

Discrete symmetries of unitary minimal conformal theories

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Abstract

We classify the possible discrete (finite) symmetries of two-dimensional critical models described by unitary minimal conformally invariant theories. We find that all but six models have the group Z_2 as maximal symmetry. Among the six exceptional theories, four have no symmetry at all, while the other two are the familiar critical and tricritical 3-Potts models, which both have an S_3 symmetry. These symmetries are the expected ones, and coincide with the automorphism groups of the Dynkin diagrams of simply-laced simple Lie algebras ADE. We note that extended chiral algebras, when present, are almost never preserved in the frustrated sectors.

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1 Introduction

Critical models with infinite correlation lengths fall in universality classes according to their long range behaviours. In the vicinity of the critical point, the long range correlations, in their scaling limit, are those of a continuous field theory, which becomes massless, conformally invariant at the critical point.

In two dimensions, conformal field theories have much more structure than in higher dimensions. In order to expose their full content, one may put the conformal field theory in various geometries, with various boundary conditions. In particular, the consistent formulation of a conformal theory on the torus has proved to be extremely constraining for the theory itself [1]. Consistent means, among other things, that the periodic partition function must take the same value on conformally equivalent tori: it must be modular invariant.

It turns out that modular invariance is formidably restrictive, putting very strong constraints on the field content of the theory. It is in fact so restrictive that a classification program of all consistent conformal theories has been initiated, following the seminal ideas of [1]. A first step towards this vastly ambitious goal was taken in [2, 3], where a complete list was given of all possible modular invariant partition functions of so-called minimal conformal theories. Remarkably, this list is structured in a few infinite series that follow an ADE pattern.

For the unitary theories, it is now established [4] that the entries in the ADE list, restricted to the unitary cases, are realized by actual lattice models [5, 6], namely the RSOS and later the dilute RSOS models, all defined in terms of ADE Dynkin diagrams. It means that for every modular invariant partition function of the list, there is (at least) one statistical lattice model whose critical partition function is the given item in the list. In some cases, different lattice models are known which have their critical partition function equal to the same modular invariant.

None of these models has a continuous symmetry group, but most of them have finite symmetries [7, 8]. Well-known cases include the Ising model, with a Z_2 symmetry, and the 3-Potts model, with an S_3 symmetry. Zuber [9] was the first to investigate, in a systematic way, the presence of a discrete symmetry from the knowledge of the critical modular invariant (*i.e.* periodic) partition function. It is our purpose to pursue his analysis, and indeed to determine the maximal symmetry group of all unitary minimal conformal theories. The importance of knowing the symmetry is two-fold.

On one hand, it helps identify the conformal theory describing the critical regime of a lattice model¹ (or vice versa). The symmetries we determine are those of continuous field theories, which are the continuum limits of critical lattice models. They are generally expected to be the symmetries already present on the lattice. Indeed in the present case, the symmetries we find are realized in the discrete RSOS models. Thus the symmetry of a given conformal theory is presumably shared by all the lattice models in the universality

¹Unfortunately almost all models we examine have the same symmetry, so its knowledge does not help much here.

class corresponding to that conformal theory.

On the other hand, the question arises precisely as to what extent a modular invariant partition function specifies a unique universality class (a conformal theory). Indeed it only fixes the content of the periodic sector, and it is not clear that this is saying anything about the non-periodic sectors that might be consistently added. Apart from inconsistencies that may come from other sources than modular covariance, we see three basic reasons why the univocity of the association of a conformal field theory with a modular invariant partition function may fail.

First, it is conceivable that a partition function be compatible with a specific symmetry, but in more than one way. In this case, one would conclude that different realizations of the symmetry lead to inequivalent conformal theories hence to different universality classes, since the field contents and the charges are different. Our results show that this situation does not occur (in the models that we have examined): when a symmetry is present, it is always realized in exactly one way.

A second situation is when the partition function is compatible with a given symmetry, which however is not fully exploited. In other words, the Hilbert spaces corresponding to twisted boundary conditions, that could, *in principle*, be adjoined to the periodic sector to form a consistent conformal theory, simply do not exist. After all, the periodic sector of any model is self-consistent, and there is no way to say whether or not the frustration operators are present in the lattice model. Indeed there is no reason as to why a model should be forced to use the maximal symmetry which is available.

The previous two situations somehow rely on the point of view that the non-periodic sectors are organized by a symmetry, which is also the stand we take in this article. That is to say, the non-periodic boundary conditions are obtained from a group operation on the microscopic variables that leaves the periodic Hamiltonian invariant. There might be a last possibility that the different sectors are not related to the existence of a symmetry.

Examples of models falling in the last two categories have been suggested in [10], where $c < 1$ unitary models were obtained through suitable projections of the spectrum of the XXZ Heisenberg chains. However no example was given of lattice models with the predicted spectra.

Finally, we would like to stress the fact that the symmetry groups we determine are computed with respect to a built-in chiral algebra, which is here the conformal algebra. This is made manifest because our analysis relies on the assumption that all Hilbert spaces decompose into Virasoro representations, so that all partition functions are expressed in terms of conformal characters. Another starting point, based for instance on an extended chiral algebra, may lead to different results. Examples of such situations will be encountered in Section 6, where block diagonal modular invariant partition functions are considered.

2 Statement of the problem and results

The definition of a lattice model requires to specify the boundary conditions. In a toroidal geometry, one identifies the opposite edges of the rectangle (soon to become a parallelogram), and the boundary conditions say how the spin configurations on these edges are related. The periodic boundary conditions are the simplest ones: the configurations on opposite edges are equal.

If the Hamiltonian (or the Boltzmann weights) for the periodic boundary conditions are invariant under a finite symmetry group G , one may impose twisted boundary conditions in which the spin variables on opposite edges are related by group operations. On an $L \times M$ rectangular lattice, they are

$$\sigma_{L+1,j} = {}^g\sigma_{1,j}, \quad \sigma_{i,M+1} = {}^{g'}\sigma_{i,1}, \quad \text{with } g, g' \in G. \quad (2.1)$$

Equivalently, twisted boundary conditions can be implemented by the insertion, in the periodic system, of frustration lines.

For each boundary condition, one can compute a partition function $Z_{g,g'}(L, M)$. Let us note that one should restrict to twisting elements g, g' that commute, because $\sigma_{L+1,M+1} = {}^{gg'}\sigma_{1,1} = {}^{g'}g\sigma_{1,1}$. Furthermore, if G is non-Abelian (like in the 3-Potts model), the symmetry of the model with the boundary conditions (g, g') is broken down² to a subgroup of G , which is the centralizer of $\{g, g'\}$. The reason is that if the Hamiltonian in the bulk has the invariance $H\{\sigma\} = H\{g\sigma\}$ for all $g \in G$, the boundary conditions (2.1) may not be invariant under all of G .

In the continuum limit, the density of lattice points increases, and one obtains a full rectangle, of sides L and iM in the complex plane. The underlying field theory being scale invariant, the partition functions can only depend on the ratio $\tau = iM/L$. Equivalently, one may normalize L to 1, and consider the rectangle with sides 1 and τ , where τ is purely imaginary and $-i\tau$ strictly positive. One can be more general and take an arbitrary torus, whose shape is a parallelogram rather than a rectangle. In the complex plane, this amounts to give the two independent periods 1 and τ , and the identifications $z \simeq z + 1$ and $z \simeq z + \tau$. The modulus τ now belongs to the upper half plane, $\text{Im } \tau > 0$. The boundary conditions then relate the field configurations on the opposite sides of the parallelogram,

$$\phi(z + 1) = {}^g\phi(z), \quad \phi(z + \tau) = {}^{g'}\phi(z), \quad (2.2)$$

and define the frustrated partition functions $Z_{g,g'}(\tau)$.

In conformal field theory, these functions can be computed in the Hamiltonian formalism (*i.e.* the transfer matrix formalism), with the following result [1, 7, 11]

$$Z_{g,g'}(\tau) = \text{Tr}_{\mathcal{H}_g} \left(q^{(L_0 - c/24)} \bar{q}^{(\bar{L}_0 - c/24)} g' \right), \quad q = e^{2i\pi\tau}. \quad (2.3)$$

²In that sense, an extended chiral algebra can also be broken in the non-periodic sectors, down to the conformal algebra. Examples include the critical 3-Potts model, where the W_3 algebra is broken by the antiperiodic boundary conditions, see Section 6.4.

In this expression, $\mathcal{H}_g$ is the Hilbert space of states that live on the fixed time slices (lines of constant $\text{Im } z$ in the complex plane) in the presence of boundary conditions twisted by g in space. It encodes the information about the boundary condition in space in the same way that a row-to-row transfer matrix includes the boundary condition along a single row. Thus $\mathcal{H}_g$ determines the state or field content of the theory in the g -sector. The boundary condition in time is effected by the insertion of g' . The operators L_0 and $\bar{L}_0$ are the generators of dilations in the z and $\bar{z}$ variables, combinations of which yield the Hamiltonian and momentum operators. Finally the number c is the central charge of the conformal theory in question, namely the central term occurring in the Virasoro algebra.

Conformal symmetry implies that each space $\mathcal{H}_g$ is made up of representations of a pair of Virasoro algebras $\text{Vir}_c \times \text{Vir}_c$ with equal central term c

$$\mathcal{H}_g = \bigoplus_{i,j} M_{ij}^{(g)} (\mathcal{R}_i \otimes \mathcal{R}_j), \quad (2.4)$$

where the numbers $M_{i,j}^{(g)}$ are multiplicities, *i.e.* they are non-negative integers. The labels i, j specify the inequivalent representations of Vir_c .

Assume now that the theory has the discrete symmetry group G . It means that its energy-momentum tensor is left invariant by G , with the consequence that the group action can only mix together equivalent representations occurring in each $\mathcal{H}_g$. So if g' has order N , and if $\mathcal{R}_i \otimes \mathcal{R}_j$ occurs in $\mathcal{H}_g$ with multiplicity $M_{i,j}^{(g)}$, the action of g' on them is by some matrix of order N . That matrix can be diagonalized, and yields a diagonal action by N -th roots of unity³. Thus one can write, in the g -sector,

$$g'(\mathcal{R}_i \otimes \mathcal{R}_j)_k = \zeta_N^{Q(g;i,j;k;g')} (\mathcal{R}_i \otimes \mathcal{R}_j)_k, \quad k = 1, 2, \dots, M_{ij}^{(g)}, \quad (2.5)$$

with $\zeta_N = e^{2i\pi/N}$. The integer $Q(g;i,j;k;g')$, defined modulo N , can be called a g' -charge (or a parity for an order 2 element).

Using now the definition of the Virasoro characters $\chi_i(q) = \text{Tr}_{\mathcal{R}_i} q^{(L_0 - c/24)}$, the decomposition of the spaces $\mathcal{H}_g$ and the action of the group on their content, one obtains the following form for the frustrated partition functions on a torus of modulus τ

$$Z_{g,g'}(\tau) = \sum_{i,j} \left[\sum_{k=1}^{M_{ij}^{(g)}} \zeta_N^{Q(g;i,j;k;g')} \right] \chi_i^*(q) \chi_j(q), \quad (2.6)$$

from which the charges of the various fields can be easily read off. This formula is the first fundamental ingredient of our analysis.

From the conformal point of view, the description of tori in terms of complex moduli τ is redundant. Indeed τ and $\tau + 1$ clearly specify the same torus because the identifications

³In concrete models, the group of symmetry has some definite action on the variables/primary fields, which leads to a preferred basis. From the pure representation theory, any choice of basis in the degenerate modules $(\mathcal{R}_i \otimes \mathcal{R}_j)_k$ is as good as any other.

$z \simeq z + 1 \simeq z + \tau + 1$ are manifestly equivalent to $z \simeq z + 1 \simeq z + \tau$. Moreover the moduli τ and $-\frac{1}{\tau}$ determine conformally equivalent tori. This may be less apparent but it follows from the conformal transformation $w = -\frac{z}{\tau}$, in terms of which the identifications become $w \simeq w - 1 \simeq w - \frac{1}{\tau}$. The corresponding two transformations, $T : \tau \rightarrow \tau + 1$ and $S : \tau \rightarrow -\frac{1}{\tau}$, generate the modular group $PSL_2(\mathbb{Z})$

$$PSL_2(\mathbb{Z}) = \left\{ \tau \rightarrow \frac{a\tau + b}{c\tau + d} : a, b, c, d \in \mathbb{Z} \text{ and } ad - bc = 1 \right\}, \quad (2.7)$$

which can also be presented as $PSL_2(\mathbb{Z}) = \langle S, T \mid S^2 = (ST)^3 = 1 \rangle$. Two tori are conformally equivalent if and only if their moduli are related by a modular transformation of $PSL_2(\mathbb{Z})$.

The existence of modular transformations has immediate and dramatic consequences on the frustrated partition functions [1, 9]. Being partition functions of conformally invariant theories, they should all be invariant under modular transformations, except for a change in the boundary conditions. Under $\tau' = \tau + 1$, the field configurations are related by $\phi(z + \tau') = g g' \phi(z)$, so that the effective twist is $g g'$ along τ' . The other transformation, $\tau' = -\frac{1}{\tau}$, is implemented by the conformal transformation $w = -\frac{z}{\tau}$, which has the property of exchanging the two periods: the identification $z \simeq z + 1$ goes to $w \simeq w + \tau'$, while $z \simeq z + \tau$ becomes $w \simeq w - 1$. Thus it follows that the two twists are in effect exchanged, with one of them inversed. Putting all together, one finds the transformation law of the partition functions under the modular group [9]:

$$Z_{g,g'}(\tau) = Z_{g,gg'}(\tau + 1) = Z_{g'^{-1},g}(-\frac{1}{\tau}) = Z_{g^a g'^c, g^b g'^d}(\frac{a\tau + b}{c\tau + d}). \quad (2.8)$$

This is the second fundamental ingredient. Let us note that the transformation $\tau \rightarrow -\frac{1}{\tau}$ being an involution implies the identities $Z_{g,g'}(\tau) = Z_{g^{-1},g'^{-1}}(\tau)$. Thus the torus partition functions transform properly under $PSL_2(\mathbb{Z})$, even in the presence of boundary conditions.

One sees that the partition functions are not individually invariant under the full modular group but only under the subgroup of $PSL_2(\mathbb{Z})$ which fixes their boundary conditions [9]. Depending on g, g' , the subgroups are various congruence subgroups of level N if the orders of the elements of G are divisors of N . For a generic pair g, g' , the individual function $Z_{g,g'}$ is invariant under the level N principal congruence subgroup $\Gamma(N)$ ($\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N}$), while $Z_{g,1}$ and $Z_{1,g'}$ are respectively invariant under $\hat{\Gamma}^0(N)$ ($\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \pmod{N}$) and $\hat{\Gamma}_0(N)$ ($\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \pmod{N}$). Finally the partition function for periodic boundary conditions, $Z_{1,1}(\tau)$, is fully modular invariant. Each of these subgroups has finite index in the modular group; the precise action of the cosets on the set of functions $Z_{g,g'}$ was discussed in [9].

The main question that now arises is whether the general form (2.6) of the partition functions is consistent with the modular transformations (2.8). In a way, the answer was the main result of Cardy in [1]: the two requirements are not compatible, unless the field content of the various sectors, the possible symmetries and the charges of the fields under that symmetry are chosen in a very specific way. This explains why the requirement of

modular invariance or covariance is so constraining and powerful. Our purpose is to find all solutions to this problem for the particular class of unitary minimal conformal theories.

Unitary minimal conformal theories are among the simplest examples of rational theories (for general background, see [12]). They have a central charge given by $c(m) = 1 - \frac{6}{m(m+1)} < 1$, where $m \geq 3$ is an integer. For fixed m , the Virasoro algebra with central term $c = c(m)$ has finitely many unitary representations, labelled by pairs (r, s) of integers satisfying $1 \leq s \leq r \leq m - 1$. The characters of these representations are known functions of τ [13], and their modular transformations can be explicitly computed (see below). Therefore for these theories, the problem is posed in concrete terms.

A first, substantial step is to look at all possible periodic partition functions. One looks for modular invariant functions, which are sesquilinear forms in the Virasoro characters, of the type

$$Z_{1,1} = \sum_{(r,s),(r',s')} [\chi_{(r,s)}(\tau)]^* M_{(r,s),(r',s')}^{(1)} [\chi_{(r',s')}(\tau)], \quad (2.9)$$

for non-negative integral coefficients $M_{(r,s),(r',s')}^{(1)}$. The representation $\mathcal{R}_{(1,1)} \otimes \mathcal{R}_{(1,1)}$ is the only one to contain the vacuum of the theory. Since the vacuum is certainly in the periodic sector $\mathcal{H}_1$, its unicity requires that none of the other spaces $\mathcal{H}_g$, $g \neq 1$, contain it. One thus imposes $M_{(1,1),(1,1)}^{(g)} = \delta_{g,1}$, which fixes the normalization of $Z_{1,1}$.

The classification of all modular invariant functions with these properties was accomplished by Cappelli, Itzykson and Zuber [2], and by Kato [3], and remains one of the most remarkable results in conformal theory. The functions $Z_{1,1}$, labelled by pairs of simply-laced Lie algebras, are listed in Table 1. By using the symmetry $\chi_{(r,s)} = \chi_{(m-r,m+1-s)}$, one has for convenience extended the range of (r, s) to $\{1, \dots, m-1\} \times \{1, \dots, m\}$, which covers twice the original set (called the Kac table).

One can see from the table that, for fixed m , there is only a very limited number of possible periodic partition functions, most often only two. Those of the series (A_{m-1}, A_m) have been identified as describing the multicritical points in RSOS models [14]. The 3-Potts model corresponds to the (A_4, D_4) theory [15, 7, 8], and the tricritical 3-Potts model to (D_4, A_6) . All of them are partition functions of dilute RSOS models [5, 6, 4].

The list of modular invariant partition functions is our starting point. For each such function $Z_{1,1}$, we want to find all possible sets of functions $Z_{g,g'}$ which satisfy the above two conditions: they must form a closed set under the action of the modular group, and they must be of the form (2.6). The known function $Z_{1,1}$ fixes the form of all $Z_{1,g'}$, which themselves yield other functions $Z_{g,g'}$ by modular transformations. All these other functions must have the required properties.

We have mentioned above that such sets are consistent only for group elements g, g' which commute, and which therefore generate an Abelian group. In practice one first looks for finite cyclic symmetry groups. The existence of a cyclic Z_N symmetry (of N^2 functions Z_{g^i, g^j} , with g the generator) implies that G contains a cyclic subgroup. Subtleties can arise when several cyclic symmetries are compatible with a given partition function. Suppose for

m	Periodic partition function $Z_{1,1}$	Name
$m \geq 3$	$\frac{1}{2} \sum_{r=1}^{m-1} \sum_{s=1}^m \chi_{(r,s)} ^2$	(A_{m-1}, A_m)
$m = 4\ell + 1$	$\frac{1}{2} \sum_{r=1}^{m-1} \left\{ \sum_{\substack{s=1 \\ \text{odd}}}^{2\ell-1} \chi_{(r,s)} + \chi_{(r,m+1-s)} ^2 + 2 \chi_{(r,2\ell+1)} ^2 \right\}$	$(A_{m-1}, D_{2\ell+2})$
$m = 4\ell + 2$	$\frac{1}{2} \sum_{s=1}^m \left\{ \sum_{\substack{r=1 \\ \text{odd}}}^{2\ell-1} \chi_{(r,s)} + \chi_{(m-r,s)} ^2 + 2 \chi_{(2\ell+1,s)} ^2 \right\}$	$(D_{2\ell+2}, A_m)$
$m = 4\ell + 3$	$\frac{1}{2} \sum_{r=1}^{m-1} \left\{ \sum_{\substack{s=1 \\ \text{odd}}}^m \chi_{(r,s)} ^2 + \chi_{(r,2\ell+2)} ^2 + \sum_{\substack{\text{even } s=2 \\ s \neq 2\ell+2}}^m \chi_{(r,s)}^* \chi_{(r,m+1-s)} \right\}$	$(A_{m-1}, D_{2\ell+3})$
$m = 4\ell + 4$	$\frac{1}{2} \sum_{s=1}^m \left\{ \sum_{\substack{r=1 \\ \text{odd}}}^{m-1} \chi_{(r,s)} ^2 + \chi_{(2\ell+2,s)} ^2 + \sum_{\substack{\text{even } r=2 \\ r \neq 2\ell+2}}^{m-1} \chi_{(r,s)}^* \chi_{(m-r,s)} \right\}$	$(D_{2\ell+3}, A_m)$
$m = 11$	$\frac{1}{2} \sum_{r=1}^{10} \chi_{(r,1)} + \chi_{(r,7)} ^2 + \chi_{(r,4)} + \chi_{(r,8)} ^2 + \chi_{(r,5)} + \chi_{(r,11)} ^2$	(A_{10}, E_6)
$m = 12$	$\frac{1}{2} \sum_{s=1}^{12} \chi_{(1,s)} + \chi_{(7,s)} ^2 + \chi_{(4,s)} + \chi_{(8,s)} ^2 + \chi_{(5,s)} + \chi_{(11,s)} ^2$	(E_6, A_{12})
$m = 17$	$\frac{1}{2} \sum_{r=1}^{16} \chi_{(r,1)} + \chi_{(r,17)} ^2 + \chi_{(r,5)} + \chi_{(r,13)} ^2 + \chi_{(r,7)} + \chi_{(r,11)} ^2$ $+ \chi_{(r,9)} ^2 + [\chi_{(r,3)} + \chi_{(r,15)}]^* \chi_{(r,9)} + \chi_{(r,9)}^* [\chi_{(r,3)} + \chi_{(r,15)}]$	(A_{16}, E_7)
$m = 18$	$\frac{1}{2} \sum_{s=1}^{18} \chi_{(1,s)} + \chi_{(17,s)} ^2 + \chi_{(5,s)} + \chi_{(13,s)} ^2 + \chi_{(7,s)} + \chi_{(11,s)} ^2$ $+ \chi_{(9,s)} ^2 + [\chi_{(3,s)} + \chi_{(15,s)}]^* \chi_{(9,s)} + \chi_{(9,s)}^* [\chi_{(3,s)} + \chi_{(15,s)}]$	(E_7, A_{18})
$m = 29$	$\frac{1}{2} \sum_{r=1}^{28} \chi_{(r,1)} + \chi_{(r,11)} + \chi_{(r,19)} + \chi_{(r,29)} ^2 + \chi_{(r,7)} + \chi_{(r,13)} + \chi_{(r,17)} + \chi_{(r,23)} ^2$	(A_{28}, E_8)
$m = 30$	$\frac{1}{2} \sum_{s=1}^{30} \chi_{(1,s)} + \chi_{(11,s)} + \chi_{(19,s)} + \chi_{(29,s)} ^2 + \chi_{(7,s)} + \chi_{(13,s)} + \chi_{(17,s)} + \chi_{(23,s)} ^2$	(E_8, A_{30})

Table 1. Complete list of modular invariant partition functions for unitary minimal conformal theories. The integer m is related to the central charge by $c = 1 - \frac{6}{m(m+1)}$.

example that a modular invariant function $Z_{1,1}$ is compatible with two symmetries Z_N and $Z_{N'}$. Namely, we have two sets

$$\{Z_{g^i, g^j} : 1 \leq i, j \leq N\}, \quad \text{and} \quad \{Z_{g'^k, g'^l} : 1 \leq k, l \leq N'\}, \quad (2.10)$$

with the *same* $Z_{1,1}$. If one could find a consistent set of functions $Z_{g^i g'^k, g^j g'^l}$, one would conclude that the two symmetries are compatible –the two types of charges are simultaneously assignable–, and that in fact the composite symmetry $Z_N \times Z_{N'}$ ($= Z_{NN'}$ if N and N' are coprime integers) is present. If not, the two cyclic groups are not commuting subgroups. This means either that G itself is a non-commutative group though containing two cyclic subgroups, or else that there exist two genuinely different models with the same periodic partition function. This second alternative would correspond to a situation where the various Hilbert spaces $\{\mathcal{H}_{g^i}\}$ and $\{\mathcal{H}_{g'^k}\}$ cannot be accommodated within a single model. In principle, this question is decidable by looking at the periodic sector, which fixes what the maximal symmetry is. One would try to find a non-commutative action on the representations of the periodic sector, which yields consistent partition functions and sensible modular transformations. That action would be a representation of G . We will illustrate this in the only two models where this situation occurs, see Section 6.4.

We have carried out this program for all unitary minimal models, with the following results. *Every model in Table 1 has a unique, maximal Z_2 symmetry, except six cases. The models (A_4, D_4) (the critical 3-Potts model) and (D_4, A_6) (the tricritical 3-Potts model) have unique Z_2 and Z_3 symmetries, which are not commuting and which combine to an S_3 symmetry. The last four models in the Table, related to E_7 and E_8 , have no symmetry. Of all partition functions that can be interpreted in terms of an extended chiral algebra (block diagonal form), only the two with an E_6 label have their extended algebra that survives all possible twisted (antiperiodic) boundary conditions. The extended algebra of the critical and tricritical 3-Potts models is preserved only by the Z_3 twisted boundary conditions.*

All explicit partition functions are given in the text. We have no explanation for the peculiarity of the two E_6 models, regarding the full compatibility of the extended algebra and the discrete symmetry group.

These results are expected. As mentioned before, the functions $Z_{1,1}$ we start from are critical partition functions of dilute RSOS models. In their very formulation, these models use data of simple Lie algebras of the type ADE, and inherit the symmetry of their Dynkin diagrams. In other words, for fixed $\mathcal{G} = A, D$ or E , the models $(A, \mathcal{G})$ and $(\mathcal{G}, A)$ do have a symmetry group equal to the automorphism group of the Dynkin diagram of $\mathcal{G}$, always equal to Z_2 , except for D_4, E_7 and E_8 . The diagram of D_4 has a symmetry S_3 , and the E_7, E_8 have no symmetry. From that point of view, our result is the statement that these known symmetries are neither broken nor enhanced by the continuum limit.

The rest of the paper is devoted to the proof of this statement. We will not follow the approach of [9], which consists in looking for submodular partition functions. Instead we make a strong use of Galois theory, which we believe allows for a simpler analysis in this context.

The relevance of Galois theory stems from the algebraic nature of the representation of the modular group acting on the characters and of the coefficients of the frustrated functions $Z_{g,g'}$. The importance in conformal field theory of Galois techniques has been recognized in recent years [18]. Section 3 recalls some general and known facts about these aspects, while Section 4 contains certain results which are specific to the present context. The subsequent sections contain the detailed analysis for the different series in Table 1.

3 Modular transformations and Galois theory

Unitary minimal conformal theories have central charges forming the infinite sequence $c = 1 - \frac{6}{m(m+1)}$ for integer $m \geq 3$. For fixed m , the Virasoro algebra has but a finite number of inequivalent representations, labelled by pairs of integers (r, s) in the Kac table

$$\text{KT} = \{(r, s) \in \mathbb{N}^2 : 1 \leq s \leq r \leq m-1\}. \quad (3.1)$$

The characters of these representations are denoted by $\chi_{(r,s)}(\tau)$.

They depend on a complex variable τ lying in the upper half plane, on which the modular group $PSL_2(\mathbb{Z})$ acts. A remarkable feature of the characters, valid much beyond the present context (see [16]), is that they transform linearly under the modular group. The (unitary) representation they carry can be given in terms of the generators S and T . Explicitly, one finds

$$\chi_i\left(\frac{-1}{\tau}\right) = \sum_{j \in \text{KT}} S_{i,j} \chi_j(\tau), \quad \chi_i(\tau+1) = \sum_{j \in \text{KT}} T_{i,j} \chi_j(\tau), \quad (3.2)$$

where the matrices S and T are given by

$$S_{(r,s),(r',s')} = \sqrt{\frac{8}{m(m+1)}} (-1)^{(r+s)(r'+s')} \sin \frac{\pi r r'}{m} \sin \frac{\pi s s'}{m+1}, \quad (3.3)$$

$$T_{(r,s),(r',s')} = e^{2i\pi(h_{r,s} - \frac{c}{24})} \delta_{r,r'} \delta_{s,s'}, \quad h_{r,s} = \frac{[(m+1)r - ms]^2 - 1}{4m(m+1)}. \quad (3.4)$$

One may check that S and T are unitary and symmetric, and that they satisfy the defining relations of the modular group, $S^2 = (ST)^3 = 1$. Moreover, one may check that S has the following symmetries, of which we will make a repeated use in subsequent sections,

$$S_{(m-r,s),(r',s')} = S_{(r,m+1-s),(r',s')} = (-1)^{(m+1)r' + ms' + 1} S_{(r,s),(r',s')}. \quad (3.5)$$

However their most distinctive feature is that their entries are all algebraic numbers (satisfy polynomial equations with rational coefficients). This also is not specific to the present situation, but is known to hold in any rational conformal theory [17], and prompts the use of Galois theory. The following Galois properties of the modular matrices S and T have been exposed in [18].

The entries of S and T belong to some cyclotomic extension of the rationals $\mathbb{Q}(\zeta_n)$, that is, they are linear combinations of n -th roots of unity with rational coefficients. The Galois

group of $\mathbb{Q}(\zeta_n)$ (over $\mathbb{Q}$), namely the group of automorphisms of $\mathbb{Q}(\zeta_n)$ which fix $\mathbb{Q}$ pointwise, is isomorphic to the group $\mathbb{Z}_n^*$ of invertible integers modulo n (the integers between 1 and n which are coprime with n). This group has order $\varphi(n)$, the Euler function. For $h \in \mathbb{Z}_n^*$, the Galois transformation σ_h acts on an n -th root of unity by raising it to the h -th power: $\sigma_h(\zeta_n) = \zeta_n^h$. This action extends to $\mathbb{Q}(\zeta_n)$ by linearity.

It follows that cyclotomic Galois groups act on the coefficients of S and T . Here we will be more concerned with the action on S . The formula (3.3) implies that the coefficients of S lie in the extension $\mathbb{Q}(\zeta_{2m(m+1)})$. It is a general fact that the action on S of the Galois group induces a permutation of the Kac table by the formula [18]

$$\sigma_h(S_{(r,s),(r',s')}) = \varepsilon_h(r,s) S_{\sigma_h(r,s),(r',s')} = \varepsilon_h(r',s') S_{(r,s),\sigma_h(r',s')}, \quad (3.6)$$

where the factors ε_h are equal to ± 1 . The formula (3.3) allows to determine explicitly the permutation $\sigma_h(r,s)$ and the signs $\varepsilon_h(r,s)$.

One first computes the residues $\langle hr \rangle_{2m}$ and $\langle hs \rangle_{2(m+1)}$ of hr modulo $2m$ and of hs modulo $2(m+1)$ (we choose the residues $\langle x \rangle_y$ between 0 and $y-1$). One then forms the unique pair of integers $(\tilde{r}_h, \tilde{s}_h)$ in the rectangle $[1, m] \times [1, m+1]$ by changing, if necessary, $\langle hr \rangle_{2m}$ by $2m - \langle hr \rangle_{2m}$, and similarly $\langle hs \rangle_{2(m+1)}$ by $2(m+1) - \langle hs \rangle_{2(m+1)}$. Up to a global sign related to the Galois action on the radical $\sqrt{8/m(m+1)}$, the sign $\varepsilon_h(r,s)$ can be determined, and is simply equal to (-1) to the number of changes needed to go from $(\langle hr \rangle_{2m}, \langle hs \rangle_{2(m+1)})$ to $(\tilde{r}_h, \tilde{s}_h)$. The pair $(\tilde{r}_h, \tilde{s}_h)$ is not in the Kac table yet, but if not, then $(m - \tilde{r}_h, m + 1 - \tilde{s}_h)$ is in the Kac table.

Putting everything together, the Galois properties of S are given by (3.6) with

$$\sigma_h(r,s) = \text{KT} \cap \left\{ (\tilde{r}_h, \tilde{s}_h), (m - \tilde{r}_h, m + 1 - \tilde{s}_h) \right\}, \quad (3.7)$$

$$\varepsilon_h(r,s) = \eta_h(m) \text{sign} \left(\sin \frac{\pi hr}{m} \sin \frac{\pi hs}{m+1} \right), \quad (3.8)$$

where $\eta_h(m) = \sigma_h(\sqrt{\frac{8}{m(m+1)}})/\sqrt{\frac{8}{m(m+1)}}$ is a sign which is independent of r, s .

The transformation law of the matrix S under the Galois group has direct and far-reaching consequences for the modular problem reviewed in the previous section. Suppose we look for a modular invariant partition function of the type (2.9),

$$Z_{1,1}(\tau) = \sum_{(r,s),(r',s')} [\chi_{(r,s)}(\tau)]^* M_{(r,s),(r',s')}^{(1)} [\chi_{(r',s')}(\tau)], \quad M_{(r,s),(r',s')}^{(1)} \in \mathbb{N}. \quad (3.9)$$

Since it is invariant under S , one obtains, by using the modular transformations of the characters, that the matrix $M^{(1)}$ must satisfy $M^{(1)} = S^\dagger M^{(1)} S$, or $S M^{(1)} = M^{(1)} S$ because S is unitary. Letting the Galois group act on this equation, a few lines calculation shows that selection rules follow [18]

$$\varepsilon_h(r,s) \neq \varepsilon_h(r',s') \text{ for some } h \implies M_{(r,s),(r',s')}^{(1)} = 0. \quad (3.10)$$

The signs $\varepsilon_h(r, s)$ are generally referred to as (Galois) parities in the literature, and the above set of conditions as the “parity rule”. It gives extremely strong constraints on $M^{(1)}$, because two pairs (r, s) and (r', s') have only rarely the same parities for all h , implying that a huge number of coefficients in $M^{(1)}$ are to vanish. Thus the matrices $M^{(1)}$ specifying modular invariant partition functions are usually sparse matrices.

The next section collects the main implications that these algebraic properties have for the existence of discrete symmetries in unitary minimal models.

4 Galois constraints on frustrated systems

This section contains four consequences of Galois theory which are particularly relevant for our problem. Denoted R1 to R4, they are the most fundamental arguments that will be used throughout the rest of the paper. We collect them here to emphasize the fact that they are independent of the details of the periodic partition functions of Table 1. Some of them are based on general Galois features only.

Let us recall that we are primarily interested in cyclic groups of symmetry Z_N . Let g be a generator of Z_N . In order to simplify a bit the notations, we define the matrices $M^{(i,j)}$ by

$$Z_{g^i, g^j}(\tau) = \sum_{(r,s), (r',s')} [\chi_{(r,s)}(\tau)]^* M_{(r,s), (r',s')}^{(i,j)} [\chi_{(r',s')}(\tau)]. \quad (4.1)$$

The periodic function $Z_{1,1}$ is taken to be one of the functions in Table 1, and corresponds to a matrix $M^{(0,0)}$ with non-negative integral entries. The matrix $M^{(0,1)}$ associated with the frustrated function $Z_{1,g}$ has the same zero pattern as $M^{(0,0)}$, and is simply obtained from it by replacing the non-zero coefficients by sums of N -th roots of unity:

$$M_{(r,s), (r',s')}^{(0,0)} = n \quad \implies \quad M_{(r,s), (r',s')}^{(0,1)} = \zeta_N^{Q_1(r,s;r',s')} + \dots + \zeta_N^{Q_n(r,s;r',s')}. \quad (4.2)$$

Similarly, the matrix $M^{(0,i)}$ is obtained from $M^{(0,1)}$ by replacing all roots of unity ζ_N by ζ_N^i , effectively multiplying all charges Q by i . The matrices $M^{(i,0)}$ are related to $M^{(0,i)}$ by the modular transformation S , according to (2.8),

$$M^{(i,0)} = S^\dagger M^{(0,i)} S, \quad (4.3)$$

and must also have non-negative integral coefficients. Finally, the matrices $M^{(i,ki)}$ are related to $M^{(i,0)}$ by a power of the modular transformation T ,

$$M^{(i,ki)} = T^k M^{(i,0)} T^{\dagger k}. \quad (4.4)$$

The first statement R1 is easy to prove and provides a simple although general means to determine the symmetry compatible with a given model. In some cases, a crude look at the S matrix allows to conclude.

We start from the relation $M^{(1,0)} = S^\dagger M^{(0,1)} S$. The matrix on the right-hand side is made up of numbers in the cyclotomic extension $\mathbb{F}$ containing the coefficients of S and the N -th

roots of unity, but, being equal to $M^{(1,0)}$, is integer-valued. Therefore it is left invariant by the Galois group of $\mathbb{F}$. Take any element σ of the Galois group that fixes the matrix $S^\dagger \otimes S$. Since it also fixes $S^\dagger M^{(0,1)} S$, the invertibility of S implies that σ actually fixes $M^{(0,1)}$, that is, the matrix elements of $M^{(0,1)}$ must belong to the field containing $S^\dagger \otimes S$.

For the models we consider here, this simple result has an easy corollary. The matrix S , given in (3.3), is real, and is therefore invariant under the specific Galois transformation corresponding to complex conjugation. Hence the matrix elements of $M^{(0,1)}$, given from (4.2) by sums of N -th roots of unity, must be real. The same is true of all $M^{(0,i)}$.

In case the matrix $M^{(0,0)}$ has all its coefficients in $\{0, 1\}$, each non-zero entry of $M^{(0,1)}$ is equal to a single N -th root of unity, which must be real, so that N can only be equal to 1 (no symmetry) or 2. It turns out that all functions in Table 1, except those of the series $(A_{m-1}, D_{2\ell+2})$ and $(D_{2\ell+2}, A_m)$, have their coefficients in $\{0, 1\}$. Thus, our first result says that

R1 *If the modular matrix S is real, a modular invariant partition function having all its coefficients in $\{0, 1\}$ is compatible with a cyclic symmetry Z_2 only. As corollary, the maximal cyclic symmetry of minimal unitary models is Z_2 , except for the models in the complementary series for $m = 4\ell + 1$ or $4\ell + 2$.*

Let us stress that this statement is not equivalent to saying that Z_2 is the maximal symmetry, since the Z_2 group could be realized in more than one way, leaving the possibility for a power $(Z_2)^k$.

The second result is a strengthened version of the previous one, and relies entirely on the Galois transformation of the matrix S , equation (3.6). Let us emphasize again that this transformation is completely general for a rational conformal theory.

Like before, we start from $M^{(i,0)} = S^\dagger M^{(0,i)} S$, and write out the transformations of the various matrices. For σ any element of the Galois group of $\mathbb{Q}(S_{ij})$, one has⁴ (summations over repeated indices)

$$(S^\dagger M^{(0,i)} S)_{kl} = S_{k,\sigma(a)}^\dagger \varepsilon_\sigma(a) (\sigma M^{(0,i)})_{ab} \varepsilon_\sigma(b) S_{\sigma(b),l} = S_{k,\sigma(a)}^\dagger (\sigma M^{(0,i)})_{ab} S_{\sigma(b),l} \quad (4.5)$$

where the last step follows from the parity rule: $M_{ab}^{(0,i)}$ is non-zero if and only if $M_{ab}^{(0,0)}$ is non-zero, which requires $\varepsilon_\sigma(a)\varepsilon_\sigma(b) = +1$ for all σ , see (3.10). Comparing the far left- and far right-hand sides leads to the conclusion that σ acts on $M^{(0,i)}$ by permutation of the indices: $(\sigma M^{(0,i)})_{ab} = (M^{(0,i)})_{\sigma(a),\sigma(b)}$.

On the other hand, if one lets σ act on the indices k and l , one obtains

$$M_{kl}^{(i,0)} = (S^\dagger M^{(0,i)} S)_{kl} = \sigma(S^\dagger M^{(0,i)} S)_{kl} = \varepsilon_\sigma(k) \varepsilon_\sigma(l) (S^\dagger \sigma M^{(0,i)} S)_{\sigma(k),\sigma(l)}. \quad (4.6)$$

⁴From the result R1, $M^{(0,i)}$ is in $\mathbb{Q}(S_{ij})$, so that its Galois group acts properly on it. As a consequence, one obtains $\zeta_N \in \mathbb{Q}(S_{ij}) = \mathbb{Q}(\zeta_{2m(m+1)})$, that is, N divides $2m(m+1)$ [9].

Since $M^{(0,i)}$ contains N -th roots of unity, σ acts on it by substituting ζ_N^h for ζ_N for some h , so in effect, $\sigma M^{(0,i)} = M^{(0,hi)}$. The previous equation then implies

$$M_{kl}^{(i,0)} = \varepsilon_\sigma(k) \varepsilon_\sigma(l) M_{\sigma(k),\sigma(l)}^{(hi,0)}, \quad \text{if } \sigma(\zeta_N) = \zeta_N^h. \quad (4.7)$$

This constraint is a generalization of the parity rule, to which it reduces when $i = 0$. Like in that case, it implies severe selection rules on the coefficients of all matrices $M^{(i,0)}$.

R2 *The matrices $M^{(i,0)}$, specifying the field content of the frustrated Hilbert spaces, must all satisfy the parity rule: $M_{kl}^{(i,0)} = 0$ unless $\varepsilon_\sigma(k) \varepsilon_\sigma(l) = +1$ for all Galois transformations. The matrices $M^{(0,i)}$, giving the charges of the periodic sector, have Galois transformations $(\sigma M^{(0,i)})_{kl} = (M^{(0,i)})_{\sigma(k),\sigma(l)}$.*

The third result we want to mention is by far the strongest: it solves the parity rule, by saying exactly which matrix elements $M_{kl}^{(i,0)}$ may be non-zero. The theorem we quote below is not properly new and has been proved in [19] (see [20] for an elementary but clever proof in the case n odd). If the first two results can rightly be called elementary, this one is not. Its use allows a straightforward proof of the ADE classifications of conformal and $su(2)$ affine modular invariant partition functions.

The basic problem is to determine all pairs $(r, s), (r', s')$ which have equal Galois parities $\varepsilon_h(r, s) = \varepsilon_h(r', s')$ for all Galois transformations σ_h , $h \in \mathbb{Z}_{2m(m+1)}^*$. A closed expression for (the r, s -dependent part of) the parities has been given in Section 3,

$$\varepsilon_h(r, s) = \text{sign} \left(\sin \frac{\pi h r}{m} \sin \frac{\pi h s}{m+1} \right). \quad (4.8)$$

Since m and $m+1$ are coprime integers and since one of them is odd, one has a canonical isomorphism $\mathbb{Z}_{2m(m+1)}^* = \mathbb{Z}_{2m}^* \times \mathbb{Z}_{2(m+1)}^*$, which induces a similar splitting of the parity

$$\varepsilon_h(r, s) = \text{sign} \left(\sin \frac{\pi h_1 r}{m} \right) \text{sign} \left(\sin \frac{\pi h_2 s}{m+1} \right), \quad h_1 \in \mathbb{Z}_{2m}^*, \quad h_2 \in \mathbb{Z}_{2(m+1)}^*. \quad (4.9)$$

Thus the problem effectively factorizes into two identical pieces. If one defines the functions $\varepsilon_n(x) = \text{sign} \left(\sin \frac{\pi x}{n} \right)$ for x not divisible by n , then

$$\varepsilon_h(r, s) = \varepsilon_h(r', s') \quad \forall h \in \mathbb{Z}_{2m(m+1)}^* \quad \Longleftrightarrow \quad \begin{cases} \varepsilon_m(hr) = \varepsilon_m(hr') & \forall h \in \mathbb{Z}_{2m}^*, \\ \varepsilon_{m+1}(hs) = \varepsilon_{m+1}(hs') & \forall h \in \mathbb{Z}_{2(m+1)}^*. \end{cases} \quad (4.10)$$

The function $\varepsilon_n(x)$ is actually the parity occurring in the Galois transformation of the matrix S relative to the characters of affine $su(2)$ algebras. As in the present situation, the affine $su(2)$ parities often appear as constitutive blocks for the parities relative to the characters of other algebras [21].

Equation (4.10) relates the conformal parity rule with the affine $su(2)$ parity rule, the solution to the latter yielding the solution to the former. It turns out that the $su(2)$ parity rule arises in mathematical questions related to complex Fermat surfaces. (The same is true

for the affine $su(3)$ parity rule. See [22] for a more precise description of the connection between parity rules and problems in the geometry of Riemann surfaces.) The results obtained by Aoki [19] in that context actually solve the $su(2)$ problem⁵. This will be our R3.

R3 Suppose that x and y , two integers between 1 and $n - 1$, satisfy $\varepsilon_n(hx) = \varepsilon_n(hy)$ for all $h \in \mathbb{Z}_{2n}^*$. Then, for $\text{GCD}(n, x, y) = 1$, x and y must be related as follows:

- (i) $y = x$ or $y = n - x$;
- (ii) $n = 6$: $x, y \in \{1, 3, 5\}$;
- $n = 10, 12$: $x, y \in \mathbb{Z}_n^*$;
- $n = 30$: $x, y \in \{1, 11, 19, 29\}$ or $x, y \in \{7, 13, 17, 23\}$.

All other solutions follow from these by multiplying x, y and n by a common integer.

Our last general result concerns the use of R3 to further constrain the possible symmetries, regardless of what the periodic partition function is. In this sense, it strengthens R1.

The basic idea is to use the equation (4.4) in conjunction with R3. Equation (4.4) says that $M^{(i, ki)} = T^k M^{(i, 0)} T^{\dagger k}$. Because T is diagonal, the entries of $M^{(i, ki)}$ and $M^{(i, 0)}$ are simultaneously zero or non-zero. The only difference between the non-zero coefficients of the two matrices are some roots of unity, specifying the type of symmetry and the charges of the sector $\mathcal{H}_{g^i}$ of the theory. The conjugation by T is what precisely produces these roots of unity since

$$M_{(r, s), (r', s')}^{(i, ki)} = e^{2i\pi k(h_{r, s} - h_{r', s'})} M_{(r, s), (r', s')}^{(i, 0)}. \quad (4.11)$$

In order to see what roots of unity can appear, it remains to compute the quantities $h_{r, s} - h_{r', s'} \bmod 1$ for those pairs $(r, s), (r', s')$ for which $M_{(r, s), (r', s')}^{(i, 0)}$ is non-zero. From R2, these pairs have to satisfy the parity rule, and are listed in R3.

In conformal models, the formula (3.4) yields

$$h_{r, s} - h_{r', s'} = \frac{(r^2 - r'^2)(m + 1)^2 + (s^2 - s'^2)m^2}{4m(m + 1)} + \frac{rs + r's'}{2} \bmod 1. \quad (4.12)$$

Let us assume that m is even, the analysis being the same for m odd.

For m even, we learn from R3 that $s' = s$ or $m + 1 - s$, and that the list of pairs r, r' satisfying the parity rule is the same as the list $r, m - r'$ that satisfy it. Since $h_{r', s'} = h_{m-r', m+1-s'}$, we can, without loss of generality, assume $s' = s$. Then, the difference of the two conformal weights simplifies

$$h_{r, s} - h_{r', s} = (r^2 - r'^2) \frac{(m + 1)}{4m} + \frac{(r + r')s}{2} \bmod 1. \quad (4.13)$$

⁵The problem is not formulated in the same way, but Theorem 2.7 of [19] answers the question, except for finitely many values of n . One can also set up a proof based on earlier results by Koblitz and Rohrlich [23].

For the generic solutions, $r' = r$ or $m - r$, we find $h_{r,s} - h_{r',s} = 0$ or $\frac{1}{2}$. For the exceptional ones at $m = 6, 10, 12$ and 30 , we simply compute the squares modulo $4m$ of the possible values of r, r' and differences thereof:

$$m = 6 : \quad r^2, r'^2 \in \{1, 9\} \quad \implies \quad r^2 - r'^2 \in \{0, \pm 8\}, \quad (4.14)$$

$$m = 10 : \quad r^2, r'^2 \in \{1, 9\} \quad \implies \quad r^2 - r'^2 \in \{0, \pm 8\}, \quad (4.15)$$

$$m = 12 : \quad r^2, r'^2 \in \{1, 25\} \quad \implies \quad r^2 - r'^2 \in \{0, 24\}, \quad (4.16)$$

$$m = 30 : \quad \begin{cases} r^2, r'^2 = 1 \\ \text{or } r^2, r'^2 = 49 \end{cases} \quad \implies \quad r^2 - r'^2 = 0. \quad (4.17)$$

In all these cases, r and r' are always both odd, so that the second term of (4.13) vanishes modulo 1, whereas the first term gives various fractions depending on m : thirds for $m = 6$, fifths for $m = 10$, halves for $m = 12$, and actually integers for $m = 30$. Finally, the solutions obtained from the preceding ones by multiplying r, r' and m by a common integral factor yield the same fractions, namely no fractions at all, halves, thirds or fifths.

From (4.11), we therefore arrive at the conclusion that the coefficients of $M^{(i,ki)}$, for all i and k , are integers times second, third and fifth roots of unity. This implies that, in the sector $\mathcal{H}_{g^i}$, the charges with respect to the power g^i of the generator are compatible with the cyclic symmetries made up of Z_2, Z_3 and Z_5 , or equivalently with a maximal cyclic symmetry equal to Z_{30} . What is missing to assert that this is the maximal cyclic symmetry the whole theory can have, is the value of the charges with respect to g , not g^i . In other words, we have to show that all $M^{(i,1)}$ contain only second, third and fifth roots of unity.

Let us suppose that there is cyclic symmetry Z_N , and let us look at the matrices $M^{(i,1)} = S^\dagger M^{(-1,i)} S$, for $i = 1, 2, \dots, N$. From what we have just proved, each matrix $M^{(-1,i)}$ contains only second, third and fifth roots of unity, and is therefore equal to $M^{(-1,i+30a)}$ for any integer a . Thus $M^{(i,1)} = M^{(i+30a,1)}$. If there exists a value of a such that $i + 30a$ is invertible modulo N (is coprime with N), then $M^{(i+30a,1)}$ hence $M^{(i,1)}$ contains only second, third and fifth roots of unity (because then $M^{(i+30a,1)}$ is in the set of $M^{(j,kj)}$). The only case where there is no such a is when N and i are not coprime with 30.

If i is multiple of 5, then $M^{(-1,i)}$ contains only second and third roots of unity. Indeed $M^{(-1,1)}$ contains second, third and fifth roots of unity, and yields $M^{(-1,i)}$ upon replacing all roots of unity by their i -th power, effectively setting to 1 all fifth roots of unity. Then $M^{(-1,i)} = M^{(-1,i+6a)}$ for any a , and running through the above argument with 6 instead of 30 shows that $M^{(i,1)} = M^{(i+6a,1)}$ contains only second, third and fifth roots of unity except if N and i are not coprime with 6. Repeating the argument for i a multiple of 3, and then for i a multiple of 2 eventually proves the statement.

R4 *The only cyclic symmetries that unitary minimal conformal theories can have are subgroups of Z_{30} .*

The above four results are rather general. The next step is to go down into the details of the various partition functions of Table 1, in order to make use of their specific form. The

easiest are the “permutation invariants”, namely those of the diagonal series (A_{m-1}, A_m) (for all $m \geq 3$), and of the complementary series $(A_{m-1}, D_{2\ell+3})$ (for $m = 4\ell + 3$) and $(D_{2\ell+3}, A_m)$ (for $m = 4\ell + 4$). We begin with these.

5 Permutation modular invariants

The modular invariant partition functions we consider in this section have the form

$$Z_{1,1}(\tau) = \sum_{(r,s) \in \text{KT}} \chi_{(r,s)}^* \chi_{\mu(r,s)}, \quad (5.1)$$

where μ is a permutation (it is also an automorphism of the fusion rules, see for instance [12]). It is the trivial permutation $\mu = 1$ for the diagonal series (A_{m-1}, A_m) , while for the two complementary series,

$$\mu(r, s) = \begin{cases} (r, s) & \text{if } s \text{ (resp. } r) \text{ is odd,} \\ (m - r, s) & \text{if } s \text{ (resp. } r) \text{ is even and } r + s \leq m, \\ (r, m + 1 - s) & \text{if } s \text{ (resp. } r) \text{ is even and } r + s > m, \end{cases} \quad (5.2)$$

for $m = 4\ell + 3$ (resp. $m = 4\ell + 4$).

For these series, R1 implies that Z_2 is the maximal cyclic symmetry, with four corresponding matrices, $M^{(0,0)} = \delta_{j,\mu(i)}$, $M^{(1,0)}$, $M^{(0,1)}$ and $M^{(1,1)}$. It remains to see how many realizations of the Z_2 are compatible with the given $M^{(0,0)}$.

5.1 The diagonal series

The matrix $M^{(0,0)}$ is the identity and thus $M^{(0,1)} = \epsilon_i \delta_{i,j}$ is diagonal with signs on the diagonal. It follows that $M^{(0,1)}$, like S , is equal to its own inverse, and the relation $M^{(1,0)} = S M^{(0,1)} S$ shows that the same is true of $M^{(1,0)}$. Since the latter has non-negative integral entries, it must be a permutation matrix. We set $M_{ij}^{(1,0)} = \delta_{j,\pi(i)}$.

Because $M^{(1,0)} S = S M^{(0,1)}$, the permutation π must satisfy

$$S_{\pi(i),j} = \epsilon_j S_{i,j}, \quad \epsilon_j = \pm 1. \quad (5.3)$$

The signs ϵ_i are the Z_2 -parities of the states in the periodic sector. The vacuum being neutral, one has $\epsilon_{(1,1)} = +1$. For $i = j = (1, 1)$, and setting $\pi(1, 1) = (r, s)$, the previous equation then says

$$S_{\pi(1,1),(1,1)} = S_{(1,1),(1,1)} \iff \sin \frac{\pi r}{m} \sin \frac{\pi s}{m+1} = \sin \frac{\pi}{m} \sin \frac{\pi}{m+1}. \quad (5.4)$$

The only integer solutions are $(r, s) = (1, 1), (m - 1, 1), (1, m)$ and $(m - 1, m)$ (use Galois !), but only $(1, 1)$ and $(m - 1, 1)$ are in the Kac table. Moreover $\pi(1, 1) = (1, 1)$ must be rejected since $M_{(1,1),(1,1)}^{(1,0)} = 1$ would mean that the vacuum belongs to two distinct sectors. Therefore $\pi(1, 1) = (m - 1, 1)$.

Inserting $i = (1, 1)$ and $j = (r, s)$ in (5.3) then yields

$$S_{(m-1,1),(r,s)} = (-1)^{(m+1)r+ms+1} S_{(1,1),(r,s)} = \epsilon_{(r,s)} S_{(1,1),(r,s)}, \quad (5.5)$$

where the first equality follows from the symmetry of S , mentioned in (3.5). Since all numbers $S_{(1,1),(r,s)}$ are non-zero, the previous equation fixes all parities to be $\epsilon_{(r,s)} = (-1)^{(m+1)r+ms+1}$. So $M^{(0,1)}$ is uniquely fixed, and yields unique $M^{(1,0)}$ and $M^{(1,1)}$ by modular transformations.

We have proved our claim: in the models of the diagonal series, there is a unique way of realizing a Z_2 symmetry, which is therefore their maximal symmetry group. There are only two sectors, which we may call periodic and antiperiodic, both carrying non-trivial Z_2 -charges. The field content and the charges of the two sectors can be read off from the partition functions, which were found in [9]. We reproduce them here for the sake of completeness,

$$Z_{PA} = Z_{1,g} = \sum_{(r,s) \in \text{KT}} (-1)^{(m+1)r+ms+1} |\chi_{(r,s)}|^2, \quad (5.6)$$

$$\begin{aligned} Z_{AA} = Z_{g,g} = \sum_{r+s \leq m} (-1)^{(m+1)(r+m/2)+ms} \chi_{(r,s)}^* \chi_{(m-r,s)} \\ + \sum_{r+s \geq m+1} (-1)^{(m+1)(r+m/2)+ms} \chi_{(r,s)}^* \chi_{(r,m+1-s)}. \end{aligned} \quad (5.7)$$

The other two functions Z_{PP} and Z_{AP} follow from these two by dropping the signs.

The first model $m = 3$ corresponds to the universality class of the Ising model, for which one recovers the well-known parity assignments [7]: in the periodic sector, the identity and the energy density are even while the spin variable is odd; in the antiperiodic sector, the disorder variable is even, while the two fermionic degrees of freedom are odd.

5.2 The complementary series

We will detail the analysis for one of the two series, say $m = 4\ell + 3$. The other, for $m = 4\ell + 4$, can be treated by the same method.

For $m = 4\ell + 3$, the matrix $M^{(0,1)}$ is a generalized permutation matrix, almost diagonal,

$$M^{(0,1)} = \begin{pmatrix} 1 & & & & & \\ & \pm 1 & & & & \\ & & \ddots & & & \\ & & & \pm 1 & & \\ & & & & \boxed{\begin{smallmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{smallmatrix}} & \\ & & & & & \boxed{\begin{smallmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{smallmatrix}} \\ & & & & & & \ddots \end{pmatrix}. \quad (5.8)$$

The diagonal coefficients refer to the indices (r, s) with s odd, and also to $(r, 2\ell + 2)$, whereas the two-by-two blocks are labelled by pairs (r, s) and $(m - r, s)$ or $(r, m + 1 - s)$ with s even (see above, equation (5.2)). The first diagonal entry, equal to 1, refers to the index $(1, 1)$ —the vacuum is neutral—, but otherwise all other signs are uncorrelated.

Whatever the signs in $M^{(0,1)}$ are, the fourth power of the matrix is equal to 1, implying the same for $M^{(1,0)}$. Thus $(M^{(1,0)})^{-1} = (M^{(1,0)})^3$ is a non-negative integral matrix, hence a permutation matrix. We set again $M_{ij}^{(1,0)} = \delta_{j, \pi(i)}$.

As before, the relation $M^{(1,0)} S = S M^{(0,1)}$ implies

$$S_{\pi(i), j} = \sum_k S_{ik} M_{kj}^{(0,1)}. \quad (5.9)$$

Letting $i = j = (1, 1)$ yields the same equation we had in the diagonal series, and the same result $\pi(1, 1) = (m - 1, 1)$. Then for $i = (1, 1)$, by using once more the symmetry (3.5) of S and the odd character of m , one finds from (5.9)

$$S_{(m-1,1), (r', s')} = (-1)^{s'+1} S_{(1,1), (r', s')} = \sum_{k \in \text{KT}} S_{(1,1), k} M_{k, (r', s')}^{(0,1)}. \quad (5.10)$$

This last equation determines all signs in $M^{(0,1)}$ in a unique way. The calculations are straightforward, so we only quote the result. All diagonal entries of $M^{(0,1)}$ must be +1, except those with indices $(r, 2\ell + 2)$ for all r , which must be -1. All two-by-two blocks must be $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. This can be summarized by saying that $M_{(r,s), (r', s')}^{(0,1)} = (-1)^{s+1} M_{(r,s), (r', s')}^{(0,0)}$.

Like in the diagonal series, $M^{(0,0)}$ fixes uniquely the other three matrices, meaning that the Z_2 has a unique realization, and is therefore the maximal symmetry. The same conclusion holds for the other complementary series, for $m = 4\ell + 4$.

We finish by displaying the partition functions. For the series $(A_{m-1}, D_{2\ell+3})$,

$$Z_{PA} = \sum_{\substack{(r,s) \in \text{KT} \\ s \text{ odd}}} |\chi_{(r,s)}|^2 - \sum_{\substack{(r,s) \in \text{KT} \\ r+s \leq m, s \text{ even}}} \chi_{(r,s)}^* \chi_{(m-r,s)} - \sum_{\substack{(r,s) \in \text{KT} \\ r+s > m, s \text{ even}}} \chi_{(r,s)}^* \chi_{(r, m+1-s)}, \quad (5.11)$$

$$Z_{AA} = \sum_{\substack{(r,s) \in \text{KT} \\ s \text{ even}}} |\chi_{(r,s)}|^2 - \sum_{\substack{(r,s) \in \text{KT} \\ r+s \leq m, s \text{ odd}}} \chi_{(r,s)}^* \chi_{(m-r,s)} - \sum_{\substack{(r,s) \in \text{KT} \\ r+s > m, s \text{ odd}}} \chi_{(r,s)}^* \chi_{(r, m+1-s)}, \quad (5.12)$$

and for the series $(D_{2\ell+3}, A_m)$,

$$Z_{PA} = \sum_{\substack{(r,s) \in \text{KT} \\ r \text{ odd}}} |\chi_{(r,s)}|^2 - \sum_{\substack{(r,s) \in \text{KT} \\ r+s \leq m, r \text{ even}}} \chi_{(r,s)}^* \chi_{(m-r,s)} - \sum_{\substack{(r,s) \in \text{KT} \\ r+s > m, r \text{ even}}} \chi_{(r,s)}^* \chi_{(r, m+1-s)}, \quad (5.13)$$

$$Z_{AA} = \sum_{\substack{(r,s) \in \text{KT} \\ r \text{ even}}} |\chi_{(r,s)}|^2 - \sum_{\substack{(r,s) \in \text{KT} \\ r+s \leq m, r \text{ odd}}} \chi_{(r,s)}^* \chi_{(m-r,s)} - \sum_{\substack{(r,s) \in \text{KT} \\ r+s > m, r \text{ odd}}} \chi_{(r,s)}^* \chi_{(r, m+1-s)}. \quad (5.14)$$

6 More complementary modular invariants

We analyze in this section the complementary series of partition functions occurring for $m = 4\ell + 1$ and $4\ell + 2$. These modular invariants are not permutation invariants, but have

a block diagonal form. They can however be considered as permutation invariants (with the trivial permutation) in terms of extended characters, written as linear combinations of conformal characters, and which are in fact the characters of a larger chiral algebra that indeed extends the Virasoro algebra.

For the problem of symmetries, these two series are more difficult. The basic reason for this is that some of the coefficients in the sesquilinear forms are bigger than 1. This means that R1 does not apply, making room for richer symmetries. Indeed at least two models are known to have the permutation group S_3 as symmetry, namely the critical and tricritical 3-Potts models, corresponding respectively to $m = 5$ ($c = \frac{4}{5}$) and $m = 6$ ($c = \frac{6}{7}$). We show below that they are the only two models to have a larger symmetry group, all others having just a Z_2 .

Again the two distinct series may be treated by the same methods, so we restrict ourselves to one of them, which we take to be $(D_{2\ell+2}, A_m)$, $m = 4\ell + 2$. From Table 1, those partition functions read, in terms of labels in the Kac table,

$$Z_{1,1}(\tau) = \sum_{\substack{s \leq r \leq 2\ell-1 \\ r \text{ odd}}} |\chi_{(r,s)} + \chi_{(m-r,s)}|^2 + \sum_{\substack{2\ell+2 \leq s \leq r \\ r \text{ odd}}} |\chi_{(r,s)} + \chi_{(r,m+1-s)}|^2 + \sum_{s=1}^{2\ell+1} 2 |\chi_{(2\ell+1,s)}|^2. \quad (6.1)$$

R4 says that these models can only have three kinds of cyclic symmetries of order equal to a prime power: Z_2 , Z_3 and Z_5 . We examine, for each symmetry in turn, the possible realizations.

6.1 Symmetry of order five

An order five symmetry requires m to be a multiple of 10. Indeed, if one looks at the way R4 was proved, the condition for a Z_5 symmetry was that there should be pairs $(r, s), (r', s')$, allowed by the parity rule, such that $h_{r,s} - h_{r',s'} = 0 \bmod \frac{1}{5}$. From the equations (4.13) to (4.17), this required m to be divisible by 10. So we take $m = 0 \bmod 5$.

From the argument that had led to R1, the matrix $M^{(0,1)}$ must be real. We also know from (4.2) that a coefficient of $M^{(0,1)}$ is a sum of n fifth roots of unity if the corresponding entry of $M^{(0,0)}$ is equal to n . Here $n = 0, 1$ and 2 , so that $M^{(0,1)}$ is actually equal to $M^{(0,0)}$ except for the diagonal terms of indices $(2\ell + 1, s)$. Thus

$$M_{(r,s),(r',s')}^{(0,1)} = M_{(r,s),(r',s')}^{(0,0)} + \left[2 \cos \frac{2\pi a_s}{5} - 2 \right] \delta_{r,2\ell+1} \delta_{r',2\ell+1} \delta_{s,s'}, \quad (6.2)$$

for some integers a_s between 0 and 4.

R2 says how this matrix must change under Galois transformations. In particular, it says that for any σ in the Galois group,

$$\sigma \left(M_{(2\ell+1,s),(2\ell+1,s)}^{(0,1)} \right) = \sigma \left(2 \cos \frac{2\pi a_s}{5} \right) = M_{\sigma(2\ell+1,s),\sigma(2\ell+1,s)}^{(0,1)}. \quad (6.3)$$

The relevant Galois group is $\text{Gal}(\mathbb{Q}(\zeta_{2m(m+1)}))$, isomorphic to $\mathbb{Z}_{2m}^* \times \mathbb{Z}_{2(m+1)}^*$. Let us consider the subgroup $\mathbb{Z}_{2m}^*$ consisting of those $\sigma = \sigma_h$ with $h = 1 \pmod{2(m+1)}$. Clearly this subgroup is the Galois group of $\mathbb{Q}(\zeta_{2m})$.

The way $\sigma(r, s)$ is computed has been recalled in Section 3, and in this particular case, one may easily check that $\sigma_h(2\ell + 1, s) = (2\ell + 1, s)$. Thus the numbers $2 \cos \frac{2\pi a_s}{5}$, lying in $\mathbb{Q}(\zeta_{2m})$ since $m = 0 \pmod{5}$, must be invariant under $\text{Gal}(\mathbb{Q}(\zeta_{2m}))$, and so must all be rational numbers, equal to 2. It means that all charges in the periodic sector are equal to zero, and that $M^{(0,i)} = M^{(0,0)}$.

The matrices $M^{(0,i)}$ are therefore both T and S -invariant, and this forces all $M^{(i,j)}$ to be equal to $M^{(0,0)}$. That is, the Z_5 can only be trivially realized.

6.2 Symmetry of order three

For an order three symmetry, m must be a multiple of 6. For the same reason as in the previous case, the form of $M^{(0,1)}$ is

$$M_{(r,s),(r',s')}^{(0,1)} = M_{(r,s),(r',s')}^{(0,0)} + \left[2 \cos \frac{2\pi a_s}{3} - 2 \right] \delta_{r,2\ell+1} \delta_{r',2\ell+1} \delta_{s,s'}, \quad (6.4)$$

where a_s are now integers taken modulo 3. We look for a set of nine matrices $M^{(i,j)}$ consistent with a Z_3 symmetry. The integers i and j are taken modulo 3.

Let us first suppose that $M^{(1,1)}$ does not contain third roots of unity. Then $M^{(2,2)} = M^{(-2,-2)} = M^{(1,1)}$ would not contain any third root of unity either, so that all charges in the sectors $\mathcal{H}_g$ and $\mathcal{H}_{g^2}$ are zero. In Section 4, we had then argued that in such circumstances, no Z_3 symmetry would be present at all. Therefore a necessary condition to have a Z_3 symmetry is that $M^{(1,1)}$ contain some third roots of unity.

Third roots of unity in $M^{(1,1)} = T M^{(1,0)} T^\dagger$ are generated from $M^{(1,0)}$ through a T transformation, and from (4.14), this happens only if some among the following entries

$$M_{(xm/6,s),(x'm/6,s)}^{(1,0)} \quad \text{or} \quad M_{(xm/6,s),(x'm/6,m+1-s)}^{(1,0)} \quad (6.5)$$

are non-zero, where $x = 1, 5$ and $x' = 3$, or vice versa. The equality $M^{(1,0)} = S M^{(0,1)} S$ implies that $M^{(1,0)}$ is symmetric, and that the coefficients in (6.5) are pairwise equal. So, without loss of generality, we may focus on $M_{(xm/6,s),(m/2,s)}^{(1,0)}$ with $x = 1, 5$. If these numbers vanish, for $x = 1, 5$ and all s , there can be no Z_3 symmetry.

They can be computed from the S matrix and $M^{(0,1)}$ given above. One finds, after a few lines calculation,

$$M_{(xm/6,s),(m/2,s)}^{(1,0)} = \frac{24}{m(m+1)} \sum_{s'=1}^{2\ell+1} [1 - \delta_{a_{s'},0}] \left(\sin \frac{\pi s s'}{m+1} \right)^2. \quad (6.6)$$

These numbers must be positive, which they are, but also integers. The case where all charges a_s are non-zero gives an upper bound

$$M_{(xm/6,s),(m/2,s)}^{(1,0)} \leq \frac{24}{m(m+1)} \sum_{s'=1}^{2\ell+1} \left(\sin \frac{\pi s s'}{m+1} \right)^2 = \frac{6}{m}. \quad (6.7)$$

One sees that unless $m = 6$, these coefficients are to vanish, and the Z_3 itself vanishes.

For $m = 6$, the upper bound is reached if all charges a_s are different from zero. This fixes $2 \cos \frac{2\pi a_s}{3} = -1$. It yields a unique $M^{(0,1)}$, and in turn, a unique set of functions $Z_{g^i, g^j}(\tau)$.

6.3 Symmetry of order two

The most general form of $M^{(0,1)}$ is more complicated than in the previous two cases. The partition function $Z_{1,1}$, given in (6.1), corresponds to a matrix $M^{(0,0)}$ which has two-by-two blocks $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and diagonal entries equal to 2 (in addition to a big substructure equal to zero, corresponding to rows and columns labelled by (r, s) with r even). In $M^{(0,1)}$, the blocks become something like $\begin{pmatrix} \epsilon_1 & \eta_1 \\ \eta_2 & \epsilon_2 \end{pmatrix}$ with $\epsilon_i, \eta_i = \pm 1$ (independent from block to block), while the diagonal entries can be equal to $-2, 0$ or 2 .

As a first step, we reduce the degrees of freedom in the signs ϵ_i, η_i by using again the part of R2 concerned with the Galois transformation of $M^{(0,1)}$. Let us consider the action of σ_h for $h = m(m+1) - 1 \in \mathbb{Z}_{2m(m+1)}^*$. Since m is even, one obtains for r odd, that $\tilde{r}_h = m - r$ and $\tilde{s}_h = s$. Therefore,

$$\begin{cases} r \text{ odd} \\ h = m(m+1) - 1 \end{cases} \implies \sigma_h(r, s) = \begin{cases} (m - r, s) & \text{if } r + s \leq m, \\ (r, m + 1 - s) & \text{if } r + s \geq m + 1. \end{cases} \quad (6.8)$$

The blocks $\begin{pmatrix} \epsilon_1 & \eta_1 \\ \eta_2 & \epsilon_2 \end{pmatrix}$ are indexed by the pairs of doublets $(r, s), (m - r, s)$ if $r + s \leq m$, and by $(r, s), (r, m + 1 - s)$ if $r + s \geq m + 1$. To permute the indices according to σ_h is to exchange $\epsilon_1 \leftrightarrow \epsilon_2, \eta_1 \leftrightarrow \eta_2$ in each block. From the Galois transformation of $M^{(0,1)}$ stated in R2, this exchange should leave the blocks invariant, since the Galois group has no action on the entries of $M^{(0,1)}$. Consequently we find $\epsilon_2 = \epsilon_1$, and $\eta_2 = \eta_1$ in each block.

In order to get further constraints, we look at the square of $M^{(1,0)}$, related by an S transformation to the square of $M^{(0,1)}$. The one thing we know about $M^{(1,0)}$ is that its $(1, 1), (1, 1)$ coefficient is equal to zero, because the representation containing the vacuum can only occur in the periodic sector. We first want to prove that the same entry vanishes in $(M^{(1,0)})^2$. For this, we use the parity rule R2 as well as R3. As a preliminary remark, we note that the symmetry (3.5) of the S matrix and the fact that $M^{(0,1)}$ is zero for rows and columns labelled by (r, s) with r even, imply that

$$M_{\mu^a(r,s), \mu^b(r',s')}^{(1,0)} = M_{(r,s), (r',s')}^{(1,0)}, \quad a, b = 0, 1, \quad (6.9)$$

where μ denotes any one of the two transformations $(r, s) \rightarrow (m - r, s)$ and $(r, s) \rightarrow (r, m + 1 - s)$.

R2 says that $M_{(1,1), (r,s)}^{(1,0)}$ is possibly non-zero only for those (r, s) satisfying $\varepsilon_\sigma(1, 1)\varepsilon_\sigma(r, s) = +1$ for all σ . Then (4.10) together with R3 imply that $(r, s) = (1, 1)$ or $(m - 1, 1)$, except possibly if $m = 6, 10$ or 30 (12 does not enter because it is not of the form $4\ell + 2$). For $m = 6$, (r, s) can also be equal $(3, 1)$ but a non-zero coefficient at that place would induce third roots of unity in $M^{(1,1)}$, which are incompatible with a Z_2 symmetry. Likewise, if

$m = 10$, (r, s) can be equal to $(3, 1)$ and $(7, 1)$, but then $M^{(1,1)}$ would contain fifth roots of unity. Finally, for $m = 30$, (r, s) can be equal to $(11, 1)$ and $(19, 1)$, but now non-zero coefficients at those places produce no root of unity in $M^{(1,1)}$. Thus $m = 30$ is a special case that needs a separate treatment.

For $m \neq 30$, the rows and columns with labels $(1, 1)$ and $(m-1, 1)$ form a potentially non-zero two-by-two block. According to (6.9), this block must have four equal elements, actually equal to zero since the $(1, 1), (1, 1)$ entry is zero. Therefore we conclude that $(M^{(1,0)})^2_{(1,1),(1,1)} = 0$.

For $m = 30$, one has instead a symmetric four-by-four block which, by using (6.9), must look like⁶

$$(M^{(1,0)})_{i,j} = \begin{pmatrix} 0 & \alpha & \alpha & 0 \\ \alpha & \beta & \beta & \alpha \\ \alpha & \beta & \beta & \alpha \\ 0 & \alpha & \alpha & 0 \end{pmatrix}, \quad i, j = (1, 1), (11, 1), (19, 1), (29, 1). \quad (6.10)$$

In this case, we find $(M^{(1,0)})^2_{(1,1),(1,1)} = 2\alpha^2$, with α integer.

We can now proceed to compute the number $(M^{(1,0)})^2_{(1,1),(1,1)}$ from the formula relating the square of $M^{(1,0)}$ to the square of $M^{(0,1)}$. The latter contain blocks equal to

$$\begin{pmatrix} \epsilon_{(r,s)} & \eta_{(r,s)} \\ \eta_{(r,s)} & \epsilon_{(r,s)} \end{pmatrix}^2 = \begin{pmatrix} 2 & 2\epsilon_{(r,s)}\eta_{(r,s)} \\ 2\epsilon_{(r,s)}\eta_{(r,s)} & 2 \end{pmatrix}, \quad (6.11)$$

and diagonal entries equal to 0 and/or +4. In contrast, $(M^{(0,0)})^2$ contains blocks $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$ and diagonal entries equal to +4. We compute

$$\begin{aligned} (M^{(1,0)})^2_{(1,1),(1,1)} &= \sum_{i,j \in \text{KT}} S_{(1,1),i} [(M^{(0,0)})^2 + (M^{(0,1)})^2 - (M^{(0,0)})^2]_{ij} S_{j,(1,1)} \\ &= (M^{(0,0)})^2_{(1,1),(1,1)} + \sum_{i,j \in \text{KT}} S_{(1,1),i} [(M^{(0,1)})^2 - (M^{(0,0)})^2]_{ij} S_{j,(1,1)} \\ &= 2 - 4 \sum_{\substack{(r,s) \in \text{KT} \\ r \neq 2\ell+1, \text{ odd}}} \delta(\epsilon_{(r,s)}\eta_{(r,s)} + 1) S_{(1,1),(r,s)}^2 - 4 \sum_{s=1}^{2\ell+1} \delta(M_{(2\ell+1,s)(2\ell+1,s)}^{(0,1)}) S_{(1,1),(2\ell+1,s)}^2. \end{aligned} \quad (6.12)$$

For $m \neq 30$, this number must be equal to zero, implying that all Kronecker deltas must be equal to 1. This means for $M^{(0,1)}$ that $\eta_{(r,s)} = -\epsilon_{(r,s)}$ in all blocks, and that the $(2\ell+1, s), (2\ell+1, s)$ diagonal entries are all equal to zero.

For $m = 30$, it can also be equal to 2 (it is manifestly smaller than 2, and should equal $2\alpha^2$), and this forces the opposite relations for $M^{(0,1)}$: $\eta_{(r,s)} = \epsilon_{(r,s)}$ in all blocks, and the $(2\ell+1, s), (2\ell+1, s)$ diagonal entries are equal to ± 2 .

⁶By using a Galois transformation, one can prove that $\beta = 0$, but this does not seem to add significant information for what follows.

The last piece of argument consists in looking at the diagonal terms in the part of $M^{(1,0)}$ indexed by pairs (r, s) with r even. This is an easy calculation which yields, for r even,

$$M_{(r,s),(r,s)}^{(1,0)} = \sum_{\substack{(r',s') \in \text{KT} \\ r' \neq 2\ell+1, \text{ odd}}} \epsilon_{(r',s')} S_{(r,s),(r',s')} [S_{(r',s'),(r,s)} \pm S_{\mu(r',s'),(r,s)}]. \quad (6.13)$$

In this expression, $\mu(r', s') = (m - r', s')$ or $(r', m + 1 - s')$ depending on which one is in the Kac table, and the $\pm$ is the sign entering the relation $\eta_{(r,s)} = \pm \epsilon_{(r,s)}$ (thus always $-$, except perhaps for $m = 30$).

Because r is even, the symmetry of S says $S_{\mu(r',s'),(r,s)} = -S_{(r',s'),(r,s)}$, so that⁷

$$M_{(r,s),(r,s)}^{(1,0)} = \sum_{\substack{(r',s') \in \text{KT} \\ r' \text{ odd}}} \epsilon_{(r',s')} [1 \pm (-1)] S_{(r,s),(r',s')}^2 \leq 2 \sum_{\substack{(r',s') \in \text{KT} \\ r' \text{ odd}}} S_{(r,s),(r',s')}^2 = 1. \quad (6.14)$$

Being integers, they are equal to 0 or 1, but the important point is that they are all simultaneously equal to 0 or 1. Indeed one of them being equal to 1 requires to take the $-$ sign and all $\epsilon_{(r,s)} = +1$.

We finally conclude by showing that $M_{(r,s),(r,s)}^{(1,0)} = 0$ for all r even, implies that there is no Z_2 at all. The proof relies once more on the parity rule. We have argued before that if $M^{(1,1)}$ contains no second roots of unity (signs !), then there is no Z_2 . As usually, we relate the signs in $M^{(1,1)}$ to the T transform of $M^{(1,0)}$:

$$M_{(r,s),(r',s')}^{(1,1)} = e^{2i\pi(h_{r,s} - h_{r',s'})} M_{(r,s),(r',s')}^{(1,0)}. \quad (6.15)$$

Since m is even, the parity rule implies $M_{(r,s),(r',s')}^{(1,0)} \neq 0$ for $s' = s$ or $m+1-s$. The possibilities for r' being the same as for $m - r'$, the identity $h_{m-r',m+1-s'} = h_{r',s'}$ allows us to assume $s' = s$, in which case

$$h_{r,s} - h_{r',s} = (r^2 - r'^2) \frac{(m+1)}{4m} + \frac{(r+r')s}{2} \pmod{1}. \quad (6.16)$$

For this difference to be equal to $\frac{1}{2}$ (and nothing else), the possible values for r' consistent with the parity rule are just $r' = m - r$ with r even. Hence we obtain that the only source of signs in $M^{(1,1)}$ is in the coefficients $M_{(r,s),(m-r,s)}^{(1,0)}$ and $M_{(r,s),(r,m+1-s)}^{(1,0)}$ for r even, which themselves are equal to $M_{(r,s),(r,s)}^{(1,0)}$ on account of the identities (6.9).

Therefore we can conclude that the vanishing of all the coefficients $M_{(r,s),(r,s)}^{(1,0)}$ prevents the existence of a non-trivial Z_2 . The only alternative is to fix them all equal to 1, which requires taking the $-$ sign in (6.13) (it rules out the exotic possibility at $m = 30$) and all $\epsilon_{(r,s)} = +1$. This fixes completely the matrix $M^{(0,1)}$, and with it, $M^{(1,0)}$ and $M^{(1,1)}$.

This finishes the analysis for one of the series. Settling the other one is just of matter of repeating the above arguments. Except for $m = 5$, which is singled out in the analysis of the

⁷One may include $r' = 2\ell + 1$ in the summation since all $S_{(r,s),(2\ell+1,s')}$ are zero anyway for r even.

Z_3 symmetry, one arrives at exactly the same conclusion, namely the maximal symmetry is Z_2 . We summarize by giving the four partitions functions for both series.

For the series $(A_{m-1}, D_{2\ell+2})$, corresponding to $m = 4\ell + 1$, they read

$$Z_{PP}(\tau) = \sum_{\substack{s \leq r \leq 2\ell \\ s \text{ odd}}} |\chi(r,s) + \chi(m-r,s)|^2 + \sum_{\substack{2\ell+3 \leq s \leq r \\ s \text{ odd}}} |\chi(r,s) + \chi(r,m+1-s)|^2 + \sum_{r=2\ell+1}^{4\ell} 2 |\chi(r,2\ell+1)|^2, \quad (6.17)$$

$$Z_{PA}(\tau) = \sum_{\substack{s \leq r \leq 2\ell \\ s \text{ odd}}} |\chi(r,s) - \chi(m-r,s)|^2 + \sum_{\substack{2\ell+3 \leq s \leq r \\ s \text{ odd}}} |\chi(r,s) - \chi(r,m+1-s)|^2, \quad (6.18)$$

$$Z_{AP}(\tau) = \sum_{\substack{s \leq r \leq 2\ell \\ s \text{ even}}} |\chi(r,s) + \chi(m-r,s)|^2 + \sum_{\substack{2\ell+2 \leq s \leq r \\ s \text{ even}}} |\chi(r,s) + \chi(r,m+1-s)|^2, \quad (6.19)$$

$$Z_{AA}(\tau) = \sum_{\substack{s \leq r \leq 2\ell \\ s \text{ even}}} |\chi(r,s) - \chi(m-r,s)|^2 + \sum_{\substack{2\ell+2 \leq s \leq r \\ s \text{ even}}} |\chi(r,s) - \chi(r,m+1-s)|^2, \quad (6.20)$$

while for the series $(D_{2\ell+2}, A_m)$, corresponding to $m = 4\ell + 2$,

$$Z_{PP}(\tau) = \sum_{\substack{s \leq r \leq 2\ell-1 \\ r \text{ odd}}} |\chi(r,s) + \chi(m-r,s)|^2 + \sum_{\substack{2\ell+2 \leq s \leq r \\ r \text{ odd}}} |\chi(r,s) + \chi(r,m+1-s)|^2 + \sum_{s=1}^{2\ell+1} 2 |\chi(2\ell+1,s)|^2, \quad (6.21)$$

$$Z_{PA}(\tau) = \sum_{\substack{s \leq r \leq 2\ell-1 \\ r \text{ odd}}} |\chi(r,s) - \chi(m-r,s)|^2 + \sum_{\substack{2\ell+2 \leq s \leq r \\ r \text{ odd}}} |\chi(r,s) - \chi(r,m+1-s)|^2, \quad (6.22)$$

$$Z_{AP}(\tau) = \sum_{\substack{s \leq r \leq 2\ell \\ r \text{ even}}} |\chi(r,s) + \chi(m-r,s)|^2 + \sum_{\substack{2\ell+2 \leq s \leq r \\ r \text{ even}}} |\chi(r,s) + \chi(r,m+1-s)|^2, \quad (6.23)$$

$$Z_{AA}(\tau) = \sum_{\substack{s \leq r \leq 2\ell \\ r \text{ even}}} |\chi(r,s) - \chi(m-r,s)|^2 + \sum_{\substack{2\ell+2 \leq s \leq r \\ r \text{ even}}} |\chi(r,s) - \chi(r,m+1-s)|^2. \quad (6.24)$$

As recalled at the beginning of this section, the fully periodic partition functions, in both series, look like diagonal modular invariants in terms of extended characters, signalling the presence of an extended symmetry. One may note that the other partition functions cannot be written in terms of those, meaning that this extended symmetry is broken by the twisted boundary conditions. Equivalently, an analysis based on the extended characters rather than on the conformal ones, would not reveal any symmetry at all.

The above two sets exhaust the frustrated partition functions in those models, except if $m = 5$ and $m = 6$, for which other frustrations are possible.

6.4 The 3-Potts models

The previous subsections show that, in addition to a Z_2 symmetry, there is room for a Z_3 symmetry when $m = 5$ or 6 . It is not difficult to see that, in both cases, the modular invariant partition functions are indeed compatible with a unique Z_3 symmetry. To find the corresponding partition functions is an easy matter, which we will not detail. However the

full symmetry group of these models is known to be the permutation group S_3 , and it is instructive to see how this conclusion can be reached in the present context.

For concreteness, we discuss the case $m = 5$, describing the critical point of the 3-Potts model, the other case being exactly similar. The content and the charges in the various sectors were determined by von Gehlen and Rittenberg [8] and by Cardy [7]. An early study of the 3-Potts model using conformal field theoretic techniques was made by Dotsenko [15].

We start by giving the partition functions pertaining to the Z_2 and Z_3 boundary conditions. For convenience, we indicate, as superscripts, the conformal weights of the various primary fields. Our notation is that g is the generator of Z_2 , and r is a generator of Z_3 . For a Z_2 frustration, they are

$$Z_{1,1} = |\chi_{(1,1)}^0 + \chi_{(4,1)}^3|^2 + |\chi_{(2,1)}^{2/5} + \chi_{(3,1)}^{7/5}|^2 + 2|\chi_{(4,3)}^{2/3}|^2 + 2|\chi_{(3,3)}^{1/15}|^2, \quad (6.25)$$

$$Z_{1,g} = |\chi_{(1,1)}^0 - \chi_{(4,1)}^3|^2 + |\chi_{(2,1)}^{2/5} - \chi_{(3,1)}^{7/5}|^2, \quad (6.26)$$

$$Z_{g,g} = |\chi_{(2,2)}^{1/40} - \chi_{(3,2)}^{21/40}|^2 + |\chi_{(4,2)}^{13/8} - \chi_{(4,4)}^{1/8}|^2. \quad (6.27)$$

Those for a Z_3 frustration read (with $\omega = e^{2i\pi/3}$)

$$Z_{1,r} = Z_{1,r^2} = |\chi_{(1,1)}^0 + \chi_{(4,1)}^3|^2 + |\chi_{(2,1)}^{2/5} + \chi_{(3,1)}^{7/5}|^2 + (\omega + \omega^2)|\chi_{(4,3)}^{2/3}|^2 + (\omega + \omega^2)|\chi_{(3,3)}^{1/15}|^2, \quad (6.28)$$

$$Z_{r,r^j} = Z_{r^2,r^j}$$

$$= \omega^j [\chi_{(1,1)}^0 + \chi_{(4,1)}^3]^* \chi_{(4,3)}^{2/3} + \omega^j [\chi_{(2,1)}^{2/5} + \chi_{(3,1)}^{7/5}]^* \chi_{(3,3)}^{1/15} + \text{c.c.} + |\chi_{(4,3)}^{2/3}|^2 + |\chi_{(3,3)}^{1/15}|^2. \quad (6.29)$$

As a first step, one may remark that the Z_2 and Z_3 symmetries are not compatible, *i.e.* the two types of charges cannot be assigned simultaneously. The quickest way to see it is to notice that the partition function $Z_{g,1}$ is a sesquilinear form with coefficients in the set $\{0,1\}$. Then the same arguments that had led to R1 show that it cannot be compatible with a Z_3 symmetry. It means that the sector of the theory which is frustrated by the Z_2 does not support a diagonal action of Z_3 (and vice versa). So one cannot make sense of partition functions like $Z_{g,r}$ or $Z_{r,g}$. It also means that the actions of r and g on the various representations do not commute.

The same conclusion follows by looking at the functions $Z_{1,g}$ and $Z_{1,r}$ giving the Z_2 and Z_3 -charges of the periodic sector. One sees that the partition function combining the Z_2 and Z_3 -charges into Z_6 -charges ought to be

$$|\chi_{(1,1)} - \chi_{(4,1)}|^2 + |\chi_{(2,1)} - \chi_{(3,1)}|^2 + (\omega - \omega^2)|\chi_{(4,3)}|^2 + (\omega - \omega^2)|\chi_{(3,3)}|^2. \quad (6.30)$$

However, its S transform is not a sesquilinear form with positive integral coefficients, as it should be. The form of this would-be partition function was dictated by the assumption that the generators g and r were diagonal in the same basis. Clearly the failure of this assumption is another hint of the non-commutativity of g and r .

One of them, say r , can still be diagonalized. Then a non-diagonal action of g , compatible with $Z_{1,g}$, is the one that exchanges the two degenerate representations,

$$(\mathcal{R}_i \otimes \mathcal{R}_i) \xleftarrow{g} (\mathcal{R}_i \otimes \mathcal{R}_i)', \quad i = (4, 3), (3, 3). \quad (6.31)$$

So, on each of the two pairs of degenerate representations, g and r act respectively as $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$, in a non-diagonal way, but consistent with the above partition functions. These two matrices clearly generate a matrix group isomorphic to S_3 .

Thus the periodic sector possesses an S_3 symmetry, which can be identified with the symmetry of the model. The six partition functions are not all distinct, as one may check that $Z_{1,1}$, $Z_{1,g} = Z_{1,gr} = Z_{1,gr^2}$ and $Z_{1,r} = Z_{1,r^2}$. The symmetry is smaller in the frustrated sectors, being broken down to Z_2 or Z_3 . A quantum chain model for them can be found in [10], along with the appropriate boundary conditions on the microscopic quantum variables.

Let us finally say a few words about the extended algebra. The periodic partition function $Z_{1,1}$ is a modular invariant that looks diagonal with respect to an extended symmetry algebra, which is in this case the W_3 algebra [24, 25]. The value $c = \frac{4}{5}$ belongs to its minimal series, where the algebra has six completely degenerate representations. The partition function $Z_{1,1}$ is exactly the diagonal combination of all six characters (two pairs of representations are inequivalent at the W_3 level, and have, within each pair, identical characters). In addition, all partition functions pertaining to the Z_3 twisted boundary conditions are written in terms of the same W_3 representations (with charges as in [25]), which is not the case for the Z_2 boundary conditions. The generator of W_3 of weight 3 is neutral with respect to the Z_3 but is odd under the Z_2 (see the sign in $Z_{1,g}$ popping up in front of $\chi_{(4,1)}^3$). The W_3 invariance of the critical 3-Potts model is therefore broken in the Z_2 twisted sectors.

The tricritical 3-Potts model, at $m = 6$, is similar. The partition functions for the Z_2 boundary conditions have been given above, Section 6.3, and those relative to the Z_3 can be found in [9]. We include them here for completeness

$$\begin{aligned} Z_{1,r} = Z_{1,r^2} = & |\chi_{(1,1)}^0 + \chi_{(5,1)}^5|^2 + |\chi_{(5,5)}^{1/7} + \chi_{(5,2)}^{22/7}|^2 + |\chi_{(5,4)}^{5/7} + \chi_{(5,3)}^{12/7}|^2 \\ & + (\omega + \omega^2)|\chi_{(3,1)}^{4/3}|^2 + (\omega + \omega^2)|\chi_{(3,2)}^{10/21}|^2 + (\omega + \omega^2)|\chi_{(3,3)}^{1/21}|^2, \end{aligned} \quad (6.32)$$

$$\begin{aligned} Z_{r,r^j} = Z_{r^2,r^j} = & \omega^{2j} [\chi_{(1,1)}^0 + \chi_{(5,1)}^5]^* \chi_{(3,1)}^{4/3} + \omega^{2j} [\chi_{(5,5)}^{1/7} + \chi_{(5,2)}^{22/7}]^* \chi_{(3,2)}^{10/21} \\ & + \omega^{2j} [\chi_{(5,4)}^{5/7} + \chi_{(5,3)}^{12/7}]^* \chi_{(3,3)}^{1/21} + \text{c.c.} + |\chi_{(3,1)}^{4/3}|^2 + |\chi_{(3,2)}^{10/21}|^2 + |\chi_{(3,3)}^{1/21}|^2. \end{aligned} \quad (6.33)$$

7 Exceptional modular invariants

We end our analysis by examining the six exceptional modular invariant partition functions, occurring for $m = 11, 12, 17, 18, 29$ and 30 . From the result R1 in Section 4, we know that their maximal cyclic symmetry is Z_2 . We want to show here that the first two models, related to the simple Lie algebra E_6 , are the only ones to possess a non-trivial symmetry,

namely a Z_2 symmetry. The other four have no symmetry at all. The same arguments may be applied to them all. For concreteness, we give some details for $m = 11$ only.

By using the symmetry of the characters, $\chi_{(r,s)} = \chi_{(m-r, m+1-s)}$, the modular invariant partition function can be written as

$$Z_{1,1} = \sum_{\substack{r=1 \\ \text{odd}}}^{10} |\chi_{(r,1)} + \chi_{(r,7)}|^2 + |\chi_{(r,5)} + \chi_{(r,11)}|^2 + |\chi_{(r,4)} + \chi_{(r,8)}|^2. \quad (7.1)$$

The pairs labelling the characters are not all in the Kac table, but for the modular transformations, it does not matter, since S also has the symmetry $S_{(r,s),(r',s')} = S_{(m-r, m+1-s),(r',s')}$. What is important is that all pairs appearing in $Z_{1,1}$ are different when brought back in the Kac table.

The corresponding matrix $M^{(0,0)}$ can thus be viewed as a direct sum of five 6-by-6 blocks, labelled by r , where each block is itself the direct sum of three 2-by-2 blocks filled up with 1's. The matrix $M^{(0,1)}$, which specifies the Z_2 -charges in the periodic sector, has the same form, with however the 1's replaced by arbitrary and uncorrelated signs.

As a first step, we use the Galois transformations in order to constrain these signs. Since the matrix elements of $M^{(0,1)}$ are invariant under the Galois group, one finds from R2 that

$$M_{(r,s),(r',s')}^{(0,1)} = M_{\sigma(r,s),\sigma(r',s')}^{(0,1)}. \quad (7.2)$$

The Galois group relevant to the present case is $\mathbb{Z}_{22 \cdot 12}^* = \mathbb{Z}_{22}^* \times \mathbb{Z}_{24}^*$. On a pair (r, s) , the first factor acts on r and the second factor acts on s . For $h \in \mathbb{Z}_{22}^*$, one computes $\langle hr \rangle_{22}$, *i.e.* the product hr taken modulo 22, and one keeps that number if it is smaller than 11, and otherwise one replaces it with $22 - \langle hr \rangle_{22}$. Since that algorithm produces the same result for h and $22 - h$, it is enough to consider $P\mathbb{Z}_{22}^* \equiv \mathbb{Z}_{22}^*/\{\pm 1\}$. The same calculations are done with the other factor $P\mathbb{Z}_{24}^*$, with all congruences taken modulo 24.

The form of $M^{(0,1)}$ makes the action of the Galois group rather transparent. The values of r , chosen to be odd between 1 and 10, form precisely the set $P\mathbb{Z}_{22}^*$. Thus the action of that factor simply maps the first block labelled by $r = 1$ onto the other blocks. The constraint (7.2) then implies that the five blocks of $M^{(0,1)}$ must be equal, say to A , a 6-by-6 matrix. Thus,

$$M_{(r,s),(r',s')}^{(0,1)} = A_{s,s'} \delta_{r,r'}, \quad \text{for } r \in P\mathbb{Z}_{22}^*. \quad (7.3)$$

The matrix $A_{s,s'}$ has indices s, s' in the set $\{1, 7, 5, 11, 4, 8\}$, in that order.

The action on the s -labels of the other factor $P\mathbb{Z}_{24}^*$ has the effect of permuting the entries of A , which must satisfy $A_{\sigma_{h'}(s), \sigma_{h'}(s')} = A_{s,s'}$ for all $h' \in P\mathbb{Z}_{24}^* = \{1, 5, 7, 11\}$. This leaves in A six undetermined signs,

$$A_{s,s'} = \begin{pmatrix} \epsilon_1 & \epsilon_2 & 0 & 0 & 0 & 0 \\ \epsilon_2 & \epsilon_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \epsilon_1 & \epsilon_2 & 0 & 0 \\ 0 & 0 & \epsilon_2 & \epsilon_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta_1 & \eta_2 \\ 0 & 0 & 0 & 0 & \eta_3 & \eta_4 \end{pmatrix}. \quad (7.4)$$

At this stage, the Galois constraints are fully satisfied.

Because ϵ_1 is eventually the Z_2 -parity of the vacuum, it must be equal to $\epsilon_1 = +1$. One can determine ϵ_2 by computing $M_{(3,3),(3,3)}^{(1,0)}$. It is independent of the signs η_i , and is equal to

$$\begin{aligned} M_{(3,3),(3,3)}^{(1,0)} &= \sum_{(r,s),(r',s')} S_{(3,3),(r,s)} M_{(r,s),(r',s')}^{(0,1)} S_{(r',s'),(3,3)} \\ &= \frac{8}{11 \cdot 12} \sum_{r \in P\mathbb{Z}_{22}^*} \left(\sin \frac{3\pi r}{11} \right)^2 \sum_{s,s'} A_{s,s'} \sin \frac{3\pi s}{12} \sin \frac{3\pi s'}{12} = \frac{1 - \epsilon_2}{3}. \end{aligned} \quad (7.5)$$

This should be a non-negative integer, which forces $\epsilon_2 = +1$.

The last step is to compute the coefficient $M_{(1,1),(1,1)}^{(1,0)}$, which must be 0 if a non-trivial symmetry is present. One finds

$$\begin{aligned} M_{(1,1),(1,1)}^{(1,0)} &= M_{(1,1),(1,1)}^{(0,0)} + \sum_{(r,s),(r',s')} S_{(1,1),(r,s)} [M_{(r,s),(r',s')}^{(0,1)} - M_{(r,s),(r',s')}^{(0,0)}] S_{(r',s'),(1,1)} \\ &= 1 + \frac{8}{11 \cdot 12} \sum_{r \in P\mathbb{Z}_{22}^*} \left(\sin \frac{\pi r}{11} \right)^2 \sum_{s,s'=4,8} (A_{s,s'} - 1) \sin \frac{\pi s}{12} \sin \frac{\pi s'}{12} = \frac{4 + \sum_i \eta_i}{8}. \end{aligned} \quad (7.6)$$

A non-trivial symmetry requires to set $\eta_i = -1$ for $i = 1, 2, 3, 4$. Then all signs in $M^{(0,1)}$ are uniquely fixed. It remains to check that the partition functions obtained by modular transformations are well-behaved, which they are.

Therefore, the model (A_{10}, E_6) , $m = 11$, has a unique Z_2 symmetry. The charges of the frustrated sectors, periodic and antiperiodic, can be read off from the two partition functions (for simplicity, we use pairs of labels which are not necessarily in the Kac table)

$$Z_{PA} = \sum_{r=1, \text{ odd}}^{10} |\chi_{(r,1)} + \chi_{(r,7)}|^2 + |\chi_{(r,5)} + \chi_{(r,11)}|^2 - |\chi_{(r,4)} + \chi_{(r,8)}|^2, \quad (7.7)$$

$$Z_{AA} = \sum_{r=1, \text{ odd}}^{10} |\chi_{(r,4)} + \chi_{(r,8)}|^2 - \left\{ [\chi_{(r,1)} + \chi_{(r,7)}]^* [\chi_{(r,5)} + \chi_{(r,11)}] + \text{c.c.} \right\}. \quad (7.8)$$

The same arguments may be repeated for the twin model (E_6, A_{12}) , at $m = 12$. One finds a unique Z_2 symmetry, and similar partition functions

$$Z_{PA} = \sum_{s=1, \text{ odd}}^{12} |\chi_{(1,s)} + \chi_{(7,s)}|^2 + |\chi_{(5,s)} + \chi_{(11,s)}|^2 - |\chi_{(4,s)} + \chi_{(8,s)}|^2, \quad (7.9)$$

$$Z_{AA} = \sum_{s=1, \text{ odd}}^{12} |\chi_{(4,s)} + \chi_{(8,s)}|^2 - \left\{ [\chi_{(1,s)} + \chi_{(7,s)}]^* [\chi_{(5,s)} + \chi_{(11,s)}] + \text{c.c.} \right\}. \quad (7.10)$$

Among all unitary minimal models, these two at $m = 11$ and 12 are the only ones for which the extended chiral symmetry is preserved in all sectors.

The analysis of the last four models proceeds the same way. The Galois symmetry leaves in $M^{(0,1)}$ five arbitrary signs for $m = 17, 18$, and four arbitrary signs for $m = 29, 30$. In each case, one sign is equal to the charge of the vacuum, and must be equal to $+1$. Then by looking at specific entries of $M^{(1,0)}$, one finds that they cannot be made non-negative integers unless all signs are equal to $+1$. It means that all charges are equal to $+1$, so there is no symmetry at all. This is what should have been expected, since the Dynkin diagrams of E_7 and E_8 have no automorphism.

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