ON FIBRATIONS BETWEEN INTERNAL GROUPOIDS

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ABSTRACT. In order to deduce the internal version of the Brown exact sequence from the internal version of the Gabriel-Zisman exact sequence, we characterize fibrations and $*$-fibrations in the 2-category of internal groupoids in terms of the comparison functor from certain pullbacks to the corresponding strong homotopy pullbacks. A similar analysis in the category of arrows allows us to give a characterization of protomodular categories using strong homotopy kernels.

Contents

1. Introduction

In [7], Gabriel and Zisman constructed a π_0-π_1 exact sequence starting from a functor $F\colon \mathbb{A} \to \mathbb{B}$ between pointed groupoids. The Gabriel-Zisman sequence involves, as first and fourth point, π_1 and π_0 of the strong homotopy kernel of F. A similar π_0-π_1 exact sequence is obtained in [4] by Brown, but now the functor F is assumed to be a fibration, and the sequence involves, instead of the strong homotopy kernel of F, its categorical kernel. Both exact sequences have been generalized in [10], replacing pointed functors by functors between groupoids internal to a pointed regular category $\mathcal{A}$ with reflexive coequalizers. In order to deduce the internal version of the Brown sequence from the internal version of the Gabriel-Zisman sequence, one needs the fact that, if the internal functor F is a fibration, then the comparison functor $J\colon \mathbb{K}\mathrm{er}(F) \to \mathbb{K}(F)$ from the kernel to the strong homotopy kernel is a weak equivalence (so that the induced arrows $\pi_0(F)$ and $\pi_1(F)$ are isomorphisms).
The first aim of this note is therefore to prove that, if F is fibration, then J is a weak equivalence. Since the converse implication is not true, we work out a more complete

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analysis of the situation getting the following results. In Section 2, we review some basic facts on homotopy pullbacks in a 2-category with invertible 2-cells, in order to conclude that $\mathbf{Grpd}(\mathcal{A})$, the 2-category of internal groupoids, has strong pullbacks and strong homotopy pullbacks. In Section 3, we assume the base category $\mathcal{A}$ to be regular and we prove that F is a fibration if and only if the comparison functor from a suitable pullback to the corresponding strong homotopy pullback is a weak equivalence. Here, the pullback and the strong homotopy pullback are those of F along the embedding of B_0, the object of objects of $\mathbb{B}$, into $\mathbb{B}$. In Section 4, we assume that $\mathcal{A}$ is regular and pointed, and we characterize $*$-fibration, that is, those functors F such that the comparison from the kernel to the strong homotopy kernel is a weak equivalence. Thanks to the regularity of $\mathcal{A}$, any fibration is a $*$-fibration, so that we get as a corollary the result needed to compare Gabriel-Zisman and Brown sequences.

The normalized version of Brown and Gabriel-Zisman sequences are the snake and snail sequences, studied in the context of regular protomodular categories in [2, 13, 9]. In this context, the condition of being a fibration is replaced by the condition that a certain arrow is a regular epimorphism. In order to have an analysis as complete as possible of the situation in the normalized context, in Section 5 we compare the notions of fibration, $*$-fibration and weak equivalence in $\mathbf{Grpd}(\mathcal{A})$ with suitable notions of fibration, $*$-fibration and weak equivalence in $\mathbf{Arr}(\mathcal{A})$, the category with null-homotopies whose objects are arrows in $\mathcal{A}$. We show that, if $\mathcal{A}$ is pointed, regular and protomodular, the normalization process $\mathcal{N}\colon \mathbf{Grpd}(\mathcal{A}) \to \mathbf{Arr}(\mathcal{A})$ preserves and reflects fibrations, $*$-fibrations and weak equivalences. Moreover, we show that, if $\mathcal{A}$ is pointed and regular, the expected implication "fibration $\Rightarrow$ $*$-fibration" in $\mathbf{Arr}(\mathcal{A})$ is in fact equivalent to the condition that $\mathcal{A}$ is protomodular.

2. Strong pullbacks and strong h-pullbacks

2.1 Let $\underline{\mathcal{B}}$ be a 2-category with invertible 2-cells, and $\mathcal{B}$ its underlying category. We adopt the following terminology :

1. A 1-cell $F\colon \mathbb{A} \to \mathbb{B}$ in $\underline{\mathcal{B}}$ is *fully faithful* if, for any $\mathbb{X}$ in $\underline{\mathcal{B}}$, the induced functor

$$- \cdot F\colon \underline{\mathcal{B}}(\mathbb{X}, \mathbb{A}) \to \underline{\mathcal{B}}(\mathbb{X}, \mathbb{B})$$

is fully faithful in the usual sense.

2. Consider 1-cells $F\colon \mathbb{A} \to \mathbb{B}$ and $G\colon \mathbb{C} \to \mathbb{B}$ in $\underline{\mathcal{B}}$. A *strong homotopy pullback* (strong h-pullback, for short) of F and G is a diagram of the form

$$\begin{tikzcd} \mathbb{P} \arrow[r, "G'"] \arrow[d, "F'"'] \arrow[dr, phantom, "\overset{\varphi}{\Rightarrow}"] & \mathbb{A} \arrow[d, "F"] \\ \mathbb{C} \arrow[r, "G"'] & \mathbb{B} \end{tikzcd}$$

satisfying the following universal property :

(a) For any diagram of the form

$$\begin{tikzcd} \mathbb{X} \arrow[r, "H"] \arrow[d, "K"'] \arrow[dr, phantom, "\overset{\mu}{\Rightarrow}"] & \mathbb{A} \arrow[d, "F"] \\ \mathbb{C} \arrow[r, "G"'] & \mathbb{B} \end{tikzcd}$$

there exists a unique 1-cell $T\colon \mathbb{X} \to \mathbb{P}$ such that $T \cdot G' = H, T \cdot F' = K$ and $T \cdot \varphi = \mu$.

(b) Given 1-cells $H, K\colon \mathbb{X} \rightrightarrows \mathbb{P}$ and 2-cells $\alpha\colon H \cdot F' \Rightarrow K \cdot F'$ and $\beta\colon H \cdot G' \Rightarrow K \cdot G'$, if

$$\begin{tikzcd} H \cdot F' \cdot G \arrow[r, Rightarrow, "\alpha \cdot G"] \arrow[d, Rightarrow, "H \cdot \varphi"'] & K \cdot F' \cdot G \arrow[d, Rightarrow, "K \cdot \varphi"] \\ H \cdot G' \cdot F \arrow[r, Rightarrow, "\beta \cdot F"'] & K \cdot G' \cdot F \end{tikzcd}$$

commutes, then there exists a unique 2-cell $\mu\colon H \Rightarrow K$ such that $\mu \cdot F' = \alpha$ and $\mu \cdot G' = \beta$.

3. We say that a pullback $\mathbb{C} \times_{G,F} \mathbb{A}$ of $F\colon \mathbb{A} \to \mathbb{B}$ and $G\colon \mathbb{C} \to \mathbb{B}$ in the category $\mathcal{B}$ is *strong* (in $\underline{\mathcal{B}}$) if

$$\begin{tikzcd} \mathbb{C} \times_{G,F} \mathbb{A} \arrow[r, "\widehat{G}"] \arrow[d, "\widehat{F}"'] \arrow[dr, phantom, "\overset{\mathrm{id}}{\Rightarrow}"] & \mathbb{A} \arrow[d, "F"] \\ \mathbb{C} \arrow[r, "G"'] & \mathbb{B} \end{tikzcd}$$

satisfies condition (b) above.

2.2 Another way to express the universal property of the strong h-pullback is to fix an object $\mathbb{X}$ in $\underline{\mathcal{B}}$ and to construct the comma-square of groupoids (which precisely is the strong h-pullback in the 2-category of groupoids)

$$\begin{tikzcd} (- \cdot G \mid - \cdot F) \arrow[r] \arrow[d] \arrow[dr, phantom, "\simeq"] & \underline{\mathcal{B}}(\mathbb{X}, \mathbb{A}) \arrow[d, "- \cdot F"] \\ \underline{\mathcal{B}}(\mathbb{X}, \mathbb{C}) \arrow[r, "- \cdot G"'] & \underline{\mathcal{B}}(\mathbb{X}, \mathbb{B}) \end{tikzcd}$$

Now the universal property of the strong h-pullback means that, for any $\mathbb{X}$, the canonical comparison functor

$$\underline{\mathcal{B}}(\mathbb{X}, \mathbb{P}) \to (- \cdot G \mid - \cdot F)$$

is bijective on objects (condition a) and fully faithful (condition b), that is, it is an isomorphism of categories. This makes evident that a strong h-pullback is determined by its universal property up to isomorphism.

A weaker universal property consists in asking that the canonical comparison functors $\underline{\mathcal{B}}(\mathbb{X}, \mathbb{P}) \to (- \cdot G \mid - \cdot F)$ are equivalences of groupoids. In this way one gets what is

sometimes called a *bipullback*, which is determined only up to equivalence.
Intermediate situations between strong homotopy pullbacks and bipullbacks are considered in the literature. For example, in [8] the comparison functors are required to be bijective on objects but not fully faithful (the name of h-pullback is used in this case), and in [7] the comparison functors are required to be surjective on objects and full.

2.3 Among strong h-pullbacks, the following one plays a special role

$$\begin{tikzcd} \vec{\mathbb{B}} \arrow[r, "\gamma"] \arrow[d, "\delta"'] \arrow[dr, phantom, "\overset{\beta}{\Rightarrow}"] & \mathbb{B} \arrow[d, "\mathrm{Id}"] \\ \mathbb{B} \arrow[r, "\mathrm{Id}"'] & \mathbb{B} \end{tikzcd}$$

Indeed, if for a category $\mathcal{A}$ we denote by $\mathbf{Arr}(\mathcal{A})$ the category having arrows of $\mathcal{A}$ as objects and commutative squares as arrows, then the universal property of $\vec{\mathbb{B}}$ gives an isomorphism of categories

$$\underline{\mathcal{B}}(\mathbb{X}, \vec{\mathbb{B}}) \to \mathbf{Arr}(\underline{\mathcal{B}}(\mathbb{X}, \mathbb{B}))$$

so that to give a 1-cell $\mathbb{X} \to \vec{\mathbb{B}}$ is the same as giving a 2-cell $\begin{tikzcd} \mathbb{X} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \mathbb{B} \end{tikzcd}$.

2.4 Pasting together two strong h-pullbacks, in general one does not get a strong h-pullback. The main interest of the notion of strong pullback relies on the following fact : given a diagram in $\underline{\mathcal{B}}$ of the form

$$\begin{tikzcd} \mathbb{D} \times_{H,F'} \mathbb{P} \arrow[r, "\widehat{H}"] \arrow[d, "\widehat{F'}"'] & \mathbb{P} \arrow[r, "G'"] \arrow[d, "F'"'] \arrow[dr, phantom, "\overset{\varphi}{\Rightarrow}"] & \mathbb{A} \arrow[d, "F"] \\ \mathbb{D} \arrow[r, "H"'] & \mathbb{C} \arrow[r, "G"'] & \mathbb{B} \end{tikzcd}$$

if the left-hand part is a strong pullback and the right-hand part is a strong h-pullback, then the total diagram is a strong h-pullback.
This fact has an interesting consequence on the existence of strong h-pullbacks : assume that $\underline{\mathcal{B}}$ has strong pullbacks and that the strong h-pullback

$$\begin{tikzcd} \vec{\mathbb{B}} \arrow[r, "\gamma"] \arrow[d, "\delta"'] \arrow[dr, phantom, "\overset{\beta}{\Rightarrow}"] & \mathbb{B} \arrow[d, "\mathrm{Id}"] \\ \mathbb{B} \arrow[r, "\mathrm{Id}"'] & \mathbb{B} \end{tikzcd}$$

exists in $\underline{\mathcal{B}}$ for any object $\mathbb{B}$. Then, for any pair of 1-cells $F\colon \mathbb{A} \to \mathbb{B}, G\colon \mathbb{C} \to \mathbb{B}$, a strong h-pullback of F and G exists and can be obtained by the following limit in $\mathcal{B}$

$$\begin{tikzcd}
 & & \mathbb{P} \arrow[dll, "F'"'] \arrow[d, "\phi"] \arrow[drr, "G'"] & & \\
\mathbb{C} \arrow[dr, "G"'] & & \vec{\mathbb{B}} \arrow[dl, "\delta"] \arrow[dr, "\gamma"'] & & \mathbb{A} \arrow[dl, "F"] \\
 & \mathbb{B} & & \mathbb{B} &
\end{tikzcd}$$

together with $\varphi = \phi \cdot \beta \colon F' \cdot G = \phi \cdot \delta \Rightarrow \phi \cdot \gamma = G' \cdot F$.

2.5 Later, we will use the following facts on strong pullbacks and strong h-pullbacks.
1. If the pullback in $\mathcal{B}$ and the strong h-pullback in $\underline{\mathcal{B}}$ of F and G exist,

$$\begin{tikzcd}
\mathbb{C} \times_{G,F} \mathbb{A} \arrow[dr, "T"] \arrow[drr, bend left, "\widehat{G}"] \arrow[ddr, bend right, "\widehat{F}"'] & & \\
 & \mathbb{P} \arrow[r, "G'"] \arrow[d, "F'"'] \arrow[dr, phantom, "\overset{\varphi}{\Rightarrow}"] & \mathbb{A} \arrow[d, "F"] \\
 & \mathbb{C} \arrow[r, "G"'] & \mathbb{B}
\end{tikzcd}$$

then the pullback $\mathbb{C} \times_{G,F} \mathbb{A}$ is strong iff the canonical comparison T is fully faithful.
2. Consider the pullback in $\mathcal{B}$ and the strong h-pullback in $\underline{\mathcal{B}}$ of F and G

$$\begin{tikzcd}
\mathbb{C} \times_{G,F} \mathbb{A} \arrow[r, "\widehat{G}"] \arrow[d, "\widehat{F}"'] & \mathbb{A} \arrow[d, "F"] \\
\mathbb{C} \arrow[r, "G"'] & \mathbb{B}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathbb{P} \arrow[r, "G'"] \arrow[d, "F'"'] \arrow[dr, phantom, "\overset{\varphi}{\Rightarrow}"] & \mathbb{A} \arrow[d, "F"] \\
\mathbb{C} \arrow[r, "G"'] & \mathbb{B}
\end{tikzcd}$$

Clearly, if F is fully faithful, then F' is fully faithful. Moreover, if the pullback $\mathbb{C} \times_{G,F} \mathbb{A}$ is strong and if F is fully faithful, then $\widehat{F}$ is fully faithful.

2.6 Now we specialize the previous discussion taking as $\underline{\mathcal{B}}$ the 2-category $\mathbf{Grpd}(\mathcal{A})$ of groupoids, functors and natural transformations internal to a category $\mathcal{A}$ with pullbacks. The notation for a groupoid $\mathbb{B}$ in $\mathcal{A}$ is

$$\mathbb{B} = \Big(\begin{tikzcd} B_1 \times_{c,d} B_1 \arrow[r, "m"] & B_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & B_0 \arrow[l, "e" description] \end{tikzcd} \;,\; \begin{tikzcd} B_1 \arrow[r, "i"] & B_1 \end{tikzcd} \Big)$$

where

$$\begin{tikzcd}
B_1 \times_{c,d} B_1 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B_1 \arrow[d, "d"] \\
B_1 \arrow[r, "c"'] & B_0
\end{tikzcd}$$

is a pullback. The notation for a natural transformation $\alpha\colon F \Rightarrow G\colon \mathbb{A} \rightrightarrows \mathbb{B}$ is

$$\begin{tikzcd}
A_1 \arrow[r, shift left, "F_1"] \arrow[r, shift right, "G_1"'] \arrow[d, shift right, "d"'] \arrow[d, shift left, "c"] & B_1 \arrow[d, shift right, "d"'] \arrow[d, shift left, "c"] \\
A_0 \arrow[r, shift left, "F_0"] \arrow[r, shift right, "G_0"'] \arrow[ur, "\alpha"] & B_0
\end{tikzcd}$$

2.7 Following 2.4, to prove that $\mathbf{Grpd}(\mathcal{A})$ has strong h-pullbacks we need two ingredients. The first one is easy, the second one is the standard construction of the groupoid of "commutative squares", and we recall it from [12] or [11].
1. Since pullbacks in $\mathbf{Grpd}(\mathcal{A})$ are constructed level-wisely, it is straightforward to check that they are strong.
2. For every internal groupoid $\mathbb{B}$, the strong h-pullback

$$\begin{tikzcd}
\vec{\mathbb{B}} \arrow[r, "\gamma"] \arrow[d, "\delta"'] \arrow[dr, phantom, "\overset{\beta}{\Rightarrow}"] & \mathbb{B} \arrow[d, "\mathrm{Id}"] \\
\mathbb{B} \arrow[r, "\mathrm{Id}"'] & \mathbb{B}
\end{tikzcd}$$

exists, and it can be described as follows :

$$\vec{\mathbb{B}} = (\ \vec{B}_1 \times_{\vec{c},\vec{d}} \vec{B}_1 \xrightarrow{\vec{m}} \vec{B}_1 \underset{\vec{c}}{\overset{\vec{d}}{\underset{\longleftarrow \vec{e}}{\rightrightarrows}}} B_1 \ \ , \ \ \vec{B}_1 \xrightarrow{\vec{i}} \vec{B}_1\ )$$

where

$$\begin{tikzcd}
\vec{B}_1 \arrow[r, "m_2"] \arrow[d, "m_1"'] & B_1 \times_{c,d} B_1 \arrow[d, "m"] \\
B_1 \times_{c,d} B_1 \arrow[r, "m"'] & B_1
\end{tikzcd}$$

is a pullback, and $\vec{d}$, $\vec{c}$ and $\vec{e}$ are defined by

$$\vec{d}\colon\ \vec{B}_1 \xrightarrow{m_1} B_1 \times_{c,d} B_1 \xrightarrow{\pi_1} B_1 \qquad \vec{c}\colon\ \vec{B}_1 \xrightarrow{m_2} B_1 \times_{c,d} B_1 \xrightarrow{\pi_2} B_1$$

$$\begin{tikzcd}
B_1 \arrow[dr, "\vec{e}"] \arrow[drr, bend left, "\langle d\cdot e,\mathrm{id}\rangle"] \arrow[ddr, bend right, "\langle \mathrm{id},c\cdot e\rangle"'] & & \\
& \vec{B}_1 \arrow[r, "m_2"] \arrow[d, "m_1"'] & B_1 \times_{c,d} B_1 \arrow[d, "m"] \\
& B_1 \times_{c,d} B_1 \arrow[r, "m"'] & B_1
\end{tikzcd}$$

The groupoid $\vec{\mathbb{B}}$ is equipped with two functors $\delta\colon \vec{\mathbb{B}} \to \mathbb{B}$ and $\gamma\colon \vec{\mathbb{B}} \to \mathbb{B}$ given by

$$\begin{tikzcd}
\vec{B}_1 \arrow[r, "\delta_1 = m_2 \cdot \pi_1"] \arrow[d, shift right, "\vec{d}"'] \arrow[d, shift left, "\vec{c}"] & B_1 \arrow[d, shift right, "d"'] \arrow[d, shift left, "c"] \\
B_1 \arrow[r, "\delta_0 = d"'] & B_0
\end{tikzcd}
\qquad
\begin{tikzcd}
\vec{B}_1 \arrow[r, "\gamma_1 = m_1 \cdot \pi_2"] \arrow[d, shift right, "\vec{d}"'] \arrow[d, shift left, "\vec{c}"] & B_1 \arrow[d, shift right, "d"'] \arrow[d, shift left, "c"] \\
B_1 \arrow[r, "\gamma_0 = c"'] & B_0
\end{tikzcd}$$

Finally, the natural transformation $\beta\colon \delta \Rightarrow \gamma$ is simply $\beta = \mathrm{id}_{B_1}\colon B_1 \to B_1$.
To help intuition, let us point out that when $\mathcal{A}$ is the category of sets, an element of the object $\vec{B}_1$ involved in the above description of the strong h-pullback is a commutative square

$$\begin{tikzcd}
\cdot \arrow[r, "g_0"] \arrow[d, "b_1"'] \arrow[dr, phantom, "S"] & \cdot \arrow[d, "b_2"] \\
\cdot \arrow[r, "f_0"'] & \cdot
\end{tikzcd}$$

with $m_1(S) = \langle b_1, f_0 \rangle, m_2(S) = \langle g_0, b_2 \rangle, \vec{d}(S) = b_1, \vec{c}(S) = b_2, \delta_1(S) = g_0, \gamma_1(S) = f_0$.

2.8 Putting together 2.4 and 2.7, we can conclude that the 2-category $\mathbf{Grpd}(\mathcal{A})$ has strong h-pullbacks. Moreover, a strong h-pullback

$$\begin{tikzcd}
\mathbb{P} \arrow[r, "G'"] \arrow[d, "F'"'] \arrow[dr, phantom, "\overset{\varphi}{\Rightarrow}"] & \mathbb{A} \arrow[d, "F"] \\
\mathbb{C} \arrow[r, "G"'] & \mathbb{B}
\end{tikzcd}$$

of $F\colon \mathbb{A} \to \mathbb{B}$ and $G\colon \mathbb{C} \to \mathbb{B}$ in $\mathbf{Grpd}(\mathcal{A})$ is described by the following limit diagram in $\mathcal{A}$:

$$\begin{tikzcd}
 & & P_1 \arrow[dll, "F'_1"'] \arrow[dr, "\varphi_1"] \arrow[drrr, "G'_1"] \arrow[ddd, shift right, "\underline{d}"'] \arrow[ddd, shift left, "\underline{c}"] & & & \\
C_1 \arrow[dr, "G_1"] \arrow[ddd, shift right, "d"'] \arrow[ddd, shift left, "c"] & & & \vec{B}_1 \arrow[dll, "m_2 \cdot \pi_1"'] \arrow[dr, "m_1 \cdot \pi_2"] \arrow[ddd, shift right, "m_1 \cdot \pi_1"'] \arrow[ddd, shift left, "m_2 \cdot \pi_2"] & & A_1 \arrow[dl, "F_1"'] \arrow[ddd, shift right, "d"'] \arrow[ddd, shift left, "c"] \\
 & B_1 \arrow[ddd, shift right, "d"'] \arrow[ddd, shift left, "c"] & & & B_1 \arrow[ddd, shift right, "d"'] \arrow[ddd, shift left, "c"] & \\
 & & P_0 \arrow[dll, "F'_0"'] \arrow[dr, "\varphi_0"'] \arrow[drrr, "G'_0"] & & & \\
C_0 \arrow[dr, "G_0"'] & & & B_1 \arrow[dll, "d"'] \arrow[dr, "c"] & & A_0 \arrow[dl, "F_0"] \\
 & B_0 & & & B_0 &
\end{tikzcd}$$

2.9 In $\mathbf{Grpd}(\mathcal{A})$, as in any 2-category, the notion of equivalence makes sense. Moreover, in $\mathbf{Grpd}(\mathcal{A})$ we also dispose of the notion of weak equivalence. From [5, 12], recall that a functor $F\colon \mathbb{A} \to \mathbb{B}$ between groupoids in $\mathcal{A}$ is :
1. fully faithful if and only if the following diagram is a limit diagram

$$\begin{tikzcd}
 & & A_1 \arrow[dll, "d"'] \arrow[d, "F_1"] \arrow[drr, "c"] & & \\
A_0 \arrow[dr, "F_0"'] & & B_1 \arrow[dl, "d"] \arrow[dr, "c"'] & & A_0 \arrow[dl, "F_0"] \\
 & B_0 & & B_0 &
\end{tikzcd}$$

2. *essentially surjective* if the first row in one (equivalently, in both) of the following diagrams is a regular epimorphism (the squares are pullbacks)

$$\begin{tikzcd}
A_0 \times_{F_0,d} B_1 \arrow[r, "\beta_d"] \arrow[d, "\alpha_d"'] & B_1 \arrow[r, "c"] \arrow[d, "d"] & B_0 \\
A_0 \arrow[r, "F_0"'] & B_0 &
\end{tikzcd}
\qquad
\begin{tikzcd}
A_0 \times_{F_0,c} B_1 \arrow[r, "\beta_c"] \arrow[d, "\alpha_c"'] & B_1 \arrow[r, "d"] \arrow[d, "c"] & B_0 \\
A_0 \arrow[r, "F_0"'] & B_0 &
\end{tikzcd}$$

3. a *weak equivalence* if it is fully faithful, and essentially surjective ;
4. an equivalence if and only if it is fully faithful and $\beta_d \cdot c$ (or equivalently, $\beta_c \cdot d$) is a split epimorphism.

3. Fibrations and strong h-pullbacks

In this section, we assume that the base category $\mathcal{A}$ is regular.

3.1 Let us recall the terminology for fibrations : consider a functor $F\colon \mathbb{A} \to \mathbb{B}$ between groupoids in $\mathcal{A}$, and the induced factorizations through the pullbacks as in the following diagrams

$$\begin{tikzcd}
A_1 \arrow[rr, "F_1"] \arrow[dd, "d"'] \arrow[dr, "\tau_d"] & & B_1 \arrow[dd, "d"] \\
 & A_0 \times_{F_0,d} B_1 \arrow[ur, "\beta_d"] \arrow[dl, "\alpha_d"] & \\
A_0 \arrow[rr, "F_0"'] & & B_0
\end{tikzcd}
\qquad
\begin{tikzcd}
A_1 \arrow[rr, "F_1"] \arrow[dd, "c"'] \arrow[dr, "\tau_c"] & & B_1 \arrow[dd, "c"] \\
 & A_0 \times_{F_0,c} B_1 \arrow[ur, "\beta_c"] \arrow[dl, "\alpha_c"] & \\
A_0 \arrow[rr, "F_0"'] & & B_0
\end{tikzcd}$$

1. F is a *fibration* when τ_d (equivalently, τ_c) is a regular epimorphism.
2. F is a *split fibration* when τ_d (equivalently, τ_c) is a split epimorphism.

3. F is a *discrete fibration* when τ_d (equivalently, τ_c) is an isomorphism.

In the next characterization of (split) fibrations, we use the functor $N\colon [B_0]_0 \to \mathbb{B}$ with domain the discrete groupoid on B_0. Explicitly,

$$\begin{tikzcd}
B_0 \arrow[r, "N_1=e"] \arrow[d, shift right, "\mathrm{id}"'] \arrow[d, shift left, "\mathrm{id}"] & B_1 \arrow[d, shift right, "d"'] \arrow[d, shift left, "c"] \\
B_0 \arrow[r, "N_0=\mathrm{id}"'] & B_0
\end{tikzcd}$$

Proposition 3.2 *Consider a functor $F\colon \mathbb{A} \to \mathbb{B}$ between groupoids in $\mathcal{A}$, and the comparison functor T from the pullback to the strong h-pullback, as in the following diagram*

$$\begin{tikzcd}
{[B_0]_0 \times_{N,F} \mathbb{A}} \arrow[dr, "T"] \arrow[drr, bend left, "\widehat{N}"] \arrow[ddr, bend right, "\widehat{F}"'] & & \\
& \mathbb{V}(F) \arrow[r, "N'"] \arrow[d, "F'"'] \arrow[dr, phantom, "v(F)\ \Rightarrow"] & \mathbb{A} \arrow[d, "F"] \\
& {[B_0]_0} \arrow[r, "N"'] & \mathbb{B}
\end{tikzcd}$$

1. F is a fibration if and only if T is a weak equivalence.

2. F is a split fibration if and only if T is an equivalence.

In fact, we are going to prove a more precise statement : the arrow attesting that T is essentially surjective is the same arrow τ_c attesting that F is a fibration.

Proof. Since pullbacks in $\mathbf{Grpd}(\mathcal{A})$ are strong (2.8), we can apply point 1 of 2.5 and we know that T is fully faithful. Now we have to compare $T_0' \cdot \underline{c}$ with τ_c

$$\begin{tikzcd}
A_0 \times_{T_0,\underline{d}} \mathbb{V}(F)_1 \arrow[r, "T_0'"] \arrow[d, "\underline{d}'"'] & \mathbb{V}(F)_1 \arrow[r, "\underline{c}"] \arrow[d, "\underline{d}"] & \mathbb{V}(F)_0 \\
A_0 \arrow[r, "T_0"'] & \mathbb{V}(F)_0 &
\end{tikzcd}
\qquad
\begin{tikzcd}
& A_1 \arrow[dl, "c"'] \arrow[d, "\tau_c"] \arrow[dr, "F_1"] & \\
A_0 & A_0 \times_{F_0,c} B_1 \arrow[l, "\alpha_c"] \arrow[r, "\beta_c"'] & B_1
\end{tikzcd}$$

The diagram giving the strong h-pullback $\mathbb{V}(F)$ is

$$\begin{tikzcd}
 & & & \mathbb{V}(F)_1 \arrow[dlll, "d^1"'] \arrow[dr, "v(F)_1"] \arrow[drrr, "V(F)_1"] \arrow[ddd, shift right, "\underline{d}"'] \arrow[ddd, shift left, "\underline{c}"] & & & \\
B_0 \arrow[dr, "e"] \arrow[ddd, shift right, "\mathrm{id}"'] \arrow[ddd, shift left, "\mathrm{id}"] & & & & \vec{B}_1 \arrow[dlll, "m_2\cdot\pi_1"'] \arrow[dr, "m_1\cdot\pi_2"] \arrow[ddd, shift right, "m_1\cdot\pi_1"'] \arrow[ddd, shift left, "m_2\cdot\pi_2"] & & A_1 \arrow[dl, "F_1"'] \arrow[ddd, shift right, "d"'] \arrow[ddd, shift left, "c"] \\
 & B_1 \arrow[ddd, shift right, "d"'] \arrow[ddd, shift left, "c"] & & & & B_1 \arrow[ddd, shift right, "d"'] \arrow[ddd, shift left, "c"] & \\
 & & & A_0 \times_{F_0,c} B_1 \arrow[dlll, "\beta_c\cdot d"] \arrow[dr, "\beta_c"'] \arrow[drrr, "\alpha_c"] & & & \\
B_0 \arrow[dr, "\mathrm{id}"'] & & & & B_1 \arrow[dlll, "d"] \arrow[dr, "c"'] & & A_0 \arrow[dl, "F_0"] \\
 & B_0 & & & & B_0 &
\end{tikzcd}$$

so that T_0 is the factorization through the pullback as in the following diagram

$$\begin{tikzcd}
A_0 \arrow[rr, "F_0"] \arrow[drr, "T_0"] \arrow[dd, "\mathrm{id}"'] & & B_0 \arrow[rr, "e"] & & B_1 \arrow[dd, "c"] \\
 & & A_0 \times_{F_0,c} B_1 \arrow[urr, "\beta_c"'] \arrow[dll, "\alpha_c"] & & \\
A_0 \arrow[rrrr, "F_0"'] & & & & B_0
\end{tikzcd}$$

Now we construct an arrow $\overline{f}\colon A_1 \to \mathbb{V}(F)_1$ in three steps

$$\begin{tikzcd}
A_1 \arrow[d, "d"'] \arrow[dr, "\langle d\cdot F_0\cdot e, F_1\rangle"] \arrow[drr, bend left, "F_1"] & & \\
A_0 \arrow[d, "F_0"'] & B_1 \times_{c,d} B_1 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B_1 \arrow[d, "d"] \\
B_0 \arrow[r, "e"'] & B_1 \arrow[r, "c"'] & B_0
\end{tikzcd}
\qquad
\begin{tikzcd}
A_1 \arrow[dr, "f"] \arrow[drr, bend left, "\langle d\cdot F_0\cdot e, F_1\rangle"] \arrow[ddr, bend right, "\langle d\cdot F_0\cdot e, F_1\rangle"'] & & \\
 & \vec{B}_1 \arrow[r, "m_2"] \arrow[d, "m_1"'] & B_1 \times_{c,d} B_1 \arrow[d, "m"] \\
 & B_1 \times_{c,d} B_1 \arrow[r, "m"'] & B_1
\end{tikzcd}$$

$$\begin{tikzcd}
 & \mathbb{V}(F)_1 \arrow[dl, "d^1"'] \arrow[dr, "v(F)_1"'] \arrow[drrr, "V(F)_1"'] & & A_1 \arrow[ll, "\overline{f}"'] \arrow[dlll, "d\cdot F_0"] \arrow[dl, "f"] \arrow[dr, "\mathrm{id}"] & \\
B_0 \arrow[dr, "e"'] & & \vec{B}_1 \arrow[dl, "m_2\cdot\pi_1"] \arrow[dr, "m_1\cdot\pi_2"'] & & A_1 \arrow[dl, "F_1"] \\
 & B_1 & & B_1 &
\end{tikzcd}$$

Finally, we get the following diagram

$$\begin{tikzcd} A_1 \arrow[r, "\overline{f}"'] \arrow[d, "d"'] \arrow[rr, bend left, "\tau_c"] & \mathbb{V}(F)_1 \arrow[r, "\underline{c}"'] \arrow[d, "\underline{d}"] & A_0 \times_{F_0,c} B_1 \\ A_0 \arrow[r, "T_0"'] & A_0 \times_{F_0,c} B_1 & \end{tikzcd}$$

and we have to check that it is commutative and that the square is a pullback. Once this done, the commutativity of the upper region immediately gives both parts of the statement.

Commutativity of the upper region :

$$\overline{f} \cdot \underline{c} \cdot \alpha_c = \overline{f} \cdot V(F)_1 \cdot c = c = \tau_c \cdot \alpha_c$$

$$\overline{f} \cdot \underline{c} \cdot \beta_c = \overline{f} \cdot v(F)_1 \cdot m_2 \cdot \pi_2 = f \cdot m_2 \cdot \pi_2 = F_1 = \tau_c \cdot \beta_c$$

Commutativity of the square :

$$\overline{f} \cdot \underline{d} \cdot \alpha_c = \overline{f} \cdot V(F)_1 \cdot d = d = d \cdot T_0 \cdot \alpha_c$$

$$\overline{f} \cdot \underline{d} \cdot \beta_c = \overline{f} \cdot v(F)_1 \cdot m_1 \cdot \pi_1 = f \cdot m_1 \cdot \pi_1 = d \cdot F_0 \cdot e = d \cdot T_0 \cdot \beta_c$$

Universality of the square : consider the factorization of the square through the pullback

$$\begin{tikzcd} A_1 \arrow[rr, "\overline{f}"] \arrow[dd, "d"'] \arrow[dr, "\langle d, \overline{f} \rangle"] & & \mathbb{V}(F)_1 \arrow[dd, "\underline{d}"] \\ & P \arrow[ur, "p_2"'] \arrow[dl, "p_1"] & \\ A_0 \arrow[rr, "T_0"'] & & A_0 \times_{F_0,c} B_1 \end{tikzcd}$$

and the arrow

$$P \xrightarrow{p_2} \mathbb{V}(F)_1 \xrightarrow{V(F)_1} A_1$$

Since $\overline{f}$ is a (split) monomorphism, in order to prove that $\langle d, \overline{f} \rangle$ and $p_2 \cdot V(F)_1$ realize an isomorphism, it is enough to check the conditions $p_2 \cdot V(F)_1 \cdot d = p_1$ and $p_2 \cdot V(F)_1 \cdot \overline{f} = p_2$. The first one is easy :

$$p_2 \cdot V(F)_1 \cdot d = p_2 \cdot \underline{d} \cdot \alpha_c = p_1 \cdot T_0 \cdot \alpha_c = p_1$$

For the second one, we compose with the three limit projections :

$$p_2 \cdot V(F) \cdot \overline{f} \cdot V(F)_1 = p_2 \cdot V(F)_1 \cdot \mathrm{id} = p_2 \cdot V(F)_1$$

$$p_2 \cdot d^1 = p_2 \cdot \underline{d} \cdot \beta_c \cdot d = p_1 \cdot T_0 \cdot \beta_c \cdot d = p_1 \cdot F_0 \cdot e \cdot d = p_1 \cdot F_0 = p_1 \cdot T_0 \cdot \alpha_c \cdot F_0 =$$
$$= p_2 \cdot \underline{d} \cdot \alpha_c \cdot F_0 = p_2 \cdot V(F)_1 \cdot d \cdot F_0 = p_2 \cdot V(F)_1 \cdot \overline{f} \cdot d^1$$

and, when composing with $v(F)_1 \colon \mathbb{V}(F)_1 \to \vec{B}_1$, we still have to compose with the four pullback projections out from $\vec{B}_1$:

$$p_2 \cdot V(F)_1 \cdot \overline{f} \cdot v(F)_1 \cdot m_1 \cdot \pi_1 = p_2 \cdot V(F)_1 \cdot f \cdot m_1 \cdot \pi_1 = p_2 \cdot V(F)_1 \cdot d \cdot F_0 \cdot e = p_2 \cdot \underline{d} \cdot \alpha_c \cdot F_0 \cdot e =$$
$$= p_1 \cdot T_0 \cdot \alpha_c \cdot F_0 \cdot e = p_1 \cdot F_0 \cdot e = p_1 \cdot T_0 \cdot \beta_c = p_2 \cdot \underline{d} \cdot \beta_c = p_2 \cdot v(F)_1 \cdot m_1 \cdot \pi_1$$
$$p_2 \cdot V(F)_1 \cdot \overline{f} \cdot v(F)_1 \cdot m_1 \cdot \pi_2 = p_2 \cdot V(F)_1 \cdot f \cdot m_1 \cdot \pi_2 = p_2 \cdot V(F)_1 \cdot F_1 = p_2 \cdot v(F)_1 \cdot m_1 \cdot \pi_2$$
$$p_2 \cdot V(F)_1 \cdot \overline{f} \cdot v(F)_1 \cdot m_2 \cdot \pi_2 = p_2 \cdot V(F)_1 \cdot f \cdot m_2 \cdot \pi_2 = p_2 \cdot V(F)_1 \cdot F_1 =$$
$$= p_2 \cdot \langle \underline{d} \cdot \alpha_c \cdot F_0 \cdot e, V(F)_1 \cdot F_1 \rangle \cdot m =?? = p_2 \cdot \langle v(F)_1 \cdot m_1 \cdot \pi_1, v(F)_1 \cdot m_1 \cdot \pi_2 \rangle \cdot m =$$
$$= p_2 \cdot v(F)_1 \cdot m_1 \cdot m = p_2 \cdot v(F)_1 \cdot m_2 \cdot m = p_2 \cdot \langle v(F)_1 \cdot m_2 \cdot \pi_1, v(F)_1 \cdot m_2 \cdot \pi_2 \rangle \cdot m =$$
$$= p_2 \cdot \langle d^1 \cdot e, v(F)_1 \cdot m_2 \cdot \pi_2 \rangle \cdot m = p_2 \cdot v(F)_1 \cdot m_2 \cdot \pi_2$$

where in the ? ?-step we use $p_2 \cdot \underline{d} \cdot \alpha_c \cdot F_0 \cdot e = p_2 \cdot v(F)_1 \cdot m_1 \cdot \pi_1$ from the first equality. As far as the last equality is concerned, observe that

$$v(F)_1 \cdot m_2 \cdot \pi_2 \cdot d \cdot e = v(F)_1 \cdot m_2 \cdot \pi_1 \cdot c \cdot e = d^1 \cdot e \cdot c \cdot e = d^1 \cdot e = v(F)_1 \cdot m_2 \cdot \pi_1.$$

Therefore, using the third equality, we have

$$p_2 \cdot V(F)_1 \cdot \overline{f} \cdot v(F)_1 \cdot m_2 \cdot \pi_1 = p_2 \cdot V(F)_1 \cdot \overline{f} \cdot v(F)_1 \cdot m_2 \cdot \pi_2 \cdot d \cdot e =$$
$$= p_2 \cdot v(F)_1 \cdot m_2 \cdot \pi_2 \cdot d \cdot e = p_2 \cdot v(F)_1 \cdot m_2 \cdot \pi_1$$

and the proof is complete. ∎

4. $*$-Fibrations and strong h-kernels

In this section, we assume that the base category $\mathcal{A}$ is regular and pointed.

4.1 Since $\mathcal{A}$ is pointed, as a special case of 2.7 we get a description of the strong h-kernel of a functor $F \colon \mathbb{A} \to \mathbb{B}$ between groupoids in $\mathcal{A}$

$$\begin{tikzcd} \mathbb{K}(F) \arrow[r, "K(F)"] \arrow[d, "0"'] \arrow[dr, phantom, "\overset{k(F)}{\Rightarrow}"] & \mathbb{A} \arrow[d, "F"] \\ {[0]_0} \arrow[r, "0"'] & \mathbb{B} \end{tikzcd}$$

Once again, let us make explicit the diagram in $\mathcal{A}$ giving the strong h-kernel $\mathbb{K}(F)$:

$$
\begin{tikzcd}
 & & \mathbb{K}(F)_1 \arrow[dll, "0"'] \arrow[dr, "k(F)_1"] \arrow[drrr, "K(F)_1"] \arrow[ddd, shift right, "\underline{d}"'] \arrow[ddd, shift left, "\underline{c}"] & & & \\
0 \arrow[dr, "0"] \arrow[ddd, shift right, "0"'] \arrow[ddd, shift left, "0"] & & & \vec{B}_1 \arrow[dll, "m_2\cdot\pi_1"'] \arrow[dr, "m_1\cdot\pi_2"] \arrow[ddd, shift right, "m_1\cdot\pi_1"'] \arrow[ddd, shift left, "m_2\cdot\pi_2"] & & A_1 \arrow[dl, "F_1"'] \arrow[ddd, shift right, "d"'] \arrow[ddd, shift left, "c"] \\
 & B_1 \arrow[ddd, shift right, "d"'] \arrow[ddd, shift left, "c"] & & & B_1 \arrow[ddd, shift right, "d"'] \arrow[ddd, shift left, "c"] & \\
 & & \mathbb{K}(F)_0 \arrow[dll, "0"'] \arrow[dr, "k(F)_0"'] \arrow[drrr, "K(F)_0"] & & & \\
0 \arrow[dr, "0"'] & & & B_1 \arrow[dll, "d"] \arrow[dr, "c"'] & & A_0 \arrow[dl, "F_0"] \\
 & B_0 & & & B_0 &
\end{tikzcd}
$$

4.2 Now we introduce fibrations and $*$-fibrations. Consider a functor $F\colon \mathbb{A} \to \mathbb{B}$ between groupoids in $\mathcal{A}$, and the induced factorizations $\widehat{\tau}_d$ and $\widehat{\tau}_c$ through the pullbacks

$$
\begin{tikzcd}
\mathrm{Ker}(F_1\cdot c) \arrow[rr, "F_1^c"] \arrow[d, "k_{F_1\cdot c}"'] \arrow[dr, "\widehat{\tau}_d"] & & \mathrm{Ker}(c) \arrow[d, "k_c"] \\
A_1 \arrow[d, "d"'] & A_0 \times_{F_0, k_c\cdot d} \mathrm{Ker}(c) \arrow[ur, "\widehat{\beta}_d"] \arrow[dl, "\widehat{\alpha}_d"'] & B_1 \arrow[d, "d"] \\
A_0 \arrow[rr, "F_0"'] & & B_0
\end{tikzcd}
\quad
\begin{tikzcd}
\mathrm{Ker}(F_1\cdot d) \arrow[rr, "F_1^d"] \arrow[d, "k_{F_1\cdot d}"'] \arrow[dr, "\widehat{\tau}_c"] & & \mathrm{Ker}(d) \arrow[d, "k_d"] \\
A_1 \arrow[d, "c"'] & A_0 \times_{F_0, k_d\cdot c} \mathrm{Ker}(d) \arrow[ur, "\widehat{\beta}_c"] \arrow[dl, "\widehat{\alpha}_c"'] & B_1 \arrow[d, "c"] \\
A_0 \arrow[rr, "F_0"'] & & B_0
\end{tikzcd}
$$

where F_1^c and F_1^d are determined by the conditions $F_1^c\cdot k_c = k_{F_1\cdot c}\cdot F_1$ and $F_1^d\cdot k_d = k_{F_1\cdot d}\cdot F_1$.

1. F is a *$*$-fibration* when $\widehat{\tau}_d$ (equivalently, $\widehat{\tau}_c$) is a regular epimorphism.
2. F is a *split $*$-fibration* when $\widehat{\tau}_d$ (equivalently, $\widehat{\tau}_c$) is a split epimorphism.

4.3 In order to justify the fact that in 4.2 one can equivalently use $\widehat{\tau}_d$ or $\widehat{\tau}_c$, consider the isomorphisms $K(i)$ and $\overline{K(i)}$ obtained by

$$
\begin{tikzcd}
\mathrm{Ker}(c) \arrow[r, "k_c"] \arrow[d, "K(i)"'] & B_1 \arrow[r, "c"] \arrow[d, "i"'] & B_0 \arrow[d, "\mathrm{id}"] \\
\mathrm{Ker}(d) \arrow[r, "k_d"'] & B_1 \arrow[r, "d"'] & B_0
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Ker}(F_1\cdot c) \arrow[r, "k_{F_1\cdot c}"] \arrow[d, "\overline{K(i)}"'] & A_1 \arrow[r, "F_1"] \arrow[d, "i"'] & B_1 \arrow[r, "c"] \arrow[d, "i"] & B_0 \arrow[d, "\mathrm{id}"] \\
\mathrm{Ker}(F_1\cdot d) \arrow[r, "k_{F_1\cdot d}"'] & A_1 \arrow[r, "F_1"'] & B_1 \arrow[r, "d"'] & B_0
\end{tikzcd}
$$

and then use them to build up the commutative diagram

$$\begin{tikzcd}
\mathrm{Ker}(F_1 \cdot c) \arrow[r, "\widehat{\tau}_d"] \arrow[d, "\overline{K(i)}"'] & A_0 \times_{F_0, k_c \cdot d} \mathrm{Ker}(c) \arrow[d, "\mathrm{id} \times K(i)"] \\
\mathrm{Ker}(F_1 \cdot d) \arrow[r, "\widehat{\tau}_c"'] & A_0 \times_{F_0, k_d \cdot c} \mathrm{Ker}(d)
\end{tikzcd}$$

4.4 Since in the diagram

$$\begin{tikzcd}
\mathrm{Ker}(F_1 \cdot c) \arrow[r, "\widehat{\tau}_d"] \arrow[d, "k_{F_1 \cdot c}"'] \arrow[dr, phantom, "(1)"] & A_0 \times_{F_0, k_c \cdot d} \mathrm{Ker}(c) \arrow[r, "\widehat{\beta}_d"] \arrow[d, "\mathrm{id} \times k_c"'] \arrow[dr, phantom, "(2)"] & \mathrm{Ker}(c) \arrow[r, "0"] \arrow[d, "k_c"'] \arrow[dr, phantom, "(3)"] & 0 \arrow[d, "0"] \\
A_1 \arrow[r, "\tau_d"'] & A_0 \times_{F_0, d} B_1 \arrow[r, "\beta_d"'] & B_1 \arrow[r, "c"'] & B_0
\end{tikzcd}$$

part (2) and part (3) are pullbacks and the whole is a pullback (because $\tau_d \cdot \beta_d = F_1$), it follows that part (1) also is a pullback. This proves that any (split) fibration is a (split) $*$-fibration.

Proposition 4.5 *Consider a functor $F \colon \mathbb{A} \to \mathbb{B}$ between groupoids in $\mathcal{A}$, and the comparison J from its kernel to its strong h-kernel as in the following diagram*

$$\begin{tikzcd}
\mathbb{K}\mathrm{er}(F) \arrow[dr, "J"] \arrow[drr, bend left, "K_F"] \arrow[ddr, bend right, "0"'] & & \\
& \mathbb{K}(F) \arrow[r, "K(F)"] \arrow[d, "0"'] \arrow[dr, phantom, "k(F)\ \Rightarrow"] & \mathbb{A} \arrow[d, "F"] \\
& {[0]_0} \arrow[r, "0"'] & \mathbb{B}
\end{tikzcd}$$

1. *F is a $*$-fibration if and only if J is a weak equivalence.*
2. *F is a split $*$-fibration if and only if J is an equivalence.*

Similarly to what we did in Proposition 3.2, we are going to prove a more precise statement : the arrow attesting that J is essentially surjective is the same arrow $\widehat{\tau}_c$ attesting that F is a $*$-fibration.

Proof. Since pullbacks in $\mathbf{Grpd}(\mathcal{A})$ are strong (2.8), we can apply point 1 of 2.5 and we know that J is fully faithful. Now we have to compare $J_0' \cdot \underline{c}$ with $\widehat{\tau}_c$

$$\begin{tikzcd}
\mathrm{Ker}(F_0) \times_{J_0, \underline{d}} \mathbb{K}(F)_1 \arrow[r, "J_0'"] \arrow[d, "\underline{d}'"'] & \mathbb{K}(F)_1 \arrow[r, "\underline{c}"] \arrow[d, "\underline{d}"] & \mathbb{K}(F)_0 \\
\mathrm{Ker}(F_0) \arrow[r, "J_0"'] & \mathbb{K}(F)_0 &
\end{tikzcd}
\qquad
\mathrm{Ker}(F_1 \cdot d) \xrightarrow{\widehat{\tau}_c} A_0 \times_{F_0, k_d \cdot c} \mathrm{Ker}(d)$$

From the explicit description of the strong h-kernel $\mathbb{K}(F)$, we see that

$$\mathbb{K}(F)_0 = A_0 \times_{F_0, k_d \cdot c} \mathrm{Ker}(d)\ ,\quad K(F)_0 = \widehat{\alpha}_c\ ,\quad k(F)_0 = \widehat{\beta}_c \cdot k_d$$

Moreover, the universal property of $\mathrm{Ker}(F_0)$ gives the following factorization $\overline{d}$:

$$\begin{tikzcd}
\mathrm{Ker}(F_1 \cdot d) \arrow[r, "k_{F_1 \cdot d}"] \arrow[d, "\overline{d}"'] & A_1 \arrow[r, "F_1"] \arrow[d, "d"] & B_1 \arrow[d, "d"] \\
\mathrm{Ker}(F_0) \arrow[r, "k_{F_0}"'] & A_0 \arrow[r, "F_0"'] & B_0
\end{tikzcd}$$

Now we construct an arrow $\overline{f} \colon \mathrm{Ker}(F_1 \cdot d) \to \mathbb{K}(F)_1$ in three steps

$$\begin{tikzcd}
\mathrm{Ker}(F_1 \cdot d) \arrow[r, "k_{F_1 \cdot d}"] \arrow[dr, "\langle 0, k_{F_1 \cdot d} \cdot F_1 \rangle"'] \arrow[ddr, bend right, "0"'] & A_1 \arrow[dr, bend left, "F_1"] & \\
& B_1 \times_{c,d} B_1 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B_1 \arrow[d, "d"] \\
& B_1 \arrow[r, "c"'] & B_0
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Ker}(F_1 \cdot d) \arrow[dr, "f"] \arrow[drr, bend left, "\langle 0, k_{F_1 \cdot d} \cdot F_1 \rangle"] \arrow[ddr, bend right, "\langle 0, k_{F_1 \cdot d} \cdot F_1 \rangle"'] & & \\
& \vec{B}_1 \arrow[r, "m_2"] \arrow[d, "m_1"'] & B_1 \times_{c,d} B_1 \arrow[d, "m"] \\
& B_1 \times_{c,d} B_1 \arrow[r, "m"'] & B_1
\end{tikzcd}$$

$$\begin{tikzcd}
& \mathbb{K}(F)_1 \arrow[dl, "0"'] \arrow[dr, "k(F)_1"] \arrow[drrr, "K(F)_1"', near end] & & \mathrm{Ker}(F_1 \cdot d) \arrow[ll, "\overline{f}"'] \arrow[dlll, "0", near end] \arrow[dl, "f"] \arrow[dr, "k_{F_1 \cdot d}"] & \\
0 \arrow[dr, "0"'] & & \vec{B}_1 \arrow[dl, "m_2 \cdot \pi_1"] \arrow[dr, "m_1 \cdot \pi_2"'] & & A_1 \arrow[dl, "F_1"] \\
& B_1 & & B_1 &
\end{tikzcd}$$

Finally, we get the following diagram

$$\begin{tikzcd}
\mathrm{Ker}(F_1 \cdot d) \arrow[r, "\overline{f}"'] \arrow[d, "\overline{d}"'] \arrow[rr, bend left, "\widehat{\tau}_c"] & \mathbb{K}(F)_1 \arrow[r, "\underline{c}"'] \arrow[d, "\underline{d}"] & A_0 \times_{F_0, k_d \cdot c} \mathrm{Ker}(d) \\
\mathrm{Ker}(F_0) \arrow[r, "J_0"'] & A_0 \times_{F_0, k_d \cdot c} \mathrm{Ker}(d) &
\end{tikzcd}$$

and we have to check that it is commutative and that the square is a pullback. Once this done, the commutativity of the upper region immediately gives both parts of the statement.
Commutativity of the upper region :

$$\overline{f} \cdot \underline{c} \cdot K(F)_0 = \overline{f} \cdot K(F)_1 \cdot c = k_{F_1 \cdot d} \cdot c = \widehat{\tau}_c \cdot \widehat{\alpha}_c = \widehat{\tau}_c \cdot K(F)_0$$

$\overline{f}\cdot \underline{c}\cdot k(F)_0 = \overline{f}\cdot k(F)_1\cdot m_2\cdot\pi_2 = f\cdot m_2\cdot \pi_2 = k_{F_1\cdot d}\cdot F_1 = F_1^d\cdot k_d = \widehat{\tau}_c\cdot\widehat{\beta}_c\cdot k_d = \widehat{\tau}_c\cdot k(F)_0$

Commutativity of the square :

$$\overline{f}\cdot \underline{d}\cdot K(F)_0 = \overline{f}\cdot K(F)_1\cdot d = k_{F_1\cdot d}\cdot d = \overline{d}\cdot k_{F_0} = \overline{d}\cdot J_0\cdot K(F)_0$$

$$\overline{f}\cdot \underline{d}\cdot k(F)_0 = \overline{f}\cdot k(F)_1\cdot m_1\cdot\pi_1 = f\cdot m_1\cdot\pi_1 = 0 = \overline{d}\cdot 0 = \overline{d}\cdot J_0\cdot k(F)_0$$

Universality of the square : consider the factorization of the square through the pullback

$$\begin{tikzcd}
\mathrm{Ker}(F_1\cdot d) \arrow[rr, "\overline{f}"] \arrow[dd, "\overline{d}"'] \arrow[dr, "\langle\overline{d},\overline{f}\rangle"] & & \mathbb{K}(F)_1 \arrow[dd, "\underline{d}"] \\
& P \arrow[ur, "p_2"'] \arrow[dl, "p_1"] & \\
\mathrm{Ker}(F_0) \arrow[rr, "J_0"'] & & \mathbb{K}(F)_0
\end{tikzcd}$$

Moreover, since

$p_2\cdot K(F)_1\cdot F_1\cdot d = p_2\cdot K(F)_1\cdot d\cdot F_0 = p_2\cdot \underline{d}\cdot K(F)_0\cdot F_0 = p_1\cdot J_0\cdot K(F)_0\cdot F_0 = p_1\cdot k_{F_0}\cdot F_0 = 0$

the universal property of $\mathrm{Ker}(F_1\cdot d)$ gives the following factorization ρ:

$$\begin{tikzcd}
P \arrow[r, "p_2"] \arrow[drr, "\rho"'] & \mathbb{K}(F)_1 \arrow[r, "K(F)_1"] & A_1 \arrow[r, "F_1"] & B_1 \arrow[r, "d"] & B_0 \\
& & \mathrm{Ker}(F_1\cdot d) \arrow[u, "k_{F_1\cdot d}"'] & &
\end{tikzcd}$$

Since $\overline{f}$ is a monomorphism (because $\overline{f}\cdot K(F)_1 = k_{F_1\cdot d}$), in order to prove that $\langle\overline{d},\overline{f}\rangle$ and ρ realize an isomorphism, it is enough to check the conditions $\rho\cdot\overline{d} = p_1$ and $\rho\cdot\overline{f} = p_2$. The first one is easy, just compose with the monomorphism k_{F_0} :

$$\rho\cdot\overline{d}\cdot k_{F_0} = \rho\cdot k_{F_1\cdot d}\cdot d = p_2\cdot K(F)_1\cdot d = p_2\cdot\underline{d}\cdot K(F)_0 = p_1\cdot J_0\cdot K(F)_0 = p_1\cdot k_{F_0}$$

For the second one, we compose with the limit projections $K(F)_1$ and $k(F)_1$ and, when composing with $k(F)_1\colon \mathbb{K}(F)_1 \to \vec{B}_1$, we still have to compose with the four pullback projections out from $\vec{B}_1$:

$$\rho\cdot\overline{f}\cdot K(F)_1 = \rho\cdot k_{F_1\cdot d} = p_2\cdot K(F)_1$$

$$\rho\cdot\overline{f}\cdot k(F)_1\cdot m_1\cdot\pi_1 = \rho\cdot f\cdot m_1\cdot\pi_1 = \rho\cdot 0 = p_1\cdot 0 = p_1\cdot J_0\cdot k(F)_0 = p_2\cdot\underline{d}\cdot k(F)_0 = p_2\cdot k(F)_1\cdot m_1\cdot\pi_1$$

$$\rho\cdot\overline{f}\cdot k(F)_1\cdot m_1\cdot\pi_2 = \rho\cdot f\cdot m_1\cdot\pi_2 = \rho\cdot k_{F_1\cdot d}\cdot F_1 = p_2\cdot K(F)_1\cdot F_1 = p_2\cdot k(F)_1\cdot m_1\cdot\pi_2$$

$$\rho\cdot\overline{f}\cdot k(F)_1\cdot m_2\cdot\pi_1 = \rho\cdot\overline{f}\cdot 0 = p_2\cdot 0 = p_2\cdot k(F)_1\cdot m_2\cdot\pi_1$$

$$\rho\cdot\overline{f}\cdot k(F)_1\cdot m_2\cdot\pi_2 = \rho\cdot f\cdot m_2\cdot\pi_2 = \rho\cdot k_{F_1\cdot d}\cdot F_1 = p_2\cdot K(F)_1\cdot F_1 =$$

$$= \langle 0, p_2\cdot K(F)_1\cdot F_1\rangle\cdot m = p_2\cdot\langle\underline{d}\cdot k(F)_0, K(F)_1\cdot F_1\rangle\cdot m = p_2\cdot\langle k(F)_1\cdot m_1\cdot\pi_1, k(F)_1\cdot m_1\cdot\pi_2\rangle\cdot m =$$

$$= p_2\cdot k(F)_1\cdot m_1\cdot m = p_2\cdot k(F)_1\cdot m_2\cdot m = p_2\cdot\langle k(F)_1\cdot m_2\cdot\pi_1, k(F)_1\cdot m_2\cdot\pi_2\rangle\cdot m =$$

$$= p_2\cdot\langle 0, k(F)_1\cdot m_2\cdot\pi_2\rangle\cdot m = p_2\cdot k(F)_1\cdot m_2\cdot\pi_2$$

■

From 4.4 and Proposition 4.5, we get the following result, announced in Proposition 4.4 of [10] :

Corollary 4.6 *Let* $F\colon \mathbb{A} \to \mathbb{B}$ *be a fibration between internal groupoids. The canonical comparison* $J\colon \mathbb{K}\mathrm{er}(F) \to \mathbb{K}(F)$ *from the kernel to the strong h-kernel is a weak equivalence. If* F *is a split fibration, then* J *is an equivalence.*

4.7 Thanks to Proposition 4.5, we can slightly improve Proposition 5.6 in [10] : assume that $\mathcal{A}$ is a pointed regular category with reflexive coequalizers and consider a $*$-fibration $F\colon \mathbb{A} \to \mathbb{B}$ in $\mathbf{Grpd}(\mathcal{A})$, with $\mathbb{A}, \mathbb{B}$ and $\mathbb{K}(F)$ proper. There exists an exact sequence

$$\pi_1(\mathbb{K}\mathrm{er}(F)) \to \pi_1(\mathbb{A}) \to \pi_1(\mathbb{B}) \to \pi_0(\mathbb{K}\mathrm{er}(F)) \to \pi_0(\mathbb{A}) \to \pi_0(\mathbb{B})$$

where $\pi_1(\mathbb{A})$ is the internal group of automorphisms on the base point of $\mathbb{A}$ and $\pi_0(\mathbb{A})$ is the object of connected components of $\mathbb{A}$. (Here, the exactness at B of

$$A \xrightarrow{f} B \xrightarrow{g} C$$

means that f factors as a regular epimorphism followed by the kernel of g). Indeed, since $J\colon \mathbb{K}\mathrm{er}(F) \to \mathbb{K}(F)$ is a weak equivalence, the arrows $\pi_0(F)\colon \pi_0(\mathbb{K}\mathrm{er}(F)) \to \pi_0(\mathbb{K}(F))$ and $\pi_1(F)\colon \pi_1(\mathbb{K}\mathrm{er}(F)) \to \pi_1(\mathbb{K}(F))$ are isomorphisms (Lemma 5.5 in [10]). Therefore, the above exact sequence immediately follows from the exact sequence

$$\pi_1(\mathbb{K}(F)) \to \pi_1(\mathbb{A}) \to \pi_1(\mathbb{B}) \to \pi_0(\mathbb{K}(F)) \to \pi_0(\mathbb{A}) \to \pi_0(\mathbb{B})$$

established in Section 4 of [10].

4.8 Corollary 4.6 can be obtained also from Proposition 3.2 without using the notion of $*$-fibration. Indeed, consider the comparison functors L and I as in the following pullback and strong h-pullback diagrams

$$\begin{tikzcd}
\mathbb{K}\mathrm{er}(F) \arrow[dr, "L"] \arrow[drr, bend left, "K_F"] \arrow[ddr, bend right, "0"'] & & \\
& {[B_0]_0 \times_{N,F} \mathbb{A}} \arrow[r, "\widehat{N}"] \arrow[d, "\widehat{F}"'] & \mathbb{A} \arrow[d, "F"] \\
& {[B_0]_0} \arrow[r, "N"'] & \mathbb{B}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{K}(F) \arrow[dr, "I"] \arrow[drr, bend left, "K(F)"] \arrow[ddr, bend right, "0"'] & & \\
& \mathbb{V}(F) \arrow[r, "N'"] \arrow[d, "F'"'] \arrow[dr, phantom, "v(F)\ \Rightarrow"] & \mathbb{A} \arrow[d, "F"] \\
& {[B_0]_0} \arrow[r, "N"'] & \mathbb{B}
\end{tikzcd}$$

We get a commutative diagram

$$\begin{tikzcd}
\mathbb{K}\mathrm{er}(F) \arrow[r, "J"] \arrow[d, "L"'] & \mathbb{K}(F) \arrow[d, "I"] \\
{[B_0]_0 \times_{N,F} \mathbb{A}} \arrow[r, "T"'] & \mathbb{V}(F)
\end{tikzcd}$$

which in fact is a pullback and where $I\colon \mathbb{K}(F) \to \mathbb{V}(F)$ is a discrete fibration. Now, the fact that J is a weak equivalence if F is a fibration (or an equivalence if F is a split fibration) follows from Proposition 3.2 and the following general lemma on pullbacks in $\mathbf{Grpd}(\mathcal{A})$.

Lemma 4.9 *Consider a pullback in* $\mathbf{Grpd}(\mathcal{A})$

$$\begin{tikzcd}
\mathbb{X} \arrow[r, "\widehat{F}"] \arrow[d, "\widehat{G}"'] & \mathbb{C} \arrow[d, "G"] \\
\mathbb{A} \arrow[r, "F"'] & \mathbb{B}
\end{tikzcd}$$

and assume that G is a discrete fibration.

1. *If F is a weak equivalence, then $\widehat{F}$ is a weak equivalence.*
2. *If F is an equivalence, then $\widehat{F}$ is an equivalence.*

Proof. Since in $\mathbf{Grpd}(\mathcal{A})$ pullbacks are strong (2.8), we know from point 2 of 2.5 that, if F is fully faithful, so is $\widehat{F}$.
Now, the universal property of the pullback $A_0 \times_{F_0,d} B_1$ gives a unique arrow λ making the following diagram commutative

$$\begin{tikzcd}
X_0 \times_{\widehat{F}_0,d} C_1 \arrow[r, "\gamma_d"] \arrow[d, "\pi_d"'] \arrow[ddr, "\lambda"] & C_1 \arrow[d, "d"] \arrow[ddr, "G_1"] & \\
X_0 \arrow[r, "\widehat{F}_0"] \arrow[ddr, "\widehat{G}_0"'] & C_0 \arrow[ddr, "G_0"] & \\
& A_0 \times_{F_0,d} B_1 \arrow[r, "\beta_d"] \arrow[d, "\alpha_d"] & B_1 \arrow[d, "d"] \\
& A_0 \arrow[r, "F_0"'] & B_0
\end{tikzcd}$$

Consider the following commutative diagrams

$$\begin{tikzcd}
X_0 \times_{\widehat{F}_0,d} C_1 \arrow[r, "\gamma_d"] \arrow[d, "\pi_d"'] \arrow[dr, phantom, "(2)"] & C_1 \arrow[d, "d"] \\
X_0 \arrow[r, "\widehat{F}_0"] \arrow[d, "\widehat{G}_0"'] \arrow[dr, phantom, "(1)"] & C_0 \arrow[d, "G_0"] \\
A_0 \arrow[r, "F_0"'] & B_0
\end{tikzcd}
\qquad
\begin{tikzcd}
X_0 \times_{\widehat{F}_0,d} C_1 \arrow[r, "\gamma_d"] \arrow[d, "\lambda"'] \arrow[dr, phantom, "(4)"] & C_1 \arrow[d, "G_1"] \\
A_0 \times_{F_0,d} B_1 \arrow[r, "\beta_d"] \arrow[d, "\alpha_d"'] \arrow[dr, phantom, "(3)"] & B_1 \arrow[d, "d"] \\
A_0 \arrow[r, "F_0"'] & B_0
\end{tikzcd}
\qquad
\begin{tikzcd}
X_0 \times_{\widehat{F}_0,d} C_1 \arrow[r, "\lambda"] \arrow[d, "\gamma_d"'] \arrow[dr, phantom, "(4)"] & A_0 \times_{F_0,d} B_1 \arrow[d, "\beta_d"] \\
C_1 \arrow[r, "G_1"] \arrow[d, "c"'] \arrow[dr, phantom, "(5)"] & B_1 \arrow[d, "c"] \\
C_0 \arrow[r, "G_0"'] & B_0
\end{tikzcd}$$

Since (1) and (2) are pullbacks, then (1)+(2) also is a pullback. But (1)+(2) = (3)+(4) and (3) is a pullback, so (4) also is a pullback. Since (5) is a pullback (because G is a

discrete fibration), we conclude that (4)+(5) is a pullback. The proof is now obviuous :
- If $\beta_d \cdot c$ is a regular epimorphism, so is $\gamma_d \cdot c$, and this proves part 1 of the statement.
- If $\beta_d \cdot c$ is a split epimorphism, so is $\gamma_d \cdot c$, and this proves part 2 of the statement. ∎

5. Normalized fibrations and normalized $*$-fibrations

5.1 From [8], recall that a *category with null-homotopies* $\underline{\mathcal{B}}$ is given by
– a category $\mathcal{B}$,
– for each morphism $f\colon A \to B$ in $\mathcal{B}$, a set $\mathcal{H}(f)$ (the set of null-homotopies on f),
– for each triple of composable morphisms $f\colon A \to B$, $g\colon B \to C$, $h\colon C \to D$, a map

$$f \circ - \circ h : \mathcal{H}(g) \to \mathcal{H}(f \cdot g \cdot h), \; \mu \mapsto f \circ \mu \circ h.$$

(If $f = \mathrm{id}_B$ or $h = \mathrm{id}_C$, we write $\mu \circ h$ or $f \circ \mu$ instead of $f \circ \mu \circ h$.)
These data have to satisfy

1. the identity condition : given a morphism $f\colon A \to B$, one has $\mathrm{id}_A \circ \mu \circ \mathrm{id}_B$ for all $\mu \in \mathcal{H}(f)$,
2. the associativity condition : given morphisms

$$A' \xrightarrow{f'} A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \xrightarrow{h'} D'$$

one has $(f' \cdot f) \circ \mu \circ (h \cdot h') = f' \circ (f \circ \mu \circ h) \circ h'$ for any $\mu \in \mathcal{H}(g)$.

5.2 The structure of category with null-homotopies is not rich enough to express the notion of strong h-pullback, but still, following [8, 13], we can express the notion of strong h-kernel : let $\underline{\mathcal{B}}$ be a category with null-homotopies and let $f\colon A \to B$ be a morphism in $\mathcal{B}$. A triple

$$\mathrm{Ker}(f),\; \mathrm{K}(f)\colon \mathrm{Ker}(f) \to A,\; k(f) \in \mathcal{H}(\mathrm{K}(f) \cdot f)$$

1. is a *h-kernel* of f if for any triple D, $g\colon D \to A$, $\mu \in \mathcal{H}(g \cdot f)$, there exists a unique morphism $g'\colon D \to \mathrm{Ker}(f)$ such that $g' \cdot \mathrm{K}(f) = g$ and $g' \circ k(f) = \mu$,
2. is a *strong h-kernel* of f if it is a homotopy kernel of f and, moreover, for any triple D, $h\colon D \to \mathrm{Ker}(f)$, $\mu \in \mathcal{H}(h \cdot \mathrm{K}(f))$ such that $\mu \circ f = h \circ k(f)$, there exists a unique $\lambda \in \mathcal{H}(h)$ such that $\lambda \circ \mathrm{K}(f) = \mu$.

Note that in [8], the identity condition have been omitted. We think it should not, in order (for instance) to be able to prove that h-kernels and strong h-kernels are determined up to isomorphism by their universal properties.

5.3 For a category $\mathcal{A}$, we consider the arrow category $\mathbf{Arr}(\mathcal{A})$: the objects are the arrows $a\colon A \to A_0$ in $\mathcal{A}$ and the arrows $(f, f_0)\colon a \to b$ are commutative squares of the form

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \\ A_0 \arrow[r, "f_0"'] & B_0 \end{tikzcd}$$

From [13], recall that $\mathbf{Arr}(\mathcal{A})$ is a category with null-homotopies : a null-homotopy for an arrow (f, f_0) is a diagonal, that is, an arrow $d \colon A_0 \to B$ such that $a \cdot d = f$ and $d \cdot b = f_0$. If $\mathcal{A}$ has finite limits and a zero object, then $\mathbf{Arr}(\mathcal{A})$ has kernels and strong h-kernels. The kernel of (f, f_0) is just the level-wise kernel

$$\begin{tikzcd}
\mathrm{Ker}(f) \arrow[r, "k_f"] \arrow[d, "\mathrm{K}(a)"'] & A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \\
\mathrm{Ker}(f_0) \arrow[r, "k_{f_0}"'] & A_0 \arrow[r, "f_0"'] & B_0
\end{tikzcd}$$

To construct the strong h-kernel of (f, f_0), consider the factorization through the pullback

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "\partial(f,f_0)_0"] \arrow[dd, "a"'] & & B \arrow[dd, "b"] \\
& A_0 \times_{f_0,b} B \arrow[ur, "f_0'"'] \arrow[dl, "b'"] & \\
A_0 \arrow[rr, "f_0"'] & & B_0
\end{tikzcd}$$

The strong h-kernel is then

$$\begin{tikzcd}
A \arrow[r, "\mathrm{id}"] \arrow[d, "\partial(f,f_0)_0"'] & A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[d, "b"] \\
A_0 \times_{f_0,b} B \arrow[r, "b'"'] \arrow[urr, "f_0'"] & A_0 \arrow[r, "f_0"'] & B_0
\end{tikzcd}$$

Moreover, we can consider the normalization functor

$$\mathcal{N} \colon \mathbf{Grpd}(\mathcal{A}) \to \mathbf{Arr}(\mathcal{A})$$

which sends a functor $F \colon \mathbb{A} \to \mathbb{B}$ in the commutative diagram

$$\begin{tikzcd}
\mathrm{Ker}(d) \arrow[r, "\mathrm{K}_d(F)"] \arrow[d, "k_d"'] & \mathrm{Ker}(d) \arrow[d, "k_d"] \\
A_1 \arrow[d, "c"'] & B_1 \arrow[d, "c"] \\
A_0 \arrow[r, "F_0"'] & B_0
\end{tikzcd}$$

5.4 In order to define fully faithful arrows in $\mathbf{Arr}(\mathcal{A})$, let us look more carefully to the situation in $\mathbf{Grpd}(\mathcal{A})$. For a functor $F\colon \mathbb{A} \to \mathbb{B}$ in $\mathbf{Grpd}(\mathcal{A})$, consider the following strong h-pullbacks

$$\begin{tikzcd} \vec{\mathbb{A}} \arrow[r, "\gamma"] \arrow[d, "\delta"'] \arrow[dr, phantom, "\overset{\alpha}{\Rightarrow}"] & \mathbb{A} \arrow[d, "\mathrm{Id}"] \\ \mathbb{A} \arrow[r, "\mathrm{Id}"'] & \mathbb{A} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{R}(F) \arrow[r, "\gamma(F)"] \arrow[d, "\delta(F)"'] \arrow[dr, phantom, "\overset{\alpha(F)}{\Rightarrow}"] & \mathbb{A} \arrow[d, "F"] \\ \mathbb{A} \arrow[r, "F"'] & \mathbb{B} \end{tikzcd}$$

and the unique functor $\partial(F)\colon \vec{\mathbb{A}} \to \mathbb{R}(F)$ such that $\partial(F) \cdot \delta(F) = \delta$, $\partial(F) \cdot \gamma(F) = \gamma$ and $\partial(F) \cdot \alpha(F) = \alpha \cdot F$. The 0-level of the functor $\partial(F)$ is precisely the unique arrow making commutative the following diagram

$$\begin{tikzcd} & A_1 \arrow[rr, "\partial(F)_0"] \arrow[dl, "d"'] \arrow[dr, "F_1"] \arrow[drrrr, "c"] & & A_0 \times_{F_0,d} B_1 \times_{c,F_0} A_0 \arrow[dlll, "\delta(F)_0"'] \arrow[dl, "\alpha(F)_0"'] \arrow[drr, "\gamma(F)_0"] & & \\ A_0 \arrow[dr, "F_0"'] & & B_1 \arrow[dl, "d"] \arrow[drr, "c"] & & & A_0 \arrow[dl, "F_0"] \\ & B_0 & & & B_0 & \end{tikzcd}$$

Therefore, we can say that the functor $F\colon \mathbb{A} \to \mathbb{B}$ is :
1. faithful if $\partial(F)_0$ is a monomorphism ;
2. full if $\partial(F)_0$ is a regular epimorphism ;
3. fully faithful if $\partial(F)_0$ is an isomorphism (accordingly to 2.9).

5.5 Now we imitate the previous argument using strong h-kernels in $\mathbf{Arr}(\mathcal{A})$. For an arrow $(f, f_0)\colon a \to b$ in $\mathbf{Arr}(\mathcal{A})$, consider the following strong h-kernels, together with the induced comparison arrow $\partial(f, f_0)$:

$$\begin{tikzcd} \mathbb{K}(\mathrm{Id}) \arrow[r] \arrow[d, "\partial(f,f_0)"'] & a \arrow[r, "\mathrm{Id}"] \arrow[d, "\mathrm{Id}"'] & a \arrow[d, "(f,f_0)"] \\ \mathbb{K}(f, f_0) \arrow[r] & a \arrow[r, "(f,f_0)"'] & b \end{tikzcd}$$

The 0-level of $\partial(f, f_0)$ is precisely the factorization $\partial(f, f_0)_0\colon A \to A_0 \times_{f_0,b} B$ through the pullback, as in 5.3. This suggests part of the following terminology.

5.6 Consider an arrow $(f, f_0)\colon a \to b$ in $\mathbf{Arr}(\mathcal{A})$ together with the induced factorization $\partial(f, f_0)_0\colon A \to A_0 \times_{f_0,b} B$ through the pullback, as in the description of the strong h-kernel in 5.3. The arrow (f, f_0) is :

1. *faithful* if $\partial(f, f_0)_0$ is a monomorphism ;

2. *full* if $\partial(f, f_0)_0$ is a regular epimorphism;
3. *essentially surjective* if f_0 and b are jointly strongly epimorphic;
4. a *weak equivalence* if it is full and faithful and essentially surjective;
5. a *fibration* if f is a regular epimorphism.

In order to compare the above terminology with the terminology for internal functors (2.9, 3.1 and 5.4), we need some intermediate steps. The first one is the version for strong h-pullbacks of the elementary fact that two opposite arrows in a pullback diagram have isomorphic kernels.

Lemma 5.7 *Consider the following diagram in* $\mathbf{Grpd}(\mathcal{A})$, *with the bottom square being a strong h-pullback, the right region being a strong h-kernel and the top square commutative.*

$$\begin{tikzcd}
\mathbb{K}(F) \arrow[r, "\mathrm{Id}"] \arrow[d, "\langle 0, K(F)\rangle"'] & \mathbb{K}(F) \arrow[d, "K(F)"'] \arrow[dd, bend left=60, "0"] \arrow[d, phantom, shift left=5ex, "\overset{k(F)}{\Leftarrow}"] \\
\mathbb{P} \arrow[r, "G'"] \arrow[d, "F'"'] \arrow[dr, phantom, "\overset{\varphi}{\Rightarrow}"] & \mathbb{A} \arrow[d, "F"'] \\
\mathbb{C} \arrow[r, "G"'] & \mathbb{B}
\end{tikzcd}$$

Then the left column is a kernel.

Proof. Using the description of the strong h-pullback (see 2.7) and of the strong h-kernel (see 4.1), we get arrows

$$\langle 0, K(F)\rangle_0 = \langle 0, k(F)_0, K(F)_0\rangle \colon \mathbb{K}(F)_0 \to P_0$$

$$\langle 0, K(F)\rangle_1 = \langle 0, k(F)_1, K(F)_1\rangle \colon \mathbb{K}(F)_1 \to P_1$$

satisfying the conditions

$$\langle 0, k(F)_0, K(F)_0\rangle \cdot F'_0 = 0\,,\ \ \langle 0, k(F)_0, K(F)_0\rangle \cdot \varphi_0 = k(F)_0\,,\ \ \langle 0, k(F)_0, K(F)_0\rangle \cdot G'_0 = K(F)_0$$

$$\langle 0, k(F)_1, K(F)_1\rangle \cdot F'_1 = 0\,,\ \ \langle 0, k(F)_1, K(F)_1\rangle \cdot \varphi_1 = k(F)_1\,,\ \ \langle 0, k(F)_1, K(F)_1\rangle \cdot G'_1 = K(F)_1$$

It is now a straightforward diagram chasing to check that in the following diagram both rows are kernels

$$\begin{tikzcd}
\mathbb{K}(F)_1 \arrow[r, "\langle 0, k(F)_1, K(F)_1\rangle"] \arrow[d, shift right, "\underline{d}"'] \arrow[d, shift left, "\underline{c}"] & P_1 \arrow[r, "F'_1"] \arrow[d, shift right, "\underline{d}"'] \arrow[d, shift left, "\underline{c}"] & C_1 \arrow[d, shift right, "d"'] \arrow[d, shift left, "d"] \\
\mathbb{K}(F)_0 \arrow[r, "\langle 0, k(F)_0, K(F)_0\rangle"'] & P_0 \arrow[r, "F'_0"'] & C_0
\end{tikzcd}$$

■

Lemma 5.8 *The normalization functor* $\mathcal{N}\colon \mathbf{Grpd}(\mathcal{A}) \to \mathbf{Arr}(\mathcal{A})$ *preserves kernels and strong h-kernels.*

Proof. Preservation of kernels is an obvious argument of exchange of limits. Consider now a strong h-kernel in $\mathbf{Grpd}(\mathcal{A})$ and the canonical comparison $T = (t, t_0)$ with the strong h-kernel in $\mathbf{Arr}(\mathcal{A})$

$$\begin{tikzcd}
\mathcal{N}(\mathbb{K}(F)) \arrow[r, "\mathcal{N}(K(F))"] \arrow[dr, "T"'] & \mathcal{N}(\mathbb{A}) \arrow[r, "\mathcal{N}(F)"] & \mathcal{N}(\mathbb{B}) \\
& \mathbb{K}(\mathcal{N}(F)) \arrow[u] &
\end{tikzcd}$$

More explicitly, we get the following diagram in $\mathcal{A}$

$$\begin{tikzcd}
\mathrm{Ker}(\underline{d}) \arrow[rr, "t"] \arrow[dr, "t"] \arrow[d, "k_{\underline{d}}"'] & & \mathrm{Ker}(d) \arrow[r, "\mathrm{K}_d(F)"] \arrow[d, "k_d"'] & \mathrm{Ker}(d) \arrow[d, "k_d"] \\
\mathbb{K}(F)_1 \arrow[d, "\underline{c}"'] & \mathrm{Ker}(d) \arrow[ur, "\mathrm{id}"'] \arrow[dd, "\partial(\mathcal{N}(F))_0"] & A_1 \arrow[d, "c"'] & B_1 \arrow[d, "c"] \\
\mathbb{K}(F)_0 \arrow[rr, "K(F)_0"] \arrow[dr, "t_0"'] & & A_0 \arrow[r, "F_0"'] & B_0 \\
& A_0 \times_{F_0, k_d \cdot c} \mathrm{Ker}(d) \arrow[ur, "\beta'"'] & &
\end{tikzcd}$$

where

$$\begin{tikzcd}
A_0 \times_{F_0, k_d \cdot c} \mathrm{Ker}(d) \arrow[r, "f_0'"] \arrow[d, "\beta'"'] & \mathrm{Ker}(d) \arrow[d, "k_d \cdot c"] \\
A_0 \arrow[r, "F_0"'] & B_0
\end{tikzcd}$$

is a pullback, and $\partial(\mathcal{N}(F))_0, t_0$ and t are determined by the conditions

$$\partial(\mathcal{N}(F))_0 \cdot \beta' = k_d \cdot c \ , \ \ \partial(\mathcal{N}(F))_0 \cdot f_0' = \mathrm{K}_d(F)$$

$$t_0 \cdot \beta' = K(F)_0 \ , \ \ t_0 \cdot f_0' \cdot k_d = k(F)_0 \ , \ \ t \cdot k_d = k_{\underline{d}} \cdot K(F)_1$$

Clearly, t_0 is an isomorphism (just look at the construction of $\mathbb{K}(F)_0$ in 4.1). It remains to show that t also is an isomorphism. Since $k_d \cdot F_1 \cdot d = k_d \cdot d \cdot F_0 = 0 \cdot F_0 = 0 = 0 \cdot c$, there exists a unique arrow $\langle 0, k_d \cdot F_1 \rangle \colon \mathrm{Ker}(d) \to B_1 \times_{c,d} B_1$ such that $\langle 0, k_d \cdot F_1 \rangle \cdot \pi_1 = 0$ and $\langle 0, k_d \cdot F_1 \rangle \cdot \pi_2 = k_d \cdot F_1$, and therefore there exists a unique arrow $\sigma \colon \mathrm{Ker}(d) \to \vec{B}_1$ such that $\sigma \cdot m_1 = \langle 0, k_d \cdot F_1 \rangle = \sigma \cdot m_2$. Now, observe that $\sigma \cdot m_2 \cdot \pi_1 = 0$ and $\sigma \cdot m_1 \cdot \pi_2 = k_d \cdot F_1$, so that there exists a unique arrow $s \colon \mathrm{Ker}(d) \to \mathbb{K}(F)_1$ such that $s \cdot k(F)_1 = \sigma$ and $s \cdot K(F)_1 = k_d$. Moreover, $s \cdot \underline{d} \cdot k(F)_0 = s \cdot k(F)_1 \cdot m_1 \cdot \pi_1 = \sigma \cdot m_1 \cdot \pi_1 = 0$ and $s \cdot \underline{d} \cdot K(F)_0 = s \cdot K(F)_1 \cdot d = k_d \cdot d = 0$, which implies that there exists a unique arrow $\tau\colon \mathrm{Ker}(d) \to \mathrm{Ker}(\underline{d})$ such that $\tau \cdot k_{\underline{d}} = s$. Now we check that t and τ realize an isomorphism :

$$\tau \cdot t \cdot k_d = \tau \cdot k_{\underline{d}} \cdot K(F)_1 = s \cdot K(F)_1 = k_d$$

$$t \cdot \tau \cdot k_{\underline{d}} \cdot K(F)_1 = t \cdot \tau \cdot t \cdot k_d = t \cdot k_d = k_{\underline{d}} \cdot K(F)_1$$

$$t \cdot \tau \cdot k_{\underline{d}} \cdot k(F)_1 = t \cdot s \cdot k(F)_1 = t \cdot \sigma$$

so that we still have to prove that $t \cdot \sigma = k_{\underline{d}} \cdot k(F)_1$. For this, we compose with the four pullback projections :

$$t \cdot \sigma \cdot m_1 \cdot \pi_1 = t \cdot 0 = 0 = k_{\underline{d}} \cdot \underline{d} \cdot k(F)_0 = k_{\underline{d}} \cdot k(F)_1 \cdot m_1 \cdot \pi_1$$

$$t \cdot \sigma \cdot m_1 \cdot \pi_2 = t \cdot k_d \cdot F_1 = k_{\underline{d}} \cdot K(F)_1 \cdot F_1 = k_{\underline{d}} \cdot k(F)_1 \cdot m_1 \cdot \pi_2$$

$$t \cdot \sigma \cdot m_2 \cdot \pi_1 = t \cdot 0 = 0 = k_{\underline{d}} \cdot 0 = k_{\underline{d}} \cdot k(F)_1 \cdot m_2 \cdot \pi_1$$

$$t \cdot \sigma \cdot m_2 \cdot \pi_2 = t \cdot k_d \cdot F_1 = k_{\underline{d}} \cdot K(F)_1 \cdot F_1 = k_{\underline{d}} \cdot k(F)_1 \cdot m_1 \cdot \pi_2 =$$

$$= k_{\underline{d}} \cdot k(F)_1 \cdot m_1 \cdot m = k_{\underline{d}} \cdot k(F)_1 \cdot m_2 \cdot m = k_{\underline{d}} \cdot k(F)_1 \cdot m_2 \cdot \pi_2$$

■

Lemma 5.9 *Let $F\colon \mathbb{A} \to \mathbb{B}$ be a functor in* **Grpd**$(\mathcal{A})$. *Consider the following diagram (notation as in 5.4)*

$$\begin{tikzcd}
\mathbb{K}(\mathrm{Id}) \arrow[r, "\langle 0,K(\mathrm{Id})\rangle"] \arrow[d, "\langle 0,\mathrm{K}(\mathrm{Id})\rangle"'] & \vec{\mathbb{A}} \arrow[r, "\delta"] \arrow[d, "\partial(F)"'] & \mathbb{A} \arrow[d, "\mathrm{Id}"] \\
\mathbb{K}(F) \arrow[r, "\langle 0,K(F)\rangle"'] & \mathbb{R}(F) \arrow[r, "\delta(F)"'] & \mathbb{A}
\end{tikzcd}$$

Then the rows are kernels and the left-hand square is a pullback.

Proof. Thanks to Lemma 5.8, $\langle 0, K(\mathrm{Id})\rangle\colon \mathbb{K}(\mathrm{Id}) \to \mathbb{K}(F)$ is sent by $\mathcal{N}$ to

$$\partial(\mathcal{N}(F))\colon \mathcal{N}(\mathbb{K}(\mathrm{Id})) = \mathbb{K}(\mathrm{Id}_{\mathcal{N}(\mathbb{A})}) \to \mathbb{K}(\mathcal{N}(F)) = \mathcal{N}(\mathbb{K}(F)).$$

The fact that the rows are kernels is a particular case of Lemma 5.7. The last part of the statement now follows easily. ■

We are ready to compare the terminology in $\mathbf{Grpd}(\mathcal{A})$ and in $\mathbf{Arr}(\mathcal{A})$. (See [1] for the notions of protomodular and homological category. Compare also with [6] for points 9 and 10 of the following result.)

Proposition 5.10 *Let $F\colon \mathbb{A} \to \mathbb{B}$ be a functor between groupoids in $\mathcal{A}$.*

1. *If F is faithful, then $\mathcal{N}(F)$ is faithful.*
2. *If F is fully faithful, then $\mathcal{N}(F)$ is fully faithful.*
3. *If $\mathcal{A}$ is regular and F is full, then $\mathcal{N}(F)$ is full.*
4. *If $\mathcal{A}$ is protomodular and $\mathcal{N}(F)$ is faithful, then F is faithful.*
5. *If $\mathcal{A}$ is protomodular and $\mathcal{N}(F)$ is fully faithful, then F is fully faithful.*
6. *If $\mathcal{A}$ is regular and protomodular and $\mathcal{N}(F)$ is full, then F is full.*
7. *If $\mathcal{A}$ is protomodular and F is essentially surjective, then $\mathcal{N}(F)$ is essentially surjective.*
8. *If $\mathcal{A}$ is regular and $\mathcal{N}(F)$ is essentially surjective, then F is essentially surjective.*
9. *If $\mathcal{A}$ is regular and F is a fibration, then $\mathcal{N}(F)$ is a fibration.*
10. *If $\mathcal{A}$ is regular and protomodular and $\mathcal{N}(F)$ is a fibration, then F is a fibration.*

Proof. From 1 to 6. The 0-level of the diagram in Lemma 5.9 gives the following diagram in $\mathcal{A}$:

$$\begin{tikzcd}
\mathrm{Ker}(d) \arrow[r, "k_d"] \arrow[d, "\partial(\mathcal{N}(F))_0"'] & A_1 \arrow[r, "d"] \arrow[d, "\partial(F)_0"'] & A_0 \arrow[d, "\mathrm{id}"] \\
A_0 \times_{F_0, k_d \cdot c} \mathrm{Ker}(d) \arrow[r, "\langle 0, K(F)\rangle_0"'] & A_0 \times_{F_0,d} B_1 \times_{c,F_0} A_0 \arrow[r, "\delta(F)_0"'] & A_0
\end{tikzcd}$$

Thanks to Lemma 5.9, the rows are kernels and the left-hand square is a pullback. Therefore :

1. If $\partial(F)_0$ is a monomorphism, then $\partial(\mathcal{N}(F))_0$ is also a monomorphism.
2. If $\partial(F)_0$ is an isomorphism, then $\partial(\mathcal{N}(F))_0$ is also an isomorphism.
3. If $\partial(F)_0$ is a regular epimorphism, then $\partial(\mathcal{N}(F))_0$ is also a regular epimorphism because the category $\mathcal{A}$ is regular.
4. If $\partial(\mathcal{N}(F))_0$ is a monomorphism, then $\partial(F)_0$ is also a monomorphism because in a protomodular category, pullbacks reflect monomorphisms (see Lemma 3.13 in [3]).
5. If $\partial(\mathcal{N}(F))_0$ is an isomorphism, then $\partial(F)_0$ is also an isomorphism because in a protomodular category, the Split Short Five Lemma holds (see Lemma 3.10 in [3]).
6. If $\partial(\mathcal{N}(F))_0$ is a regular epimorphism, then $\partial(F)_0$ is also a regular epimorphism by Proposition 8 in [2] (see also Proposition 2.4 in [13]), which can be applied here because $\mathcal{A}$ is regular and protomodular and $d\colon A_1 \to A_0$ is a split and then regular epimorphism.

7 and 8. Consider the following commutative diagram

$$\begin{tikzcd}
A_0 \arrow[r, "\langle \mathrm{id}, F_0 \cdot e\rangle"] \arrow[dr, "F_0"'] & A_0 \times_{F_0,d} B_1 \arrow[d, "\beta_d \cdot c"] & \mathrm{Ker}(d) \arrow[l, "\langle 0, k_d\rangle"'] \arrow[dl, "k_d \cdot c"] \\
& B_0 &
\end{tikzcd}$$

7. If $\beta_d \cdot c$ is a regular epimorphism, it is a strong one. Therefore, it suffices to prove that $\langle \mathrm{id}, F_0 \cdot e\rangle$ and $\langle 0, k_d\rangle$ are jointly strongly epimorphic. This is the case since $\mathcal{A}$ is protomodular and $\langle 0, k_d\rangle$ and $\langle \mathrm{id}, F_0 \cdot e\rangle$ are respectively a kernel and a section of α_d (see [2, 3]).

$$\begin{tikzcd}
\mathrm{Ker}(d) \arrow[r, "\langle 0,k_d\rangle"] \arrow[d] & A_0 \times_{F_0,d} B_1 \arrow[d, shift right, "\alpha_d"'] \\
0 \arrow[r] & A_0 \arrow[u, shift right, bend right, "\langle \mathrm{id},F_0\cdot e\rangle"']
\end{tikzcd}$$

8. If F_0 and $k_d \cdot c$ are jointly strongly epimorphic, $\beta_d \cdot c$ is a strong epimorphism and so a regular epimorphism since $\mathcal{A}$ is regular.
9 and 10. Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Ker}(d) \arrow[r, "k_d"] \arrow[d, "\mathrm{K}_d(F)"'] & A_1 \arrow[r, "d"] \arrow[d, "\tau_d"'] & A_0 \arrow[d, "\mathrm{id}"] \\
\mathrm{Ker}(d) \arrow[r, "\langle 0,k_d\rangle"'] & A_0 \times_{F_0,d} B_1 \arrow[r, "\alpha_d"'] & A_0
\end{tikzcd}$$

Since $\mathrm{id}\colon A_0 \to A_0$ is a monomorphism, the left-hand square is a pullback. Therefore :
9. If τ_d is a regular epimorphism, then $\mathrm{K}_d(F)$ is also a regular epimorphism because the category $\mathcal{A}$ is regular.
10. If $\mathrm{K}_d(F)$ is a regular epimorphism, then τ_d is also a regular epimorphism by Proposition 8 in [2]. ∎

We have not yet discussed $*$-fibrations in $\mathbf{Arr}(\mathcal{A})$. For this, we need a last preparatory step.

Lemma 5.11 *In the category with null-homotopies* $\mathbf{Arr}(\mathcal{A})$, *kernels are strong : for any arrow* $(f, f_0)\colon a \to b$, *the canonical comparison* $J\colon \mathbb{K}\mathrm{er}(f, f_0) \to \mathbb{K}(f, f_0)$ *is full and faithful.*

Proof. Using the descriptions of the kernel and of the strong h-kernel given in 5.3, the comparison J turns out to be the left-hand square in the following diagram

$$\begin{tikzcd}
\mathrm{Ker}(f) \arrow[r, "k_f"] \arrow[d, "\mathrm{K}(a)"'] & A \arrow[r, "f"] \arrow[d, "\partial(f,f_0)_0"'] & B \arrow[d, "\mathrm{id}"] \\
\mathrm{Ker}(f_0) \arrow[r, "\langle k_{f_0},0\rangle"'] & A_0 \times_{f_0,b} B \arrow[r, "f_0'"'] & B
\end{tikzcd}$$

Since both rows are kernels and $\mathrm{id}\colon B \to B$ is a monomorphism, the left-hand square is a pullback, which means that J is full and faithful. ∎

5.12 Having in mind Proposition 4.5, we could now define a $*$-*fibration* in $\mathbf{Arr}(\mathcal{A})$ as an arrow $(f, f_0)\colon a \to b$ such that the canonical comparison $J\colon \mathbb{K}\mathrm{er}(f, f_0) \to \mathbb{K}(f, f_0)$ is essentially surjective (and then, by Lemma 5.11, a weak equivalence). Thanks to Proposition 5.10, if $\mathcal{A}$ is homological, with such a notion of $*$-fibration we have that a functor F is a $*$-fibration in $\mathbf{Grpd}(\mathcal{A})$ if and only if the arrow $\mathcal{N}(F)$ is a $*$-fibration in $\mathbf{Arr}(\mathcal{A})$. Now that we dispose of the notions of fibration and $*$-fibration in $\mathbf{Arr}(\mathcal{A})$, we can ask if every fibration is a $*$-fibration (this is the case in $\mathbf{Grpd}(\mathcal{A})$, as observed in 4.4). The surprise is that not only the answer is negative, but the expected implication "fibration $\Rightarrow$ $*$-fibration" is in fact equivalent to the condition of protomodularity.

Proposition 5.13 *The following conditions on a pointed regular category $\mathcal{A}$ are equivalent :*

1. *$\mathcal{A}$ is protomodular (and then homological).*
2. *For every fibration $(f, f_0)\colon a \to b$ in $\mathbf{Arr}(\mathcal{A})$, the canonical comparison*

$$J\colon \mathbb{K}\mathrm{er}(f, f_0) \to \mathbb{K}(f, f_0)$$

is a weak equivalence.

Proof. $1 \Rightarrow 2$. Suppose $\mathcal{A}$ is homological and (f, f_0) is a fibration in $\mathbf{Arr}(\mathcal{A})$. Thanks to Lemma 5.11, we already know that J is full and faithful. Consider now the following diagram where $\mathrm{id} \times f$ is a regular epimorphism since so is f.

$$\begin{tikzcd} \mathrm{Ker}(f_0) \arrow[r, "\langle k_{f_0},0\rangle"] \arrow[dr, "\langle k_{f_0},0\rangle"'] & A_0 \times_{f_0, a\cdot f_0} A \arrow[d, "\mathrm{id}\times f"] & A \arrow[l, "\langle a,\mathrm{id}\rangle"'] \arrow[dl, "\partial(f,f_0)_0"] \\ & A_0 \times_{f_0,b} B & \end{tikzcd}$$

Thus, in order to prove that J is essentially surjective, it suffices to notice that the protomodularity of $\mathcal{A}$ implies that $\langle k_{f_0}, 0\rangle$ and $\langle a, \mathrm{id}\rangle$ are jointly strongly epimorphic since they are respectively the kernel and a section of the second projection $A_0 \times_{f_0,a\cdot f_0} A \to A$.
$2 \Rightarrow 1$. Firstly, let us prove that if the kernel of a morphism $f_0 : A_0 \to B_0$ is the zero object, then it is a monomorphism. In order to do so, consider the fibration $(\mathrm{id}, f_0)\colon \mathrm{id} \to f_0$ in $\mathbf{Arr}(\mathcal{A})$.

$$\begin{tikzcd} A_0 \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & A_0 \arrow[d, "f_0"] \\ A_0 \arrow[r, "f_0"'] & B_0 \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] \arrow[d] & A_0 \arrow[d, "\langle \mathrm{id},\mathrm{id}\rangle"] \\ 0 \arrow[r] & A_0 \times_{f_0,f_0} A_0 \end{tikzcd}$$

The diagram on the right represents the canonical comparison $J\colon \mathbb{K}\mathrm{er}(\mathrm{id}, f_0) \to \mathbb{K}(\mathrm{id}, f_0)$. By the assumption, we know that $0 \to A_0 \times_{f_0,f_0} A_0$ and $\langle \mathrm{id}, \mathrm{id}\rangle$ are jointly strongly epimorphic. This is equivalent to the fact that $\langle \mathrm{id}, \mathrm{id}\rangle$ is a regular epimorphism. Since it is also a split monomorphism, it is an isomorphism, which means that f_0 is a monomorphism.

Let us now prove that the Short Five Lemma holds in $\mathcal{A}$. Consider the following diagram where both rows are kernel of regular epimorphisms and $\mathrm{K}(a)$ and b are isomorphisms.

$$\begin{tikzcd}
\mathrm{Ker}(f) \arrow[r, "k_f"] \arrow[d, "\mathrm{K}(a)"'] & A \arrow[r, two heads, "f"] \arrow[d, "a"] & B \arrow[d, "b"] \\
\mathrm{Ker}(f_0) \arrow[r, "k_{f_0}"'] & A_0 \arrow[r, two heads, "f_0"'] & B_0
\end{tikzcd}$$

Since b is a monomorphism, the left-hand square is a pullback. Hence, $\mathrm{K}(a)$ and a have isomorphic kernels. Since $\mathrm{K}(a)$ is a monomorphism, this kernel is the zero object. But, by the first part of the proof, this means that a is a monomorphism. So, it remains to prove that it is a regular epimorphism. The morphism $(f, f_0)\colon a \to b$ is a fibration in $\mathbf{Arr}(\mathcal{A})$. Thus, the comparison morphism $J\colon \mathbb{K}\mathrm{er}(f, f_0) \to \mathbb{K}(f, f_0)$ is a weak equivalence. Since b is an isomorphism, this implies that k_{f_0} and a are jointly strongly epimorphic. But since $\mathrm{K}(a)$ is an isomorphism, k_{f_0} factors through a, so that a is a regular epimorphism. ■

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