VALUATION OF MIXED LIFE INSURANCE CONTRACTS UNDER STOCHASTIC CORRELATED MORTALITY AND INTEREST RATES

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REPRINT | 2023 / 15

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https://uclouvain.be/en/research-institutes/lidam/isba/publication.html

European Actuarial Journal
https://doi.org/10.1007/s13385-023-00354-4

ORIGINAL RESEARCH PAPER

Valuation of mixed life insurance contracts under stochastic correlated mortality and interest rates

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Received: 9 September 2022 / Revised: 7 February 2023 / Accepted: 2 June 2023

Abstract
The need to find a good balance between attractive returns and long-term guarantees has motivated the development of hybrid life insurance products. In these contracts, the premium payments are distributed between a participating and a unit-linked fund, where an additional guarantee fee is applied to the unit-linked return in order to increase the investment guarantee of the participating fund. Moreover, the traditional pricing approach in life insurance is based on models for the financial elements and the mortality elements without any correlations between them. However, recent trends, such as the recent COVID-19 pandemic or the effect of ageing on stock market preferences, motivate us to account for the correlation between changes in demographic trends and the value of financial assets. The purpose of this paper is to price the mixed insurance contract based on longevity and to propose a multi-dimensional approach to explicitly studying the dependence between financial and mortality risks in a joint stochastic continuous time model of interest rates, stock returns, and mortality.

Keywords Mixed insurance contracts · Stochastic interest rates · Stochastic mortality · Dependence finance-mortality · Investment guarantee · Change of measure

1 Introduction

It is well known that the life insurance sector is facing a major challenge in combining a reasonable balance between the desire for potential returns and the ability to generate long-term guarantees in this difficult financial environment. Various insurance

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markets tend to remove all forms of guarantees and move to new types of life insurance products with no investment guarantees, which effectively reduce the insurer's risk and transfer it to the policyholder. In general, the traditional form of insurance products, so-called participating life insurance contracts, in which the insurer guarantees a return with a possible bonus depending on the investment performance, have been replaced by so-called unit-linked life insurance contracts that allow the client to opt for investment funds with no guaranteed return, according to an investment strategy adapted to his risk profile. A main difference between the two products is the investment strategy, as no financial decision can be made by the customer in the traditional product because the participating contract fund is internal and controlled by the insurer, whereas the unit-linked fund is based on the investor's choice. Nonetheless, the customer's interest in downside protection is highly present, and removing guarantees entirely is not an appealing trend from the client's perspective, as many empirical and experimental studies have shown; see, e.g., Gatzert et al. [17] and Ruß and Schelling [26]. Based on behavioral models, products with guarantees are a preferred choice for investor (Døskeland and Nordahl [15]). Therefore, the development of attractive life and pension insurance product designs has become necessary but, at the same time, very difficult due to demographic changes and volatile equity and interest rate markets.

In this context, hybrid products can be a good solution for both the insurance industry and the customer. For the insurer, it is less risky than the participating product, for which the insurance company bears the entire risk, and for the customer, it is more attractive than the pure unit-linked product that does not offer any investment guarantees (Reuß et al. [25]). In particular, dynamic hybrid products are innovative life insurance products that provide guarantees through a periodic rebalancing process between the policy reserves, a guarantee fund, and an equity fund. Bohnert et al. [7] shows that in the case of dynamic hybrid contracts, a minimum guarantee level should be provided to ensure that the willingness-to-pay is greater than the minimum premium that the insurer must charge to sell the contract. Moreover, a study was conducted by Bohnert and Gatzert [8] to examine the fair valuation and risk condition of an insurer that offers both dynamic hybrid and traditional participating life insurance contracts. The results highlight the benefits and risks of offering dynamic hybrids by revealing significant interaction effects between the two contract types within the portfolio that are highly dependent on the portfolio composition. So, hybrid life insurance products are designed to fulfill customers' demands for guaranteed stability as well as adequate upside potential, and they have risen in popularity since their introduction to the German market by a number of life insurance companies. For example, there is the introduction of "Select Products" in the German market as a version of deferred equity-indexed annuity contracts that replaced the participating life insurance in 2013 (Alexandrova et al. [1]) or the development of participating products with alternative forms of guarantees (Reuß et al. [24]). Let us remark that even if these mixed contracts have appeared in many markets during a period of low interest rates, their design remains relevant in other circumstances as a natural way to incorporate sound principles of diversification inside a life insurance product.

In this paper, we consider mixed life insurance contracts as proposed by Hanna et al. [18]. These products combine features of participating and pure unit-linked life insurance contracts to achieve a compromise between a guarantee from the participating part and no guarantee from the pure unit-linked part, where a constant fee is extracted from the pure unit-linked account value to finance the guaranteed rates in the participating contract with a maturity or annual guarantee. We emphasize that the pure unit-linked product considered in this paper has no maturity guarantee and is referred to hereafter as a "unit-linked product". In Hanna et al. [18], these products were studied without mortality (pure financial products) and in a classical Black–Scholes environment with a constant risk-free rate. The purpose of this paper is to extend this model in various directions: including the mortality component, and considering different products (pure endowment for the entire contract or combination of a pure endowment and a pure investment), modelling the movement and evolution of interest rates by allowing for stochastic interest rates using the Hull and White model, and finally studying the correlation between all the risk factors, including correlations between financial and actuarial risks.

Indeed, following the improvement in health sciences, human mortality has increased significantly, which puts both the life actuary and the elderly under considerable stress. Apart from the financial risk that notably affects life insurance contracts, mortality risk is an important factor that has been accounted for in the pricing and valuation of life and pension contracts for a long time now. For instance, Bacinello [4] prices a classical life insurance policy given a deterministic life table and a constant interest rate for a single premium and a sequence of periodical premium cases. Ultimately, deterministic mortality does not really change the structure of the fairness relation of the contract. So, mortality risk was typically modelled deterministically, but stochastic mortality models later showed up and are widely accepted as significant improvements and uncertainty have been detected in mortality trends. In addition, various papers admit that interest and mortality rates share a lot in common under the modelling framework (see, e.g., Milevsky and Promislow [23], Biffis [6] and Cairns et al. [10]). As a result, we can model the force of mortality for mortality risk in the same way that we model the short term as the interest rate term structure. In this regard, Zeddouk and Devolder [27] prove that considering a continuous-time mean-reverting model with a time-dependent target is suitable to model mortality. Accordingly, a paper held by Devolder and Azizieh [13] valued a life insurance product with a surplus return based on the insurer's overall profit, in which financial and longevity risks are modelled in a stochastic framework, assuming independence between these two components.

Generally, mortality and finance are separated in life insurance pricing, but this assumption of independence is no longer acceptable in the long term, and it was further strengthened after the recent COVID-19 pandemic that intuitively has an influence on the financial markets. A number of studies show that changes in demographic trends can affect the value of financial assets. For instance, Dacorogna and Cadena [11] study the relationship between mortality and financial risks in different countries. They show a weak effect on the financial markets by taking the full sample into account and a higher effect by taking the 10 worst years of the changes in mortality. Moreover, they highlight that extreme downturns in the mortality indices increase the size of this dependence. Analogously, Jalen and Mamon [20] cover the dependency between interest and mortality rates over the long term and note the impact of a catastrophic event on interest rates as well as population size in the short term. According to Maurer

[22], most of the long-term time series variation in interest rates and stock market returns can be attributed to demographic shifts. He provides enough conditions for the interest rate to drop in the birth rate and rise in the death rate. In addition, Dhaene et al. [14] state that an independence assumption between financial and actuarial risks in the pricing world may be convenient but challenging to comprehend intuitively, and they show that it is often not maintained in the risk-neutral world.

In this paper, we move from a pure financial contract in Hanna et al. [18] to a real life insurance contract with a mortality component. We consider stochastic models for mortality and interest rates, and we introduce some form of correlation between mortality evolution and the financial markets to obtain a whole view of the risks of an insurance contract. The aim of the paper is to price and evaluate different contracts in this stochastic environment, assuming a dependence between financial and demographic risk factors. Some papers have already studied other products and have also tried to introduce the idea of correlation. Our paper follows the classical method used by these papers, which is the change of measure technique (also known as the change of numéraire), as it leads to simple formulas in the pricing of insurance contracts under the dependence assumption (see for example, Jalen and Mamon [20], Dhaene et al. [14], Liu et al. [21], Gao et al. [16], Deelstra et al. [12], Zhao and Mamon [28] and Arık et al. [2]). Accordingly, we obtain an explicit form of the fair valuation for the maturity guarantee contract and analyze the impact of the correlation on the fair guaranteed rates. Our results emphasize the advantage of these mixed contracts, in particular their ability to offer better guaranteed rates. We also show that the introduction of correlations can have various impacts on the price and the level of a fair guarantee. Furthermore, we point out that the impact of the correlation parameters on the fair guaranteed rates is affected by the level of volatility in the model.

The remainder of this paper is organized as follows: The financial market and the stochastic longevity model are introduced in Sect. 2. More precisely, the well-known Hull and White model is used to model the evolution of both mortality and interest rate processes. Furthermore, the correlation parameters between the interest rate, stock returns, and mortality are defined. The structure of the hybrid product is described in Sect. 3. In Sect. 4, we introduce the change of measure technique to compute fair values and obtain an explicit form of the fairness relation of a maturity guarantee contract. Section 5 deals with other kinds of contracts and particularly the annual guarantee contract. Section 6 illustrates a numerical example for the different products examined in Sects. 4 and 5. Finally, we conclude in Sect. 7.

2 Financial and mortality assumptions

Life insurance policies, as long-term financial products, are particularly exposed to a variety of risks. The main risks that these contracts face are those linked to uncertain evolution in financial markets, such as market risk and interest rate risk, characterized recently by volatile stock and interest rate markets, as well as those associated with uncertain evolution in longevity, called mortality risk, which appears with the rapid ageing of the population. In order to model these risks, we allow ourselves to introduce the various stochastic assumptions of the financial market model and the longevity model. Furthermore, we attempt to represent the correlations between the assets defined in the market to explicitly study the dependence between financial and mortality risks.

We work on a finite time interval $[0, T]$, and we consider a global filtered probability space $(\Omega, \mathcal{M}, (\mathcal{M}_t)_t)$ to reflect all the possible financial and mortality events. We introduce two sub-filtrations of $(\mathcal{M}_t)_{t\in[0,T]}$: the financial filtration $(\mathcal{F}_t)_t$ generated by the information on the financial market and the actuarial filtration $(\mathcal{A}_t)_t$ generated by the survival information. Therefore, $\mathcal{M}_t := \mathcal{F}_t \vee \mathcal{A}_t$ is the sigma-algebra generated by $\mathcal{F}_t$ and $\mathcal{A}_t$ on the global space Ω. In this space, we introduce a probability measure $\mathbb{Q}$, which is our global valuation probability. Referring to Artzner et al. [3], this probability when restricted to the financial filtration $\mathcal{F}$, corresponds to the financial risk-neutral measure $\mathbb{Q}^f$. More precisely, the $\mathbb{Q}^f$ is obtained when applying to events that are just generated by the financial filtration. Therefore, the valuation in this paper will be computed using this risk-neutral probability measure $\mathbb{Q}$.

2.1 The financial market

We assume a complete financial market (existence and uniqueness of $\mathbb{Q}^f$) on which the insurer trades three basic processes: the term structure of interest rates and two risky assets, one of which is more volatile than the other. In the following, we display the financial assumptions related to each process.

1. The term structure of interest rates evolves according to the one-factor Hull and White model [19]. Then, the risk-free rate r_t is a solution of the following stochastic equation:

$$\mathrm{d}r_t = \theta\,(\nu_t - r_t)\,\mathrm{d}t + \sigma_r\,\mathrm{d}W_t^r, \quad t \in [0, T] \text{ and } T \in \mathbb{N}, \tag{1}$$

where W^r is a standard Brownian motion under the risk-neutral measure $\mathbb{Q}^f$. θ and σ_r are positive constants, and ν_t is a deterministic function chosen to ensure the fit of the observed yields and expressed as $\nu_t = \frac{1}{\theta} \cdot \frac{\partial f(0,t)}{\partial t} + f(0, t) + \frac{\sigma_r^2}{2\theta^2}(1 - e^{-2\theta t})$ with $f(0, t) = -\frac{\partial \ln P(0,t)}{\partial t}$, the instantaneous forward rate. ν_t is the mean reversion level toward which r_t converges, and θ is the speed of mean reversion.
In an arbitrage free environment, the price of a zero-coupon bond at time t with maturity T is defined by the following expression:

$$P(t, T) = \mathbb{E}^{\mathbb{Q}^f}\left[e^{-\int_t^T r(s)ds} \mid \mathcal{F}_t\right].$$

This price has an explicit form under Hull and White and is given by:

$$P(t, T) = \exp\bigg(- r_t \cdot B_r(t, T, \theta) - \theta \int_t^T \nu_s \cdot B_r(s, T, \theta)ds + \frac{\sigma_r^2}{2} \int_t^T B_r^2(s, T, \theta)ds\bigg), \tag{2}$$

where $B_r(t,T,\theta)=\frac{1-e^{-\theta(T-t)}}{\theta}$, and using the expression of ν_t, the explicit form of the future price $P(t,T)$ becomes:

$$P(t,T)=\frac{P(0,T)}{P(0,t)}\cdot\exp\bigg(-r_t\cdot B_r(t,T,\theta)+B_r(t,T,\theta)\cdot f(0,t)\\ -\frac{\sigma_r^2}{2}\cdot B_r(0,t,2\theta)\cdot B_r^2(t,T,\theta)\bigg).$$

The risk-neutral dynamic of $P(t,T)$ is given by:

$$\mathrm{d}P(t,T)=P(t,T)\,[r_t\,\mathrm{d}t-\sigma_r\,B_r(t,T,\theta)\,\mathrm{d}W_t^r].$$

2. Two risky funds, G and H, where G is a fund with low volatility and H is a fund with high volatility. Their dynamics are described as follows:

$$\mathrm{d}G_t=r_t\,G_t\,\mathrm{d}t+\sigma_G\,G_t\,\mathrm{d}W_t^G,\quad G_0=1,\ t\in[0,T] \tag{3}$$
$$\mathrm{d}H_t=r_t\,H_t\,\mathrm{d}t+\sigma_H\,H_t\,\mathrm{d}W_t^H,\quad H_0=1,\ t\in[0,T]. \tag{4}$$

The two funds follow a geometric Brownian motion where W^G and W^H are standard Brownian motions under the risk-neutral measure $\mathbb{Q}^f$. We state that the fund G is the insurer's general fund used as a reference for the participating contract, while the fund H is an investment fund used to compute the value of the unit-linked contract. We assume then that $\sigma_H>\sigma_G>0$.

Recalling Eqs. (1), (3) and (4), we note that W^r, W^G and W^H are three correlated Brownian motions. We define $\rho_{r,G}$ as the correlation parameter that links the asset G with the interest rate, $\rho_{r,H}$ as the correlation parameter that links the asset H with the interest rate, and $\rho_{G,H}$ as the correlation parameter between the two assets, such that
$\mathbb{E}^{\mathbb{Q}^f}[\mathrm{d}W_t^r\mathrm{d}W_t^G]=\rho_{r,G}$, $\mathbb{E}^{\mathbb{Q}^f}[\mathrm{d}W_t^r\mathrm{d}W_t^H]=\rho_{r,H}$ and $\mathbb{E}^{\mathbb{Q}^f}[\mathrm{d}W_t^G\mathrm{d}W_t^H]=\rho_{G,H}$.
The aforementioned stochastic differential equations (SDEs) are written as follows using the Cholesky decomposition for correlated Brownian motions:

$$\mathrm{d}G_t=r_t\,G_t\,\mathrm{d}t+\sigma_G\,\rho_{r,G}\,G_t\,\mathrm{d}W_t^r+\sigma_G\sqrt{1-\rho_{r,G}^2}\,G_t\,\mathrm{d}\overline{W}_t^G,\quad G_0=1,\ t\in[0,T] \tag{5}$$
$$\mathrm{d}H_t=r_t\,H_t\,\mathrm{d}t+\sigma_H\,\rho_{r,H}\,H_t\,\mathrm{d}W_t^r+\sigma_H\,\overline{\rho}_{G,H}\sqrt{1-\rho_{r,H}^2}\,H_t\,\mathrm{d}\overline{W}_t^G\\ +\sigma_H\sqrt{(1-\rho_{r,H}^2)(1-\overline{\rho}_{G,H}^2)}\,H_t\,\mathrm{d}\overline{\overline{W}}_t^H,\quad H_0=1,\ t\in[0,T], \tag{6}$$

with

$$W_t^G=\rho_{r,G}\cdot W_t^r+\sqrt{1-\rho_{r,G}^2}\cdot\overline{W}_t^G, \tag{7}$$
$$W_t^H=\rho_{r,H}\cdot W_t^r+\overline{\rho}_{G,H}\cdot\sqrt{1-\rho_{r,H}^2}\cdot\overline{W}_t^G+\sqrt{(1-\rho_{r,H}^2)(1-\overline{\rho}_{G,H}^2)}\cdot\overline{\overline{W}}_t^H \tag{8}$$

and

$$\overline{\rho}_{G,H} = \frac{\rho_{G,H} - \rho_{r,G} \cdot \rho_{r,H}}{\sqrt{(1-\rho_{r,G}^2)(1-\rho_{r,H}^2)}},$$

where W^r, $\overline{W}^G$ and $\overline{\overline{W}}^H$ are three uncorrelated Brownian motions.
We further exhibit the annual returns of G and H, respectively:

$$g_t = \frac{G_t}{G_{t-1}} - 1 \quad \text{and} \quad h_t = \frac{H_t}{H_{t-1}} - 1, \quad t = 1, 2, \ldots, T, \tag{9}$$

where the solutions of the SDEs (5) and (6), under the risk-neutral measure, are given by:

$$\begin{aligned}
G_t &= G_0 \exp\left(\int_0^t r_s ds - \frac{\sigma_G^2}{2} \cdot t + \sigma_G\, \rho_{r,G} \int_0^t \mathrm{d}W_s^r + \sigma_G \sqrt{1-\rho_{r,G}^2} \int_0^t \mathrm{d}\overline{W}_s^G \right) \\
H_t &= H_0 \exp\left(\int_0^t r_s ds - \frac{\sigma_H^2}{2} \cdot t + \sigma_H\, \rho_{r,H} \int_0^t \mathrm{d}W_s^r + \sigma_H\, \overline{\rho}_{G,H} \sqrt{1-\rho_{r,H}^2} \int_0^t \mathrm{d}\overline{W}_s^G \right. \\
&\quad \left. + \sigma_H \sqrt{(1-\rho_{r,H}^2)(1-\overline{\rho}_{G,H}^2)} \int_0^t \mathrm{d}\overline{\overline{W}}_s^H \right).
\end{aligned}$$

2.2 The stochastic longevity model

In addition to the financial risk, we include in this paper the mortality risk, which is still a significant concern for life actuaries. According to Devolder and Azizieh [13], there are numerous criteria to consider when selecting a stochastic mortality model. We will use the Hull and White model, which is a plausible mortality model. This is a continuous-time model using mean-reverting with a floating target, as suggested by Zeddouk and Devolder [27]. The main benefit is that this model provides for a complete and elegant equivalence between interest rate and mortality rate.

As a result, we define a new process μ_x called intensity of mortality, which is assumed to be a solution of an affine equation under the risk-neutral measure $\mathbb{Q}$. So, the intensity of mortality $\mu_x(t)$ follows a stochastic process solution of the following equation:

$$\mathrm{d}\mu_x(t) = b\,(\xi_t - \mu_x(t))\,\mathrm{d}t + \sigma_\mu\,\mathrm{d}W_t^\mu, \quad t \in [0, T], \tag{10}$$

where W^μ is a standard Brownian motion under the risk-neutral measure $\mathbb{Q}$. b and σ_μ are positive constants, and ξ_t is a deterministic function. x is the initial age of the policyholder.

This stochastic intensity of mortality $\mu_x(t)$ generates a stochastic probability of survival process, denoted $p_x(t)$, that is the solution of a stochastic differential equation of the form:

$$\mathrm{d}p_x(t, T) = p_x(t, T)\,[\mu_x(t)\,\mathrm{d}t - \sigma_\mu\, B_\mu(t, T, b)\,\mathrm{d}W_t^\mu],$$

where $B_\mu(t,T,b) = \frac{1-e^{-b(T-t)}}{b}$.

As well, the survival probability at time t, under the risk-neutral measure $\mathbb{Q}$, is written as

$$p_x(t,T) = \mathbb{E}^{\mathbb{Q}^a}\big[e^{-\int_t^T \mu_x(s)ds} \mid \mathcal{A}_t\big],$$

with an explicit formula under Hull and White:

$$p_x(t,T) = \exp\bigg(-\mu_x(t)\cdot B_\mu(t,T,b) \\ -b\int_t^T \xi_s \cdot B_\mu(s,T,b)ds + \frac{\sigma_\mu^2}{2}\int_t^T B_\mu^2(s,T,b)ds\bigg). \tag{11}$$

In recent decades, following many natural disasters, particularly the recent COVID-19 pandemic, it is illogical to ignore how their occurrence affects demographic tables as well as financial markets and, accordingly, the correlation between changes in demographic trends and the value of financial assets. This dependence is captured by the correlation parameters $\rho_{r,\mu}$, $\rho_{G,\mu}$ and $\rho_{H,\mu}$ such that $\mathbb{E}^{\mathbb{Q}}[dW_t^r dW_t^\mu] = \rho_{r,\mu}$, $\mathbb{E}^{\mathbb{Q}}[dW_t^G dW_t^\mu] = \rho_{G,\mu}$ and $\mathbb{E}^{\mathbb{Q}}[dW_t^H dW_t^\mu] = \rho_{H,\mu}$. Using Cholesky decomposition, one can write

$$W_t^\mu = \rho_{r,\mu}\cdot W_t^r + \sqrt{1-\rho_{r,\mu}^2}\cdot \overline{W}_t^\mu. \tag{12}$$

Recalling Eqs. (7) and (12), the correlation between $\overline{W}_t^G$ and $\overline{W}_t^\mu$ is determined by $\overline{\rho}_{G,\mu}$ as follows

$$\overline{\rho}_{G,\mu} = \frac{\rho_{G,\mu} - \rho_{r,G}\cdot\rho_{r,\mu}}{\sqrt{(1-\rho_{r,G}^2)(1-\rho_{r,\mu}^2)}}.$$

So, Eq. (12) is expressed as

$$W_t^\mu = \rho_{r,\mu}\cdot W_t^r + \overline{\rho}_{G,\mu}\cdot\sqrt{1-\rho_{r,\mu}^2}\cdot\overline{W}_t^G + \sqrt{(1-\rho_{r,\mu}^2)(1-\overline{\rho}_{G,\mu}^2)}\cdot\overline{\overline{W}}_t^\mu, \tag{13}$$

where W_t^r, $\overline{W}_t^G$ and $\overline{\overline{W}}_t^\mu$ are uncorrelated Brownian motions.

Again, recalling Eqs. (8) and (13), the correlation between $\overline{\overline{W}}_t^H$ and $\overline{\overline{W}}_t^\mu$ is determined by $\overline{\overline{\rho}}_{H,\mu}$ as follows

$$\overline{\overline{\rho}}_{H,\mu} = \frac{\rho_{H,\mu} - \rho_{r,H}\cdot\rho_{r,\mu} - \overline{\rho}_{G,H}\cdot\overline{\rho}_{G,\mu}\sqrt{(1-\rho_{r,H}^2)(1-\rho_{r,\mu}^2)}}{\sqrt{(1-\rho_{r,H}^2)(1-\rho_{r,\mu}^2)(1-\overline{\rho}_{G,H}^2)(1-\overline{\rho}_{G,\mu}^2)}}.$$

Consequently, we write Eq. (13) as follows

$$
\begin{aligned}
W_t^{\mu} = \rho_{r,\mu} \cdot W_t^r + \overline{\rho}_{G,\mu} \cdot \sqrt{1-\rho_{r,\mu}^2} \cdot \overline{W}_t^G + \overline{\overline{\rho}}_{H,\mu} \cdot \sqrt{(1-\rho_{r,\mu}^2)(1-\overline{\rho}_{G,\mu}^2)} \cdot \overline{\overline{W}}_t^H \\
+ \sqrt{(1-\rho_{r,\mu}^2)(1-\overline{\rho}_{G,\mu}^2)(1-\overline{\overline{\rho}}_{H,\mu}^2)} \cdot \overline{\overline{\overline{W}}}_t^{\mu} .
\end{aligned} \tag{14}
$$

Eventually, we exhibit the dynamic of the intensity of the mortality $\mu_x(t)$, under $\mathbb{Q}$, with independent Brownian motions:

$$
\begin{aligned}
\mathrm{d}\mu_x(t) = b\,(\xi_t - \mu_x(t))\,\mathrm{d}t + \sigma_\mu\, \rho_{r,\mu}\, \mathrm{d}W_t^r + \sigma_\mu\, \overline{\rho}_{G,\mu} \sqrt{1-\rho_{r,\mu}^2}\, \mathrm{d}\overline{W}_t^G \\
+ \sigma_\mu\, \overline{\overline{\rho}}_{H,\mu} \cdot \sqrt{(1-\rho_{r,\mu}^2)(1-\overline{\rho}_{G,\mu}^2)}\, \mathrm{d}\overline{\overline{W}}_t^H \\
+ \sigma_\mu \sqrt{(1-\rho_{r,\mu}^2)(1-\overline{\rho}_{G,\mu}^2)(1-\overline{\overline{\rho}}_{H,\mu}^2)}\, \mathrm{d}\overline{\overline{\overline{W}}}_t^{\mu} ,
\end{aligned}
$$

where W_t^r is the noise associated with the interest rate, $\overline{W}_t^G$ is the specific risk associated with the insurer's general fund that is independent of the interest rate, $\overline{\overline{W}}_t^H$ is the risk of the investment fund that is not correlated with the interest rate and with the general fund, and finally, $\overline{\overline{\overline{W}}}_t^{\mu}$ is the specific noise associated with mortality that is not correlated with the finance.

3 Design of the basic contract

We extend the model of Hanna et al. [18] to include stochastic interest rates and mortality risks. As a result, we have chosen a pure endowment life insurance contract with a single premium P paid at inception as the basic contract in order to examine the effect of mortality on the entire contract; However, we will look at other alternative forms of contracts in Sect. 5.

In a pure endowment type of policy, the insurer commits to pay the policyholder the sum assured if he survives to the end of the policy term. The sum assured is paid in a single instalment as a lump sum at the policy's maturity $T > 0$.

The insurance company invests the customers' contributions in a mixed contract such that a part $\pi \in]0, 1]$ of the premium is invested in a participating life insurance contract with participation in the financial profits on the investments of the company, and the remaining part $1-\pi$ is invested in a pure unit-linked contract. In the participating part, we look first at the guarantee at maturity, so the basic contract that we study in Sect. 4 is a maturity guarantee contract. An annual guarantee contract will be considered as a generalization in Sect. 5.

The structure of the mixed contract as well as the mentioned sub-contracts are discussed in detail in Hanna et al. [18]. Shortly, the participating contract invests in the fund G defined in Sect. 2.1, and as is known, this type of contract guarantees the protection of the asset and provides the insured with a surplus return on top of the guaranteed benefit. So, the total return on the participating fund is the guaranteed interest rate in addition to a surplus return.

On the other side, the pure unit-linked contract invests in the investment fund H without any guarantee and is subject to a continuous guarantee fee $\varepsilon \in \mathbb{R}^+$ to form a new process F such that $F_t := e^{-\varepsilon t} \cdot H_t$ and given by:

$$\begin{aligned} \mathrm{d}F_t &= (r_t - \varepsilon)\, F_t\, \mathrm{d}t + \sigma_H\, \rho_{r,H}\, F_t\, \mathrm{d}W_t^r + \sigma_H\, \overline{\rho}_{G,H} \sqrt{1-\rho_{r,H}^2}\, F_t\, \mathrm{d}\overline{W}_t^G \\ &\quad + \sigma_H \sqrt{(1-\rho_{r,H}^2)(1-\overline{\rho}_{G,H}^2)}\, F_t\, \mathrm{d}\overline{\overline{W}}_t^H, \quad F_0 = 1,\ \varepsilon \geq 0,\ t \in [0, T]. \end{aligned}$$

We focus in the following on the fair valuation of the mixed contract, where the guarantee is given only at maturity and so the financial profit, under a stochastic interest rate and a stochastic mortality.

In order to price and evaluate the contract, the insurance company relies on two basis. The first-order basis is the basis for the pricing and computation of the premium. More precisely, this is what the insurer guarantees to the client: an interest rate i and a life table ${}_T p_x$. However, for the provision, the insurer uses a second-order basis to compute the real valuation and check if the contract is fair across all the parameters. These two basis are discussed further below, where we link the first-order and second-order basis to the retrospective and prospective reserves, respectively.

1. The retrospective reserve that corresponds to the initial reserve (which is the initial premium) is capitalized to the maturity of the policy. The capitalization is based on the total returns of the participating and risky funds.
   As a result, the initial retrospective reserve is equal to the premium, so $V_0 = P$, and the final retrospective reserve is equal to the benefit at maturity and is written as follows:

$$\begin{aligned} V_T = \frac{P}{{}_T p_x} \cdot \Bigg[\pi \cdot \left((1+i)^T + \beta \cdot \left(\frac{G_T}{G_0} - \left(1 + \frac{(1+i)^T - 1}{\beta} \right) \right)^+ \right) \\ + (1-\pi) \cdot \frac{F_T}{F_0} \Bigg], \end{aligned} \tag{15}$$

   where i is the annual guaranteed interest rate, $\beta \in [0, 1]$ is the participation level, assumed to be constant over time, and ${}_T p_x$ is the probability, knowing that one is alive at age x, of still being alive after T years. We point out that no payments, apart from the premium, occur until the maturity of the contract.
   In summary, Eq. (15) represents the final reserve, which is the initial premium invested in the different financial instruments, taking into account the initial guaranteed probability of survival.
2. The prospective reserve, which is the fair valuation price, is equal to the risk-neutral expectation of the discounted final retrospective reserve, taking into account the real mortality. So, the fair value at t is written as:

$$FV_t = \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_t^T r_s \mathrm{d}s} \cdot e^{-\int_t^T \mu_x(s) \mathrm{d}s} \cdot V_T \mid \mathcal{M}_t \right].$$

So, the contract is fairly priced if

$$FV_0 = P = V_0.$$

4 Valuation of the basic contract

As previously mentioned, the dependence between the financial markets and demographic trends is a very important aspect to examine nowadays. According to several papers (see, e.g., Devolder and Azizieh [13] and Bernard et al. [5]), the intuitive way to price life insurance contracts in a stochastic environment is to use the change of measure technique, as it is consistent with finance. The advantage of this technique in our framework is that different changes of numéraire can be applied.

To price the basic contract, we split Eq. (15) into three components as follows:

$$V_T = \frac{P}{{}_T p_x} \cdot \Big[C_i(T) + C_G(T) + C_F(T) \Big],$$

where,

$$\begin{aligned} C_i(T) &= \pi \cdot (1+i)^T \quad \text{is the guaranteed part,} \\ C_G(T) &= \pi \cdot \beta \cdot \left(\frac{G_T}{G_0} - K \right)^+ \quad \text{is the profit sharing part,} \\ C_F(T) &= (1-\pi) \cdot \frac{F_T}{F_0} \quad \text{is the unit-linked part;} \end{aligned}$$

with $K = 1 + \frac{(1+i)^T - 1}{\beta}$.

The valuation of these three components is given by these propositions:

Proposition 1 *The time* 0 *price of* $C_i(T)$ *is derived as follows*

$$C_i(0) = \pi \cdot (1+i)^T \cdot P(0,T) \cdot p_x(0,T) \cdot \rho(0,T),$$

where,

$$\rho(0,T) = \exp\left(\frac{\sigma_r\, \sigma_\mu\, \rho_{r,\mu}}{\theta \cdot b} \big[T + B_{r,\mu}(0,T,\theta+b) - B_r(0,T,\theta) - B_\mu(0,T,b) \big] \right) \tag{16}$$

is the term of correlation that arises only from the dependence between the interest rate and the mortality rate $\rho_{r,\mu}$,
with $B_{r,\mu}(t,T,\theta+b) = \frac{1-e^{-(\theta+b)(T-t)}}{\theta+b}$.

Proposition 2 *The time* 0 *price of* $C_G(T)$ *is given by the following form*

$$C_G(0) = \pi \cdot \beta \cdot p_x(0,T) \cdot \rho(0,T) \cdot c_0^T,$$

where,

$$c_0^T = P(0,T)\cdot\left[e^{\hat{m}_G+\frac{\hat{v}_G}{2}}\cdot\Phi\left(\frac{\hat{m}_G+\hat{v}_G-\ln K}{\sqrt{\hat{v}_G}}\right)-K\cdot\Phi\left(\frac{\hat{m}_G-\ln K}{\sqrt{\hat{v}_G}}\right)\right].$$

The expectation $\hat{m}_G$ and the variance $\hat{v}_G$ being given by:

$$\begin{aligned}\hat{m}_G = E\left[\ln\left(\frac{G_T}{G_0}\right)\right] = \ln\left(\frac{1}{P(0,T)}\right) - \frac{\sigma_G^2}{2}\cdot T - \frac{\sigma_G\,\sigma_r\,\rho_{r,G}}{\theta}\,[T - B_r(0,T,\theta)]\\ - \frac{\sigma_G\,\sigma_\mu\,\rho_{G,\mu}}{b}\,[T - B_\mu(0,T,b)]\\ -\frac{\sigma_r\,\sigma_\mu\,\rho_{r,\mu}}{\theta\cdot b}\,[T + B_{r,\mu}(0,T,\theta+b) - B_r(0,T,\theta) - B_\mu(0,T,b)]\\ -\frac{\sigma_r^2}{2\theta^2}\,[T + B_r(0,T,2\theta) - 2\,B_r(0,T,\theta)]\end{aligned} \tag{17}$$

$$\begin{aligned}\hat{v}_G = V\left[\ln\left(\frac{G_T}{G_0}\right)\right] = \sigma_G^2\cdot T + 2\cdot\frac{\sigma_G\,\sigma_r\,\rho_{r,G}}{\theta}\,[T - B_r(0,T,\theta)]\\ + \frac{\sigma_r^2}{\theta^2}\,[T + B_r(0,T,2\theta) - 2\,B_r(0,T,\theta)].\end{aligned} \tag{18}$$

We note that c_0^T is implicitly affected by $\rho_{r,\mu}$ and $\rho_{G,\mu}$, the correlations between the interest and the mortality rates and between the fund G and the mortality rate, through the expectation term $\hat{m}_G$.

Proposition 3 *The time 0 price of $C_F(T)$ is obtained as*

$$C_F(0) = (1-\pi)\cdot p_x(0,T)\cdot\rho'(0,T)\cdot e^{-\varepsilon\cdot T},$$

where,

$$\rho'(0,T) = \exp\left(-\frac{\sigma_H\,\sigma_\mu\,\rho_{H,\mu}}{b}\left[T - B_\mu(0,T,b)\right]\right) \tag{19}$$

is the term of correlation that arises from the dependence between the investment fund and the mortality rate.

Therefore, the total valuation of the contract is given by

$$\begin{aligned}FV_0 = P\cdot\frac{p_x(0,T)}{{}_Tp_x}\cdot\Big(\pi\cdot\rho(0,T)\cdot\big((1+i)^T\cdot P(0,T)+\beta\cdot c_0^T\big)\\ +(1-\pi)\cdot\rho'(0,T)\cdot e^{-\varepsilon\cdot T}\Big).\end{aligned} \tag{20}$$

In the following, we display the proofs for these three propositions explicitly:

Proof 1. We start by estimating the time 0 price of $C_i(T)$, so we write:

$$C_i(0) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r_s ds} \cdot e^{-\int_0^T \mu_x(s)ds} \cdot C_i(T)\right].$$

To solve this expectation, we choose the zero-coupon bond as the numéraire for the T-forward measure defined from the risk-neutral measure using Girsanov's theorem as follows:

$$\begin{aligned}\frac{\mathrm{d}\mathbb{Q}^T}{\mathrm{d}\mathbb{Q}} &= \frac{M_0}{M_T} \cdot \frac{P(T,T)}{P(0,T)} = \frac{e^{-\int_0^T r_s ds}}{\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r_s ds}\right]} \\ &= \exp\left(-\sigma_r \int_0^T B_r(s,T,\theta)\,\mathrm{d}W_s^r - \frac{1}{2}\sigma_r^2 \int_0^T B_r^2(s,T,\theta)\,\mathrm{d}s\right),\end{aligned}$$

where $M_t = e^{\int_0^t r_s ds}$ is the risk-free asset known as the money market account.

Therefore, the fair value of $C_i(T)$, under the T-forward measure, is

$$\begin{aligned}C_i(0) =&\ P(0,T) \cdot \mathbb{E}^{\mathbb{Q}^T}\left[e^{-\int_0^T \mu_x(s)ds} \cdot C_i(T)\right] \\ =&\ \pi \cdot (1+i)^T \cdot P(0,T) \cdot \mathbb{E}^{\mathbb{Q}^T}\left[e^{-\int_0^T \mu_x(s)ds}\right].\end{aligned}$$

We can prove that $\mathbb{E}^{\mathbb{Q}^T}\left[e^{-\int_0^T \mu_x(s)ds}\right] = p_x(0,T) \cdot \rho(0,T)$, see Appendix A.

Consequently,

$$C_i(0) = \pi \cdot (1+i)^T \cdot P(0,T) \cdot p_x(0,T) \cdot \rho(0,T). \tag{21}$$

□

Proof 2. We move to the second term of the contract, $C_G(T)$, and we also compute its value at 0 under the T-forward measure as

$$\begin{aligned}C_G(0) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r_s ds} \cdot e^{-\int_0^T \mu_x(s)ds} \cdot C_G(T)\right] \\ &= P(0,T) \cdot \mathbb{E}^{\mathbb{Q}^T}\left[e^{-\int_0^T \mu_x(s)ds} \cdot C_G(T)\right] \\ &= \pi \cdot \beta \cdot P(0,T) \cdot \mathbb{E}^{\mathbb{Q}^T}\left[\left(\frac{G_T}{G_0} - K\right)^+ \cdot e^{-\int_0^T \mu_x(s)ds}\right].\end{aligned}$$

Due to the dependence between the value of financial assets and the mortality rate, we define a new measure from the T-forward measure called the forward-mortality measure, denoted by $\mathbb{Q}^\mu$. Using Girsanov's theorem, the Radon–Nikodym derivative is, in this case

$$\frac{\mathrm{d}\mathbb{Q}^\mu}{\mathrm{d}\mathbb{Q}^T} = \frac{e^{-\int_0^T \mu_x(s)ds}}{\mathbb{E}^{\mathbb{Q}^T}\left[e^{-\int_0^T \mu_x(s)ds}\right]}.$$

Under this new measure, the expectation in the valuation of $C_G(T)$ becomes

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}^T}\left[\left(\frac{G_T}{G_0}-K\right)^+\cdot e^{-\int_0^T \mu_x(s)ds}\right] &= \mathbb{E}^{\mathbb{Q}^T}\left[e^{-\int_0^T \mu_x(s)ds}\right]\cdot\mathbb{E}^{\mathbb{Q}^\mu}\left[\left(\frac{G_T}{G_0}-K\right)^+\right]\\ &= p_x(0,T)\cdot\rho(0,T)\cdot\mathbb{E}^{\mathbb{Q}^\mu}\left[\left(\frac{G_T}{G_0}-K\right)^+\right].\end{aligned}$$

We show in Appendix B that

$$\mathbb{E}^{\mathbb{Q}^\mu}\left[\left(\frac{G_T}{G_0}-K\right)^+\right] = e^{\hat{m}_G+\frac{\hat{v}_G}{2}}\cdot\Phi\left(\frac{\hat{m}_G+\hat{v}_G-\ln K}{\sqrt{\hat{v}_G}}\right)-K\cdot\Phi\left(\frac{\hat{m}_G-\ln K}{\sqrt{\hat{v}_G}}\right),$$

where Φ is the cumulative distribution function of a standard normal variable and $\hat{m}_G$ and $\hat{v}_G$ are respectively the expectation and the variance of $\ln\left(\frac{G_T}{G_0}\right)$ under the forward-mortality measure, defined in Eqs. (17) and (18).

Accordingly, we obtain the fair value of $C_G(T)$ at 0 as follows:

$$C_G(0) = \pi\cdot\beta\cdot p_x(0,T)\cdot\rho(0,T)\cdot c_0^T, \tag{22}$$

where $c_0^T = P(0,T)\cdot\left[e^{\hat{m}_G+\frac{\hat{v}_G}{2}}\cdot\Phi\left(\frac{\hat{m}_G+\hat{v}_G-\ln K}{\sqrt{\hat{v}_G}}\right)-K\cdot\Phi\left(\frac{\hat{m}_G-\ln K}{\sqrt{\hat{v}_G}}\right)\right]$.

We note that c_0^T is implicitly affected by $\rho_{G,\mu}$, the correlation between the fund G and the mortality rate, through the expectation term $\hat{m}_G$. □

Proof 3. Ultimately, we attempt to price the third term of the contract, $C_F(T)$. For this as well, we move first to the T-forward measure, then to the forward-mortality measure:

$$\begin{aligned}C_F(0) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r_s ds}\cdot e^{-\int_0^T \mu_x(s)ds}\cdot C_F(T)\right]\\ &= P(0,T)\cdot\mathbb{E}^{\mathbb{Q}^T}\left[e^{-\int_0^T \mu_x(s)ds}\cdot C_F(T)\right]\\ &= (1-\pi)\cdot P(0,T)\cdot\mathbb{E}^{\mathbb{Q}^T}\left[\frac{F_T}{F_0}\cdot e^{-\int_0^T \mu_x(s)ds}\right]\\ &= (1-\pi)\cdot P(0,T)\cdot p_x(0,T)\cdot\rho(0,T)\cdot\mathbb{E}^{\mathbb{Q}^\mu}\left[\frac{F_T}{F_0}\right].\end{aligned}$$

We prove that $P(0,T)\cdot\rho(0,T)\cdot\mathbb{E}^{\mathbb{Q}^\mu}\left[\frac{F_T}{F_0}\right] = \rho'(0,T)\cdot e^{-\varepsilon\cdot T}$, see Appendix C.

As a result,

$$C_F(0) = (1-\pi)\cdot p_x(0,T)\cdot\rho'(0,T)\cdot e^{-\varepsilon\cdot T}, \tag{23}$$

where $\rho'(0,T)$ is the term of correlation that arises from the dependence between the investment fund and the mortality rate, defined in Eq. (19).

Corollary 1 *The premium equivalence of the maturity guarantee contract follows from Eq.* (20) *under the condition that* $FV_0 = P$:

$$p_x(0,T)\cdot\left[\pi\cdot\rho(0,T)\cdot\left(P(0,T)\cdot(1+i)^T+\beta\cdot c_0^T\right)+(1-\pi)\cdot\rho'(0,T)\cdot e^{-\varepsilon\cdot T}\right]={}_Tp_x, \tag{24}$$

where $\rho(0,T)$ *and* $\rho'(0,T)$ *are the terms of correlation exhibited in Eqs.* (16) *and* (19), *respectively.*

This relation establishes a fair connection between all of the parameters, allowing us to create a fair contract.

It is interesting to shed light, in Eq. (24), on the terms of correlation and their impact on the guaranteed interest rate and the value of the contract. The two terms of correlation, $\rho(0,T)$ and $\rho'(0,T)$, are shown explicitly in Eq. (24). Nevertheless, the term c_0^T is impacted by the term of correlation that reflects the dependence between the general fund G and the mortality rate. For instance, an increase in the correlation between interest and mortality rates is followed by an increase in the term of the correlation $\rho(0,T)$ that results in an increase in the price of the contract and a decrease in the guaranteed interest rate. On the other hand, the increase in the correlation between the risky fund H and the mortality rate leads to a decrease in the term of the correlation $\rho'(0,T)$ that lowers the price of the contract and increases the guaranteed rate i. This impact of the terms of correlations on the fair value of contracts could be explained by the fact that moving the interest rate r and the mortality rate μ in the same direction has the same effect on the cost of the product, whereas moving the fund H and the mortality rate μ in the same direction has the opposite effect.

5 Other forms of pension products

In Sects. 3 and 4, we have seen a product with longevity conditions on the two parts, which is a combination of two pure endowments, one for the participating product and the other for the unit-linked product. In this section, we consider other kinds of products and derive explicitly the premium equivalence of each product.

1. *Pure financial product*:
   This form of product is a direct generalization of Hanna et al. [18] in which no mortality is introduced; however, a stochastic interest rate is considered. So, this case can be simply extracted from Eq. (24) by ignoring the presence of the mortality element in the life insurance contract. This leads to the following fair relation:

$$\pi\cdot\left(P(0,T)\cdot(1+i)^T+\beta\cdot C_0^{\prime T}\right)+(1-\pi)\cdot e^{-\varepsilon\cdot T}=1. \tag{25}$$

   This relation is similar to the result obtained in Hanna et al. [18]. The only difference lies in the price of a zero coupon and the call option, which are now valued with a stochastic interest rate instead of a constant interest rate.

2. *Combination of pure endowment and pure investment product:*
This kind of product entails paying the sum assured on the participating part of the contract if the policyholder survives to the end of the policy term, in addition to the return of the investment fund on the risky part (minus the guarantee fee) at the end of the year of death or at maturity if he survives to that point. In this kind of contract, the terminal cash flow is written as follows:

$$V_T = \frac{P}{{}_T p_x} \cdot \left[\pi \cdot \left((1+i)^T + \beta \cdot \left(\frac{G_T}{G_0} - \left(1 + \frac{(1+i)^T - 1}{\beta}\right)\right)^+\right)\right] + (1-\pi) \cdot P \cdot \frac{F_T}{F_0}. \tag{26}$$

Compared to the cash flow of Eq. (15), the risky part of Relation (26) is not affected by survival probabilities.

Proposition 4 *The total valuation of this contract is given by*:

$$\begin{aligned} FV_0 &= \pi \cdot P \cdot \frac{p_x(0,T)}{{}_T p_x} \cdot \rho(0,T) \cdot \left(P(0,T) \cdot (1+i)^T + \beta \cdot c_0^T\right) \\ &+ (1-\pi) \cdot P \cdot \Big(p_x(0,T) \cdot \rho'(0,T) \cdot e^{-\varepsilon \cdot T} \\ &+ \sum_{j=0}^{T-1} \Big[p_x(0,j) \cdot \rho'(0,j) - p_x(0,j+1) \cdot \rho'(0,j+1)\Big] \cdot e^{-\varepsilon(j+1)}\Big) \end{aligned} \tag{27}$$

Proof 4. The fair valuation of V_T is expressed as follows:

$$\begin{aligned} FV_0 &= \mathbb{E}^{\mathbb{Q}}\Big[V_T \cdot e^{-\int_0^T r_s ds} \cdot e^{-\int_0^T \mu_x(s) ds} \mid \mathcal{M}_0\Big] \\ &+ \mathbb{E}^{\mathbb{Q}}\Big[(1-\pi) \cdot P \cdot \sum_{j=0}^{T-1} \frac{F_{j+1}}{F_0} \cdot e^{-\int_0^{j+1} r_s ds} \left(e^{-\int_0^j \mu_x(s) ds} - e^{-\int_0^{j+1} \mu_x(s) ds}\right) \mid \mathcal{M}_0\Big], \end{aligned}$$

where the second expectation takes into account the payment of the investment fund in case of death between the start of the contract and the maturity date. We divide this expectation into two halves and write down the fair value:

$$\begin{aligned} FV_0 &= \mathbb{E}^{\mathbb{Q}}\Big[V_T \cdot e^{-\int_0^T r_s ds} \cdot e^{-\int_0^T \mu_x(s) ds} \mid \mathcal{M}_0\Big] \\ &+ (1-\pi) \cdot P \cdot \sum_{j=0}^{T-1} \mathbb{E}^{\mathbb{Q}}\Big[\frac{F_{j+1}}{F_0} \cdot e^{-\int_0^{j+1} r_s ds} \cdot e^{-\int_0^j \mu_x(s) ds} \mid \mathcal{M}_0\Big] \\ &- (1-\pi) \cdot P \cdot \sum_{j=0}^{T-1} \mathbb{E}^{\mathbb{Q}}\Big[\frac{F_{j+1}}{F_0} \cdot e^{-\int_0^{j+1} r_s ds} \cdot e^{-\int_0^{j+1} \mu_x(s) ds} \mid \mathcal{M}_0\Big]. \end{aligned}$$

The computations of the three expectations are detailed in Appendix D, which allows us to obtain the following result:

$$\begin{aligned} FV_0 = \pi \cdot P \cdot \frac{p_x(0,T)}{{}_T p_x} \cdot \rho(0,T) \cdot \Big(P(0,T) \cdot (1+i)^T + \beta \cdot c_0^T \Big) \\ + (1-\pi) \cdot P \cdot \Big(p_x(0,T) \cdot \rho'(0,T) \cdot e^{-\varepsilon \cdot T} \\ + \sum_{j=0}^{T-1} \Big[p_x(0,j) \cdot \rho'(0,j) - p_x(0,j+1) \cdot \rho'(0,j+1) \Big] \cdot e^{-\varepsilon(j+1)} \Big). \end{aligned}$$

□

Corollary 2 *The premium equivalence of a maturity guarantee contract under this special combination of products is given as follows*:

$$\begin{aligned} \pi \cdot \frac{p_x(0,T)}{{}_T p_x} \cdot \rho(0,T) \cdot \Big(P(0,T) \cdot (1+i)^T + \beta \cdot c_0^T \Big) + (1-\pi) \cdot \\ \Big(p_x(0,T) \cdot \rho'(0,T) \cdot e^{-\varepsilon \cdot T} \\ + \sum_{j=0}^{T-1} \Big[p_x(0,j) \cdot \rho'(0,j) - p_x(0,j+1) \cdot \rho'(0,j+1) \Big] \cdot e^{-\varepsilon(j+1)} \Big) = 1. \end{aligned} \tag{28}$$

Relation (28) gives a fair condition between all the parameters and differs from Eq. (24) in the fact that the insured has the right to a return on the risky part if he dies before the maturity of the contract. We note that even if the final benefit is independent of mortality (Relation 26), we can observe here that the valuation of the unit-linked part depends on the survival probabilities, and this is induced by the effect of the guarantee fee ε because by setting $\varepsilon = 0$, we obtain 1 as the value of the unit-linked fund.

3. *Annual guarantee*:

We recall our basic contract in which we combine two pure endowments; but in this form of contract, we look at the guarantee and the financial profit each year and not only at maturity, so we can distinguish between a rebalanced and a non-rebalanced portfolio. This means that we either create an automatic link between the participating and unit-linked accounts or that the proportion π invested in the participating part stays constant throughout the duration of the contract.

The terminal cash flow of a non-rebalanced contract (in which the proportion π is invested in the participating part at the beginning of the contract and no further action is taken) is written as follows:

$$V_T = \frac{P}{{}_T p_x} \cdot \left[\pi \cdot \prod_{t=1}^{T} \Big(1 + i + (\beta \cdot g_t - i)^+ \Big) + (1-\pi) \cdot \prod_{t=1}^{T} \frac{F_t}{F_{t-1}} \right], \tag{29}$$

however, the terminal cash flow of a rebalanced contract (in which the investment share is rebalanced so that, once more, the constant proportion π is invested in the participating contract each year) is written as follows:

$$V_T = \frac{P}{{}_T p_x} \cdot \prod_{t=1}^{T} \left[\pi \cdot \left(1 + i + \left(\beta \cdot g_t - i\right)^+\right) + (1-\pi) \cdot \frac{F_t}{F_{t-1}} \right]. \tag{30}$$

Following the same logic as Sect. 4, we use the change of measure technique to compute the real valuation of Eqs. (29) and (30) at the inception of the contract.

Proposition 5 *The total valuations of annual guarantee contracts without and with rebalancing are given, respectively, by Eqs.* (31) *and* (32):

$$FV_0 = P \cdot \frac{p_x(0,T)}{{}_T p_x} \cdot \left[\pi \cdot P(0,T) \cdot \rho(0,T) \cdot \mathbb{E}^{\mathbb{Q}^\mu}\left[\prod_{t=1}^{T} \left(1 + i + \left(\beta \cdot g_t - i\right)^+\right) \right] \right.$$
$$\left. + (1-\pi) \cdot \rho'(0,T) \cdot e^{-\varepsilon T} \right] \tag{31}$$

$$FV_0 = P \cdot \frac{p_x(0,T)}{{}_T p_x} \cdot P(0,T) \cdot \rho(0,T) \cdot \mathbb{E}^{\mathbb{Q}^\mu}\left[\prod_{t=1}^{T} \left[\pi \cdot \left(1 + i + \left(\beta \cdot g_t - i\right)^+\right) \right.\right.$$
$$\left.\left. + (1-\pi) \cdot \frac{F_t}{F_{t-1}} \right]\right]. \tag{32}$$

Proof 5. The fair valuation of Eqs. (29) and (30) is stated as follows:

$$\begin{aligned} FV_0 &= \mathbb{E}^{\mathbb{Q}}\left[V_T \cdot e^{-\int_0^T r_s ds} \cdot e^{-\int_0^T \mu_x(s) ds} \mid \mathcal{M}_0 \right] \\ &= P(0,T) \cdot p_x(0,T) \cdot \rho(0,T) \cdot \mathbb{E}^{\mathbb{Q}^\mu}\left[V_T \right]. \end{aligned} \tag{33}$$

This is obtained by applying the change of measures defined in Sect. 4; from the risk-neutral measure to a T-forward measure, then to a forward-mortality measure. □

For the rebalanced contract, it is not possible to go further because of the dependence between the annual returns of the funds G and F. Thus, the fair value of the non-rebalanced contract (29) follows from Eq. (33):

$$FV_0 = V_0 \cdot P(0,T) \cdot \frac{p_x(0,T)}{{}_T p_x} \cdot \rho(0,T) \cdot \left[\pi \cdot \mathbb{E}^{\mathbb{Q}^\mu}\left[\prod_{t=1}^{T} \left(1 + i + \left(\beta \cdot g_t - i\right)^+\right) \right] \right.$$
$$\left. + (1-\pi) \cdot \mathbb{E}^{\mathbb{Q}^\mu}\left[\underbrace{\prod_{t=1}^{T} \frac{F_t}{F_{t-1}}}_{=\frac{F_T}{F_0}} \right]\right].$$

Due to the dependence between the annual returns of the fund G, the term $\mathbb{E}^{\mathbb{Q}^{\mu}}\left[\prod_{t=1}^{T}\left(1+i+\left(\beta\cdot g_t-i\right)^{+}\right)\right]$ could not be solved explicitly. However, the second term $\mathbb{E}^{\mathbb{Q}^{\mu}}\left[\frac{F_T}{F_0}\right]$ gives, referring to the computations in Sect. 4, $\frac{\rho'(0,T)\cdot e^{-\varepsilon T}}{\rho(0,T)\cdot P(0,T)}$.

Corollary 3 *The fairness relations of annual guarantee contracts without and with rebalancing are given, respectively, in Eqs.* (34) *and* (35):

$$p_x(0,T)\Bigg[\pi\cdot P(0,T)\cdot\rho(0,T)\cdot\mathbb{E}^{\mathbb{Q}^{\mu}}\left[\prod_{t=1}^{T}\left(1+i+\left(\beta\cdot g_t-i\right)^{+}\right)\right] + (1-\pi)\cdot\rho'(0,T)\cdot e^{-\varepsilon\cdot T}\Bigg]={}_{T}p_x \tag{34}$$

$$p_x(0,T)\cdot P(0,T)\cdot\rho(0,T)\cdot\mathbb{E}^{\mathbb{Q}^{\mu}}\left[\prod_{t=1}^{T}\left[\pi\cdot\left(1+i+\left(\beta\cdot g_t-i\right)^{+}\right)+(1-\pi)\cdot\frac{F_t}{F_{t-1}}\right]\right]={}_{T}p_x. \tag{35}$$

We state in the following some special cases deducted directly from Eq. (24) of our basic contract:

Remark 1 (*Independence finance-mortality*). We assume independence between the financial and mortality elements. In many studies, this case is considered when pricing a life insurance contract for ease of computation on the one hand and assuming that a correlation between the demographic trend and the financial market is almost negligible in the absence of an actual pandemic on the other.

Practically, the independence leads to $\rho(0,T)=\rho'(0,T)=1$, resulting in a fair relation of the form

$$p_x(0,T)\cdot\left[\pi\cdot\left(P(0,T)\cdot(1+i)^T+\beta\cdot C_0^{'T}\right)+(1-\pi)\cdot e^{-\varepsilon\cdot T}\right]={}_{T}p_x, \tag{36}$$

where $C_0^{'T}=\Phi\left(\frac{-\ln P(0,T)+\frac{\hat{v}_G}{2}-\ln K}{\sqrt{\hat{v}_G}}\right)-K\cdot P(0,T)\cdot\Phi\left(\frac{-\ln P(0,T)-\frac{\hat{v}_G}{2}-\ln K}{\sqrt{\hat{v}_G}}\right)$.

Remark 2 (*Pure endowment contract without participation*). Another special case is presented as a comparison of the type of pure endowment for a classic life insurance contract, between a classical valuation and a stochastic framework.

The classical actuarial valuation of a pure endowment without profit sharing ($\pi=1,\ \beta=0$) is:

$${}_{T}E_x=\frac{1}{(1+i)^T}\cdot{}_{T}p_x.$$

However, under the framework of this paper, the valuation of the classical product becomes:

$$FV = P(0, T) \cdot p_x(0, T) \cdot \rho(0, T),$$

in which case we need three factors instead of two to represent the correlation between financial and longevity risks.

6 Numerical illustrations

In this section, we present some numerical applications for Eqs. (24), (28), (36) and (25) to compute the fair guaranteed interest rate i, and we attempt to compare these results to Hanna et al. [18] after accounting for stochastic interest rates and mortality rates. In this framework, we will find solutions for fairness relations with respect to i based on different values of the investment share π.

So, we consider a pure endowment contract, and we display in Table 1 the parameter values chosen to be used in this section.

We point out that we chose low and high volatility levels for the two funds G and H, respectively, using values from the paper by Hanna et al. [18]. Moreover, based on the same paper, we use as an example a participation level of $\beta = 70\%$ and a guarantee fee of $\varepsilon = 0.25\%$ for the valuation of the contract.

In order to simulate the price of a zero-coupon bond $P(0, T)$ in Eq. (2), the calibration of the instantaneous forward rate $f(0, T)$ is given by the restricted exponential model defined by Cairns [9]:

$$f(0, t) = \beta_0 + \sum_{j=1}^{n} \beta_j \cdot e^{-c_j t}.$$

We fitted this model to the curve of Belgian zero-coupon rates on January 6, 2021, by using non-linear regression through the non-linear least squares function; we use a virtual adjustment with $n = 3$ and the following values for the parameters β_j and c_j:

$\beta_0 = -0.0039$, $\beta_1 = 0.001$, $\beta_2 = -0.0165$, $\beta_3 = 0.012$, $c_1 = 0.1385$, $c_2 = 0.0385$ and $c_3 = -0.02$.

Regarding the simulation of the survival probability $p_x(0, T)$ in Eq. (11), we refer to Zeddouk and Devolder [27], which states that the target is given by the Gompertz law with the following expression:

$$\xi_t = \frac{A}{b} e^{Bt},$$

where A and B are two strictly positive constants and refer, respectively, to the baseline mortality at age x and the senescent component. Zeddouk and Devolder [27] show that using a mean-reverting model with an exponential target consistent with Gompertz law is appropriate for describing mortality over time, as they confirm its adequacy.

Table 1 Model parameters' values

Parameters	Symbols	Contract parameters	Parameters	Symbols	Market parameters (%)
Maturity	T	15	G-volatility	σ_G	3
Initial age	x	50	H-volatility	σ_H	15
Survival probability	${}_T p_x$	0.95745364[a]			
Parameters	**Symbols**	**Interest rates parameters (%)**	**Parameters**	**Symbols**	**Mortality parameters (%)**
Interest rate volatility	σ_r	0.8	Mortality volatility	σ_μ	0.0519610
Speed of mean reversion	θ	15	Speed of mean reversion	b	13.8550587
Initial interest rate	r_0	0.5	Initial mortality force	$\mu_x(0)$	0.2600332

[a] The numbers of survivals at the corresponding ages are based on the Belgian official XR table, for 1 million of births

We note that the optimal parameters for the mortality model chosen in Table 1 are derived from the calibration performed by Zeddouk and Devolder [27], with an initial age of $x = 50$ and considering generation 1970. We add to these parameters the values of A and B, which are, respectively, 0.000307570 and 0.100627916.

We note that in the first three subsections, we analyze the results of Eq. (24).

6.1 The effect of the correlation parameters

To study the correlation parameters clearly, we display graphically the effect of each parameter on the fair guaranteed rates. We assume that the insurer invests equal parts in the participating and the unit-linked contracts ($\pi = 50\%$). In this setting, we study each parameter separately by putting the two other correlation parameters at zero and varying the one in question in the range of values $[-20\%, 20\%]$. We note that, in the case where $\rho_{r,\mu}$ is different from zero, we have an indirect correlation between the funds G and F and the mortality through the interest rate by imposing $\rho_{G,\mu} = \rho_{r,\mu} \cdot \rho_{r,G}$ and $\rho_{H,\mu} = \rho_{r,\mu} \cdot \rho_{r,H}$.

The first graph in Fig. 1 shows a decreasing effect on the guaranteed rate since for $\rho_{r,\mu} \in [-20\%, 20\%]$, the guaranteed interest rate i takes values between $[0.473\%, 0.479\%]$. So, a negative correlation between interest and mortality rates boosts the guaranteed rate.

In contrast to the effect of $\rho_{r,\mu}$, the guaranteed interest rate i remains relatively stable for the different values of $\rho_{G,\mu}$ and increases with the increase of $\rho_{H,\mu}$, as shown in graphs 2 and 3 in Fig. 1. Clearly, $\rho_{G,\mu}$ has no significant impact on i that takes values between 0.476% and 0.477%, whereas $\rho_{H,\mu}$ has a greater impact since the values of i emerge between 0.467% and 0.485%. As a result, stronger and positive correlations between the fund H and the mortality rate improve the guarantee on the participating part.

The empirical evidence provided by Dacorogna and Cadena [11] to explain the change in the financial market due to the mortality factor shows that the influence is quite weak in a normal situation and that the correlation between interest rates and mortality rates is higher in most countries than the correlation between stocks and the mortality index. For this reason, it is worth noting that the values of the correlation parameters, $\rho_{r,\mu}$, $\rho_{G,\mu}$, and $\rho_{H,\mu}$, are chosen relatively small, with $\rho_{r,\mu} > \rho_{G,\mu}$ and $\rho_{H,\mu}$. Therefore, we state in the following subsections that $|\rho_{r,\mu}| = 7\%$ and $|\rho_{G,\mu}| = |\rho_{H,\mu}| = 5\%$.

6.2 Sign sensitivity

Table 2 shows the sensitivity of the signs of the correlation parameters to the fair guaranteed rates i. We assume that the allocation level π is equal to 50% and that $\rho_{r,\mu}$, $\rho_{G,\mu}$ and $\rho_{H,\mu}$ are equal, in absolute value, to 7%, 5% and 5%, respectively, where each row in the table indicates the sign of these parameters. For instance, the sign "+" under $\rho_{r,\mu}$ means that $\rho_{r,\mu} = +7\%$, while the sign "−" means that $\rho_{r,\mu} = -7\%$. Moreover, the relations $\rho_{G,\mu} = \rho_{r,\mu} \cdot \rho_{r,G}$ and $\rho_{H,\mu} = \rho_{r,\mu} \cdot \rho_{r,H}$ in Table 2 imply

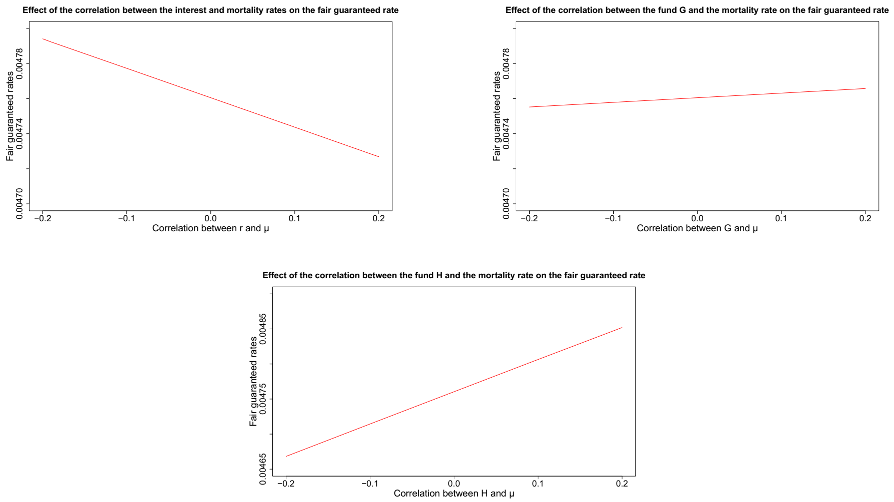

Fig. 1 Effect of the different correlation parameters

Table 2 Fair guaranteed rates i (expressed in %) depending on the sign of the correlation parameters

$\lvert\rho_{r,\mu}\rvert = 7\%,\ \lvert\rho_{G,\mu}\rvert = 5\%,\ \lvert\rho_{H,\mu}\rvert = 5\%,$ $\rho_{r,G} = -20\%,\ \rho_{r,H} = -15\%,\ \pi = 50\%$			
$\rho_{r,\mu}$	$\rho_{G,\mu}$	$\rho_{H,\mu}$	$i(\%)$
$+$	$\rho_{r,\mu} \cdot \rho_{r,G}$	$\rho_{r,\mu} \cdot \rho_{r,H}$	0.475
$+$	$+$	$+$	0.478
$+$	$-$	$-$	0.473
$-$	$\rho_{r,\mu} \cdot \rho_{r,G}$	$\rho_{r,\mu} \cdot \rho_{r,H}$	0.477
$-$	$+$	$+$	0.479
$-$	$-$	$-$	0.474

that the correlation between the funds and the mortality rate arises only from their correlations with the interest rate.

We observe an increase in the fair guaranteed rate by switching to a negative correlation between the interest and mortality rates and a positive correlation between the funds and the mortality rate. In fact, a negative correlation between the two risk factors r and μ operates as a hedging mechanism against the overall unpredictability of the contract, which reduces its price and increases the guaranteed rate.

From this analysis, we present two central scenarios for the correlation parameters, and we find the combination of signs for the three parameters that gives the lowest as well as the highest values for the guaranteed rates i that maintain the fair relation of the maturity guarantee determined by Eq. (24). The results are illustrated in Table 3.

The first and third columns of Table 3 are two central scenarios since they indicate the signs of the set of correlations $(\rho_{r,\mu}, \rho_{G,\mu}, \rho_{H,\mu})$ that give the lowest and highest fair guaranteed rates. However, the middle column refers to the case of independence between mortality and financial elements, for which we recall Eq. (36). We notice that the highest values of guaranteed rates are obtained by considering a negative

Table 3 Fair guaranteed rates i (expressed in %) for three extreme scenarios of the correlations

π	$\rho_{r,G} = -20\%$		
	$\rho_{r,\mu} = 7\%$ $\rho_{G,\mu} = -5\%$ $\rho_{H,\mu} = -5\%$	$\rho_{r,\mu} = 0\%$ $\rho_{G,\mu} = 0\%$ $\rho_{H,\mu} = 0\%$	$\rho_{r,\mu} = -7\%$ $\rho_{G,\mu} = 5\%$ $\rho_{H,\mu} = 5\%$
0.10	2.700	2.711	2.722
0.30	0.978	0.983	0.987
0.50	0.473	0.476	0.479
0.70	0.175	0.178	0.180
1.00	−0.144	−0.142	−0.141

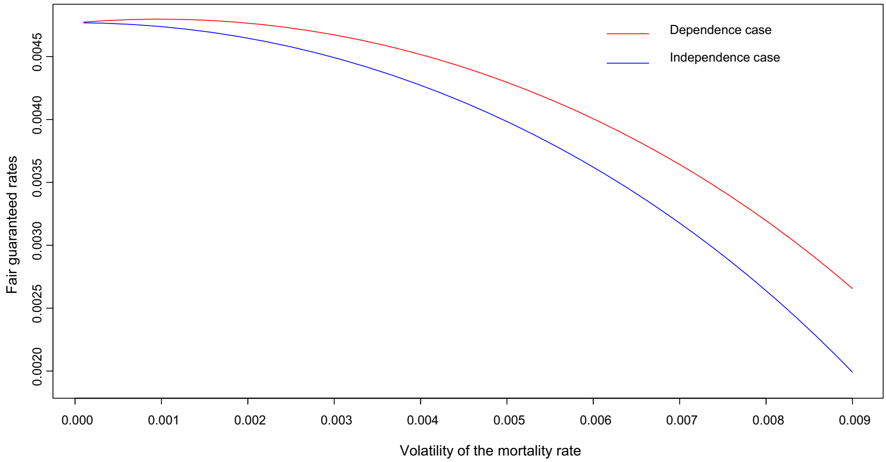

Fig. 2 Sensitivity of the volatility σ_μ

correlation between interest and mortality rates together with positive values between equities and mortality rates, which is in line with the graphs in Fig. 1. However, the impact of the correlation parameters is not significant with respect to the case where no correlation is considered, especially with a higher investment in the participating part.

In addition, we observe in Table 3 that the guaranteed rates increase the more we invest in the risky fund. So, the results show that a mixed product is preferred over a participating life insurance product in terms of the guaranteed interest rate, regardless of the values of the correlation parameters.

6.3 Sensitivity analysis with respect to the volatility of mortality σ_μ

The weak effect of the correlation parameters is due to the choice of very small values for the volatilities. Hence, we are interested in showing how the volatility σ_μ affects a product with a dependency between financial and mortality elements and another where no dependency is taken into account. A higher choice of volatility has an influence on the terms of correlation that, in turn, make all the difference.

Table 4 Fair guaranteed rates i (expressed in %) for three extreme scenarios of the correlations with $\sigma_\mu = 0.5\%$

π	$\rho_{r,G} = -20\%$					
	$\sigma_\mu = 0.05\%$			$\sigma_\mu = 0.5\%$		
	$\rho_{r,\mu} = 7\%$ $\rho_{G,\mu} = -5\%$ $\rho_{H,\mu} = -5\%$	$\rho_{r,\mu} = 0\%$ $\rho_{G,\mu} = 0\%$ $\rho_{H,\mu} = 0\%$	$\rho_{r,\mu} = -7\%$ $\rho_{G,\mu} = 5\%$ $\rho_{H,\mu} = 5\%$	$\rho_{r,\mu} = 7\%$ $\rho_{G,\mu} = -5\%$ $\rho_{H,\mu} = -5\%$	$\rho_{r,\mu} = 0\%$ $\rho_{G,\mu} = 0\%$ $\rho_{H,\mu} = 0\%$	$\rho_{r,\mu} = -7\%$ $\rho_{G,\mu} = 5\%$ $\rho_{H,\mu} = 5\%$
0.10	2.700	2.711	2.722	2.417	2.524	2.629
0.30	0.978	0.983	0.987	0.844	0.890	0.935
0.50	0.473	0.476	0.479	0.366	0.398	0.430
0.70	0.175	0.178	0.180	0.078	0.102	0.126
1.00	−0.144	−0.142	−0.141	−0.241	−0.226	−0.211

Figure 2 displays the fair guaranteed rates of a maturity guarantee product with and without dependence between financial and mortality elements for different values of the volatility σ_μ by investing equal parts in the participating and the unit-linked contracts ($\pi = 50\%$). The values of the correlation parameters used in the dependence case are as follows: $\rho_{r,\mu} = -7\%$, $\rho_{G,\mu} = 5\%$ and $\rho_{H,\mu} = 5\%$. The graph proves that by increasing the volatility, the impact on the fair guaranteed rates becomes more and more important, and consequently, taking the dependency into account in the model is more prudent. In Table 4, we compare the fair guaranteed rates for the three extreme correlation scenarios with two different volatility levels, $\sigma_\mu = 0.05\%$ and $\sigma_\mu = 0.5\%$.

Table 4 shows that with a higher value of σ_μ, the impact on the fair guaranteed rates is more significant between these three extreme scenarios; however, the fair guaranteed rates decrease with increasing volatility, and that's because with a higher risk, the cost of the guarantee increases, which decreases the guaranteed rate. In addition, we notice that a positive value of $\rho_{r,\mu}$ along with negative values of $\rho_{G,\mu}$ and $\rho_{H,\mu}$ deliver guaranteed rates that even fall below the case where no correlation is considered. As a result, the contribution of the correlations inside the model is prudent in terms of their signs since they can be optimistic as well as pessimistic.

In the following, we would like to present the consequences of taking these two parameters with the wrong sign or size when pricing the contract. More precisely, we will show the size of the loss in percentage of the premium in the worst-case scenario.

The first step is to remove any correlation between financial and mortality factors and to look at the fair guaranteed rate obtained by investing equal shares in both parts of the contract, which is 0.398%, referring to Table 4 (for $\sigma_\mu = 0.5\%$). The second step is to include the correlation parameters in order to measure the size of the loss as a percentage of the premium, which is supposed to be 1000. Hence, we evaluate the contract again with a guaranteed rate of $i = 0.398\%$, but we incorporate the correlation parameters to look at the level of the provision.

Table 5 shows the results for including a correlation between interest and mortality rates and a correlation between the risky fund H and the mortality rate. We note that the integration of $\rho_{r,\mu}$ and $\rho_{H,\mu}$ into the contract is significant because the insurer incurs a loss when the correlation between finance and mortality is at its worst. In

Table 5 Size of loss in terms of the premium

$\pi = 0.50$, $i = 0.398\%$, $\rho_{G,\mu} = \rho_{r,\mu} \cdot \rho_{r,G}$, $\rho_{r,G} = -20\%$, $\rho_{r,H} = -15\%$							
$\rho_{H,\mu}$	$\rho_{r,\mu}$						
	0%	5%	7%	10%	15%	20%	50%
$\rho_{r,\mu} \cdot \rho_{r,H}$	0	0.042	0.058	0.084	0.125	0.167	0.420
−3%	0.070	0.094	0.104	0.119	0.143	0.167	0.314
−5%	0.117	0.141	0.151	0.166	0.190	0.214	0.361
−7%	0.164	0.188	0.198	0.213	0.237	0.261	0.408
−10%	0.235	0.259	0.268	0.283	0.307	0.332	0.478
−15%	0.352	0.376	0.386	0.401	0.425	0.449	0.596
−20%	0.470	0.494	0.504	0.519	0.543	0.567	0.714
−50%	1.184	1.209	1.218	1.233	1.257	1.281	1.428

Table 6 Fair guaranteed rates i (expressed in %) with respect to the share in the participating contract π for different forms of products

π	Basic contract	Pure endowment and pure investment product	Pure financial product
0.10	2.722	2.254	2.221
0.30	0.987	0.807	0.735
0.50	0.479	0.372	0.264
0.70	0.180	0.119	−0.032
1.00	−0.141	−0.141	−0.385

addition, the impact of $\rho_{H,\mu}$ is greater as the percentage of loss on the level of the provision is higher (see first column, Table 5).

6.4 Numerical comparison for other products

In this section, we look at the impact of each of the forms of product represented in Sect. 5 on the fair guaranteed rates i for the different values of the investment share π.

The third column of Table 6 displays the results of Eq. (28), in which we consider a combination between a pure endowment and a pure financial contract. However, in the last column, we provide solutions for the guaranteed rates in order to maintain the fair relation determined in Eq. (25). This allows us to look at our model without mortality. We compare these cases to the dependence case that is reflected by Eq. (24). The results are displayed in Table 6 by choosing $\rho_{r,G} = -20\%$, $\rho_{r,\mu} = -7\%$, $\rho_{G,\mu} = 5\%$, $\rho_{H,\mu} = 5\%$ and $\sigma_\mu = 0.05\%$.

We notice in Table 6 that the case of a pure financial product delivers the worst values of i, which means that mortality has to be taken into account in life insurance pricing. In addition, we have to consider a correlation between financial and mortality elements since the best results with respect to the guaranteed rates are obtained in the dependence case. Eventually, the cost of the guarantee is more expensive in the

Table 7 Fair guaranteed rates i (expressed in %) for three extreme scenarios of the correlations—annual guarantee without rebalancing (1,000,000 simulations)

π	$\rho_{r,G} = -20\%$, $\rho_{r,H} = -15\%$, $\sigma_\mu = 0.5\%$		
	$\rho_{r,\mu} = 7\%$ $\rho_{G,\mu} = -5\%$ $\rho_{H,\mu} = -5\%$	$\rho_{r,\mu} = 0\%$ $\rho_{G,\mu} = 0\%$ $\rho_{H,\mu} = 0\%$	$\rho_{r,\mu} = -7\%$ $\rho_{G,\mu} = 5\%$ $\rho_{H,\mu} = 5\%$
0.10	2.167	2.301	2.431
0.30	−0.125	−0.040	0.042
0.50	−1.049	−0.981	−0.916
0.70	−1.679	−1.623	−1.569
1.00	−2.457	−2.418	−2.380

contract in which we consider a combination of a pure endowment and a pure financial return (column 3) than in a pure endowment contract (column 2).

Ultimately, the results in Table 6 confirm that introducing a correlation between financial and mortality elements inside the model can be important in the case where the signs of the correlation parameters increase the guaranteed interest rate, as described in Sect. 6.2.

6.5 Annual guarantee

We close the numerical part with results from the fairness relations of Eqs. (34) and (35) with respect to the guaranteed interest rate for an annual guarantee contract with and without rebalancing. The results are based on Monte Carlo simulations with 1,000,000 iterations. In order to find the guaranteed interest rate in each equation, we simulate the Monte Carlo estimates for the expectations in Eqs. (34) and (35), respectively.

Tables 7 and 8 present the effect of the correlation parameters on the fair guaranteed rates, which is coherent with the results obtained in Table 4 for a $\sigma_\mu = 0.5\%$; a negative value of $\rho_{r,\mu}$ and positive values of $\rho_{G,\mu}$ and $\rho_{H,\mu}$ lead to the highest guaranteed rates. Moreover, we notice that the guaranteed rates are higher in the maturity guarantee case (Table 4), and this makes sense because in a maturity guarantee contract, the guarantee and the financial profit are computed only at maturity, so the good years of investment returns finance the bad years.

Furthermore, we also highlight here the increase in the guaranteed interest rate the more we invest in the unit-linked part, which makes the mixed product more attractive than the pure participating product.

Eventually, we perform the same analysis held in Sect. 6.3, which is the size of the loss in percentage of the premium in case the correlation parameters were not taken into account or considered with the wrong sign or size. The results are displayed in Appendix E (Tables 9 and 10), and they are in the same direction as the base case.

Table 8 Fair guaranteed rates i (expressed in %) for three extreme scenarios of the correlations—annual guarantee with rebalancing (1,000,000 simulations)

π	$\rho_{r,G} = -20\%$, $\rho_{r,H} = -15\%$, $\rho_{G,H} = 10\%$, $\sigma_\mu = 0.5\%$		
	$\rho_{r,\mu} = 7\%$ $\rho_{G,\mu} = -5\%$ $\rho_{H,\mu} = -5\%$	$\rho_{r,\mu} = 0\%$ $\rho_{G,\mu} = 0\%$ $\rho_{H,\mu} = 0\%$	$\rho_{r,\mu} = -7\%$ $\rho_{G,\mu} = 5\%$ $\rho_{H,\mu} = 5\%$
0.10	1.714	1.911	2.102
0.30	−0.494	−0.384	−0.279
0.50	−1.321	−1.237	−1.157
0.70	−1.855	−1.790	−1.727
1.00	−2.457	−2.418	−2.380

7 Conclusion

In this paper, we have priced a mixed life insurance product in stochastic models of mortality, interest rates, and stock returns, and we have presented a multi-dimensional approach to explicitly examine the dependence between financial and mortality elements. The mixed product is a combination of a participating contract with a minimum guaranteed rate and a unit-linked life insurance contract with no guarantee, where a guarantee fee is extracted continuously to boost the guarantee on the participating part. Different forms of products have been considered, on the one hand by varying the guarantee period (maturity guarantee or annual guarantee contracts) and on the other hand by differing between pure endowment contracts or mixing pure endowment and pure investment. Using the change of numéraire technique, we obtain a closed form of the premium equivalence for the different forms of products except for the annual guarantee contract due to the dependence between the annual returns.

It is clear from the numerical results that the mixed insurance contract is a more attractive product for the insurer than a pure investment in the participating contract in this complete stochastic environment. Indeed, the concept of a partial guarantee can be an alternative trend for the insurance industry to provide higher investment guarantees.

In this stochastic framework, including the dependence assumption, we have focused on the maturity guarantee contract, analyzed the effect of the correlation parameters, and developed fair combinations, in terms of the parameter's signs, that boost the fair guaranteed rates. The model shows that a negative correlation between interest and mortality rates and positive correlations between the funds and mortality increase the fair guaranteed rates. Hence, we present two central scenarios, based on the signs of the correlation parameters, that give the lowest as well as the highest values of fair guaranteed rates. We show that the impact of the correlation parameters is more pronounced with a higher value of the volatility of mortality. Basically, higher volatility increases the terms of correlations, allowing them to have a greater impact on the fairness relation. In addition, we have studied the percentage of loss on the level of the provision for different values of the correlation parameters, and we see that by considering the wrong sign of the correlation, the loss increases with the size of this parameter.

In future work, we will consider other forms of mortality models as well as calibrate the correlation parameters. We also plan to develop this kind of methodology for other forms of life insurance products (for instance, annuities, endowment insurance, and whole life insurance).

Acknowledgements We want to thank the anonymous reviewers and participants of the 10th International Hybrid Conference on Mathematical and Statistical Methods for Actuarial Sciences and Finance for many insightful remarks and discussions.

Funding This work was supported by the chair Fondation Louvain - Ethias.

Declarations

Conflict of interest No potential conflict of interest was reported by the author(s).

Appendix A: Proof of Proposition 1

After defining the T-forward measure, $\mathrm{d}\tilde{W}_t^r = \mathrm{d}W_t^r + \int_0^t \sigma_r\, B_r(s, T, \theta)ds$ is a $\mathbb{Q}^T$-Brownian motion.

We need then to write the dynamic of the intensity of mortality $\mu_x(t)$ under the T-forward measure, that is,

$$\begin{aligned}
\mathrm{d}\mu_x(t) =& \, b\,(\tilde{\xi}_t - \mu_x(t))\,\mathrm{d}t + \sigma_\mu\, \rho_{r,\mu}\,\mathrm{d}\tilde{W}_t^r + \sigma_\mu\, \overline{\rho}_{G,\mu}\sqrt{1-\rho_{r,\mu}^2}\,\mathrm{d}\tilde{W}_t^G \\
& + \sigma_\mu\, \overline{\overline{\rho}}_{H,\mu}\sqrt{(1-\rho_{r,\mu}^2)(1-\overline{\rho}_{G,\mu}^2)}\,\mathrm{d}\tilde{W}_t^H \\
& + \sigma_\mu\sqrt{(1-\rho_{r,\mu}^2)(1-\overline{\rho}_{G,\mu}^2)(1-\overline{\overline{\rho}}_{H,\mu}^2)}\,\mathrm{d}\tilde{W}_t^\mu; \\
\tilde{\xi}_t =& \, \xi_t - \sigma_\mu\, \sigma_r\, \rho_{r,\mu}\frac{B_r(t, T, \theta)}{b},
\end{aligned}$$

where $\mathrm{d}\tilde{W}_t^G = \mathrm{d}W_t^G$, $\mathrm{d}\tilde{W}_t^H = \mathrm{d}W_t^H$ and $\mathrm{d}\tilde{W}_t^\mu = \mathrm{d}W_t^\mu$ are $\mathbb{Q}^T$-Brownian motions.

As a result,

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}^T}\left[e^{-\int_0^T \mu_x(s)ds}\right] =& \exp\Big(-\mu_x(0)\cdot B_\mu(0, T, b) \\
& -b\int_0^T \tilde{\xi}_s \cdot B_\mu(s, T, b)ds + \frac{\sigma_\mu^2}{2}\int_0^T B_\mu^2(s, T, b)ds\Big) \\
=& \exp\Big(- \mu_x(0)\cdot B_\mu(0, T, b) - b\int_0^T \xi_s \cdot B_\mu(s, T, b)ds \\
& + \sigma_r\, \sigma_\mu\, \rho_{r,\mu}\int_0^T B_r(s, T, \theta)\cdot B_\mu(s, T, b)ds + \frac{\sigma_\mu^2}{2}\int_0^T B_\mu^2(s, T, b)ds\Big) \\
=& \exp\Big(-\mu_x(0)\cdot B_\mu(0, T, b) \\
& - b\int_0^T \xi_s \cdot B_\mu(s, T, b)ds + \frac{\sigma_\mu^2}{2}\int_0^T B_\mu^2(s, T, b)ds\Big)
\end{aligned}$$

$$
\begin{aligned}
&\cdot \exp\left(\sigma_r\,\sigma_\mu\,\rho_{r,\mu}\int_0^T B_r(s,T,\theta)\cdot B_\mu(s,T,b)ds\right)\\
&= p_x(0,T)\cdot\rho(0,T),
\end{aligned}
$$

where $\rho(0,T)$ is the term of correlation that arises from the dependence only between the interest rate and the mortality rate, defined in Eq. (16).

Appendix B: Proof of Proposition 2

By computing the Radon–Nikodym derivative, we obtain

$$
\begin{aligned}
\frac{\mathrm{d}\mathbb{Q}^\mu}{\mathrm{d}\mathbb{Q}^T} &= \exp\Bigg(-\sigma_\mu\,\rho_{r,\mu}\int_0^T B_\mu(s,T,b)\,\mathrm{d}\tilde{W}_s^r - \sigma_\mu\,\overline{\rho}_{G,\mu}\sqrt{1-\rho_{r,\mu}^2}\int_0^T B_\mu(s,T,b)\,\mathrm{d}\tilde{W}_s^G\\
&- \sigma_\mu\,\overline{\overline{\rho}}_{H,\mu}\sqrt{(1-\rho_{r,\mu}^2)(1-\overline{\rho}_{G,\mu}^2)}\int_0^T B_\mu(s,T,b)\,\mathrm{d}\tilde{W}_s^H\\
&- \sigma_\mu\sqrt{(1-\rho_{r,\mu}^2)(1-\overline{\rho}_{G,\mu}^2)(1-\overline{\overline{\rho}}_{H,\mu}^2)}\int_0^T B_\mu(s,T,b)\,\mathrm{d}\tilde{W}_s^\mu\\
&- \frac{1}{2}\sigma_\mu^2\,\rho_{r,\mu}^2\int_0^T B_\mu^2(s,T,b)\,\mathrm{d}s - \frac{1}{2}\sigma_\mu^2\,\overline{\rho}_{G,\mu}^2(1-\rho_{r,\mu}^2)\int_0^T B_\mu^2(s,T,b)\,\mathrm{d}s\\
&- \frac{1}{2}\sigma_\mu^2\,\overline{\overline{\rho}}_{H,\mu}^2(1-\rho_{r,\mu}^2)(1-\overline{\rho}_{G,\mu}^2)\int_0^T B_\mu^2(s,T,b)\,\mathrm{d}s\\
&- \frac{1}{2}\sigma_\mu^2(1-\rho_{r,\mu}^2)(1-\overline{\rho}_{G,\mu}^2)(1-\overline{\overline{\rho}}_{H,\mu}^2)\int_0^T B_\mu^2(s,T,b)\,\mathrm{d}s\Bigg).
\end{aligned}
$$

Then, $\begin{cases} \mathrm{d}\hat{W}_t^r = \mathrm{d}\tilde{W}_t^r + \sigma_\mu\,\rho_{r,\mu}\int_0^t B_\mu(s,T,b)\mathrm{d}s\\ \mathrm{d}\hat{W}_t^G = \mathrm{d}\tilde{W}_t^G + \sigma_\mu\,\overline{\rho}_{G,\mu}\sqrt{1-\rho_{r,\mu}^2}\int_0^t B_\mu(s,T,b)\mathrm{d}s\\ \mathrm{d}\hat{W}_t^H = \mathrm{d}\tilde{W}_t^H + \sigma_\mu\,\overline{\overline{\rho}}_{H,\mu}\sqrt{(1-\rho_{r,\mu}^2)(1-\overline{\rho}_{G,\mu}^2)}\int_0^t B_\mu(s,T,b)\mathrm{d}s\\ \mathrm{d}\hat{W}_t^\mu = \mathrm{d}\tilde{W}_t^\mu + \sigma_\mu\sqrt{(1-\rho_{r,\mu}^2)(1-\overline{\rho}_{G,\mu}^2)(1-\overline{\overline{\rho}}_{H,\mu}^2)}\int_0^t B_\mu(s,T,b)\mathrm{d}s \end{cases}$
are $\mathbb{Q}^\mu$-Brownian motions.

The objective is to compute the expectation of $\left(\frac{G_T}{G_0}-K\right)^+$ under the new measure. So, we will express this expression as follows

$$
\left(\frac{G_T}{G_0}-K\right)^+ = \frac{G_T}{G_0}\cdot\mathbb{1}_{\frac{G_T}{G_0}>K} - K\cdot\mathbb{1}_{\frac{G_T}{G_0}>K}.
$$

Hence,

$$
\mathbb{E}^{\mathbb{Q}^\mu}\left[\left(\frac{G_T}{G_0}-K\right)^+\right] = \mathbb{E}^{\mathbb{Q}^\mu}\left[\frac{G_T}{G_0}\cdot\mathbb{1}_{\frac{G_T}{G_0}>K}\right] - K\cdot\mathbb{E}^{\mathbb{Q}^\mu}\left[\mathbb{1}_{\frac{G_T}{G_0}>K}\right]. \tag{37}
$$

Let's express the dynamic of the fund G under $\mathbb{Q}^{\mu}$ in order to get the solution of its differential equation. We move first from the risk-neutral measure to the T-forward measure, then to the forward-mortality measure.

Under the forward measure:

$$\begin{aligned}\mathrm{d}G_t &= (r_t - \sigma_G\,\sigma_r\,\rho_{r,G}\,B_r(t,T,\theta))\,G_t\,\mathrm{d}t + \sigma_G\,\rho_{r,G}\,G_t\,\mathrm{d}\tilde{W}_t^r \\ &+ \sigma_G\sqrt{1-\rho_{r,G}^2}\,G_t\,\mathrm{d}\tilde{W}_t^G.\end{aligned}$$

Under the forward-mortality measure:

$$\begin{aligned}\mathrm{d}G_t =& \Big(r_t - \sigma_G\,\sigma_r\,\rho_{r,G}\,B_r(t,T,\theta) - \sigma_G\,\sigma_\mu\,\rho_{G,\mu}\,B_\mu(t,T,b)\Big)\,G_t\,\mathrm{d}t \\ &+ \sigma_G\,\rho_{r,G}\,G_t\,\mathrm{d}\hat{W}_t^r + \sigma_G\sqrt{1-\rho_{r,G}^2}\,G_t\,\mathrm{d}\hat{W}_t^G.\end{aligned}$$

After integration, we obtain the SDE of the latter expression as follows:

$$\begin{aligned}G_t = G_0\cdot\exp\Bigg(&\int_0^t r_s\mathrm{d}s - \sigma_G\,\sigma_r\,\rho_{r,G}\int_0^t B_r(s,T,\theta)\mathrm{d}s - \sigma_G\,\sigma_\mu\,\rho_{G,\mu}\int_0^t B_\mu(s,T,b)\mathrm{d}s \\ &- \frac{\sigma_G^2}{2}\cdot t + \sigma_G\,\rho_{r,G}\int_0^t \mathrm{d}\hat{W}_s^r \\ &+ \sigma_G\sqrt{1-\rho_{r,G}^2}\int_0^t \mathrm{d}\hat{W}_s^G\Bigg).\end{aligned}$$

This can also be written as

$$\begin{aligned}\frac{G_T}{G_0} = \frac{1}{P(0,T)}\cdot\exp\Bigg(&-\sigma_G\,\sigma_r\,\rho_{r,G}\int_0^T B_r(s,T,\theta)\mathrm{d}s - \sigma_G\,\sigma_\mu\,\rho_{G,\mu}\int_0^T B_\mu(s,T,b)\mathrm{d}s \\ &- \frac{\sigma_r^2}{2}\int_0^T B_r^2(s,T,\theta)\mathrm{d}s - \sigma_r\,\sigma_\mu\,\rho_{r,\mu}\int_0^T B_r(s,T,\theta)\cdot B_\mu(s,T,b)\mathrm{d}s \\ &- \frac{\sigma_G^2}{2}\cdot T + \int_0^T (\sigma_G\,\rho_{r,G} + \sigma_r\,B_r(s,T,\theta))\mathrm{d}\hat{W}_s^r + \sigma_G\sqrt{1-\rho_{r,G}^2}\int_0^T \mathrm{d}\hat{W}_s^G\Bigg).\end{aligned}$$

Under $\mathbb{Q}^{\mu}$, $\frac{G_T}{G_0}$ is log-normally distributed which allows us to write Equation (37) as

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}^{\mu}}\left[\left(\frac{G_T}{G_0}-K\right)^+\right] &= \mathbb{E}^{\mathbb{Q}^{\mu}}\left[\frac{G_T}{G_0}\Big|\frac{G_T}{G_0}>K\right]\cdot\mathbb{Q}^{\mu}\left[\frac{G_T}{G_0}>K\right] - K\cdot\mathbb{Q}^{\mu}\left[\frac{G_T}{G_0}>K\right] \\ &= e^{\hat{m}_G+\frac{\hat{v}_G}{2}}\cdot\Phi\left(\frac{\hat{m}_G+\hat{v}_G-\ln K}{\sqrt{\hat{v}_G}}\right) - K\cdot\Phi\left(\frac{\hat{m}_G-\ln K}{\sqrt{\hat{v}_G}}\right),\end{aligned}$$

where Φ is the cumulative distribution function of a standard normal variable and $\hat{m}_G$ and $\hat{v}_G$ are, respectively, the expectation and the variance of $\ln\left(\frac{G_T}{G_0}\right)$ under the forward-mortality measure, defined in Eqs. (17) and (18).

Appendix C: Proof of Proposition 3

We start by writing the dynamic of the fund F under the forward-mortality measure

$$\begin{aligned}
\mathrm{d}F_t = & \big(r_t - \varepsilon - \sigma_H\,\sigma_r\,\rho_{r,H}\,B_r(t,T,\theta) - \sigma_H\,\sigma_\mu\,\rho_{H,\mu}\,B_\mu(t,T,b)\big)\,F_t\,\mathrm{d}t \\
& + \sigma_H\,\rho_{r,H}\,F_t\,\mathrm{d}\hat{W}_t^r + \sigma_H\,\overline{\rho}_{G,H}\sqrt{1-\rho_{r,H}^2}\,F_t\,\mathrm{d}\hat{W}_t^G \\
& + \sigma_H\sqrt{(1-\rho_{r,H}^2)(1-\overline{\rho}_{G,H}^2)}\,F_t\,\mathrm{d}\hat{W}_t^H.
\end{aligned}$$

After integration,

$$\begin{aligned}
F_t = F_0\cdot\exp\Bigg(& \int_0^t r_s\mathrm{d}s - \varepsilon\cdot t - \sigma_H\,\sigma_r\,\rho_{r,H}\int_0^t B_r(s,T,\theta)\mathrm{d}s \\
& - \sigma_H\,\sigma_\mu\,\rho_{H,\mu}\int_0^t B_\mu(s,T,b)\mathrm{d}s \\
& - \frac{\sigma_H^2}{2}\cdot t + \sigma_H\,\rho_{r,H}\int_0^t \mathrm{d}\hat{W}_s^r + \sigma_H\,\overline{\rho}_{G,H}\sqrt{1-\rho_{r,H}^2}\int_0^t \mathrm{d}\hat{W}_s^G \\
& + \sigma_H\sqrt{(1-\rho_{r,H}^2)(1-\overline{\rho}_{G,H}^2)}\int_0^t \mathrm{d}\hat{W}_s^H\Bigg),
\end{aligned}$$

hence, the expression of $\frac{F_T}{F_0}$ can be written as

$$\begin{aligned}
\frac{F_T}{F_0} = \frac{1}{P(0,T)}\cdot\exp\Bigg(& -\varepsilon\cdot T - \sigma_H\,\sigma_r\,\rho_{r,H}\int_0^T B_r(s,T,\theta)\mathrm{d}s \\
& - \sigma_H\,\sigma_\mu\,\rho_{H,\mu}\int_0^T B_\mu(s,T,b)\mathrm{d}s \\
& - \frac{\sigma_r^2}{2}\int_0^T B_r^2(s,T,\theta)\mathrm{d}s \\
& - \sigma_r\,\sigma_\mu\,\rho_{r,\mu}\int_0^T B_r(s,T,\theta)\cdot B_\mu(s,T,b)\mathrm{d}s - \frac{\sigma_H^2}{2}\cdot T \\
& + \int_0^T (\sigma_H\,\rho_{r,H} + \sigma_r\,B_r(s,T,\theta))\mathrm{d}\hat{W}_s^r \\
& + \sigma_H\,\overline{\rho}_{G,H}\sqrt{1-\rho_{r,H}^2}\int_0^T \mathrm{d}\hat{W}_s^G \\
& + \sigma_H\sqrt{(1-\rho_{r,H}^2)(1-\overline{\rho}_{G,H}^2)}\int_0^T \mathrm{d}\hat{W}_s^H\Bigg).
\end{aligned}$$

From this end, we can compute $\mathbb{E}^{\mathbb{Q}^\mu}\left[\frac{F_T}{F_0}\right] = e^{\hat{m}_F + \frac{\hat{v}_F}{2}}$ with

$$\hat{m}_F = E\left[\ln\left(\frac{F_T}{F_0}\right)\right] = \ln\left(\frac{1}{P(0,T)}\right) - \varepsilon\cdot T - \frac{\sigma_H^2}{2}\cdot T - \frac{\sigma_H\,\sigma_r\,\rho_{r,H}}{\theta}\,[T - B_r(0,T,\theta)]$$

$$
\begin{aligned}
&- \frac{\sigma_H\, \sigma_\mu\, \rho_{H,\mu}}{b} \left[T - B_\mu(0,T,b)\right] \\
&- \frac{\sigma_r\, \sigma_\mu\, \rho_{r,\mu}}{\theta \cdot b} \left[T + B_{r,\mu}(0,T,\theta + b) - B_r(0,T,\theta) - B_\mu(0,T,b)\right] \\
&- \frac{\sigma_r^2}{2\theta^2} \left[T + B_r(0,T,2\theta) - 2\, B_r(0,T,\theta)\right]
\end{aligned}
$$

$$
\begin{aligned}
\hat{\upsilon}_F = V\left[\ln\left(\frac{F_T}{F_0}\right)\right] &= \sigma_H^2 \cdot T + 2 \cdot \frac{\sigma_H\, \sigma_r\, \rho_{r,H}}{\theta} \left[T - B_r(0,T,\theta)\right] \\
&+ \frac{\sigma_r^2}{\theta^2} \left[T + B_r(0,T,2\theta) - 2\, B_r(0,T,\theta)\right],
\end{aligned}
$$

where $\hat{m}_F$ and $\hat{\upsilon}_F$ are, respectively, the expectation and the variance of $\ln\left(\frac{F_T}{F_0}\right)$ under the forward-mortality measure.

Appendix D: Proof of Proposition 4

Referring to Sect. 4, the first and the third expectations can be solved explicitly as follows:

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[V_T \cdot e^{-\int_0^T r_s ds} \cdot e^{-\int_0^T \mu_x(s)ds}\right] &= \pi \cdot P \cdot \frac{p_x(0,T)}{{}_T p_x} \cdot \rho(0,T) \cdot \left(P(0,T) \cdot (1+i)^T + \beta \cdot c_0^T\right) \\
&+ (1-\pi) \cdot P \cdot p_x(0,T) \cdot \rho'(0,T) \cdot e^{-\varepsilon \cdot T},
\end{aligned}
$$

and

$$
\mathbb{E}^{\mathbb{Q}}\left[\frac{F_{j+1}}{F_0} \cdot e^{-\int_0^{j+1} r_s ds} \cdot e^{-\int_0^{j+1} \mu_x(s)ds}\right] = p_x(0, j+1) \cdot \rho'(0, j+1) \cdot e^{-\varepsilon \cdot (j+1)}.
$$

Whereas the expectation in the middle can be written as

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[\frac{F_{j+1}}{F_0} \cdot e^{-\int_0^{j+1} r_s ds} \cdot e^{-\int_0^{j} \mu_x(s)ds}\right] \\
&= \mathbb{E}^{\mathbb{Q}}\Bigg[\underbrace{\frac{F_j}{F_0} \cdot e^{-\int_0^{j} r_s ds} \cdot e^{-\int_0^{j} \mu_x(s)ds}}_{\mathbf{I}} \cdot \underbrace{\frac{F_{j+1}}{F_j} \cdot e^{-\int_j^{j+1} r_s ds}}_{\mathbf{II}}\Bigg]. \qquad (38)
\end{aligned}
$$

We then develop the two processes I and II:

$$
\begin{aligned}
\mathrm{I} &= \frac{F_j}{F_0} \cdot e^{-\int_0^{j} r_s ds} \cdot e^{-\int_0^{j} \mu_x(s)ds} \\
&= \exp\Bigg(-\varepsilon \cdot j - \frac{\sigma_H^2}{2} \cdot j - \mu_x(0) \cdot B_\mu(0, j, b)
\end{aligned}
$$

$$
\begin{aligned}
&- b\int_0^j \xi_s \cdot B_\mu(s,j,b)\mathrm{d}s + \int_0^j (\sigma_H\rho_{r,H} - \sigma_\mu\rho_{r,\mu}B_\mu(s,j,b))\mathrm{d}W_s^r \\
&+ \int_0^j (\sigma_H\overline{\rho}_{G,H}\sqrt{1-\rho_{r,H}^2} - \sigma_\mu\overline{\rho}_{G,\mu}\sqrt{1-\rho_{r,\mu}^2}B_\mu(s,j,b))\mathrm{d}\overline{W}_s^G \\
&+ \int_0^j (\sigma_H\sqrt{(1-\rho_{r,H}^2)(1-\overline{\rho}_{G,H}^2)} - \sigma_\mu\overline{\overline{\rho}}_{H,\mu}\sqrt{(1-\rho_{r,\mu}^2)(1-\overline{\rho}_{G,\mu}^2)}B_\mu(s,j,b))\mathrm{d}\overline{\overline{W}}_s^H \\
&- \sigma_\mu\sqrt{(1-\rho_{r,\mu}^2)(1-\overline{\rho}_{G,\mu}^2)(1-\overline{\overline{\rho}}_{H,\mu}^2)}\int_0^j B_\mu(s,j,b)\mathrm{d}\overline{\overline{\overline{W}}}_s^\mu\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{II} &= \frac{F_{j+1}}{F_j}\cdot e^{-\int_j^{j+1} r_s ds} \\
&= \exp\Big(-\varepsilon - \frac{\sigma_H^2}{2} + \sigma_H\rho_{r,H}\int_j^{j+1}\mathrm{d}W_s^r + \sigma_H\overline{\rho}_{G,H}\sqrt{1-\rho_{r,H}^2}\int_j^{j+1}\mathrm{d}\overline{W}_s^G \\
&\quad + \sigma_H\sqrt{(1-\rho_{r,H}^2)(1-\overline{\rho}_{G,H}^2)}\int_j^{j+1}\mathrm{d}\overline{\overline{W}}_s^H\Big).
\end{aligned}
$$

The independence between these two processes allows us to split Eq. (38) to

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[\frac{F_j}{F_0}\cdot e^{-\int_0^j r_s ds}\cdot e^{-\int_0^j \mu_x(s)ds}\right]\cdot\mathbb{E}^{\mathbb{Q}}\left[\frac{F_{j+1}}{F_j}\cdot e^{-\int_j^{j+1} r_s ds}\right] \\
&\quad = p_x(0,j)\cdot\rho'(0,j)\cdot e^{-\varepsilon(j+1)}.
\end{aligned}
$$

Appendix E: Size of the loss in percentage of the premium—annual guarantee contract

See Tables 9 and 10.

Table 9 Size of loss in terms of the premium—annual guarantee without rebalancing

$\pi = 0.50,\ i = -0.981\%,\ \rho_{G,\mu} = \rho_{r,\mu}\cdot\rho_{r,G},\ \rho_{r,G} = -20\%,\ \rho_{r,H} = -15\%$							
$\rho_{H,\mu}$	$\rho_{r,\mu}$						
	0%	5%	7%	10%	15%	20%	50%
$\rho_{r,\mu}\cdot\rho_{r,H}$	0	0.036	0.051	0.073	0.109	0.146	0.365
−3%	0.070	0.089	0.097	0.108	0.127	0.146	0.259
−5%	0.117	0.136	0.143	0.155	0.174	0.196	0.306
−7%	0.164	0.183	0.190	0.202	0.221	0.239	0.353
−10%	0.235	0.253	0.261	0.272	0.291	0.310	0.424
−15%	0.352	0.371	0.379	0.390	0.409	0.428	0.541
−20%	0.470	0.489	0.497	0.508	0.527	0.546	0.659
−50%	1.184	1.203	1.211	1.222	1.241	1.260	1.373

Table 10 Size of loss in terms of the premium—annual guarantee with rebalancing

$\pi = 0.50,\quad i = -1.237\%,\quad \rho_{G,\mu} = \rho_{r,\mu} \cdot \rho_{r,G},\quad \rho_{r,G} = -20\%,\quad \rho_{r,H} = -15\%$

$\rho_{H,\mu}$	$\rho_{r,\mu}$						
	0%	5%	7%	10%	15%	20%	50%
$\rho_{r,\mu} \cdot \rho_{r,H}$	0	0.036	0.050	0.071	0.107	0.143	0.357
−3%	0.073	0.091	0.097	0.108	0.125	0.143	0.247
−5%	0.122	0.139	0.146	0.157	0.174	0.192	0.296
−7%	0.171	0.188	0.195	0.206	0.223	0.240	0.345
−10%	0.244	0.262	0.268	0.279	0.296	0.314	0.419
−15%	0.366	0.384	0.391	0.401	0.419	0.436	0.541
−20%	0.489	0.506	0.513	0.524	0.541	0.559	0.664
−50%	1.227	1.245	1.252	1.262	1.280	1.297	1.403

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