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Dynamics of ideal efforts and consensus in a multi-layer network game

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ABSTRACT

We study a network game on a fixed multi-layer network of two types of relationships. The social interactions in the first layer carry a pressure to conform with the social norm within the layer. The second layer provides additional strategic complementarities from players' interaction. Players are endowed with personal ideal efforts and are heterogeneous in their ideal efforts and productivity. Each player repeatedly chooses her effort level in the network game and updates her ideal effort based on the new effort choice. Each player suffers disutility when her effort differs from her neighbors' efforts or is inconsistent with her ideal effort. We find the pure Nash equilibrium of the game in each period and provide conditions for the convergence of efforts and ideals to a steady state. Furthermore, we find that the sensitivity to cognitive dissonance and the taste for conformity have opposing effects on the speed of convergence to a consensus and the steady state.

1. Introduction

Individual choices are often affected by the social environment.[1] The behaviors and the attitudes of social contacts affect individuals' well-being, driving their choices and leading them to adapt or adopt their opinions leading to conformity.[2] However, individual beliefs on many issues vary significantly in society and individuals differ in their willingness to change those beliefs.[3] Despite the importance of a consensus in problems where coordination matters, the presence of diverse opinions about some issue is an important part in the evolution of the beliefs and the dynamics of individual choices.

Recent research has shown that studying the social and economic relations in the context of network interactions can provide valuable insights about group outcomes and individual choices.[4] The literature on network games provides a game-theoretic framework for analyzing the behavior of players embedded in a network in a variety of economic settings. The structure of the network, players' characteristics and economic incentives affect the individuals' choices.[5] The topology of the network impacts the efforts dynamics and learning in the population (see e.g Golub and Jackson (2010) or Acemoglu and Ozdaglar (2011)). Most of the network games commonly studied assume that individuals are embedded in a network structure representing one type of relationships.[6] However, very often individuals are engaged in multiple types of relationships that influence individual choices regarding a single issue. Individuals can belong to both social and professional networks,

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[1] Behavioral spillovers and peer effects through social interaction are present in education, labor markets, environmental choices, crime, etc. For surveys on theory and empirical evidence see Durlauf (2004), Ioannides and Datcher Loury (2004), Ioannides and Ioannides (2012).

[2] See the pioneering work by Asch (1955) on opinions and social pressure.

[3] Bednar and Page (2007) define *behavioral stickiness* as one of the elements of cultural behavior and show that the individuals may not alter their behavior despite changes in incentives.

[4] There is a wide range of applications of network theory in social and economic problems, such as in labor markets (e.g. Calvó-Armengol (2004), Calvó-Armengol and Jackson (2004)), financial markets (e.g. Gale and Kariv (2007), Elliott et al. (2014)), R&D collaborations (e.g. Goyal and Moraga-González (2001), Dawid and Hellmann (2014), Mauleon et al. (2023)), as well as peer effects in networks (see the recent survey by Bramoullé et al. (2020)), opinion diffusion (e.g. Golub and Jackson (2010), Jackson and Yariv (2011)), adoption of environmentally friendly behavior (Currarini et al. (2016)) and technology adoption (e.g. Beaman et al. (2021)).

[5] See Jackson and Zenou (2015) for a comprehensive introduction to network games.

[6] Few exceptions are the papers of Belhaj and Deroïan (2014) and Chen et al. (2018) that capture multiple activities on a single-layer network, Foerster et al. (2021) that analyzes a framework where players can form two types of links, public and shadow, on a single-layer network, and Walsh (2019) who studies a public good game on a two-layer network where players choose to invest in one of the layers.

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firms can be involved in several R&D collaborative relationships, and countries can sign multiple economic and/or military alliances. This raises an important question: When agents interact in several dimensions of a multi-layer network, how are individuals' choices affected by the presence of spillovers, social norms and peer pressure in the network? In this paper, we study the consequences of social norms and conformity on both, individuals' efforts and the long run consensus of ideals in the society, when agents interact in a two-layer network.

We analyze a network game with social and personal norms regarding the effort choice among a population of players embedded in a two-layer network and exerting costly efforts. The presence of social interactions in the first layer carries a pressure to conform with the social norm within the layer and players suffer disutility when their efforts are different from those exerted by their neighbors. This disutility can be interpreted as a conflict cost for non-compliance with the social norm.[7] Players are heterogeneous in their *ideal efforts*[8] and productivity, and incur disutility when there is a discrepancy between their beliefs about the ideal effort and their actual efforts. This disutility can be described as cognitive dissonance or inner conflict.[9] The novelty of this paper is that players suffer from these two different types of disutilities in a two-layer network where the interaction with neighbors generates strategic complementarity of efforts in the second layer.

Apart from individual efforts influencing the decisions of other players through spillovers, the efforts of all players in a network are additionally affected by their idiosyncratic productivities and ideal efforts, their position in the network and the costs from miscoordination in the first-layer network and from the inconsistencies with their ideal efforts. However, players are myopic in the sense that they do not consider how their effort choice affects those of other players in the network. We derive the unique Nash equilibrium of the network game. We find that the equilibrium efforts of the players are proportional to their weighted Katz–Bonacich centrality in the combined network of two-layers,[10] that is, the network centrality measure indicating the weighted number of paths in the network stemming from a given player, additionally weighted by idiosyncratic productivities and ideal efforts of each player on the paths.[11]

We assume that the norms, both social and personal, tend to change. After each period network game, players update their ideal efforts by adapting those to their actual efforts in the network game. The ideal effort is updated as a linear combination of the ideal effort in the previous period and the actual effort. This form of updating implies that the players adjust their ideals to justify their actions and to manage their cognitive dissonance.[12] Such linear updating method is most similar to the well-known DeGroot (1974)'s updating that is widely used in opinion formation and social learning models. Compared to DeGroot updating where the beliefs are revised according to the social norm, in our model the ideal effort is updated according to the actual effort that is, in turn, a linear combination of social norm, the current ideal effort of the player, her productivity, and effort complementarities from the additional network layer. A similar approach to DeGroot updating has been adopted by DeMarzo et al. (2003) who define the notion of persuasion bias as a failure to adjust for possible repetitions of information they receive and instead treat all information as new. Correctly adjusting for repetitions of information would become extremely complicated when individuals interact repeatedly on complex social networks.[13] Such bounded rationality of players is assumed in most non-Bayesian learning models.

Similarly to subjective beliefs, we assume that ideal efforts are a subjective measure idiosyncratic to the players. We study the dynamics of ideal efforts given the multi-layer network structure, the taste for conformity, the sensitivity to cognitive dissonance in the network, and players' productivity coefficients. We show that under certain conditions on the network layer of strategic complementarities, the steady state exists. Moreover, in presence of structural equivalence of players in the given two-layer network (i.e., players similar in their relational patterns and neighborhoods), the ideals of players converge to a consensus. Referring to ideal efforts as to an individual subjective matter, we refrain from studying the correctness of it. Thus, the consensus over the ideals in this setting represents the possibility of coordination in the network, rather than the "correctness" of learning and convergence to true action. A number of works have studied the ability of correct aggregation of information and conditions to converge to the truth. Golub and Jackson (2010) characterize the population and the learning process as wise if, given the existence of consensus, the influence of the most influential player vanishes as the network grows. Bala and Goyal (1998) study a non-Bayesian model of social learning in the context of technology adoption and show that the structure of the network has important implications in the adoption of new technologies, and their diffusion in society. Foerster et al. (2016) show that manipulation can connect a segregated society and thus lead to mutual consensus in a model of boundedly rational opinion dynamics. Acemoglu and Ozdaglar (2011) provide an overview of works on opinion dynamics and social learning. In this paper, we study the dynamics of ideal efforts and we show that the sensitivity to cognitive dissonance and the taste for conformity have opposing effects on the speed of convergence to a consensus and the steady state. The sensitivity to cognitive dissonance, that captures the behavioral stickiness of the players, slows down the speed of convergence to a steady state but does not affect the long-run efforts. The taste for conformity, on the other hand, affects the steady state efforts' levels and accelerates the consensus over the ideals.

This paper is closely related to the recent work by Olcina et al. (2024) on the long-run integration behavior of ethnic minorities. They study a network game in which there is a conflict between an individual's integration choice and that of her friends and between an individual's assimilation choice and that of her family and community. They show that the key determinant of the different integration choices is their position in the network and the initial preferences of the persons they are connected to.[14] Similarly, we study a network game where the economic incentives, together with the ideal efforts and the social

[7] That is, individuals are penalized if they deviate from the effort level of their network in the first network layer. See e.g. Ballester et al. (2010), Patacchini and Zenou (2012), Liu et al. (2014), Boucher (2016), Atay et al. (2023) about peer effects and conformism in social networks.

[8] The notion of ideal efforts in network games is used in recent works by Olcina et al. (2024) and Galeotti et al. (2021).

[9] The *Theory of Cognitive Dissonance* introduced by social psychologists dates back to Festinger (1957). It suggests that people strive for internal consistency continuously aligning the cognitions with their actions in order to minimize the cognitive dissonance. Experimental work by Elliot and Devine (1994) proves this result. Akerlof and Dickens (1982) and Rabin (1994) incorporate the psychological theory of cognitive dissonance into theoretical economic models of rational choice. They show that such approach provides a better explanation of the welfare consequences of some economic phenomena.

[10] This network centrality measure, introduced first by Katz (1953) and reestablished later by Bonacich (1987), captures the power or the status as a relative degree of influence of the player in the network.

[11] Following the seminal work by Ballester et al. (2006), the link between the Katz–Bonacich centrality of a player and her effort is an established result in the literature on network games.

[12] An early experimental study by Brehm (1956) shows that people attempt to increase the desirability of their choices to stay consistent with their decisions. In the literature on psychology, such ex-post rationalization of decisions and choice-supportive bias is present in experimental results (see e.g. Mather and Johnson (2000), Mather et al. (2000)).

[13] See DeMarzo et al. (2003) for a detailed discussion of this assumption and Choi et al. (2012) and Bolte and Fan (2024) for experimental evidence.

[14] Recently, Luo et al. (2024) analyze how the presence of farsighted individuals can affect the long-run integration outcome and under which circumstances this can destabilize segregation in a friendship network formation model with two different communities.

norms, affect the effort choice of the players. The novelty of this paper is that we allow players to be embedded in a two-layer network.[15] We show that the layer of network complementarities affects the structural similarities of players in the network providing conditions for the consensus in the ideal efforts of the players.

We can find several examples of social and economic issues that can be described by games of spillovers and conflicts. The adoption of technology when compatibility matters is an example where spillovers and miscoordination, together with private dissonance, affect individual choices. For instances, the choice of technology such as messaging applications, software for video conferences, cloud computing services, or file transfer services is strongly affected by the choice of peers (or firms) that are also collaborating in R&D in another network and benefiting from strategic complementarities in their interactions. A failure to coordinate creates incompatibility and one reason for such miscoordination can be personal preferences or ideal choices of individuals which can be based on trust or mistrust in the specific service provider, data protection issues, or simply personal habits.

Another example of a network game with spillovers and conflict is the environmentally friendly behavior. Studies show that social norms and private values are strong determinants of the pro-environmental decisions of individuals. The discomfort from acting differently from the neighbors, or the willingness to fit the social norms can drive the attitudes of individuals regardless of any monetary benefit.[16] In addition, private values or norms are another driving force of individual behavior.[17] Individuals who consider themselves environmentalists are more likely to be concerned about private and external norms experiencing green guilt from acting less environmentally friendly compared to their neighbors. Moreover, these individuals are often involved in organizations and movements (group and network activities) that intent to promote green behavior and that allow them to benefit from strategic complementarities in their interactions.

The paper is organized as follows. In Section 2 we describe the network spillover game with conformism and private dissonance and we characterize the unique Nash equilibrium efforts of the network game. In Section 3 we characterize the convergence to the steady state and the consensus. Some numerical examples are used to study the dynamics of the ideal efforts. In Section 4 we conclude the paper.

2. Model

2.1. The network game

We have a set $N = \{1, 2, \dots, n\}$ of players embedded in a multi-layer network of social influences. The network consists of two layers, where one layer represents the network $\boldsymbol{g}$ of social connections, and the other is the network of complementarities $\boldsymbol{l}$. The network $\boldsymbol{g}$ is a directed weighted network described by the weighted adjacency matrix $\mathbf{G}^*$ with elements g^*_{ij} for any neighbors $i, j \in N$ in the network $\boldsymbol{g}$. The weight g^*_{ij} on the link $i \to j$ is normalized by the degree of player i on $\boldsymbol{g}$, such that $\sum_{j\in N} g^*_{ij} = 1$ for all $i \in N$. The network $\boldsymbol{l}$ is represented by the matrix $\boldsymbol{\Lambda}$ with elements $\lambda_{ij} > 0$ as coefficients of effort complementarity from player j given the existence of a directed link $i \to j$ in the network $\boldsymbol{l}$ for all $i, j \in N$. By convention, we assume there are no self-loops in the network, i.e. $g^*_{ii} = 0$ and $\lambda_{ii} = 0$ for all $i \in N$. Networks $\boldsymbol{g}$ and $\boldsymbol{l}$ are not necessarily connected networks.[18]

Players are initially endowed with *personal ideals*, or *personal norms*, y^0_i.[19] Each period t they choose their efforts x^t_i based on the network game on the layers $\boldsymbol{g}$ and $\boldsymbol{l}$. The choice of effort for each player i is based on her idiosyncratic productivity θ_i, complementarities from their neighbors in the network $\boldsymbol{l}$, the cost from miscoordination due to the difference between her effort and those of each of her neighbors in the network $\boldsymbol{g}$,[20] as well as the cost from the discrepancy between her actual effort x^t_i and her *ideal behavior* y^t_i.[21] After each period network game, players update their ideal efforts as a linear combination of their ideal efforts and their actual efforts, with a speed of updating $\xi \in [0; 1]$.

The timing of the model is the following for each period t.

1. *Network game and effort selection.* Players $i \in N$ choose the best response efforts x^t_i in the two-layer network game by maximizing the following utility:

$$u^t_i(x) = \theta_i x^t_i + x^t_i \sum_{j\in N} \lambda_{ij} x^t_j \underbrace{- \frac{\omega_1}{2} \sum_{j\in N} g^*_{ij}(x^t_i - x^t_j)^2}_{\text{Conformism or conflict}} - \frac{\omega_2}{2}(x^t_i)^2 \underbrace{- \frac{\omega_3}{2}(x^t_i - y^t_i)^2}_{\substack{\text{Consistency or} \\ \text{cognitive dissonance}}}, \tag{1}$$

$$x^t_i = BR_i(\mathbf{x}^t_{-i}; \mathbf{G}^*, \boldsymbol{\Lambda}, y^t_i, \theta_i, \omega_1, \omega_2, \omega_3),$$

where $\mathbf{x}^t_{-i}$ is the list of efforts of players $N \setminus i$.

2. *Updating ideal efforts.* Players update their ideal efforts based on the actual effort x^t_i and ideal effort y^t_i of the current period:

$$y^{t+1}_i = \xi x^t_i + (1-\xi) y^t_i. \tag{2}$$

The coefficients ω_1, ω_2 and ω_3 in the utility of player i in (1) are the cost coefficients of the model. The difference in efforts of neighbors in the network $\boldsymbol{g}$ is amplified by the coefficient ω_1, that is, the cost of conflict. The cost of conflict may also be regarded as a taste for conformity. The cost of unit effort is given by the coefficient $\frac{\omega_2}{2}$. In addition, ω_3 is the cost of moral conflict or sensitivity to cognitive dissonance. When the cost of conflict ω_1 is high, the disutility from miscoordination in the network $\boldsymbol{g}$ is high, forcing the neighbors to coordinate around social norm faster, resulting in conformity. If the sensitivity to cognitive dissonance ω_3 is high, the players remain consistent with their ideal efforts longer.[22]

2.2. Equilibrium

Solving for the first-order conditions on the utility (1) of player i, we find her best response effort in period t:

$$x^t_i = \frac{1}{\omega_1 + \omega_2 + \omega_3}\left(\theta_i + \omega_3 y^t_i + \sum_{j\in N} \lambda_{ij} x^t_j + \omega_1 \sum_{j\in N} g^*_{ij} x^t_j\right).$$

[15] Joshi et al. (2020) and Billand et al. (2023) propose two different models on the endogenous formation of multilayer networks.

[16] Ando et al. (2010) study the determinants of individual and collective environmentally conscious behaviors. They find that subjective norms from the network impact substantially the collective pro-environmental behaviors.

[17] For instance, Viscusi et al. (2011) find that individuals who consider themselves environmentalists are more likely to be concerned about private and external norms experiencing green guilt from acting less environmentally friendly compared to their neighbors. See e.g. Gifford and Nilsson (2014) for an extensive survey of studies about personal and social determinants of pro-environmental behavior.

[18] A directed network is weakly connected if for every pair of players i and j, there is a path connecting them when the link direction is ignored. A directed network is strongly connected if for every pair of players i and j, there is a directed path from i to j and another directed path from j to i.

[19] The ideal efforts can be perceived as a personal norm of a player, personal standard, an attitude toward the specific issue, feeling of moral obligation, or self-expectations (Ajzen and Fishbein, 1970; Schwartz, 1973; Schwartz and Howard, 1980).

[20] Minimization of this cost of conflict leads to conformism similarly to conventional local-average models with descriptive social norms.

[21] In network games literature, the notion of ideal efforts and consistency is present in works by Olcina et al. (2024) and Galeotti et al. (2021).

[22] The cost of private dissonance indicates the level of flexibility of the players. This leaves room for extending the model by introducing heterogeneity in players' types. Particularly, dividing the players into stubborn (or persistent) and flexible agents, with the first group having higher coefficient ω_3 compared to the flexible agents, and thus *sticky* ideals.

One can see that the best response of the player is the linear combination of her productivity θ_i, the ideal effort y_i^t, complementarities $\sum_{j\in N} \lambda_{ij} x_j^t$ from the network $\boldsymbol{l}$, and the social norm $\sum_{j\in N} g_{ij}^* x_j^t$ in the network $\boldsymbol{g}$.

Let $\mathbf{x}^t$, $\mathbf{y}^t$ be the vectors of efforts and ideal efforts in period t respectively, and let $\boldsymbol{\theta}$ denote the vector of idiosyncratic elements θ_i, for all $i \in N$. Then, the best response in matrix form will be given by:

$$\mathbf{x}^t = \frac{1}{\omega_1 + \omega_2 + \omega_3} \left(\boldsymbol{\theta} + \omega_3 \mathbf{y}^t + \boldsymbol{\Lambda} \mathbf{x}^t + \omega_1 \mathbf{G}^* \mathbf{x}^t \right). \tag{3}$$

We characterize the conditions for the existence of unique Nash equilibrium in the following proposition.

Proposition 1. *Assuming the spectral radius of $\boldsymbol{\Lambda} + \omega_1 \mathbf{G}^*$ is smaller than $(\omega_1 + \omega_2 + \omega_3)$, the unique Nash equilibrium in pure strategies is then given by:*[23]

$$\mathbf{x}^t = \frac{1}{\omega_1 + \omega_2 + \omega_3} \left(\mathbf{I} - \frac{1}{\omega_1 + \omega_2 + \omega_3} (\boldsymbol{\Lambda} + \omega_1 \mathbf{G}^*) \right)^{-1} (\boldsymbol{\theta} + \omega_3 \mathbf{y}^t).$$

One can see that the unique Nash equilibrium of the game is proportional to the weighted Katz–Bonacich centrality of the players on a network represented by the matrix $(\boldsymbol{\Lambda} + \omega_1 \mathbf{G}^*)$ and weighted by the linear combination of productivites and ideal efforts at time $(\boldsymbol{\theta} + \omega_3 \mathbf{y}^t)$.[24] The matrix $(\boldsymbol{\Lambda} + \omega_1 \mathbf{G}^*)$ represents a network formed by projecting the links of the layer $\boldsymbol{l}$ on $\boldsymbol{g}$, where the adjacency matrix of the layer $\boldsymbol{g}$ is weighted by the coefficient of conflict cost. Since the largest eigenvalue (the spectral radius) of $(\boldsymbol{\Lambda} + \omega_1 \mathbf{G}^*)$ is smaller than $(\omega_1 + \omega_2 + \omega_3)$, $\left(\mathbf{I} - \frac{1}{\omega_1 + \omega_2 + \omega_3} (\boldsymbol{\Lambda} + \omega_1 \mathbf{G}^*) \right)$ is invertible and with non-negative entries. Thus, the equilibrium effort for each agent i at time t is unique and interior.

The expression for the equilibrium efforts does not enable us to understand the relationship between the dynamics of ideal efforts and the topological properties of the network $(\boldsymbol{\Lambda} + \omega_1 \mathbf{G}^*)$. Following Olcina et al. (2024), we define the following vectors of length $2n$.

$$\hat{\mathbf{x}}^t := \begin{bmatrix} \boldsymbol{\theta} \\ \mathbf{x}^t \end{bmatrix} \qquad \hat{\mathbf{y}}^t := \begin{bmatrix} \boldsymbol{\theta} \\ \mathbf{y}^t \end{bmatrix}$$

The vectors $\hat{\mathbf{x}}^t$ and $\hat{\mathbf{y}}^t$ are constructed using the vector $\boldsymbol{\theta}$ as first n elements, and vectors $\mathbf{x}^t$ and $\mathbf{y}^t$ as the second half of the respective vector. Additionally, we define the following matrix.

$$\bar{\mathbf{G}} := \frac{1}{\omega_1 + \omega_2} \left[\begin{array}{c|c} (\omega_1 + \omega_2)\mathbf{I} & \mathbf{O} \\ \hline \mathbf{I} & (\boldsymbol{\Lambda} + \omega_1 \mathbf{G}^*) \end{array} \right] \tag{4}$$

Notice that $\bar{\mathbf{G}}$ is a stochastic matrix where the first n rows represent the links of fictitious agents while the last n rows represent the links of real agents.[25] Using the definitions above and Eq. (3) we can write the augmented vector of efforts $\hat{\mathbf{x}}^t$ as the weighted average of $\hat{\mathbf{y}}^t$ and the social norm in the network described by the augmented adjacency matrix $\bar{\mathbf{G}}$.

$$\hat{\mathbf{x}}^t = \frac{\omega_3}{\omega_1 + \omega_2 + \omega_3} \hat{\mathbf{y}}^t + \frac{\omega_1 + \omega_2}{\omega_1 + \omega_2 + \omega_3} \bar{\mathbf{G}} \hat{\mathbf{x}}^t$$

Notice that the first n elements of the vector $\hat{\mathbf{x}}^t$, the vector $\boldsymbol{\theta}$, does not change over time and does not carry any economic meaning. However, the elements from $n+1$ to $2n$ correspond to (3). This allows us to rewrite the equilibrium efforts and characterize new conditions for equilibrium existence.

Proposition 2. *Let δ_1 be the spectral radius of the matrix $\bar{\mathbf{G}}$. If $\delta_1 < \frac{\omega_1 + \omega_2 + \omega_3}{\omega_1 + \omega_2}$, then the unique Nash equilibrium in pure strategies is given by:*

$$\hat{\mathbf{x}}^t = \frac{\omega_3}{\omega_1 + \omega_2 + \omega_3} \left(\mathbf{I} - \frac{\omega_1 + \omega_2}{\omega_1 + \omega_2 + \omega_3} \bar{\mathbf{G}} \right)^{-1} \hat{\mathbf{y}}^t.$$

Let

$$\overline{\omega}_2 \equiv \max_i \sum_{j\in N} \lambda_{ij},$$

$$\overline{\overline{\omega}}_2 \equiv \max_j \sum_{i\in N\setminus j} (\lambda_{ij} + \omega_1 g_{ij}^*) - \omega_1 - 1.$$

We have that 1 is always an eigenvalue of $\bar{\mathbf{G}}$ by construction, whereas for the parameter values satisfying $\omega_2 \geq 1 + \overline{\omega}_2$, it is also the spectral radius. Then, $\left(\mathbf{I} - \frac{\omega_1 + \omega_2}{\omega_1 + \omega_2 + \omega_3} \bar{\mathbf{G}} \right)$ is invertible and with non-negative entries guaranteeing a unique and interior solution. The above equation shows that the equilibrium effort of agent i at time t depends on her taste for conformity, her ideal effort at time t, and her position in the network, as captured by her Katz–Bonacich centrality.

Lemma 1 provides conditions on parameter values and combinations that ensure the equilibrium existence in Proposition 2.

Lemma 1. *Let δ_1 be the spectral radius of the matrix $\bar{\mathbf{G}}$. The condition $\delta_1 < \frac{\omega_1 + \omega_2 + \omega_3}{\omega_1 + \omega_2}$ is satisfied if one of the conditions below holds:*

I. *Given $\overline{\omega}_2 > \overline{\overline{\omega}}_2$, $\omega_2 < \overline{\overline{\omega}}_2$ and $\omega_3 > \overline{\overline{\omega}}_2 + 1 - \omega_2$;*
II. *Given $\overline{\omega}_2 > \overline{\overline{\omega}}_2$, $\omega_2 \in [\overline{\overline{\omega}}_2; \overline{\omega}_2)$ and $\omega_3 > 1$;*
III. *Given $\overline{\omega}_2 \leq \overline{\overline{\omega}}_2$, $\omega_2 < \overline{\omega}_2$ and $\omega_3 > 1 + \overline{\omega}_2 - \omega_2$;*
IV. *$\omega_2 \in [\overline{\omega}_2; 1 + \overline{\omega}_2)$ and $\omega_3 > 1 + \overline{\omega}_2 - \omega_2$;*
V. *$\omega_2 \geq 1 + \overline{\omega}_2$ and $\omega_3 > 0$.*

Fig. 1 illustrates the conditions provided in Lemma 1. The five shaded areas correspond to the five conditions on the parameters of Lemma 1. The combination of parameter values of ω_2 and ω_3 in the gray shaded area above the black line in Fig. 1 ensures the existence of equilibrium given that $\overline{\omega}_2 > \overline{\overline{\omega}}_2$. The green shaded area above the green line in Fig. 1 (from $\overline{\omega}_2 + 1$ to $\overline{\omega}_2 + 1$) displays the satisfying parameter combinations, when $\overline{\omega}_2 \leq \overline{\overline{\omega}}_2$.

When the coefficients of cost and cognitive dissonance are too low relative to complementarities in the network, the positive feedback loops in players' efforts escalate infinitely, so the Nash equilibrium does not exist. Thus, the cost coefficients have to be high enough to curb the complementarities in the network measured by the spectral radius of the matrix $\bar{\mathbf{G}}$. The combinations of parameters in Fig. 1 ensure this requirement and the existence of Nash equilibria.

3. Dynamics and convergence of ideal efforts

3.1. Updating ideal efforts

Let us now characterize the dynamics of ideal efforts using the augmented vector of equilibrium efforts in Proposition 2. Let $\phi = \frac{\omega_1 + \omega_2}{\omega_1 + \omega_2 + \omega_3}$. We can then rewrite the augmented vector of equilibrium efforts as follows:

$$\hat{\mathbf{x}}^t = (1 - \phi) \left(\mathbf{I} - \phi \bar{\mathbf{G}} \right)^{-1} \hat{\mathbf{y}}^t. \tag{5}$$

Similarly to $\hat{\mathbf{x}}^t$, we can apply the updating rule of ideal efforts on the augmented vector $\hat{\mathbf{y}}^t$. The first n elements of $\hat{\mathbf{y}}^t$ remain unchanged and equal to $\boldsymbol{\theta}$ over time, while the elements $n+1$ to n correspond to ideal efforts $\mathbf{y}^t$ in time period t. Applying (2) to $\hat{\mathbf{y}}^t$ we obtain:

$$\hat{\mathbf{y}}^{t+1} = \xi \hat{\mathbf{x}}^t + (1 - \xi) \hat{\mathbf{y}}^t \tag{6}$$

[23] The spectral radius of a matrix is its largest absolute eigenvalue. Debreu and Herstein (1953) show that for a square matrix A, $(sI - A)^{-1}$ is well-defined and non-negative if and only if the spectral radius of A is smaller than s.

[24] The link between the equilibrium efforts of the network game and the Katz–Bonacich centralities has been first established in the seminal work by Ballester et al. (2006). See Remark 1 in Ballester et al. (2006) for the relationship between the equilibrium effort and the weighted Bonacich centrality.

[25] These fictitious agents do not have a real counterpart and can be interpreted as stubborn agents who always choose θ_i.

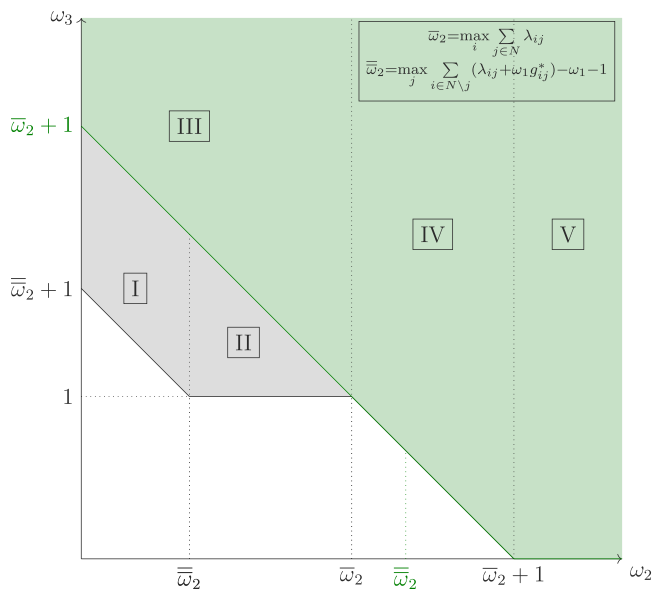

Fig. 1. The gray and green shaded areas cover the combination of parameter values according to the conditions on $\bar{\mathbf{G}}$ provided by Lemma 1. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

By plugging the value of the equilibrium effort $\hat{\mathbf{x}}^t$ given by (5) into Eq. (6) we have:

$$\hat{\mathbf{y}}^{t+1} = [\xi(1-\phi)\left(\mathbf{I} - \phi\bar{\mathbf{G}}\right)^{-1} + (1-\xi)\mathbf{I}]\hat{\mathbf{y}}^t.$$

The transition to a new ideal effort depends on player's current ideal effort and the equilibrium effort, with the latter determined by her position in the network, as captured by her Katz–Bonacich centrality.

3.2. Convergence and consensus

Let us define the following matrices:

$$\mathbf{M} := \left(\mathbf{I} - \phi\bar{\mathbf{G}}\right)^{-1} \tag{7}$$

$$\mathbf{T} := \xi(1-\phi)\mathbf{M} + (1-\xi)\mathbf{I} \tag{8}$$

We can rewrite the ideal efforts on the augmented vector $\hat{\mathbf{y}}^t$ using the notations in (7) and (8) as follows:

$$\hat{\mathbf{y}}^{t+1} = \mathbf{T}\hat{\mathbf{y}}^t = \mathbf{T}^{t+1}\hat{\mathbf{y}}^0 \tag{9}$$

Given the initial ideal efforts $\hat{\mathbf{y}}^0$ and the matrix $\mathbf{T}$, we can find the ideal efforts at any time period t, as well as the limit beliefs $\hat{\mathbf{y}}^\infty$. One can notice that if $\sum_{j\in N} \lambda_{ij} = \omega_2 - 1$ for all $i \in N$, then the matrix of weights $\bar{\mathbf{G}}$, and thus, the matrix $(1-\phi)\mathbf{M}$ are row-normalized matrices.[26] As a result, the matrix $\mathbf{T}$ is row-normalized as a convex combination of a row-stochastic matrix and an identity. With this setting, the dynamics described in (9) is a time-homogeneous Markov process with transition matrix $\mathbf{T}$.

Notice, however, that the transition matrix $\mathbf{T}$ represents a non-trivial transformation of the augmented adjacency matrix $\bar{\mathbf{G}}$ that encompasses information regarding the equilibrium efforts, the updating process, and the ideal efforts of all agents involved in the network of interactions. Proposition 3 provides the conditions for existence of the limit ideal efforts. Moreover, it shows that in order to find the ideal effort and the equilibrium effort in the limit it is sufficient to use the augmented matrix of weights $\bar{\mathbf{G}}$ instead of $\mathbf{T}$ provided $\bar{\mathbf{G}}$ is diagonalizable.

Proposition 3. *For a given network with layers g and l such that $\max_i \sum_{j\in N} \lambda_{ij} \leq \omega_2 - 1$, and $\bar{\mathbf{G}}$ is diagonalizable, there exist $\bar{\mathbf{G}}^\infty = \lim_{t\to\infty} \bar{\mathbf{G}}^t$ and $\mathbf{T}^\infty = \lim_{t\to\infty} \mathbf{T}^t$ such that:*

$$\hat{\mathbf{y}}^\infty = \mathbf{T}^\infty\hat{\mathbf{y}}^0 = \bar{\mathbf{G}}^\infty\hat{\mathbf{y}}^0$$

and

$$\hat{\mathbf{x}}^\infty = (1-\phi)\left(\mathbf{I} - \phi\bar{\mathbf{G}}\right)^{-1}\bar{\mathbf{G}}^\infty\hat{\mathbf{y}}^0.$$

This result about the common asymptotic properties of matrices $\mathbf{T}$ and $\bar{\mathbf{G}}$ is based on the fact that these matrices are commuting matrices, that is $\bar{\mathbf{G}}\mathbf{T} = \mathbf{T}\bar{\mathbf{G}}$.[27] Since commuting matrices have the same eigenvector associated to the maximum eigenvalue, which is 1, the convergence result holds. This result is significant because it shows that the steady state vector of ideals, defined by the matrix $\mathbf{T}$, can be analyzed solely by examining the augmented adjacency matrix $\bar{\mathbf{G}}$.

A special case fulfilling the conditions of Proposition 3 are the networks with undirected layers g and l described by symmetric matrices $\mathbf{G}^*$ and $\boldsymbol{\Lambda}$. This makes the augmented matrix $\bar{\mathbf{G}}$ symmetric and thus diagonalizable. Therefore, for symmetric matrices $\mathbf{G}^*$ and $\boldsymbol{\Lambda}$, the existence of limit ideal efforts and actions depends on the maximum row-sum norm of the matrix of weights $\boldsymbol{\Lambda}$.

Given the convergence of $\bar{\mathbf{G}}$ in Proposition 3 we can find the steady state equilibrium efforts of the players from the expression in Proposition 1 once we notice that $\mathbf{x}^\infty = \mathbf{y}^\infty$:

$$\mathbf{x}^\infty = \mathbf{y}^\infty = \frac{1}{\omega_1 + \omega_2}\left(\mathbf{I} - \frac{1}{\omega_1 + \omega_2}(\boldsymbol{\Lambda} + \omega_1\mathbf{G}^*)\right)^{-1}\boldsymbol{\theta}. \tag{10}$$

Indeed, when $\hat{\mathbf{y}}^{t+1} = \hat{\mathbf{y}}^t = \hat{\mathbf{y}}^\infty$, the equilibrium efforts are identical to the ideals, $\hat{\mathbf{y}}^\infty = \hat{\mathbf{x}}^\infty$. From equilibrium efforts in (5) in the steady state we have:

$$\hat{\mathbf{x}}^\infty = (1-\phi)\left(\mathbf{I} - \phi\bar{\mathbf{G}}\right)^{-1}\hat{\mathbf{x}}^\infty \quad \Longrightarrow \quad \hat{\mathbf{x}}^\infty = \bar{\mathbf{G}}\hat{\mathbf{x}}^\infty,$$

which indicates that the steady state effort vector is such that each agent chooses an effort that corresponds to the average of their neighbors' effort in the augmented matrix.

Recalling the definition of $\bar{\mathbf{G}}$ in 4 we find the steady state equilibrium efforts of players from the best response expression in Eq. (3):

$$\mathbf{x}^\infty = \frac{1}{\omega_1 + \omega_2}\left(\boldsymbol{\theta} + \left(\boldsymbol{\Lambda} + \omega_1\mathbf{G}^*\right)\mathbf{x}^\infty\right).$$

Thus, the long-term efforts are uniquely determined by the ex-ante productivities, the complementarities from the network l and their position in the network g.

We study now the dynamics of ideal efforts and the conditions on the network layer of strategic complementarities, the taste for conformity, the sensitivity to cognitive dissonance and players' productivity coefficients that guarantee the existence of the steady state level of ideals efforts. Assume there exists a row vector $\mathbf{e}^\intercal$, such that $y_i^\infty = \mathbf{e}^\intercal\mathbf{y}^\infty$ for all players $i \in N$. Denote $y^\infty := \mathbf{e}^\intercal\mathbf{y}^\infty$. This implies, that in the steady state the ideal efforts of the players converge to a consensus y^∞, in which case we have:

$$\mathbf{y}^\infty = \frac{1}{\omega_1 + \omega_2}\left(\boldsymbol{\theta} + \left(\boldsymbol{\Lambda} + \omega_1\right)\mathbf{y}^\infty\right),$$

$$y_i^\infty = \frac{\theta_i}{\omega_2 - \sum_{j\in N}\lambda_{ij}} \quad \text{s.t.} \quad y_i^\infty = y^\infty \quad \text{for all} \quad i \in N. \tag{11}$$

Notice that while the taste for conformity affects both the steady state effort levels and the speed of convergence to a consensus, the

[26] For a row-normalized $\bar{\mathbf{G}}$ and $\phi < 1$, the matrix $\mathbf{M} = \sum_{k=0}^{\infty}\phi^k\bar{\mathbf{G}}^k$ is a converging Neumann series with sum of elements in each row equal to $\frac{1}{1-\phi}$. Therefore, the matrix $(1-\phi)\mathbf{M}$ is row-normalized.

[27] See Step 1 in the proof of Proposition 3 in the Appendix.

sensitivity to cognitive dissonance, that captures the behavioral stickiness of the players, affects the speed of convergence to a consensus but does not affect the long-run levels of ideals efforts. Moreover, the consensus over the ideals depends on the network layer of strategic complementarities. In the following, we characterize the consensus among a certain group of players in a two-layer network such that the conditions for the consensus of the ideal efforts are guaranteed.

A consensus over ideal efforts occurs among *structurally equivalent players* (Lorrain and White, 1971), that is, the players that share the same neighborhood and similar relationship patterns in the network, and have the same productivity.[28] Such strict form of players' equivalence ensures the consensus. A more general definition of equivalent structure of player is the *regular equivalence* (Sailer, 1978; White and Reitz, 1983).[29]

Definition 1. Let $C \subset N$ be a subset of players in the network. Given the layers $\boldsymbol{g}$ and $\boldsymbol{l}$, and productivity coefficients $\boldsymbol{\theta}$, we say that the players in C are regularly equivalent if their neighborhoods are equivalent to each other, and for all $i \in C$ there exists a consensus such that $y_i^\infty = y_C^\infty$.

Unlike structurally equivalent players, regularly equivalent players do not necessarily share the same neighbors in the network, but they have similar connection patterns and are connected to similar neighbors. The neighbors, in turn, are also regularly equivalent. Using the definition of ideal efforts in the steady state in (10) and assuming there exists consensus among structurally equivalent players in C, such that $y_i^\infty = y_C^\infty =$ for all $i \in C$, the consensus ideal effort of regularly equivalent players in C is given by[30]:

$$y_C^\infty = \frac{\theta_i + \sum_{j\in N\setminus C}(\lambda_{ij} + \omega_1 g_{ij}^*)y_j^\infty}{\omega_2 - \sum_{j\in C}\lambda_{ij} + \omega_1(1 - \sum_{j\in C} g_{ij}^*)}. \tag{12}$$

The regular equivalence is associated with similarities in networks characteristics of the players, such as the neighborhood and the weights on all layers of the network, the idiosyncratic productivity parameters of the neighbors, as well as the productivity of the player herself. From Eq. (12), we observe that, among regularly equivalent players, the complementarities in network $\boldsymbol{l}$ increase the ideal efforts of players in the steady state. Moreover, while the taste for conformity affects the steady state effort levels, the sensitivity to cognitive dissonance does not affect the long-run levels of ideals efforts. Finally, we observe that the sensitivity to cognitive dissonance and the taste for conformity have opposing effects on the speed of convergence to a steady state (see the next numerical examples). The higher the sensitivity to cognitive dissonance (ω_3), the longer the players will remain consistent with their ideal efforts and the slower the convergence to the steady state. On the contrary, the higher the taste for conformity (ω_1), the higher the disutility from miscoordination in the network $\boldsymbol{g}$ and the faster the convergence to the steady state.

3.3. Numerical examples

Consider a network of $n = 10$ players connected through a social network $\boldsymbol{g}$ (Fig. 2) with normalized equal weights on the directed links, and a network of additional complementarities $\boldsymbol{l}$ (Fig. 3). In each period the players engage in the two-layer network game in (1) where the productivity of effort is $\theta_i = 1$ for all players, and the effort cost parameter is $\omega_2 = 2$. In the initial period 0, the players are endowed with ideal efforts $y_i = 0.2i$:

$$\mathbf{y}^{0\intercal} = [0.2\ 0.4\ 0.6\ 0.8\ 1\ 1.2\ 1.4\ 1.6\ 1.8\ 2].$$

After each period game the players update their ideal efforts according to Eq. (2) with the speed of updating $\xi = 0.2$. We consider an arbitrary set of combinations for parameters of conflict cost and sensitivity to cognitive dissonance with $\omega_1, \omega_3 \in \{2, 8, 15\}$.

Consider the network of social interactions in Fig. 2(a). In the absence of complementarity layer $\boldsymbol{l}$, the ideal efforts in all three components of the network $\boldsymbol{g}$ converge to the consensus $y^\infty = 0.5$ by expression (11). The layer of complementarities in the network affects the steady state level of efforts and ideals, as well as the existence of the consensus. Let us now add the layer $\boldsymbol{l}$ in Fig. 3(a) described by the following adjacency matrix:

$$\boldsymbol{\Lambda}_{(a)} = \begin{pmatrix} 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

For this and further examples, we distinguish the following sets of players that form disconnected components in both networks in Figs. 2(a) and 3(a):

$$C_1 = \{1, 2, 3, 4, 5\}, \quad C_2 = \{6, 7\}, \quad C_3 = \{8, 9, 10\}.$$

Moreover, we assume that the players in each set C_i are homogeneous in their aggregate complementarity coefficients such that

$$\sum_{j\in N} \lambda_{ij} = 0.5\ \forall i \in C_1, \quad \sum_{j\in N} \lambda_{ij} = 0.7\ \forall i \in C_2 \quad \text{and} \quad \sum_{j\in N} \lambda_{ij} = 1\ \forall i \in C_3. \tag{13}$$

It is easy to see that the players within each component $C_{i,i\in\{1,2\}}$ are regularly equivalent. The players in C_3 are regularly equivalent since player 8 receives the same aggregate complementarity from the regularly equivalent players 9 and 10. Moreover, the components are disconnected in both layers of the network (represented by Figs. 2(a) and 3(a)), thus the steady state efforts and ideals can be found using Eq. (11). The following vector shows the ideal efforts of players in the steady state:

$$\mathbf{y}^{\infty\intercal} = [0.667\ 0.667\ 0.667\ 0.667\ 0.667\ 0.769\ 0.769\ 1\ 1\ 1].$$

Unsurprisingly, we can see that, the higher the complementarities in network $\boldsymbol{l}$ (Fig. 3(a)) among a group of regularly equivalent players, the higher the ideal efforts of players of this group in the steady state. Fig. 4 shows the dynamics of the ideal efforts of the players located in the network of complementarities in Fig. 3(a) and the network of social interactions in Fig. 2(a). Notice that each small graphic in Fig. 4 represents in the vertical axis the initial and steady state ideal efforts and in the horizontal axis the time it takes for each group of regularly equivalent players to converge to the steady state, given the cost of conflict ω_1 (above the figure) and the cost of cognitive dissonance ω_3 (at the left of the figure). One can observe that the higher the cost of conflict ω_1, the faster the convergence to the steady state ideal efforts within each component C_i. On the contrary, the higher the cost of cognitive dissonance ω_3, the slower the convergence to the steady state ideal efforts within each component C_i.

Assume now a complete network of social interactions $\boldsymbol{g}$ in Fig. 2(b) combined with the network of complementarities in Fig. 3(a). The regular equivalence of players within each set C_i remains. Thus, the consensus over ideal efforts among players in C_i satisfies the expression (12). For instance, for any player $i \in C_3$, given $\omega_1 = 15$, we find:

$$y_{C_3}^\infty = \frac{\theta_i + \omega_1(\frac{5}{9}y_{C_1}^\infty + \frac{2}{9}y_{C_2}^\infty)}{\omega_2 - \sum_{j\in C_3}\lambda_{ij} + \omega_1\frac{7}{9}} = 0.778.$$

[28] Similarity occurs in network analysis when two nodes (or other structures) fall in the same equivalent class. There are three fundamental approaches for constructing measures of network similarity: structural equivalence, automorphic equivalence, and regular equivalence.

[29] Any set of structural equivalences are also regular equivalences. However, not all regular equivalences are necessarily structural.

[30] See the Appendix for the proof.

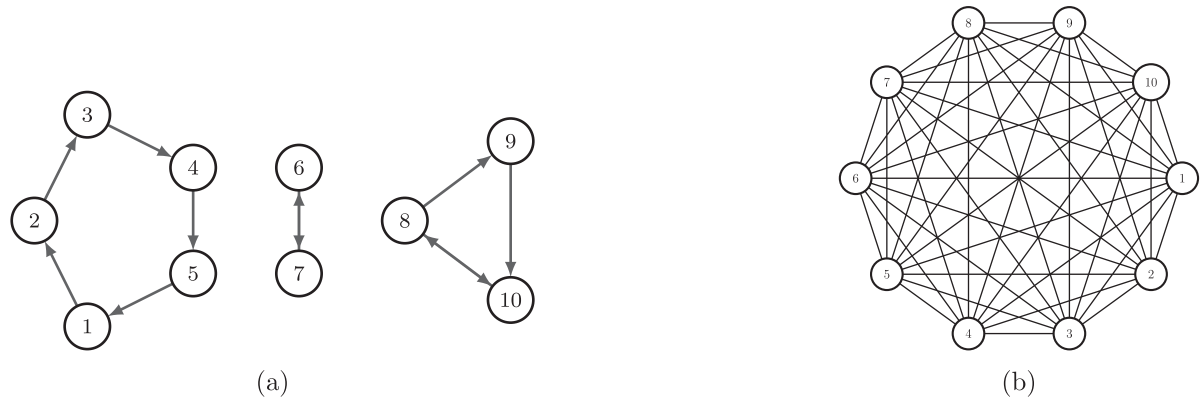

Fig. 2. Social network g.

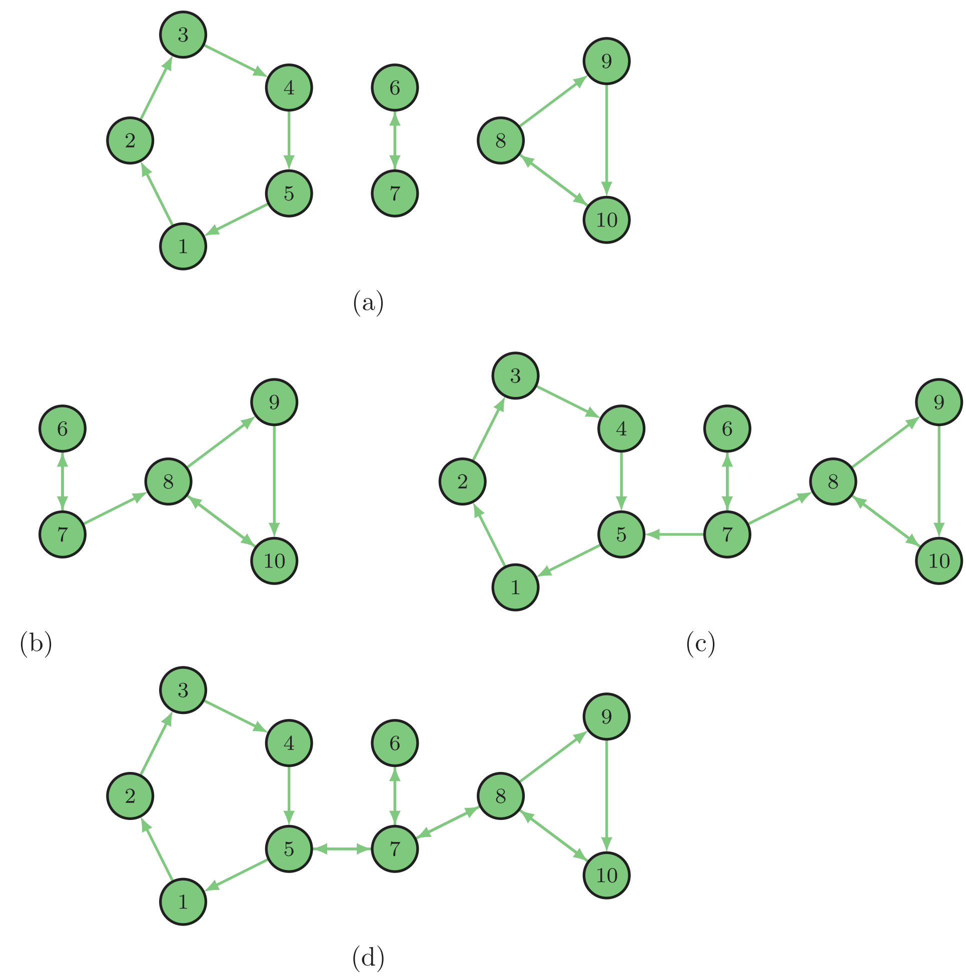

Fig. 3. Network of complementarities l.

The ideal efforts of players in the steady state, given the cost of conflict $\omega_1 = 15$, are the following:

$$\mathbf{y}^{\infty\intercal} = [0.756\ 0.756\ 0.756\ 0.756\ 0.756\ 0.765\ 0.765\ 0.778\ 0.778\ 0.778].$$

The dynamics of ideal effort with the complete network g in Fig. 2(b) are displayed in Fig. 5. Comparing the dynamics of ideal efforts in Figs. 4 and 5, we can see that the variance between the steady state ideals of the three different disconnected components in the network of complementarities in Fig. 3(a) is lower in the complete network of social interactions represented in Fig. 2(b) than in the network of social interactions represented in Fig. 2(a). This result seems to indicate that a more dense and equally connected layer of social interactions among the players will reduce the variance between the steady state ideals of the different regularly equivalent sets of players.

For the rest of the examples below, we fix the social network g depicted in Fig. 2(a). Further, let us consider the layer of effort complementarities l in Fig. 3(b). We assume that the aggregate of the complementarity coefficients of players do not change by adding a new link. Thus, by adding a connection in the network layer l, we redistribute the complementarity coefficients of outgoing links maintaining

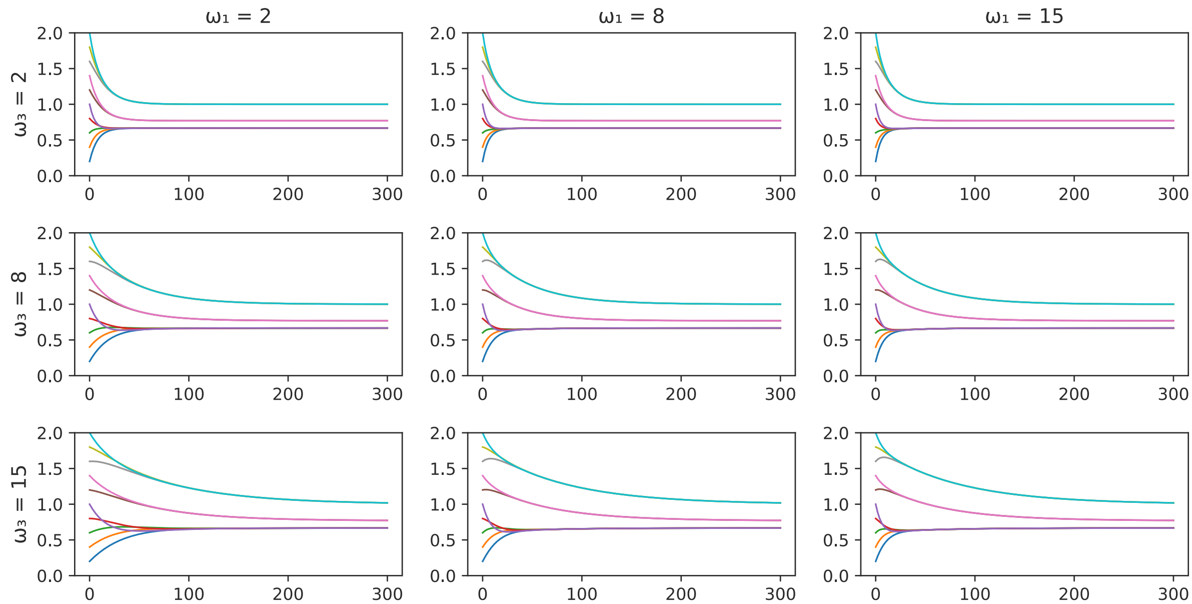

Fig. 4. Dynamics of ideal efforts in the network of social interactions in Fig. 2(a) and the network of complementarities in Fig. 3(a).

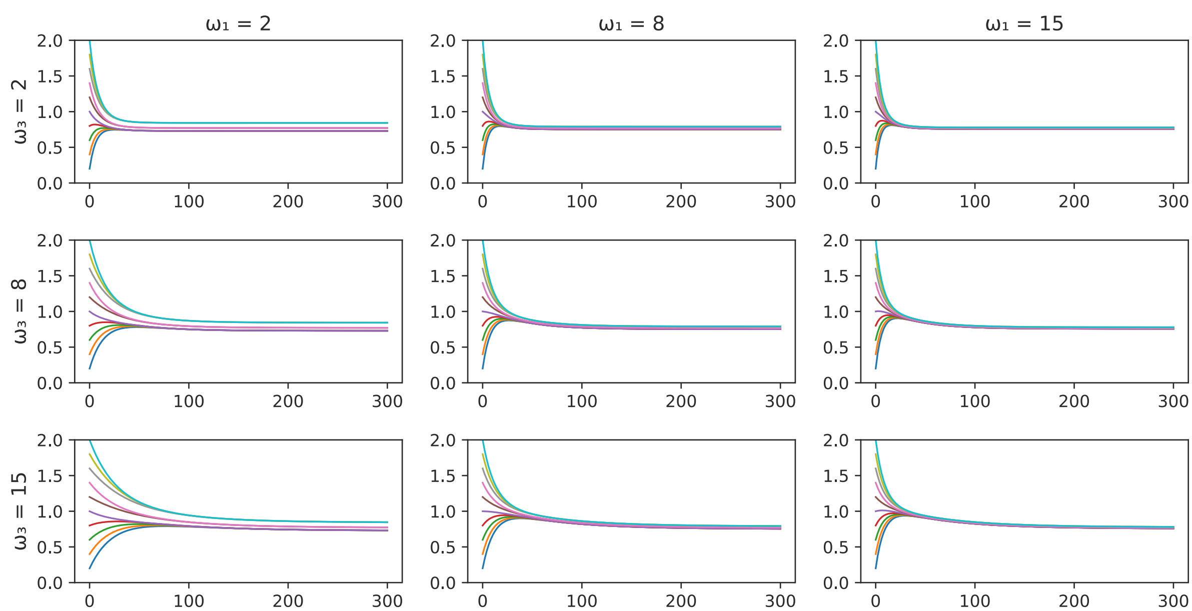

Fig. 5. Dynamics of ideal efforts in the complete network of social interactions in Fig. 2(b) and the network of complementarities in Fig. 3(a).

the characteristics described in (13). By doing so, we can analyze the effect of the change in the structure of the layer $\boldsymbol{l}$ without increasing the complementarity levels in the network. Assume, the layer $\boldsymbol{l}$ is described by the following matrix of complementarity coefficients:

$$\boldsymbol{\Lambda}_{(b)} = \begin{pmatrix} 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

The additional link $7 \to 8$ in $\boldsymbol{l}$ makes the player 7 influenced by the component C_3 driving the ideal efforts of players 6 and 7 closer to $y^{\infty}_{C_3}$. Unlike the case with $\boldsymbol{l}$ in Fig. 3(a), players 6 and 7 are not regularly equivalent when the layer of effort complementarities $\boldsymbol{l}$ is given in Fig. 3(b). Then, as shown in Table 1, the consensus over ideal effort among the players in C_2 is not achieved.

Let us now consider the layer of effort complementarities $\boldsymbol{l}$ in Fig. 3(c) where we add a complementarity link connecting player 7 to player 5 without changing the aggregate of the complementarity coefficients of players in (13). In this case, the component C_2 is then additionally influenced by the efforts in the component C_1. Assume the link $7 \to 5$ with $\lambda_{75} = 0.2$ in the matrix of weights $\boldsymbol{\Lambda}_{(c)}$ describing the layer in Fig. 3(c):

$$\boldsymbol{\Lambda}_{(c)} = \begin{pmatrix} 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0.3 & 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

As shown in Table 1, the steady state consensus of ideal effort in the component C_1 is 0.667 and is lower than that of player 6 and component C_3 in the previous example where the layer of effort complementarities $\boldsymbol{l}$ was given in Fig. 3(b). Receiving complementarity from the lower-effort component C_1, the ideal efforts of player 7 decreases. At the same time, the pressure to conform in layer $\boldsymbol{g}$ forces player 6 to adjust her effort according to the new social norm.

Table 1
Steady state consensus over ideal efforts.

		y_1^∞	y_2^∞	y_3^∞	y_4^∞	y_5^∞	y_6^∞	y_7^∞	y_8^∞	y_9^∞	y_{10}^∞
	$\boldsymbol{\Lambda}_{(a)}$	0.667	0.667	0.667	0.667	0.667	0.769	0.769	1	1	1
$\omega_1 = 2$	$\boldsymbol{\Lambda}_{(b)}$	0.667	0.667	0.667	0.667	0.667	0.783	0.79	1	1	1
	$\boldsymbol{\Lambda}_{(c)}$	0.667	0.667	0.667	0.667	0.667	0.777	0.78	1	1	1
	$\boldsymbol{\Lambda}_{(d)}$	0.668	0.669	0.67	0.671	0.673	0.776	0.779	0.98	0.989	0.985
$\omega_1 = 8$	$\boldsymbol{\Lambda}_{(b)}$	0.667	0.667	0.667	0.667	0.667	0.785	0.787	1	1	1
	$\boldsymbol{\Lambda}_{(c)}$	0.667	0.667	0.667	0.667	0.667	0.778	0.779	1	1	1
	$\boldsymbol{\Lambda}_{(d)}$	0.6687	0.669	0.6694	0.667	0.671	0.777	0.778	0.983	0.986	0.984
$\omega_1 = 15$	$\boldsymbol{\Lambda}_{(b)}$	0.667	0.667	0.667	0.667	0.667	0.786	0.787	1	1	1
	$\boldsymbol{\Lambda}_{(c)}$	0.667	0.667	0.667	0.667	0.667	0.778	0.779	1	1	1
	$\boldsymbol{\Lambda}_{(d)}$	0.669	0.6692	0.6695	0.6698	0.67	0.777	0.778	0.983	0.985	0.984

Consider now the layer of effort complementarities $\boldsymbol{l}$ in Fig. 3(d) where players 5 and 8 reciprocate player 7 with complementarity links, without changing the aggregate of the complementarity coefficients of players in (13). The matrix of weights is the following:

$$\boldsymbol{\Lambda}_{(d)} = \begin{pmatrix} 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0.3 & 0 & 0 & 0 & 0 & 0 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0.3 & 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.2 & 0 & 0.4 & 0.4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

The new links make the network $\boldsymbol{l}$ connected. There is no regular equivalence between players anymore, which results in all players having different steady state ideal efforts in Table 1. The ideal efforts of the players in C_1 are positively affected by the ideal efforts in the rest of the network. This positive effect is weaker for players that are further from player 5. On the contrary, the ideal efforts of the players in C_3 are negatively affected by the ideal efforts in the rest of the network.

The steady state ideal efforts in previous examples are summarized in Table 1.

Several conclusions could then be derived from the numerical examples. First, the higher the complementarities in network $\boldsymbol{l}$ among a group of regularly equivalent players, the higher their the ideal efforts in the steady state. Second, while the sensitivity to cognitive dissonance ω_3 does not affect the steady state levels of ideals efforts, an increase in the cost of cognitive dissonance slows down the emergence of the consensus among players in the steady state. Third, an increase in the taste for conformity ω_1, accelerates the consensus over the ideals among players connected in the layer of social influences $\boldsymbol{g}$. Fourth, the variance between the steady state ideals of different groups of regularly equivalent players is lower in a more dense and connected network of social interactions $\boldsymbol{g}$ than in a less dense and disconnected network of social influences.

4. Conclusion

In this paper, we have proposed a network game model with multi-dimensionality of network relations, where people repeatedly choose their efforts and update their ideal beliefs based on their current choices. Specifically, we study a model with a two-layer network where the first layer dictates the social norm, and the second is a network of interactions with strategic complementarities in efforts. The players in the network are initially endowed with heterogeneous ideal efforts and differ in their productivity. In line with the network game literature, we find that the pure Nash equilibrium of the game is proportional to Katz–Bonacich centrality in the combined network of the two layers. Additionally, we find the steady state equilibrium efforts and ideals. We derive the conditions for the existence of consensus about ideal efforts in the whole population and among structurally and regularly equivalent players.

The dynamics of ideal efforts in our model show that the sensitivity to cognitive dissonance and the taste for conformity have opposing effects on the speed of convergence to a consensus and the steady state. The sensitivity to cognitive dissonance, that captures the behavioral stickiness of the players, slows down the speed of convergence to a steady state but does not change the long-run efforts. The taste for conformity, on the other hand, affects the steady state efforts' levels and accelerates the consensus over the ideals. Consideration of possible heterogeneity in the cost of the cognitive dissonance, as well as the taste for conformity, is left for future research.

We show that the multidimensionality of network interactions may change the overall effort levels and the beliefs in society. While focusing on the network layer with strategic complementarities we leave the consideration of other types of relations for future research. In particular, the direct extension of the model provided in this paper is the consideration of strategic substitutability along with complementarities on the same network layer. In the paper, we have assumed a given two-layer network. Endogenizing the formation of the network layer with complementarities is left for future research.[31]

CRediT authorship contribution statement

Ana Mauleon: Writing – review & editing, Writing – original draft, Methodology, Investigation, Formal analysis, Conceptualization. **Mariam Nanumyan:** Writing – review & editing, Writing – original draft, Methodology, Investigation, Formal analysis, Conceptualization. **Vincent Vannetelbosch:** Writing – review & editing, Writing – original draft, Methodology, Investigation, Funding acquisition, Formal analysis, Conceptualization.

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Appendix

Proof of Proposition 1. Given x_i^* is the optimal effort choice for i

$$\frac{\partial u_i^t}{\partial x_i^t}(x_i^{t*}) \equiv 0$$

$$\frac{\partial u_i^t(x^t)}{\partial x_i^t} = \theta_i + \sum_{j\in N\setminus i} \lambda_{ij} x_j^t - \omega_1 \sum_{j\in N} g_{ij}^*(x_i^t - x_j^t) - \omega_2 x_i^t - \omega_3(x_i^t - y_i^t) =$$

$$= \theta_i + \sum_{j\in N\setminus i} \lambda_{ij} x_j^t - \omega_1 x_i^t \underbrace{\sum_{j\in N} g_{ij}^*}_{=1} + \omega_1 \sum_{j\in N} g_{ij}^* x_j^t - \omega_2 x_i^t - \omega_3 x_i^t + \omega_3 y_i^t =$$

[31] Mauleon and Vannetelbosch (2016) provide a comprehensive introduction to network formation games.

$$= (\theta_i + \omega_3 y_i^t) + \sum_{j\in N\setminus i} \lambda_{ij} x_j^t + \omega_1 \sum_{j\in N} g_{ij}^* x_j^t - (\omega_1+\omega_2+\omega_3) x_i^t$$
$$= (\theta_i + \omega_3 y_i^t) + \sum_{j\in N\setminus i} (\lambda_{ij} + \omega_1 g_{ij}^*) x_j^t - (\omega_1+\omega_2+\omega_3) x_i^t$$

Thus the equilibrium effort choice of i is:

$$x_i^{t*} = \frac{\theta_i + \omega_3 y_i^t}{(\omega_1+\omega_2+\omega_3)} + \sum_{j\in N\setminus i} \frac{\lambda_{ij} + \omega_1 g_{ij}^*}{(\omega_1+\omega_2+\omega_3)} x_j^t,$$

for all $i \in N$, or:

$$x_i^{t*} = \frac{\theta_i + \omega_3 y_i^t}{(\omega_1+\omega_2+\omega_3)} + \frac{1}{(\omega_1+\omega_2+\omega_3)} \sum_{j\in N\setminus i} \lambda_{ij} x_j^t + \frac{\omega_1}{(\omega_1+\omega_2+\omega_3)} \sum_{j\in N} g_{ij}^* x_j^t,$$

We can rewrite it in a matrix form:

$$\mathbf{x}^t = \frac{1}{(\omega_1+\omega_2+\omega_3)}\boldsymbol{\theta} + \frac{\omega_3}{(\omega_1+\omega_2+\omega_3)}\mathbf{y}^t + \frac{1}{(\omega_1+\omega_2+\omega_3)}\boldsymbol{\Lambda}\mathbf{x}^t + \frac{\omega_1}{(\omega_1+\omega_2+\omega_3)}\mathbf{G}^*\mathbf{x}^t,$$

So the matrix form solution of the equilibrium efforts is the following:

$$\left(\mathbf{I} - \frac{1}{(\omega_1+\omega_2+\omega_3)}(\boldsymbol{\Lambda} + \omega_1 \mathbf{G}^*)\right)\mathbf{x}^t = \frac{1}{(\omega_1+\omega_2+\omega_3)}(\boldsymbol{\theta} + \omega_3 \mathbf{y}^t)$$

And thus the vector of equilibrium efforts is:

$$\mathbf{x}^t = \frac{1}{(\omega_1+\omega_2+\omega_3)}\left(\mathbf{I} - \frac{1}{(\omega_1+\omega_2+\omega_3)}(\boldsymbol{\Lambda} + \omega_1 \mathbf{G}^*)\right)^{-1}(\boldsymbol{\theta} + \omega_3 \mathbf{y}^t) \quad \square$$

Proof of Lemma 1. In order to ensure the condition of Proposition 2 on the spectral radius δ_1 of the matrix $\bar{\mathbf{G}}$ in 4, we use maximum row-sum and column-sum norms of the matrix, $|||\bar{\mathbf{G}}|||_\infty$ and $|||\bar{\mathbf{G}}|||_1$, to find an upper bound for the spectral radius.[32] We get that the row-sum for each of the first n rows of the matrix $\bar{\mathbf{G}}$ is equal to 1. While for the consecutive n rows, the sum of row elements of $(n+i)$th row is equal to $\frac{1+\omega_1+\sum_{j\in N}\lambda_{ij}}{\omega_1+\omega_2}$, for each $i \in N$. As a result, the maximum row-sum norm of the matrix is the maximum of given values:

$$|||\bar{\mathbf{G}}|||_\infty = \max\left\{1, \frac{1}{\omega_1+\omega_2}(1+\omega_1+\overline{\omega}_2)\right\}$$

Therefore, from Gershgorin circle theorem we obtain that 1 is one of the eigenvalues of the matrix $\bar{\mathbf{G}}$. Similarly, the sum for each of the first n column elements of the matrix $\bar{\mathbf{G}}$ is $\frac{1+\omega_1+\omega_2}{\omega_1+\omega_2}$, and the sum of column elements for columns $n+j$ for each $j \in N$ is $\frac{\sum_{i\in N\setminus j}(\lambda_{ij}+\omega_1 g_{ij}^\star)}{\omega_1+\omega_2}$. Thus, the maximum column-sum norm of the matrix is given by:

$$|||\bar{\mathbf{G}}|||_1 = \max\left\{1 + \frac{1}{\omega_1+\omega_2}, \frac{1}{\omega_1+\omega_2}\max_j \sum_{i\in N\setminus j}(\lambda_{ij} + \omega_1 g_{ij}^\star)\right\}.$$

Now we look for a better proxy for the upper bound $UB(\delta_1)$ of the spectral radius by finding $\min\{|||\bar{\mathbf{G}}|||_\infty, |||\bar{\mathbf{G}}|||_1\}$ for all parameter values that satisfy $\omega_2 \geq \overline{\omega}_2$. Notice that $|||\bar{\mathbf{G}}|||_1$ refines the upper bound for δ_1 when $\omega_2 < \overline{\omega}_2$.

Case 1: Assume $|||\bar{\mathbf{G}}|||_\infty = 1$. Thus $1 \geq \frac{1+\omega_1+\overline{\omega}_2}{\omega_1+\omega_2}$, and $\omega_2 \geq 1 + \overline{\omega}_2$.

(a) If $|||\bar{\mathbf{G}}|||_1 = 1 + \frac{1}{\omega_1+\omega_2} > 1$.

(b) If $|||\bar{\mathbf{G}}|||_1 = \frac{\max_j \sum_{i\in N\setminus j}(\lambda_{ij}+\omega_1 g_{ij}^\star)}{\omega_1+\omega_2} \geq 1 + \frac{1}{\omega_1+\omega_2} > 1$.

Case 2: Assume $|||\bar{\mathbf{G}}|||_\infty = \frac{1+\omega_1+\overline{\omega}_2}{\omega_1+\omega_2}$. This means that $\frac{1+\omega_1+\overline{\omega}_2}{\omega_1+\omega_2} > 1$. Then, we have $\omega_2 < 1 + \overline{\omega}_2$.

(a) If $|||\bar{\mathbf{G}}|||_1 = 1 + \frac{1}{\omega_1+\omega_2}$. Then, $\omega_2 \geq \overline{\overline{\omega}}_2$, and $1 + \frac{1}{\omega_1+\omega_2} \geq \frac{1+\omega_1+\overline{\omega}_2}{\omega_1+\omega_2}$ only when $\omega_2 \geq \overline{\omega}_2$.

(b) If $|||\bar{\mathbf{G}}|||_1 = \frac{\max_j \sum_{i\in N\setminus j}(\lambda_{ij}+\omega_1 g_{ij})}{\omega_1+\omega_2} \geq 1 + \frac{1}{\omega_1+\omega_2}$. Then, $\omega_2 < \overline{\overline{\omega}}_2$. Thus, $|||\bar{\mathbf{G}}|||_\infty$ is the upper bound for the spectral radius if $\frac{\max_j \sum_{i\in N\setminus j}(\lambda_{ij}+\omega_1 g_{ij}^\star)}{\omega_1+\omega_2} \geq \frac{1+\omega_1+\overline{\omega}_2}{\omega_1+\omega_2}$.

To sum up, the upper bound $UB(\delta_1)$ of the spectral radius is given by:

$$\begin{array}{llllll}
UB(\delta_1) & = |||\bar{\mathbf{G}}|||_\infty & = 1 & \text{when} & \omega_2 \geq \overline{\omega}_2 + 1. \\
UB(\delta_1) & = |||\bar{\mathbf{G}}|||_\infty & = \frac{1+\omega_1+\overline{\omega}_2}{\omega_1+\omega_2} & \text{when} & \omega_2 \in [\overline{\omega}_2; \overline{\omega}_2 + 1). \\
UB(\delta_1) & = |||\bar{\mathbf{G}}|||_1 & = 1 + \frac{1}{\omega_1+\omega_2} & \text{when} & \omega_2 \in [\overline{\overline{\omega}}_2; \overline{\omega}_2) \\
 & & & \text{and} & \overline{\overline{\omega}}_2 < \overline{\omega}_2. \\
UB(\delta_1) & = |||\bar{\mathbf{G}}|||_1 & = \frac{\overline{\overline{\omega}}_2+\omega_1+1}{\omega_1+\omega_2} & \text{when} & \omega_2 < \overline{\overline{\omega}}_2 \\
 & & & \text{and} & \overline{\overline{\omega}}_2 < \overline{\omega}_2. \\
UB(\delta_1) & = |||\bar{\mathbf{G}}|||_\infty & = \frac{1+\omega_1+\overline{\omega}_2}{\omega_1+\omega_2} & \text{when} & \omega_2 < \overline{\overline{\omega}}_2 \\
 & & & \text{and} & \overline{\overline{\omega}}_2 \geq \overline{\omega}_2.
\end{array}$$

For $\delta_1 < \frac{\omega_1+\omega_2+\omega_3}{\omega_1+\omega_2}$ it is sufficient that $UB(\delta_1) < \frac{\omega_1+\omega_2+\omega_3}{\omega_1+\omega_2}$. Thus:

- For $UB(\delta_1) = |||\bar{\mathbf{G}}|||_\infty = 1$, then $1 < \frac{\omega_1+\omega_2+\omega_3}{\omega_1+\omega_2}$ when $\omega_3 > 0$ (Area V).
- For $UB(\delta_1) = |||\bar{\mathbf{G}}|||_\infty = \frac{1+\omega_1+\overline{\omega}_2}{\omega_1+\omega_2}$, then $\frac{1+\omega_1+\overline{\omega}_2}{\omega_1+\omega_2} < \frac{\omega_1+\omega_2+\omega_3}{\omega_1+\omega_2}$ when $\omega_3 > 1 + \overline{\omega}_2 - \omega_2$ (Areas III and IV).
- For $UB(\delta_1) = |||\bar{\mathbf{G}}|||_1 = 1 + \frac{1}{\omega_1+\omega_2}$, then $1 + \frac{1}{\omega_1+\omega_2} < \frac{\omega_1+\omega_2+\omega_3}{\omega_1+\omega_2}$ when $\omega_3 > 1$ (Area II).
- For $UB(\delta_1) = |||\bar{\mathbf{G}}|||_1 = \frac{\overline{\overline{\omega}}_2+\omega_1+1}{\omega_1+\omega_2}$, then $\frac{\overline{\overline{\omega}}_2+\omega_1+1}{\omega_1+\omega_2} < \frac{\omega_1+\omega_2+\omega_3}{\omega_1+\omega_2}$ when $\omega_3 > \overline{\overline{\omega}}_2 + 1 - \omega_2$ (Area I). $\square$

Proof of Proposition 3. Similarly to Proposition 1 in Olcina et al. (2024) we prove the proposition in two steps.

Step 1. $\bar{\mathbf{G}}$ and $\mathbf{T}$ are commuting matrices: $\bar{\mathbf{G}}\mathbf{T} = \mathbf{T}\bar{\mathbf{G}}$.
Recalling the notations in (7) and (8) we have that:

$$\bar{\mathbf{G}}\mathbf{T} = \xi(1-\phi)\bar{\mathbf{G}}\mathbf{M} + (1-\xi)\bar{\mathbf{G}}.$$

To show that $\bar{\mathbf{G}}$ and $\mathbf{M}$ are commuting matrices we first show that $\bar{\mathbf{G}}$ and $\mathbf{M}^{-1}$ commute.

$$\bar{\mathbf{G}}\mathbf{M}^{-1} = \bar{\mathbf{G}}\left(\mathbf{I} - \phi\bar{\mathbf{G}}\right) = \left(\mathbf{I} - \phi\bar{\mathbf{G}}\right)\bar{\mathbf{G}} = \mathbf{M}^{-1}\bar{\mathbf{G}}.$$

Multiplying the above equality by $\mathbf{M}$ from left and right we have that $\bar{\mathbf{G}}\mathbf{M} = \mathbf{M}\bar{\mathbf{G}}$. From which it directly follows that $\bar{\mathbf{G}}\mathbf{T} = \mathbf{T}\bar{\mathbf{G}}$.

Step 2. From diagonalizability of $\bar{\mathbf{G}}$ we can infer that $\mathbf{T}$ is also diagonalizable given that $\mathbf{M}$ is a polynomial of $\bar{\mathbf{G}}$, and $\mathbf{T}$ is a linear convex combination of $\mathbf{M}$ and $\mathbf{I}$. As commuting matrices, $\bar{\mathbf{G}}$ and $\mathbf{T}$ share the same set of eigenvectors, thus also the eigenvector associated with the eigenvalue equal to 1. We can rewrite the two matrices as

$$\bar{\mathbf{G}} = \mathbf{E}\boldsymbol{\Delta}\mathbf{E}^{-1} \quad \text{and} \quad \mathbf{T} = \mathbf{E}\boldsymbol{\Sigma}\mathbf{E}^{-1},$$

where $\mathbf{E}$ is the matrix of eigenvectors of $\bar{\mathbf{G}}$ and $\mathbf{T}$ as columns, $\boldsymbol{\Delta}$ is the diagonal matrix of eigenvalues of $\bar{\mathbf{G}}$, and $\boldsymbol{\Sigma}$ is the diagonal matrix of eigenvalues of $\mathbf{T}$.[33] The eigenvalues in $\boldsymbol{\Delta}$ and $\boldsymbol{\Sigma}$ are ordered in the decreasing manner, with the order of eigenvectors in $\mathbf{E}$ corresponding to associated eigenvalues. From the above-mentioned it directly follows that:

$$\bar{\mathbf{G}}^t = \mathbf{E}\boldsymbol{\Delta}^t\mathbf{E}^{-1} \quad \text{thus} \quad \bar{\mathbf{G}}^\infty = \mathbf{E}\boldsymbol{\Delta}^\infty\mathbf{E}^{-1} \quad \text{with} \quad \boldsymbol{\Delta}^\infty = \lim_{t\to\infty}\boldsymbol{\Delta}^t$$

and

$$\mathbf{T}^t = \mathbf{E}\boldsymbol{\Sigma}^t\mathbf{E}^{-1} \quad \text{thus} \quad \mathbf{T}^\infty = \mathbf{E}\boldsymbol{\Sigma}^\infty\mathbf{E}^{-1} \quad \text{with} \quad \boldsymbol{\Sigma}^\infty = \lim_{t\to\infty}\boldsymbol{\Sigma}^t.$$

[32] This follows common linear algebra results, e.g see Theorem 5.6.9 in Horn and Johnson (2012).

[33] Since $\bar{\mathbf{G}}$ is diagonalizable, it has $2n$ eigenvectors.

The condition $\overline{\omega}_2 \leq \omega_2 - 1$ ensures that $\delta_1 = 1$.[34] From Lemma 2 that establishes the relationship between the corresponding eigenvalues of the two matrices $\bar{\mathbf{G}}$ and $\mathbf{T}$, we get that $\tau_1 = 1$. For any other eigenvalue $\delta_i < 1$ of $\bar{\mathbf{G}}$, we have that $\tau_i < 1$. It follows that

$$\boldsymbol{\Delta}^{\infty} = \boldsymbol{\Sigma}^{\infty} \qquad \text{and} \qquad \bar{\mathbf{G}}^{\infty} = \mathbf{T}^{\infty}.$$

Using the results above in (9) we find $\hat{\mathbf{y}}^{\infty}$. Plugging it into the equilibrium efforts in (5) we prove the second part of the proposition. □

Lemma 2. *Consider the distinct and ordered sets of eigenvalues δ_i of $\bar{\mathbf{G}}$, and τ_i of $\mathbf{T}$. Then for all $i \in [1, n]$:*

$$\tau_i = \xi \frac{1-\phi}{1-\phi\delta_i} + (1-\xi).$$

Proof of Lemma 2.

Notice the following:

$$\bar{\mathbf{G}} = \mathbf{E}\boldsymbol{\Delta}\mathbf{E}^{-1} \implies \boldsymbol{\Delta} = \mathbf{E}^{-1}\bar{\mathbf{G}}\mathbf{E}$$
$$\mathbf{T} = \mathbf{E}\boldsymbol{\Sigma}\mathbf{E}^{-1} \implies \boldsymbol{\Sigma} = \mathbf{E}^{-1}\mathbf{T}\mathbf{E}$$

multiplying $\mathbf{T}$ in (8) by $\mathbf{E}^{-1}$ from the left and by $\mathbf{E}$ from the right we get:

$$\begin{aligned}
\mathbf{E}^{-1}\mathbf{T}\mathbf{E} &= \mathbf{E}^{-1}\left(\xi(1-\phi)\mathbf{M} + (1-\xi)\mathbf{I}\right)\mathbf{E}\\
\boldsymbol{\Sigma} &= \xi(1-\phi)\mathbf{E}^{-1}\mathbf{M}\mathbf{E} + (1-\xi)\mathbf{I}\\
&= \xi(1-\phi)\mathbf{E}^{-1}\left(\mathbf{I}-\phi\bar{\mathbf{G}}\right)^{-1}\mathbf{E} + (1-\xi)\mathbf{I}\\
&= \xi(1-\phi)\left(\mathbf{E}^{-1}(\mathbf{I}-\phi\bar{\mathbf{G}})\mathbf{E}\right)^{-1} + (1-\xi)\mathbf{I}\\
&= \xi(1-\phi)\left(\mathbf{I}-\phi\mathbf{E}^{-1}\bar{\mathbf{G}}\mathbf{E}\right)^{-1} + (1-\xi)\mathbf{I}\\
&= \xi(1-\phi)\left(\mathbf{I}-\phi\boldsymbol{\Delta}\right)^{-1} + (1-\xi)\mathbf{I}.
\end{aligned}$$

We find the vector of eigenvalues $\boldsymbol{\tau}$ of $\mathbf{T}$ by multiplying the expression for Σ by a vector of ones, $\mathbf{1}$, such that $\boldsymbol{\tau} = P(\boldsymbol{\delta})$. Since

$$(\mathbf{I}-\phi\boldsymbol{\Delta})^{-1} = \sum_{t=0}^{\infty}\phi^t\boldsymbol{\Delta}^t,$$

we get:

$$\begin{aligned}
\boldsymbol{\Sigma}\mathbf{1} &= \xi(1-\phi)\left(\mathbf{I}-\phi\boldsymbol{\Delta}\right)^{-1}\mathbf{1} + (1-\xi)\mathbf{I}\mathbf{1},\\
\boldsymbol{\tau} &= \xi(1-\phi)\sum_{t=0}^{\infty}\phi^t\boldsymbol{\Delta}^t\mathbf{1} + (1-\xi)\mathbf{1}.
\end{aligned}$$

Then, for each distinct eigenvalue we get:

$$\begin{aligned}
\tau_i &= \xi(1-\phi)\sum_{t=0}^{\infty}\phi^t\delta_i^t + (1-\xi),\\
&= \xi\frac{1-\phi}{1-\phi\delta_i} + (1-\xi).
\end{aligned}$$

Remember that $\phi = \dfrac{\omega_1+\omega_2}{\omega_1+\omega_2+\omega_3}$ and from the condition in Proposition 2 we have that $\delta_1 < \dfrac{\omega_1+\omega_2+\omega_3}{\omega_1+\omega_2}$ which implies that $\phi\delta_i < 1$. Notice that when $\delta_i < 1$ then $\tau_i < 1$, as a convex combination of 1 and a value below 1, for all $i \in [1, n]$. Additionally, this is always true when $\omega_3 = 0$. □

Proof of Eq. (12). Using the definition of ideal efforts in the steady state in (10) and assuming there exists consensus among structurally equivalent players in C, such that $y_i^{\infty} = y_C^{\infty} =$ for all $i \in C$, we can find the consensus ideal for players in C.

$$\left(\mathbf{I} - \frac{1}{\omega_1+\omega_2}(\boldsymbol{\Lambda} + \omega_1\mathbf{G}^*)\right)\mathbf{y}^{\infty} = \frac{1}{\omega_1+\omega_2}\boldsymbol{\theta}$$

For player $i \in C$ with $y_i^{\infty} = y_C^{\infty}$ we have:

$$\begin{aligned}
&y_C^{\infty} - \frac{1}{\omega_1+\omega_2}\left(\sum_{j\in N\setminus C}\lambda_{ij}y_j^{\infty} + \underbrace{\sum_{j\in C}\lambda_{ij}y_j^{\infty}}_{y_C^{\infty}\sum_{j\in C}\lambda_{ij}} + \omega_1\sum_{j\in N\setminus C}g_{ij}^*y_j^{\infty} + \omega_1\underbrace{\sum_{j\in C}g_{ij}^*y_j^{\infty}}_{y_C^{\infty}\sum_{j\in C}g_{ij}^*}\right)\\
&= \frac{1}{\omega_1+\omega_2}\theta_i\\
&y_C^{\infty} - \frac{1}{\omega_1+\omega_2}\left(y_C^{\infty}\Big(\sum_{j\in C}\lambda_{ij} + \omega_1\sum_{j\in C}g_{ij}^*\Big) + \sum_{j\in N\setminus C}(\lambda_{ij}+\omega_1 g_{ij}^*)y_j^{\infty}\right)\\
&= \frac{1}{\omega_1+\omega_2}\theta_i\\
&y_C^{\infty}(\omega_1+\omega_2) - y_C^{\infty}\Big(\sum_{j\in C}\lambda_{ij} + \omega_1\sum_{j\in C}g_{ij}^*\Big) - \sum_{j\in N\setminus C}(\lambda_{ij}+\omega_1 g_{ij}^*)y_j^{\infty}\\
&= \theta_i\\
&y_C^{\infty}\left(\omega_2 - \sum_{j\in C}\lambda_{ij} + \omega_1\Big(1-\sum_{j\in C}g_{ij}^*\Big)\right) = \sum_{j\in N\setminus C}(\lambda_{ij}+\omega_1 g_{ij}^*)y_j^{\infty} + \theta_i
\end{aligned}$$

Thus, the consensus effort of regularly equivalent group of players is given by (12). □

Data availability

No data was used for the research described in the article.

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[34] Recall that, by construction of the matrix in 4, 1 is always an eigenvalue of $\bar{\mathbf{G}}$.

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