Inequality-averse well-being measurement*

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Abstract

We construct individual well-being measures that respect individual preferences and depend on the bundles of goods consumed by the individual. Building on Fleurbaey and Maniquet (2017a) where general families of well-being measures are identified, we introduce basic transfer principles that apply either to bundles or directly to indifference sets and we characterize specific well-being measures that involve either the ray utility or the money-metric utility.

JEL Classification: D63, I32.

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1 Introduction

The measurement of individual well-being is one of the most vexing issues in welfare economics. In previous works (Fleurbaey and Maniquet 2011, 2017a), we have argued that it is possible to make progress by relying on fairness principles that can guide interpersonal comparisons. Fairness principles tell us that an individual, in a particular situation depicted by a commodity bundle and a preference ordering, should be considered better off than another individual in a situation also described by his bundle and preferences.

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That is, fairness principles allow us to replace interpersonal *utility* comparison with a combination of commodity bundle comparisons and the use of ordinal "non-comparable" preferences as the only data on subjective well-being. It may seem counter-intuitive that comparability can be constructed with non-comparable data on preferences, but it directly follows from fairness principles that do allow interpersonal comparisons of relative advantage with such information. And it aligns very well with the common practice of interpersonal comparisons in terms of income or wealth, which has received support from philosophers of equality of resources such as Rawls (1982) and Dworkin (2000).

Let us note that the informational basis of our well-being judgements, commodity bundles and ordinal non-comparable preferences, is the same as that under which the Kaldor-Hicks-Scitovsky compensation principle has been developed (see Suzumura 1980 for a discussion of this approach). On the other hand, our informational basis is poorer than that of the extended sympathy approach, in which individuals are allowed to make judgements comparing any two other individuals in two different situations, an approach which Kotaro Suzumura has thoroughly analyzed (Suzumura 1981a and b). Fairness requirements, such as no-envy, which Suzumura has examined in the extended-sympathy context, are more traditionally defined in our more parsimonious framework.

In Fleurbaey and Maniquet (2017a), two general families of well-being measures were characterized on the basis of properties that considered how to compare indifference sets —note that if one respects individual preferences, only indifference sets matter, not the particular bundle that is effectively consumed. The most basic property is that an indifference set that is everywhere above another one corresponds to a better situation. It is less obvious to deal with crossing indifference sets, and in that earlier paper we proposed two principles dealing with crossings, that implied two different approaches to well-being measurement. However, these results did not pin down specific measures and remained at the level of general classes of measures.

In this paper, we propose an extension of the analysis that introduces a new set of properties having to do with inequality aversion. These properties require that, the better off an individual in a specific situation, the less of a well-being improvement follows from any given commodity bundle increment to it. These properties turn out to pin down specific members of the families of well-being measures, namely, the ray utility and the money-metric utility. The ray utility measures well-being by the fraction of a reference bundle that is as good as the current bundle for the individual's preferences. The money-metric utility measures it by the income needed to obtain the current satisfaction of the individual, at given reference market prices. These measures are in fact standard, but have generally been considered to be rather arbitrary and merely used as examples. For instance, Samuelson (1977) mentions the ray utility as an example of a utility function that could be used in a Bergson-Samuelson social welfare function. Money-metric utility has been criticized by Donaldson (1992) and recently defended by Fleurbaey and Blanchet (2013).

The well-being measures we justify in this paper are intended to be arguments of social welfare functions, but the precise way of aggregating individual well-being remains outside the current analysis. Remember, however, that considering indifference sets as we do here is sufficient to escape Arrow's impossibility and build Paretian and fair social welfare functions, as it has been established by several authors, including Suzumura (See Fleurbaey, Suzumura and Tadenuma, 2005a and b).

The paper is organized as follows. In the next section, we introduce the model and recall the results obtained by Fleurbaey and Maniquet (2017a) with properties about the comparison of crossing and non-crossing indifference sets. Then, in section 3, we introduce the inequality aversion properties and state the results characterizing the ray utility and the money-metric utility. Section 4 gathers the proofs. Section 5 contains a brief conclusion.

2 Two general families

In our model, identical to Fleurbaey and Maniquet (2017a), there are K divisible goods. The consumption set is $X = \mathbb{R}_+^K$ and over this set, individual preferences are assumed to be continuous, convex and monotonic.[1] Let $\mathcal{R}$ denote the set of all such preferences. A well-being measure is a function $W : X \times \mathcal{R} \to \mathbb{R}$, such that $W(x, R)$ is the well-being level of an agent consuming bundle x with preferences R. Observe that W does not depend on any additional data; in particular, it does not depend on subjective well-being, as the only subjective information used by W is contained in the ordinal preference ordering R.

[1] We use $>, \geqslant$ and $\gg$ to denote the vector inequalities. Preferences R are monotonic if and only if $x > x'$ implies $x \, R \, x'$ and $x \gg x'$ implies $x \, P \, x'$.

Throughout the paper, we require W to be continuous in x, and to respect individual preferences, in the sense that for all $x, x' \in X$, $R \in \mathcal{R}$,

$$x\, R\, x' \Leftrightarrow W(x,R) \geq W(x',R).$$

The latter condition is reminiscent of Pareto efficiency in the social choice literature. Here, it represents our desire to define well-being in a way that is consistent with what agents themselves think about how the different dimensions of life should be aggregated.

For $x \in \mathbb{R}_+^K$, $R \in \mathcal{R}$, we let $L(x,R), U(x,R)$ and $I(x,R)$ denote the lower, upper and indifference contour of R at x, respectively:

$$\begin{aligned} L(x,R) &= \{x' \in \mathbb{R}_+^K | x\, R\, x'\}, \\ U(x,R) &= \{x' \in \mathbb{R}_+^K | x'\, R\, x\}, \\ I(x,R) &= L(x,R) \cap U(x,R). \end{aligned}$$

We now present the two basic properties that we would like to impose on well-being measures. *Supremum Nested Contour* requires that if the lower contour set of one agent lies in the interior of the union of the lower contour sets of two other agents, then the well being of the former agent is strictly lower than that of at least one of the latter agents.[2]

Axiom 1 Supremum Nested Contour
For all $x, x', x'' \in X$, $R, R', R'' \in \mathcal{R}$, if $L(x,R) \subset$ interior$[L(x',R') \cup L(x'',R'')]$, then $W(x,R) < \max\{W(x',R'), W(x'',R'')\}$.

Fig. 1 illustrates this property with three individuals. According to *Supremum Nested Contour*, the situation (x,R) cannot be at least as good as the two other situations.

The characterization of well-being measures satisfying *Supremum Nested Contour* is recalled in the following lemma (Th. 1 in Fleurbaey and Maniquet 2017a). We let $\mathcal{R}^w$ denote the subdomain of preferences exhibiting the

[2] In Fleurbaey and Maniquet (2017a), a stronger version of the axiom is defined. It requires the property to hold even if $L(x',R')$ and $L(x'',R'')$ are replaced with a countable sequence of lower contour sets. Existing proofs in the literature, such as the one developed in Chambers and Miller (2014), however, suggest that the version of Fleurbaey and Maniquet (2017a) is unduly strong. This point is acknowledged in Fleurbaey and Maniquet (2017b), where the Chambers and Miller (2014) proof is adapted to the present context. This is why we stick to the weaker (binary) version of *Supremum* and *Infimum Nested Contour* in this paper.

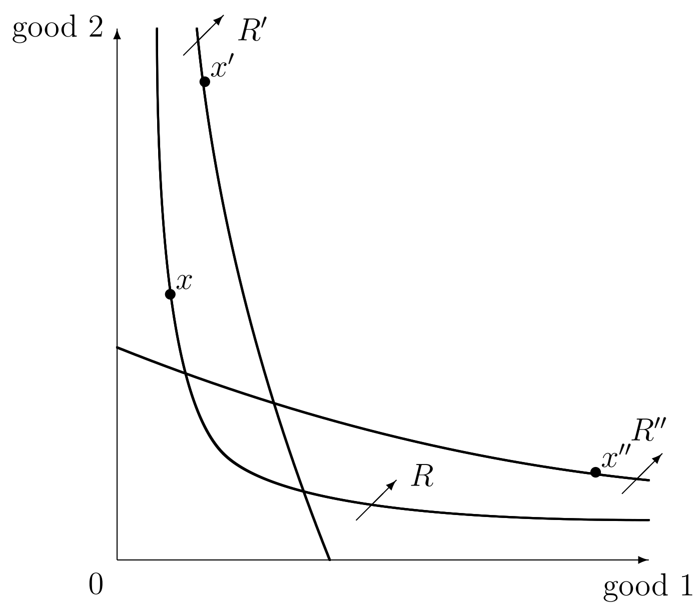

Figure 1: *Supremum Nested Contour*: $W(x, R) < \max\{W(x', R'), W(x'', R'')\}$.

property that all $R^w \in \mathcal{R}^w$ have a maximal bundle over each lower contour set of each $R \in \mathcal{R}$, and that $W(\cdot, R^w)$ increases continuously with these lower contour sets.

Lemma 1 *A well-being measure W over X satisfies* Supremum Nested Contour *if and only if there exists $R^w \in \mathcal{R}^w$ such that W satisfies: for all $x \in X$ and $R \in \mathcal{R}$:*

$$W(x, R) = \max_{x' \in L(x,R)} W(x', R^w).$$

The second property, *Infimum Nested Contour*, also refers to the idea that some preferences are intermediary. Consider $x, x', x'' \in X$ and $R, R', R'' \in \mathcal{R}$. We say that (x, R) is intermediary between (x', R') and (x'', R'') if for all $y \in X$ such that $y\, I\, x$, there exist $y', y'' \in X$ such that $y'\, I'\, x'$ and $y''\, I''\, x''$ and $y = \alpha y' + (1-\alpha) y''$, for some $\alpha \in [0, 1]$. Observe that this is equivalent to requiring that $U(x, R)$ be included in the convex hull of the union of $U(x', R')$ and $U(x'', R'')$.[3] We can then state the axiom that if (x, R) is intermediary between (x', R') and (x'', R'') in this sense, then the well-being at (x, R) must be greater than either that of (x', R'), or that of (x'', R''), or both. Let CH denote the convex hull operator.

Axiom 2 Infimum Nested Contour
For all $x, x', x'' \in X$, $R, R', R'' \in \mathcal{R}$, if $U(x, R) \subset interior[CH(U(x', R') \cup U(x'', R''))]$, then $W(x, R) > \min\{W(x', R'), W(x'', R'')\}$.

Fig. 2 illustrates the property. The situation (x, R) cannot be at least as bad as the two other situations.

For a similar reason as for *Supremum Nested Contour* above, the characterization of well-being measures satisfying *Infimum Nested Contour* requires that we define a subdomain of preferences. We denote $\mathcal{R}^b$ the subdomain of preferences R^b exhibiting the property that all $L(x, R^b)$ are compact. The following lemma is Theorem 2 in Fleurbaey and Maniquet (2017a).

Lemma 2 *A well-being measure W over X satisfies* Infimum Nested Contour *if and only if there exists $R^b \in \mathcal{R}^b$ such that W satisfies: for all $x \in X$ and $R \in \mathcal{R}$:*

$$W(x, R) = \min_{x' \in U(x,R)} W(x', R^b).$$

[3]The convex hull of a set is the smallest convex set containing that set.

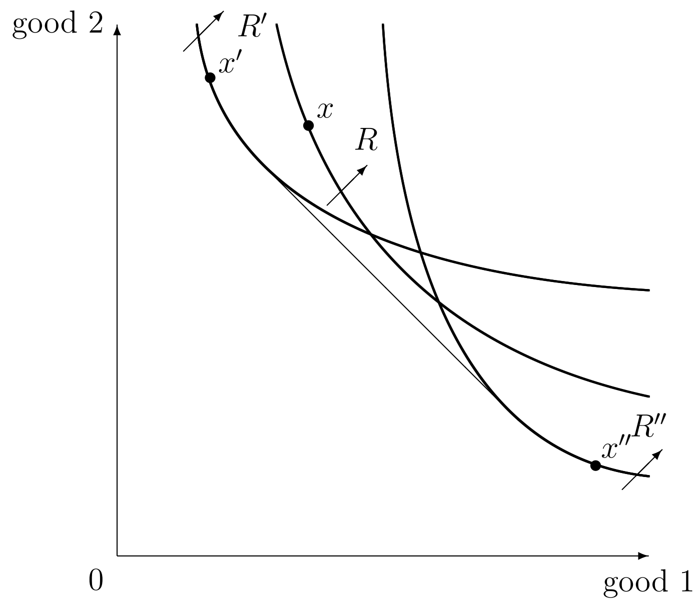

Figure 2: *Infimum Nested Contour*: $W(x,R) > \min\{W(x',R'), W(x'',R'')\}$.

It is useful to notice that Supremum Nested Contour and Infimum Nested Contour both imply the very basic property that an indifference set that is everywhere above another corresponds to a greater level of well-being:

Axiom 3 Nested Contour
For all $x, x' \in X$, $R, R' \in \mathcal{R}$, if $U(x,R) \cap L(x',R') = \emptyset$, then $W(x,R) > W(x',R')$.

Let us briefly comment on the fact that W does not depend on subjective well-being beyond ordinal preferences. One could of course develop a theory where subjective well-being, say U (understood as a function of x), is recorded in the data, with a well-being function $W(x,U)$. Then the properties introduced in this section would imply that actually only R is really used, and not U. Indeed, consider Nested Contour. By continuity of W, it would imply that $W(x,U) = W(x,U')$ whenever U and U' represent the same ordering R. Therefore only R is relevant and other information contained in U is simply ignored by W.

3 Refining the basic families of measures

In this section, we study the further requirement that if an additional consumption becomes available, it should be assigned in priority to the agent whose consumption and well-being are lower. More precisely, we study properties that well-being measures must satisfy so that any egalitarian way of aggregating individual well-being would satisfy such a requirement. It turns out to be difficult to combine this requirement with the properties we have already defined, as we now show.

The first axiom we define requires that if we look at two bundles x and x' such that $x' \gg x$, and if a fraction of the difference between the two bundles becomes available, then assigning this fraction of bundles to the agent consuming x increases her well-being more than if it is assigned to the other agent, provided their indifference curves do not intersect (which guarantees that the agent consuming x has a lower well-being).

Axiom 4 Nested Priority
For all $x, x' \in X$, $R, R' \in \mathcal{R}$, $\lambda \in (0,1)$, if $x' \gg x$ and $L(x + \lambda(x'-x), R) \cap U(x', R') = \emptyset$, then

$$W(x + \lambda(x'-x), R) - W(x, R) > W(x' + \lambda(x'-x), R') - W(x', R').$$

Nested Priority is a way to capture the value of resource inequality aversion in this framework of well-being measurement. Of course, inequality aversion is a property bearing on allocations of resources and not on the well-being reached by single individuals. The relationship between the two notions can be seen as follows. Assume an amount $(1-\lambda)x + (1+\lambda)x'$ of resources, $x \ll x'$, has to be allocated between two individuals. Allocation $(x + \lambda(x'-x), x')$ is clearly less unequal than allocation $(x, x' + \lambda(x'-x))$, so that an inequality averse social welfare function should strictly prefer the former allocation. If such a social welfare function can be written as the sum of individual well-being measures, denoted w, this amounts to require that w satisfies the property that $w(x+\lambda(x'-x)) + w(x') > w(x) + w(x'+\lambda(x'-x))$. Rearranging the terms of the sums and adding the restriction on individuals' preferences, we obtain *Nested Priority*.

This property turns out to be too strong, in the following sense.

Lemma 3 *No well-being measure satisfies* Nested Priority *and* Nested Contour*.*

This incompatibility is different from seemingly similar results such as the Fleurbaey and Trannoy (2003) impossibility of a Paretian egalitarian, or related results (for a review of these results, see Weymark, 2013). Indeed, our proviso that indifference curves should not intersect is the typical way of escaping the difficulties highlighted in these papers.

The impossibility stated in the Lemma comes from the conflict between insisting on indifference surfaces, as required by *Nested Contour*, and insisting on bundles, as required by *Nested Priority*. This conflict is illustrated in Fig. 3 and 4. Let us first look at Fig. 3. An agent with preferences R consumes bundle x, and her indifference curve lies below that of an agent with preferences R' consuming bundle x'. According to *Nested Contour*, the well-being of the former agent is lower than that of the latter. If we only look at bundles, we conclude that allocating additional resources to x increases the well-being of the agent more than allocating them to x'. If we only look at indifference surfaces, we see that the difference between the indifference curves through x and $x + \lambda(x' - x)$ is smaller than that between the indifference curves through x' and $x' + \lambda(x' - x)$, however we define this difference.

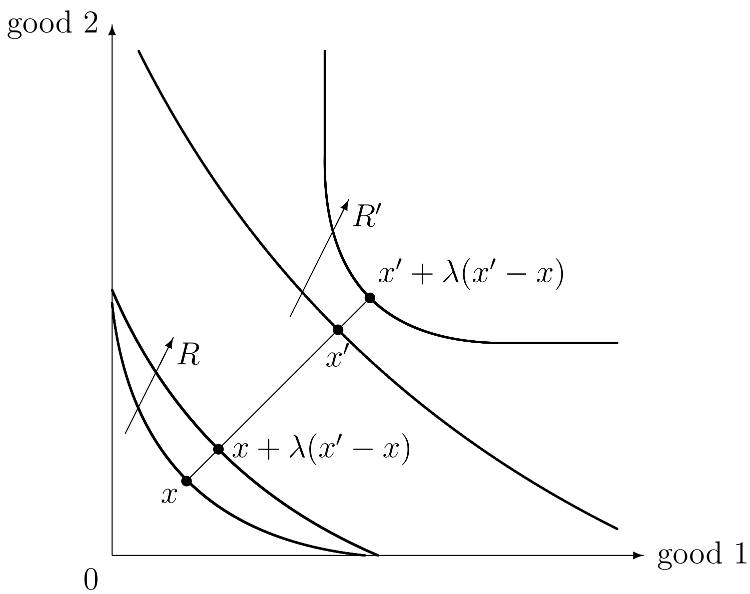

Figure 3: Intuition of Lemma 3, part 1.

The consequence of the tension between bundle differences and indifference set differences is illustrated in Fig. 4, a simplified version of Fig. 8 illustrating the formal proof of the Lemma. Fig. 4 represents preferences R' having the following property. Two indifference curves are arbitrarily far from each other, the ones containing x^1 and x^2. In spite of that large distance, it is possible to go from the former to the latter by a sequence of (in the figure, two) arbitrarily small decreases in commodity bundles, the one from x^1 to y^1 and the one from y^2 to x^2. As a result, it is impossible to guarantee that the well-being difference between x^1 and x^2 reflects how far away from each other the two indifference curves are because it is bounded above by the sum of the small differences between x^1 and y^1 and the one hand and between y^2 and x^2 on the other. The incompatibility comes from the fact that *Nested Priority* and *Nested Contour*, together with the continuity of W, actually do require that the difference between the well-being at x^1 and the well-being at x^2 be sufficiently large.

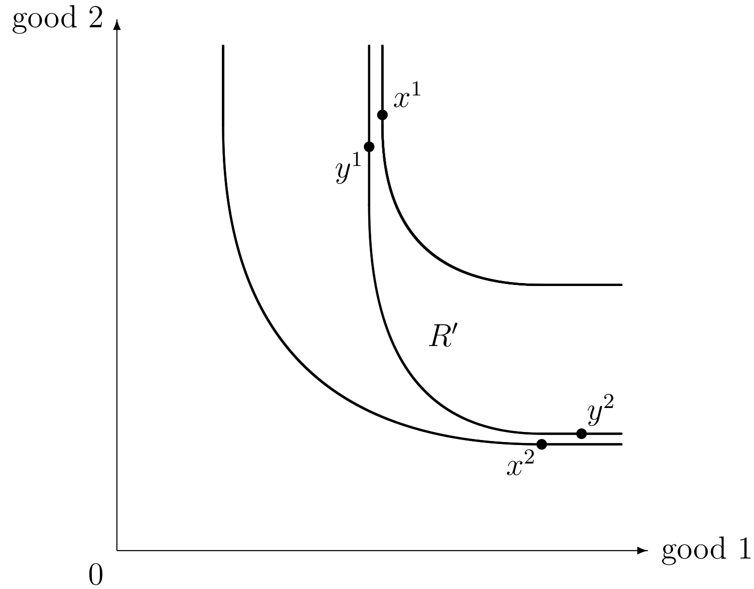

Figure 4: Intuition of Lemma 3, part 2.

To recover a way to construct inequality averse well-being measures satisfying *Nested Contour*, we weaken *Nested Priority*. We restrict the requirement of *Nested Priority* only to cases when the distance between the two indifference surfaces of the poorer agent is identical to the distance between the two indifference surfaces of the richer agent in the following very strict sense: the latter must be an upward translation of the former. The resulting property, which we call *Diminishing Priority*, is illustrated in Fig. 5.

Let us note that the translation operation imposes a further restriction on the indifference surfaces of the poorer agents, as they cannot touch the axes (otherwise the translated preferences would not be continuous). As a result, *Diminishing Priority* may look like a very weak property. It will turn out, on the contrary, to be very demanding.

This property is a direct multidimensional extension of the usual notion of diminishing marginal value that is most common in the one-dimensional context. When there is only one good, it can be formulated without reference to derivatives by saying that a utility function u has this property when $u(y) - u(x) > u(y + \Delta) - u(x + \Delta)$ for all $x < y$, $\Delta > 0$. *Diminishing Priority* is a most immediate extension of this to the multiple-good case.

We need the following notation. For $A, B \subset X$ and $\Delta \in \mathbb{R}^L_{++}$, we write that $B = A + \Delta$ if and only if

$$B = \{b \in X \mid \exists a \in A, b = a + \Delta\}.$$

Axiom 5 Diminishing Priority
For all $x, x', y, y' \in X$, $R, R' \in \mathcal{R}$, $\Delta \in \mathbb{R}^L_{++}$ *if*

$$\begin{aligned} y &= x + \lambda\Delta, 0 < \lambda < 1, &(1)\\ U(x', R') &= U(x, R) + \Delta, \text{ and} &(2)\\ U(y', R') &= U(y, R) + \Delta, &(3)\\ L(y, R) \cap U(x', R') &= \emptyset, &(4)\end{aligned}$$

then

$$W(y, R) - W(x, R) > W(y', R') - W(x', R').$$

A consequence of translated transfers as defined in the axiom is that the sum of the upper contour sets through x' and y is the same as the sum of the upper contour sets through x and y'.[4] As a result, *Diminishing Priority* is weaker than the Transfer axiom studied by Bosmans, Decancq and Ooghe (2015), which only requires the sum to be constant. These authors obtain a similar result to the theorem below.

[4]Equivalently, the Scitovsky set at allocation (x', y) is the same as at allocation (x, y').

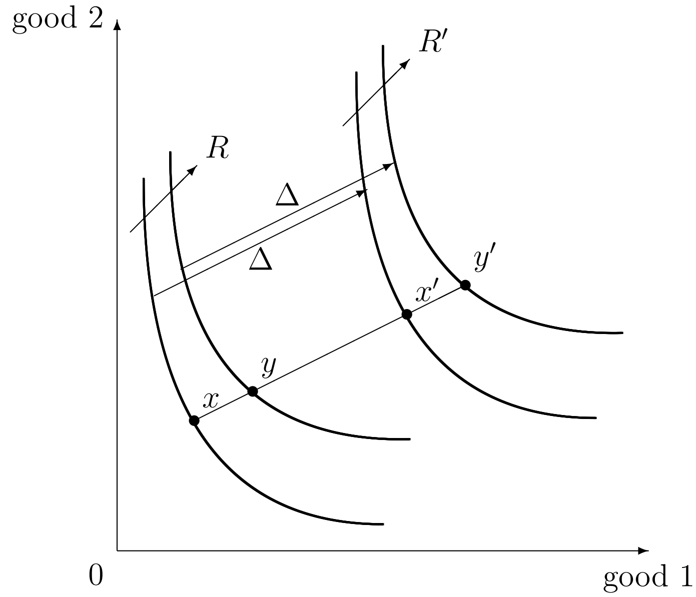

Figure 5: *Diminishing Priority*: $W(y, R) - W(x, R) > W(y', R') - W(x', R')$.

We begin by combining *Diminishing Priority* with *Infimum Nested Contour*. This restricts the family of well-being measures characterized in Lemma 2 in two ways. First, the reference preferences R^b need to be linear. Preferences R^p are linear if there exists $p \in interior[S^{K-1}]$ (where S^{K-1} denotes the $K-1$-dimensional simplex) such that

$$x \, R^p \, x' \Leftrightarrow \sum_{k \in K} p_k x_k \geq \sum_{k \in K} p_k x'_k.$$

If we take linear preferences for the reference preferences, then the well-being measures are ordinally equivalent to the money-metric utility, introduced by Samuelson (1974) and Samuelson and Swamy (1974). In their definition, the p vector stands for a vector of prices, and the money-metric utility at (x, R) is the minimal expenditure a consumer with preferences R would incur, facing price vector p, to reach the same satisfaction as at x. Instead on relying on the expenditure function terminology, we can define that well-being measure W^p by using the following function: for a set of bundles $B \subset X$, for $R \in \mathcal{R}$, we write $\max(R, B)$ to denote any bundle in B that maximizes R over B, that is, $\max(R, B) = x$ only if $x \in B$ and $x \, R \, x'$ for all

$x' \in B$. For all $x \in X$, all $R \in \mathcal{R}$,

$$W^p(x, R) = w \Leftrightarrow x\, I\, \max(R, \{x' \in X | px' \leq w\}).$$

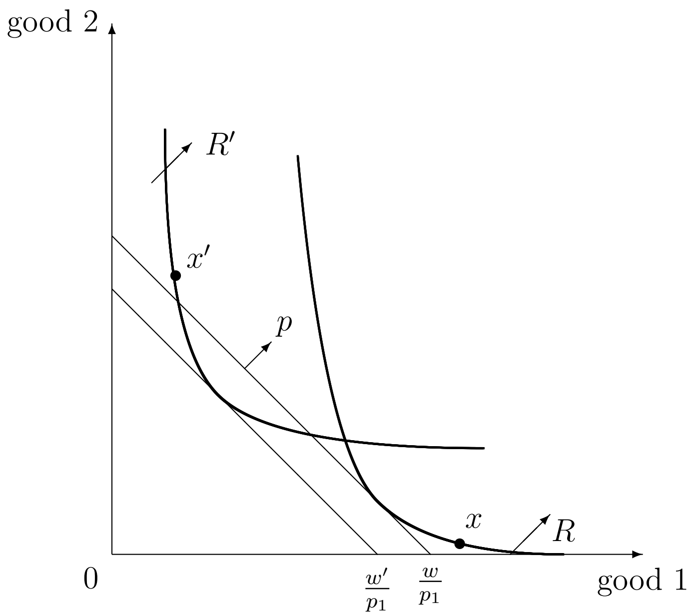

Figure 6: Illustration of W^p: $W^p(x, R) = w, W^p(x', R') = w'$.

The second restriction imposed by *Diminishing Priority* on the family of well-being measures characterized in Lemma 2 is that the well-being measures need to be strictly concave transformations of the money-metric utility.

Theorem 1 *A well-being measure W satisfies* Infimum Nested Contour *and* Diminishing Priority *if and only if there exists a vector $p \in interior[S^{K-1}]$ and a strictly concave function $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that $W = g \circ W^p$.*

In contrast, if we try to combine *Diminishing Priority* with *Supremum Nested Contour*, we reach the following impossibility.

Lemma 4 *No well-being measure satisfies* Supremum Nested Contour *and* Diminishing Priority*.*

This incompatibility can be bypassed by allowing *Diminishing Priority* to apply only to one specific direction.

Axiom 6 Weak Diminishing Priority
There exists $\Delta \in \mathbb{R}^L_{++}$ *such that for all* $x, x', y, y' \in X,\ R, R' \in \mathcal{R},\ \lambda \in \mathbb{R}_{++}$ *if*

$$\begin{aligned} y &= x + \lambda\Delta, 0 < \lambda < 1, && (5)\\ U(x', R') &= U(x, R) + \lambda\Delta, \ \textit{and} && (6)\\ U(y', R') &= U(y, R) + \lambda\Delta, && (7)\\ L(y, R) \cap U(x', R') &= \emptyset, && (8) \end{aligned}$$

then

$$W(y, R) - W(x, R) > W(y', R') - W(x', R').$$

Combining *Weak Diminishing Priority* with *Supremum Nested Contour* forces us to restrict the family of well-being measures characterized in Lemma 1 in two ways. First, the reference preferences R^w need to be Leontief. Preferences R^ℓ are Leontief if there exist $\ell \in interior[S^{K-1}]$, such that

$$x\, R^\ell\, x' \Leftrightarrow \min_{k \in K} \frac{x_k}{\ell_k} \geq \min_{k \in K} \frac{x'_k}{\ell_k}.$$

If a well-being measure satisfies *Supremum Nested Contour* with $R^w = R^\ell$ for some $\ell \in interior[S^{K-1}]$, then the well-being of an agent is measured by the bundle that is proportional to ℓ and to which this agent is indifferent. To put it differently, we say that $W(x, R) = W(x', R')$ if and only if there exists some number $\lambda \in \mathbb{R}_+$ such that $x\, I\, \lambda\ell$ and $x'\, I'\, \lambda\ell$ as well. All the well-being measures satisfying this property are ordinally equivalent to the ray utility W^ℓ, defined by: for all $x \in X$, all $R \in \mathcal{R}$,

$$W^\ell(x, R) = w \Leftrightarrow x\, I\, w\ell.$$

Observe that the set of bundles $r \in \mathbb{R}^K_+$ such that $r = \lambda\ell$ for some $\lambda \in \mathbb{R}_+$ is a ray in $\mathbb{R}^K_+$. This way of measuring well-being was suggested by Samuelson (1977), Pazner (1979) and Deaton (1979). If it is combined with an egalitarian aggregator, it comes close to allocation rules and social ordering functions axiomatized in the literature on fair allocation (see, for instance, Pazner and Schmeidler 1978, and Fleurbaey 2005).

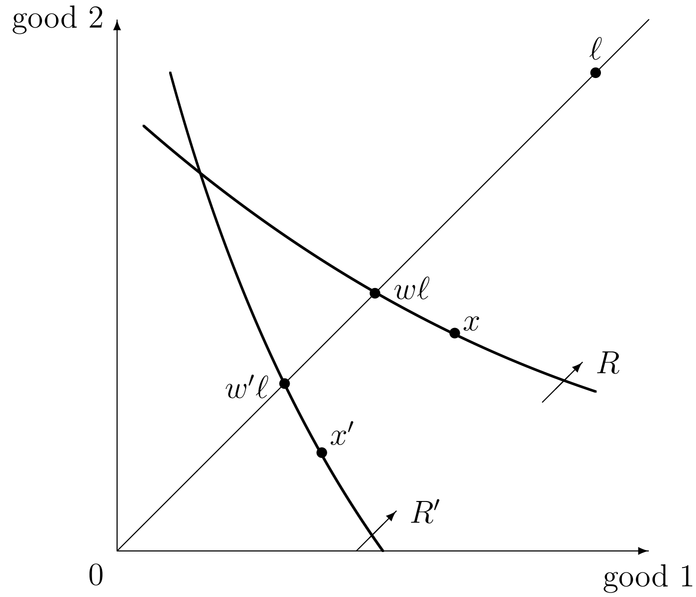

Figure 7: Illustration of W^ℓ: $W^\ell(x, R) = w, W^\ell(x', R') = w'$.

The second restriction imposed by *Weak Diminishing Priority* on the family of well-being measures characterized in Lemma 2 is that the well-being measures need to be strictly concave transformations of the ray utility.

Theorem 2 *A well-being measure W satisfies* Supremum Nested Contour *and* Diminishing Priority *if and only if there exist a bundle $\ell \in interior[X]$ and a strictly concave function $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that $W = g \circ W^\ell$.*

In conclusion, the results of this section teach us that it is possible to define inequality averse well-being measures satisfying our two main properties, *Supremum* and *Infimum Nested Contour*, but the latter can interestingly be combined with a stronger property of inequality aversion.

4 Proofs

Proof of Lemma 3: Let W satisfy the axioms. By Fleurbaey and Maniquet (2017a), we know that if W satisfies *Nested Contour*, then it satisfies *Unchanged Indifference Independence*, requiring that the well-being of an agent consuming a given bundle only depend on her indifference set at that bundle.

Axiom 7 Unchanged Indifference Independence
For all $x \in X$, $R, R' \in \mathcal{R}$, if $I(x,R) = I(x,R')$, then $W(x,R) = W(x,R')$.

The construction of the preferences and the bundles needed for the proof is illustrated in Figure 8. Let $R \in \mathcal{R}$ be such that there exist $x^1, x^2, x^3 \in X$ such that $x^1 \, P \, x^2$, $x^1 \not\geq x^2$, $x^3 \ll x^1$, $x^3 \ll x^2$, and there exist $k, k' \in \{1, \ldots, K\}$ such that

$$\forall x, y \in X, \text{ if } y_k = x_k, y'_k = x'_k \text{ and } y_l > x_l \;\; \forall l \neq k, k' \text{ then } y \, I \, x.$$

Let $x^4 \in X$ be such that $x^3 \, P \, x^4$ but

$$W(x^3, R) - W(x^4, R) < W(x^1, R) - W(x^2, R). \tag{9}$$

We can find four bundles $y^1, y^2, y^3, y^4 \in X$ and two numbers $\alpha, \alpha' > 0$ such that $y^4 \, I \, x^4$ and

$$\begin{array}{rcl} y^1 & = & x^1 - \alpha(x^1 - x^3), \\ y^2 & = & x^2 + \alpha'(x^2 - y^4), \\ y^3 & = & x^3 - \alpha(x^1 - x^3), \\ y^3 & = & y^4 + \alpha'(x^2 - y^4). \end{array}$$

Let $R' \in \mathcal{R}$ be such that $I(x^i, R') = I(x^i, R)$, $i \in \{1, \ldots, 4\}$ and $y^1 \, I' \, y^2$. By *Unchanged Indifference Independence*,

$$W(x^i, R'^i) = W(x^i, R), \forall i \in \{1, \ldots, 4\}.$$

By *Nested Priority* applied twice,

$$\begin{array}{rcl} W(x^3, R') - W(y^3, R') & > & W(x^1, R') - W(y^1, R') \\ W(y^3, R') - W(y^4, R') & > & W(y^2, R') - W(x^2, R'). \end{array}$$

Summing up the two inequalities, and using the facts that $W(y^1, R') = W(y^2, R')$ and $W(y^4, R') = W(x^4, R')$, we get

$$W(x^3, R') - W(x^4, R') > W(x^1, R') - W(x^2, R'),$$

contradicting eq. 9.

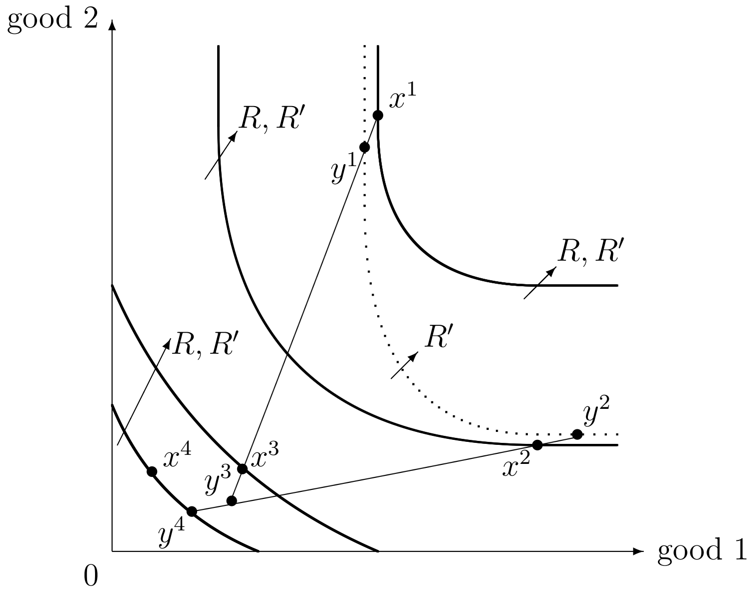

Figure 8: Illustration of Lemma 3 for $k = 2$ and $k' = 1$.

Proof of Theorem 1: Let X satisfy *Infimum Nested Contour* and *Diminishing Priority*. We develop the proof by restricting our attention to $X = \mathbb{R}^2_+$. This is without loss of generality, as we can generalize the proof to a K-dimensional X by assuming that the preferences we deal with in the proof are all indifferent to the consumption of all but two goods. By Theorem 2, W satisfies *Best Preferences* for some $R^b \in \mathcal{R}^b$. We need to prove that $R^b = R^p$ for some linear preferences R^p. We use the following characterization of linear preferences: R^b is linear if and only if for all $x, x' \in X$, $x \ll x'$, $\Delta = x' - x$, $I(x, R^b) + \Delta \subset I(x'^b)$. Assume R^b is not linear. Then there exists $x, x', x'' \in X$ and $\Delta \in \mathbb{R}^2_{++}$ such that $x' \gg x$, $U(x, R^b) + \Delta$ is tangent to $L(x'^b)$, $x'' = x + \Delta$, and $x'' \notin L(x'^b)$. Let $x''' = x'' + (\epsilon, \epsilon)$. Let $R, R' \in \mathcal{R}$ be such that

$$\begin{aligned}
U(x, R) &= U(x, R^b), \\
U(x'', R') &= U(x, R) + \Delta, \\
U(x''', R') &= \{x \in X \mid x \geq x'''\}, \\
U(x''' - \Delta, R) &= U(x''', R') - \Delta.
\end{aligned}$$

By a direct application of Theorem 2,

$$\begin{aligned} W(x,R) &= W(x,R^b), \\ W(x'''-\Delta,R) &= W(x''',R^b), \\ W(x'',R') &= W(x',R^b), \\ W(x''',R') &= W(x''',R^b). \end{aligned}$$

As a result, $W(x'''-\Delta,R)-W(x,R)$ tends to 0 as ϵ tends to 0, whereas $W(x''',R')-W(x'',R')$ tends to $W(x''^b)-W(x'^b)$, which is bounded below away from 0. Consequently, for ϵ sufficiently small,

$$W(x'''-\Delta,R)-W(x,R) < W(x''',R')-W(x'',R'),$$

contradicting *Diminishing Priority.*

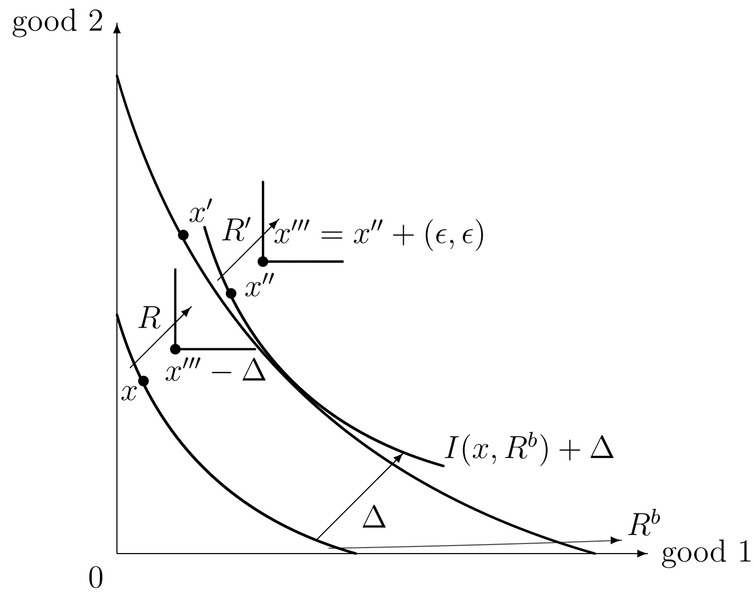

Figure 9: Illustration of the proof of Theorem 1.

That proves that $R^b = R^p$. Now in order to satisfy *Diminishing Priority* (which is defined with a strict inequality), W needs to be a strictly concave transform of W^p.

Proof of Lemma 4: This is actually a corollary of Theorem 2, but a separate proof may have some value.

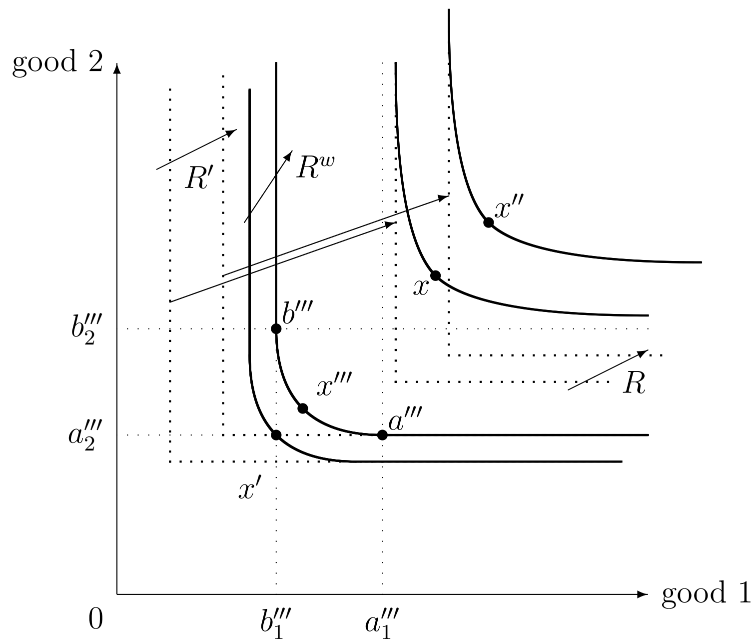

Figure 10: Illustration of Lemma 4 for $k = 2$.

As in the previous proof, we assume $X = \mathbb{R}^2_+$. Let W satisfy the axioms. By Theorem 1, W satisfies *Worst Preferences*. Let $R^w \in \mathcal{R}^w$ be the worst preferences. The construction of the bundles and the indifference curves is illustrated in Fig. 10. Let $x''' \in X$ be such that there exist $a''' = (a_1''', a_2'''), b''' = (b_1''', b_2''') \in X$, $a_1''' > b_1'''$ and $a_2''' < b_2'''$, such that for all $y = (y_1, y_2) \in X$ such that $y_1 \geq a_1'''$ and $y_2 = a_2'''$ or $y_1 = b_1'''$ and $y_2 \geq b_2'''$, $y\, I^w\, x'''$. Let $x \in X$ be such for all $y \in U(x, R^w)$, $y \geq a'''$ and $y \geq b'''$. Such x and x''' always exist because $R^w \in \mathcal{R}^w$. Then, we take $x', x'' \in X$ such that

$$W(x'', R^w) - W(x, R^w) = W(x''', R^w) - W(x', R^w).$$

The intuition is that the indifference curves containing x and x'' should contain portions that are sufficiently close to each other, in the sense that the distance between them should be smaller than the corresponding distance between the indifference curve containing x' and the axis. As a result, x and x'' should be close to each other whereas x' and x''' can be arbitrarily closer to each other than x and x''. Indeed, in order to satisfy *Diminishing Priority*, a well-being measure is likely to be as concave as possible. The key point is that it is always possible to construct preferences R and R' with translated indifference curves that are tangent from below to the indifference curves of R^w through x' and x''' on the one hand and x and x'' on the other.

Let us consider $y, y', y'', y''' \in X$ such that the indifference curves drawn in the figure are $I(y', R')$, $I(y''', R')$, $I(y, R)$ and (y'', R). For the sake of clarity, we did not add these bundles on the figure. In the figure, the vertical parts of the indifference curves of R' lie left of the indifference curve of R^w through x' and they can be drawn arbitrarily close to each other. Because $I(x, R^w)$ lies everywhere above (a_1''', b_2'''), it is then possible to translate the indifference curves of R' so as to define R in a way that the indifference curves through y and y'' are tangent from below to the indifference curves of R^w through x and x''. As a result,

$$W(y'', R) - W(y, R) = W(y''', R') - W(y', R'),$$

violating *Diminishing Priority.*

Proof of Theorem 2:

Let $\Delta \in \mathbb{R}^K_{++}$ be such that *Weak Diminishing Priority* is satisfied w.r.t. it. By *Supremum Nested Contour*, well-being satisfies the property

$$W(x, R) = \max_{x' \in L(x,R)} W(x', R^w).$$

If R^w are the Leontief preferences R^Δ, then indeed *Weak Diminishing Priority* is satisfied. We will show that this is the only possibility for R^w. The proof is illustrated in Fig. 11.

By Fleurbaey and Maniquet (2017a), the lowest indifference set of R^w must be $\{x \in X \mid \exists k \in \{1, ..., K\},\ x_k = 0\}$. Therefore, R^Δ are the only preferences such that all upper contour sets are translated by some proportion of Δ.

If R^w is a different preference ordering, then there is $x, y \in X, R \in \mathcal{R}$ and $\lambda > 0$ such that

$$\begin{array}{rcl} U(y, R^w) & = & U(x, R) + \lambda\Delta, \\ U(y, R) & = & U(x, R^w) + \lambda\Delta, \\ U(x, R) & \subset & U(x, R^w), \\ U(x, R) & \neq & U(x, R^w). \end{array}$$

The reason why such configuration exists is that there must be $U(y, R^w)$ such that some downward translation in the Δ direction is not an upper contour set for R^w.

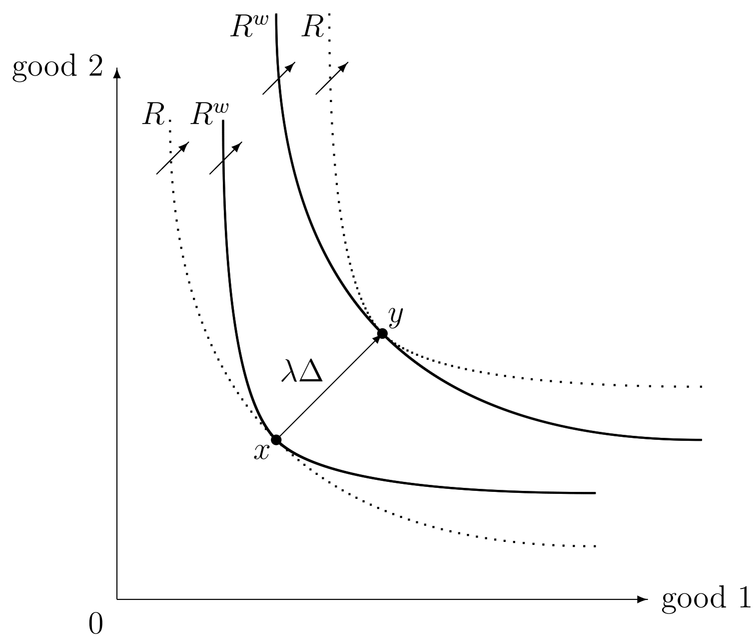

Figure 11: Illustration of the proof of Theorem 2 for $k = 2$.

Observe that by construction,

$$\begin{aligned} W(x, R) &= W(x, R^w), \\ W(y, R) &> W(y, R^w). \end{aligned}$$

This means that *Weak Diminishing Priority* cannot be satisfied, because by continuity of W, taking the limit of sequences of bundles converging toward x and y in a similar argument as in the proof of Th. 1, one should have

$$W(x, R^w) - W(x, R) \geq W(y, R) - W(y, R^w).$$

This contradiction shows that all upper contour sets of R^w are translated in the Δ direction, implying that $R^w = R^\Delta$. Now, in order to satisfy *Weak Diminishing Priority* (which is defined with a strict inequality), W needs to be a strictly concave transform of W^Δ.

Since this property can only hold for one direction, this also shows that the full *Diminishing Priority* cannot be satisfied, so that Lemma 4 is a corollary of this theorem.

5 Conclusion

In this paper, we have shown how transfer principles, combined with axioms bearing on the comparison of crossing indifference sets, pin down concave transformations of the ray utility and the money-metric utility as well-being measures. The underlying fairness principles are clear in the transfer axioms. One can be less sanguine about the Supremum and Infimum Nested Transfer axioms, which do not transparently involve ethical values, and appear to refer more to the idea that given preferences cannot be better or worse than the whole set of preferences in which they occupy an intermediary position. This may offer a space for the exploration of alternative approaches. However, we believe that the weaker Nested Contour axiom, which bears on non-crossing configurations of indifference sets, should remain a pillar of any approach that takes the respect of preferences as a main value. As noted in the literature (e.g., Fleurbaey and Blanchet 2013), respecting preferences is not just about intrapersonal comparisons but also about interpersonal comparisons: When indifference sets do not cross the two individuals strongly agree about who has a better situation, therefore how could an observer make the opposite judgment?

Another direction of research that is worth considering is to explore more concrete and specific settings, for which the model of divisible commodities is not adapted. For instance, if health or leisure is a dimension, one dimension in the space is bounded and preferences for leisure may not be monotonic. We leave the exploration of such different settings for future research.

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