case is special because the auxiliary field φ_1 that is usually mapped into the conformal mode cannot be defined due to the noninvertibility of the Legendre transformation for χ_1 . Thus, we can only introduce the field χ_3 ,

and it is related to φ_3 by $-\varphi_3 = -m\beta\chi_3^{m-1}$. Again, for $m = 1$, this transformation is not invertible and must be treated separately (see the end of this section). The action in terms of the auxiliary fields now yields

$$S_{\text{II}} = \int d^4x \sqrt{-g} \left[\mathcal{U}(\varphi_3) - \left(\frac{1}{16\pi G} + \varphi_3 V(\phi) \right) R - \varphi_3 G^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \mathcal{L}_m(\phi, g_{\mu\nu}) \right], \quad (50)$$

with $\mathcal{U}(\varphi_3) = \beta(1-m)\left(\frac{-\varphi_3}{m\beta}\right)^{\frac{m}{m-1}}$. Accordingly, we can see that the conformal mode will still be excited, but now it will be originated from the auxiliary field φ_3 . Therefore, the auxiliary field φ_3 will disappear from the action after performing the conformal transformation. In fact, the required conformal factor is now given by $\varphi_3 = \frac{e^{-2\Omega}-1}{16\pi G V(\phi)}$, and the resulting action reads

$$S_{\text{II}} = \int d^4x \sqrt{-\tilde{g}} \left\{ e^{4\Omega} \mathcal{U}(\Omega, \phi) - \frac{1}{16\pi G} (\tilde{R} + 6(\partial\Omega)^2) + \frac{1 - e^{-2\Omega}}{16\pi G V(\phi)} [\tilde{G}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + 2(\partial^\alpha \Omega \partial_\alpha \phi)^2 + (\partial\Omega)^2 (\partial\phi)^2 + 2(\tilde{g}^{\mu\nu} (\partial\phi)^2 - \partial^\mu \phi \partial^\nu \phi) \tilde{\nabla}_\mu \tilde{\nabla}_\nu \Omega] + e^{4\Omega} \mathcal{L}_m(\phi, e^{2\Omega} \tilde{g}_{\mu\nu}) \right\}, \quad (51)$$

where we can see that all the interactions containing second-order derivatives have the appropriate form to avoid higher-order equations of motion. Thus, even though the starting action is not linear in $R_{\mu\nu} T^{\mu\nu}$, the fact that it is linear in the scalar curvature makes the theory free from Ostrogradski instabilities. The underlying reason, as we have shown, is that one cannot define the auxiliary field φ_1 , and this actually ensures that it is the auxiliary field φ_3 that is the one that will be mapped into the conformal mode, and therefore it will not be present in the final action after the conformal transformation.

The case with $m = 1$ was studied in Ref. [12] and corresponds to the particular model described by the action

$$S_{\text{II}} = \int d^4x \sqrt{-g} \{ [1 + \alpha V(\phi)] R + \alpha G^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \mathcal{L}_m(\phi, g_{\mu\nu}) \}, \quad (52)$$

where we see that the theory belongs to the Horndeski class for the scalar field ϕ and the conformal mode is not excited.

C. Model III: $f(R, T, R_{\mu\nu} T^{\mu\nu}) = \alpha R(1 + \beta R_{\mu\nu} T^{\mu\nu})$ case

Now, we will consider a slightly more involved model in which the different arguments of the function f appear mixed. The field redefinition from the fields χ_i to the fields φ_i is given (notice that we only have χ_1 and χ_3 but not χ_2

because there is no dependence on T in the action): $\varphi_1 = -\alpha(1 + \beta\chi_3)$ and $\varphi_3 = -\alpha\beta\chi_1$. This field redefinition is actually an injective transformation for $\alpha \neq 0$ and $\beta \neq 0$ so that it is perfectly valid. Moreover, notice that the matrix $\partial^2 f / \partial \chi_i \partial \chi_j$ has determinant $\det(\partial^2 f / \partial \chi_i \partial \chi_j) = -(\alpha\beta)^2$, which is nonvanishing, and therefore the Legendre transformation is legitimate. After introducing the auxiliary fields φ_i , the action reads

$$S_{\text{III}} = \int d^4x \sqrt{-g} [\mathcal{U}(\varphi_1, \varphi_3) - (\varphi_1 + \varphi_3 V(\phi)) R - \varphi_3 G^{\mu\nu} \partial_\mu \phi \partial_\nu \phi] \quad (53)$$

with $\mathcal{U}(\varphi_1, \varphi_3) = \varphi_3 \chi_3 = -\frac{1}{\beta} \varphi_3 (1 + \frac{1}{\alpha} \varphi_1)$. We see in this case that both φ_1 and φ_3 appear linearly, so they could be seen as Lagrange multiplier fields. However, since $\partial^2 \mathcal{U} / \partial \varphi_i \partial \varphi_j$ is a nondegenerate matrix (which coincides with the inverse of $-\partial^2 f / \partial \chi_i \partial \chi_j$, as we explained above), the equations of motion for φ_1 and φ_3 actually allow us to algebraically solve them, so they are indeed auxiliary fields. This will be the main obstruction for the stability of this theory because that will make the term $G^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$ appear nonlinearly in the action, and therefore the Ostrogradski instability will be present. Notice also that the conformal mode will be excited and can be associated with φ_1 so that we do not encounter the situation that allowed us to assure the stability of model II in which the field φ_3 could be used to excite the conformal mode and therefore disappeared from the transformed action.

VII. CONCLUSIONS

In this work, we have considered a class of universal nonminimally coupled theories of gravity in which the nonminimal coupling is achieved through couplings of the energy-momentum tensor of the matter sector to the curvature; i.e., the gravitational Lagrangian is of the form $f(R, T, R_{\mu\nu} T^{\mu\nu})$. These theories have received some attention in recent literature and might offer interesting cosmological applications. The aim of this work has been to clarify some issues concerning the consistency and stability of such theories. First of all, we discussed the fact that the energy-momentum tensor appears at the level of the action, and this might pose consistency problems since the energy-momentum tensor is a quantity to be derived from the action. Another way of expressing the potential inconsistency is that the energy-momentum tensor is the conserved current associated with infinitesimal translations. However, if plugged back in the action, this statement is no longer true, and, in fact, such a quantity will not be conserved anymore. As we argued in Sec. II, for the considered theories, the object entering the action is not really the conserved energy-momentum tensor and should be regarded simply as an operational manner to define the theory.

Second, another potential problem with these theories is the presence of second-order derivatives in the Lagrangian that will typically lead to higher-order equations of motion with the associated Ostrogradski instability. The study of this issue has comprised the majority of this work. We have first shown the problem for a general matter Lagrangian. Moreover, we have also allowed for cases in which the conformal mode can also be excited. We have found two sources from which the Ostrogradski might appear. The first one is the generation of derivative nonminimal couplings of the matter fields to curvature, while the second one corresponds to the conformal mode entering with second derivatives in the action. After presenting the general framework to analyze the presence of Ostrogradski instabilities for these types of theories, we considered specific cases for the matter sector, namely, a canonical scalar field, a K -essence field, and, finally, a vector field. For the canonical scalar field, we have found conditions under which both the scalar field and the conformal mode lead to second-order equations of motion. For that, we have imposed that the corresponding terms in the action are of the Horndeski form, as those are the most general terms for scalar-tensor theories yielding second-order equations of motion. However, as we have already commented, those do not correspond to the most general scalar-tensor theory free from the Ostrogradski instability so that our results can be viewed as sufficient conditions, although more general theories might still be possible. Subsequently to the detailed analysis for the canonical scalar field case, we have considered general K -essence models. In these theories, the presence of one extra free function opens new possibilities. Thus, we have obtained the conditions to avoid higher-order equations of motion for general K -essence models.

Finally, after studying the case of scalar fields, we have considered the case of a Proca vector field. Unlike

the scalar field case, we have found that it is not possible to obtain viable models free from the Ostrogradski instability. This led us to conclude that the universal nature of the nonminimal coupling should be abandoned because, although it is possible to obtain stable models for scalar fields, it is troublesome to have couplings to vector fields.

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