

UNIVERSITÉ CATHOLIQUE DE LOUVAIN

Near extreme eigenvalue behaviour in random matrix theory

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Introduction

Motivations

Random matrix theory belongs to the kind of mathematics that I find are the most beautiful, and here is why. It is the kind of mathematics that can start from an everyday life observation, allows you to develop a rich and deep theory and, at the end of the day, rewards you with plenty of connections with other areas of mathematics, physics, economy, computer science...

The core of the thesis is devoted to the middle part—developing the theory—, but in this introduction, I would like to spend some time on the other two parts.

Random matrix theory deals with phenomena where “particles” are caught in the crossfire of two opposing “forces”, one of which having a tendency to push the particles together, and the second acting as a repulsion between them. My favourite every day life example of that situation is people sitting in a film theatre. In this example, the particles are the people, the force that pushes people together is the fact that every one wants to sit in the middle, and the forces that repel people from each other is the fact that nobody wants to sit next to a stranger. When the film starts, one can take a picture of the room and look at the locations where all the people are seated. Forgetting about people coming together (i.e., considering them as one “particle”), the situation that one sees is the kind of phenomenon that we study in random matrix theory. I like this example because it manages to catch the core principle of random matrix theory and it even shows us the limitations of such models. If there are hardly a few people in the room, then there is lots of

space for everybody, and the gathering force does not really matter. On the contrary, if there are plenty of people, so that the room is crowded, then the repulsive force has no effect since people need to be seated anyway. This suggests that the two forces must be balanced in order to find an interesting mathematical model.

Although I am not aware of any real world data sustaining the fact that people in the theatres actually behave the same way as particles in random matrix theory, there are numerical evidence of connections between random matrix theory and real world phenomena as varied as car parked in a street, birds on an electric wire [Seb09], or even buses passage times in a Mexican city [KS02, KS00] (see also [BBDS06] where this remarkable connection is attempted to be explained).

History, of course, is somewhat less romantic, as random matrix theory did not begin by watching people in a film theatre, or pigeons on an electric wire. As one can learn from the introduction of [Meh04], one of the most classical reference for random matrix theory, the birth and the expansion of random matrix theory dates back to the fifties, where random matrices were used to model the statistical behaviour of energy levels of neutrons.

Connections with other areas

Roughly speaking, in random matrix theory, we study sets of matrices equipped with a probability measure, and we are interested in the statistical properties of the eigenvalues of large such matrices. As mentioned at the very beginning, random matrix theory is connected to many areas of science, where we find similar kinds of behaviour. Here we present two connections in details: the statistics of the dominoes in the Aztec diamond, and the statistics of the non-trivial zeroes of the Riemann zeta function.

The Aztec diamond A first connection exists with random domino tilings of the Aztec diamond. The Aztec diamond is a region in the plane made of one by one squares put in a double pyramid shape depicted on the first row of Figure 1 for diamonds of “orders” 3 and 4. These shapes

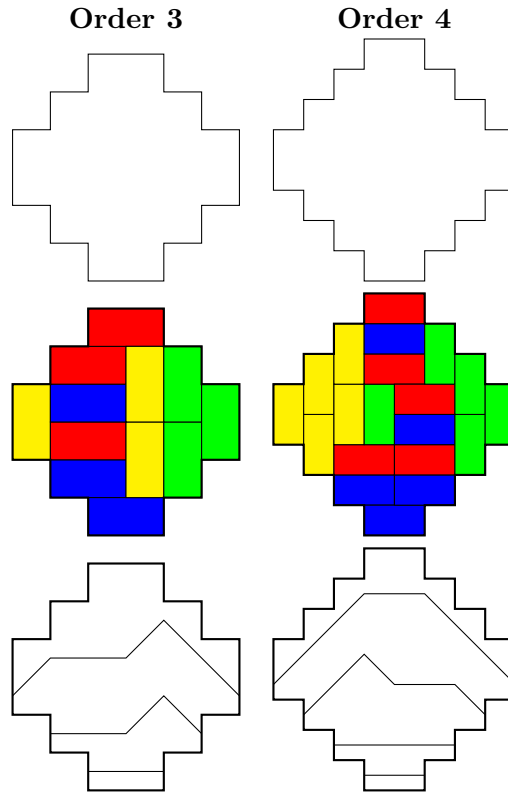


Figure 1: On the first row, the shapes of the Aztec diamond for orders 3 (to the left) and 4 (to the right). The order is the number of rows in the upper half part. On the second row, we have drawn two particular tilings of these Aztec diamonds. Horizontal dominoes are coloured in red and blue and vertical dominoes are coloured in yellow and green. On the last row, we can see the bijection between tilings of the Aztec diamond and non-intersecting paths on a lattice with constrained starting and ending positions. Blue, yellow and green dominoes are sent respectively to east, north-east and south-east lines

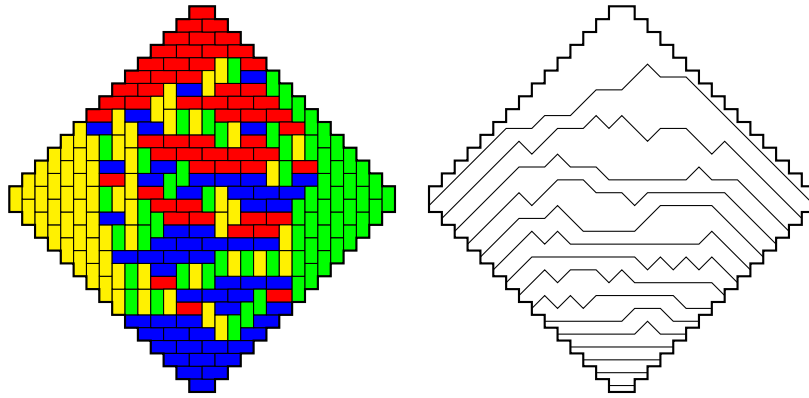


Figure 2: A tiling of an Aztec diamond of order 15 and its non-intersecting paths representation.

can be tiled with 1 by 2 or 2 by 1 dominoes, in such a way that there are no overlaps between dominoes (except perhaps at their boundaries), see the second row on Figure 1.

When taking a tiling of an Aztec diamond of large order uniformly at random among all the possible tilings, the “Arctic circle” starts to appear. Dominoes outside of the inscribed circle are “frozen” in only one direction (with high probability) and dominoes inside have a chaotic behaviour, see Figure 3. In these figures, there are four different colours for the dominoes. Two for horizontal dominoes and two for vertical dominoes. There are good reasons for using four colours instead of two, and one of them comes from the “domino shuffling” algorithm that is used to generate random tilings of Aztec diamonds. The interested reader can find the explanation on the website [Doe]. Another reason comes from the bijection between the set of tilings and a set of non-intersecting paths on a lattice, allowed to go east, north-east or south-east, see the last row of Figure 1 and Figure 2.

The particles in this model can be taken as one of the four types of dominoes. The force that pushes the particles together comes from the sawtooth boundary feature of the Aztec shape. Each little corner pushes the dominoes to the center. This is one of the reasons why random tilings of rectangles do not have a frozen region, as there is only one corner per side, see Figure 4. The repulsive forces between the

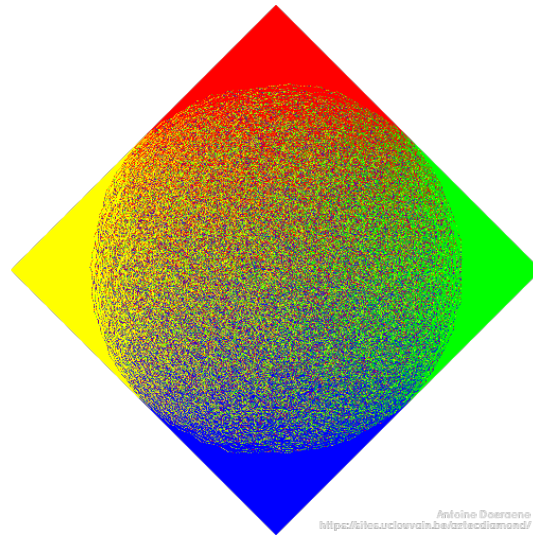


Figure 3: A tiling of an Aztec diamond of order 1000 drawn uniformly at random. We can witness the Arctic circle within the diamond. The four corners of the diamond are frozen in only one direction. The diamond was generated using the domino shuffling algorithm, developed by Kuperberg and Propp, later generalized by Propp [Pro03] for non uniform probability measure on the set of tilings. This figure was generated using the website [Doe].

dominoes is less clear, and comes from the combinatorics properties of the tilings. They can however be intuitively understood when looking at the non-intersecting paths the Aztec diamonds are in bijection with. Since paths are not allowed to intersect, they somehow don't like to be close to each other.

A link between Aztec diamonds and random matrix theory actually lies on the Arctic circle. In [Joh05], the author proved that the fluctuations of the dominoes near the Arctic circle are governed by the “Airy Ensemble”. The Airy Ensemble is a point process that appears in various places in random matrix theory where a macroscopic phase transition takes place. This ensemble will be fully described in subsection 1.2.5 in Chapter 1. See also [ACJvM15] for an other connection in a double Aztec diamond.

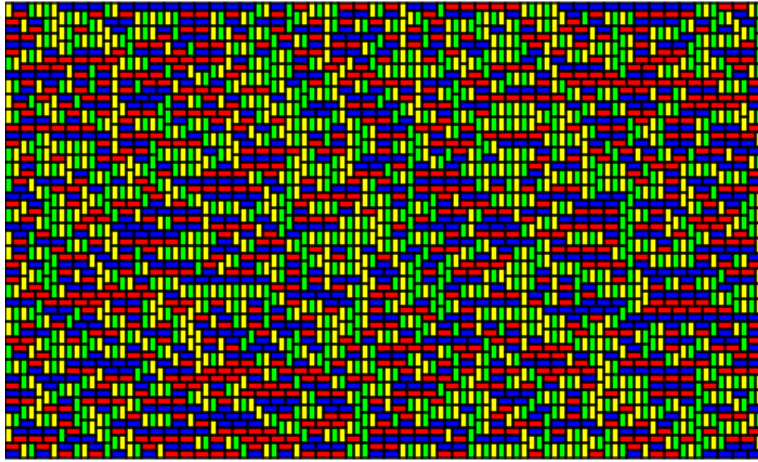


Figure 4: Uniform random tiling of a 100 by 60 rectangle. Unlike the Aztec diamond, there is no frozen region. The tiling was also generated using [Doe], but in this case a wider generalization of the algorithm is necessary [JdlRV06].

The zeros of the Riemann zeta function A second –perhaps more surprising– area of mathematics having connection with random matrix theory comes from number theory, and in particular the distribution of the zeros of the Riemann zeta function ζ on the critical line. In 1974, Montgomery [Mon73] unveiled the first hint of a link between the Riemann zeros and the eigenvalues of random matrices. Let $(\gamma_n)_n$ be the increasing sequence of the imaginary parts of the zeros of ζ on the critical line, so that all the zeros on that axis are given by $\frac{1}{2} \pm i\gamma_n$. Define also rescaled zeros ω_n by the formula

$$\omega_n = \frac{\gamma_n}{2\pi} \ln \frac{\gamma_n}{2\pi}. \quad (0.0.1)$$

Montgomery’s result states that for any test function f whose Fourier transform is smooth and supported in $(-1, 1)$, we have

$$\lim_{t \rightarrow +\infty} \frac{1}{N(t)} \sum_{1 \leq j, k \leq N(t), j \neq k} f(\omega_j - \omega_k) = \int_{\mathbb{R}} f(y) r_2(y) dy,$$

where

$$r_2(y) = 1 - \left(\frac{\sin(\pi y)}{\pi y} \right)^2$$

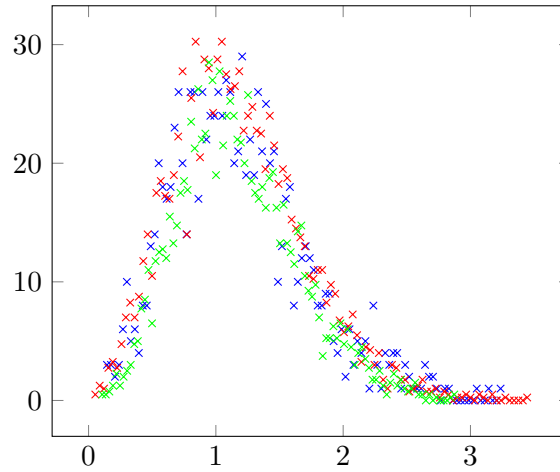


Figure 5: Numerical comparison between the nearest neighbours distances of the sine process (blue), the eigenvalues of random unitary matrix with respect to the Haar measure (red), and the rescaled zeros $\{\omega_n\}_{100 \leq n \leq 4000}$ (0.0.1) of the Riemann zeta function (green). The zeros of ζ are computed using Mathematica, and the random points of the sine process and the unitary Haar matrix are generated using Matlab.

and $N(t)$ is the number of zeros up to imaginary height t . The function r_2 is precisely the 2-point correlation function of the “sine process”, a point process ubiquitous in random matrix theory, see subsection 1.2.4 from Chapter 1. First numerical evidence came more than a decade later [Od187]. In order to witness the comparison between zeros of ζ and eigenvalues in random matrix theory, one can for example compute nearest neighbour distances, see Figure 5.

More recent evidence of connections between Riemann zeros and random matrix theory can be found in the review [KS11].

Other connections Random matrix theory also plays a role in wireless telecommunications [Mou17] or analysing the stock market [Bou17]. In the latter, for example, random matrix theory is used for “cleaning” corrupted matrices arising when looking at correlation of stocks. The corrupted matrix is modelled as the “data” matrix, containing the actual information, plus a “noise” perturbing the data. This is not too far from the models that we are going to study in Chapter 2.

Outline

We give a brief outline of the following chapters.

Determinantal Point Processes Chapter 1 will be devoted to the basics of “determinantal point processes” (DPPs). DPPs are a class of point processes which exhibit repulsion between particles, and which are particularly well suited for studying large particle number asymptotics. Results exposed in that chapter are classical from the theory, and only serve to summarize the relevant known results.

Gaussian perturbations of hard edge random matrix ensembles In Chapter 2, we will consider random matrix ensembles where the eigenvalues are constrained to be positive, and perturb them by an additive entry-wise Gaussian noise, hence breaking the “physical” boundary at 0. We will study how the local statistics near 0 are affected by such a perturbation, with respect to the strength of this perturbation when the size of the matrices tends to infinity. We will give a description of the phenomenon when the strength of the perturbation is either very small, or reaches a critical value. At this point, we will show that a new type of behaviour appears. These observations will be the consequences of a technical lemma (Lemma 2.1.2), proved for a wide class of random matrix ensembles. It will be applied to some random matrix ensembles considered in the literature. We also study the average characteristic polynomials of such ensembles, for which we prove the convergence of their zero counting measures to a limit expressed as a free convolution. This second result is not very surprising in view of free probability theory, but came as a by-product of our research.

The generating function for the Airy point process and a system of coupled Painlevé II equations The two last chapters are quite similar, however they focus on two different point processes. Chapter 3 studies generating functions of occupancy numbers in the Airy Ensemble, which will be described in subsection 1.2.5 in Chapter 1. Occupancy numbers, for a point process, are random variables defined

as the number of particles that fall into particular sets (in our cases, subsets of the real line). In determinantal point processes such as the Airy ensemble, these generating functions are expressed as “Fredholm determinants”, formula (1.1.7) in Chapter 1.

The main result of Chapter 3, Theorem 3.1.1, gives a description of these generating functions in terms of solutions to a system of coupled Painlevé II-like differential equations. This result nicely generalizes a classical result from Tracy and Widom, see the end of subsection 1.2.5. In particular, our result allow to express the joint probability density of any choice of k ordered particles, for any $k = 1, 2, \dots$, while Tracy and Widom’s result only applies for $k = 1$.

The generating function for the Bessel point process and a system of coupled Painlevé V equations In Chapter 4, we derive the analogues of the results in Chapter 3 for the “Bessel point process”. The Bessel point process is also a determinantal point process, where particles live on the positive half line. This point process for example appears as limiting point process of ensembles reviewed in Chapter 2.

The main result of the chapter will be Theorem 4.1.1. It is the analogue of Theorem 3.1.1 from Chapter 3. In the case of the Bessel point process, the generating functions that we consider will be expressed in terms of solutions to coupled Painlevé V-like equations, instead of II.

Chapter 1

Determinantal Point Processes

Random matrix theory (RMT) began to expand widely during the fifties. Early on, people realized the link between RMT and orthogonal polynomials. Indeed, when studying the eigenvalues of the Gaussian Unitary Ensemble, a random matrix ensemble where matrices have entry-wise iid complex Gaussian random variables, they discovered that any correlation function of the eigenvalues can be expressed in terms of a “kernel” built out of Hermite polynomials.

Over the years, the ability to express correlation functions of a point process in terms of the determinant of some kernel appeared in many other models related to RMT. The understanding of these connections led Borodin and Olshanski [BO00] to give the name “determinantal point processes” (DPPs) to this formalism in 2000. Their first appearance, however, dates back to 1975 in a paper by Macchi [Mac75], where they were referred to as fermion point processes, given the repulsive feature that they exhibit.

Since then, special treatments and reviews on these processes have been made, see e.g., [BDS16, Bor11, Joh06, ST03, Sos00]. A somewhat soft introduction to DPPs on discrete spaces can also be found in [Kea12], with a focus on applications of DPPs to machine learning.

DPPs also appear naturally in the study of non-intersecting Brownian motions, Aztec diamonds (see, e.g., [BDS16, Dei99, Joh06, Meh04])...

In this chapter, we review some of the basics of DPPs, as well as some examples related to RMT. We will consider only point processes on $\mathbb{R}$, and we mostly focus on the properties that will be useful in the subsequent chapters.

1.1 Definition and basic properties

DPPs are a special kind of point process in which all correlation functions are expressed in terms of the determinant of some integral operator kernel K .

Let X be the set of sums of Dirac measures on $\mathbb{R}$. That is, if $\xi \in X$, then we may write ξ as

$$\xi = \sum_j \delta_{x_j},$$

for j in a (possibly infinite) discrete set. A point process is simply a probability measure P on the set X . We consider only point processes that are locally finite, meaning that for any bounded Borel set $A \subseteq \mathbb{R}$, the random variable $n_A := \xi(A)$, $\xi \in X$, is almost surely finite. This number n_A is called the occupancy number of A . If the point process is regarded as a random configuration of points on the real line, then the occupancy number of A is simply the number of particles that fall into A . The probability on X is on the smallest σ -algebra such that the functions n_A are measurable for all Borel sets A . Occupancy numbers are important quantities related to DPPs.

Other important quantities are correlation functions. We define the correlation functions $\rho_k : \mathbb{R}^k \rightarrow \mathbb{R}$ for $k = 1, 2, \dots$ such that for any disjoint Borel sets $A_1, \dots, A_m \subseteq \mathbb{R}$, $k_1, \dots, k_m \in \mathbb{N}$ such that $\sum_{j=1}^m k_j = k$, we have

$$\mathbb{E} \left(\prod_{j=1}^m \frac{n_{A_j}!}{(n_{A_j} - k_j)!} \right) = \int_{A_1^{k_1} \times \dots \times A_m^{k_m}} \rho_k(x_1, \dots, x_k) dx_1 \dots dx_k,$$

where the expectation is taken over the probability measure of the point process. A point process will be called *determinantal* if there exists a locally trace class operator $\mathcal{K}$ acting on $L^2(\mathbb{R})$ such that for all $k =$

$1, 2, \dots, x_1, \dots, x_k \in \mathbb{R}$, we have

$$\rho_k(x_1, \dots, x_k) = \det \left(K(x_j, x_\ell) \right)_{j, \ell=1}^k,$$

where K is the integral kernel of the operator $\mathcal{K}$, called the correlation kernel. That is, for $u \in L^2(\mathbb{R})$, $\mathcal{K}[u] \in L^2(\mathbb{R})$ is defined by

$$\mathcal{K}[u](x) = \int_{\mathbb{R}} K(x, y) u(y) dy.$$

Locally trace class simply means that for every bounded Borel set $A \subseteq \mathbb{R}$,

$$\int_A K(x, x) dx < +\infty. \quad (1.1.1)$$

A first use of the correlation kernel is to compute the average of occupancy numbers. For every Borel set $A \subseteq \mathbb{R}$, the (possibly infinite) average of n_A is given by

$$\mathbb{E}(n_A) = \int_A K(x, x) dx. \quad (1.1.2)$$

In view of (1.1.1) and (1.1.2), we see that $\mathcal{K}$ being locally trace class is equivalent to the fact that the process is locally finite.

The repulsive nature of determinantal point processes is for example encoded in the fact that, for every $k \in \mathbb{N}_*$, $x_1, \dots, x_k \in \mathbb{R}$, we have [ST03, Proposition 4.3]

$$\rho_k(x_1, \dots, x_k) \leq \rho_1(x_1) \dots \rho_1(x_k).$$

A correlation kernel determines the process uniquely, but it is not unique. If K is the correlation kernel of some determinantal point process, then for any non vanishing function f the new kernel

$$K_f(x, y) = \frac{f(x)}{f(y)} K(x, y) \quad (1.1.3)$$

is again a correlation kernel for the same determinantal point process. The function f is sometimes called a “gauge”, and formula (1.1.3) is called a “gauge transformation”.

Generating function of occupancy numbers The correlation kernel of a determinantal point process is central in its study, in that it contains all the information and characterizes the process. For example, given $A_1, \dots, A_k \subseteq \mathbb{R}$ k disjoint Borel sets, the generating function for the joint occupancy numbers $n_{A_1}, \dots, n_{A_k}$ is expressed in terms of a Fredholm determinant involving the kernel $\mathcal{K}$. Precisely, we have for $s_1, \dots, s_k \in \mathbb{C}$ [Sos00, Theorem 2]

$$\mathbb{E} \left(\prod_{j=1}^k s_j^{n_{A_j}} \right) = \det \left(1 - \chi_{\cup_j A_j} \sum_{j=1}^k (1 - s_j) \mathcal{K} \chi_{A_j} \right), \quad (1.1.4)$$

where χ_A is the projection operator from $L^2(\mathbb{R})$ to $L^2(A)$, or the characteristic function of A at the level of the kernels. This expectation can of course be rewritten as

$$\mathbb{E} \left(\prod_{j=1}^k s_j^{n_{A_j}} \right) = \sum_{\ell_1, \dots, \ell_k \geq 0} s_1^{\ell_1} \dots s_k^{\ell_k} \mathbb{P}(n_{A_1} = \ell_1 \cap \dots \cap n_{A_k} = \ell_k).$$

The Fredholm determinant appearing in (1.1.4) can be expressed in terms of an infinite series involving the kernel. If $\mathcal{J}$ is a trace class integral operator acting on $L^2(A \subseteq \mathbb{R})$ with integral kernel J , then the Fredholm determinant $\det(1 - \mathcal{J})$ is given by

$$\det(1 - \mathcal{J}) = 1 + \sum_{j=1}^{+\infty} \frac{1}{j!} \int_{A^j} \det(J(x_\ell, x_m))_{\ell, m=1}^j dx_1 \dots dx_j.$$

Largest particle property In the next chapters, we will study DPPs that exhibit the property of having a *largest particle*, or a *smallest particle*. The largest particle property is the fact that the random variable

$$\sup \left\{ x_\ell : \xi = \sum_j x_j \right\}$$

is finite almost surely. At the level of the kernel, this is actually asking that $\mathcal{K}$ not only is trace class on bounded sets, but also on sets of the form $(t, +\infty)$ for any (or equivalently one, since $\mathcal{K}$ is locally trace class)

$t \in \mathbb{R}$, i.e.,

$$\mathbb{E}(n_{(t,+\infty)}) = \int_t^{+\infty} K(x, x) dx < +\infty. \quad (1.1.5)$$

The smallest particle property is the obvious adaptation to the definition of largest particle.

In processes with the largest particle property, the generating function of the occupancy numbers has even more importance in that it allows to express the joint probability distribution of any choice of k ordered particles. The largest particle property allows to write

$$\zeta_1 \geq \zeta_2 \geq \dots$$

for the ordered particles of the point process. Let $+\infty := x_0 > x_1 > \dots > x_k > -\infty$ and choose the sets $A_1, \dots, A_k$ in (1.1.4) to be

$$A_j = (x_j, x_{j-1}), \quad j = 1, \dots, k. \quad (1.1.6)$$

Define the function $F(\vec{x}, \vec{s}) := F(x_1, \dots, x_k; s_1, \dots, s_k)$ by

$$F(\vec{x}; \vec{s}) = \det \left(1 - \chi_{(x_k, +\infty)} \sum_{j=1}^k (1 - s_j) \mathcal{K} \chi_{(x_j, x_{j-1})} \right). \quad (1.1.7)$$

For $k = 1$, the function F allows to express the probability density of any of the ζ -particles. The probability density for the largest particle is for example given by

$$\mathbb{P}(\zeta_1 < x_1) = \mathbb{P}(n_{(x_1, +\infty)} = 0) = \mathbb{E} \left(s_1^{n_{(x_1, +\infty)}} \right) \Big|_{s_1=0} = F(x_1; 0).$$

More generally, the probability density of the ℓ -th largest eigenvalue is also available from F with $k = 1$, with the formula

$$\mathbb{P}(\zeta_\ell < x_1) = \sum_{j=0}^{\ell-1} \mathbb{P}(n_{(x_1, +\infty)} = j) = \sum_{j=0}^{\ell-1} \frac{1}{j!} \frac{d^j}{ds_1^j} F(x_1; s_1) \Big|_{s_1=0}.$$

Similarly as for $\mathbb{P}(\zeta_\ell < x)$, we can express the joint probability of two particles $\zeta_{m_1} > \zeta_{m_2}$ with $m_1 < m_2$ as

$$\begin{aligned}
\mathbb{P}(\zeta_{m_1} < x_1, \zeta_{m_2} < x_2) &= \sum_{\substack{j_1 < m_1 \\ j_1 + j_2 < m_2}} \mathbb{P}(n_{(x_1, +\infty)} = j_1, n_{(x_2, x_1)} = j_2) \\
&= \sum_{\substack{j_1 < m_1 \\ j_1 + j_2 < m_2}} \frac{1}{j_1! j_2!} \frac{\partial^{j_1 + j_2}}{\partial s_1^{j_1} \partial s_2^{j_2}} F(x_1, x_2; s_1, s_2) \Big|_{s_1 = s_2 = 0}. \quad (1.1.8)
\end{aligned}$$

More generally, the joint distribution of k particles $\zeta_{m_1} > \dots > \zeta_{m_k}$ with $m_1 < \dots < m_k$ is given by

$$\begin{aligned}
\mathbb{P}\left(\cap_{j=1}^k (\zeta_{m_j} < x_j)\right) &= \sum \mathbb{P}\left(\cap_{j=1}^k (n_{A_j} = m_j)\right) \\
&= \sum \frac{1}{j_1! j_2! \dots j_k!} \frac{\partial^{j_1 + j_2 + \dots + j_k}}{\partial s_1^{j_1} \partial s_2^{j_2} \dots \partial s_k^{j_k}} F(\vec{x}; \vec{s}) \Big|_{\vec{s}=0}, \quad (1.1.9)
\end{aligned}$$

where A_j is as in (1.1.6), and where the sum is taken over all indices $j_1, \dots, j_k$ such that

$$j_1 < m_1, \quad j_1 + j_2 < m_2, \quad \dots \quad \sum_{i=1}^k j_i < m_k. \quad (1.1.10)$$

Thinning of Determinantal Point Processes A natural way of producing new point processes is a procedure called “thinning”, see, e.g., [DVJ07]. Given $0 < s < 1$, a point process and $(X_j)_j$ a family of independent and identically distributed Bernoulli random variables such that $\mathbb{P}(X_j = 0) = s$ and $\mathbb{P}(X_j = 1) = 1 - s$, we define an other point process by applying the random map

$$\xi = \sum_j \delta_{x_j} \mapsto \xi_s = \sum_j X_j \delta_{x_j}.$$

The procedure can be interpreted as follows. For each particle ζ_j of the point process, we toss a (biased) coin and remove that particle with probability s . The particles that remain can be regarded as “observed” particles.

In [BP06], Bohigas and Pato began to consider thinning of DPPs. It has the interesting property that a thinned DPP is again determinantal, with an easy transformation of the kernel. If K is the correlation kernel of the starting DPP, then $(1-s)K$ is the correlation kernel of the thinned

DPP [LMR15].

1.2 Examples

In this section, we present some examples of determinantal point processes, many of which will be important in the next chapters.

1.2.1 Gaussian Unitary Ensemble

Probably the most studied DPP in the field of random matrix theory is the process of the eigenvalues of the Gaussian Unitary Ensemble (GUE).

The GUE is the set of Hermitian $n \times n$ matrices equipped with the probability measure

$$\frac{1}{Z_n^{\text{GUE}}} e^{-\frac{n}{2} \text{tr}(H^2)} dH, \quad dH = \prod_{j=1}^n dH_{jj} \prod_{1 \leq i < j \leq n} d\text{Re } H_{ij} d\text{Im } H_{ij}. \quad (1.2.1)$$

Equivalently, the diagonal entries of H are independent identically distributed (iid) normal random variables $\mathcal{N}(0, 1/n)$ with mean 0 and variance $1/n$ and the upper triangular entries are iid complex normal random variables $\mathcal{N}(0, \frac{1}{2n}) + i\mathcal{N}(0, \frac{1}{2n})$.

For such matrices, the eigenvectors and the eigenvalues are independent. When we average out the eigenvectors, we are left with an induced joint probability density on the eigenvalues given by

$$p(x_1, \dots, x_n) = \frac{1}{\tilde{Z}_n^{\text{GUE}}} \Delta(x)^2 \prod_{j=1}^n e^{-\frac{n}{2} x_j^2} dx_j, \quad (1.2.2)$$

where

$$\Delta(x) = \prod_{1 \leq j < k \leq n} (x_k - x_j) \quad (1.2.3)$$

is the Vandermonde determinant.

From the density (1.2.2), we can already see the repulsive nature of the model, since the Vandermonde determinant vanishes when two particles are identical (see Figure 1.1 for a comparison of the distances in the GUE with the distances of iid uniform random variables where repulsion is absent). On the other hand, the potential $n \frac{x^2}{2}$ pushes the

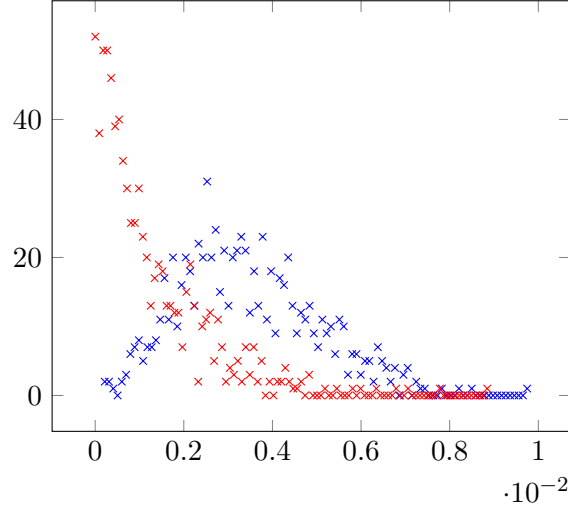


Figure 1.1: Numerical comparison between nearest neighbour distances for eigenvalues of a 1000×1000 matrix from the GUE (blue) and 1000 iid uniform random variables on $[0, 1]$ (red). Theory shows that the limit distribution of the blue crosses vanishes quadratically when approaching 0, which is a manifestation of the repulsion: small distances are unlikely. Samples were generated using Matlab.

particles to the origin. On the physical point of view, we see that the n -factor in the potential makes both the repulsive force and the potential to be of the same order, leading to non trivial limits for large n , as we shall see.

Correlation kernel The correlation kernel of this determinantal point process can be constructed explicitly in terms of Hermite polynomials. We call the Hermite polynomials the sequence $(p_j)_j$ of polynomials, p_j of degree j , such that for any $j, k \in \mathbb{N}$, we have

$$\int_{\mathbb{R}} p_j(x) p_k(x) e^{-\frac{n}{2}x^2} dx = \delta_{jk}.$$

With these polynomials, the correlation kernel may be given by

$$K_n(x, y) = e^{-\frac{n}{4}(x^2+y^2)} \sum_{j=0}^{n-1} p_j(x) p_j(y). \quad (1.2.4)$$

The kernel K_n can also be rewritten thanks to the Christoffel-Darboux formula,

$$K_n(x, y) = e^{-\frac{n}{4}(x^2+y^2)} \frac{p_n(x)p_{n-1}(y) - p_{n-1}(x)p_n(y)}{x - y}, \quad (1.2.5)$$

which is actually valid for any family of orthogonal polynomials.

If one wants to re-derive the formula for the kernel (1.2.4), they should use Andreiev's identity and the fact that the kernel K_n is “reproducing”, i.e., for all $x, z \in \mathbb{R}$,

$$K_n(x, z) = \int_{\mathbb{R}} K_n(x, y) K_n(y, z) dy.$$

Wigner semi-circle law When studying the GUE, one is often interested in computing asymptotics of quantities when $n \rightarrow +\infty$, and the formalism of DDPs is especially well suited for that.

A first consequence of the formula (1.2.5) and the large degree asymptotics of Hermite polynomials is the celebrated Wigner semi-circle law. When integrating out $n - 1$ particles and keeping only the (normalized) overall density $\frac{1}{n}\rho_1^{(n)}(x)$, we have for all $x \in \mathbb{R}$

$$\lim_{n \rightarrow +\infty} \frac{1}{n}\rho_1^{(n)}(x) = \lim_{n \rightarrow +\infty} \frac{1}{n}K_n(x, x) = \frac{1}{2\pi}\sqrt{4 - x^2}\chi_{[-2, 2]}(x). \quad (1.2.6)$$

This result means that as n grows, the eigenvalues of the GUE tend to accumulate around the interval $[-2, 2]$ and are distributed according to a semi-circle. A stronger result actually holds. We have indeed that as $n \rightarrow +\infty$, the random variables $\xi_n = \frac{1}{n} \sum_{j=1}^n \delta_{x_j}$ converge weakly almost surely to the semi-circle probability measure. This fact can be concretely witnessed when numerically generating a random matrix from the GUE, and representing its eigenvalues in a histogram that counts the number of particles in several small intervals, see Figure 1.2.

The microscopic regimes of the GUE are also well known. As someone could guess from the semi-circle law, two very different types of behaviour emerge when looking at local statistics. The first one appears in the “bulk”, around each point $x^* \in (-2, 2)$ and the second at the “edges”, i.e., around 2 and -2 .

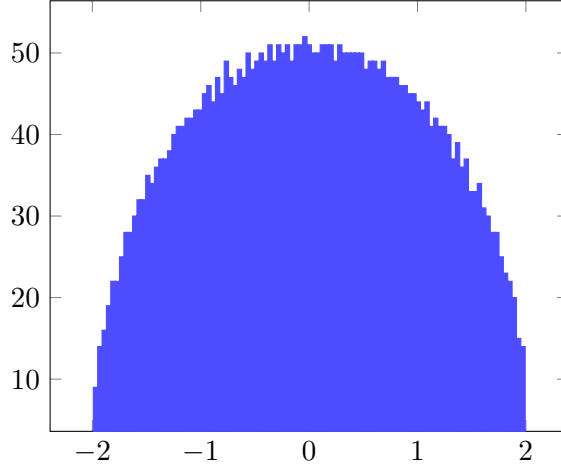


Figure 1.2: Histogram of the eigenvalues of a 4000×4000 matrix drawn randomly from the GUE with probability density given by (1.2.1). This is a manifestation of the Wigner semi-circle law: all eigenvalues accumulate around $[-2, 2]$ and are distributed according to a semi-circle. Here the interval $[-2, 2]$ is cut into a hundred sub-intervals, and each bar in the plot counts the number of eigenvalues in it.

In the bulk In order to study the microscopic behaviour near a point $x^* \in (-2, 2)$, we need to rescale the process in such a way that the distances between particles will be of order 1, and to that end we need to compute the order of the distances for our process. It can actually be obtained using a simple “physical” argument. Around $x^* \in (-2, 2)$, we search for a sequence $(d_n)_n$ such that the average number of particles inside $(x^*, x^* + d_n)$ is of order one. Let λ denote the semi-circle density,

$$\lambda(x) = \frac{1}{2\pi} \sqrt{4 - x^2} \chi_{[-2, 2]}(x). \quad (1.2.7)$$

Using formulas (1.1.2) and (1.2.6), we can compute

$$\begin{aligned} \mathbb{E} \left(n_{(x^*, x^* + d_n)} \right) &= \int_{x^*}^{x^* + d_n} K_n(x, x) dx \sim \\ &\int_{x^*}^{x^* + d_n} n \lambda(x) dx \sim n d_n \lambda(x^*). \end{aligned} \quad (1.2.8)$$

Since this must be of order 1 as $n \rightarrow +\infty$, we have that d_n must be of order $\frac{1}{n}$. Although this argument is not rigorous, it still allows to predict the right scale and indeed we have a non-trivial limit [Meh04, Formula (6.3.13)]

$$\lim_{n \rightarrow +\infty} \frac{1}{n\lambda(x^*)} K_n \left(x^* + \frac{\zeta}{n\lambda(x^*)}, x^* + \frac{\theta}{n\lambda(x^*)} \right) = K^{\text{SIN}}(\zeta, \theta) = \frac{\sin \pi(\zeta - \theta)}{\pi(\zeta - \theta)},$$

uniformly for ζ and θ in compact subsets of $\mathbb{C}$. The kernel K^{SIN} is the correlation kernel of another determinantal point process, called the sine process, discussed below in sub-section 1.2.4. The sine process neither has the largest nor the smallest particle property. This intuitively can be seen by the fact that, having zoomed inside the support of the limiting density, as n grows towards infinity, there are an infinite number of particles both to the left and to the right of x^* .

The fact of rescaling the variables by a factor $\frac{1}{n}$ in the kernel is simply because

$$\frac{1}{n\lambda(x^*)} K_n \left(x^* + \frac{\zeta}{n\lambda(x^*)}, x^* + \frac{\theta}{n\lambda(x^*)} \right)$$

is the kernel that one would obtain by multiplying the matrices of the GUE by $\lambda(x^*)n$, hence “zooming” by the right factor so that the typical distance between two particles is now of order 1. Note that the computations in (1.2.8) not only gave us the right scale, but also the correct constant.

At the edges Around 2 or -2 , the microscopic behaviour of the particles is not governed by the sine process. When zooming near the (say right) edge, the particles to the right and to the left of 2 should behave differently.

The order of the typical distance can once again be obtained using the same physical argument as in (1.2.8). The main difference, however, lies in the fact that the density λ (1.2.7) vanishes at the end point. The

computations now are

$$\mathbb{E}\left(n_{(2-d_n, 2)}\right) \sim \int_{2-d_n}^2 n\lambda(x)dx \sim \int_{2-d_n}^2 \frac{n}{\pi}\sqrt{2-x}dx = \frac{2n}{3\pi}d_n^{\frac{3}{2}}, \quad (1.2.9)$$

and the typical distance between the eigenvalues at the edge should thus be of order $\frac{1}{n^{2/3}}$.

These heuristics are confirmed by the limit [Meh04, Formula (24.2.1)]

$$\begin{aligned} \lim_{n \rightarrow +\infty} \frac{2}{n^{2/3}} K_n \left(2 + \frac{2\zeta}{n^{2/3}}, 2 + \frac{2\theta}{n^{2/3}} \right) = \\ K^{\text{AIRY}}(\zeta, \theta) = \frac{\text{Ai}(\zeta)\text{Ai}'(\theta) - \text{Ai}'(\zeta)\text{Ai}(\theta)}{\zeta - \theta}, \end{aligned} \quad (1.2.10)$$

uniformly for ζ and θ in compact subsets of $\mathbb{C}$, and Ai is the Airy function. The correlation kernel K^{AIRY} is the one of a determinantal point process called the Airy ensemble, or the Airy point process, and is discussed in sub-section 1.2.5. The Airy point process has the largest particle property, which confirms the intuition that the right of 2 is different from the left in the GUE.

1.2.2 Polynomial Ensembles

The GUE is actually a special case of a bigger class of determinantal point processes, called polynomial ensembles (PEs) and introduced in [KS14]. A PE is a probability density on $\mathbb{R}^n$ of the form

$$p_n(x_1, \dots, x_n) = \frac{1}{Z_n} \Delta(x) \det \left[f_{k-1}(x_j) \right]_{j,k=1}^n, \quad (1.2.11)$$

where Δ is again the Vandermonde determinant (1.2.3), for certain functions $f_0, \dots, f_{n-1}$ and where Z_n is a normalizing constant. The GUE can for example be viewed as a PE with $f_j(x) = x^j e^{-n\frac{x^2}{2}}$. Conditions on the functions f_j 's are willingly left vague, but it suffices to say that we only need functions such that (1.2.11) is indeed a probability density on $\mathbb{R}^n$. The particles in a DPP live on the union $A \subseteq \mathbb{R}$ of the supports of the functions f_j 's, and in that case we say that the PE is on A .

The correlation kernel of a PE can be written in the form

$$K_n(x, y) = \sum_{j=0}^{n-1} p_j(x) q_j(y), \quad (1.2.12)$$

where p_j is a polynomial of degree j , and q_j is a linear combination of the functions $f_0, \dots, f_{n-1}$ such that the orthogonality conditions

$$\int_{\mathbb{R}} p_j(x) q_k(x) dx = \delta_{jk}, \quad j, k = 0, \dots, n-1 \quad (1.2.13)$$

are satisfied.

The joint probability density of the eigenvalues of a wide range of random matrix ensembles are PEs, and they are preserved under certain transformations [CKW15, KKS16]. In Chapter 2, we will investigate how certain limiting kernels appearing in PEs on $\mathbb{R}^+$ are affected by adding a “small” GUE matrix.

1.2.3 Biorthogonal ensembles

A bigger class of determinantal point process are biorthogonal ensembles [Bor99] that are probability densities on $\mathbb{R}^n$ given by

$$p_n(x_1, \dots, x_n) = \frac{1}{Z_n} \det \left[f_{k-1}(x_j) \right]_{j,k=1}^n \times \det \left[g_{k-1}(x_j) \right]_{j,k=1}^n,$$

for certain functions $f_0, \dots, f_{n-1}$ and $g_0, \dots, g_{n-1}$. PEs are a special case of biorthogonal ensembles where $g_j(x) = x^j$, i.e., the determinant on the right is the Vandermonde determinant (1.2.3).

1.2.4 Sine process

The sine process is a determinantal point process that we may define through its correlation kernel

$$K^{\text{sin}}(\zeta, \theta) = \frac{\sin \pi(\zeta - \theta)}{\pi(\zeta - \theta)}.$$

As we saw, the sine process appears as limiting point process in the bulk of the GUE. This is actually ubiquitous in random matrix theory, and is



Figure 1.3: Comparison between particles drawn from the sine process (above) and iid uniform random variables (below). Both are normalized in $(-1, 1)$. Clustering of particles happens less frequently for the sine process.

part of what is called “universality”. Universality principles for example suggest that the sine process behaviour will appear in a wide range of models and does not depend on the particular features of the model.

Unlike the GUE, the number of particles in the sine process is (almost surely) infinite. Moreover, it neither has the largest nor the smallest particle property, resulting in a process that occupies the whole real line. We made a comparison between the particles of the sine process and iid uniform random variables on $(-1, 1)$, see Figure 1.3.

1.2.5 Airy point process

The Airy ensemble (sometimes called the Airy point process) is the determinantal point process that appears at the edges of the GUE (1.2.10). Its kernel is given for $\zeta, \theta \in \mathbb{C}$ by

$$K^{\text{AIRY}}(\zeta, \theta) = \frac{\text{Ai}(\zeta)\text{Ai}'(\theta) - \text{Ai}'(\zeta)\text{Ai}(\theta)}{\zeta - \theta}, \quad (1.2.14)$$

where Ai is the Airy function of the first kind defined for $z \in \mathbb{C}$ by

$$\text{Ai}(z) = \frac{1}{\pi} \int_{e^{-\pi i/3}\infty}^{e^{\pi i/3}\infty} e^{\frac{w^3}{3} - zw} dw, \quad (1.2.15)$$

see [OLBC10, Chapter 9] for plethora of formulas involving the Airy function. The graph of Ai is represented in Figure 1.4. The decay of $\text{Ai}(x)$ for $x \rightarrow +\infty$ ensures that the Airy point process has the largest particle property. It however does not have the smallest particle property, as the number of particles for the Airy ensemble is infinite with probability one.

When looking at the Airy point process from the perspective of the

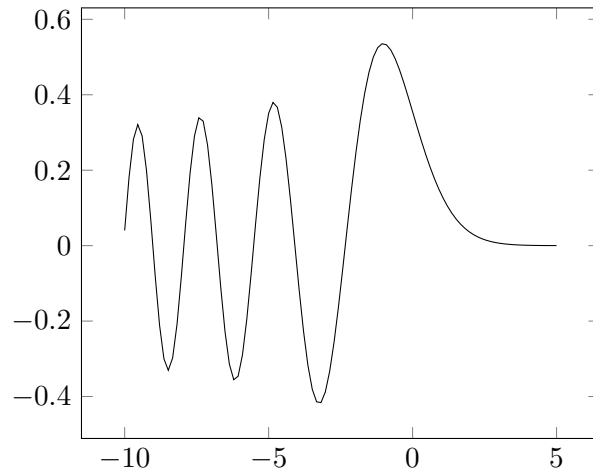


Figure 1.4: Graph of the Airy function Ai of the first kind from -10 to 5 . The Airy function has two very different types of behaviour at plus and minus infinity. For large positive values of the argument, the Airy function has super-exponential decay, while it is oscillatory for negative values of the argument. Numerical values for the plot were computed using Matlab's Airy function.

limiting kernel for the GUE at the edge, it is not surprising to see that $\text{Ai}(x)$ has an oscillatory behaviour for $x \rightarrow +\infty$, since it is the same kind of behaviour as the sine process in the bulk of the spectrum has.

The Airy ensemble is actually yet another witness of the Universality principle of random matrix theory. It typically appears when a macroscopic phase transition is “soft”, in the sense that no physical boundary restrains the particles. From that perspective, it is indeed not surprising that the Airy point process also governs the behaviour of dominoes in the Aztec diamond near the Arctic circle (see the introduction of the thesis). Other examples of models where the Airy ensemble comes into play are non-intersecting Brownian paths, and random partitions or Young diagrams with respect to the Plancherel measure, see e.g. [Joh06] for an overview.

In 1992, Tracy and Widom [TW94a] gave a description of the generating function $F(\vec{x}, \vec{s})$ (1.1.7) in the special case where $k = 1$ in terms

of solutions to the Painlevé II equation: for $0 \leq s < 1$, $x \in \mathbb{R}$, we have

$$F(x; s) = \exp \left(- \int_x^{+\infty} (\xi - x) q^2(\xi; s) d\xi \right), \quad (1.2.16)$$

with $q(\xi; s)$ the solution to the homogeneous Painlevé II equation

$$q'' = \xi q + 2q^3 \quad (1.2.17)$$

which is characterized by the following asymptotic behavior at $+\infty$,

$$q(\xi; s) = \sqrt{1-s} \operatorname{Ai}(\xi)(1 + o(1)), \quad \xi \rightarrow +\infty. \quad (1.2.18)$$

These solutions $q(\xi; s)$ are known as the Ablowitz-Segur solutions [AS77] of Painlevé II for $0 < s < 1$ and as the Hastings-McLeod solution [HM80] if $s = 0$. The probability distribution of the largest particle of the Airy ensemble is now called the Tracy-Widom distribution.

In Chapter 3, we will give a description of the generating function F for any $k > 0$ in terms of solutions to a system of coupled Painlevé II equations, hence extending Tracy-Widom's result.

Chapter 2

Gaussian perturbations of hard edge random matrix ensembles

The content of this chapter has been taken from the eponymous paper with Tom Claeys [CD16].

2.1 Introduction

We consider a class of Hermitian random matrices which are perturbed by additive Gaussian noise, and investigate to what extent the microscopic behaviour of the eigenvalues is affected by such a perturbation, in the limit where the size of the matrices tends to infinity. We take the Gaussian noise to be a small multiple of a matrix from the Gaussian Unitary Ensemble (GUE), which was introduced in sub-section 1.2.1 in Chapter 1. Recall that a matrix drawn from the GUE can be constructed by putting independent identically distributed (iid) normal random variables $\mathcal{N}(0, 1/n)$ with mean 0 and variance $1/n$ on the diagonal, and iid complex normal random variables $\mathcal{N}(0, \frac{1}{2n}) + i\mathcal{N}(0, \frac{1}{2n})$ in the upper triangular part. Adding a small multiple of a GUE matrix, ϵH , to another random matrix M can thus be viewed as an entry-wise Gaussian perturbation of M .

Gaussian perturbations of random matrices are closely related to random matrices with external source. To see this, we note that, given M , the probability distribution of the random matrix $S = M + \epsilon H$ can be written as

$$\frac{1}{Z_n} e^{-\frac{n}{2\epsilon^2} \text{Tr}((S-M)^2)} dS, \quad dS = \prod_{j=1}^n dS_{jj} \prod_{1 \leq i < j \leq n} d\text{Re} S_{ij} d\text{Im} S_{ij}.$$

This is known as the GUE with external source M [BH98, ZJ98]. In our setting, the external source matrix M is not deterministic but is itself a random matrix.

If M is a unitary invariant random matrix with probability measure

$$\frac{1}{Z_n} e^{-n \text{Tr} V(M)} dM, \quad dM = \prod_{j=1}^n dM_{jj} \prod_{1 \leq i < j \leq n} d\text{Re} M_{ij} d\text{Im} M_{ij},$$

for some potential V , then our model is equivalent to a special case of the two-matrix model [BEH02], which is defined as a probability measure on pairs of Hermitian matrices (M_1, M_2) , given by

$$\frac{1}{Z_n} e^{-n \text{Tr}(V_1(M_1) + V_2(M_2) - \tau M_1 M_2)} dM_1 dM_2,$$

for certain potentials V_1, V_2 . If we take

$$V_2(x) = x^2/2, \quad V_1(x) = V(x) + \frac{\tau^2}{2} x^2, \quad \tau = 1/\epsilon,$$

then it is straightforward to verify that M_1 and $M_1 - \tau M_2$ are independent, that M_1 has the same distribution as M , and that $\frac{1}{\tau} M_2$ has the same distribution as the sum $S = M + \epsilon H$, see also [Dui14, Section 5].

The eigenvalues of Gaussian perturbations of (deterministic or random) matrices can alternatively be realized as the positions of n non-intersecting Brownian paths with a common endpoint and with (deterministic or random) starting points, see e.g. [Joh02] and the recent work [FG15].

In what follows, the random Hermitian $n \times n$ matrix M has to be independent of the GUE matrix H and such that the joint probability density function of the eigenvalues is a Polynomial Ensemble, see sub-

section 1.2.2 in Chapter 1.

Later on, we will focus on polynomial ensembles defined by functions f_j supported on $[0, +\infty)$, but for now, they can be general.

If the joint eigenvalue density of a random matrix M is a polynomial ensemble with correlation kernel K_n , then it was shown in [CKW15] that the eigenvalues of $S = M + \epsilon H$, with H a GUE matrix independent of M and $\epsilon > 0$, also follow a polynomial ensemble, with the transformed eigenvalue correlation kernel

$$K_n^S(x, y) = \frac{n}{2\pi i \epsilon^2} \int_{i\mathbb{R}} \int_{\mathbb{R}} K_n(s, t) e^{\frac{n}{2\epsilon^2}((x-s)^2 - (y-t)^2)} dt ds. \quad (2.1.1)$$

In addition, if $p_n(x) = \mathbb{E}[\det(xI - M)]$ is the average characteristic polynomial of M , then the average characteristic polynomial of $S = M + \epsilon H$ is given by

$$P_n(x) = \mathbb{E}[\det(xI - S)] = \frac{\sqrt{n}}{\sqrt{2\pi i \epsilon}} \int_{i\mathbb{R}} p_n(s) e^{\frac{n}{2\epsilon^2}(x-s)^2} ds. \quad (2.1.2)$$

The formulas (2.1.1) and (2.1.2) follow from [CKW15, formulas (2.6) and (2.8)] after a simple re-scaling argument, and they will be the starting point of our analysis.

Macroscopic eigenvalue behaviour

The macroscopic large n behaviour of the eigenvalues of $M + \epsilon H$ is well understood thanks to free probability theory: if $M = M_n$ is a sequence of random $n \times n$ matrices whose eigenvalue distributions converge almost surely to a measure μ and if M is independent of the GUE matrix $H = H_n$, then M and H are asymptotically free and we can apply results from free probability theory [NS06, Spe11] to describe the limiting eigenvalue distribution of $S = M + \epsilon H$. Writing λ_ϵ for the rescaled semi-circle law,

$$d\lambda_\epsilon(x) = \frac{1}{2\pi\epsilon^2} \sqrt{4\epsilon^2 - x^2} dx, \quad x \in [-2\epsilon, 2\epsilon], \quad (2.1.3)$$

which is the limiting macroscopic density of the eigenvalues of ϵH , it is well-known that the limiting eigenvalue distribution of S is almost surely

given by the free additive convolution $\mu \boxplus \lambda_\epsilon$ of μ and λ_ϵ , see [Bia97] for the definition and properties of the free convolution of a measure μ with λ_ϵ .

Another quantity containing global information about random matrix eigenvalues is the zero counting measure of the average characteristic polynomial. The zeros of the average characteristic polynomial are real and simple (see Lemma 2.3.1), and the zero counting measure can heuristically be interpreted as a typical eigenvalue configuration. It can therefore be expected that its large n limit coincides with the limiting (mean) eigenvalue distribution in many cases. This is well-known for classical random matrix ensembles and was investigated in a more general framework in [Har15]. The following result about convergence of the zero counting measure of the average characteristic polynomial of S is not surprising in view of the above-mentioned results from free probability. We will prove it in Section 2.3 directly using the integral representation (2.1.2) and without relying on the more sophisticated results from free probability theory.

Theorem 2.1.1. *Let M be an $n \times n$ Hermitian random matrix such that its eigenvalue density is a polynomial ensemble (1.2.11), let H be an $n \times n$ GUE matrix independent of M , and let $\epsilon > 0$. Write μ_n for the zero counting measure of the average characteristic polynomial of M , and ν_n for the zero counting measure of the average characteristic polynomial of $S = M + \epsilon H$.*

If, for sufficiently large n , the support of μ_n is contained in some n -independent compact K , and if μ_n converges weakly to a probability measure μ , then ν_n converges weakly to $\mu \boxplus \lambda_\epsilon$, where λ_ϵ is given by (2.1.3).

Microscopic eigenvalue behaviour

From now on, we consider polynomial ensembles supported on $[0, +\infty)$ or on an interval of the form $[0, b]$. We mean by this that the functions f_j in (1.2.11) are supported on $[0, +\infty)$ or on $[0, b]$. Ensembles of this kind are said to have a *hard edge* at zero, as opposed to the soft edge appearing in the GUE. In the hard edge case, particles are indeed physically

constrained to be positive. Classical examples of random matrix ensembles with a hard edge are the Laguerre Unitary Ensemble and the Jacobi Unitary Ensemble. In these ensembles, the microscopic eigenvalue correlations near 0 are described in terms of Bessel functions. As we will see below, other ensembles may lead to other types of microscopic eigenvalue correlations, described in terms of other functions, such as Meijer G-functions or Wright's generalized Bessel functions. A common feature of all hard edge random matrix ensembles which we will study below, is the existence of a scaling limit near the hard edge of the following form:

$$\lim_{n \rightarrow +\infty} \frac{1}{cn^\gamma} K_n \left(\frac{u}{cn^\gamma}, \frac{v}{cn^\gamma} \right) = \mathbb{K}(u, v), \quad v > 0, u \in \mathbb{C}, \quad (2.1.4)$$

for some values of $c, \gamma > 0$, and for some limiting kernel $\mathbb{K}(u, v)$, which depend on the particular choice of random matrix ensemble.

If we consider a Gaussian perturbation of M of the form $S = M + \epsilon H$, even if $\epsilon > 0$ is small, the matrix S is typically not positive-definite, in other words the hard edge at 0 is removed by the perturbation. It is our aim to understand how scaling limits of the eigenvalue correlation kernel near 0 of the form (2.1.4) change after the Gaussian perturbation. In particular, we want to see what happens in double scaling limits where the constant ϵ goes to 0 as n goes to infinity, as this is the limit in which the soft edge of the spectrum (which we have for fixed $\epsilon > 0$) turns into a hard edge at the origin. We now present a general auxiliary result, which we will apply to several concrete examples later on. Given a scaling limit of the form (2.1.4), it states that the scaling limit is preserved for the eigenvalue correlation kernel of S provided that $\epsilon \rightarrow 0$ sufficiently fast with $n \rightarrow \infty$. If $\epsilon \rightarrow 0$ at a critical speed, the limiting kernel $\mathbb{K}$ is deformed.

Lemma 2.1.2. *Consider a sequence of $n \times n$ random matrices M such that their eigenvalue densities are polynomial ensembles on $[0, +\infty)$ or on $[0, b]$. We assume there exist constants $\gamma > 1/2$, $c, c_1, c_2, n_0 > 0$ and $\beta \in [0, 1)$ such that the associated correlation kernels K_n satisfy the following conditions:*

1. there exists a function $\mathbb{K}(u, v)$ such that

$$\lim_{n \rightarrow +\infty} v^\beta \frac{1}{cn^\gamma} K_n \left(\frac{u}{cn^\gamma}, \frac{v}{cn^\gamma} \right) = v^\beta \mathbb{K}(u, v), \quad (2.1.5)$$

uniformly for u in any compact subset of $\mathbb{C}$ and v in any compact subset of $[0, +\infty)$. Here $v^\beta \mathbb{K}(u, v)$ and $v^\beta K_n \left(\frac{u}{cn^\gamma}, \frac{v}{cn^\gamma} \right)$ for $v = 0$ have to be understood as the limits as $v \rightarrow 0$ and it is supposed that these limits exist,

2. for every $(u, v) \in i\mathbb{R} \times [0, +\infty)$ and $n > n_0$,

$$|K_n(u, v)| \leq c_1 v^{-\beta} n^{\gamma(1-\beta)} e^{c_2 n^\gamma (|u| + |v|)}. \quad (2.1.6)$$

Let $S = M + \epsilon_n H$ where H is a GUE matrix independent of M , and let K_n^S be the eigenvalue correlation kernel for S . Then,

1. if ϵ_n is such that $\lim_{n \rightarrow +\infty} \epsilon_n n^{\gamma - \frac{1}{2}} = 0$, we have

$$\lim_{n \rightarrow +\infty} \frac{1}{cn^\gamma} K_n^S \left(\frac{x}{cn^\gamma}, \frac{y}{cn^\gamma} \right) = \mathbb{K}(x, y), \quad x, y > 0, \quad (2.1.7)$$

2. if ϵ_n is such that $\lim_{n \rightarrow +\infty} c\epsilon_n n^{\gamma - \frac{1}{2}} = \sigma > 0$, we have

$$\begin{aligned} \lim_{n \rightarrow +\infty} \frac{1}{cn^\gamma} K_n^S \left(\frac{x}{cn^\gamma}, \frac{y}{cn^\gamma} \right) = \\ \frac{1}{2\pi i \sigma^2} \int_{i\mathbb{R}} \int_{\mathbb{R}^+} \mathbb{K}(s, t) e^{\frac{1}{2\sigma^2}((x-s)^2 - (y-t)^2)} dt ds, \end{aligned} \quad (2.1.8)$$

uniformly for (x, y) in compact subsets of $\mathbb{C}^2$.

Remark 2.1.3. The rate of decay $\mathcal{O}\left(n^{\frac{1}{2}-\gamma}\right)$ for ϵ_n appears as a critical speed at which the local eigenvalue behaviour changes. When ϵ_n goes to 0 faster than the critical speed, the eigenvalues of the perturbed random matrix $M + \epsilon_n H$ behave locally near 0 as if there were no perturbation. At the critical speed, a new limiting kernel appears at 0, given by (2.1.8). By a saddle point approximation, it is easy to verify that

$$\lim_{\sigma \rightarrow 0} \frac{1}{2\pi i \sigma^2} \int_{i\mathbb{R}} \int_{\mathbb{R}^+} \mathbb{K}(s, t) e^{\frac{1}{2\sigma^2}((x-s)^2 - (y-t)^2)} dt ds = \mathbb{K}(x, y), \quad (2.1.9)$$

which means that (2.1.7) and (2.1.8) are consistent.

Remark 2.1.4. Conditions 1 and 2 in the above lemma are designed in such a way that they hold for a large class of random matrix ensembles. In some cases, we can just take $\beta = 0$. However, it may happen that the functions $f_j(x)$ defining the polynomial ensemble (1.2.11) blow up as $x \rightarrow 0$. This implies that the kernel $K_n(x, y)$ blows up as $y \rightarrow 0$, and thus one cannot expect (2.1.5) and (2.1.6) to hold for $\beta = 0$. This is why we allow $\beta \in [0, 1)$.

In the next section, we discuss several concrete examples of random matrix ensembles to which we can apply Lemma 2.1.2.

2.2 Examples

2.2.1 Perturbed Laguerre/Wishart random matrices

We define the generalized Laguerre Unitary Ensemble (LUE) as the set of $n \times n$ positive-definite Hermitian matrices equipped with the probability measure

$$\frac{1}{Z_{n,\alpha,k}^{\text{LUE}}} (\det M)^\alpha e^{-n\text{Tr}(M^k)} dM, \\ dM = \prod_{j=1}^n dM_{jj} \prod_{1 \leq i < j \leq n} d\text{Re} M_{ij} d\text{Im} M_{ij}, \quad \alpha > -1, k \in \mathbb{N}. \quad (2.2.1)$$

Similarly as for the GUE, the factor n in the exponential ensures the eigenvalues to remain bounded as $n \rightarrow +\infty$ with probability 1. For $\alpha \in \mathbb{N}$ and $k = 1$, a random LUE matrix can be realized as $M = G^* G$, where G is a $(n + \alpha) \times n$ complex Ginibre matrix, which has independent identically distributed complex normal entries $\mathcal{N}(0, \frac{1}{2n}) + i\mathcal{N}(0, \frac{1}{2n})$. In Figure 2.1, we present numerical samples of the perturbed LUE for different values of ϵ .

The eigenvalues of a random matrix M with probability distribution (2.2.1) have the joint probability distribution

$$\frac{1}{\tilde{Z}_{n,\alpha,k}^{\text{LUE}}} \Delta(x)^2 \prod_{j=1}^n x_j^\alpha e^{-n x_j^k} dx_j, \quad x_1, \dots, x_n > 0, \quad (2.2.2)$$

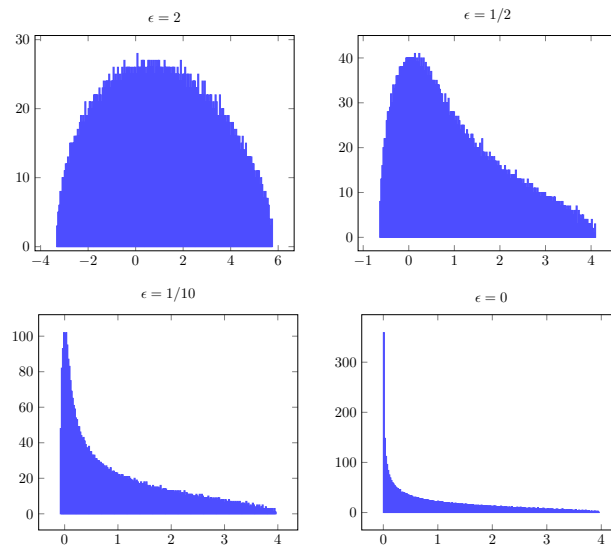


Figure 2.1: Numerical samples of the eigenvalues of $M + \epsilon H$, with M a 4000×4000 matrix drawn from the LUE (with $\alpha = 0$ and $k = 1$), and H a 4000×4000 matrix drawn from the GUE. The chosen values for ϵ are 2, $1/2$, $1/10$ and 0. The case $\epsilon = 0$ is the unperturbed LUE. The eigenvalues are represented in histograms of 200 intervals.

which is a polynomial ensemble (1.2.11) with $f_j(x) = x^{j+\alpha}e^{-nx^k}$ on $\mathbb{R}^+$. The limiting eigenvalue distribution μ in this ensemble is almost surely given by a (generalized) Marchenko-Pastur law of the form

$$d\mu(x) = \frac{1}{2\pi} \sqrt{\frac{b-x}{x}} h(x) dx, \quad x \in (0, b), \quad (2.2.3)$$

for some n -independent $b > 0$ and polynomial h , positive on $[0, b]$. We denote

$$G_\mu(z) = \int \frac{d\mu(x)}{z-x}, \quad z \in \mathbb{C} \setminus [0, b] \quad (2.2.4)$$

for the Stieltjes transform of μ . The limiting eigenvalue distribution of a Gaussian perturbation $S = M + \epsilon H$ is almost surely the free convolution $\mu \boxplus \lambda_\epsilon$. Its density can be shown to have the form [OR12]

$$\frac{d(\mu \boxplus \lambda_\epsilon)(x)}{dx} = h_\epsilon(x) \sqrt{(x-a_\epsilon)(b_\epsilon-x)}, \quad x \in [a_\epsilon, b_\epsilon], \quad (2.2.5)$$

with h_ϵ positive on $[a_\epsilon, b_\epsilon]$. The density vanishes like a square root at both edges for any $\epsilon > 0$, whereas the density of μ blows up at the left edge like an inverse square root.

In [Van07], large n asymptotics for the eigenvalue correlation kernel $K_n(x, y)$ of M have been obtained using the Deift/Zhou steepest descent method [DZ93] applied to the Riemann-Hilbert problem for generalized Laguerre polynomials. In particular, by [Van07, Theorem 2.10 (a)], we have

$$\lim_{n \rightarrow +\infty} \frac{1}{cn^2} K_n\left(\frac{x}{cn^2}, \frac{y}{cn^2}\right) = K_\alpha^{\text{BESSEL}}(x, y), \quad (2.2.6)$$

uniformly for x, y in compact subsets of $(0, +\infty)$, with $c = bh(0)^2$. The limiting kernel $\mathbb{K}_\alpha^{\text{BESSEL}}$ is expressed in terms of Bessel functions of the first kind J_α and takes the explicit form

$$K_\alpha^{\text{BESSEL}}(x, y) = x^{-\frac{\alpha}{2}} y^{\frac{\alpha}{2}} \frac{J_\alpha(\sqrt{x}) \sqrt{y} J'_\alpha(\sqrt{y}) - J_\alpha(\sqrt{y}) \sqrt{x} J'_\alpha(\sqrt{x})}{2(x-y)}. \quad (2.2.7)$$

The Bessel kernel is usually defined without the factor $x^{-\frac{\alpha}{2}} y^{\frac{\alpha}{2}}$ in front, such that it is symmetric in x and y . The pre-factor is present in our situation because the polynomial ensemble kernel is not symmetric. Recall however that correlation kernels may be transformed via a gauge

conjugation (see (1.1.3) in Chapter 1). Using Lemma 2.1.2 and results from [Van07], we will prove the following result.

Theorem 2.2.1. *Let M be an $n \times n$ random matrix with probability measure (2.2.1), and let H be an $n \times n$ GUE matrix with probability measure (1.2.1), independent of M . Write K_n^S for the eigenvalue correlation kernel of $S = M + \epsilon_n H$.*

- (i) **(Sub-critical perturbation)** *If $\lim_{n \rightarrow +\infty} \epsilon_n n^{\frac{3}{2}} = 0$, then for $x, y > 0$, we have*

$$\lim_{n \rightarrow +\infty} \frac{1}{cn^2} K_n^S \left(\frac{x}{cn^2}, \frac{y}{cn^2} \right) = K_\alpha^{\text{BESSEL}}(x, y), \quad (2.2.8)$$

with $c = bh(0)^2$, where h and b are defined by (2.2.3).

- (ii) **(Critical perturbation)** *If $\lim_{n \rightarrow +\infty} c\epsilon_n n^{\frac{3}{2}} = \sigma > 0$, then for $x, y \in \mathbb{C}$, we have*

$$\begin{aligned} \lim_{n \rightarrow +\infty} \frac{1}{cn^2} K_n^S \left(\frac{x}{cn^2}, \frac{y}{cn^2} \right) = \\ \frac{1}{2\pi i \sigma^2} \int_{i\mathbb{R}} \int_{\mathbb{R}^+} K_\alpha^{\text{BESSEL}}(s, t) e^{\frac{1}{2\sigma^2}((s-x)^2 - (t-y)^2)} dt ds. \end{aligned} \quad (2.2.9)$$

- (iii) **(Super-critical perturbation)** *If $\epsilon_n \rightarrow 0$ in such a way that $\epsilon_n n^{\frac{3}{2}} \rightarrow +\infty$ as $n \rightarrow +\infty$, then for $x, y \in \mathbb{C}$, we have*

$$\begin{aligned} \lim_{n \rightarrow +\infty} \frac{e^{\varphi_{\epsilon_n, n}(y)}}{e^{\varphi_{\epsilon_n, n}(x)}} \frac{1}{c_{\epsilon_n} n^{\frac{2}{3}}} K_n^S \left(a_{\epsilon_n} - \frac{x}{c_{\epsilon_n} n^{\frac{2}{3}}}, a_{\epsilon_n} - \frac{y}{c_{\epsilon_n} n^{\frac{2}{3}}} \right) = \\ K^{\text{AIRY}}(x, y), \end{aligned} \quad (2.2.10)$$

with a_ϵ as in (2.2.5), $c_\epsilon = \epsilon^{-2} 6^{1/3} G_\mu''(u_\epsilon)^{-1/3}$, where u_ϵ is the unique negative solution of the equation

$$1 + \epsilon^2 G_\mu'(u) = 0,$$

and with

$$\varphi_{\epsilon, n}(z) = \frac{n^{1/3} z}{c_\epsilon \epsilon^2} (u_\epsilon - a_\epsilon).$$

Recall that the limiting kernel K^{AIRY} is the Airy kernel (1.2.14).

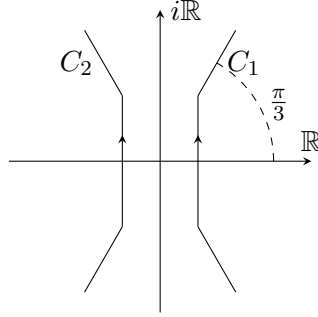


Figure 2.2: Contours in the definition of the Airy kernel K^{AIRY} .

Remark 2.2.2. The natural interpretation of these results is as follows: if ϵ_n tends to 0 sufficiently fast, then the perturbation is too weak to have an effect on the large n behaviour of the eigenvalues near 0. In this case we have the same Bessel kernel limit as for the unperturbed LUE, even though 0 is not a hard edge any longer (for any $\epsilon > 0$ and n fixed, there is a non-zero probability of having negative eigenvalues). On the other hand, if ϵ_n tends to zero slowly, one is close to the fixed ϵ case where one has, macroscopically, soft edges, which suggests Airy behaviour. In (2.2.10), one should note that c_ϵ blows up as $\epsilon \rightarrow 0$: it is of the order $\epsilon^{-8/9}$. The intuition behind this, is that the typical distance between eigenvalues near a_ϵ is of the order $\epsilon^{8/9} n^{-2/3}$. If $\epsilon_n \rightarrow 0$ at the critical speed, the typical distance between eigenvalues is of the order n^{-2} and it is on this scale that the actual transition between the Bessel and the Airy kernel takes place.

Remark 2.2.3. One could consider more general LUE type ensembles where the monomial M^k in (2.2.1) is replaced by a polynomial $V(M)$. As long as V is such that the limiting eigenvalue density blows up like an inverse square root near 0, it leads no doubt that similar scaling limits can be obtained, but we do not investigate this further.

Parts (i) and (ii) of Theorem 2.2.1 will be direct consequences of Lemma 2.1.2. We need the results from [Van07] to show that the conditions of Lemma 2.1.2 are fulfilled, and also to prove part (iii). Here, instead of (1.2.14), we will arrive at a different representation of the

Airy kernel:

$$K^{\text{AIRY}}(u, v) = \frac{1}{4\pi^2} \int_{C_2} ds \int_{C_1} dt \frac{1}{s-t} \frac{e^{t^3-vt}}{e^{s^3-us}}, \quad (2.2.11)$$

with C_1 going from $e^{-\frac{i\pi}{3}}\infty$ to $e^{\frac{i\pi}{3}}\infty$ and C_2 its reflexion through the vertical axis, as in Figure 2.2. Using integration by parts on the integral representation of the Airy function (1.2.15), one can easily check that both kernels (1.2.14) and (2.2.11) are indeed the same.

2.2.2 Perturbed products of Ginibre matrices

Products of Ginibre matrices have been studied intensively during the last years, see e.g. [AI15, AIK13, AKW13, For14, KS14, KZ14]. The squared singular values of products of $m > 0$ independent complex Ginibre matrices follow also a polynomial ensemble with a hard edge at 0. Let $Y_m = X_m X_{m-1} \dots X_1$, with X_j an $(n + \nu_j) \times (n + \nu_{j-1})$ matrix with complex standard Gaussian iid entries, and with the X_j 's independent. The ν_j 's are assumed to be non-negative integers, and $\nu_0 = 0$. For n fixed, it was proved in [KS14] that the joint density of the squared singular values of Y_m is a polynomial ensemble. For $m = 1$, we recover the LUE with $k = 1$ and $\alpha = \nu_1$ after rescaling. Numerical samples for perturbed products of Ginibre matrices are presented in Figure 2.3.

The correlation kernel for the squared singular values of Y_m , or the eigenvalues of $Y_m^* Y_m$, is given by [KZ14, Formula (5.1)]

$$K_n(x, y) = \frac{1}{(2\pi i)^2} \int_{-\frac{1}{2} + i\mathbb{R}} ds \int_{\Sigma_n} dt \prod_{j=0}^m \frac{\Gamma(s + \nu_j + 1)}{\Gamma(t + \nu_j + 1)} \frac{\Gamma(t - n + 1)}{\Gamma(s - n + 1)} \frac{x^t y^{-s-1}}{s - t}, \quad (2.2.12)$$

where Γ denotes the Euler Gamma-function and where Σ_n is a closed contour encircling $0, 1, \dots, n$ in the positive direction in such a way that $\text{Re } t > -\frac{1}{2}$ for $t \in \Sigma_n$. The largest eigenvalue of this ensemble is typically of order n^m [PZ11], and it is therefore more natural for us to rescale the kernel in the following way

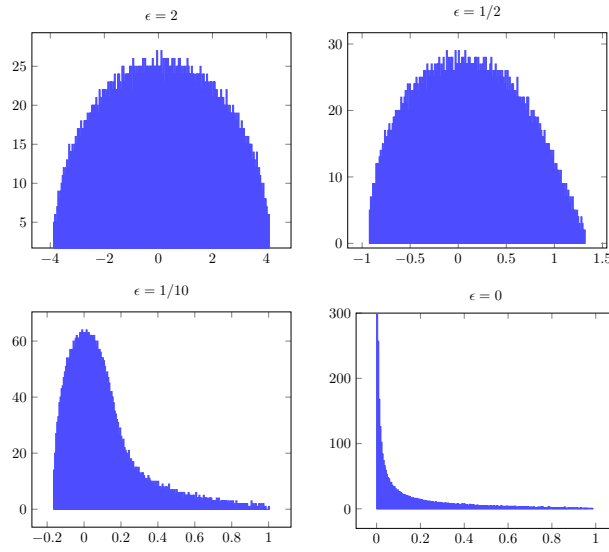


Figure 2.3: Numerical samples of the eigenvalues of $\frac{3^3}{4^4 4000^3} Y_3^* Y_3 + \epsilon H$, where Y_3 is a product of three 4000×4000 (i.e., $\nu_j = 0$ for $j = 1, 2, 3$) Ginibre matrices, and H is a 4000×4000 matrix drawn from the GUE. Because of the factor in front of Y_3 , the eigenvalues accumulate on $[0, 1]$. The chosen values for ϵ are 2, $1/2$, $1/10$ and 0. The eigenvalues are represented in histograms of 200 intervals. The first column for the $\epsilon = 0$ case has been truncated to 300 eigenvalues, its actual value being around 1500.

$$\begin{aligned} \tilde{K}_n(x, y) := n^m K_n(n^m x, n^m y) &= \frac{1}{(2\pi i)^2} \int_{-\frac{1}{2} + i\mathbb{R}} ds \int_{\Sigma_n} dt \\ &\prod_{j=0}^m \frac{\Gamma(s + \nu_j + 1)}{\Gamma(t + \nu_j + 1)} \frac{\Gamma(t - n + 1)}{\Gamma(s - n + 1)} \frac{x^t y^{-s-1} n^{m(t-s)}}{s - t}. \end{aligned} \quad (2.2.13)$$

This is the correlation kernel for the eigenvalues of $\frac{1}{n^m} Y_m^* Y_m$. Using this normalization, it has been shown [Neu13, Theorem 3.2] that the zero counting measures of the average characteristic polynomials converge (in the weak-* sense) to the Fuss-Catalan distribution [PZ11]. We may apply Theorem 2.1.1, and this implies that the counting measures of the average characteristic polynomials of the perturbed random matrix $\frac{1}{n^m} Y_m^* Y_m + \epsilon H$ converges to the free additive convolution of the Fuss-Catalan distribution with the semi-circle law λ_ϵ for any $\epsilon > 0$.

The microscopic behaviour of the eigenvalues near the origin is described by the following scaling limit: we have [KZ14]

$$\lim_{n \rightarrow +\infty} \frac{1}{n^{m+1}} \tilde{K}_n \left(\frac{x}{n^{m+1}}, \frac{y}{n^{m+1}} \right) = \mathbb{K}_\nu^G(x, y), \quad (2.2.14)$$

for $x, y > 0$, where

$$\mathbb{K}_\nu^G(x, y) = \frac{1}{(2\pi i)^2} \int_{-\frac{1}{2} + i\mathbb{R}} ds \int_{\Sigma} dt \prod_{j=0}^m \frac{\Gamma(s + \nu_j + 1)}{\Gamma(t + \nu_j + 1)} \frac{\sin \pi s}{\sin \pi t} \frac{x^t y^{-s-1}}{s - t}. \quad (2.2.15)$$

The contour Σ comes from $+\infty$ in the upper half plane, encircles the positive real axis and goes back to $+\infty$ in the lower half plane, in such a way that $\operatorname{Re} t > -\frac{1}{2}$ for $t \in \Sigma$. The kernel $\mathbb{K}_\nu^G$ can also be expressed in terms of Meijer G -functions. Recently, sine and Airy kernel limits were confirmed rigorously in the bulk and at the right edge [LWZ14].

Using Lemma 2.1.2, we will prove the following result in Section 2.5.3.

Theorem 2.2.4. *Let $Y_m = X_m X_{m-1} \dots X_1$, with the X_j 's independent $(n + \nu_j) \times (n + \nu_{j-1})$ complex Ginibre matrices, $\nu_0 = 0$, and let H be an $n \times n$ GUE matrix independent of Y_m . Write K_n^S for the eigenvalue correlation kernel of $S = \frac{1}{n^m} Y_m^* Y_m + \epsilon_n H$.*

(i) **(Sub-critical perturbation)** *If $\lim_{n \rightarrow +\infty} \epsilon_n n^{m+\frac{1}{2}} = 0$, then for*

$x, y > 0$, we have

$$\lim_{n \rightarrow +\infty} \frac{1}{n^{m+1}} K_n^S \left(\frac{x}{n^{m+1}}, \frac{y}{n^{m+1}} \right) = \mathbb{K}_\nu^G(x, y). \quad (2.2.16)$$

(ii) **(Critical perturbation)** If $\lim_{n \rightarrow +\infty} \epsilon_n n^{m+\frac{1}{2}} = \sigma > 0$, then for $x, y \in \mathbb{C}$, we have

$$\begin{aligned} \lim_{n \rightarrow +\infty} \frac{1}{n^{m+1}} K_n^S \left(\frac{x}{n^{m+1}}, \frac{y}{n^{m+1}} \right) = \\ \frac{1}{2\pi i \sigma^2} \int_{i\mathbb{R}} \int_{\mathbb{R}^+} \mathbb{K}_\nu^G(s, t) e^{\frac{1}{2\sigma^2}((s-x)^2 - (t-y)^2)} dt ds. \end{aligned} \quad (2.2.17)$$

Remark 2.2.5. In the super-critical regime, one expects Airy behaviour just like in the LUE case. To prove this, one could try to follow the same steps as for the LUE, but this will become considerably harder because the kernel K_n is now expressed in terms of multiple orthogonal polynomials instead of (generalized) Laguerre polynomials. We will come back to this issue later on in Remark 2.5.3.

Remark 2.2.6. The limiting kernel (2.2.15), in the case $m = 2$, appears also at the hard edge of a matrix from the Cauchy-Laguerre two-matrix model [BGS14]. This is the space of pairs of positive-definite Hermitian $n \times n$ matrices with probability measure

$$\frac{1}{Z_n} \frac{\det(M_1)^a \det(M_2)^b e^{-\text{Tr}(c_1 M_1 + c_2 M_2)}}{\det(M_1 + M_2)^n} dM_1 dM_2, \quad (2.2.18)$$

with $a, b > -1$, $a + b > -1$ and $c_1, c_2 > 0$. The correlation kernel for the eigenvalues of one of the matrices is given as a double contour integral similar to (2.2.12), and we expect that Lemma 2.1.2 can be applied to this case as well.

Remark 2.2.7. A different type of (deterministic) perturbation of products of Ginibre matrices has been studied in [FL15].

2.2.3 Perturbed products of truncated unitary matrices

Another example is given by the squared singular values of products of $m > 0$ truncated unitary matrices. As for the previous case, we form

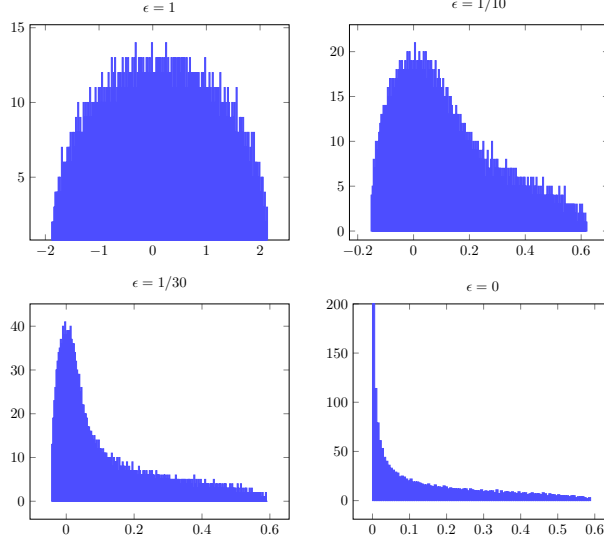


Figure 2.4: Numerical distribution of the eigenvalues of $Y_3^* Y_3 + \epsilon H$, for H a 2000×2000 GUE matrix and Y_3 is the product of three 2000×2000 truncations of three 4005×4005 Haar matrices. The chosen values for ϵ are 1, $1/10$, $1/30$ and 0. The eigenvalues are represented in a histogram of 200 intervals. The first column for the $\epsilon = 0$ case has been truncated to 200, the actual value being around 550.

$Y_m = T_m \dots T_1$, but now T_j is the upper left $(n + \nu_j) \times (n + \nu_{j-1})$ truncation of a unitary matrix of size $\ell_j \times \ell_j$ drawn randomly from the unitary group $\mathcal{U}(\ell_j)$ equipped with the Haar measure, as in [ŽS00]. We assume that $\nu_0 = 0$, that $\nu_1, \dots, \nu_m$ are non-negative integers, and that $\ell_j \geq n + \nu_j + 1$ for $j = 1, \dots, m$. See Figure 2.4 for numerical realizations of this ensemble perturbed by additive Gaussian noise.

If $\ell_1 \geq 2n + \nu_1$, it was shown in [KKS16] that the joint probability density of the squared singular values is a polynomial ensemble, whose kernel is given by

$$K_n(x, y) = \frac{1}{(2\pi i)^2} \int_C ds \int_{\Sigma_n} dt \prod_{j=0}^m \frac{\Gamma(s+1+\nu_j) \Gamma(t+1+\ell_j-n)}{\Gamma(t+1+\nu_j) \Gamma(s+1+\ell_j-n)} \frac{x^t y^{-s-1}}{s-t}. \quad (2.2.19)$$

The contour C leaves at $-\infty$ in the lower half plane, encircles the semi

axis $(-\infty, -1)$ and returns to $-\infty$ in the positive half plane, Σ_n being the same contour as in Section 2.2.2. Moreover, the contours C and Σ_n are not allowed to intersect. In [CKW15], (2.2.19) was proved under the weaker assumption $\sum_{j=1}^m (\ell_j - n - \nu_j) \geq n$, instead of $\ell_1 \geq 2n + \nu_1$.

This kernel also has a limiting kernel appearing near the hard edge. As n goes to infinity, we also have to let $\ell_1, \dots, \ell_m$ go to infinity. For each ℓ_j , we may choose either to let $\ell_j - n$ go to infinity, or to keep $\ell_j - n$ fixed. For each of these choices, the scaling leads to a different limiting kernel. We thus take $J \subseteq \{2, \dots, m\}$ a subset of indices. We then let $\ell_1, \dots, \ell_m$ go to infinity in such a way that

$$\ell_k - n \rightarrow +\infty, \quad \text{if } k \notin J, \quad (2.2.20)$$

$$\ell_k - n = \mu_k, \quad \text{if } k = j_k \in J. \quad (2.2.21)$$

Define finally $c_n = n \prod_{j \notin J} (\ell_j - n)$. The kernel (2.2.19) then has the following scaling limit [KKS16, Theorem 2.8] for $x \in \mathbb{C}, y > 0$,

$$\lim_{n \rightarrow +\infty} \frac{1}{c_n} K_n \left(\frac{x}{c_n}, \frac{y}{c_n} \right) = \mathbb{K}_{\nu, \mu}^T(x, y) := \frac{1}{(2\pi i)^2} \int_{-\frac{1}{2} + i\mathbb{R}} ds \int_{\Sigma} dt \prod_{j=0}^m \frac{\Gamma(s+1+\nu_j)}{\Gamma(t+1+\nu_j)} \frac{\sin \pi s}{\sin \pi t} \prod_{k \in J} \frac{\Gamma(t+1+\mu_k)}{\Gamma(s+1+\mu_k)} \frac{x^t y^{-s-1}}{s-t}. \quad (2.2.22)$$

The contour Σ is the same as in Section 2.2.2. Note that if J is empty, the limiting kernel (2.2.22) reduces to the kernel (2.2.15). As eigenvalues of a product of truncated unitary matrices, the eigenvalues of M remain bounded as $n \rightarrow +\infty$. It can be verified, in a similar way as we will do in the case of products of Ginibre matrices, that the eigenvalue correlation kernel for M satisfies conditions similar to those of Lemma 2.1.2, if we replace cn^γ by c_n (see Remark 2.5.4). This will allow us to prove the following.

Theorem 2.2.8. *Let Y_m be a product of truncations of unitary Haar distributed matrices as described above, such that the eigenvalue correlation kernel of $Y_m^* Y_m$ is given by (2.2.19), and let H be an $n \times n$ GUE matrix independent of Y_m . Write K_n^S for the eigenvalue correlation kernel of $S = Y_m^* Y_m + \epsilon_n H$. Then, we have*

(i) **(Sub-critical perturbation)** If $\lim_{n \rightarrow +\infty} \epsilon_n c_n n^{-\frac{1}{2}} = 0$, then for $x, y > 0$, we have

$$\lim_{n \rightarrow +\infty} \frac{1}{c_n} K_n^S \left(\frac{x}{c_n}, \frac{y}{c_n} \right) = \mathbb{K}_{\nu, \mu}^T(x, y). \quad (2.2.23)$$

(ii) **(Critical perturbation)** If $\lim_{n \rightarrow +\infty} \epsilon_n c_n n^{-\frac{1}{2}} = \sigma > 0$, then for $x, y \in \mathbb{C}$, we have

$$\begin{aligned} \lim_{n \rightarrow +\infty} \frac{1}{c_n} K_n^S \left(\frac{x}{c_n}, \frac{y}{c_n} \right) = \\ \frac{1}{2\pi i \sigma^2} \int_{i\mathbb{R}} \int_{\mathbb{R}^+} \mathbb{K}_{\nu, \mu}^T(s, t) e^{\frac{1}{2\sigma^2}((s-x)^2 - (t-y)^2)} dt ds. \end{aligned} \quad (2.2.24)$$

Remark 2.2.9. If $m = 1$, we have a single truncation of a unitary Haar matrix. Its squared singular values are in the Jacobi Unitary Ensemble, and in this case the kernel $\mathbb{K}_{\nu, \mu}^T$ reduces to the Bessel kernel K_ν^{BESSEL} .

2.2.4 Perturbed Muttalib-Borodin biorthogonal ensembles

The last example consists of random matrices for which the joint probability density of eigenvalues is the Muttalib-Borodin Laguerre ensemble [Bor99, Mut95]

$$\frac{1}{Z_n} \prod_{j < k} (\lambda_k - \lambda_j) (\lambda_k^\theta - \lambda_j^\theta) \prod_{j=1}^n \lambda_j^\alpha e^{-n\lambda_j}, \quad \theta > 0, \alpha > -1. \quad (2.2.25)$$

Such densities can be realized as eigenvalue densities of random matrices, see [AvMW13, Che14, FW15]. In [FW15], the authors constructed a random matrix with this eigenvalue density in the following way, in the case where θ is a positive integer and α a non-negative integer. Define α_j , $j = 1, \dots, n$ by $\alpha_j = \theta(j-1) + \alpha$. Then, consider the matrix X of size $m \times n$, with $m \geq n + (n-1)\theta + \alpha$, whose (j, k) entry is 0 if $j - k > \alpha_k$, and following independent standard complex Gaussian distributions otherwise. In other words, X is a complex Ginibre matrix, but with the entries in a certain region in the lower left corner of the matrix replaced by zeros. The eigenvalues of such matrices are shown

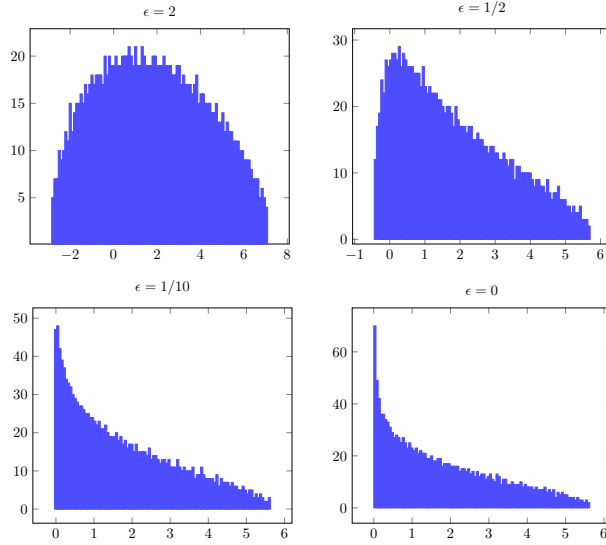


Figure 2.5: Numerical samples of eigenvalues of $\frac{1}{1500}X^*X + \epsilon H$, for H a GUE matrix and X a 1500×1500 matrix as described below (2.2.25), with $\theta = 3$ and $\alpha = 1$. Values for ϵ have been taken as 2, $1/2$, $1/10$ and 0. The eigenvalues are represented in histograms with 100 intervals.

for $\theta = 3$ and $\alpha = 1$ in Figure 2.5. The density of the eigenvalues of $\frac{1}{n}X^*X$ is then given by (2.2.25). The eigenvalue correlation kernel of $\frac{1}{n}X^*X$ can be expressed as [FW15]

$$K_n(x, y) = \frac{1}{(2\pi i)^2} e^{n(x-y)} \int_C ds \int_{C_\alpha} dt \frac{x^{-s-1} y^t}{s-t} n^{-s+t} \frac{\Gamma(s+1)}{\Gamma(t+1)} \prod_{k=1}^n \frac{s - \alpha_k}{t - \alpha_k}, \quad (2.2.26)$$

with C_α a contour enclosing $\alpha_1, \dots, \alpha_n$ and C starting at $-\infty$ in the lower half plane, enclosing C_α and going back to $-\infty$ in the upper half plane. An alternative expression was given in [Zha15]:

$$K_n(x, y) = \frac{\theta}{(2\pi i)^2} \int_{c+i\mathbb{R}} ds \int_{\Sigma_n} dt \frac{x^{-\theta s-1} y^{\theta t}}{s-t} n^{-\theta s+\theta t} \times \frac{\Gamma(s+1)\Gamma(\alpha+1+\theta s)\Gamma(t-n+1)}{\Gamma(t+1)\Gamma(\alpha+1+\theta t)\Gamma(s-n+1)}, \quad (2.2.27)$$

with $c = -\frac{1}{2} + \frac{1}{2} \max\{0, 1 - \frac{\alpha+1}{\theta}\}$, and Σ a closed counter-clockwise contour going around $0, 1, \dots, n-1$ and for which $\operatorname{Re} t > c$.

It admits the hard edge scaling limit [Bor99, FW15]

$$\lim_{n \rightarrow +\infty} \frac{1}{n^{1+\frac{1}{\theta}}} K_n \left(\frac{x}{n^{1+\frac{1}{\theta}}}, \frac{y}{n^{1+\frac{1}{\theta}}} \right) = \mathbb{K}_{\alpha, \theta}^{\text{MB}}(x, y), \quad x \in \mathbb{C}, y > 0, \quad (2.2.28)$$

where the limiting kernel $\mathbb{K}_{\alpha, \theta}^{\text{MB}}$ is given by

$$\mathbb{K}_{\alpha, \theta}^{\text{MB}}(x, y) := \frac{\theta}{(2\pi i)^2} \left(\frac{y}{x} \right)^\alpha \int_{C^\delta} ds \int_\Sigma dt \frac{x^{-\theta s - 1} y^{\theta t} \Gamma(\theta s + \alpha + 1) \Gamma(s + 1) \sin \pi s}{s - t \Gamma(\theta t + \alpha + 1) \Gamma(t + 1) \sin \pi t}, \quad (2.2.29)$$

with Σ starting from $+\infty$ in the upper half plane, enclosing the positive real axis and going back to $+\infty$ in the lower half plane, and C^δ consisting of two rays starting from $-\frac{1}{2}$ and making an angle $0 < \delta < \frac{\pi}{2}$ with the vertical axis, oriented upwards.

Similarly as in the previous cases, we have the following result.

Theorem 2.2.10. *Let M be a random matrix with eigenvalue density (2.2.25) and correlation kernel (2.2.26), and let H be an $n \times n$ GUE matrix independent of X . Write K_n^S for the eigenvalue correlation kernel of $S = M + \epsilon_n H$.*

(i) **(Sub-critical perturbation)** *If $\lim_{n \rightarrow +\infty} \epsilon_n n^{\frac{1}{\theta} + \frac{1}{2}} = 0$, then for $x, y > 0$, we have*

$$\lim_{n \rightarrow +\infty} \frac{1}{n^{\frac{1}{\theta} + 1}} K_n^S \left(\frac{x}{n^{\frac{1}{\theta} + 1}}, \frac{y}{n^{\frac{1}{\theta} + 1}} \right) = \mathbb{K}_{\alpha, \theta}^{\text{MB}}(x, y). \quad (2.2.30)$$

(ii) **(Critical perturbation)** *If $\lim_{n \rightarrow +\infty} \epsilon_n n^{\frac{1}{\theta} + \frac{1}{2}} = \sigma > 0$, then for $x, y \in \mathbb{C}$, we have*

$$\lim_{n \rightarrow +\infty} \frac{1}{n^{\frac{1}{\theta} + 1}} K_n^S \left(\frac{x}{n^{\frac{1}{\theta} + 1}}, \frac{y}{n^{\frac{1}{\theta} + 1}} \right) = \frac{1}{2\pi i \sigma^2} \int_{i\mathbb{R}} \int_{\mathbb{R}_+} \mathbb{K}_{\alpha, \theta}^{\text{MB}}(s, t) e^{\frac{1}{2\sigma^2}((s-x)^2 - (t-y)^2)} dt ds. \quad (2.2.31)$$

Remark 2.2.11. The hard edge scaling limit (2.2.28) was derived in

[Bor99] with a different expression for the limiting kernel,

$$\mathbb{K}_{\alpha,\theta}^{\text{MB}}(x, y) = \theta y^\alpha \int_0^1 J_{(\alpha+1)/\theta, 1/\theta}(xu) J_{\alpha+1,\theta}(yu)^\theta u^\alpha du, \quad (2.2.32)$$

where $J_{a,b}$ is Wright's generalization of Bessel functions given by

$$J_{a,b}(x) = \sum_{j=0}^{+\infty} \frac{(-x)^j}{j! \Gamma(a + jb)}. \quad (2.2.33)$$

If $1/\theta \in \mathbb{N}$, this limiting kernel can be expressed in terms of the kernel $\mathbb{K}_\nu^{\text{G}}$ appearing for products of Ginibre matrices, see [KS14, Section 5].

Yet another hard edge limiting kernel was obtained in [ACM15, XDZ14] in random matrix ensembles with singularities of the form

$$\frac{1}{Z_n} e^{-n\text{Tr}(M+t^k/M^k)} dM, \quad k \in \mathbb{N}, \quad (2.2.34)$$

on the space of $n \times n$ positive-definite Hermitian matrices. A limiting kernel was obtained which can be expressed in terms of the Painlevé III hierarchy, the corresponding value of γ in (2.1.4) is $\gamma = 2$. We believe that Lemma 2.1.2 can also be applied to this ensemble, but a detailed study would lead us too far.

Outline

In Section 2.3, we prove Theorem 2.1.1 on the convergence of the zero counting measures of the average characteristic polynomials. In Section 2.4, we prove the central auxiliary result of this chapter, Lemma 2.1.2. Parts (i) and (ii) of Theorem 2.2.1 on perturbed LUE matrices are proved in Section 2.5.1, and part (iii) in Section 2.5.2. In Section 2.5.3, the proof of Theorem 2.2.4 on perturbed Ginibre products is given. The proofs of Theorem 2.2.8 and Theorem 2.2.10 on perturbed products of truncated unitary matrices and Muttalib-Borodin ensembles are similar to the Ginibre case, as we explain in Remark 2.5.4 without giving details.

2.3 Proof of Theorem 2.1.1

In this section we give the proof of Theorem 2.1.1. We first need a general result about the zeros of the average characteristic polynomial.

Lemma 2.3.1. *In a polynomial ensemble of the form (1.2.11), the average characteristic polynomial $p_n(z) = \mathbb{E} \left(\prod_{j=1}^n (z - x_j) \right)$ has n simple real zeros.*

Proof. The average characteristic polynomial has real coefficients and satisfies the orthogonality conditions

$$\int_{\mathbb{R}} p_n(x) f_j(x) dx = 0, \quad j = 0, \dots, n-1.$$

If p_n would have a non-simple or non-real zero z_0 , we can write $p_n(z) = (z - z_0)(z - \bar{z}_0) \left(\sum_{k=0}^{n-2} a_k z^k \right)$ with $a_{n-2} = 1$. By the orthogonality conditions, we have

$$\sum_{k=0}^{n-2} a_k \int_{\mathbb{R}} x^k (|x - z_0|^2 f_j(x)) dx = 0, \quad j = 0, \dots, n-1.$$

If this homogeneous linear system has a non-zero solution $(a_0, \dots, a_{n-2})$, then the coefficient matrix is of rank $\leq n-2$, so the extended $n \times n$ matrix $(\int_{\mathbb{R}} |x - z_0|^2 x^k f_j(x) dx)_{j,k=0,\dots,n-1}$ is at most of rank $n-1$ and has zero determinant.

But on the other hand, by the Andreief identity, we have

$$\det \left(\int_{\mathbb{R}} |x - z_0|^2 x^k f_j(x) dx \right)_{j,k=0,\dots,n-1} = \frac{1}{n!} \int_{\mathbb{R}^n} \Delta(x) \det (f_{j-1}(x_k))_{j,k=1,\dots,n} \prod_{j=1}^n |x_j - z_0|^2 dx_j, \quad (2.3.1)$$

which is strictly positive as it is equal to $\frac{Z_n}{n!} \mathbb{E} \left(\prod_{j=1}^n |x_j - z_0|^2 \right)$. This is a contradiction, so p_n has only simple real zeros. $\square$

The proof of Theorem 2.1.1 now relies on the following two lemmas.

Lemma 2.3.2. *Under the conditions of Theorem 2.1.1, the sequence of functions*

$$f_n(z) := \int \log(1 - s/z) d(\mu_n - \mu)(s) \quad (2.3.2)$$

converges to 0, uniformly for $|z| > r$, where r is such that $r > \max\{|w| : w \in K\}$.

Proof. Point-wise convergence of f_n to 0 follows from the weak-* convergence of μ_n to μ , since $\log(1 - s/z)$ is continuous for $|z| > r$. To prove uniform convergence, we note first that $f_n(z)$ is uniformly bounded: we have

$$|f_n(z)| \leq \int |\log(1 - s/z)| d\mu_n(s) + \int |\log(1 - s/z)| d\mu(s) \leq 2 \max_{|z| \geq r, s \in K} |\log(1 - s/z)|. \quad (2.3.3)$$

If we define $h_n(z) = f_n(1/z)$, $h_n(z)$ is a uniformly bounded sequence of analytic functions on $|z| < 1/r$, which converges point-wise to 0. By Vitali's theorem, it follows that h_n converges to 0 uniformly for $|z| < 1/r$, and hence $f_n(z)$ converges to 0 uniformly for $|z| > r$. $\square$

For any compactly supported probability measure μ on $\mathbb{R}$, we define

$$g_\mu(z) = \int \log(z - s) d\mu(s), \quad G_\mu(z) = g'_\mu(z) = \int \frac{1}{z - s} d\mu(s), \quad (2.3.4)$$

where we choose the logarithm corresponding to arguments between $-\pi$ and π .

Lemma 2.3.3. *Under the conditions of Theorem 2.1.1, let $P_n(z)$ be the average characteristic polynomial of S . For $|\operatorname{Re} z|$ large enough, we have*

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \log P_n(z) = g_\mu(s_c(z)) + \frac{1}{2\epsilon^2} (z - s_c(z))^2, \quad (2.3.5)$$

where $s_c(z)$ is the solution in s of $G_\mu(s) + \epsilon^{-2}(s - z) = 0$.

Proof. Writing p_n for the average characteristic polynomial of M and

μ_n for its zero counting measure, we have the identity

$$p_n(z) = \prod_{j=1}^n (z - z_j^{(n)}) = e^{ng_{\mu_n}(z)}, \quad (2.3.6)$$

where $z_j^{(n)}$, $j = 1, \dots, n$ are the zeros of p_n .

From the transformation formula (2.1.2) for P_n in terms of p_n , we have

$$P_n(z) = \frac{\sqrt{n}}{\sqrt{2\pi i \epsilon}} \int_{i\mathbb{R}} e^{nF_n(s; z)} ds, \quad F_n(s; z) := g_{\mu_n}(s) + \frac{1}{2\epsilon^2}(z - s)^2. \quad (2.3.7)$$

Since $g'_{\mu_n}(s) = \mathcal{O}(1/s)$ as $s \rightarrow \infty$, uniformly for n large enough, the saddle point equation $F'_n(s; z) = 0$ has, for z sufficiently large, a unique solution $s_c(z)$ such that $s_c(z) \sim z$ as $z \rightarrow \infty$, by the inverse function theorem.

In order to obtain large n asymptotics for $P_n(z)$, we deform the integration contour $i\mathbb{R}$ in (2.3.7) to the steepest descent path $\gamma = \gamma_n$ passing through $s_c(z)$ and on which the imaginary part of $F_n(s; z)$ is constant. For $|\operatorname{Re} z|$ large, γ makes a small angle with the vertical line through $s_c(z)$, and therefore it remains outside of the compact K . We have

$$\operatorname{Re} F''_n(s; z) = \operatorname{Re} g''_{\mu_n}(s) + \epsilon^{-2} = \epsilon^{-2} + \mathcal{O}(s^{-2}), \quad s \rightarrow \infty,$$

and this implies that $\operatorname{Re} F_n$ achieves its unique local maximum on γ at $s_c(z)$. We can use the saddle point method to approximate the integral in (2.3.7) in the following way, for z sufficiently large, as $n \rightarrow +\infty$.

We may choose an implicit parametrization $\gamma_n(t)$ of the steepest descent path γ_n by imposing

$$F_n(\gamma_n(t); z) - F_n(s_c(z); z) = -t^2, \quad t \in [-\delta, \delta], \quad \operatorname{Im} \dot{\gamma}_n(0) > 0,$$

for some sufficiently small $\delta > 0$, and such that $|\dot{\gamma}_n(t)| = 1$ for $|t| > \delta$. We can then write the integral in (2.3.7) as

$$\int_{\gamma} e^{nF_n(s;z)} ds = e^{nF(s_c(z);z)} \int_{-\delta}^{\delta} \dot{\gamma}_n(t) e^{-nt^2} dt + \int_{\mathbb{R} \setminus [-\delta, \delta]} \dot{\gamma}_n(t) e^{nF_n(\gamma_n(t);z)} dt. \quad (2.3.8)$$

Since $\operatorname{Re} F_n(s; z)$ has its unique global maximum on γ_n at $t = 0$ and grows as $t \rightarrow \pm\infty$, the second term is $\mathcal{O}(e^{n[F_n(s_c(z)) - \eta^2]})$ as $n \rightarrow +\infty$. For the first term, note that $\dot{\gamma}_n(0) = i\sqrt{\frac{2}{F_n''(s_c(z);z)}}$. Moreover, there is a constant $C > 0$ independent of n such that

$$|\dot{\gamma}_n(t) - \dot{\gamma}_n(0)| \leq C|t|, \quad t \in [-\delta, \delta].$$

From (2.3.8), we now get

$$\begin{aligned} \int_{\gamma} e^{nF_n(s;z)} ds &= i\sqrt{\frac{2}{F_n''(s_c(z);z)}} e^{nF(s_c(z);z)} \int_{-\delta}^{\delta} e^{-nt^2} dt \\ &\quad + \mathcal{O}\left(\int_{-\delta}^{\delta} |t| e^{-nt^2} dt\right) + \mathcal{O}\left(e^{n[F_n(s_c(z)) - \eta^2]}\right), \end{aligned} \quad (2.3.9)$$

as $n \rightarrow +\infty$. Evaluating the integrals as $n \rightarrow +\infty$ and substituting in (2.3.7), we finally obtain

$$P_n(z) \sim \frac{1}{\epsilon} F_n''(s_c(z); z)^{-\frac{1}{2}} e^{nF(s_c(z);z)}, \quad n \rightarrow +\infty, \quad (2.3.10)$$

for $|\operatorname{Re} z|$ sufficiently large. It follows that

$$\frac{1}{n} \log P_n(z) = g_{\mu_n}(s_c(z)) + \frac{1}{2\epsilon^2} (z - s_c(z))^2 + o(1), \quad n \rightarrow +\infty.$$

Using the fact that

$$g_{\mu_n}(z) = g_{\mu}(z) + f_n(z), \quad \text{with } f_n(z) = \int \log(1-s/z) d(\mu_n - \mu)(s) \quad (2.3.11)$$

and Lemma 2.3.2, (2.3.5) now follows easily. $\square$

Proof of Theorem 2.1.1. Using the general definition of the free convolution, it was noted in [Bia97] that the free convolution of a compactly supported probability measure μ with the semi-circle λ_{ϵ} satisfies the

equation

$$G_{\mu \boxplus \lambda_\epsilon}(s + \epsilon^2 G_\mu(s)) = G_\mu(s), \quad (2.3.12)$$

for s sufficiently large. In other words, if $s_c(z)$ is the solution of $G_\mu(s) + \epsilon^{-2}(s - z) = 0$, we have

$$G_{\mu \boxplus \lambda_\epsilon}(z) = G_\mu(s_c(z)). \quad (2.3.13)$$

This implies, after a straightforward calculation,

$$\frac{d}{dz} g_{\mu \boxplus \lambda_\epsilon}(z) = \frac{d}{dz} \left(g_\mu(s_c(z)) + \frac{1}{2\epsilon^2} (z - s_c(z))^2 \right), \quad (2.3.14)$$

and upon integrating, we obtain

$$g_{\mu \boxplus \lambda_\epsilon}(z) = g_\mu(s_c(z)) + \frac{1}{2\epsilon^2} (z - s_c(z))^2, \quad (2.3.15)$$

since both the left and the right hand side behave like $\log z + \mathcal{O}(z^{-1})$ as $z \rightarrow \infty$.

By Lemma 2.3.3, we have

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \log P_n(z) = g_{\mu \boxplus \lambda_\epsilon}(z) \quad (2.3.16)$$

for $|\operatorname{Re} z|$ sufficiently large. This implies that there exists an n -independent compact $\tilde{K}$ such that P_n has no zeros outside $\tilde{K}$, for n sufficiently large. By Helly's theorem, $(\nu_n)_n$, and every subsequence of it, has a weak-* converging subsequence. We claim that every such converging subsequence $(\nu_{n_k})_k$ converges to $\mu \boxplus \lambda_\epsilon$. If so, it is easily seen by contraposition that the whole sequence ν_n converges to $\mu \boxplus \lambda_\epsilon$.

To prove the claim, we suppose that a subsequence $(\nu_{n_k})_k$ converges in weak-* sense to some measure $\tilde{\nu}$. We then have

$$\lim_{k \rightarrow +\infty} \frac{1}{n_k} \log P_{n_k}(z) = \lim_{k \rightarrow +\infty} \int \log(z - s) d\nu_{n_k}(s) = g_{\tilde{\nu}}(z), \quad (2.3.17)$$

for z outside $\tilde{K}$, and combining this with (2.3.16), it follows that $g_{\tilde{\nu}} = g_{\mu \boxplus \lambda_\epsilon}$. Since the supports of $\tilde{\nu}$ and $\mu \boxplus \lambda_\epsilon$ are both contained in $\mathbb{R}$, $g_{\tilde{\nu}}$ and $g_{\mu \boxplus \lambda_\epsilon}$ are analytic in $\mathbb{C} \setminus \mathbb{R}$, and by analytic continuation we have in particular that $\operatorname{Re} g_{\tilde{\nu}} = \operatorname{Re} g_{\mu \boxplus \lambda_\epsilon}$ everywhere in $\mathbb{C}$ except possibly on

a set of 2-dimensional Lebesgue measure 0. We can then use the unicity theorem [ST97, Theorem II.2.1] to conclude that $\tilde{\nu} = \mu \boxplus \lambda_\epsilon$. $\square$

2.4 Proof of Lemma 2.1.2

We will now give the proof of the central lemma of this chapter. We assume that the conditions (2.1.5) and (2.1.6) hold.

Replacing x and y by $\frac{x}{cn^\gamma}$ and $\frac{y}{cn^\gamma}$ in (2.1.1), and substituting $s = \frac{u}{cn^\gamma}$ and $t = \frac{v}{cn^\gamma}$ in the integrals, we obtain the identity

$$\begin{aligned} & \frac{1}{cn^\gamma} K_n^S \left(\frac{x}{cn^\gamma}, \frac{y}{cn^\gamma} \right) = \\ & \frac{n^{1-2\gamma}}{2\pi i c^2 \epsilon_n^2} \int_{i\mathbb{R}} \int_{\mathbb{R}^+} \frac{1}{cn^\gamma} K_n \left(\frac{u}{cn^\gamma}, \frac{v}{cn^\gamma} \right) e^{\frac{n^{1-2\gamma}}{2c^2 \epsilon_n^2} ((x-u)^2 - (y-v)^2)} dv du. \end{aligned} \quad (2.4.1)$$

By (2.1.6), we have

$$\left| \frac{1}{cn^\gamma} K_n \left(\frac{u}{cn^\gamma}, \frac{v}{cn^\gamma} \right) \right| \leq \frac{c_1}{c^{1-\beta}} v^{-\beta} e^{\frac{c_2}{c} (|u|+|v|)},$$

which implies that we can use the dominated convergence theorem in (2.4.1) if $n \rightarrow +\infty$, $\epsilon_n \rightarrow 0$ in such a way that $\lim_{n \rightarrow +\infty} c \epsilon_n n^{\gamma-\frac{1}{2}} = \sigma > 0$. Using (2.1.5), we immediately obtain the limit (2.1.8), point-wise for $x, y \in \mathbb{C}$.

To see that the limit is uniform for (x, y) in compact sets and to treat the case where $\lim_{n \rightarrow +\infty} \epsilon_n n^{\gamma-\frac{1}{2}} = 0$, we need the following technical estimates.

Lemma 2.4.1. *Let $K_n : \mathbb{C} \times \mathbb{R}^+ \rightarrow \mathbb{C}$ be a sequence of kernels satisfying condition (2.1.6). Assume that $|x|, |y| \leq r$ and let $\omega > 0$. Then, for n and R sufficiently large, the following estimates hold:*

$$\begin{aligned} & \left| \int_{i\mathbb{R} \setminus [-iR, iR]} \int_0^R \frac{1}{cn^\gamma} K_n \left(\frac{s}{cn^\gamma}, \frac{t}{cn^\gamma} \right) e^{\frac{1}{2\omega^2} ((x-s)^2 - (y-t)^2)} dt ds \right| \\ & \leq \frac{2c_1}{c^{1-\beta}} \frac{R^{1-\beta}}{1-\beta} \frac{e^{2\frac{Rr}{\omega^2} + \frac{r^2}{\omega^2} + 2\frac{c_2}{c} R}}{R - r - \frac{c_2}{c} \omega^2} \omega^2 e^{-\frac{R^2}{2\omega^2}}; \end{aligned} \quad (2.4.2)$$

$$\left| \int_{-iR}^{iR} \int_R^{+\infty} \frac{1}{cn^\gamma} K_n \left(\frac{s}{cn^\gamma}, \frac{t}{cn^\gamma} \right) e^{\frac{1}{2\omega^2}((x-s)^2 - (y-t)^2)} dt ds \right| \leq \frac{c_1 \sqrt{2\pi}}{c^{1-\beta}} e^{\frac{r^2}{\omega^2} + 2\frac{Rr}{\omega^2} + 2\frac{c_2}{c}R} \omega^2 e^{-\frac{R^2}{2\omega^2}}; \quad (2.4.3)$$

$$\left| \int_{i\mathbb{R} \setminus [-iR, iR]} \int_R^{+\infty} \frac{1}{cn^\gamma} K_n \left(\frac{s}{cn^\gamma}, \frac{t}{cn^\gamma} \right) e^{\frac{1}{2\omega^2}((x-s)^2 - (y-t)^2)} dt ds \right| \leq \frac{2c_1}{c^{1-\beta}} \frac{e^{2\frac{c_2}{c}R + 2\frac{Rr}{\omega^2} - \frac{R^2}{\omega^2}}}{(R - r - \frac{c_2}{c}\omega^2)^2} \omega^2. \quad (2.4.4)$$

Proof. To prove inequality (2.4.2), we first use condition (2.1.6) to get

$$\begin{aligned} & \left| \int_{i\mathbb{R} \setminus [-iR, iR]} \int_0^R \frac{1}{cn^\gamma} K_n \left(\frac{s}{cn^\gamma}, \frac{t}{cn^\gamma} \right) e^{\frac{1}{2\omega^2}((x-s)^2 - (y-t)^2)} dt ds \right| \\ & \leq \frac{c_1}{c^{1-\beta}} \int_{\mathbb{R} \setminus [-R, R]} e^{\frac{c_2}{c}|s|} \left| e^{\frac{1}{2\omega^2}(x-is)^2} \right| ds \int_0^R t^{-\beta} e^{\frac{c_2}{c}t} e^{-\frac{1}{2\omega^2}(y-t)^2} dt. \end{aligned} \quad (2.4.5)$$

The choices of R and r then allow us to estimate

$$\left| e^{\frac{1}{2\omega^2}(x-is)^2} \right| \leq e^{\frac{r^2}{2\omega^2}} e^{\frac{|s|r}{\omega^2}} e^{-\frac{s^2}{2\omega^2}} \text{ and } e^{-\frac{1}{2\omega^2}(y-t)^2} \leq e^{\frac{r^2}{2\omega^2}} e^{\frac{Rr}{\omega^2}} e^{-\frac{t^2}{2\omega^2}}. \quad (2.4.6)$$

Using these two things, and the symmetry along the imaginary axis, we have

$$\begin{aligned} & \left| \int_{i\mathbb{R} \setminus [-iR, iR]} \int_0^R \frac{1}{cn^\gamma} K_n \left(\frac{s}{cn^\gamma}, \frac{t}{cn^\gamma} \right) e^{\frac{1}{2\omega^2}((x-s)^2 - (y-t)^2)} dt ds \right| \\ & \leq \frac{2c_1}{c^{1-\beta}} \frac{R^{1-\beta}}{1-\beta} e^{\frac{r^2}{\omega^2}} e^{\frac{Rr}{\omega^2}} e^{\frac{c_2}{c}R} \int_R^{+\infty} e^{\frac{c_2}{c}s} e^{\frac{1}{2\omega^2}(2sr-s^2)} ds. \end{aligned} \quad (2.4.7)$$

We may integrate by parts the integral at the right and doing so, we obtain

$$\int_R^{+\infty} e^{\frac{c_2}{c}s} e^{\frac{1}{2\omega^2}(2sr-s^2)} ds =$$

$$\frac{\omega^2 e^{\frac{c_2}{c}R + \frac{Rr}{\omega^2} - \frac{R^2}{2\omega^2}}}{R - r - \frac{c_2}{c}\omega^2} - \int_R^{+\infty} \frac{\omega^4}{(s - r - \frac{c_2}{c}\omega^2)^2} e^{\frac{c_2}{c}s + \frac{sr}{\omega^2} - \frac{s^2}{2\omega^2}} ds \leq \frac{\omega^2 e^{\frac{c_2}{c}R + \frac{Rr}{\omega^2} - \frac{R^2}{2\omega^2}}}{R - r - \frac{c_2}{c}\omega^2}, \quad (2.4.8)$$

which yields (2.4.2).

The proofs for inequalities (2.4.3) and (2.4.4) are very similar. For (2.4.3), the only thing that changes is that the roles of s and t are interchanged. For (2.4.4), we need to use inequality (2.4.8) twice. $\square$

We now proceed with the proof of Lemma 2.1.2. To see that the limit (2.1.8) is uniform for x, y in compact sets, we assume that $|x|, |y| < r$ and choose $R > r$, as in Lemma 2.4.1. Define the three sets

$$U_1 := [-iR, iR] \times [R, +\infty), \quad U_2 := (i\mathbb{R} \setminus [-iR, iR]) \times [0, R]$$

and

$$U_3 := (i\mathbb{R} \setminus [-iR, iR]) \times [R, +\infty),$$

and write $\tilde{\epsilon}_n := c\epsilon_n n^{\gamma - \frac{1}{2}}$. By (2.4.1), we have

$$\begin{aligned} & \left| \frac{1}{cn^\gamma} K_n^S \left(\frac{x}{cn^\gamma}, \frac{y}{cn^\gamma} \right) - \frac{1}{2\pi i \sigma^2} \int_{i\mathbb{R}} \int_{\mathbb{R}^+} \mathbb{K}(s, t) e^{\frac{1}{2\sigma^2}((x-s)^2 - (y-t)^2)} dt ds \right| \\ & \leq \left| \frac{1}{2\pi i \tilde{\epsilon}_n^2} \int_{-iR}^{iR} \int_0^R \left(v^\beta \frac{1}{cn^\gamma} K_n \left(\frac{s}{cn^\gamma}, \frac{t}{cn^\gamma} \right) - v^\beta \mathbb{K}(s, t) \right) \right. \\ & \quad \left. \times v^{-\beta} e^{\frac{1}{2\tilde{\epsilon}_n^2}((x-s)^2 - (y-t)^2)} dt ds \right| + \\ & \quad \left| \int_{-iR}^{iR} \int_0^R \mathbb{K}(s, t) \left(\frac{1}{2\pi i \tilde{\epsilon}_n^2} e^{\frac{1}{2\tilde{\epsilon}_n^2}((x-s)^2 - (y-t)^2)} - \frac{1}{2\pi i \sigma^2} e^{\frac{1}{2\sigma^2}((x-s)^2 - (y-t)^2)} \right) dt ds \right| + \\ & \quad \frac{1}{2\pi \tilde{\epsilon}_n^2} \sum_{j=1}^3 \left| \frac{1}{cn^\gamma} \iint_{U_j} K_n \left(\frac{s}{cn^\gamma}, \frac{t}{cn^\gamma} \right) e^{\frac{1}{2\tilde{\epsilon}_n^2}((x-s)^2 - (y-t)^2)} ds dt \right| \\ & \quad + \frac{1}{2\pi \sigma^2} \sum_{j=1}^3 \left| \iint_{U_j} \mathbb{K}(s, t) e^{\frac{1}{2\sigma^2}((x-s)^2 - (y-t)^2)} ds dt \right|. \quad (2.4.9) \end{aligned}$$

In the limit where $n \rightarrow +\infty$ and $\tilde{\epsilon}_n = c\epsilon_n n^{\gamma-1/2} \rightarrow \sigma > 0$, the first term at the right hand side of the above expression tends to 0 uniformly for $|x|, |y| < r$ because of (2.1.5). The second term at the right tends to 0 as $n \rightarrow \infty$ by the dominated convergence theorem. The remaining terms at the right can be estimated using Lemma 2.4.1 with $\omega = \tilde{\epsilon}_n$ and become small (uniformly in x and y) as R gets large. Since the left hand side does not depend on R , we can take R as large as we want, and this implies that we have (2.1.8) uniformly for $|x|$ and $|y|$ smaller than r .

We now deal with the case where $\tilde{\epsilon}_n \rightarrow 0$ as $n \rightarrow +\infty$. By (2.4.1), we have

$$\begin{aligned} & \frac{1}{cn^\gamma} K_n^S\left(\frac{x}{cn^\gamma}, \frac{y}{cn^\gamma}\right) \\ &= \frac{1}{2\pi i \tilde{\epsilon}_n^2} \int_{[-iR, iR]} \int_{[0, R]} \frac{1}{cn^\gamma} K_n\left(\frac{u}{cn^\gamma}, \frac{v}{cn^\gamma}\right) e^{\frac{1}{2\tilde{\epsilon}_n^2}((x-u)^2 - (y-v)^2)} dv du \\ & \quad + \sum_{j=1}^3 \frac{1}{2\pi i \tilde{\epsilon}_n^2} \iint_{U_j} \frac{1}{cn^\gamma} K_n\left(\frac{u}{cn^\gamma}, \frac{v}{cn^\gamma}\right) e^{\frac{1}{2\tilde{\epsilon}_n^2}((x-u)^2 - (y-v)^2)} dv du, \end{aligned} \quad (2.4.10)$$

where we have cut the integral in the same 4 parts as before. We may use Lemma 2.4.1 with $\omega = \tilde{\epsilon}_n$ and we get that the three terms in the sum on the last line tend to 0 as $n \rightarrow +\infty$, provided R is sufficiently large. In the first term, we can deform the integration contour for u to $\square := [-iR, x - iR] \cup [x - iR, x + iR] \cup [x + iR, iR]$. This does not change the integral since K_n is analytic in u . We use the convergence in (2.1.5), which is uniform for u and v on their respective integration contours, and obtain

$$\begin{aligned} & \frac{1}{2\pi i \tilde{\epsilon}_n^2} \int_{[-iR, iR]} \int_{[0, R]} \frac{1}{cn^\gamma} K_n\left(\frac{u}{cn^\gamma}, \frac{v}{cn^\gamma}\right) e^{\frac{1}{2\tilde{\epsilon}_n^2}((x-u)^2 - (y-v)^2)} dv du = \\ & \frac{1}{2\pi i \tilde{\epsilon}_n^2} \int_{\square} \int_{[0, R]} \left(\mathbb{K}(u, v) + o(1)v^{-\beta} \right) e^{\frac{1}{2\tilde{\epsilon}_n^2}((x-u)^2 - (y-v)^2)} dv du, \end{aligned} \quad (2.4.11)$$

as $n \rightarrow +\infty$. Since R is arbitrary, we can take it large enough so that the contribution to the integral on the paths $[-iR, x - iR]$ and $[x + iR, iR]$ is small in n . Then, the contours $[x - iR, x + iR]$ and $[0, R]$ are the

steepest descent contours of the u - and v -phase functions, respectively. This allows us to apply the usual saddle point method to the integral, which gives

$$\frac{1}{2\pi i \tilde{\epsilon}_n^2} \int_{x-iR}^{x+iR} \int_{[0,R]} \left(\mathbb{K}(u, v) + o(1)v^{-\beta} \right) e^{\frac{1}{2\tilde{\epsilon}_n^2}((x-u)^2 - (y-v)^2)} dv du = \mathbb{K}(x, y) + o(1), \text{ as } n \rightarrow +\infty, \quad (2.4.12)$$

for $x, y > 0$ fixed. Note that the error term is not uniform as $y \rightarrow 0$. This completes the proof of Lemma 2.1.2.

2.5 Proofs of applications

2.5.1 Proof of Theorem 2.2.1: sub-critical and critical cases

For the proof of parts (i) and (ii) of Theorem 2.2.1, we only need to verify that the conditions of Lemma 2.1.2, namely (2.1.5)-(2.1.6), are satisfied. It should be noted that it is important to have uniformity of (2.1.5) for u in any compact subset of $\mathbb{C}$ and v in any compact subset of $[0, +\infty)$, and therefore (2.1.5) is not a direct consequence of (2.2.6).

Define p_n , $n = 0, 1, \dots$ to be the normalized orthogonal polynomials with respect to the generalized Laguerre weight $w(x) = x^\alpha e^{-nx^k}$. Let Y_n be the matrix-valued function

$$Y_n(z) = \begin{pmatrix} \frac{1}{\kappa_n} p_n(z) & \frac{1}{\kappa_n} C(p_n w)(z) \\ -2\pi i \kappa_{n-1} p_{n-1}(z) & -2\pi i \kappa_{n-1} C(p_{n-1} w)(z) \end{pmatrix}, \quad (2.5.1)$$

where $\kappa_j > 0$ is the leading coefficient of p_j and Cf is the Cauchy transform

$$Cf(z) = \frac{1}{2\pi i} \int_0^{+\infty} \frac{f(s)}{s-z} ds, \text{ for } z \notin [0, +\infty). \quad (2.5.2)$$

The matrix Y_n is the solution to the usual Fokas-Its-Kitaev Riemann-Hilbert problem for orthogonal polynomials [FIK92]. The eigenvalue correlation kernel K_n for a (generalized) LUE matrix is expressed in

terms of Y_n as

$$K_n(x, y) = w(y) \frac{1}{2\pi i(x-y)} \begin{pmatrix} 0 & 1 \end{pmatrix} Y_n^{-1}(y) Y_n(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (2.5.3)$$

which is easily verified by the Christoffel-Darboux formula and the fact that $\det Y_n = 1$.

Define the re-scaled matrix

$$U_n(z) = \left(\frac{2}{kA_k} \right)^{\frac{1}{k}(-n-\frac{\alpha}{2})\sigma_3} Y_n \left(\left(\frac{2}{kA_k} \right)^{\frac{1}{k}} z \right) \left(\frac{2}{kA_k} \right)^{\frac{1}{k}\frac{\alpha}{2}\sigma_3}, \quad (2.5.4)$$

with

$$A_k = \prod_{j=1}^k \frac{2j-1}{2j},$$

and where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. This re-normalization is the one used in formula (3.14) in [Van07]. The rescaled kernel $\tilde{K}_n$ defined by

$$\tilde{K}_n(x, y) = \left(\frac{2}{kA_k} \right)^{\frac{\alpha}{k}} y^\alpha e^{-\frac{2n}{kA_k}y^k} \frac{1}{2\pi i(x-y)} \begin{pmatrix} 0 & 1 \end{pmatrix} U_n^{-1}(y) U_n(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (2.5.5)$$

corresponds to a re-scaled LUE in which the limiting mean eigenvalue distribution is supported on $[0, 1]$, and given by [Van07]

$$d\tilde{\mu}(x) = \frac{1}{2\pi} \sqrt{\frac{1-x}{x}} \tilde{h}(x) dx, \quad \tilde{h}(x) = 2 \sum_{j=0}^{k-1} \frac{A_{k-1-j}}{A_k} x^j, \quad x \in (0, 1). \quad (2.5.6)$$

We have

$$\tilde{K}_n(x, y) = \left(\frac{2}{kA_k} \right)^{\frac{\alpha+1}{k}} K_n \left(\left(\frac{2}{kA_k} \right)^{\frac{1}{k}} x, \left(\frac{2}{kA_k} \right)^{\frac{1}{k}} y \right), \quad (2.5.7)$$

where the $\left(\frac{2}{kA_k} \right)^{\frac{\alpha}{k}}$ in front comes from the $\left(\frac{2}{kA_k} \right)^{\frac{1}{k}\frac{\alpha}{2}\sigma_3}$ factor at the right in (2.5.4).

We now describe the asymptotics for $U_n(z)$, which were obtained in [Van07]. Let $\delta > 0$ be sufficiently small and consider the following four

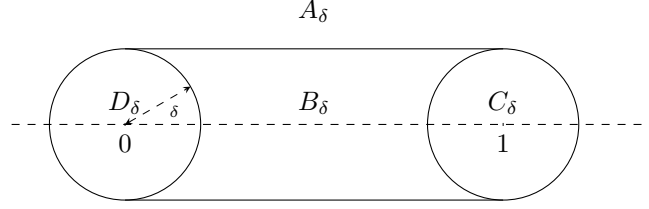


Figure 2.6: The four regions in which the asymptotics of U_n are expressed differently.

regions, as illustrated in Figure 2.6:

$$\begin{cases} D_\delta = \{z \in \mathbb{C} : |z| < \delta\}, \\ C_\delta = \{z \in \mathbb{C} : |z - 1| < \delta\}, \\ B_\delta = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1, |\operatorname{Im}(z)| < \delta, z \notin D_\delta \cup C_\delta\}, \\ A_\delta = \mathbb{C} \setminus (B_\delta \cup C_\delta \cup D_\delta). \end{cases}$$

The nature of the asymptotics for $U_n(z)$ is different in each of those regions.

For $z \in A_\delta$, we have

$$U_n(z) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{z^{-\frac{\alpha}{2}}}{2z^{\frac{1}{4}}(z-1)^{\frac{1}{4}}} e^{ng_{\tilde{\mu}}(z)} \begin{pmatrix} 2^{-\alpha} \varphi(z)^{\frac{1}{2}(\alpha+1)} (1 + \mathcal{O}(1/n)) \\ -ie^{-n\ell} 2^\alpha \varphi(z)^{\frac{1}{2}(\alpha-1)} (1 + \mathcal{O}(1/n)) \end{pmatrix}, \quad (2.5.8)$$

as $n \rightarrow \infty$, where $\ell = -\frac{2}{k} - 4 \log 2$,

$$g_{\tilde{\mu}}(z) = \int \log(z-s) d\tilde{\mu}(s) \sim \log z, \quad \text{as } z \rightarrow \infty, \quad (2.5.9)$$

and

$$\varphi(z) = 2(z-1/2) + 2z^{1/2}(z-1)^{1/2}, \quad (2.5.10)$$

with the square roots defined such that φ is analytic except on $[0, 1]$. It follows that there exists a constant c_1 such that

$$\left\| U_n(z) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\| \leq e^{c_1 n |z|}, \quad z \in A_\delta, \quad (2.5.11)$$

for n sufficiently large. This is a very rough estimate but it will be enough for our purposes.

For $z \in B_\delta$, we have

$$U_n(z) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{z^{-\frac{\alpha}{2}}}{z^{\frac{1}{4}}(z-1)^{\frac{1}{4}}} e^{n \frac{z^k}{kA_k}} \times \begin{pmatrix} e^{\frac{1}{2}n\ell} 2^{-\alpha} \cos(\eta_1(z) - in\xi(z) - \frac{\pi}{4})(1 + \mathcal{O}(1/n)) \\ e^{-\frac{1}{2}n\ell} 2^\alpha \cos(\eta_2(z) - in\xi(z) - \frac{\pi}{4})(1 + \mathcal{O}(1/n)) \end{pmatrix}, \quad (2.5.12)$$

as $n \rightarrow +\infty$, for some continuous functions η_1, η_2 independent of n , and with

$$\xi(z) = -\pi i \int_1^z \frac{1}{2\pi i} \frac{(s-1)^{1/2}}{s^{1/2}} \tilde{h}(s) ds, \quad (2.5.13)$$

with the square roots such that ξ is analytic in $\mathbb{C} \setminus [0, 1]$. It follows that there exists a constant c_2 such that

$$\left\| U_n(z) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\| \leq e^{c_2 n |z|}, \quad z \in B_\delta, \quad (2.5.14)$$

for n sufficiently large.

For $z \in C_\delta$,

$$U_n(z)_{11} = \frac{\sqrt{\pi} z^{-\frac{\alpha}{2}}}{z^{1/4}(z-1)^{1/4}} e^{n \frac{z^k}{kA_k}} e^{\frac{1}{2}n\ell} 2^{-\alpha} \times \left(\cos(\eta_1(z)) f_n(z)^{1/4} \text{Ai}(f_n(z)) - i \sin(\eta_1(z)) f_n(z)^{-1/4} \text{Ai}'(f_n(z)) \right) \times (1 + \mathcal{O}(1/n))$$

and

$$U_n(z)_{21} = \frac{\sqrt{\pi} z^{-\frac{\alpha}{2}}}{z^{1/4}(z-1)^{1/4}} e^{n \frac{z^k}{kA_k}} e^{-\frac{1}{2}n\ell} 2^\alpha \times \left(-i \cos(\eta_2(z)) f_n(z)^{1/4} \text{Ai}(f_n(z)) - \sin(\eta_2(z)) f_n(z)^{-1/4} \text{Ai}'(f_n(z)) \right) \times (1 + \mathcal{O}(1/n)),$$

as $n \rightarrow +\infty$, with Ai the Airy function, and f_n a conformal map defined in a neighborhood of 1, satisfying $f_n(1) = 0$, $f'_n(1) > 0$ and

$$\frac{2}{3} f_n(z)^{\frac{3}{2}} = n\xi(z). \quad (2.5.15)$$

Using the asymptotics for the Airy function at large arguments and the fact that $|f_n(z)| \leq Cn^{2/3}|z-1|$ for some constant $C > 0$ independent of n , for z sufficiently close to 1, it follows again that there exists a constant c_3 such that

$$\left\| U_n(z) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\| \leq e^{c_3 n |z|}, \quad z \in C_\delta, \quad (2.5.16)$$

for n sufficiently large.

Finally, we have for $z \in D_\delta$,

$$\begin{aligned} U_n(z)_{11} &= \frac{(-1)^n \sqrt{\pi} (-\tilde{f}_n(z))^{1/4} z^{-\frac{\alpha}{2}}}{z^{1/4} (1-z)^{1/4}} e^{n \frac{z^k}{k A_k}} e^{\frac{1}{2} n \ell} 2^{-\alpha} \\ &\quad \times \left(\sin(\zeta_1(z)) J_\alpha(2(-\tilde{f}_n(z))^{1/2}) + \cos(\zeta_1(z)) J'_\alpha(2(-\tilde{f}_n(z))^{1/2}) \right) \\ &\quad \times (1 + \mathcal{O}(1/n)), \end{aligned} \quad (2.5.17)$$

and

$$\begin{aligned} U_n(z)_{21} &= \frac{(-1)^n \sqrt{\pi} (-\tilde{f}_n(z))^{1/4} z^{-\frac{\alpha}{2}}}{z^{1/4} (1-z)^{1/4}} e^{n \frac{z^k}{k A_k}} - i e^{-\frac{1}{2} n \ell} 2^\alpha \\ &\quad \times \left(\sin(\zeta_2(z)) J_\alpha(2(-\tilde{f}_n(z))^{1/2}) + \cos(\zeta_2(z)) J'_\alpha(2(-\tilde{f}_n(z))^{1/2}) \right) \\ &\quad \times (1 + \mathcal{O}(1/n)), \end{aligned} \quad (2.5.18)$$

as $n \rightarrow +\infty$, with $\tilde{f}_n$ a conformal map defined in a neighbourhood of 0 and satisfying

$$2\tilde{f}_n(z)^{\frac{1}{2}} = n\xi(z), \quad \tilde{f}_n(0) = 0, \quad \tilde{f}'_n(0) < 0. \quad (2.5.19)$$

The functions ζ_1 and ζ_2 are independent of n and such that $\zeta_1(0) = -\zeta_2(0) = \pm \frac{\pi}{2}$. We have $|\tilde{f}_n(z)| \leq Cn^2|z|$ for some positive n -independent constant C , if z is sufficiently close to 0. The asymptotics [OLBC10, Formula 10.7.8] of the Bessel function J_α and the fact that $z^{-\alpha} J_\alpha(z)$ is bounded allow us to conclude that there exist constants $c_4, C > 0$ such that

$$\left\| e^{-\frac{n}{2} \ell \sigma_3} U_n(z) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\| \leq C e^{c_4 n |z|}, \quad z \in D_\delta. \quad (2.5.20)$$

for n sufficiently large.

Proof of Theorem 2.2.1, parts (i) and (ii). The hard edge scaling limit (2.2.6) was proved in [Van07] (using (2.5.17) and (2.5.18)) to hold point-wise for $u, v > 0$. From the proof, or from (2.5.17) and (2.5.18), it is however readily seen that it holds point-wise for any $u \in \mathbb{C}$. For $\alpha \geq 0$, the Bessel function J_α is bounded near 0, and then it is straightforward to show, again using (2.5.17) and (2.5.18), that (2.1.5) with $\beta = 0$ is uniform for u in any compact in $\mathbb{C}$ and for v in any compact in $[0, +\infty)$. If $-1 < \alpha < 0$, $J_\alpha(z)$ blows up as $z \rightarrow 0$, but we have that $z^{-\alpha}J_\alpha(z)$ is analytic at 0, and this can be used to show that (2.1.5) with $\beta = -\alpha$ holds uniformly for u in any compact in $\mathbb{C}$ and for v in any compact in $[0, +\infty)$.

To verify condition (2.1.6) for $(u, v) \in i\mathbb{R} \times \mathbb{R}^+$, we have to consider eight different regions for (u, v) : $A_\delta \times A_\delta$, $A_\delta \times B_\delta$, $A_\delta \times C_\delta$, $A_\delta \times D_\delta$, $D_\delta \times A_\delta$, $D_\delta \times B_\delta$, $D_\delta \times C_\delta$, and $D_\delta \times D_\delta$. In each of these cases, we can bound the kernel $\tilde{K}_n$ defined in (2.5.5) using the estimates (2.5.11), (2.5.14), (2.5.16), and (2.5.20) corresponding to the different regions. Substituting them in (2.5.7), we obtain the desired estimate (2.1.6) in each of the regions.

The results now follow directly from Lemma 2.1.2. $\square$

2.5.2 Proof of Theorem 2.2.1: super-critical case

For part (iii) of Theorem 2.2.1, we will use saddle point methods and a modified version of the integral representation (2.1.1) for the correlation kernel K_n^S . We first need some technical results.

Lemma 2.5.1. *Let μ be the limiting mean eigenvalue distribution given by (2.2.3), and let λ_ϵ be the rescaled semicircle law (2.1.3). The free additive convolution $\mu \boxplus \lambda_\epsilon$ is supported on an interval $[a_\epsilon, b_\epsilon]$, and a_ϵ is given by*

$$a_\epsilon = u_\epsilon - \epsilon^2 \int \frac{d\mu(x)}{x - u_\epsilon}, \quad (2.5.21)$$

where u_ϵ is the unique negative solution of the equation

$$\int \frac{d\mu(x)}{(x - u_\epsilon)^2} = \frac{1}{\epsilon^2}. \quad (2.5.22)$$

Moreover, for some $\kappa, \widehat{\kappa} > 0$,

$$u_\epsilon = -\kappa\epsilon^{4/3}(1 + o(1)), \quad a_\epsilon = -\widehat{\kappa}\epsilon^{4/3}(1 + o(1)), \quad \epsilon \rightarrow 0. \quad (2.5.23)$$

Proof. It follows from the general results in [Bia97] on the free additive convolution of a measure with a semi-circle law that the left endpoint a_ϵ is given by (2.5.21), with u_ϵ solving (2.5.22), provided that such a solution u_ϵ exists. For general μ and $\epsilon > 0$, it is not always true that (2.5.22) has a unique solution. However, with μ given by (2.2.3), the left hand side of (2.5.22) tends to 0 as $u_\epsilon \rightarrow -\infty$, is increasing, and tends to $+\infty$ as $u_\epsilon \rightarrow 0$. This implies that (2.5.22) has indeed a unique solution for any $\epsilon > 0$. Small ϵ asymptotics for u_ϵ and a_ϵ can be obtained from (2.5.22) and (2.5.21) using residue arguments. This leads to (2.5.23). $\square$

The following result provides us with suitable paths in the complex plane where the real part of a certain phase function Ψ_ϵ , to be used in the saddle point analysis later on, is monotone.

Lemma 2.5.2. *Let μ be given by (2.2.3), let $\epsilon > 0$, and let a_ϵ and u_ϵ be as in Lemma 2.5.1. Define*

$$\Psi_\epsilon(z) = \frac{1}{2\epsilon^2}(z - a_\epsilon)^2 + \int \log(z - x)d\mu(x), \quad z \in \mathbb{C} \setminus (-\infty, b], \quad (2.5.24)$$

and define two paths γ_1 and γ_2 by

$$\gamma_1(t) = u_\epsilon + e^{\frac{\pi i}{3}}|u_\epsilon| + t, \quad \gamma_2(t) = u_\epsilon + e^{\frac{2\pi i}{3}}|u_\epsilon| + it, \quad t > 0. \quad (2.5.25)$$

Then, for $\epsilon > 0$ sufficiently small, the functions

$$t \in \mathbb{R}^+ \mapsto \operatorname{Re}(\Psi_\epsilon(\gamma_1(t))) \text{ and } t \in \mathbb{R}^+ \mapsto \operatorname{Re}(\Psi_\epsilon(\bar{\gamma}_1(t)))$$

are increasing and the functions

$$t \in \mathbb{R}^+ \mapsto \operatorname{Re}(\Psi_\epsilon(\gamma_2(t))) \text{ and } t \in \mathbb{R}^+ \mapsto \operatorname{Re}(\Psi_\epsilon(\bar{\gamma}_2(t)))$$

are decreasing.

Proof. Because of conjugational symmetry, it suffices to check that

$$\frac{d}{dt} \operatorname{Re} (\Psi_\epsilon(\gamma_1(t))) > 0 \text{ and } \frac{d}{dt} \operatorname{Re} (\Psi_\epsilon(\gamma_2(t))) < 0, \quad \text{for } t > 0.$$

On γ_2 , we have after a straightforward computation,

$$\begin{aligned} \frac{d}{dt} \operatorname{Re} (\Psi_\epsilon(\gamma_2(t))) &= \left(t + \frac{\sqrt{3}}{2} |u_\epsilon| \right) \\ &\times \left(\frac{-1}{\epsilon^2} + \int \frac{d\mu(s)}{\left(\frac{3}{2} u_\epsilon - s \right)^2 + \left(t + \frac{\sqrt{3}}{2} |u_\epsilon| \right)^2} \right). \end{aligned} \quad (2.5.26)$$

For $t \geq 0$, we have

$$0 \leq \int \frac{d\mu(s)}{\left(\frac{3}{2} u_\epsilon - s \right)^2 + \left(t + \frac{\sqrt{3}}{2} |u_\epsilon| \right)^2} \leq \int \frac{d\mu(s)}{\left(\frac{3}{2} u_\epsilon - s \right)^2 + \frac{3}{4} u_\epsilon^2}. \quad (2.5.27)$$

The integral at the right is easily computable using the residue theorem, and by (2.5.23), it is readily seen that it is of order $\epsilon^{-4/3}$ as $\epsilon \rightarrow 0$. Hence, by (2.5.26), for ϵ small enough,

$$\frac{d}{dt} \operatorname{Re} (\Psi_\epsilon(\gamma_2(t))) < 0, \quad \text{for all } t \geq 0. \quad (2.5.28)$$

On γ_1 , we have

$$\frac{d}{dt} \operatorname{Re} (\Psi_\epsilon(\gamma_1(t))) = \frac{1}{\epsilon^2} \left(\frac{u_\epsilon}{2} + t - a_\epsilon \right) + \operatorname{Re} \left(\int \frac{d\mu(x)}{u_\epsilon + e^{\frac{\pi i}{3}} |u_\epsilon| + t - x} \right). \quad (2.5.29)$$

By Lemma 2.5.1, we can replace a_ϵ by its expression in terms of u_ϵ given in (2.5.21), and we get

$$\begin{aligned} \frac{d}{dt} \operatorname{Re} (\Psi_\epsilon(\gamma_1(t))) &= \\ &\frac{1}{\epsilon^2} \left(-\frac{u_\epsilon}{2} + t \right) + \int \frac{d\mu(x)}{x - u_\epsilon} + \operatorname{Re} \left(\int \frac{d\mu(x)}{u_\epsilon + e^{\frac{\pi i}{3}} |u_\epsilon| + t - x} \right). \end{aligned} \quad (2.5.30)$$

This is easily seen to be positive if $t \geq b - \frac{u_\epsilon}{2}$. For t smaller, we need to be more careful and exploit the fact that ϵ is small. The Stieltjes

transform can be computed explicitly by (2.5.6) and a residue calculus (or alternatively by observing that μ is an equilibrium measure which satisfies Euler-Lagrange variational conditions): for $z \in \mathbb{C} \setminus [0, b]$,

$$\int \frac{d\mu(x)}{z-x} = \frac{k}{2} z^{k-1} - \frac{h(z)}{2} \frac{\sqrt{z-b}}{\sqrt{z}}, \quad (2.5.31)$$

with principal branches of the square roots. Evaluating at $z = u_\epsilon$ and at $z = t + u_\epsilon + e^{\frac{\pi i}{3}} |u_\epsilon|$, and substituting in (2.5.30), we obtain

$$\begin{aligned} \frac{d}{dt} \operatorname{Re} (\Psi_\epsilon(\gamma_1(t))) &= \frac{1}{\epsilon^2} \left(-\frac{u_\epsilon}{2} + t \right) + \frac{h(0)}{2} \sqrt{\frac{b}{|u_\epsilon|}} \\ &\quad - \operatorname{Re} \left(\frac{h(t)}{2} \frac{\sqrt{t + u_\epsilon + e^{\frac{\pi i}{3}} |u_\epsilon| - b}}{\sqrt{t + u_\epsilon + e^{\frac{\pi i}{3}} |u_\epsilon|}} \right) + \mathcal{O}(1), \quad \text{as } \epsilon \rightarrow 0. \end{aligned} \quad (2.5.32)$$

For $t \gg \epsilon^{4/3}$, the leading order term in this expansion is $\frac{t}{\epsilon^2}$, and it then follows that $\frac{d}{dt} \operatorname{Re} (\Psi_\epsilon(\gamma_1(t))) > 0$ for ϵ sufficiently small.

Writing $t = T|u_\epsilon| \sim \kappa T \epsilon^{4/3}$ as $\epsilon \rightarrow 0$, we have

$$\begin{aligned} \frac{d}{dt} \operatorname{Re} (\Psi_\epsilon(\gamma_1(t))) &= \\ &\epsilon^{-2/3} \left(\frac{\kappa}{2} + T\kappa + \frac{h(0)\sqrt{b}}{2\sqrt{\kappa}} \left(1 - \operatorname{Re} \frac{i\sqrt{T-1+e^{-\frac{\pi i}{3}}}}{|T-1+e^{\frac{\pi i}{3}}|} \right) \right) \\ &\quad + o(\epsilon^{-2/3}). \end{aligned} \quad (2.5.33)$$

It is straightforward to check that $\operatorname{Re} \frac{i\sqrt{T-1+e^{-\frac{\pi i}{3}}}}{|T-1+e^{\frac{\pi i}{3}}|} < 1$ for $T > 0$, and this implies by (2.5.33) that $\frac{d}{dt} \operatorname{Re} (\Psi_\epsilon(\gamma_1(t))) > 0$ for $t > 0$. $\square$

We can now prove part (iii) of Theorem 2.2.1. The general strategy of the proof is similar to the one of [LWZ14, Theorem 1.2].

Proof of Theorem 2.2.1, part (iii). We use an alternative expression for K_n^S given by [CKW15, Formula (3.23)]

$$K_n^S(x, y) = \frac{n}{(2\pi i)^2 \epsilon^2} \int_C ds \int_\Sigma d\zeta$$

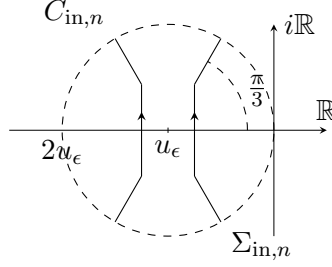


Figure 2.7: Local contours around the saddle point of the phase function Ψ_ϵ .

$$\frac{1}{s - \zeta} \begin{pmatrix} 0 & 1 \end{pmatrix} Y_n^{-1}(s) Y_n(\zeta) \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{\frac{n}{2\epsilon^2}((x-s)^2 - (y-\zeta)^2)}, \quad (2.5.34)$$

where Y_n is defined in (2.5.1). Here, Σ is a contour leaving from $+\infty$ in the upper half plane, encircling the positive real axis and going back to $+\infty$ in the lower half plane, as in Section 2.2.2. The second contour C does not intersect with Σ but can otherwise be any path going from $c_1 - i\infty$ to $c_2 + i\infty$. The fact that the two contours do not intersect, explains why the first term of [CKW15, Formula (3.23)] cancels out here. We will choose the contours C and Σ such that they are suitable for a saddle point analysis of the integral in (2.5.34).

We let Ψ_ϵ , which will serve as a phase function in the saddle point analysis, be defined as in (2.5.24), and we let $u_\epsilon < 0$ be as in Lemma 2.5.1. By (2.5.22) and (2.5.21), we have that $\Psi'_\epsilon(u_\epsilon) = \Psi''_\epsilon(u_\epsilon) = 0$ and since $u_\epsilon < 0$, $\Psi'''_\epsilon(u_\epsilon) = G''_\mu(u_\epsilon) < 0$. Thus the steepest descent paths for Ψ_ϵ emanating from u_ϵ make angles respectively $0, \frac{2\pi}{3}$ and $-\frac{2\pi}{3}$. By (2.5.23) and (2.5.31), we have

$$\Psi_\epsilon^{(j)}(u_\epsilon) = -\kappa_j \epsilon^{-\frac{2}{3}(2j-1)}(1 + o(1)), \text{ as } \epsilon \rightarrow 0, \quad (2.5.35)$$

for $j \geq 3$, where $\kappa_3, \kappa_4, \dots$ are positive constants independent of ϵ .

For the rest of the proof, we set $q_\epsilon = \left| \frac{G''_\mu(u_\epsilon)}{6} \right|$ and we define the two contours $\Sigma_{\text{in},n}$ and $C_{\text{in},n}$ as in Figure 2.7 by

$$\begin{aligned} \Sigma_{\text{in},n} = & \left\{ u_\epsilon + \frac{1}{4\sqrt{3}n^{\frac{1}{3}}q_\epsilon^{\frac{1}{3}}} + it : t \in \left[-\frac{1}{4n^{\frac{1}{3}}q_\epsilon^{\frac{1}{3}}}, \frac{1}{4n^{\frac{1}{3}}q_\epsilon^{\frac{1}{3}}} \right] \right\} \\ & \cup \left\{ u_\epsilon + te^{i\pi/3} : t \in \left[\frac{1}{2\sqrt{3}n^{\frac{1}{3}}q_\epsilon^{\frac{1}{3}}}, |u_\epsilon| \right] \right\} \\ & \cup \left\{ u_\epsilon + te^{-i\pi/3} : t \in \left[\frac{1}{2\sqrt{3}n^{\frac{1}{3}}q_\epsilon^{\frac{1}{3}}}, |u_\epsilon| \right] \right\}, \quad (2.5.36) \end{aligned}$$

$C_{\text{in},n}$ being the mirror image of $\Sigma_{\text{in},n}$ with respect to the vertical line passing through u_ϵ . Note that the condition $\epsilon n^{\frac{3}{2}} \rightarrow +\infty$ as $n \rightarrow +\infty$ is needed for the path $\Sigma_{\text{in},n}$ not to cross the positive real axis.

We will show that the leading behaviour of K_n^S in (2.5.34) as $n \rightarrow +\infty$ comes from the integration over these two local paths, and that it converges to the Airy kernel.

Define

$$\begin{aligned} K_n^{\text{in}}(x, y) = & -\frac{n}{(2\pi i)^2 \epsilon^2} \int_{C_{\text{in},n}} ds \int_{\Sigma_{\text{in},n}} d\zeta \\ & \frac{1}{s - \zeta} \begin{pmatrix} 0 & 1 \end{pmatrix} Y_n^{-1}(s) Y_n(\zeta) \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{\frac{n}{2\epsilon^2}((x-s)^2 - (y-\zeta)^2)}. \end{aligned}$$

We first note that the asymptotic expansion (2.5.8) holds in fact not only in the outer region A_δ , but also when z approaches 0 at a sufficiently slow rate as $n \rightarrow +\infty$. If $n \rightarrow +\infty$ and $z \rightarrow 0$ in such a way that $n^2 z \rightarrow \infty$, (2.5.8) remains valid, but with a weaker error term $o(1)$ instead of $\mathcal{O}(1/n)$. This follows after a comparison of (2.5.8) and (2.5.17), (2.5.18), by the asymptotic behaviour of the Bessel function $J_\alpha(z)$ as $z \rightarrow \infty$. Since $\epsilon n^{\frac{3}{2}} \rightarrow +\infty$ as $n \rightarrow +\infty$, this means that we can use (2.5.8), with weaker error term, for all points of the contour $C_{\text{in},n} \times \Sigma_{\text{in},n}$. We obtain

$$\begin{aligned} K_n^{\text{in}}(x, y) = & -\frac{n}{(2\pi i)^2 \epsilon^2} \int_{C_{\text{in},n}} ds \int_{\Sigma_{\text{in},n}} d\zeta \\ & e^{\frac{n}{2\epsilon^2}((x-s)^2 - (y-\zeta)^2)} e^{n(g_\mu(s) - g_\mu(\zeta))} f(s, \zeta) (1 + o(1)), \end{aligned}$$

as $n \rightarrow +\infty$. Here f is a certain function independent of n which can be expressed in terms of φ , which satisfies $f(z, z) = 1$ since $\det Y_n = 1$,

but whose precise form is not important for us. We define

$$c_\epsilon = \epsilon^{-2} q_\epsilon^{-\frac{1}{3}}, \quad \varphi_{\epsilon,n}(z) = \frac{n^{1/3} z}{c_\epsilon \epsilon^2} (u_\epsilon - a_\epsilon). \quad (2.5.37)$$

We now let x and y approach the left edge a_ϵ at an appropriate speed and we get

$$\begin{aligned} & \frac{1}{c_\epsilon n^{\frac{2}{3}}} K_n^{\text{in}} \left(a_\epsilon - \frac{x}{c_\epsilon n^{\frac{2}{3}}}, a_\epsilon - \frac{y}{c_\epsilon n^{\frac{2}{3}}} \right) = -\frac{n^{\frac{1}{3}}}{c_\epsilon (2\pi i)^2 \epsilon^2} e^{\varphi_{\epsilon,n}(x) - \varphi_{\epsilon,n}(y)} \\ & \times \int_{C_{\text{in},n}} ds \int_{\Sigma_{\text{in},n}} d\zeta \frac{e^{n\Psi_\epsilon(s) + \frac{xn^{\frac{1}{3}}}{\epsilon^2 c_\epsilon}(s-u_\epsilon)}}{e^{n\Psi_\epsilon(\zeta) + \frac{yn^{\frac{1}{3}}}{\epsilon^2 c_\epsilon}(\zeta-u_\epsilon)}} \frac{1}{s-\zeta} f(s, \zeta) (1 + o(1)), \end{aligned} \quad (2.5.38)$$

as $n \rightarrow +\infty$. Using the asymptotics for the derivatives (2.5.35) and the fact that $f(z, z) = 1$, we may expand the phase functions and f around u_ϵ to get

$$\begin{aligned} & \frac{1}{c_\epsilon n^{\frac{2}{3}}} K_n^{\text{in}} \left(a_\epsilon - \frac{x}{c_\epsilon n^{\frac{2}{3}}}, a_\epsilon - \frac{y}{c_\epsilon n^{\frac{2}{3}}} \right) = -\frac{n^{\frac{1}{3}}}{c_\epsilon (2\pi i)^2 \epsilon^2} e^{\varphi_{\epsilon,n}(x) - \varphi_{\epsilon,n}(y)} \\ & \times \int_{C_{\text{in},n}} ds \int_{\Sigma_{\text{in},n}} d\zeta \frac{e^{-nq_\epsilon(s-u_\epsilon)^3 + n^{\frac{1}{3}} q_\epsilon^{\frac{1}{3}} x(s-u_\epsilon) + r_4^{n,\epsilon}(s)}}{e^{-nq_\epsilon(\zeta-u_\epsilon)^3 + n^{\frac{1}{3}} q_\epsilon^{\frac{1}{3}} y(\zeta-u_\epsilon) + r_4^{n,\epsilon}(\zeta)}} \frac{1}{s-\zeta} (1 + o(1)), \end{aligned} \quad (2.5.39)$$

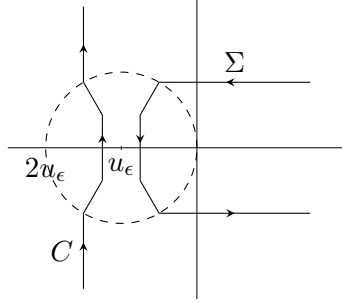
as $n \rightarrow +\infty$, since $s - \zeta \rightarrow 0$, where

$$r_4^{n,\epsilon}(z) = \frac{n}{2\epsilon^2} \sum_{j=4}^{+\infty} \frac{\Psi_\epsilon^{(j)}(u_\epsilon)}{j!} (z - u_\epsilon)^j. \quad (2.5.40)$$

We now apply the changes of variables $s \mapsto u$ and $\zeta \mapsto v$ defined implicitly by

$$\begin{aligned} -nq_\epsilon(s - u_\epsilon)^3 + r_4^{n,\epsilon}(s) &= -u^3, \\ -nq_\epsilon(\zeta - u_\epsilon)^3 + r_4^{n,\epsilon}(\zeta) &= -v^3, \end{aligned}$$

such that $u \sim q_\epsilon^{1/3} n^{1/3} (s - u_\epsilon)$ and $v \sim q_\epsilon^{1/3} n^{1/3} (\zeta - u_\epsilon)$ as $n \rightarrow +\infty$.

Figure 2.8: Choices for the contours C and Σ .

We then get

$$\begin{aligned}
 & e^{-\varphi_{\epsilon,n}(x)+\varphi_{\epsilon,n}(y)} \frac{1}{c_{\epsilon} n^{\frac{2}{3}}} K_n^{\text{in}} \left(a_{\epsilon} - \frac{x}{c_{\epsilon} n^{\frac{2}{3}}}, a_{\epsilon} - \frac{y}{c_{\epsilon} n^{\frac{2}{3}}} \right) = \\
 & - \frac{1}{(2\pi i)^2} \int_{C_n} du \int_{\Sigma_n} dv \frac{e^{-u^3+xu}}{e^{-v^3+yv}} \frac{1}{u-v} (1+o(1)) = K^{\text{AIRY}}(x,y) + o(1),
 \end{aligned} \tag{2.5.41}$$

as $n \rightarrow +\infty$, C_n and Γ_n being contours that grow to the contours of the Airy integral formula as $n \rightarrow +\infty$.

Let γ_1 and γ_2 be the contours as in Lemma 2.5.2. We take the contours C and Σ , depending on n , to be $C = \bar{\gamma}_2 \cup C_{\text{in},n} \cup \gamma_2$ and $\Sigma = \bar{\gamma}_1 \cup \Sigma_{\text{in},n} \cup \gamma_1$ as pictured in Figure 2.8. We have

$$\begin{aligned}
 & \left| \int_{C_{\text{in},n}} ds \int_{\gamma_1} d\zeta \frac{f(s,\zeta)}{s-\zeta} \frac{e^{n\Psi_{\epsilon}(s)+n^{\frac{1}{3}}q_{\epsilon}^{\frac{1}{3}}x(s-u_{\epsilon})}}{e^{n\Psi_{\epsilon}(\zeta)+n^{\frac{1}{3}}q_{\epsilon}^{\frac{1}{3}}y(\zeta-u_{\epsilon})}} \right| \leq \\
 & \int_{C_{\text{in},n}} |ds| \int_{\gamma_1} |d\zeta| \left| \frac{f(s,\zeta)}{s-\zeta} \right| \frac{e^{n\text{Re}(\Psi_{\epsilon}(s))+n^{\frac{1}{3}}q_{\epsilon}^{\frac{1}{3}}\text{Re}(x(s-u_{\epsilon}))}}{e^{n\text{Re}(\Psi_{\epsilon}(\zeta))+n^{\frac{1}{3}}q_{\epsilon}^{\frac{1}{3}}\text{Re}(y(\zeta-u_{\epsilon}))}}. \tag{2.5.42}
 \end{aligned}$$

In view of (2.5.35), we know that for n sufficiently large, the real part of the phase function is increasing on $\Sigma_{\text{in},n} \cap \{\text{Im } z > 0\}$ and decreasing on $C_{\text{in},n} \cap \{\text{Im } z > 0\}$. Moreover, $n^{\frac{1}{3}}q_{\epsilon}^{\frac{1}{3}}$ becomes small compared to $n\text{Re}(\Psi_{\epsilon}(s))$. The function $\text{Re}(\Psi_{\epsilon}(\gamma_1(t)))$ is increasing on $(0, +\infty)$ because of Lemma 2.5.2. Moreover, the contour γ_1 is designed in such a way that its growth satisfies $\text{Re}(\Psi_{\epsilon}(\gamma_1(t))) \sim t^2$ as $t \rightarrow +\infty$, which means

that the integral over γ_1 is exponentially small. In a similar way one shows that all parts of the integration contour $(C \times \Sigma) \setminus (C_{\text{in},n} \times \Sigma_{\text{in},n})$ give exponentially small contributions.

The limit (2.2.10) now follows from (2.5.41). $\square$

2.5.3 Proofs of Theorem 2.2.4, Theorem 2.2.8, and Theorem 2.2.10

To prove the results, once more we only need to verify that the conditions of Lemma 2.1.2 are satisfied.

We start with the Ginibre case, Theorem 2.2.4. Let K_n be the correlation kernel for the squared singular values of a product of Ginibre matrices, given by (2.2.12), and let $\tilde{K}_n$ be given by (2.2.13).

We will first prove that there exists $c_1 > 0$ independent of n such that

$$\left| \frac{1}{n^{m+1}} \tilde{K}_n \left(\frac{x}{n^{m+1}}, \frac{y}{n^{m+1}} \right) \right| \leq c_1 y^{-1/2} e^{|x|} \text{ on } i\mathbb{R} \times \mathbb{R}^+. \quad (2.5.43)$$

This implies condition (2.1.6) with $\beta = 1/2$. Using the fact that

$$\frac{\Gamma(t - n + 1)}{\Gamma(s - n + 1)} = \frac{\Gamma(n - s)}{\Gamma(n - t)} \frac{\sin \pi s}{\sin \pi t} \quad (2.5.44)$$

(which follows from the reflection formula of the Γ function), we obtain

$$\begin{aligned} \frac{1}{n^{m+1}} \tilde{K}_n \left(\frac{x}{n^{m+1}}, \frac{y}{n^{m+1}} \right) &= \frac{1}{(2\pi i)^2} \int_{-\frac{1}{2} + i\mathbb{R}} ds \int_{\Sigma_n} dt \\ &\prod_{j=0}^m \frac{\Gamma(s + \nu_j + 1)}{\Gamma(t + \nu_j + 1)} \frac{\Gamma(n - s)}{\Gamma(n - t)} \frac{\sin \pi s}{\sin \pi t} \frac{x^t y^{-s-1}}{s - t} n^{s-t}. \end{aligned} \quad (2.5.45)$$

We can compute the t -integral using the residue theorem. The only poles of the t -integrand are the ones of $1/\sin \pi t$ since $1/\Gamma$ is entire, and the s - and t -contours do not intersect. We obtain

$$\frac{1}{n^{m+1}} \tilde{K}_n \left(\frac{x}{n^{m+1}}, \frac{y}{n^{m+1}} \right) = \frac{1}{2\pi i} \sum_{k=0}^{n-1} (-1)^k \frac{n^{-k} x^k}{\Gamma(n - k)} \prod_{j=0}^m \frac{1}{\Gamma(k + \nu_j + 1)}$$

$$\times \int_{-\frac{1}{2}+i\mathbb{R}} \prod_{j=0}^m \Gamma(s + \nu_j + 1) \Gamma(n - s) (\sin \pi s) y^{-s-1} n^s \frac{1}{s - k} ds. \quad (2.5.46)$$

The integral is absolutely convergent since $|\sin \pi (\frac{1}{2} + it)| \sim \frac{e^{\pi|t|}}{2}$ and [OLBC10, Formula 5.11.9]

$$\left| \Gamma \left(\frac{1}{2} + \nu_j + it \right) \right| \sim \sqrt{2\pi} |t|^{\nu_j} e^{-\frac{\pi|t|}{2}}, \quad (2.5.47)$$

as $t \rightarrow \pm\infty$. For s on the integration contour, we have $|y^{-s-1} n^s| = (ny)^{-\frac{1}{2}}$. Moreover, we have [OLBC10, Inequality 5.6.6] $|\Gamma(x + iy)| \leq |\Gamma(x)|$ and thus

$$\begin{aligned} & \left| \frac{1}{n^{m+1}} \tilde{K}_n \left(\frac{x}{n^{m+1}}, \frac{y}{n^{m+1}} \right) \right| \leq \\ & \frac{1}{\pi} y^{-\frac{1}{2}} \int_{-\frac{1}{2}+i\mathbb{R}} \prod_{j=0}^m |\Gamma(s + \nu_j + 1)| |\sin \pi s| |ds| \sum_{k=0}^{n-1} \frac{|x|^k}{k!} \frac{n^{-k-\frac{1}{2}} \Gamma(n + 1/2)}{\Gamma(k + 1) \Gamma(n - k)}. \end{aligned} \quad (2.5.48)$$

It follows from Stirling's Inequality [OLBC10, Inequality 5.6.1] that there exists a constant $C > 0$ such that for $n \in \mathbb{N}$, $k = 0, \dots, n - 1$,

$$\frac{n^{-k-\frac{1}{2}} \Gamma(n + 1/2)}{\Gamma(k + 1) \Gamma(n - k)} \leq C. \quad (2.5.49)$$

In $|x|$, we are left with a truncated exponential series. This implies (2.5.43).

To prove (2.1.5) with $\beta = 1/2$, we follow the proof in [KZ14, Section 5.2], where (2.2.14) was proved uniformly for x, y in compact subsets of $(0, +\infty)$. We need uniformity of (2.1.5) for u in any compact subset of $\mathbb{C}$ and v in any compact subset of $[0, +\infty)$. Therefore, we fix $R > 0$ and assume that $|u| < R$ and $v \in [0, R]$. Using the asymptotics $\frac{\Gamma(n-s)}{\Gamma(n-t)} = n^{t-s} (1 + \mathcal{O}(n^{-1}))$ as $n \rightarrow \infty$, we obtain from (2.5.45) that

$$\frac{v^{1/2}}{n^{m+1}} \tilde{K}_n \left(\frac{u}{n+1}, \frac{v}{n+1} \right)$$

$$\begin{aligned}
&= \frac{1}{(2\pi i)^2} \int_{-\frac{1}{2}+i\mathbb{R}} ds \int_{\Sigma_n} dt \prod_{j=0}^m \frac{\Gamma(s+\nu_j+1)}{\Gamma(t+\nu_j+1)} \frac{\sin \pi s}{\sin \pi t} \frac{u^t v^{-s-\frac{1}{2}}}{s-t} \\
&\quad \times (1 + \mathcal{O}(n^{-1})), \quad (2.5.50)
\end{aligned}$$

as $n \rightarrow \infty$. Using Stirling's formula and the inequality on sines $|\sin \pi t| \geq |\sinh(\pi \operatorname{Im} t)|$, we can bound $\left| \frac{u^t}{\sin \pi t \prod_{j=0}^m \Gamma(t+\nu_j+1)} \right|$ by a function independent of u which decays rapidly as $t \rightarrow +\infty$. It follows that we can deform the integration contour Σ_n to Σ , provided Σ lies not too close to the real line. For the s -integrand, we can use (2.5.47) and the fact that $|v^{-s-\frac{1}{2}}| = 1$ to bound it uniformly. By the dominated convergence theorem, we can take $(1 + \mathcal{O}(n^{-1}))$ (with error term independent of u and v) out of the integral, and we obtain (2.1.5) uniformly in u and v .

Remark 2.5.3. Although the kernel for the squared singular values of products of Ginibre matrices cannot be expressed in terms of orthogonal polynomials, it can be expressed in terms of the solution to a Riemann-Hilbert problem for multiple orthogonal polynomials [KZ14, Section 2.2], and it has a Christoffel-Darboux type formula. If large n asymptotics for this Riemann-Hilbert problem were available, one could be optimistic that similar techniques as in Section 2.5.2 can be used in the supercritical case to prove convergence to the Airy kernel.

Remark 2.5.4. The proofs of Theorem 2.2.8 and Theorem 2.2.10 are very similar to the proof of Theorem 2.2.4. For Theorem 2.2.8, one has to start with the expression (2.2.19) for the kernel K_n instead of (2.2.12). Then one shows in a similar way as in the Ginibre case that

$$\lim_{n \rightarrow +\infty} v^{1/2} \frac{1}{c_n} K_n \left(\frac{u}{c_n}, \frac{v}{c_n} \right) = v^{1/2} \mathbb{K}_{\nu, \mu}^T(u, v), \quad (2.5.51)$$

uniformly for u in any compact subset of $\mathbb{C}$ and v in any compact subset of $[0, +\infty)$, and that there are constants C_1, C_2 such that

$$|K_n(u, v)| \leq C_1 v^{-1/2} c_n^{1/2} e^{C_2 c_n(|u|+|v|)}, \quad (2.5.52)$$

for every $(u, v) \in i\mathbb{R} \times [0, +\infty)$ and $n > n_0$. The two above conditions are almost the same as the assumptions in Lemma 2.1.2, except for the

fact that c_n now plays the role of cn^γ . From the proof of Lemma 2.1.2, it is straightforward to check that conditions (2.5.51) and (2.5.52) imply (2.1.7) and (2.1.8) with cn^γ replaced by c_n . This leads to the proof of Theorem 2.2.8 without further complications

For Theorem 2.2.10, one has to use the expression (2.2.26) or (2.2.27) instead. The rest of the proof is a straightforward adaptation of the proof of Theorem 2.2.4, and we do not give the details here.

Chapter 3

The generating function for the Airy point process and a system of coupled Painlevé II equations

This chapter is based on the eponymous paper with Tom Claeys [CD17b].

3.1 Introduction

Subsection 1.2.5 in Chapter 1 introduced the Airy point process (or Airy ensemble) that appeared as the limiting point process at the edge of the GUE, see (1.2.10). This chapter is devoted to the study of the generating function $F(\vec{x}; \vec{s})$ (1.1.7) of the occupancy numbers for the Airy ensemble. We will thus be focusing on the formula (1.1.7) with $\mathcal{K}$ being the integral operator corresponding to the Airy kernel K^{AIRY} (1.2.14),

$$F(\vec{x}; \vec{s}) = \det \left(1 - \chi_{(x_k, +\infty)} \sum_{j=1}^k (1 - s_j) \mathcal{K}^{\text{AIRY}} \chi_{(x_j, x_{j-1})} \right), \quad (3.1.1)$$

with $+\infty = x_0 > x_1 > x_2 > \cdots > x_k$.

As discussed at the end of sub-section 1.2.5, the simplest case $k = 1$ is

well understood in terms of solution to the Painlevé II equation (1.2.17).

A Hamiltonian and integrable structure associated to $F(\vec{x}; \vec{s})$ has also been established in [HTW93], and it was linked to a Lie algebra, see also [AvM80] for a similar approach in a more general context. Given the integrable structure, it is natural to ask whether an explicit formula for $F(\vec{x}; \vec{s})$ in terms of solutions to Painlevé-type equations exists.

Tracy-Widom formula. The main result of this chapter is a Tracy-Widom type formula for the multi-interval Airy kernel Fredholm determinant $F(x_1, \dots, x_k; s_1, \dots, s_k)$ for general $k > 1$, similar to (1.2.16).

Theorem 3.1.1. *Let $s_1, \dots, s_k \in [0, 1]$ be such that $s_j \neq s_{j+1}$ for $j = 1, \dots, k$ with $s_{k+1} = 1$. Let $x_1, \dots, x_k$ be as in (1.1.6), and let $F(\vec{x}; \vec{s})$ be defined by (3.1.1), where here $\mathcal{K}$ is the integral operator related to the Airy kernel K^{AIRY} . We have*

$$F(\vec{x}; \vec{s}) = \prod_{j=1}^k \exp \left(- \int_0^{+\infty} \xi u_j(\xi; \vec{x}, \vec{s})^2 d\xi \right), \quad (3.1.2)$$

where $u_1(\xi; \vec{x}, \vec{s}), \dots, u_k(\xi; \vec{x}, \vec{s})$ satisfy the following system of ordinary differential equations,

$$\begin{cases} u_1'' = (\xi + x_1)u_1 + 2u_1 \sum_{j=1}^k u_j^2 \\ u_2'' = (\xi + x_2)u_2 + 2u_2 \sum_{j=1}^k u_j^2 \\ \vdots \\ u_k'' = (\xi + x_k)u_k + 2u_k \sum_{j=1}^k u_j^2 \end{cases} \quad (3.1.3)$$

and have the following behavior at $+\infty$,

$$u_j(\xi; \vec{x}, \vec{s}) = \sqrt{s_{j+1} - s_j} \text{Ai}(\xi + x_j)(1 + o(1)), \quad \text{as } \xi \rightarrow +\infty. \quad (3.1.4)$$

If $s_{j+1} > s_j$, $u_j(\xi; \vec{x}, \vec{s})$ is real-valued for real ξ ; if $s_{j+1} < s_j$, $u_j(\xi; \vec{x}, \vec{s})$ is purely imaginary for real ξ .

Remark 3.1.2. If $s_1 < s_2 < \dots < s_k$, all u_j 's are real, and then (3.1.2) can be written as

$$F(\vec{x}; \vec{s}) = \exp \left(- \int_0^{+\infty} \xi \|\vec{u}(\xi; \vec{x}, \vec{s})\|^2 d\xi \right), \quad (3.1.5)$$

where

$$u_j'' = (\xi + x_j)u_j + 2u_j\|\vec{u}\|^2, \quad j = 1, \dots, k, \quad (3.1.6)$$

with $\|\vec{u}\|$ the 2-norm of the vector $\vec{u} = (u_1, \dots, u_k)$.

Remark 3.1.3. For $k = 1$, the system of ODEs is simply the (shifted) Painlevé II equation

$$u_1'' = (\xi + x_1)u_1 + 2u_1^3,$$

and then $q(\xi; s) = u_1(\xi - x_1; x_1, s)$ is the Ablowitz-Segur (for $s \neq 0$) or Hastings-McLeod solution (for $s = 0$) of Painlevé II. We then easily recover the classical Tracy-Widom formula (1.2.16). For $k > 1$, (3.1.3) can be seen as a system of Painlevé II equations coupled by the last term at the right hand side. We will show in Section 3.3 that it is the compatibility condition of a Lax pair of size 2×2 with k regular singularities and one irregular singularity at infinity, see (3.3.19) and (3.3.23) below.

Remark 3.1.4. In the case where $s_1 < \dots < s_k$, the system (3.1.3) of coupled Painlevé II equations is a *traveling wave reduction* of the integrable system of PDEs

$$i\vec{q}_t = \vec{q}_{\xi\xi} - \xi\vec{q} - 2\vec{q}\|\vec{q}\|^2, \quad \vec{q} = (q_1, \dots, q_k), \quad (3.1.7)$$

which is known as the spatially inhomogeneous defocusing *vector nonlinear Schrödinger equation* with linear potential, see [BBPGB11, Equations (2a)–(2b)] and also [Hon01, VMC03] for similar reductions. For $k = 2$, the above system of PDEs is known as the Manakov system [Man74]. For traveling wave solutions to (3.1.7) of the form

$$q_j(\xi; t) = u_j(\xi)e^{-ix_j t}, \quad j = 1, \dots, k,$$

(3.1.7) is easily seen to be equivalent to the system of coupled Painlevé II equations (3.1.6). Also for $k = 2$, this system appeared in Hamiltonian form in a study by Sasano [Sas07] of fourth order extensions of the Painlevé II equation.

Remark 3.1.5. Theorem 3.1.1 establishes existence of a solution to the

system (3.1.3) with boundary conditions (3.1.4), but not uniqueness. The existence is a consequence of the fact that we will construct the functions $u_j(\xi; \vec{x}, \vec{s})$ explicitly in terms of the solution of a Riemann-Hilbert (RH) problem, and use this representation to show that they satisfy the required boundary conditions. RH characterizations of solutions of Painlevé type equations are in general very useful in the study of asymptotics, but are not effective to show uniqueness of solutions with a specific asymptotic behavior.

Remark 3.1.6. Our system of differential equations shows remarkable similarities with systems of differential equations derived in [TW03] for the time-dependent dynamical version of the Airy process in the case of $k = 1$ discontinuity, see also [TW04] for the GUE process. In [BC12a], a matrix non-commutative version of the Painlevé II equation was studied, which can be seen as a different system of coupled Painlevé II equations. Unique smooth solutions to this system with Airy-type asymptotics were proven to exist in [BC12b].

Theorem 3.1.1 reveals a connection between the Fredholm determinant $F(\vec{x}; \vec{s})$ and an integrable system of differential equations. One may hope that this relation will help to understand asymptotic properties of the Fredholm determinant $F(\vec{x}; \vec{s})$ in various limits, as it did for $k = 1$ in the past. Indeed, (1.2.16) together with the asymptotics of the Painlevé II solutions $q(\xi; s)$ as $\xi \rightarrow -\infty$ provided a simple way to derive $x \rightarrow -\infty$ asymptotics for the Fredholm determinant $F(x; s)$, up to a multiplicative constant. In this perspective, it is a natural first step to understand the asymptotic behavior of the functions $u_j(\xi; \vec{x}, \vec{s})$, in particular in limits where the system of k coupled Painlevé II equations gets reduced to a system of $k - 1$ coupled Painlevé II equations.

Asymptotics for the solutions of the system of coupled Painlevé II equations. The vector solution $\vec{u}(\xi; \vec{x}, \vec{s})$ exhibits several interesting degenerate cases. If consecutive values of the parameters s_j, s_{j+1} or x_j, x_{j+1} coalesce, the system of k equations simplifies and it formally reduces to the system of $k - 1$ coupled Painlevé II equations. More precisely, if two s -values s_j and s_{j+1} approach each other, then the

component u_j will become small. If we delete the j -th equation from the system (3.1.3), the remaining system is again of the same form, but with k reduced by 1. If we consecutively let all s_j 's for $j = 1, \dots, k$ approach each other, the number of equations in the system gets reduced at each step, and finally asymptotics can be expressed in terms of the Ablowitz-Segur or Hastings-McLeod solutions of the Painlevé II equation. A similar phenomenon takes place if two x -values x_{j+1} and x_j coalesce, or if $x_1 \rightarrow +\infty$.

Before stating these results in detail, we introduce the following notation: given a vector $\vec{v} = (v_1, v_2, \dots, v_k)$, we denote by $\vec{v}^{[j]} = (v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_k)$, for $j = 1, \dots, k$, the vector $\vec{v}$ without its j -th component.

Theorem 3.1.7. *Let $s_1, \dots, s_k \in [0, 1]$ be such that $s_j \neq s_{j+1}$ for $j = 1, \dots, k$ with $s_{k+1} = 1$, and let $x_1, \dots, x_k$ be as in (1.1.6). Let $u_1, \dots, u_k$ be the solutions appearing in Theorem 3.1.1 of the system of equations (3.1.3). We have the following asymptotic results, which are uniform in $\xi > M$ for any $M \in \mathbb{R}$.*

1. Let $j \in \{1, \dots, k\}$. As $s_{j+1} - s_j \rightarrow 0$ with $s_{j-1} \neq s_{j+1}$ (if $j \neq 1$), we have

$$u_j(\xi; \vec{x}, \vec{s}) = \mathcal{O}\left(|s_j - s_{j+1}|^{1/2}\right), \quad (3.1.8)$$

$$u_\ell^2(\xi; \vec{x}, \vec{s}) = u_\ell^2(\xi; \vec{x}^{[j]}, \vec{s}^{[j]}) + \mathcal{O}\left(|s_j - s_{j+1}|^{1/2}\right), \quad \ell < j, \quad (3.1.9)$$

$$u_\ell^2(\xi; \vec{x}, \vec{s}) = u_{\ell-1}^2(\xi; \vec{x}^{[j]}, \vec{s}^{[j]}) + \mathcal{O}\left(|s_j - s_{j+1}|^{1/2}\right), \quad \ell > j. \quad (3.1.10)$$

2. Let $j \in \{2, \dots, k\}$. As $x_{j-1} - x_j \rightarrow 0$ with $s_{j-1} \neq s_{j+1}$, we have

$$u_{j-1}^2(\xi; \vec{x}, \vec{s}) + u_j^2(\xi; \vec{x}, \vec{s}) = u_{j-1}^2(\xi; \vec{x}^{[j]}, \vec{s}^{[j]}) + \mathcal{O}(x_{j-1} - x_j), \quad (3.1.11)$$

$$u_\ell^2(\xi; \vec{x}, \vec{s}) = u_\ell^2(\xi; \vec{x}^{[j]}, \vec{s}^{[j]}) + \mathcal{O}(x_{j-1} - x_j), \quad \ell < j - 1, \quad (3.1.12)$$

$$u_\ell^2(\xi; \vec{x}, \vec{s}) = u_{\ell-1}^2(\xi; \vec{x}^{[j]}, \vec{s}^{[j]}) + \mathcal{O}(x_{j-1} - x_j), \quad \ell > j. \quad (3.1.13)$$

3. If $x_1 \rightarrow +\infty$, we have

$$u_1(\xi; \vec{x}, \vec{s}) = \sqrt{s_2 - s_1} \text{Ai}(\xi + x_1) + o(1), \quad (3.1.14)$$

$$u_\ell^2(\xi; \vec{x}, \vec{s}) = u_{\ell-1}^2(\xi; \vec{x}^{[1]}, \vec{s}^{[1]}) + \mathcal{O}(x_1^{-1/2}), \quad \ell > 1. \quad (3.1.15)$$

Remark 3.1.8. For $k = 2$, the above results show that, for $s_2 > s_1$, the function $\|\vec{u}\| = \sqrt{u_1^2 + u_2^2}$ reduces to Ablowitz-Segur or Hastings-McLeod solutions to Painlevé II in either of the limits $x_1 \rightarrow x_2$, $x_1 \rightarrow +\infty$, $s_2 \rightarrow s_1$, $s_2 \rightarrow 1$. We indeed have

$$\|\vec{u}(\xi; x_1, x_2, s_1, s_2)\| \rightarrow \begin{cases} q(\xi + x_2; s_1), & \text{as } s_2 \rightarrow s_1, \\ q(\xi + x_1; s_1), & \text{as } s_2 \rightarrow 1. \end{cases} \quad (3.1.16)$$

In other words, if we let s_2 increase from s_1 to 1, $\|\vec{u}(\xi; x_1, x_2, s_1, s_2)\|$ shows a transition from the Painlevé II solution $q(\xi + x_2; s_1)$ to its shifted version $q(\xi + x_1; s_1)$.

Similarly, if we let x_1 increase from x_2 to $+\infty$, a cross-over between the Painlevé II solutions $q(\xi + x_2, s_1)$ and $q(\xi + x_2, s_2)$ takes place:

$$\|\vec{u}(\xi; x_1, x_2, s_1, s_2)\| \rightarrow \begin{cases} q(\xi + x_2; s_1), & \text{as } x_1 \rightarrow x_2, \\ q(\xi + x_2; s_2), & \text{as } x_1 \rightarrow +\infty. \end{cases} \quad (3.1.17)$$

Here, the increase of x_1 does not cause a shift in the Painlevé variable ξ , but in the parameter s which specifies the boundary condition of the Painlevé II solution.

On the level of the Fredholm determinant $F(\vec{x}; \vec{s})$ and on the level of an associated RH problem, these shifts were already observed in 2001 by Baik, Deift, and Rains [BDR01, Section 6], who noticed that these phenomena are reminiscent of the behavior of superposed soliton solutions to nonlinear wave equations after they collided. The terminology *Painlevé-tons*, or *multi-Painlevé* functions, was suggested in [BDR01].

Remark 3.1.9. In principle, the asymptotic analysis in Section 3.4

allows with some more effort to compute further subleading terms in the asymptotic expansions from Theorem 3.1.7, expressed in terms of the solution to the RH problem for Ψ stated below. However, it is not clear that this would lead to simple explicit expressions for these subleading terms.

Outline. In Section 3.2, we will describe some concrete examples of probability distributions which one can compute using the Fredholm determinants $F(\vec{x}; \vec{s})$, and we express those distributions explicitly in terms of the solutions $\vec{u}(\xi; \vec{x}, \vec{s})$ of the system of coupled Painlevé II equations. In Section 3.3, we will express logarithmic derivatives of the Fredholm determinants in terms of a RH problem, and we will derive an associated Lax pair. By manipulating the Lax pair in a convenient way, we will prove Theorem 3.1.1, except for the asymptotic formula (3.1.4). In Section 3.4, we will analyze the RH problem associated to the Fredholm determinants and the functions u_j asymptotically, and this will allow us to prove (3.1.4) and Theorem 3.1.7.

3.2 Examples and applications

Near-extreme GUE eigenvalues. As was discussed in sub-section 1.2.1 in Chapter 1, the prototype example of a model in which the Airy point process appears as a limiting process is the GUE. The eigenvalues $\lambda_1 > \lambda_2 > \dots > \lambda_n$ of an $n \times n$ GUE matrix follow a determinantal point process with kernel K_n built out of Hermite polynomials,

$$K_n^{\text{GUE}}(x, y) = e^{-\frac{1}{4}(x^2 + y^2)} \sum_{j=0}^{n-1} p_j(x) p_j(y), \quad (3.2.1)$$

where $(p_j)_{j=0,1,\dots}$ is the sequence of normalized Hermite polynomials, orthogonal with respect to the weight $e^{-x^2/2}$ on $\mathbb{R}$.

After the re-scaling

$$x_j = n^{1/6}(\lambda_j - 2\sqrt{n}), \quad (3.2.2)$$

this kernel converges to the Airy kernel,

$$\lim_{n \rightarrow \infty} n^{-1/6} K_n^{\text{GUE}}(2\sqrt{n} + xn^{-1/6}, 2\sqrt{n} + yn^{-1/6}) = K^{\text{AIRY}}(x, y), \quad (3.2.3)$$

uniformly for $x, y \in [M, +\infty)$ for any $M \in \mathbb{R}$. This implies also trace-norm convergence of the associated operators when acting on bounded intervals or unbounded intervals of the form $[M, +\infty)$. Therefore, with $\vec{x}$ and $\vec{\lambda}$ related as in (3.2.2), we have the limit

$$F_n^{\text{GUE}}(\vec{\lambda}; \vec{s}) := \det \left(1 - \chi_{(\lambda_k, +\infty)} \sum_{j=1}^k (1 - s_j) K_n \chi_{(\lambda_j, \lambda_{j-1})} \right) \rightarrow F(\vec{x}; \vec{s}), \quad (3.2.4)$$

as $n \rightarrow +\infty$, see e.g. [Sos00]. It is worth noting that F_n^{GUE} can also be expressed as a ratio of Hankel determinants with respect to a discontinuous Gaussian weight,

$$F_n^{\text{GUE}}(\vec{\lambda}; \vec{s}) = \frac{\det \left(\int_{\mathbb{R}} x^{\ell+m} e^{-\frac{x^2}{2}} \sum_{j=1}^{k+1} s_j \chi_{(x_j, x_{j-1})}(x) dx \right)_{\ell, m=0}^{n-1}}{\det \left(\int_{\mathbb{R}} x^{\ell+m} e^{-\frac{x^2}{2}} dx \right)_{\ell, m=0}^{n-1}}, \quad (3.2.5)$$

with $x_0 = +\infty$, $x_{k+1} = -\infty$, $s_{k+1} = 1$. Since the denominator is a Selberg integral which can be evaluated explicitly, Theorem 3.1.1 together with (3.2.4) also implies an asymptotic expansion for the Hankel determinant in the numerator of (3.2.5), which is a generalization of the asymptotic expansion from [BCI16] in the case of 1 jump discontinuity.

The joint probability distribution of k near-extreme eigenvalues in the GUE can be expressed exactly in terms of $F_n^{\text{GUE}}(\vec{\lambda}; \vec{s})$ in a similar way as in formula (1.1.9) for the Airy point process. In particular, gap probabilities near the edge, the distribution of spacings between the k -th and the ℓ -th largest eigenvalue, the sum of the k largest eigenvalues, and the distribution of truncated linear statistics of the form $\sum_{j=1}^k f(\lambda_j)$ (as studied recently in [GMT17]) can be expressed identically in terms of $F_n(\vec{\lambda}; \vec{s})$. In the large n limit, after the re-scaling (3.2.3), limit distributions arise which are given in terms of $F(\vec{x}; \vec{s})$.

Gap probabilities. The probability to find no GUE eigenvalues λ_j in a scaled interval $(2\sqrt{n} + x_2 n^{-1/6}, 2\sqrt{n} + x_1 n^{-1/6})$ near the edge of the spectrum converges as $n \rightarrow +\infty$ to the probability of having no particles ζ_j of the Airy point process in a finite interval (x_2, x_1) , and this is given by

$$\mathbb{P}(n_{(x_2, x_1)} = 0) = F(x_1, x_2; 1, 0) = \exp\left(\int_0^{+\infty} \xi |u_1(\xi)|^2 d\xi\right) \exp\left(-\int_0^{+\infty} \xi |u_2(\xi)|^2 d\xi\right) \quad (3.2.6)$$

with

$$u_j''(x) = (x + x_j)u_j(x) + 2u_j(x)(u_1(x)^2 + u_2(x)^2), \quad j = 1, 2 \quad (3.2.7)$$

and

$$u_j(x)^2 \sim (-1)^j \text{Ai}(x + x_j)^2, \quad x \rightarrow +\infty, \quad j = 1, 2.$$

As $x_1 \rightarrow +\infty$, it is obvious that this gap probability converges to the largest eigenvalue distribution $F(x_2; 0)$, which can also be seen easily from Theorem 3.1.7. Setting $k \geq 2$ in Theorem 3.1.1, we can find expressions for gap probabilities on any finite union of intervals.

Thinning and a conditional largest eigenvalue distribution.

The thinned Airy point process is the process obtained by removing each particle $\zeta_1, \zeta_2, \dots$ independently with a given probability $s \in (0, 1)$. The parameter s can be seen as a measure for repulsion: $s = 1$ corresponds to a repulsive point process and the limit $s \rightarrow 0$ to a Poisson process. In [BdCP09, BB17], the largest particle distribution in this process was studied and a transition between the Tracy-Widom distribution and the Weibull distribution was observed as s tends to 1. One can interpret the removed particles in this process as unobserved and the remaining ones as observed [BP06]. It is then natural to ask what the distribution of the largest particle ζ_1 of the Airy point process is, given information about the position of the largest observed particle, which we denote by ξ_1 and which is equal in distribution to ζ_Y , with Y a geometric random variable defined by $\mathbb{P}(Y = k) = (1 - s)s^{k-1}$. More precisely, we consider

the conditional probability distribution of ζ_1 , conditioning on the event that the largest observed particle ξ_1 is less than a given value. We note in this context that the observed particles also form a determinantal point process, with correlation kernel $(1-s)K^{\text{AIRY}}$ [LMR15], and therefore the distribution of ξ_1 is given by $\mathbb{P}(\xi_1 < x_2) = F(x_2; s)$.

For $x_1 < x_2$, the conditional probability that $\zeta_1 < x_1$, given that $\xi_1 < x_2$ is clearly equal to the ratio $F(x_1; 0)/F(x_2; s)$. For $x_1 > x_2$, it can be expressed in terms of $F(x_1, x_2; 0, s)$ as follows,

$$\begin{aligned} \mathbb{P}(\zeta_1 < x_1 | \xi_1 < x_2) &= \frac{\mathbb{P}(\zeta_1 < x_1, \xi_1 < x_2)}{\mathbb{P}(\xi_1 < x_2)} = \\ &= \frac{\sum_{j=0}^{\infty} s^j \mathbb{P}(n_{(x_1, +\infty)} = 0, n_{(x_2, x_1)} = j)}{\mathbb{P}(\xi_1 < x_2)} = \\ &= \frac{\mathbb{E}(s^{n_{(x_2, x_1)}})}{\mathbb{P}(\xi_1 < x_2)} = \frac{F(x_1, x_2; 0, s)}{F(x_2; s)}. \end{aligned}$$

Using (1.2.16) and (3.1.2), we can write this as

$$\begin{aligned} \mathbb{P}(\zeta_1 < x_1 | \xi_1 < x_2) &= \exp\left(\int_0^{+\infty} \xi q(\xi + x_2; s)^2 d\xi\right) \\ &\times \exp\left(-\int_0^{+\infty} \xi u_1(\xi; x_1, x_2, 0, s)^2 d\xi\right) \\ &\times \exp\left(-\int_0^{+\infty} \xi u_2(\xi; x_1, x_2, 0, s)^2 d\xi\right), \quad (3.2.8) \end{aligned}$$

where q is the Ablowitz-Segur solution of Painlevé II, and where u_1 and u_2 solve the system (3.2.7) with asymptotic behaviour

$$\begin{aligned} u_1(\xi)^2 &\sim s \text{Ai}(\xi + x_1)^2, & \xi \rightarrow +\infty \\ u_2(\xi)^2 &\sim (1-s) \text{Ai}(\xi + x_2)^2, & \xi \rightarrow +\infty. \end{aligned}$$

Simultaneously with our work, a determinant closely related to $F(x_1, x_2; 0, s)$ but with an extra spectral singularity was studied by Xu and Dai in [XD17, Theorem 2], where they obtained a Tracy-Widom formula which is equivalent to ours if the spectral singularity is absent.

As $x_1 \rightarrow x_2$, one can use Theorem 3.1.7 to confirm that (3.2.8) tends to $\frac{P(\zeta_1 < x_1)}{P(\xi_1 < x_1)} = F(x_1; 0)/F(x_2; s)$. As $s \rightarrow 0$, Theorem 3.1.7 implies

that (3.2.8) converges to 1, which is natural since almost all eigenvalues are observed. If $s \rightarrow 1$ on the other hand, Theorem 3.1.7 implies that (3.2.8) converges to the Tracy-Widom distribution $F(x_1; 0)$, which is again what could be expected: since almost no eigenvalues are observed, we condition on a very likely event, and the conditional largest particle distribution converges to the unconditioned largest particle distribution.

The thinned Airy point process arises not only as the large n limit of the thinned GUE, but also as limit point process in domino tilings of Aztec diamonds with different weights for horizontal and vertical dominoes [CJY15].

Distribution of the spacing between the largest two particles.

The distribution of the spacing between the largest two eigenvalues of a GUE matrix has attracted a lot of interest recently, see [PS14, WBF13] and also [DT18] where it was shown that this distribution is directly related to the distribution of the first halting time of the Toda eigenvalue algorithm applied to random matrices. The limit distribution of this spacing, after re-scaling, is given by $\mathbb{P}(\zeta_1 - \zeta_2 > \sigma)$, with $\zeta_1 > \zeta_2$ the largest two particles in the Airy point process. This distribution can be expressed as

$$\begin{aligned} \mathbb{P}(\zeta_1 - \zeta_2 > \sigma) &= \int_{\mathbb{R}} \frac{\partial}{\partial x_1} \mathbb{P}(\zeta_1 < x_1, \zeta_2 < \zeta - \sigma) \Big|_{x_1=\zeta} d\zeta \\ &= \int_{\mathbb{R}} \frac{\partial}{\partial x_1} \left(\mathbb{P}(n_{(x_1, +\infty)} = 0, n_{(\zeta - \sigma, \zeta)} = 0) + \right. \\ &\quad \left. \mathbb{P}(n_{(x_1, +\infty)} = 0, n_{(\zeta - \sigma, \zeta)} = 1) \right) \Big|_{x_1=\zeta} d\zeta \\ &= \int_{\mathbb{R}} \frac{\partial^2}{\partial s_2 \partial x_1} (F(x_1, \zeta - \sigma; 0, s_2)) \Big|_{x_1=\zeta, s_2=0} d\zeta, \end{aligned}$$

by (1.1.8). Using (3.1.2), this can be rewritten as

$$\mathbb{P}(\zeta_1 - \zeta_2 > \sigma) = \int_{\mathbb{R}} v(\zeta + \sigma, \zeta) F^{\text{TW}}(\zeta; 0) d\zeta, \quad (3.2.9)$$

where

$$v(x_1, x_2) = \int_{\mathbb{R}} \xi \frac{-\partial^2}{\partial s_2 \partial x_1} (u_1^2(\xi; x_1, x_2; 0, s_2) + u_2^2(\xi; x_1, x_2; 0, s_2)) \Big|_{s_2=0} d\xi, \quad (3.2.10)$$

and u_1, u_2 solve the system of equations (3.2.7) with asymptotic behavior (3.1.4).

Different expressions for this distribution were obtained in [PS14] and [WBF13], in terms of fundamental solutions of the Lax pair associated to the Hastings-McLeod solution of the Painlevé II equation. The large and small σ asymptotics of the distribution (3.2.9) are of particular interest and were obtained in [PS14]. To derive them from our formula, a more detailed understanding of the asymptotic behavior of $v(x_1, x_2)$ would be needed as $x_1 - x_2 \rightarrow +\infty$ or as $x_1 - x_2 \rightarrow 0$.

Plancherel measure and maximal sum of lengths of k disjoint increasing subsequences of a random permutation. Another model in which the Airy point process arises naturally consists of random partitions with respect to the Plancherel measure. Consider the joint distribution of the k first components $\lambda_1, \dots, \lambda_k$ of a random partition of n or a Young diagram of n boxes following the Plancherel measure. If we set $\zeta_j = n^{-1/6}(\lambda_j - \sqrt{2n})$, then the joint distribution of $\zeta_1, \dots, \zeta_k$ converges as $n \rightarrow +\infty$ to the joint distribution of the k largest particles in the Airy point process [BDJ99, Oko00, BOO00, Joh06]. From the Robinson-Schensted-Knuth correspondence, it follows that the sum of the first k components $S_k := \lambda_1 + \lambda_2 + \dots + \lambda_k$ has the same distribution as the maximal sum of the lengths of k disjoint increasing subsequences of a random permutation (see e.g. [BDS16, Rom15]).

For $k = 2$, a straightforward calculation similar to the one for the spacing leads to

$$\begin{aligned} \lim_{n \rightarrow +\infty} \mathbb{P} \left(n^{1/6} (S_2 - 4\sqrt{n}) < \sigma \right) &= \mathbb{P}(\zeta_1 + \zeta_2 < \sigma) = \\ &= \int_{\mathbb{R}} \frac{\partial}{\partial x_1} \mathbb{P}(\zeta_1 < x_1, \zeta_2 < \sigma - \zeta) \Big|_{x_1=\zeta} d\zeta = \\ &= \int_{\mathbb{R}} \frac{\partial^2}{\partial s_2 \partial x_1} (F(x_1, \sigma - \zeta; 0, s_2)) \Big|_{x_1=\zeta, s_2=0} d\zeta. \end{aligned}$$

Using (3.1.2), we can write this after a straightforward calculation as

$$\mathbb{P}(\zeta_1 + \zeta_2 < \sigma) = \int_{\mathbb{R}} v(\sigma - \zeta, \zeta) F^{\text{TW}}(\zeta; 0) d\zeta, \quad (3.2.11)$$

with v as defined in (3.2.10).

Hence, by (3.2.11), it follows that

$$\lim_{n \rightarrow \infty} \mathbb{P}\left(n^{1/6}(S_2 - 4\sqrt{n}) < \sigma\right) = \int_{\mathbb{R}} v(\sigma - \zeta, \zeta) F^{\text{TW}}(\zeta; 0) d\zeta, \quad (3.2.12)$$

where S_2 is the maximal total length of two disjoint increasing subsequences of a random permutation of n . Similar but lengthier formulas can be obtained for general k .

3.3 Differential identities and Lax pair

3.3.1 Differential identities in terms of RH problem

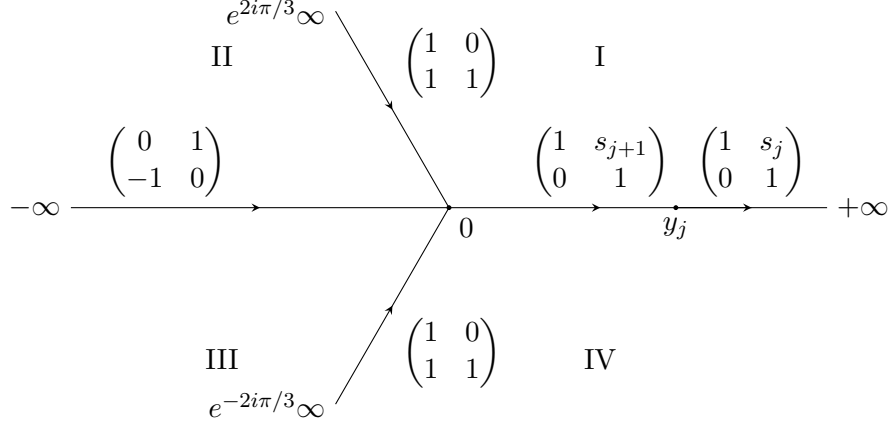
In this section, we will relate the Fredholm determinants $F(\vec{x}; \vec{s})$ with the functions $u_j(\xi; \vec{x}, \vec{s})$ via a RH problem. The RH problem depends on parameters $x, s_1, \dots, s_k$ and $y_1 > y_2 > \dots > y_{k-1} > 0$, where y_j will be identified with $x_j - x_k$, and x with $\xi + x_k$. We first state an identity which expresses logarithmic derivatives of the Fredholm determinants F in terms of a RH problem.

Proposition 3.3.1. *Let $F(\vec{x}; \vec{s})$ be as in (3.1.1), with $\vec{s}$ and $\vec{x}$ as in Theorem 3.1.1. Define $\vec{y} = (y_1, \dots, y_{k-1})$ with $y_j = x_j - x_k > 0$. We have the differential identities*

$$\frac{\partial}{\partial x_j} \log F(\vec{x}; \vec{s}) = \frac{s_{j+1} - s_j}{2\pi i} \lim_{\zeta \rightarrow y_j} (\Psi^{-1} \Psi_{\zeta})_{2,1}(\zeta; x = x_k, \vec{y}, \vec{s}), \quad (3.3.1)$$

$j = 1, \dots, k$, with $\Psi(\zeta; x, \vec{y}, \vec{s})$ the unique solution of the RH problem below, which depends on parameters $x \in \mathbb{R}$, $y_{k-1} < y_{k-2} < \dots < y_1$ and $s_1, \dots, s_k$, and where Ψ_{ζ} is the derivative of Ψ with respect to ζ .

RH problem for Ψ .

Figure 3.1: Jump contours for the model RH problem for Ψ .

(a) $\Psi : \mathbb{C} \setminus \Gamma \rightarrow \mathbb{C}^{2 \times 2}$ is analytic, with

$$\Gamma = \mathbb{R} \cup e^{\pm \frac{2\pi i}{3}}(0, +\infty) \quad (3.3.2)$$

and Γ oriented as in Figure 3.1.

(b) $\Psi(\zeta)$ has continuous boundary values as $\zeta \in \Gamma \setminus \{y_1, \dots, y_k\}$ is approached from the left (+ side) or from the right (− side) and they are related by

$$\begin{cases} \Psi_+(\zeta) = \Psi_-(\zeta) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} & \text{for } \zeta \in e^{\pm \frac{2\pi i}{3}}(0, +\infty), \\ \Psi_+(\zeta) = \Psi_-(\zeta) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{for } \zeta \in (-\infty, 0), \\ \Psi_+(\zeta) = \Psi_-(\zeta) \begin{pmatrix} 1 & s_j \\ 0 & 1 \end{pmatrix} & \text{for } \zeta \in (y_j, y_{j-1}), j = 1, \dots, k, \end{cases}$$

where we write $y_k = 0$, $y_0 = +\infty$.

(c) As $\zeta \rightarrow \infty$, there exist functions $p = p(x; \vec{y}, \vec{s})$, $q = q(x; \vec{y}, \vec{s})$ and $r = r(x; \vec{y}, \vec{s})$ such that Ψ has the asymptotic behaviour

$$\Psi(\zeta) = \left(I + \frac{1}{\zeta} \begin{pmatrix} q & -ir \\ ip & -q \end{pmatrix} + \mathcal{O}\left(\frac{1}{\zeta^2}\right) \right)$$

$$\zeta^{\frac{1}{4}\sigma_3} M^{-1} e^{-(\frac{2}{3}\zeta^{3/2} + x\zeta^{1/2})\sigma_3}, \quad (3.3.3)$$

$$\text{where } M = (I + i\sigma_1)/\sqrt{2}, \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(d) $\Psi(\zeta) = \mathcal{O}(\log(\zeta - y_j))$ as $\zeta \rightarrow y_j$, $j = 1, \dots, k$.

When all the y -values are equal, i.e. when $k = 1$, then the RH problem for Ψ simply reduces to the Painlevé II RH problem. We proceed with the proof of Proposition 3.3.1, which relies on a well-known procedure developed by Its, Izergin, Korepin, and Slavnov ([IKS90], see also [DIZ97] in general and [BC12b, CIK10] for applications of this procedure in cases which are similar to ours).

Proof. Let K be the integral operator kernel

$$K(u, v) = \chi_{(x_k, +\infty)}(u) \sum_{j=1}^k (1 - s_j) K^{\text{AIRY}}(u, v) \chi_{(x_j, x_{j-1})}(v),$$

and $\mathcal{K}$ the operator it represents. The kernel K is integrable in the sense of Its, Izergin, Korepin, and Slavnov since it can be written in the form

$$K(u, v) = \frac{f^t(u)h(v)}{u - v}$$

with

$$\begin{aligned} f(z) &= \begin{pmatrix} \text{Ai}(z)\chi_{(x_k, +\infty)}(z) \\ \text{Ai}'(z)\chi_{(x_k, +\infty)}(z) \end{pmatrix}, \\ h(z) &= \begin{pmatrix} \sum_{j=1}^k (1 - s_j) \text{Ai}'(z)\chi_{(x_j, x_{j-1})}(z) \\ -\sum_{j=1}^k (1 - s_j) \text{Ai}(z)\chi_{(x_j, x_{j-1})}(z) \end{pmatrix}, \end{aligned} \quad (3.3.4)$$

so that $f^t(u)h(u) = 0$. By (1.1.4), and the fact that $s_j \geq 0$ for all $j = 1, \dots, k$, we have that the operator $I - \mathcal{K}$ is invertible. Using standard properties of trace-class operators, we have for any $j = 1, \dots, k$,

$$\frac{\partial}{\partial x_j} \ln \det(I - \mathcal{K}) = -\text{tr} \left((I - \mathcal{K})^{-1} \frac{\partial \mathcal{K}}{\partial x_j} \right) = \frac{s_{j+1} - s_j}{1 - s_j} R(x_j, x_j), \quad (3.3.5)$$

where R is the integral operator kernel of the resolvent $\mathcal{R}$ of $\mathcal{K}$, defined by the operator relation

$$I + \mathcal{R} = (I - \mathcal{K})^{-1},$$

and $R(x_j, x_j)$ has to be understood as the limit

$$R(x_j, x_j) = \lim_{x \searrow x_j} R(x, x).$$

Here we suppose that $s_j \neq 1$ for each $j = 1, \dots, k$. If $s_j = 1$, then the limit has to be taken from the left instead, and $1 - s_j$ has to be replaced by $1 - s_{j+1}$ which will be different from 0, since $s_j \neq s_{j+1}$.

If we define

$$Y(\zeta) = I - \int_{x_k}^{+\infty} \frac{F(\mu)h^t(\mu)}{\mu - \lambda} d\mu, \quad F = \begin{pmatrix} (1 - \mathcal{K})^{-1}f_1 \\ (1 - \mathcal{K})^{-1}f_2 \end{pmatrix},$$

then Y satisfies the following RH problem, see [IIKS90].

RH problem for Y

- (a) $Y : \mathbb{C} \setminus [x_k, +\infty) \rightarrow \mathbb{C}^{2 \times 2}$ is analytic
- (b) Y has continuous boundary values when $\zeta \in (x_k, +\infty) \setminus \{x_1, \dots, x_k\}$ is approached from the left (+ side) or right (− side) and both limits are related via

$$Y_+(\zeta) = Y_-(\zeta)J(\zeta), \quad J(\zeta) = 1 - 2\pi i f(\zeta)h^t(\zeta).$$

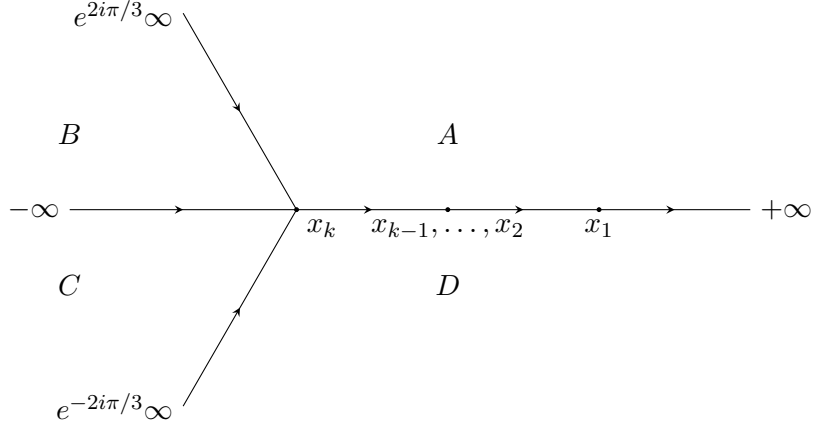
- (c) $Y(\zeta) = I + \mathcal{O}(1/\zeta)$, as $\zeta \rightarrow \infty$,
- (d) $Y(\zeta) = \mathcal{O}(\log(|\zeta - x_j|))$ as $\zeta \rightarrow x_j$, $j = 1, \dots, k$.

In terms of the solution Y , we have (see [DIZ97, Lemmas 2.8 and 2.12]),

$$R(u, v) = \frac{F^t(u)H(v)}{u - v}, \tag{3.3.6}$$

with

$$F(u) = Y_+(u)f(u), \quad H(v) = (Y_+^{-1}(v))^t h(v).$$

Figure 3.2: Regions for the definition of Φ .

In other words, the integral kernel R is also integrable, and can be expressed explicitly in terms of the unique solution to the above RH problem.

Consider the regions A , B , C and D in Figure 3.2 and define matrices $\Phi_A, \Phi_B, \Phi_C, \Phi_D$ in terms of the Airy function as follows,

$$\Phi_A(z) = \sqrt{2\pi}e^{-\pi i/12} \begin{pmatrix} \text{Ai}(z) & \text{Ai}(\omega^2 z) \\ \text{Ai}'(z) & \omega^2 \text{Ai}'(\omega^2 z) \end{pmatrix} e^{-\pi i/6\sigma_3}, \quad (3.3.7)$$

$$\Phi_D(z) = \sqrt{2\pi}e^{-\pi i/12} \begin{pmatrix} \text{Ai}(z) & -\omega^2 \text{Ai}(\omega z) \\ \text{Ai}'(z) & -\text{Ai}'(\omega z) \end{pmatrix} e^{-\pi i/6\sigma_3}, \quad (3.3.8)$$

$$\Phi_B(z) = \Phi_A(z) \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad (3.3.9)$$

$$\Phi_C(z) = \Phi_D(z) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad (3.3.10)$$

with $\omega = e^{\frac{2\pi i}{3}}$, and let

$$\Phi(z) = \Phi_*(z), \quad \text{for } z \text{ in region } * = A, B, C \text{ or } D. \quad (3.3.11)$$

Φ is the solution to the standard Airy model RH problem (see e.g.

[DKM⁺99a]): it satisfies the jump relations

$$\begin{aligned}\Phi_+(z) &= \Phi_-(z) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} && \text{for } z \in x_k + e^{\pm \frac{2\pi i}{3}}(0, +\infty), \\ \Phi_+(z) &= \Phi_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} && \text{for } z \in (-\infty, x_k), \\ \Phi_+(z) &= \Phi_-(z) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} && \text{for } z \in (x_k, +\infty),\end{aligned}$$

and it has the asymptotics

$$\Phi(z) = z^{-\sigma_3/4} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \left(I + \mathcal{O}\left(\frac{1}{z^{3/2}}\right) \right) e^{-\pi i \sigma_3/4} e^{-\frac{2}{3} z^{3/2} \sigma_3}, \quad (3.3.12)$$

as $z \rightarrow \infty$.

We define

$$\Psi(z) = \begin{pmatrix} 1 & i \frac{x_k^2}{4} \\ 0 & 1 \end{pmatrix} e^{\frac{i\pi}{4} \sigma_3} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} Y(z + x_k) \Phi(z + x_k),$$

and it is then straightforward, by the RH conditions for Φ , to verify that Ψ is the (unique) solution to the RH problem for Ψ stated before. $\square$

Remark 3.3.2. The existence of a solution to the RH problem for Ψ is a consequence of the above proof. Indeed, we started with a Fredholm determinant $\det(I - \mathcal{K})$ in (3.3.4) which is non-zero by (1.1.4), and the solutions to the RH problems for Y and Ψ are constructed explicitly in terms of the inverse of the operator $I - \mathcal{K}$, hence those RH problems are solvable. In more general situations, one cannot always rely on a probabilistic interpretation like (1.1.4). In such cases, an alternative is to use the vanishing lemma approach. This was done for instance in [XZ11, Lemma 1] for $k = 1$, and the proof generalizes in a straightforward way to the case of general k . Uniqueness of the solution follows from standard arguments (see, e.g., [Dei99]).

The next proposition relates the RH solution Ψ , and in particular the right hand side of the differential identities in Proposition 3.3.1, to the system of differential equations (3.1.3).

Proposition 3.3.3. *Let $\Psi(\zeta; x, \vec{y}, \vec{s})$ be the solution of the above RH problem, with $\vec{s}$ and $\vec{y}$ as in Proposition 3.3.1. If we define*

$$u_j^2(\xi; \vec{x}, \vec{s}) = -\frac{s_{j+1} - s_j}{2\pi} \lim_{\zeta \rightarrow y_j} \left(\Psi_{2,1}(\zeta; \xi + x_k, \vec{y}, \vec{s}) \right)^2, \quad j = 1, \dots, k \quad (3.3.13)$$

with the limit $\zeta \rightarrow y_j$ taken as ζ approaches y_j from region I , then

1. $\vec{u}(\xi; \vec{x}, \vec{s})$ satisfies the system of equations (3.1.3),

2. we have the identity

$$u_j^2(x - x_k; \vec{x}, \vec{s}) = -\frac{s_{j+1} - s_j}{2\pi i} \frac{\partial}{\partial x} (\Psi^{-1} \Psi_\zeta)_{2,1} (\zeta = y_j; x, \vec{y}, \vec{s}). \quad (3.3.14)$$

The proof of this result is given in the next subsection, and is based on Lax pair techniques.

3.3.2 Proof of Proposition 3.3.3

In this subsection, the vectors $\vec{y}$ and $\vec{s}$ will be considered as fixed parameters, whereas ζ and x will be variables. For the ease of notation, we omit the dependence on the parameters in our notations and write, for instance, $\Psi(\zeta; x)$ instead of $\Psi(\zeta; x, \vec{y}, \vec{s})$.

Derivation of the Lax pair. We first show that the solution $\Psi(\zeta; x)$ of the RH problem solves systems of linear differential equations in ζ and x . To that end, we first need to refine the asymptotic behavior of $\Psi(\zeta; x)$ for ζ near the singularities $0 = y_k < \dots < y_1$. Define, for $j = 1, \dots, k$, F_j by

$$\Psi(\zeta; x) = F_j(\zeta; x) \left(I + \frac{s_{j+1} - s_j}{2\pi i} \sigma_+ \log(\zeta - y_j) \right) W_j(\zeta), \quad (3.3.15)$$

for ζ near y_j , with $\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and with W_j given by

$$W_j(\zeta) = \begin{cases} I, & \text{if } \zeta \in \text{I}, \\ \begin{pmatrix} 1 & -s_j \\ 0 & 1 \end{pmatrix}, & \text{if } \zeta \in \text{IV}, \end{cases} \quad (3.3.16)$$

if $j < k$, and

$$W_k(\zeta) = \begin{cases} I, & \text{if } \zeta \in \text{I}, \\ \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, & \text{if } \zeta \in \text{II}, \\ \begin{pmatrix} 1-s_k & -s_k \\ 1 & 1 \end{pmatrix}, & \text{if } \zeta \in \text{III}, \\ \begin{pmatrix} 1 & -s_k \\ 0 & 1 \end{pmatrix}, & \text{if } \zeta \in \text{IV}, \end{cases} \quad (3.3.17)$$

where I, II, III and IV are sectors delimited by the jump contours of Ψ , as illustrated in Figure 3.1. Using the jump relations for Ψ , it is straightforward to verify that F_j is an analytic function of ζ near y_j . Indeed, the factors at the right of $F_j(\zeta; x)$ in (3.3.15) are designed precisely to model the correct jumps for Ψ .

In order to simplify the Lax pair, we left-multiply Ψ by a convenient upper-triangular matrix independent of ζ and define

$$\Phi(\zeta; x) = e^{\frac{1}{4}\pi i \sigma_3} \begin{pmatrix} 1 & -ip(x) \\ 0 & 1 \end{pmatrix} \Psi(\zeta; x), \quad (3.3.18)$$

where p comes from the asymptotic behavior of Ψ at ∞ , see (3.3.3). Since Φ has the same jump matrices as Ψ , piecewise constant in ζ and independent of x , it follows that the matrices A and B defined by

$$\begin{cases} \Phi_\zeta(\zeta; x) = A(\zeta; x)\Phi(\zeta; x), \\ \Phi_x(\zeta; x) = B(\zeta; x)\Phi(\zeta; x), \end{cases} \quad (3.3.19)$$

where the indices ζ and x denote derivatives with respect to ζ and x , are meromorphic functions of ζ in the whole complex plane, with possible poles only at the points $y_1, \dots, y_k$.

It follows from (3.3.15) that $\Psi_x(\zeta; x)\Psi^{-1}(\zeta; x)$ and $\Phi_x(\zeta; x)\Phi^{-1}(\zeta; x)$ have removable singularities at $y_1, \dots, y_k$, hence B is an entire function of ζ . From the asymptotics for Ψ at infinity, see (3.3.3) for the formula and definitions of p and q , we deduce that $\Psi_x\Psi^{-1}$ takes the form

$$\Psi_x(\zeta; x)\Psi^{-1}(\zeta; x) = -i\zeta\sigma_+ + \begin{pmatrix} -p(x) & -2iq(x) \\ i & p(x) \end{pmatrix}.$$

By (3.3.18), we get

$$B(\zeta; x) = \zeta\sigma_+ + \begin{pmatrix} 0 & 2q(x) + p^2(x) + p_x(x) \\ 1 & 0 \end{pmatrix}.$$

From the $1/\zeta$ -term in the large ζ expansion for B obtained using (3.3.3), we get in addition the identity

$$p_x(x) - 2q(x) - p^2(x) = 0. \quad (3.3.20)$$

We may thus write B as

$$B(\zeta; x) = \zeta\sigma_+ + \begin{pmatrix} 0 & 2p_x(x) \\ 1 & 0 \end{pmatrix}. \quad (3.3.21)$$

The form of the matrix $A(\zeta; x)$ is a bit more involved, as it follows from (3.3.15) that A has simple poles at the points $y_1, \dots, y_k$, because of the logarithmic singularities. Using the asymptotics (3.3.3) of Ψ at infinity, we get

$$\Psi_\zeta(\zeta; x)\Psi^{-1}(\zeta; x) = -i\zeta\sigma_+ + i \begin{pmatrix} ip(x) & -2q(x) - \frac{x}{2} \\ 1 & -ip(x) \end{pmatrix} + \mathcal{O}(\zeta^{-1}),$$

$\zeta \rightarrow \infty$. Therefore, using (3.3.18), we have that the matrix A has the form

$$A(\zeta; x) = \zeta\sigma_+ + \begin{pmatrix} 0 & 2q(x) + \frac{x}{2} + p^2(x) \\ 1 & 0 \end{pmatrix} + \sum_{j=1}^k \frac{1}{\zeta - y_j} A_j(x) \quad (3.3.22)$$

for matrices $A_1, \dots, A_k$ independent of ζ . Using (3.3.20) again, we have

$$A(\zeta; x) = \zeta \sigma_+ + \begin{pmatrix} 0 & p_x(x) + \frac{x}{2} \\ 1 & 0 \end{pmatrix} + \sum_{j=1}^k \frac{1}{\zeta - y_j} A_j(x). \quad (3.3.23)$$

Compatibility condition. The Lax pair (3.3.19) and the condition $\Phi_{x\zeta} = \Phi_{\zeta x}$ imply the compatibility condition

$$B_\zeta - A_x + [B, A] = 0. \quad (3.3.24)$$

This gives us several useful relations between p and the matrices A_j . Since the determinant of Φ is identically equal to 1, the trace of A is zero, and we can write

$$A(\zeta; x) = \begin{pmatrix} a(\zeta; x) & b(\zeta; x) \\ c(\zeta; x) & -a(\zeta; x) \end{pmatrix}. \quad (3.3.25)$$

Using this parametrization, the compatibility condition reads

$$\begin{cases} b = c(\zeta + 2p_x) - \frac{1}{2}c_{xx} \\ a = \frac{1}{2}c_x \\ b_x = 1 - 2a(\zeta + 2p_x). \end{cases} \quad (3.3.26)$$

We get from (3.3.23) that c is of the form

$$c(\zeta; x) = 1 + \sum_{j=1}^k \frac{c_j(x)}{\zeta - y_j}, \quad (3.3.27)$$

and that b is of the form

$$b(\zeta; x) = \zeta + \frac{2p_x(x) + x}{2} + \sum_{j=1}^k \frac{b_j(x)}{\zeta - y_j}.$$

We obtain from the first equation in (3.3.26), in particular from the constant term in the expansion as $\zeta \rightarrow \infty$, that

$$\sum_{j=0}^k c_j(x) = \frac{x}{2} - p_x(x). \quad (3.3.28)$$

We now expand $(\zeta - y_j)^2 \det(A(\zeta; x))$ for ζ near y_j , and we get from the constant term that

$$-\frac{1}{4} \frac{c_j'(x)^2}{c_j(x)^2} + \frac{1}{2} \frac{c_j''(x)}{c_j(x)} - y_j - 2p_x(x) = 0.$$

The fact that the right hand side is 0 follows from the asymptotics of Ψ as $\zeta \rightarrow y_j$, see (3.3.15). The relation (3.3.28) yields the following system of equations for $c_1, \dots, c_k$,

$$\begin{cases} -\frac{1}{4} \frac{(c_1')^2}{c_1^2} + \frac{1}{2} \frac{c_1''}{c_1} + 2 \sum_{j=1}^k c_j - x - y_1 = 0, \\ \vdots \\ -\frac{1}{4} \frac{(c_k')^2}{c_k^2} + \frac{1}{2} \frac{c_k''}{c_k} + 2 \sum_{j=1}^k c_j - x - y_k = 0. \end{cases} \quad (3.3.29)$$

We now make the transformation

$$c_j(\xi + x_k; \vec{y}, \vec{s}) = -u_j^2(\xi; \vec{x}, \vec{s}), \quad j = 1, \dots, k, \quad y_j = x_j - x_k, \quad (3.3.30)$$

and the system of equations for $c_1, \dots, c_k$ then transforms to the system (3.1.3) for $u_1, \dots, u_k$.

Expressions for u_j in terms of Ψ . Next, we want to relate the functions $u_1, \dots, u_k$ directly to the RH solution Ψ . By (3.3.15) and (3.3.18), we can expand Φ as $\zeta \rightarrow y_j$ in the following way,

$$\begin{aligned} \Phi(\zeta; x) &= E_{0,j}(x)(I + E_{1,j}(x)(\zeta - y_j) + \mathcal{O}((\zeta - y_j)^2)) \\ &\quad \times \left(I + \frac{s_{j+1} - s_j}{2\pi i} \sigma_+ \log(\zeta - y_j) \right) W_j(\zeta), \end{aligned}$$

with $E_{0,j}$ and $E_{1,j}$ depending on x but not on ζ .

From the definition of $c_j(\xi + x_k) = -u_j^2(\xi)$, see (3.3.19), (3.3.25), and (3.3.27), we also have the following expression for u_j in terms of Φ ,

$$u_j^2(x - x_k) = - \lim_{\zeta \rightarrow y_j} (\zeta - y_j) \left((\Phi_\zeta(\zeta; x)) \Phi^{-1}(\zeta; x) \right)_{2,1}. \quad (3.3.31)$$

This allows us to express u_j as

$$u_j^2(x - x_k) = \frac{s_{j+1} - s_j}{2\pi i} (E_{0,j})_{2,1}^2(x) = -\frac{s_{j+1} - s_j}{2\pi} \Psi_{2,1}^2(y_j; x), \quad (3.3.32)$$

where we used (3.3.18).

On the other hand, exploiting the constant and linear terms as $\zeta \rightarrow y_j$ in the equation $\Phi_x = B\Phi$, we find that

$$(E'_{1,j})_{2,1} = -(E_{0,j})_{2,1}^2,$$

where $'$ is the x -derivative. It can be verified that

$$(\Psi^{-1}\Psi_\zeta)_{2,1}(y_j; x) = (E_{1,j})_{2,1}(x),$$

and we finally get the following relation,

$$\frac{\partial}{\partial x}(\Psi^{-1}\Psi_\zeta)_{2,1}(y_j; x) = -(E_{0,j})_{2,1}^2(x) = -\frac{2\pi i}{s_{j+1} - s_j} u_j^2(x - x_k). \quad (3.3.33)$$

By (3.3.32) and (3.3.33), Proposition 3.3.3 is proved.

Symmetry relation. One verifies easily that $\sigma_3 \overline{\Psi(\bar{\zeta}; x)} \sigma_3$ satisfies the same RH conditions as Ψ . Since the solution to the RH problem for Ψ is unique, we get the relation

$$\sigma_3 \overline{\Psi(\bar{\zeta}; x)} \sigma_3 = \Psi(\zeta; x). \quad (3.3.34)$$

This implies that $\Psi_{2,1}(\zeta; x)$ is purely imaginary for real ζ , hence $u_j^2(x)$ is always real by (3.3.32). If $s_{j+1} > s_j$, we have that $u_j(x)$ is real; if $s_{j+1} < s_j$, $u_j(x)$ is purely imaginary.

3.3.3 Proof of (3.1.2)

Define

$$\tilde{F}(x_k; \vec{y}, \vec{s}) = F(\vec{x}; \vec{s}), \quad y_j = x_j - x_k. \quad (3.3.35)$$

As $x_k \rightarrow \infty$ with $\vec{y}, \vec{s}$ fixed, $\tilde{F}(x; \vec{y}, \vec{s})$ converges to 1, and is increasing. Therefore, by Proposition 3.3.1, we have

$$\begin{aligned}
\ln \tilde{F}(\tilde{x}_k; \vec{y}, \vec{s}) &= - \int_{\tilde{x}_k}^{+\infty} \frac{\partial}{\partial x_k} \ln \tilde{F}(x_k; \vec{y}, \vec{s}) dx_k = \\
&= - \int_{\tilde{x}_k}^{+\infty} \sum_{j=1}^k \frac{\partial}{\partial x_j} \ln F(\vec{x}, \vec{s}) dx_k = \\
&= - \int_{\tilde{x}_k}^{+\infty} \left(\sum_{j=1}^k \frac{s_{j+1} - s_j}{2\pi i} (\Psi^{-1} \Psi_\zeta)_{2,1}(\zeta = y_j; x, \vec{y}, \vec{s}) \right) dx. \quad (3.3.36)
\end{aligned}$$

We now use Proposition 3.3.3 to integrate by parts, and obtain

$$\begin{aligned}
\ln F(\vec{x}; \vec{s}) &= \ln \tilde{F}(x_k; \vec{y}, \vec{s}) = - \int_{x_k}^{+\infty} (\xi - x_k) \sum_{j=1}^k u_j^2(\xi - x_k; \vec{x}, \vec{s}) d\xi \\
&= - \int_0^{+\infty} \xi \sum_{j=1}^k u_j^2(\xi; \vec{x}, \vec{s}) d\xi, \quad (3.3.37)
\end{aligned}$$

and we get formula (3.1.2). To complete the proof of Theorem 3.1.1, it only remains to prove the asymptotics (3.1.4).

3.4 Asymptotic analysis of the RHP for Ψ

In this section, we study the RH problem for Ψ , which depends on x , $\vec{y}$, $\vec{s}$, asymptotically. There are three asymptotic regimes that we will study:

1. $x \rightarrow +\infty$ while keeping $\vec{y}$, $\vec{s}$ fixed,
2. $x_j - x_{j+1} \rightarrow 0$ with all other parameters fixed, for $j = 0, \dots, k-1$, where we set $x_0 = +\infty$,
3. $s_j \rightarrow s_{j+1}$ with all other parameters fixed, for $j = 1, \dots, k$, with $s_{k+1} = 1$.

Each of these cases will require a separate analysis of the RH problem for Ψ , using Deift-Zhou steepest descent techniques [DZ92]. The RH analysis in the first case $x \rightarrow +\infty$ requires several transformations and the construction of parametrices; the other cases are more straightforward. The asymptotic analysis of Ψ will enable us to understand the

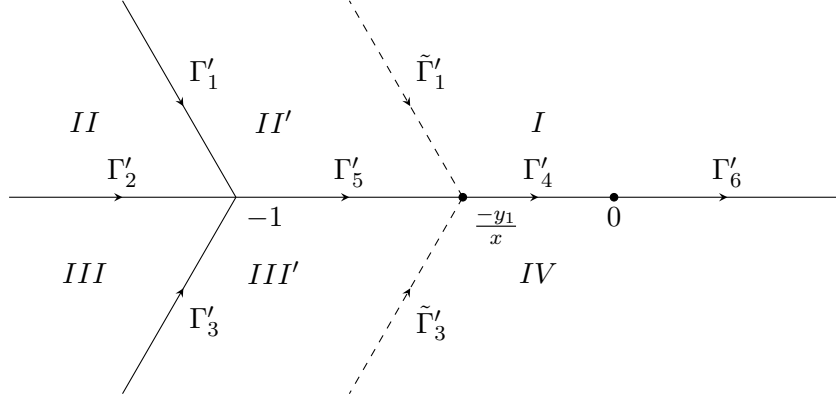


Figure 3.3: The shifted jump contour Γ' for C (solid lines) compared to the contour Γ for B (dashed lines). The dashed lines $\tilde{\Gamma}'_1$ and $\tilde{\Gamma}'_3$ meet at $\frac{-y_1}{x}$. The rays Γ'_1 and $\tilde{\Gamma}'_1$ both have an angle $\frac{2\pi}{3}$ with respect to the real line. The region II' (resp. III') is delimited by $\Gamma'_1, \tilde{\Gamma}'_1, \Gamma'_5$ (resp. $\Gamma'_3, \tilde{\Gamma}'_3, \Gamma'_5$).

asymptotic behavior of the solutions u_j to the system (3.1.3), and to prove (3.1.4) and Theorem 3.1.7.

3.4.1 Asymptotic analysis as $x \rightarrow +\infty$

We will perform a series of invertible transformations $\Psi \mapsto A \mapsto B \mapsto C \mapsto D$ in order to obtain a matrix D close to the identity matrix.

Re-scaling and shift of the contour

The first transformation $\Psi \mapsto A$ consists of a re-scaling of the ζ -variable: we define

$$A(\lambda; x, \vec{y}, \vec{s}) = x^{-\frac{\sigma_3}{4}} \Psi(x\lambda; x, \vec{y}, \vec{s}). \quad (3.4.1)$$

The second transformation $A \mapsto B$ shifts the two non-real parts of the jump contour to the left, in order to separate them from the discontinuities in the jump matrices which are now located at $\lambda_j := y_j/x$, $j = 1, \dots, k$, with $\lambda_k = y_k = 0$. To do this, we need to continue A analytically from sectors I, IV to regions II' , III' , see Figure 3.3. The region II' is the region enclosed respectively by Γ'_1, Γ'_5 and $\tilde{\Gamma}'_1$, the region III' being the symmetric of II' with respect to the real line.

More precisely, we define the matrix B by

$$B(\lambda) = B(\lambda; x, \vec{y}, \vec{s}) := A(\lambda; x, \vec{y}, \vec{s}) \begin{cases} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & \text{for } \lambda \in \text{II}', \\ \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, & \text{for } \lambda \in \text{III}', \text{ and} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \text{everywhere else.} \end{cases} \quad (3.4.2)$$

The function B satisfies the following RH problem for B .

RH problem for B .

- (a) $B : \mathbb{C} \setminus \Gamma' \rightarrow \mathbb{C}^{2 \times 2}$ is analytic, with $\Gamma' = \mathbb{R} \cup \left(-1 + e^{\pm \frac{2\pi i}{3}} \mathbb{R}^+\right)$,
- (b) B has continuous boundary values on $\Gamma' \setminus \{-1, \lambda_1, \dots, \lambda_k\}$ with $\lambda_j = y_j/x$, and $B_+(\lambda) = B_-(\lambda)J_B(\lambda)$, where J_B is given by

$$J_B(\lambda) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} & \text{on } -1 + e^{\pm \frac{2\pi i}{3}} \mathbb{R}^+, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{on } (-\infty, -1), \\ \begin{pmatrix} 1 & s_j \\ 0 & 1 \end{pmatrix} & \text{on } (\lambda_j, \lambda_{j-1}), \ j = 1, \dots, k, \ \lambda_0 = +\infty, \\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \text{on } (-1, 0), \end{cases}$$

- (c) B has the following asymptotics:

$$B(\lambda) = (I + \mathcal{O}(\lambda^{-1})) \lambda^{\sigma_3/4} M^{-1} e^{-x^{3/2}(\frac{2}{3}\lambda^{3/2} + \lambda^{1/2})}, \text{ as } \lambda \rightarrow \infty,$$

- (d) B has the following behavior at λ_j for $j = 1, \dots, k$:

$$B(\lambda) = \widehat{F}_j(\lambda) \left(I + \frac{s_{j+1} - s_j}{2\pi i} \sigma_+ \log(\lambda - \lambda_j) \right) \widehat{W}_j(\lambda), \text{ as } \lambda \rightarrow \lambda_j,$$

where $\widehat{F}_j$ is analytic in a neighborhood of λ_j , $\widehat{W}_j = I$ for λ in

the upper half plane, and $\widehat{W}_j = \begin{pmatrix} 1 & -s_j \\ 0 & 1 \end{pmatrix}$ for λ in the lower half plane, similarly to (3.3.15).

Normalization at infinity

The goal of the next transformation is to normalize the RH problem at infinity, in such a way that the jump matrices behave nicely. To that end, we define

$$g(\lambda) = \frac{2}{3}(\lambda + 1)^{3/2}, \quad (3.4.3)$$

with branch cut on $(-\infty, -1]$, such that $g_-(\lambda) = -g_+(\lambda)$ on $(-\infty, -1)$. At infinity, g behaves as

$$g(\lambda) = \frac{2}{3}\lambda^{3/2} + \lambda^{1/2} + \frac{1}{4}\lambda^{-1/2} + \mathcal{O}(\lambda^{-3/2}), \text{ as } \lambda \rightarrow \infty, \quad (3.4.4)$$

with all branch cuts at the right hand side on $(-\infty, 0]$. We now perform the transformation $B \mapsto C$ by defining

$$C(\lambda) = \begin{pmatrix} 1 & -\frac{ix^{3/2}}{4} \\ 0 & 1 \end{pmatrix} B(\lambda) e^{x^{3/2}g(\lambda)\sigma_3}. \quad (3.4.5)$$

The constant upper-triangular pre-factor in the definition of C is needed to ensure that C has a suitable behavior at infinity. C satisfies the following RH problem.

RH problem for C .

(a) $C : \mathbb{C} \setminus \Gamma' \rightarrow \mathbb{C}^{2 \times 2}$ is analytic,

(b) $C_+ = C_- J_C$ on $\Gamma' \setminus \{-1, \lambda_1, \dots, \lambda_k\}$ with the jump matrices J_C given by

$$J_C(\lambda) = e^{-x^{3/2}g_-(\lambda)\sigma_3} J_B(z) e^{x^{3/2}g_+(\lambda)\sigma_3}, \quad (3.4.6)$$

i.e.

$$J_C(\lambda) = \begin{cases} \begin{pmatrix} 1 & 0 \\ e^{2x^{3/2}g(\lambda)} & 1 \end{pmatrix} & \text{on } -1 + e^{\pm \frac{2\pi i}{3}} \mathbb{R}^+, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{on } (-\infty, -1), \\ \begin{pmatrix} 1 & e^{-2x^{3/2}g(\lambda)} \\ 0 & 1 \end{pmatrix} & \text{on } (-1, 0), \\ \begin{pmatrix} 1 & s_j e^{-2x^{3/2}g(\lambda)} \\ 0 & 1 \end{pmatrix} & \text{on } (\lambda_j, \lambda_{j-1}), \quad j = 1, \dots, k, \end{cases}$$

(c) C has the following asymptotics at ∞ :

$$C(\lambda) = (I + \mathcal{O}(\lambda^{-1})) \lambda^{\sigma_3/4} M^{-1}, \text{ as } \lambda \rightarrow \infty,$$

(d) C has the same logarithmic behavior near λ_j as B has, multiplied to the right by $e^{x^{3/2}g(\lambda)\sigma_3}$.

We now need to construct a global parametrix and local parametrices near the special points -1 and 0 .

Global parametrix

Away from $-1, 0$, all jumps of C except the one on $(-\infty, -1)$ tend to the identity matrix as $x \rightarrow +\infty$. Ignoring small jumps and small neighbourhoods of $-1, 0$, we get the following.

RH problem for $C^{(\infty)}$.

(a) $C^{(\infty)}$ is analytic everywhere in the complex plane except on $(-\infty, -1]$,

(b) $C_+^{(\infty)} = C_-^{(\infty)} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on $(-\infty, -1)$,

(c) $C^{(\infty)}$ has the following asymptotics:

$$C^{(\infty)}(\lambda) = (I + \mathcal{O}(\lambda^{-1})) \lambda^{\sigma_3/4} M^{-1}, \text{ as } \lambda \rightarrow \infty.$$

A simple solution to this problem is given by

$$C^{(\infty)}(\lambda) = (\lambda + 1)^{\sigma_3/4} M^{-1}, \quad (3.4.7)$$

with the branch cut of $(\lambda + 1)^{\sigma_3/4}$ along $(-\infty, -1]$.

Local parametrix near -1

Near -1 , the RH problem for C resembles the standard model RH problem built out of the Airy function, and we will use this model problem to build the local parametrix. This construction is fairly standard, and was also used in the proof of Proposition 3.3.1, see (3.3.7)–(3.3.12). We will construct the local parametrix in the form

$$C^{(-1)}(\lambda) = E(\lambda) \widehat{\Phi}(x^{3/2}(\lambda + 1)) e^{x^{3/2}g(\lambda)\sigma_3}, \quad (3.4.8)$$

where E is analytic around -1 and is chosen below, and $\widehat{\Phi}$ is, analogously to Φ defined in (3.3.11) but with a slightly different jump contour, the solution to the standard Airy model RH problem, given by

$$\widehat{\Phi}(z) = \begin{cases} \Phi_A(z), & \text{for } z \text{ in region I,} \\ \Phi_B(z), & \text{for } z \text{ in region II,} \\ \Phi_C(z), & \text{for } z \text{ in region III,} \\ \Phi_D(z), & \text{for } z \text{ in region IV,} \end{cases} \quad (3.4.9)$$

where the regions I, II, III and IV are the ones described in Figure 3.1. The asymptotics for $\widehat{\Phi}$ as $z \rightarrow \infty$ are the same as the ones for Φ given in (3.3.12).

Let $0 < \delta < 1/2$, $U_{-1,\delta}$ the disk of radius δ around -1 , and define $C^{(-1)} : U_{-1,\delta} \setminus \Gamma' \rightarrow \mathbb{C}^{2 \times 2}$ by

$$C^{(-1)}(\lambda) = e^{\frac{i\pi}{4}\sigma_3} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (x^{3/2})^{\sigma_3/4} \widehat{\Phi}(x(\lambda + 1)) e^{x^{3/2}g(\lambda)\sigma_3}.$$

On the boundary of $U_{-1,\delta}$, we have

$$C^{(-1)}(\lambda) = (I + \mathcal{O}(x^{-1})) C^{(\infty)}(\lambda), \quad (3.4.10)$$

uniformly for $\lambda \in \partial U_{-1,\delta}$ as $x \rightarrow +\infty$. Moreover, $C^{(-1)}$ has the same jumps as C on $\Gamma' \cap U_{-1,\delta}$.

Local parametrix near 0

When x gets large and the values y_j remain fixed, then the discontinuities λ_j are all close to 0. We can construct a local parametrix near 0 which models the jumps near 0 in an elementary way. Consider the function v which takes the value s_j on $(\lambda_j, \lambda_{j-1})$ for $j = 1, \dots, k+1$, with the convention that $\lambda_0 = +\infty$, $\lambda_{k+1} = -1/2$ and $s_{k+1} = 1$. Next we define the matrix-valued function $C^{(\text{loc})}$ by

$$C^{(\text{loc})}(\lambda) = \begin{pmatrix} 1 & \int_{-\frac{1}{2}}^{+\infty} \frac{v(w)}{w-\lambda} e^{-2x^{3/2}g(w)} dw \\ 0 & 1 \end{pmatrix}. \quad (3.4.11)$$

This function has the same jumps as C across the real axis, but does not have the correct matching with the parametrix at infinity $C^{(\infty)}$ on the boundary of $U_{0,\delta}$. The matrix $C^{(\text{loc})}$ is however exponentially close to the identity matrix on this boundary, and we may therefore construct the local parametrix near 0 by defining

$$C^{(0)}(\lambda) = C^{(\infty)}(\lambda)C^{(\text{loc})}(\lambda), \quad \lambda \in U_{0,\delta}. \quad (3.4.12)$$

In this way, since $C^{(\infty)}$ is analytic near 0, $C^{(0)}$ has the same jumps as S has near 0 and it matches with $C^{(\infty)}$ on the boundary of $U_{0,\delta}$,

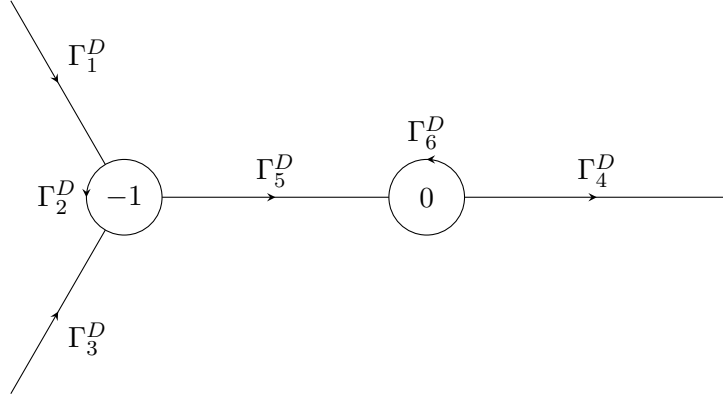
$$C^{(-1)}(\lambda) = (I + \mathcal{O}(x^{-1}))C^{(\infty)}(\lambda), \quad x \rightarrow +\infty, \quad (3.4.13)$$

uniformly for λ on the boundary of $U_{0,\delta}$.

Last transformation $C \mapsto D$

We take $0 < \delta < 1/2$ and x large enough, such that $\lambda_j < \delta$ for all j . We define the matrix-valued function D by

$$D(\lambda) = \begin{cases} C(\lambda)C^{(-1)}(\lambda)^{-1} & \text{for } \lambda \in U_{-1,\delta}, \\ C(\lambda)C^{(0)}(\lambda)^{-1} & \text{for } \lambda \in U_{0,\delta}, \\ C(\lambda)C^{(\infty)}(\lambda)^{-1} & \text{otherwise.} \end{cases} \quad (3.4.14)$$

Figure 3.4: Jump contours for D

The matrix D satisfies the following RH problem. The contours $\Gamma_1^D, \dots, \Gamma_6^D$ are depicted on Figure 3.4.

RH problem for D .

- (a) $D : \mathbb{C} \setminus \cup_{j=1}^6 \Gamma_j^D \rightarrow \mathbb{C}^{2 \times 2}$ is analytic, with Γ_j^D , $j = 1, \dots, 6$, as in Figure 3.4,
- (b) D has the jumps $D_+ = D_- J_D$ on $\cup_{j=1}^6 \Gamma_j^D$, with

$$J_D(\lambda) = \begin{pmatrix} 1 & 0 \\ e^{2x^{3/2}g(\lambda)} & 1 \end{pmatrix} \quad \text{on } \Gamma_1^D \text{ and } \Gamma_3^D, \quad (3.4.15)$$

$$J_D(\lambda) = C^{(\infty)}(\lambda) C^{(-1)}(\lambda)^{-1} \quad \text{on } \Gamma_2^D, \quad (3.4.16)$$

$$J_D(\lambda) = \begin{pmatrix} 1 & s_k e^{-2x^{3/2}g(\lambda)} \\ 0 & 1 \end{pmatrix} \quad \text{on } \Gamma_4^D, \quad (3.4.17)$$

$$J_D(\lambda) = \begin{pmatrix} 1 & e^{-2x^{3/2}g(\lambda)} \\ 0 & 1 \end{pmatrix} \quad \text{on } \Gamma_5^D, \quad (3.4.18)$$

$$J_D(\lambda) = C^{(\infty)}(\lambda) C^{(\text{loc})}(\lambda)^{-1} C^{(\infty)}(\lambda)^{-1} \quad \text{on } \Gamma_6^D, \quad (3.4.19)$$

- (c) D has the following asymptotic behavior at infinity:

$$D(\lambda) = I + \mathcal{O}(\lambda^{-1}), \quad \text{as } \lambda \rightarrow \infty.$$

Since $g(\lambda) < 0$ for $\lambda \in \Gamma_1^D \cup \Gamma_3^D$, and $g(\lambda) > 0$ for $\lambda \in \Gamma_4^D \cup \Gamma_5^D$, we have that all the jumps of D are $I + \mathcal{O}(x^{-1}(\lambda_k + 1)^{-1})$, uniformly for λ on the jump contours, as $x(\lambda_k + 1) \rightarrow +\infty$. This implies, using usual small-norm arguments for RH problems (see, e.g., [Dei99]) that the function D satisfies

$$D(\lambda) = I + \mathcal{O}(x^{-1}), \quad \text{as } x \rightarrow +\infty, \quad (3.4.20)$$

uniformly for $\lambda \in \mathbb{C} \setminus \bigcup_{j=1}^6 \Gamma_j^D$.

Asymptotic behavior of $u_0, \dots, u_k$.

By inverting the transformations $\Psi \mapsto A \mapsto B \mapsto C \mapsto D$, the large x asymptotics (3.4.20) for D enable us to find large x asymptotics for Ψ in terms of the parametrices $C^{(\infty)}$, $C^{(-1)}$, and $C^{(0)}$. Moreover, using the identity (3.3.13), we can express the asymptotics of u_j , $j = 1, \dots, k$ in terms of the parametrices.

Using (3.4.1) and (3.4.2), we have

$$\begin{aligned} u_j^2(x - x_k; \vec{x}, \vec{s}) &= -\frac{s_{j+1} - s_j}{2\pi} \lim_{\zeta \rightarrow y_j} \Psi_{2,1}^2(\zeta; x, \vec{y}, \vec{s}) = \\ &= -\frac{s_{j+1} - s_j}{2\pi\sqrt{x}} \lim_{\lambda \rightarrow \lambda_j} A_{2,1}^2(\lambda; x, \vec{y}, \vec{s}) = \\ &= -\frac{s_{j+1} - s_j}{2\pi\sqrt{x}} \lim_{\lambda \rightarrow \lambda_j} B_{2,1}^2(\lambda; x, \vec{y}, \vec{s}), \end{aligned} \quad (3.4.21)$$

where we adopt the convention that the limits $\lambda \rightarrow \lambda_j$ are taken from region I, see Figure 3.3. By (3.4.3), (3.4.5), and (3.4.14),

$$\begin{aligned} u_j^2(x - x_k; \vec{x}, \vec{s}) &= -\frac{s_{j+1} - s_j}{2\pi\sqrt{x}} \lim_{\lambda \rightarrow \lambda_j} C_{2,1}^2(\lambda; x, \vec{y}, \vec{s}) e^{-\frac{4}{3}(y_j+x)^{3/2}} \\ &= -\frac{s_{j+1} - s_j}{2\pi\sqrt{x}} \lim_{\lambda \rightarrow \lambda_j} \left(DC^{(0)} \right)_{2,1}^2(\lambda; x, \vec{y}, \vec{s}) e^{-\frac{4}{3}(y_j+x)^{3/2}}. \end{aligned} \quad (3.4.22)$$

Now we use the asymptotics (3.4.20) and the expression (3.4.12) (see also (3.4.7) and (3.4.11)) for the local parametrix to obtain

$$u_j^2(x - x_k; \vec{x}, \vec{s}) =$$

$$\begin{aligned}
& -\frac{s_{j+1}-s_j}{2\pi\sqrt{x}} \left(D_{2,1}(\lambda_j)C_{1,1,+}^{(0)}(\lambda_j) + D_{2,2}(\lambda_j)C_{2,1,+}^{(0)}(\lambda_j) \right) e^{-\frac{4}{3}(y_j+x)^{3/2}} \\
& = \frac{s_{j+1}-s_j}{4\pi\sqrt{y_j+x}} e^{-\frac{4}{3}(y_j+x)^{3/2}} (1 + \mathcal{O}(x^{-1})), \quad x \rightarrow +\infty. \quad (3.4.23)
\end{aligned}$$

Using the fact that $y_j = x_j - x_k$ and the standard large argument asymptotics for the Airy function, we obtain (3.1.4) as $x \rightarrow +\infty$.

3.4.2 Asymptotic analysis as $x_j \rightarrow x_{j-1}$

Fix $j \in \{2, \dots, k\}$, and denote $\vec{y}^{[j]}, \vec{s}^{[j]}$ for the vectors $\vec{y}, \vec{s}$ without their j -th components. We also define

$$\Psi^{[j]}(\zeta; x, \vec{y}, \vec{s}) = \Psi(\zeta; x, \vec{y}^{[j]}, \vec{s}^{[j]}), \quad (3.4.24)$$

i.e., $\Psi^{[j]}$ is the solution to the model RH problem for Ψ with $k-1$ singularities located at $y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_k = 0$. The RH conditions for Ψ and $\Psi^{[j]}$ are the same, except on $[x_j, x_{j-1}]$. Therefore, it is not surprising that if x_j and x_{j-1} are close to each other, the function $\Psi^{[j]}(\zeta; x, \vec{y}, \vec{s})$ will be a good approximation for $\Psi(\zeta; x, \vec{y}, \vec{s})$ as $x_j \rightarrow x_{j-1}$ for ζ not too close to x_j, x_{j-1} .

Near x_j, x_{j-1} , we need to build a local parametrix, for which we need to define (similarly as in (3.3.15)) the function $F_{j-1}^{[j]}(\zeta)$ by the equation

$$\Psi^{[j]}(\zeta; x, \vec{y}, \vec{s}) = F_{j-1}^{[j]}(\zeta) \left(I + \frac{s_{j+1}-s_{j-1}}{2\pi i} \sigma_+ \log(\zeta - y_{j-1}) \right) W_{j-1}(\zeta), \quad (3.4.25)$$

for ζ near y_{j-1} , with W_{j-1} given as in (3.3.16)–(3.3.17), and with principal branches of the logarithms. Then, one verifies, in a similar way as we did for (3.3.15), that $F_{j-1}^{[j]}$ is analytic at y_{j-1} . We now define the local parametrix P in the form

$$\begin{aligned}
P(\zeta; x, \vec{y}, \vec{s}) &= F_{j-1}^{[j]}(\zeta) \\
&\left(I + \frac{s_{j+1}-s_j}{2\pi i} \sigma_+ \log(\zeta - y_j) + \frac{s_j-s_{j-1}}{2\pi i} \sigma_+ \log(\zeta - y_{j-1}) \right) W_{j-1}(\zeta), \quad (3.4.26)
\end{aligned}$$

for ζ in a fixed sufficiently small open set U which contains y_j, y_{j-1} but

which does not contain any of the other singularities y_ℓ (recall that, in this regime, $y_j \rightarrow y_{j-1}$, while all the other y -values are fixed). It is then straightforward to check that P has exactly the same jump relations as Ψ has inside U , and by (3.4.25) and (3.4.26), we get the matching condition

$$P(\zeta; x, \vec{y}, \vec{s}) \Psi^{[j]}(\zeta; x, \vec{y}, \vec{s})^{-1} = I + \mathcal{O}(x_{j-1} - x_j), \quad x_j \rightarrow x_{j-1}, \quad (3.4.27)$$

uniformly for $\zeta \in \partial U$.

Define

$$R(\zeta) = \begin{cases} \Psi(\zeta; x, \vec{x}, \vec{s}) \Psi^{[j]}(\zeta; x, \vec{y}, \vec{s})^{-1}, & \zeta \in \mathbb{C} \setminus U, \\ \Psi(\zeta; x, \vec{x}, \vec{s}) P(\zeta; x, \vec{y}, \vec{s})^{-1}, & \zeta \in U. \end{cases} \quad (3.4.28)$$

Then R is analytic everywhere in the complex plane, except on ∂U , where we have $R_+ = R_- (I + \mathcal{O}(x_{j-1} - x_j))$ as $x_j \rightarrow x_{j-1}$ by (3.4.27). As $\zeta \rightarrow \infty$, $R(\zeta) \rightarrow I$ because Ψ and $\Psi^{[j]}$ have the same asymptotics (3.3.3). It follows again from usual small-norm arguments for RH problems that

$$R(\zeta) = I + \mathcal{O}(x_{j-1} - x_j), \quad x_j \rightarrow x_{j-1}, \quad (3.4.29)$$

uniformly for $\zeta \in \mathbb{C} \setminus \partial U$. Hence, as $x_j \rightarrow x_{j-1}$, we have for $\ell < j - 1$,

$$\begin{aligned} u_\ell^2(x - x_k; \vec{x}, \vec{s}) &= -\frac{s_{\ell+1} - s_\ell}{2\pi} \lim_{\zeta \rightarrow y_\ell} \Psi_{2,1}(\zeta; x, \vec{y}, \vec{s})^2 \\ &= -\frac{s_{\ell+1} - s_\ell}{2\pi} \lim_{\zeta \rightarrow \zeta_\ell} \Psi_{2,1}^{[j]}(\zeta; x, \vec{y}, \vec{s})^2 + \mathcal{O}(x_{j-1} - x_j) \\ &= u_\ell^2(x - x_k; \vec{x}^{[j]}, \vec{s}^{[j]}) + \mathcal{O}(x_{j-1} - x_j), \end{aligned}$$

and similarly for $\ell > j$,

$$u_\ell^2(x - x_k; \vec{x}, \vec{s}) = u_{\ell-1}^2(x - x_k; \vec{x}^{[j]}, \vec{s}^{[j]}) + \mathcal{O}(x_{j-1} - x_j).$$

For $\ell = j, j - 1$, we have

$$u_\ell^2(x - x_k; \vec{x}, \vec{s}) = -\frac{s_{\ell+1} - s_\ell}{2\pi} \lim_{\zeta \rightarrow y_\ell} \Psi_{2,1}(\zeta; x, \vec{y}, \vec{s})^2$$

$$\begin{aligned}
&= -\frac{s_{\ell+1} - s_\ell}{2\pi} \lim_{\zeta \rightarrow y_\ell} P_{2,1}(\zeta; x, \vec{y}, \vec{s})^2 + \mathcal{O}(x_{j-1} - x_j) \\
&= -\frac{s_{\ell+1} - s_\ell}{2\pi} F_{j-1}^{[j]}(y_\ell)_{2,1}^2 + \mathcal{O}(x_{j-1} - x_j),
\end{aligned}$$

which implies that

$$\begin{aligned}
u_{j-1}^2(x - x_k; \vec{x}, \vec{s}) + u_j^2(x - x_k; \vec{x}, \vec{s}) &= \\
&= -\frac{s_{j+1} - s_{j-1}}{2\pi} F_{j-1}^{[j]}(y_j)_{2,1}^2 + \mathcal{O}(x_{j-1} - x_j) = \\
&= u_j^2(x - x_k; \vec{x}^{[j]}, \vec{s}^{[j]}) + \mathcal{O}(x_{j-1} - x_j)
\end{aligned}$$

as $x_j \rightarrow x_{j-1}$. This proves Theorem 3.1.7, part 2.

Part 3 of Theorem 3.1.7 can be seen as a special case of the previous with $j = 1$, if one identifies x_0 with $+\infty$. Then, U has to be chosen as a neighborhood of $[y_1, +\infty)$, and the parametrix P in U has to be defined in a slightly different manner as

$$\begin{aligned}
P(\zeta; x, \vec{y}, \vec{s}) &= \\
\Psi^{[1]}(\zeta; x, \vec{y}, \vec{s}) W_2(\zeta)^{-1} \left(I + \frac{s_2 - s_1}{2\pi i} \sigma_+ \log(\zeta - y_1) \right) W_2(\zeta), \quad (3.4.30)
\end{aligned}$$

with W_2 as in (3.3.16)–(3.3.17), and with the global parametrix still given by $\Psi^{[1]}$. R is defined in the same way as before, and has a jump $I + \mathcal{O}(x_1^{-1})$ on the boundary of U as $x_1 \rightarrow +\infty$. A straightforward calculation similar to the one for $j > 1$ now leads to the proof of part 3 of Theorem 3.1.7.

3.4.3 Asymptotic analysis of the model RH problem for Ψ as $s_{j+1} - s_j \rightarrow 0$

Fix $j \in \{1, \dots, k\}$, and define as before

$$\Psi^{[j]}(\zeta; x, \vec{y}, \vec{s}) = \Psi(\zeta; x, \vec{y}^{[j]}, \vec{s}^{[j]}), \quad (3.4.31)$$

with $\vec{y}^{[j]}, \vec{s}^{[j]}$ the vectors $\vec{y}, \vec{s}$ without their j -th components. This function will again serve as the global parametrix for Ψ as $s_{j+1} - s_j \rightarrow 0$. The local parametrix in an open set U containing both y_j and y_{j-1} , but

no other singularities, is given by (3.4.26) if $j \neq 1$, and by (3.4.30) if $j = 1$, exactly as in the analysis for $x_j \rightarrow x_{j-1}$.

We define R as in (3.4.28), and R is analytic everywhere in the complex plane, except on ∂U , where we have $R_+ = R_- (I + \mathcal{O}(s_j - s_{j+1}))$ as $s_{j+1} - s_j \rightarrow 0$ by (3.4.26). As $\zeta \rightarrow \infty$, $R(\zeta) \rightarrow I$ because Ψ and $\Psi^{[j]}$ have the same asymptotics (3.3.3). It follows that

$$R(\zeta) = I + \mathcal{O}(s_{j+1} - s_j), \quad s_{j+1} - s_j \rightarrow 0. \quad (3.4.32)$$

It follows that, for $\ell < j$,

$$\begin{aligned} u_\ell^2(x - x_k; \vec{x}, \vec{s}) &= -\frac{s_{\ell+1} - s_\ell}{2\pi} \lim_{\zeta \rightarrow y_\ell} \Psi_{2,1}(\zeta; x, \vec{y}, \vec{s})^2 \\ &= -\frac{s_{\ell+1} - s_\ell}{2\pi} \lim_{\zeta \rightarrow y_\ell} \Psi_{2,1}^{[j]}(\zeta; x, \vec{y}, \vec{s})^2 + \mathcal{O}(s_{j+1} - s_j) \\ &= u_\ell^2(x - x_k; \vec{x}^{[j]}, \vec{s}^{[j]}) + \mathcal{O}(s_{j+1} - s_j), \end{aligned}$$

as $s_{j+1} - s_j \rightarrow 0$, and a similar calculation applies to the case $\ell > j$. For $\ell = j$, we have

$$u_j^2(x - x_k; \vec{x}, \vec{s}) = -\frac{s_{j+1} - s_j}{2\pi} \lim_{\zeta \rightarrow y_j} \Psi_{2,1}(\zeta; x, \vec{y}, \vec{s})^2 = \mathcal{O}(s_{j+1} - s_j),$$

as $s_{j+1} - s_j \rightarrow 0$. This completes the proof of part 1 of Theorem 3.1.7.

Chapter 4

The generating function for the Bessel point process and a system of coupled Painlevé V equations

This chapter is based on the eponymous paper with Christophe Charlier [CD17a].

4.1 Introduction

The Bessel point process is a determinantal point process on $\mathbb{R}^+$ arising as a limit point process of a wide range of mathematical models in random matrix theory [For93, FN95]. A celebrated toy example is the behaviour near 0 of the squared singular values of Ginibre matrices, also known as the Laguerre Unitary Ensemble, see sub-section 2.2.1 and in particular (2.2.6) in Chapter 2. Other examples include non-intersecting squared Bessel paths [KMFW09], and the conditional Circular Unitary Ensemble near the edges [CC17].

The correlation kernel of the Bessel point process is given by

$$K^{\text{BESSEL}}(x, y) = \frac{J_\alpha(\sqrt{x})\sqrt{y}J'_\alpha(\sqrt{y}) - \sqrt{x}J'_\alpha(\sqrt{x})J_\alpha(\sqrt{y})}{2(x - y)}, \quad \alpha > -1, \quad (4.1.1)$$

where J_α stands for the Bessel function of the first kind of order α (see [OLBC10, Chapter 10] for definition and properties). Note that the definition in this chapter is different from (2.2.7) in Chapter 2, as we take the symmetric definition here, which will be much more convenient for our purposes.

The goal of this chapter is to obtain the analogous results of Chapter 3. The Bessel point process being “physically” constrained to the positive half line, it has the smallest particle property instead of the largest particle property (see Chapter 1 for their definitions). Zero being a physical wall also makes ratios of particles more natural to study, instead of differences as it was the case for the Airy Ensemble. The system of coupled differential equations (4.1.9) that we are going to obtain this time corresponds to the Painlevé V equation.

The generating function that we will study in this chapter is thus slightly different from the one defined at (1.1.7) in Chapter 1. We will now be looking at the following. Let $k = 1, 2, \dots$ and consider $0 = x_0 < x_1 < \dots < x_k < \infty$ and $s_1, s_2, \dots, s_k \in \mathbb{C}$. We are interested in

$$F(\vec{x}, \vec{s}) = F(x_1, \dots, x_k; s_1, \dots, s_k) := \det \left(1 - \chi_{(0, x_k)} \sum_{j=1}^k (1 - s_j) K^{\text{BESSEL}} \chi_{(x_{j-1}, x_j)} \right). \quad (4.1.2)$$

Tracy-Widom formula for $k = 1$ In [TW94b], Tracy and Widom have studied $F(x_1, s_1)$, i.e., the case $k = 1$. This is the generating function of the occupancy number of the interval $(0, x_1)$. In particular, we can deduce from $F(x_1, s_1)$ the probability distribution of the ℓ -th smallest particle ζ_ℓ as follows

$$\mathbb{P}(\zeta_\ell > x_1) = \mathbb{P}(n_{(0, x_1)} < \ell) = \sum_{j=0}^{\ell-1} \frac{1}{j!} \partial_{s_1}^j F(x_1, s_1) \Big|_{s_1=0}. \quad (4.1.3)$$

Their theorem states that for $0 \leq s_1 < 1$ and $x_1 > 0$,

$$F(x_1, s_1) = \exp \left(-\frac{1}{4} \int_0^{x_1} \log \left(\frac{x_1}{\xi} \right) q^2(\xi; s_1) d\xi \right), \quad (4.1.4)$$

where $q = q(x; s_1)$ satisfies the Painlevé V equation given by

$$xq(1-q^2)(xqq')' + x(1-q^2)^2 \left((xq')' + \frac{q}{4} \right) + x^2q(qq')^2 = \alpha^2 \frac{q(x)}{4}, \quad (4.1.5)$$

with boundary condition $q(x) \sim \sqrt{1-s_1} J_\alpha(\sqrt{x})$ as $x \rightarrow 0$.

Tracy-Widom type formula for F The Tracy-Widom formula (4.1.4) characterized F in the case $k = 1$ in terms of a function q which satisfies the Painlevé V equation (4.1.5). The main result of this chapter gives a generalisation of this result for a general $k > 1$. We find that F can be expressed in terms of k functions $q_1, \dots, q_k$, which satisfy a system of k coupled Painlevé V equations with Bessel boundary conditions at 0. The following theorem is the analogous of Theorem 3.1.1 from Chapter 3.

Theorem 4.1.1. *Let $\vec{r} = (r_1, \dots, r_k) \in (\mathbb{R}^+)^k$ and $\vec{s} = (s_1, \dots, s_k) \in [0, 1]^k$ be such that*

$$r_j > r_{j-1}, \text{ for } j = 1, \dots, k, \text{ where } r_0 := 0. \quad (4.1.6)$$

$$s_j \neq s_{j+1}, \text{ for } j = 1, \dots, k, \text{ where } s_{k+1} := 1, \quad (4.1.7)$$

The joint generating function $F(\vec{r}x, \vec{s})$, defined at (4.1.2), is given by

$$F(x\vec{r}, \vec{s}) = \prod_{j=1}^k \exp \left(-\frac{r_j}{4} \int_0^x \log \left(\frac{x}{\xi} \right) q_j^2(\xi; \vec{r}, \vec{s}) d\xi \right), \quad (4.1.8)$$

where the functions $q_1, \dots, q_k$ satisfy the system of k equations given by

$$xq_j \left(1 - \sum_{\ell=1}^k q_\ell^2 \right) \sum_{\ell=1}^k (xq_\ell q'_\ell)' + x \left(1 - \sum_{\ell=1}^k q_\ell^2 \right)^2 \left((xq'_j)' + \frac{r_j q_j}{4} \right) + x^2 q_j \left(\sum_{\ell=1}^k q_\ell q'_\ell \right)^2 = \alpha^2 \frac{q_j}{4}, \quad (4.1.9)$$

and for every $j \in \{1, \dots, k\}$, $q_j^2(x) \in \mathbb{R}$ for $x > 0$ and satisfies the boundary condition $q_j(x; \vec{r}, \vec{s}) = \sqrt{s_{j+1} - s_j} J_\alpha(\sqrt{r_j x})(1 + \mathcal{O}(x))$ as $x \rightarrow 0$.

Remark 4.1.2. As mentioned earlier, ratios of particles are more natural to study in the Bessel point process than differences. This is the reason why we write $F(x\vec{r}, \vec{s})$ here instead of $F(\vec{x}, \vec{s})$ in Theorem 3.1.1 from Chapter 3.

Remark 4.1.3. If $k = 1$ and $r_1 = 1$, the above formulas (4.1.8) and (4.1.9) are reduced to (4.1.4) and (4.1.5), and q_1 given in Theorem 4.1.1 and q given by (4.1.5) satisfy the same boundary condition at 0.

Remark 4.1.4. The system (4.1.9) given in Theorem 4.1.1 has at least one solution, but there is no guarantee that this solution is unique. Therefore, the functions $q_1, \dots, q_k$ that appear in (4.1.8) are not defined through the system (4.1.9), but they are explicitly constructed from the solution of a Riemann-Hilbert problem for Φ . This RH problem is presented in Section 4.3.

The asymptotic behaviour as $x \rightarrow 0$ for the q_j 's allow to compute directly the small x asymptotics of $F(\vec{r}x, \vec{s})$, and is given in the following corollary.

Corollary 4.1.5. Let $x > 0$, fix $\vec{r} = (r_1, \dots, r_k) \in (\mathbb{R}^+)^k$ independent of x such that $0 < r_1 < \dots < r_k$, and fix $\vec{s} = (s_1, \dots, s_k) \in [0, 1]^k$ independent of x such that $s_j \neq s_{j+1}$ for $j = 1, \dots, k$ with $s_{k+1} = 1$. We have

$$F(\vec{r}x, \vec{s}) = 1 - \sum_{j=1}^k \frac{s_{j+1} - s_j}{2^{2+2\alpha} \Gamma(2+\alpha)^2} (r_j x)^{1+\alpha} + \mathcal{O}(x^{2+\alpha}), \quad \text{as } x \rightarrow 0. \quad (4.1.10)$$

Proof. This is a direct consequence of Theorem 4.1.1 together with the formula $z\Gamma(z) = \Gamma(z+1)$ and the limiting behaviour

$$J_\alpha(x) = \left(\frac{x}{2}\right)^\alpha \frac{1}{\Gamma(\alpha+1)} (1 + \mathcal{O}(x^2))$$

as $x \rightarrow 0$ (see [OLBC10, formula 10.7.3]). □

Asymptotics for $q_1, \dots, q_k$ as $s_j \rightarrow s_{j+1}$ or as $r_j \rightarrow r_{j-1}$ In Theorem 4.1.1, it is essential that the conditions (4.1.6) and (4.1.7) hold. Suppose that one of these conditions is not satisfied, i.e., suppose we have $s_j = s_{j+1}$ or $r_j = r_{j-1}$ for a certain $j \in \{1, \dots, k\}$. Then, from (4.1.2), we have

$$F(\vec{r}x, \vec{s}) = F(\vec{r}^{[j]}x, \vec{s}^{[j]}), \quad (4.1.11)$$

where for given a vector $\vec{v} = (v_1, \dots, v_k)$, we use the notation $\vec{v}^{[j]} = (v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_k)$, i.e., $\vec{v}^{[j]}$ is the vector $\vec{v}$ with its j -th component removed. Theorem 4.1.1 applied to the right-hand side of (4.1.11) allows to rewrite $F(\vec{r}x, \vec{s})$ in terms of a solution of a system of $k - 1$ coupled Painlevé equations.

Thus, if $\vec{r}$ and $\vec{s}$ satisfy (4.1.6) and (4.1.7), we should observe a transition in the solutions $q_1, \dots, q_k$ if $s_j \rightarrow s_{j+1}$ or $r_j \rightarrow r_{j-1}$ for a certain $j \in \{1, \dots, k\}$. Theorem 4.1.6 below, which is analogous to Theorem 3.1.7 from Chapter 3, gives asymptotics for the solutions $q_1, \dots, q_k$ in the above degenerate cases.

Theorem 4.1.6. *Fix $x > 0$ and let $\vec{r} = (r_1, \dots, r_k) \in (\mathbb{R}^+)^k$ and $\vec{s} = (s_1, \dots, s_k) \in [0, 1]^k$ be such that (4.1.6) and (4.1.7) are satisfied.*

1. *Let $j \in \{1, \dots, k\}$. As $s_j \rightarrow s_{j+1}$, we have*

$$q_j(x; \vec{r}, \vec{s}) = \mathcal{O}\left(\sqrt{|s_j - s_{j+1}|}\right), \quad (4.1.12)$$

$$\left|q_\ell(x; \vec{r}, \vec{s}) - q_\ell(x; \vec{r}^{[j]}, \vec{s}^{[j]})\right| = \mathcal{O}\left(\sqrt{|s_j - s_{j+1}|}\right), \quad (4.1.13)$$

for any $\ell \neq j$.

2. *Let $j \in \{2, \dots, k\}$. If $s_{j+1} \neq s_{j-1}$, as $r_j \rightarrow r_{j-1}$, we have*

$$q_{j-1}^2(x; \vec{r}, \vec{s}) = \frac{s_j - s_{j-1}}{s_{j+1} - s_{j-1}} q_{j-1}^2(x; \vec{r}^{[j]}, \vec{s}^{[j]}) + \mathcal{O}(r_j - r_{j-1}), \quad (4.1.14)$$

$$q_j^2(x; \vec{r}, \vec{s}) = \frac{s_{j+1} - s_j}{s_{j+1} - s_{j-1}} q_{j-1}^2(x; \vec{r}^{[j]}, \vec{s}^{[j]}) + \mathcal{O}(r_j - r_{j-1}), \quad (4.1.15)$$

$$\left|q_\ell^2(x; \vec{r}, \vec{s}) - q_\ell^2(x; \vec{r}^{[j]}, \vec{s}^{[j]})\right| = \mathcal{O}(r_j - r_{j-1}), \quad (4.1.16)$$

and (4.1.16) holds for any $\ell \neq j$, and where $\tilde{\ell} = \ell$ if $\ell < j - 1$ and $\tilde{\ell} = \ell - 1$ if $\ell > j$.

3. As $r_1 \rightarrow 0$, we have

$$q_1^2(x; \vec{r}, \vec{s}) = \mathcal{O}(r_1^\alpha), \quad (4.1.17)$$

$$\left| q_\ell^2(x; \vec{r}, \vec{s}) - q_{\tilde{\ell}}^2(x; \vec{r}^{[j]}, \vec{s}^{[j]}) \right| = \mathcal{O}(r_1), \quad (4.1.18)$$

and (4.1.18) holds for any $\ell \geq 2$, and where $\tilde{\ell} = \ell - 1$.

4.2 Examples and applications

4.2.1 Gap probability on a union of disjoint intervals

A gap in a point process is the event of finding no particle in a certain set. The gap probability for a single interval (a, b) in the Bessel point process is given by

$$\begin{aligned} \mathbb{P}(n_{(a,b)} = 0) &= F((a, b), (1, 0)) = \\ &\exp\left(\frac{a}{4} \int_0^1 \log(\xi) q_1^2(\xi) d\xi\right) \exp\left(\frac{b}{4} \int_0^1 \log(\xi) q_2^2(\xi) d\xi\right), \end{aligned}$$

where in the above expression we have used the definition of F given by (4.1.2) for the first equality, and we have applied Theorem 4.1.1 (with $k = 2$, $x = 1$, $\vec{r} = (a, b)$, $\vec{s} = (1, 0)$) for the second equality. The two functions q_1 and q_2 satisfy the system (4.1.9).

This computation can be generalized for the gap probability of any finite union of disjoint bounded intervals. Let $\ell = 1, 2, \dots$ be the number of intervals and $a_1 < b_1 < a_2 < \dots < b_\ell < +\infty$, we have

$$\begin{aligned} \mathbb{P}\left(\bigcap_{j=1}^{\ell} n_{(a_j, b_j)} = 0\right) &= F((a_1, b_1, \dots, a_\ell, b_\ell), (1, 0, \dots, 1, 0)) = \\ &\exp\left(\frac{1}{4} \int_0^1 \log(\xi) \sum_{j=1}^{\ell} (a_j q_{2j-1}^2(\xi) + b_j q_{2j}^2(\xi)) d\xi\right), \end{aligned}$$

where we have applied Theorem 4.1.1 with $k = 2\ell$, $x = 1$, $\vec{r} = (a_1, b_1, \dots, a_\ell, b_\ell)$ and $\vec{s} = (1, 0, \dots, 1, 0)$, and where the 2ℓ functions $q_1, \dots, q_{2\ell}$ satisfy the system (4.1.9).

4.2.2 Distribution of the smallest particle in the thinned and conditional Bessel point process

We consider a constant and independent thinning of the Bessel point process. Given a realization $\zeta_1 < \zeta_2 < \dots$ of the Bessel point process, it consists of removing each of these particles with the same probability $s \in (0, 1)$. The thinned point process is composed of the remaining particles $\xi_1 < \xi_2 < \dots$, and is again a determinantal point process, whose correlation kernel is given by $(1 - s)K^{\text{BESSEL}}$ (see [LMR15]). For a given Borel set $B \subset \mathbb{R}^+$, we denote $\tilde{n}_B$ for the occupancy number of B in the thinned point process. The probability distribution of ξ_1 can be deduced from F with $k = 1$, since by (4.1.2) we have

$$\begin{aligned} \mathbb{P}(\xi_1 > x) &= \sum_{j=0}^{+\infty} \mathbb{P}\left(n_{(0,x)} = j \cap \tilde{n}_{(0,x)} = 0\right) = \\ &= \sum_{j=0}^{+\infty} \mathbb{P}\left(n_{(0,x)} = j\right) s^j = F(x, s). \end{aligned} \quad (4.2.1)$$

We can also consider another situation, where we have information about the thinned point process. Suppose that we observe the event $\tilde{n}_{(0,x_2)}$ with $x_2 > 0$. We want to retrieve information on ζ_1 conditionally on this event. The distribution of the smallest particle in this conditional point process is given by

$$\mathbb{P}(\zeta_1 > x_1 \mid \xi_1 > x_2) = \frac{\mathbb{P}(\zeta_1 > x_1 \cap \xi_1 > x_2)}{\mathbb{P}(\xi_1 > x_2)}, \quad 0 < x_1 < x_2. \quad (4.2.2)$$

The denominator in the above expression is just given by $F(x_2, s)$, as shown in (4.2.1). The numerator is slightly more involved, and can be expressed in terms of F with $k = 2$ as follows

$$\mathbb{P}(\zeta_1 > x_1 \cap \xi_1 > x_2) =$$

$$\sum_{j=0}^{+\infty} s^j \mathbb{P}\left(n_{(0,x_1)} = 0 \cap n_{(x_1,x_2)} = j\right) = F((x_1, x_2), (0, s)). \quad (4.2.3)$$

4.2.3 Near-smallest LUE eigenvalues

The Laguerre Unitary Ensemble (LUE) was introduced in sub-section 2.2.1 in Chapter 2. At its hard edge 0, the limiting behaviour of the LUE is governed by the Bessel point process (2.2.6).

Let m be a positive integer. We consider the symmetric kernel for the LUE given by

$$K_n^{\text{LUE}}(x, y) = \sqrt{w(x)w(y)} \sum_{j=0}^{n-1} p_j(x)p_j(y),$$

where $w(x) = x^\alpha e^{-nx^m} \chi_{\mathbb{R}^+}(x)$ and $(p_j)_j$ is the sequence of orthonormal polynomials with respect to w , p_j of degree j .

The convergence to the Bessel kernel (2.2.6) actually implies trace-norm convergence of the associated operators, we have that the generating functions of the occupancy numbers of the LUE converge with the one of the Bessel point process, after proper rescaling. Precisely, we have

$$F_n^{\text{LUE}}\left(\frac{1}{cn^2}\vec{x}, \vec{s}\right) := \det\left(1 - \chi_{(0,\lambda_k)} \sum_{j=1}^k (1 - s_j) K_n^{\text{LUE}} \chi_{(\lambda_{j-1}, \lambda_j)}\right) = F(\vec{x}, \vec{s}) + o(1), \quad \text{as } n \rightarrow +\infty \quad (4.2.4)$$

where c is defined the line after (2.2.6), and $x_0 = 0$. On the other hand, $F_n^{\text{LUE}}(\vec{x}/cn^2, \vec{s})$ can also be written as the following ratio of Hankel determinants, writing $\lambda_j = \frac{x_j}{cn^2}$,

$$F_n^{\text{LUE}}(\vec{\lambda}, \vec{s}) = \frac{\det\left(\int_0^\infty w(x) \left(1 - \sum_{j=1}^k (1 - s_j) \chi_{(\lambda_{j-1}, \lambda_j)}(x)\right) x^{i+j-2} dx\right)_{i,j=1}^n}{\det\left(\int_0^\infty w(x) x^{i+j-2} dx\right)_{i,j=1}^n}. \quad (4.2.5)$$

In particular, Theorem 4.1.1 together with (4.2.4) implies large n asymptotics for the ratio (4.2.5) up to constant term, but this does not provide an estimate for error term $o(1)$ in (4.2.4).

4.2.4 Ratio and gap probability between the two smallest particles

We are interested in the ratio and the gap probabilities between the two smallest particles in the Bessel point process, i.e., in $Q_\alpha(r) = \mathbb{P}\left(\frac{\zeta_2}{\zeta_1} > r\right)$ and $G_\alpha(d) = \mathbb{P}(\zeta_2 - \zeta_1 > d)$, where $r > 1$ is the size of the ratio and $d > 0$ is the size of the gap.

Since the LUE kernel converges to the Bessel kernel near the hard edge (see (2.2.6)), the quantities $Q_\alpha(r)$ and $G_\alpha(d)$ can be obtained as the limiting probabilities of the ratio and the rescaled gap between the first two smallest eigenvalues in the LUE. Using this approach, the gap $G_\alpha(d)$ was obtained in [FW07] and the ratio $Q_\alpha(r)$ was obtained in [ACZ16].

In this section, we express these quantities in terms of F . By definition, we have

$$\begin{aligned} Q_\alpha(r) &= \int_0^\infty \partial_\xi \mathbb{P}(\zeta_1 \leq \xi \cap \zeta_2 > r\xi) \Big|_{\xi=x} dx, \\ &= \int_0^\infty \partial_\xi \mathbb{P}(n_{(0,\xi)} = 1 \cap n_{(\xi,rx)} = 0) \Big|_{\xi=x} dx. \end{aligned} \quad (4.2.6)$$

The probability in the integrand can be obtained from the generating function (4.1.2) as follows

$$\partial_s F((\xi, rx), (s, 0))|_{s=0} = \mathbb{P}(n_{(0,\xi)} = 1 \cap n_{(\xi,rx)} = 0), \quad (4.2.7)$$

and thus

$$\begin{aligned} Q_\alpha(r) &= \int_0^\infty \partial_\xi \partial_s F(\xi, rx, s, 0)|_{s=0} \Big|_{\xi=x} dx \\ &= \int_0^\infty \partial_{r_1} \partial_s F(r_1 x, rx, s, 0)|_{s=0} \Big|_{r_1=1} \frac{dx}{x}. \end{aligned} \quad (4.2.8)$$

Remark 4.2.1. It is worth comparing in details our result with the result obtained in [ACZ16].

For $\alpha \geq 0$ they obtained [ACZ16, Theorem 1.3] that

$$Q_\alpha(r) = \frac{2^{2+2\alpha}}{\Gamma(1+\alpha)\Gamma(2+\alpha)} \int_0^\infty x^\alpha e^{I(4x;r)} dx, \quad (4.2.9)$$

where $I(4x;r)$ is given by [ACZ16, equations (1.21), (4.16)–(4.22)] (which are equivalent with our equations (4.3.25)–(4.3.27))

$$I(4x;r) = -\frac{1}{4} \int_0^x (w(\xi;r) - w(0)) \frac{d\xi}{\xi} = -\frac{1}{4} \int_0^x \log\left(\frac{x}{\xi}\right) w'(\xi;r) d\xi, \quad (4.2.10)$$

where $w(x;r) = \tilde{q}_1(x) + r\tilde{q}_2(x)$, and $\tilde{q}_1, \tilde{q}_2(x)$ satisfy precisely the same system of ODEs (4.1.9) with $k = 2$, $r_1 = 1$, $r_2 = r$, but the small x asymptotics are not the same. Instead, we have [ACZ16, equations (1.17), (1.18), (1.22)]

$$\begin{aligned} \tilde{q}_1^2(x) &= \mathcal{O}(x^{-1}), & \tilde{q}_2^2(x) &= \mathcal{O}(x^{-1}), & \text{as } x \rightarrow 0, \\ \tilde{q}_1^2(x) &= \frac{2}{\sqrt{r-1}\sqrt{x}} + \mathcal{O}(x^{-1}), & & \text{as } x \rightarrow +\infty, \\ \tilde{q}_2^2(x) &= \sqrt{x} - \frac{\alpha}{\sqrt{rx}} - \frac{2}{\sqrt{(r-1)x}} + \mathcal{O}(x^{-1}), & & \text{as } x \rightarrow +\infty, \\ w(x;r) &= \frac{1}{4} - (\alpha+2)^2 + \mathcal{O}(x), & & \text{as } x \rightarrow 0, \\ w(x;r) &= rx - 2\alpha\sqrt{rx} - 4\sqrt{(r-1)x} + \mathcal{O}(1), & & \text{as } x \rightarrow +\infty. \end{aligned}$$

In particular, we have the explicit formula

$$\begin{aligned} \partial_{r_1} \partial_s F(r_1 x, rx, s, 0) \Big|_{s=0} \Big|_{r_1=1} &= \\ &= \frac{2^{2+2\alpha} x^{1+\alpha}}{\Gamma(1+\alpha)\Gamma(2+\alpha)} \exp\left(\frac{-1}{4} \int_0^x \log\left(\frac{x}{\xi}\right) (\tilde{q}_1^2(x) + r\tilde{q}_2^2(x)) dx\right). \end{aligned}$$

The calculations for the probability of the gap is similar, and lead to the following expression

$$G_\alpha(d) = \int_0^\infty \partial_{r_1} \partial_s F\left(r_1 x, \left(1 + \frac{d}{x}\right) x\right) \Big|_{s=0} \Big|_{r_1=1} \frac{dx}{x}. \quad (4.2.11)$$

Outline We introduce a model Riemann-Hilbert problem, whose solution is denoted Φ , in Section 4.3. The system of ODEs for the q_j ,

given by (4.1.9), is obtained from a Lax pair of this RH problem and is also obtained in Section 4.3. In Section 4.4, we prove Theorem 4.1.1 with the integration of a differential identity which relate F with Φ . In Section 4.5, we perform the Deift/Zhou [DZ92, DZ93] steepest descent method on the model RH problem to obtain the asymptotic behaviour of $q_j(x)$ as $x \rightarrow 0$. The first and second part of Theorem 4.1.6 are obtained in Section 4.6 and Section 4.7, respectively. These are obtained via a more direct steepest descent method on the RH problem.

4.3 Model RH problem

Let $0 < x_1 < \dots < x_k$ and $s_1, \dots, s_k \in [0, 1]$, $s_{k+1} = 1$ be such that $s_{j+1} \neq s_j$ for $j \in \{1, \dots, k\}$. For convenience, we also define $x_0 = 0$ and $x_{k+1} = +\infty$.

In order to have compact notations in coming sections, we also define for $y \in \mathbb{R}$ the following piecewise constant matrix:

$$H_y(z) = \begin{cases} I, & \text{for } -\frac{2\pi}{3} < \arg(z - y) < \frac{2\pi}{3}, \\ \begin{pmatrix} 1 & 0 \\ -e^{\pi i \alpha} & 1 \end{pmatrix}, & \text{for } \frac{2\pi}{3} < \arg(z - y) < \pi, \\ \begin{pmatrix} 1 & 0 \\ e^{-\pi i \alpha} & 1 \end{pmatrix}, & \text{for } -\pi < \arg(z - y) < -\frac{2\pi}{3}. \end{cases} \quad (4.3.1)$$

The solution of our model RH problem will be denoted by $\Phi(z; \vec{x}, \vec{s})$, where $\vec{x} = (x_1, \dots, x_k)$ and $\vec{s} = (s_1, \dots, s_k)$. If it is clear from the context, we will just denote it by $\Phi(z; \vec{x})$ where the dependence in $\vec{s}$ is omitted and furthermore $\Phi(z)$ when $\vec{x}$ is fixed.

Let $\Sigma_1 = \{-x_3 + te^{2\pi i/3}, t \in (0, +\infty)\}$ and $\Sigma_2 = \{-x_3 + te^{-2\pi i/3}, t \in (0, +\infty)\}$.

RH problem for Φ

- (a) $\Phi : \mathbb{C} \setminus \Sigma_\Phi \rightarrow \mathbb{C}^{2 \times 2}$ is analytic, where the contour $\Sigma_\Phi = ((-\infty, 0] \cup \Sigma_1 \cup \Sigma_2)$ is oriented as shown in Figure 4.1.
- (b) The limits of $\Phi(z)$ as z approaches $\Sigma_\Phi \setminus \{0, -x_1, \dots, -x_k\}$ from the

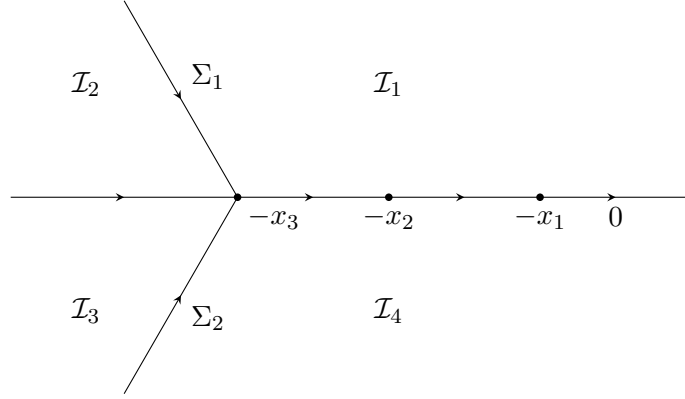


Figure 4.1: The jump contour for Φ with $k = 3$, and the four sectors $\mathcal{I}_i$, $i = 1, 2, 3, 4$.

left (+ side) and from the right (− side) exist, are continuous on $\Sigma_\Phi \setminus \{0, -x_1, \dots, -x_k\}$ and are denoted by Φ_+ and Φ_- respectively. Furthermore they are related by:

$$\Phi_+(z) = \Phi_-(z) \begin{pmatrix} 1 & 0 \\ e^{\pi i \alpha} & 1 \end{pmatrix}, \quad z \in \Sigma_1, \quad (4.3.2)$$

$$\Phi_+(z) = \Phi_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad z \in (-\infty, -x_k), \quad (4.3.3)$$

$$\Phi_+(z) = \Phi_-(z) \begin{pmatrix} 1 & 0 \\ e^{-\pi i \alpha} & 1 \end{pmatrix}, \quad z \in \Sigma_2, \quad (4.3.4)$$

$$\Phi_+(z) = \Phi_-(z) \begin{pmatrix} e^{\pi i \alpha} & s_j \\ 0 & e^{-\pi i \alpha} \end{pmatrix}, \quad z \in (-x_j, -x_{j-1}), \quad (4.3.5)$$

where $j = 1, \dots, k$.

(c) There exist functions q , p and t such that as $z \rightarrow \infty$, we have

$$\Phi(z; \vec{x}) = \left(I + \Phi_1(\vec{x}) z^{-1} + \mathcal{O}(z^{-2}) \right) z^{-\frac{\sigma_3}{4}} N e^{z^{\frac{1}{2}} \sigma_3}, \quad (4.3.6)$$

where the principal branch is chosen for the square root, and $\Phi_1(\vec{x})$ is traceless and given by

$$\Phi_1(\vec{x}) = \begin{pmatrix} q(\vec{x}) & -it(\vec{x}) \\ ip(\vec{x}) & -q(\vec{x}) \end{pmatrix}. \quad (4.3.7)$$

As z tends to $-x_j$, $j \in \{1, \dots, k\}$, Φ takes the form

$$\Phi(z) = \Phi_{0,j}(z) \begin{pmatrix} 1 & \frac{s_{j+1}-s_j}{2\pi i} \log(z+x_j) \\ 0 & 1 \end{pmatrix} V_j(z) e^{\frac{\pi i \alpha}{2} \theta(z) \sigma_3} H_{-x_k}(z), \quad (4.3.8)$$

where $\Phi_{0,j}(z)$ is analytic in a neighbourhood of $(-x_{j+1}, -x_{j-1})$, and $\theta(z)$ and $V_j(z)$ are piecewise constant defined by

$$\theta(z) = \begin{cases} +1, & \text{Im } z > 0, \\ -1, & \text{Im } z < 0, \end{cases} \quad V_j(z) = \begin{cases} I, & \text{Im } z > 0, \\ \begin{pmatrix} 1 & -s_j \\ 0 & 1 \end{pmatrix}, & \text{Im } z < 0. \end{cases} \quad (4.3.9)$$

As z tends to 0, the behaviour of Φ is

$$\Phi(z) = \Phi_{0,0}(z) z^{\frac{\alpha}{2} \sigma_3} \begin{pmatrix} 1 & s_1 h(z) \\ 0 & 1 \end{pmatrix}, \quad (4.3.10)$$

where $\Phi_{0,0}(z)$ is analytic in a neighbourhood of 0 and

$$h(z) = \begin{cases} \frac{1}{2i \sin(\pi \alpha)}, & \alpha \notin \mathbb{Z}, \\ \frac{(-1)^\alpha}{2\pi i} \log z, & \alpha \in \mathbb{Z}. \end{cases} \quad (4.3.11)$$

Remark 4.3.1. The solution of the RH problem for Φ is unique. This follows from the fact that $\det \Phi(z) \equiv 1$ and by standard arguments, see e.g., [Dei99]. We will prove the existence of the solution in Section 4.4 below using a probabilistic argument.

Remark 4.3.2. We can verify that $\sigma_3 \overline{\Phi(\bar{z})} \sigma_3$ is also a solution of the RH problem for Φ . Thus by uniqueness of the solution, see Remark 4.3.1, we have

$$\Phi(z) = \sigma_3 \overline{\Phi(\bar{z})} \sigma_3. \quad (4.3.12)$$

4.3.1 Lax pair

In this section, we obtain a system of k ODEs for k functions associated to Φ . We derive these equations using Lax Pair techniques. The

following computations are similar to those done in [ACZ16] for the distribution of the ratio between the two smallest eigenvalues in the Laguerre Unitary Ensemble. We introduce some notations. Let $x > 0$ and $0 = r_0 < r_1 < r_2 < \dots < r_k$ be such that $\vec{x} = (x_1, \dots, x_k) = \vec{r}x := (r_1x, \dots, r_kx)$. We begin with a transformation on Φ , in order to have a jump contour independent of x :

$$\tilde{\Phi}(z; x) = \tilde{E}(x)\Phi(x^2z; \vec{r}x^2), \quad \tilde{E}(x) = \begin{pmatrix} 1 & 0 \\ \frac{t(\vec{r}x^2)}{x} & 1 \end{pmatrix} e^{\frac{\pi i}{4}\sigma_3 x^{\frac{\sigma_3}{2}}}, \quad (4.3.13)$$

where t comes from the asymptotics of Φ at infinity, see (4.3.7). By standard arguments, $\tilde{\Phi}(z; x)$ is analytic in x for x in a compact subset of $(0, \infty)$. Since the jump contour and the jumps of $\tilde{\Phi}$ are independent of both z and x , by differentiating $\tilde{\Phi}$ with respect to z and x , we obtain a Lax pair of the form

$$\begin{cases} \partial_z \tilde{\Phi}(z; x) = A(z; x) \tilde{\Phi}(z; x), \\ \partial_x \tilde{\Phi}(z; x) = B(z; x) \tilde{\Phi}(z; x). \end{cases} \quad (4.3.14)$$

Since $\tilde{\Phi}$, $\partial_z \tilde{\Phi}(z; x)$ and $\partial_x \tilde{\Phi}(z; x)$ have the same jumps, A and B are meromorphic in $z \in \mathbb{C}$. From (4.3.8) and (4.3.10), B is an entire function in z and A has simple poles in z at $0, -r_1, \dots, -r_k$. We can use (4.3.6) to obtain an explicit expression for B :

$$B(z; x) = B_0(x) + zB_1, \quad B_0(x) = \begin{pmatrix} 0 & 1 \\ u(x) & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (4.3.15)$$

where $u(x) = \frac{2t'(\vec{r}x^2)x^2 + t(\vec{r}x^2)^2 - 2q(\vec{r}x^2) - t(\vec{r}x^2)}{x^2}$, and $t'(\vec{r}x^2) = \partial_y t(\vec{r}y)|_{y=x^2}$. On the other hand, A can be written as

$$A(z; x) = A_\infty(x) + \sum_{j=0}^k \frac{A_k(x)}{z + r_k}. \quad (4.3.16)$$

The matrix A_∞ can also be explicitly evaluated by using (4.3.6), we have

$$A_\infty(x) = \begin{pmatrix} 0 & 0 \\ \frac{x}{2} & 0 \end{pmatrix}. \quad (4.3.17)$$

Since $\det \tilde{\Phi}(z)$ is constant, A is traceless and we can also write

$$A(z; x) = \begin{pmatrix} a(z; x) & b(z; x) \\ c(z; x) & -a(z; x) \end{pmatrix}, \quad b(z; x) = \sum_{j=0}^k \frac{b_j(x)}{z + r_j}. \quad (4.3.18)$$

We will derive a system of ODEs for $b_0(x), b_1(x), \dots, b_k(x)$ and $u(x)$ from the compatibility condition

$$\partial_x \partial_z \tilde{\Phi}(z; x) = \partial_z \partial_x \tilde{\Phi}(z; x), \quad (4.3.19)$$

which is equivalent to

$$\partial_x A - \partial_z B + AB - BA = 0. \quad (4.3.20)$$

This condition gives rise to three equations for a, b, c , and u .

$$0 = c - b(z + u) - \partial_x a, \quad (4.3.21)$$

$$0 = 2a + \partial_x b, \quad (4.3.22)$$

$$0 = 2a(z + u) - \partial_x c + 1. \quad (4.3.23)$$

In particular a and c can be expressed in terms of b . Thus we can write the determinant of A as

$$\det A = -b^2(z + u) + \frac{\partial_{xx}(b^2)}{4} - \frac{3}{4}(\partial_x b)^2. \quad (4.3.24)$$

Expanding $\det A(z)$ around $z = 0, -r_1, \dots, -r_k$ and ∞ using on one hand (4.3.18) and (4.3.24), and on the other hand (4.3.6), (4.3.8) and (4.3.10), and by expanding $A_{12}(z) = b(z)$ around $z = \infty$, we obtain

$$\sum_{j=0}^k b_j(x) = \frac{x}{2}, \quad (4.3.25)$$

$$(u(x) - r_j)b_j(x)^2 + \frac{1}{4}b_j'(x)^2 - \frac{1}{2}b_j(x)b_j''(x) = 0, \quad j = 1, \dots, k \quad (4.3.26)$$

$$u(x)b_0(x)^2 + \frac{1}{4}b_0'(x)^2 - \frac{1}{2}b_0(x)b_0''(x) = \frac{\alpha^2}{4}. \quad (4.3.27)$$

Definition 4.3.3. We define q_j in terms of b_j as follows:

$$q_j^2(x) = \frac{2b_j(\sqrt{x})}{\sqrt{x}}, \quad j = 1, \dots, k. \quad (4.3.28)$$

We can use (4.3.25) and (4.3.27) to express u in terms of $b_1, \dots, b_k$, and therefore in terms of $q_1, \dots, q_k$. By substituting this expression in (4.3.26), we obtain the system of ODEs (4.1.9). Also, from (4.3.12), if $z \in \mathbb{R} \setminus \{-r_k, \dots, -r_1, 0\}$, we have $b(z; x) = \overline{b(z; x)}$ and thus $q_j^2(x) \in \mathbb{R}$.

Proposition 4.3.4. *There holds the following relation*

$$\partial_x \left(x \lim_{z \rightarrow -r_j x} [\Phi^{-1}(z; \vec{r}x) \Phi'(z; \vec{r}x)]_{21} \right) = \frac{2\pi i e^{-\pi i \alpha}}{s_{j+1} - s_j} \frac{q_j^2(x)}{4}, \quad (4.3.29)$$

where the limit is taken from the side $\text{Im } z < 0$ if $j = 1, \dots, k-1$ and from $z \in \mathcal{I}_4$ if $j = k$ with $\mathcal{I}_4$ as on Figure 4.1.

Proof. Since $\Phi_{0,j}(z)$ (defined at (4.3.8)) is analytic in z in a neighbourhood of $-x_j$, we have

$$\Phi_{0,j}(z) = E_j(x)(I + F_j(x)(z + x_j) + \mathcal{O}((z + x_j)^2)), \quad \text{as } z \rightarrow -x_j. \quad (4.3.30)$$

Therefore, we have

$$\begin{aligned} \lim_{z \rightarrow -r_j x} [\Phi^{-1}(z; \vec{r}x) \Phi'(z; \vec{r}x)]_{21} = \\ e^{-\pi i \alpha} [\Phi_{0,j}^{-1}(-r_j x) \Phi'_{0,j}(-r_j x)]_{21} = e^{-\pi i \alpha} F_{j,21}(x), \end{aligned} \quad (4.3.31)$$

where the limit is taken from the side $\text{Im } z < 0$ if $j = 1, \dots, k-1$ and from $z \in \mathcal{I}_4$ if $j = k$. On the other hand, taking the limit $z \rightarrow -r_j$ in the B -equation in the Lax pair (4.3.14) leads to

$$\partial_x (\tilde{E}(x) E_j(x^2)) = (B_0(x) - r_j B_1) \tilde{E}(x) E_j(x^2) - \tilde{E}(x) E_j(x^2) K(x), \quad (4.3.32)$$

$$\partial_x (x^2 F_j(x^2)) = (\tilde{E}(x) E_j(x^2))^{-1} B_1 \tilde{E}(x) E_j(x^2) + x^2 [K(x), F_j(x^2)], \quad (4.3.33)$$

where $K(x) = \begin{pmatrix} 0 & \frac{s_{j+1} - s_j}{2\pi i} \frac{2}{x} \\ 0 & 0 \end{pmatrix}$. In particular, taking the $(2, 1)$ entry in

(4.3.33) and using the fact that $\det E_j(x) = 1$ leads to

$$\partial_x(x^2 F_{j,21}(x^2)) = ix E_{j,11}(x^2)^2. \quad (4.3.34)$$

By a change of the variable x , this can be rewritten as

$$\partial_x(x F_{j,21}(x)) = \frac{i}{2} E_{j,11}(x)^2. \quad (4.3.35)$$

We also have, by the A -equation in the Lax pair (4.3.14), as $z \rightarrow -r_j$

$$\begin{aligned} A(z; x) &= \frac{1}{z + r_j} \frac{s_{j+1} - s_j}{2\pi i} \tilde{E}(x) \\ &\times \begin{pmatrix} -E_{j,11}(x^2) E_{j,21}(x^2) & E_{j,11}(x^2)^2 \\ -E_{j,21}(x^2)^2 & E_{j,11}(x^2) E_{j,21}(x^2) \end{pmatrix} \tilde{E}(x)^{-1} + \mathcal{O}(1). \end{aligned} \quad (4.3.36)$$

Equation (4.3.36) then implies

$$b_j(x) = \frac{s_{j+1} - s_j}{2\pi i} ix E_{j,11}(x^2)^2. \quad (4.3.37)$$

Thus by (4.3.31), (4.3.35) and the definition of q (4.3.28), we obtain the claim. $\square$

4.4 Proof of Theorem 4.1.1

The idea of the proof of Theorem 4.1.1 is to compute the logarithmic derivatives of the Fredholm determinant (1.1.7) with respect to the parameters x_j . As in Chapter 3, the proof uses the theory of integrable kernels developed by Its, Izergin, Korepin and Slavnov [IIKS90], this derivative is linked to the model RH problem described in Section 4.3.

Here, the Bessel kernel K^{BESSEL} has to be viewed as the integral kernel operator of an integral operator acting on $L^2(\mathbb{R}^+)$. In order to emphasize this difference, we will write $\mathcal{K}$ for the integral operator whose kernel is given by

$$K(u, v) = \chi_{(0, x_k)}(u) \sum_{j=1}^k (1 - s_j) K^{\text{BESSEL}}(u, v) \chi_{(x_{j-1}, x_j)}(v).$$

The operator $\mathcal{K}$ is thus a trace class operator acting on $L^2((0, x_k))$. The kernel K , for its part, is integrable in the sense of Its, Izergin, Korepin and Slavnov, in that it may be written in the form

$$K(z, w) = \frac{f^T(z)g(w)}{z - w}, \quad z, w > 0, \quad (4.4.1)$$

where $f(z)$ and $g(w)$ are given by

$$\begin{aligned} f(z) &= \frac{1}{2} \begin{pmatrix} \chi_{(0, x_k)}(z) J_\alpha(\sqrt{z}) \\ \chi_{(0, x_k)}(z) \sqrt{z} J'_\alpha(\sqrt{z}) \end{pmatrix}, \\ g(w) &= \begin{pmatrix} \sum_{j=1}^k (1 - s_j) \sqrt{w} J'_\alpha(\sqrt{w}) \chi_{(x_{j-1}, x_j)}(w) \\ - \sum_{j=1}^k (1 - s_j) J_\alpha(\sqrt{w}) \chi_{(x_{j-1}, x_j)}(w) \end{pmatrix}. \end{aligned} \quad (4.4.2)$$

Consider the matrix valued function $\tilde{P}_{\text{Be}}$ defined by formula (4.8.7) in section 4.8. Using the connection formula $I_\alpha(e^{\frac{\pi i}{2}} \sqrt{z}) = e^{\frac{\alpha \pi i}{2}} J_\alpha(\sqrt{z})$ for $z > 0$ (see [OLBC10, formula 10.27.6]), $f(z)$ and $g(w)$ can be rewritten in terms of $\tilde{P}_{\text{Be}}$ as follows:

$$f(z) = \frac{e^{-\frac{\alpha \pi i}{2}} e^{\frac{\pi i}{4}}}{2\sqrt{\pi}} \chi_{(0, x_k)}(z) \sigma_3 \tilde{P}_{\text{Be},+}(-z) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{for } z > 0, \quad (4.4.3)$$

$$\begin{aligned} g(w) &= \frac{e^{-\frac{\alpha \pi i}{2}} e^{\frac{\pi i}{4}}}{\sqrt{\pi}} \sum_{j=1}^k (1 - s_j) \chi_{(x_{j-1}, x_j)}(w) \sigma_3 \tilde{P}_{\text{Be},+}^{-1}(-w)^T \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\ &\quad \text{for } w > 0. \end{aligned} \quad (4.4.4)$$

From (4.1.2) and (1.1.7), we have that for $s_1, \dots, s_k \in [0, 1]$, $\det(1 - \mathcal{K}) > 0$ — this is actually equivalent to the fact that for all bounded Borel set with non zero Lebesgue measure, $P(n_B = 0) > 0$ holds, which is the case for the Bessel point process. We can then take the logarithmic derivative of $\det(1 - \mathcal{K})$ with respect to any of the x_j . For trace class operators in general, this gives

$$\partial_{x_j} \log \det(1 - K) = -\text{Tr}((1 - K)^{-1} \partial_{x_j} K), \quad j \in \{1, \dots, k\}. \quad (4.4.5)$$

In our case, it becomes

$$\begin{aligned}
 \operatorname{Tr}((1-K)^{-1}\partial_{x_j}K) &= (s_{j+1}-s_j)\operatorname{Tr}((1-K)^{-1}K^{\text{BESSEL}}\delta_{x_j}) \\
 &= \frac{s_{j+1}-s_j}{1-s_j} \lim_{z \nearrow x_j} [(1-K)^{-1}K](z, z) \\
 &= \frac{s_{j+1}-s_j}{1-s_j} \lim_{z \nearrow x_j} R(z, z)
 \end{aligned} \tag{4.4.6}$$

where R is kernel for the resolvent operator $\mathcal{R}$ defined by

$$1 + \mathcal{R} = (1 - \mathcal{K})^{-1}$$

(note again that $1 - \mathcal{K}$ is invertible, for $\det(1 - \mathcal{K}) > 0$). If $s_j = 1$, then we take the limit $z \searrow x_j$ instead, and the above formula is replaced by

$$\operatorname{Tr}((1-K)^{-1}\partial_{x_j}K) = \frac{s_{j+1}-s_j}{1-s_{j+1}} \lim_{z \rightarrow x_{j,+}} R(z, z), \tag{4.4.7}$$

which is well defined since $s_{j+1} \neq s_j$. Now we relate R to a RH problem for the Y matrix defined by

$$Y(z) = 1 - \int_0^{x_k} \frac{F(\mu)h^t(\mu)}{z - \mu} d\mu, \quad F(\mu) = \begin{pmatrix} (1 - \mathcal{K})^{-1}f_1(\mu) \\ (1 - \mathcal{K})^{-1}f_2(\mu) \end{pmatrix}.$$

The function Y satisfies the following RH problem for Y [DIZ97].

RH problem for Y

- (a) $Y : \mathbb{C} \setminus [0, x_k] \rightarrow \mathbb{C}^{2 \times 2}$ is analytic
- (b) For $z \in (0, x_k) \setminus \{x_1, \dots, x_k\}$, the limits $\lim_{\epsilon \rightarrow 0+} Y(z \pm i\epsilon)$ exist, are denoted $Y_+(z)$ and $Y_-(z)$ respectively, are continuous as functions of z along $(0, x_k)$, and satisfy furthermore the jump relation

$$Y_+(z) = Y_-(z)J_Y(z), \quad J_Y(z) = I - 2\pi i f(z)g^T(z). \tag{4.4.8}$$

- (c) $Y(z) = I + \mathcal{O}(z^{-1})$ as $z \rightarrow \infty$.
- (d) $Y(z) = \mathcal{O}(\log(z - x_j))$ for each $j = 0, \dots, k$ (with $x_0 = 0$).

For $z, w \in (0, x_k)$, the resolvent can now be written as [DIZ97]

$$R(z, w) = \frac{F^T(z)G(w)}{z - w},$$

where $F(z) = Y_+(z)f(z)$ and $G(z) = (Y_+^{-1}(z))^T g(z)$. (4.4.9)

Now we want to relate Y with our model RH problem Φ . Let us consider $X(z) = \tilde{Y}(z)\tilde{P}_{\text{Be}}(z)$, where $\tilde{Y}(z) = \sigma_3 Y(-z)\sigma_3$ and $\tilde{P}_{\text{Be}}$ is a modified Bessel model RH problem, presented in section 4.8. Since $\tilde{Y}$ is analytic on $\Sigma_1 \cup \Sigma_2 \cup (-\infty, -x_k)$, from the jumps of $\tilde{P}_{\text{Be}}$, it is direct that X has exactly the same jumps as Φ on $\Sigma_1 \cup \Sigma_2 \cup (-\infty, -x_k)$. The jumps J_X of X are *a priori* more involved on $(-x_k, 0)$. They are given by

$$J_X(z) = \begin{pmatrix} e^{\pi i \alpha} & 1 \\ 0 & e^{-\pi i \alpha} \end{pmatrix} \tilde{P}_{\text{Be},+}^{-1}(z) J_{\tilde{Y}}(z) \tilde{P}_{\text{Be},+}(z), \quad z \in (-x_k, 0),$$

(4.4.10)

where $J_{\tilde{Y}}$ is the jump of $\tilde{Y}$, given by

$$J_{\tilde{Y}}(z) = \sigma_3 J_Y(-z)^{-1} \sigma_3. \quad (4.4.11)$$

For $z \in (-x_k, -x_0)$, by (4.4.3) and (4.4.4), we have

$$J_{\tilde{Y}}(z) = \tilde{P}_{\text{Be},+}(z) \begin{pmatrix} 1 & -e^{-\pi i \alpha} \sum_{j=1}^k (1 - s_j) \chi_{(x_{j-1}, x_j)}(-z) \\ 0 & 1 \end{pmatrix} \tilde{P}_{\text{Be},+}^{-1}(z).$$

(4.4.12)

By plugging it into (4.4.10), J_X is simply reduced to

$$J_X(z) = \begin{pmatrix} e^{\pi i \alpha} & \sum_{j=1}^k s_j \chi_{(x_{j-1}, x_j)}(-z) \\ 0 & e^{-\pi i \alpha} \end{pmatrix}, \quad z \in (-x_k, 0), \quad (4.4.13)$$

which is precisely the same jump as Φ on $(-x_j, -x_{j-1})$. On the other hand, from (4.8.9), as $z \rightarrow \infty$ we have

$$X(z) = e^{-\frac{\pi i}{4} \sigma_3} \begin{pmatrix} 1 & 0 \\ \frac{-i}{8}(4\alpha^2 + 3) & 1 \end{pmatrix} (I + \mathcal{O}(z^{-1})) z^{\frac{-\sigma_3}{4}} N e^{z^{\frac{1}{2}} \sigma_3}. \quad (4.4.14)$$

Thus by uniqueness of the solution of the RH problem for Φ , see Remark

4.3.1, we have

$$\Phi(z) = \begin{pmatrix} 1 & 0 \\ \frac{i}{8}(4\alpha^2 + 3) & 1 \end{pmatrix} e^{\frac{\pi i}{4}\sigma_3} \tilde{Y}(z) \tilde{P}_{\text{Be}}(z). \quad (4.4.15)$$

Since from our proof, the matrix on the right hand side is constructed explicitly in terms of $(1 - \mathcal{K})^{-1}$, it also proves the existence of a solution for the RH problem for Φ .

Note that (4.4.3) and (4.4.4) can equivalently be written as

$$\begin{aligned} \tilde{P}_{\text{Be},-}^{-1}(-z)\sigma_3 f(z) &= \frac{c}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ \tilde{P}_{\text{Be},-}(-z)^T \sigma_3 g(z) &= c \sum_{j=1}^k (1 - s_j) \chi_{(x_{j-1}, x_j)}(z) \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \end{aligned} \quad (4.4.16)$$

where $c = \frac{1+i}{\sqrt{2\pi}} e^{\frac{\alpha\pi i}{2}}$. Thus for $z, w \in (x_{j-1}, x_j)$, we have

$$R(z, w) = \frac{c^2}{2} (1 + s_j) \frac{[\Phi_-^{-1}(-w)\Phi_-(-z)]_{21}}{z - w}. \quad (4.4.17)$$

By taking the limit $w \rightarrow z$ in the above expression, we obtain

$$R(z, z) = -\frac{c^2}{2} (1 - s_j) [\Phi_-^{-1}(-z)\Phi'_-(-z)]_{21}. \quad (4.4.18)$$

Taking now the limit $z \nearrow x_j$ and substituting it in (4.4.5) and (4.4.6), we obtain an explicit differential identity in terms of Φ :

$$\partial_{x_j} \log F(\vec{x}, \vec{s}) = (s_{j+1} - s_j) \frac{e^{\pi i \alpha}}{2\pi i} [\Phi_-(-x_j; \vec{x})^{-1} \Phi'_-(-x_j; \vec{x})]_{21}, \quad (4.4.19)$$

$j = 1, \dots, k$. By simple compositions, we can use the above identities to get

$$\begin{aligned} \partial_x \log F(\vec{r}x, \vec{s}) &= \sum_{j=1}^k r_j \partial_{x_j} \log F(\vec{x}, \vec{s})|_{\vec{x}=\vec{r}x} = \\ &= \sum_{j=1}^k r_j (s_{j+1} - s_j) \frac{e^{\pi i \alpha}}{2\pi i} [\Phi_-(-r_j x; \vec{r}x)^{-1} \Phi'_-(-r_j x; \vec{r}x)]_{21}. \end{aligned} \quad (4.4.20)$$

Let ϵ be such that $0 < \epsilon < x$. Since Φ is analytic in x in compact subsets of $(0, \infty)$, we can integrate the above expression from ϵ to x . This gives

$$\log \frac{F(\vec{r}x, \vec{s})}{F(\vec{r}\epsilon, \vec{s})} = \sum_{j=1}^k r_j(s_{j+1} - s_j) \frac{e^{\pi i \alpha}}{2\pi i} \int_{\epsilon}^x [\Phi_{-}(-r_j x; \vec{r}\xi)^{-1} \Phi'_{-}(-r_j x; \vec{r}\xi)]_{21} d\xi. \quad (4.4.21)$$

Integrating it by parts and using Proposition 4.3.4, one has

$$\begin{aligned} \int_{\epsilon}^x [\Phi_{-}(-r_j x; \vec{r}\xi)^{-1} \Phi'_{-}(-r_j x; \vec{r}\xi)]_{21} d\xi = \\ -\epsilon [\Phi_{-}(-r_j \epsilon; \vec{r}\epsilon)^{-1} \Phi'_{-}(-r_j \epsilon; \vec{r}\epsilon)]_{21} \log \frac{\epsilon}{x} \\ - e^{-\pi i \alpha} \frac{2\pi i}{s_{j+1} - s_j} \int_{\epsilon}^x \log \left(\frac{\xi}{x} \right) \frac{q_j^2(\xi)}{4} d\xi. \end{aligned} \quad (4.4.22)$$

We will prove in the next section that

$$\lim_{\epsilon \rightarrow 0} -\epsilon [\Phi_{-}(-r_j \epsilon; \vec{r}\epsilon)^{-1} \Phi'_{-}(-r_j \epsilon; \vec{r}\epsilon)]_{21} \log \frac{\epsilon}{x} = 0, \quad (4.4.23)$$

$$\text{and that } \int_0^x \log \left(\frac{\xi}{x} \right) q_j^2(\xi) d\xi \text{ exists for every } j \in \{1, \dots, k\}. \quad (4.4.24)$$

Thus, taking the limit $\epsilon \rightarrow 0$ in (4.4.22) leads to

$$\log F(\vec{r}x, \vec{s}) = - \sum_{k=1}^k \frac{r_j}{4} \int_0^x \log \left(\frac{\xi}{x} \right) q_j^2(\xi) d\xi. \quad (4.4.25)$$

This finishes the proof of Theorem 4.1.1, except for the small x asymptotics of the functions q_j , that we are going to do next.

4.5 Small x asymptotics

In this section, we perform a Deift/Zhou steepest descent method on Φ defined in section 4.3 [DZ92, DZ93, DKM⁺99b, DKM⁺99a] to obtain small x asymptotics for $\Phi(z; \vec{r}x)$ uniformly in z .

4.5.1 First transformation $\Phi \rightarrow W$

The first transformation consists of making the rays Σ_1 and Σ_2 , see Figure 4.1, ending at 0 instead of $-r_k x$, we define

$$W(z) = \Phi(z) H_{-r_k x}(z)^{-1} H_0(z). \quad (4.5.1)$$

It satisfies the following RH problem.

RH problem for W

- (a) $W : \mathbb{C} \setminus \left((-\infty, 0] \cup e^{\pm \frac{2\pi i}{3}} \mathbb{R}^+ \right) \rightarrow \mathbb{C}^{2 \times 2}$ is analytic, where the rays $e^{\pm \frac{2\pi i}{3}} \mathbb{R}^+$ are oriented from $e^{\pm \frac{2\pi i}{3}} \infty$ to 0.

- (b) The jumps for W are given by

$$W_+(z) = W_-(z) \begin{pmatrix} 1 & 0 \\ e^{\pi i \alpha} & 1 \end{pmatrix}, \quad z \in e^{\frac{2\pi i}{3}} \mathbb{R}^+,$$

$$W_+(z) = W_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad z \in (-\infty, -r_k x),$$

$$W_+(z) = W_-(z) \begin{pmatrix} 1 & 0 \\ e^{-\pi i \alpha} & 1 \end{pmatrix}, \quad z \in e^{-\frac{2\pi i}{3}} \mathbb{R}^+,$$

$$W_+(z) = W_-(z) \begin{pmatrix} e^{\pi i \alpha}(1 - s_j) & s_j \\ s_j - 2 & e^{-\pi i \alpha}(1 - s_j) \end{pmatrix},$$

$$z \in (-r_j x, -r_{j-1} x),$$

where $j = 1, \dots, k$.

- (c) As $z \rightarrow \infty$, we have

$$W(z) = (I + \mathcal{O}(z^{-1})) z^{-\frac{\sigma_3}{4}} N e^{z^{1/2} \sigma_3}. \quad (4.5.2)$$

As z tends to $-x_j$, $j \in \{1, \dots, k\}$, the behaviour of W is

$$W(z) = \Phi_{0,j}(z) \begin{pmatrix} 1 & \frac{s_{j+1}-s_j}{2\pi i} \log(z+x_j) \\ 0 & 1 \end{pmatrix} V_j(z) e^{\frac{\pi i \alpha}{2} \theta(z) \sigma_3} H_0(z). \quad (4.5.3)$$

As z tends to 0, the behaviour of W is

$$W(z) = \Phi_{0,0}(z) z^{\frac{\alpha}{2} \sigma_3} \begin{pmatrix} 1 & s_1 h(z) \\ 0 & 1 \end{pmatrix} H_0(z). \quad (4.5.4)$$

4.5.2 Global parametrix

Ignoring a small neighbourhood of 0, we are left with a Riemann-Hilbert problem which is independent of x . We denote the solution of this RH problem $P^{(\infty)}$. The jumps of $P^{(\infty)}$, as well as its asymptotic behaviour at ∞ (4.5.2), are the same of those of the Bessel model RH problem presented in section 4.8. Nevertheless, we see that we have some freedom, i.e., we can choose $P^{(\infty)}(z) = P_{\text{Be}}(z; \alpha + 2n)$ for any $n \in \mathbb{N}$. In order to have later the matching condition with the local parametrix, see (4.5.6), we need to choose

$$P^{(\infty)}(z) = P_{\text{Be}}(z; \alpha). \quad (4.5.5)$$

4.5.3 Local parametrix

Inside a fixed disk D_0 around 0, we want the local parametrix P to satisfy the following RH problem:

RH problem for P

- (a) $P : D_0 \setminus \left((-\infty, 0] \cup e^{\pm \frac{2\pi i}{3}} \mathbb{R}^+ \right) \rightarrow \mathbb{C}^{2 \times 2}$ is analytic.
- (b) For $z \in D_0 \cap \left((-\infty, 0] \cup e^{\pm \frac{2\pi i}{3}} \mathbb{R}^+ \right)$, $P(z)$ has the same jumps as $W(z)$, i.e. we have $P_-^{-1}(z)P_+(z) = W_-^{-1}(z)W_+(z)$.
- (c) As $x \rightarrow 0$, we have

$$P(z) = (I + \mathcal{O}(x))P^{(\infty)}(z), \quad (4.5.6)$$

uniformly for $z \in \partial D_0$.

As z tends to $-r_j x$, $j \in \{0, 1, \dots, k\}$, we have

$$W(z)P^{-1}(z) = \mathcal{O}(1). \quad (4.5.7)$$

The unique solution of the above RH problem is given by

$$P(z) = P_{\text{Be},0}(z; \alpha) \begin{pmatrix} 1 & f(z; x) \\ 0 & 1 \end{pmatrix} z^{\frac{\alpha}{2}\sigma_3} \begin{pmatrix} 1 & h(z) \\ 0 & 1 \end{pmatrix} H_0(z), \quad (4.5.8)$$

where $P_{\text{Be},0}(z; \alpha)$ is analytic in a neighbourhood of 0 and defined in (4.8.4), and where f is given by

$$f(z; x) = \frac{-1}{2\pi i} \sum_{j=1}^k (1 - s_j) \int_{-r_j x}^{-r_{j-1} x} \frac{|s|^\alpha}{s - z} ds. \quad (4.5.9)$$

From (4.8.4) and (4.5.5), we have the matching condition (4.5.6).

4.5.4 Small norm RH problem

Now, we proceed to the final transformation, in order to obtain a small-norm RH problem, whose solution will be denoted R . We define

$$R(z; x) = \begin{cases} W(z)P^{(\infty)}(z)^{-1}, & z \in \mathbb{C} \setminus D_0, \\ W(z)P(z)^{-1}, & z \in D_0. \end{cases} \quad (4.5.10)$$

By construction, R has non trivial jumps only on ∂D_0 , where it satisfies

$$R_+(z; x) = (I + \mathcal{O}(x))R_-(z; x), \quad \text{for } z \in \partial D_0, \quad \text{as } x \rightarrow 0. \quad (4.5.11)$$

Furthermore, from (4.5.7), R is bounded near $0, -r_1 x, \dots, -r_k x$ and thus $0, -r_1 x, \dots, -r_k x$ are removable singularities. It follows that R is analytic on $\mathbb{C} \setminus \partial D_0$. Also, from (4.5.2), (4.5.5) and (4.8.2), we have that $R(z) = I + \mathcal{O}(z^{-1})$ as $z \rightarrow \infty$. It follows from standard theory for small norm RH problem that R exists for sufficiently small x and satisfies

$$R(z; x) = I + \mathcal{O}(x), \quad R'(z; x) = \mathcal{O}(x), \quad (4.5.12)$$

uniformly for $z \in \mathbb{C} \setminus \partial D_0$, as $x \rightarrow 0$.

We are now in a position to compute the small x asymptotics of $b_0(x), \dots, b_k(x)$. Inverting the transformations $R \mapsto W \mapsto \Phi$, we obtain for $z \in D_0$ that

$$\Phi(z; \vec{r}x) = R(z; x) P_{\text{Be},0}(z; \alpha) \begin{pmatrix} 1 & f(z; x) \\ 0 & 1 \end{pmatrix} z^{\frac{\alpha}{2}\sigma_3} \begin{pmatrix} 1 & h(z) \\ 0 & 1 \end{pmatrix} H_{-r_k x}(z). \quad (4.5.13)$$

By definition of b (4.3.18), we have, for any $j \in \{1, \dots, k\}$,

$$b(z; \sqrt{x}) = i\sqrt{x} \lim_{z \rightarrow -r_j} (z + r_j) [\partial_z(\Phi(xz; \vec{r}x)) \Phi^{-1}(xz; \vec{r}x)]_{12}. \quad (4.5.14)$$

Using the explicit form for Φ (4.5.13), and the small x asymptotics for R , after some calculations we obtain for $j \in \{1, \dots, k\}$ that

$$b_j(\sqrt{x}) = i\sqrt{x}(1 + \mathcal{O}(x)) P_{\text{Be},0,11}^2(0; \alpha) \lim_{z \rightarrow -r_j} (z + r_j) \partial_z(f(xz; x)). \quad (4.5.15)$$

For $j \in \{1, \dots, k-1\}$, only two terms in the sum (4.5.9) contribute to this limit. Note that after a straightforward calculation we have that

$$\begin{aligned} \lim_{z \rightarrow -r_j} (z + r_j) \int_{-r_j x}^{-r_{j-1} x} \frac{x|s|^\alpha}{(s - xz)^2} ds &= -(r_j x)^\alpha, \\ \lim_{z \rightarrow -r_j} (z + r_j) \int_{-r_{j+1} x}^{-r_j x} \frac{x|s|^\alpha}{(s - xz)^2} ds &= (r_j x)^\alpha. \end{aligned}$$

For $j = k$, the analysis is similar except that only one term in the sum (4.5.9) contributes to the limit (4.5.14). Thus, we obtain as $x \rightarrow 0$

$$b_j(\sqrt{x}) = \frac{\sqrt{x}}{2\pi} P_{\text{Be},0,11}^2(0; \alpha) (r_j x)^\alpha (s_{j+1} - s_j) (1 + \mathcal{O}(x)), \quad j \in \{1, \dots, k\}. \quad (4.5.16)$$

To obtain an explicit form for $P_{\text{Be},0,11}(0; \alpha)$, we use the explicit form of P_{Be} given in the section 4.8, see (4.8.5). Since $P_{\text{Be},0,11}(z; \alpha)$ is an entire function of z , we can take the limit from any region. In particular, for $z \in \{z \in \mathbb{C} : \frac{2\pi}{3} < \arg z < \pi\}$, we have

$$P_{\text{Be},0,11}(z; \alpha) = e^{\frac{\pi i \alpha}{2}} z^{-\frac{\alpha}{2}} \frac{1}{2} \left(H_\alpha^{(1)}(2\sqrt{-z}) + H_\alpha^{(2)}(2\sqrt{-z}) \right). \quad (4.5.17)$$

By the connection formula [OLBC10, 10.4.4], it simplifies to

$$P_{\text{Be},0,11}(z; \alpha) = \sqrt{\pi} e^{\frac{\pi i \alpha}{2}} z^{-\frac{\alpha}{2}} J_\alpha(\sqrt{-z}), \quad (4.5.18)$$

and by taking the limit $z \rightarrow 0$ we have $P_{\text{Be},0,11}(0; \alpha) = \frac{\sqrt{\pi}}{2^\alpha \Gamma(\alpha+1)}$. The equation (4.5.16) can now be rewritten as

$$b_j(\sqrt{x}) = \frac{\sqrt{x}(s_{j+1} - s_j)}{2} J_\alpha(\sqrt{r_j x})^2 (1 + \mathcal{O}(x)), \quad \text{as } x \rightarrow 0, \quad (4.5.19)$$

where $j \in \{1, \dots, k\}$. By the definition of q_j , see (4.3.28), we have

$$q_j(x) = \sqrt{s_{j+1} - s_j} J_\alpha(\sqrt{r_j x}), \quad \text{as } x \rightarrow 0, \quad (4.5.20)$$

which is precisely the boundary conditions of the system (4.1.9) and in particular this proves (4.4.24). Also, (4.5.13) implies for every $j \in \{1, \dots, k\}$ that as $x \rightarrow 0$ we have

$$\begin{aligned} [\Phi_-^{-1}(-r_j x; \vec{r}x) \Phi'_-(-r_j x; \vec{r}x)]_{21} = \\ (-r_j x)^\alpha \left([P_{\text{Be},0}^{-1}(-r_j x; \alpha) P'_{\text{Be},0}(-r_j x)]_{21} + \mathcal{O}(x) \right) = \mathcal{O}(x^\alpha), \end{aligned}$$

which proves (4.4.23).

4.6 Asymptotics for $s_j \rightarrow s_{j+1}$, $j \in \{1, \dots, k\}$

Let $j \in \{1, \dots, k\}$ and if $j \neq 1$, we assume $s_{j+1} \neq s_{j-1}$. When $s_j \rightarrow s_{j+1}$, the jumps on $(-r_j x, -r_{j-1} x)$ tend to be the same as those on $(-r_{j+1} x, -r_j x)$ and therefore the logarithmic singularity at $z = -r_j x$ for $\Phi(z; \vec{r}x, \vec{s})$ tends to disappear. Consider U_j , a fixed open neighbourhood of $[-r_j x, -r_{j-1} x]$ with smooth boundaries, sufficiently small such that it does not include any $-r_\ell x$, $\ell \neq j$, $\ell \neq j-1$. Outside U_j , the model RH problem $\Phi(z; \vec{r}^{[j]}x, \vec{s}^{[j]})$ possesses exactly the same jumps and the same large z asymptotics as $\Phi(z; \vec{r}x, \vec{s})$, and thus heuristically it is a good approximation of $\Phi(z; \vec{r}x, \vec{s})$ for $z \in \mathbb{C} \setminus U_j$. Furthermore, for $z \in U_j$, from (4.3.8) and (4.3.10), $\Phi(z; \vec{r}^{[j]}x, \vec{s}^{[j]})$ can be written as

$$\Phi(z; \vec{r}^{[j]}x, \vec{s}^{[j]}) =$$

$$\Phi_{0,j-1}^*(z) \begin{pmatrix} 1 & \frac{s_{j+1}-s_{j-1}}{2\pi i} \log(z+r_{j-1}x) \\ 0 & 1 \end{pmatrix} V_{j-1}(z) e^{\frac{\pi i \alpha}{2} \theta(z) \sigma_3} H_{-r_k x}(z), \quad (4.6.1)$$

if $j \in \{2, \dots, k\}$ and as

$$\Phi(z; \vec{r}^{[j]}x, \vec{s}^{[j]}) = \Phi_{0,0}^*(z) z^{\frac{\alpha}{2} \sigma_3} \begin{pmatrix} 1 & s_2 h(z) \\ 0 & 1 \end{pmatrix}, \quad \text{if } j = 1. \quad (4.6.2)$$

Therefore, we define the local parametrix inside U_j by

$$P(z; x) = \Phi_{0,j-1}^*(z) \begin{pmatrix} 1 & \frac{s_{j+1}-s_j}{2\pi i} \log(z+r_j x) + \frac{s_j-s_{j-1}}{2\pi i} \log(z+r_{j-1}x) \\ 0 & 1 \end{pmatrix} \times V_{j-1}(z) e^{\frac{\pi i \alpha}{2} \theta(z) \sigma_3} H_{-r_k x}(z), \quad (4.6.3)$$

if $j \in \{2, \dots, k\}$ and

$$P(z; x) = \Phi_{0,0}^*(z) \begin{pmatrix} 1 & \tilde{f}(z; x) \\ 0 & 1 \end{pmatrix} z^{\frac{\alpha}{2} \sigma_3} \begin{pmatrix} 1 & s_2 h(z) \\ 0 & 1 \end{pmatrix}, \quad \text{if } j = 1, \quad (4.6.4)$$

where $\tilde{f}(z; x) = -\frac{s_2-s_1}{2\pi i} \int_{-r_1 x}^0 \frac{|s|^\alpha}{s-z} ds$. It is direct to check that $P(z)$ has exactly the same jumps as $\Phi(z; \vec{r}x, \vec{s})$ inside U_j . We define

$$R(z; x) = \begin{cases} \Phi(z; \vec{r}x, \vec{s}) \Phi(z; \vec{r}^{[j]}x, \vec{s}^{[j]})^{-1}, & \text{for } z \in \mathbb{C} \setminus U_j, \\ \Phi(z; \vec{r}x, \vec{s}) P(z)^{-1}, & \text{for } z \in U_j. \end{cases} \quad (4.6.5)$$

From the above remarks, it follows that $R(z; x) = I + \mathcal{O}(z^{-1})$ as $z \rightarrow \infty$ and $R(z; x)$ has no jump at all inside and outside U_j , and has removable singularities at $0, -x_1, \dots, -x_k$. Let us denote the boundary of U_j by ∂U_j , whose orientation is chosen to be clockwise. For $z \in \partial U_j$, the jumps J_R

of R satisfy $J_R(z) = P(z)\Phi(z; x, \vec{r}^{[j]}, \vec{s}^{[j]})^{-1}$, or more explicitly

$$J_R(z) = \begin{cases} \Phi_{0,j-1}^*(z) \begin{pmatrix} 1 & \frac{s_{j+1}-s_j}{2\pi i} \log \left(\frac{z+r_j x}{z+r_{j-1} x} \right) \\ 0 & 1 \end{pmatrix} \Phi_{0,j-1}^*(z)^{-1}, \\ \quad \text{if } j \in \{2, \dots, k\}, \\ \Phi_{0,0}^*(z) \begin{pmatrix} 1 & \tilde{f}(z; x) \\ 0 & 1 \end{pmatrix} \Phi_{0,0}^*(z)^{-1}, \\ \quad \text{if } j = 1. \end{cases} \quad (4.6.6)$$

In all the cases, we thus have $J_R(z) = I + \mathcal{O}(s_{j+1} - s_j)$ as $s_{j+1} - s_j \rightarrow 0$ uniformly for $z \in \partial U_j$. It follows from standard analysis for small norm RH problem that R exists for sufficiently small $s_{j+1} - s_j$ and satisfies

$$R(z; x) = I + \mathcal{O}(s_{j+1} - s_j), \quad R'(z; x) = \mathcal{O}(s_{j+1} - s_j), \quad (4.6.7)$$

uniformly for $z \in \mathbb{C} \setminus \partial U_j$, as $s_{j+1} - s_j \rightarrow 0$. By definition of b , we have (4.5.14), i.e.

$$b_\ell(\sqrt{x}; \vec{r}, \vec{s}) = i\sqrt{x} \lim_{z \rightarrow -r_\ell} (z + r_\ell) [\partial_z(\Phi(xz; \vec{r}x, \vec{s}))\Phi^{-1}(xz; \vec{r}x, \vec{s})]_{12}. \quad (4.6.8)$$

If $\ell \neq j$, $\ell \neq j-1$, then $-r_j x \notin U_j$, and in the above limit, from (4.6.5) we have $\Phi(xz; \vec{r}x, \vec{s}) = R(xz; x)\Phi(xz; \vec{r}^{[j]}x, \vec{s}^{[j]})$, and thus

$$\begin{aligned} \partial_z(\Phi(xz; \vec{r}x, \vec{s}))\Phi^{-1}(xz; \vec{r}x, \vec{s}) &= \partial_z(R(xz; x))R(xz; x) \\ &+ R(xz; x)\partial_z(\Phi(xz; \vec{r}^{[j]}x, \vec{s}^{[j]}))\Phi^{-1}(xz; \vec{r}^{[j]}x, \vec{s}^{[j]})R(xz; x)^{-1}. \end{aligned} \quad (4.6.9)$$

Thus by (4.6.8), we have

$$b_\ell(\sqrt{x}, \vec{r}, \vec{s}) = b_{\tilde{\ell}}(\sqrt{x}, \vec{r}^{[j]}, \vec{s}^{[j]})(1 + \mathcal{O}(s_{j+1} - s_j)), \quad \ell \notin \{j, j-1\}, \quad (4.6.10)$$

where $\tilde{\ell} = \ell$ if $\ell < j-1$ and $\tilde{\ell} = \ell-1$ if $\ell > j$. If $\ell \in \{j, j-1\}$, then $-r_\ell x \in U_j$ and we have to use the local parametrix

$$\begin{aligned} b_\ell(\sqrt{x}, \vec{r}, \vec{s}) &= i\sqrt{x} \lim_{z \rightarrow -r_\ell} (z + r_\ell) [\partial_z(R(xz; x))R(xz; x)^{-1} \\ &+ R(xz; x)\partial_z(P(xz; x))P^{-1}(xz; x)R^{-1}(xz; x)]_{12}. \end{aligned} \quad (4.6.11)$$

Note that from (4.6.1) and (4.6.3), we have as $z \rightarrow -r_{j-1}$ that

$$\frac{[\partial_x(P(xz))P^{-1}(xz)]_{12}}{[\partial_z(\Phi(xz; \vec{r}^{[j]}, \vec{s}^{[j]}))\Phi^{-1}(xz; \vec{r}^{[j]}x, \vec{s}^{[j]})]_{12}} \sim 1 - \frac{s_{j+1} - s_j}{s_{j+1} - s_{j-1}},$$

and for $j = 1$, from (4.6.2) and (4.6.4), we have as $z \rightarrow 0$ that

$$\frac{[\partial_x(P(xz))P^{-1}(xz)]_{12}}{[\partial_z(\Phi(xz; \vec{r}^{[j]}, \vec{s}^{[j]}))\Phi^{-1}(xz; \vec{r}^{[j]}x, \vec{s}^{[j]})]_{12}} \sim \frac{\left[\Phi_{0,0}^*(0) \begin{pmatrix} 1 & \tilde{f}(0) \\ 0 & 1 \end{pmatrix} \sigma_3 \begin{pmatrix} 1 & -\tilde{f}(0) \\ 0 & 1 \end{pmatrix} \Phi_{0,0}^*(0)^{-1} \right]_{12}}{[\Phi_{0,0}^*(0)\sigma_3\Phi_{0,0}^*(0)^{-1}]_{12}}.$$

Thus we also obtain $b_{j-1}(\sqrt{x}; \vec{r}, \vec{s}) = b_{j-1}(\sqrt{x}; \vec{r}^{[j]}, \vec{s}^{[j]}) + \mathcal{O}(s_{j+1} - s_j)$ as $s_j \rightarrow s_{j+1}$. When $\ell = j \in \{2, \dots, k\}$, from (4.6.3) and (4.6.8), we have as $s_j \rightarrow s_{j+1}$

$$b_j(\sqrt{x}; \vec{r}, \vec{s}) = i\sqrt{x} \frac{s_{j+1} - s_j}{2\pi i} (\Phi_{0,j-1}^*(-r_j x))_{11}^2 (1 + \mathcal{O}(s_{j+1} - s_j)) = \mathcal{O}(s_{j+1} - s_j).$$

For $\ell = j = 1$, from (4.6.4), we obtain similarly that

$$b_1(\sqrt{x}; \vec{r}, \vec{s}) = i\sqrt{x} \frac{s_{j+1} - s_j}{2\pi i} (r_1 x)^\alpha (\Phi_{0,0}^*(-r_1 x))_{11}^2 (1 + \mathcal{O}(s_2 - s_1)) = \mathcal{O}(s_2 - s_1). \quad (4.6.12)$$

This finishes the proof of part 1 of Theorem 4.1.6

4.7 Asymptotics for $r_j \rightarrow r_{j-1}$, $j \in \{2, \dots, k\}$

As $r_j \rightarrow r_{j-1}$, the jumps along $(-r_j x, -r_{j-1} x)$ tend to disappear. Thus, we do exactly the same steepest descent as in the previous section. The computations are very similar and we will give less detail. We define R exactly as in (4.6.5). By (4.6.6), we have $J_R(z) = \mathcal{O}(r_j - r_{j-1})$ as $r_j \rightarrow r_{j-1}$ uniformly for $z \in \partial U_j$. Thus, by standard theory for small

norm RH problems, R exists for sufficiently small $r_j - r_{j-1}$ and satisfies

$$R(z) = I + \mathcal{O}(r_j - r_{j-1}), \quad R'(z) = \mathcal{O}(r_j - r_{j-1}), \quad (4.7.1)$$

uniformly for $z \in \mathbb{C} \setminus \partial U_j$. From (4.6.8), (4.6.9) together with (4.7.1), we have

$$b_\ell(\sqrt{x}; \vec{r}, \vec{s}) = b_\ell(\sqrt{x}; \vec{r}^{[j]}, \vec{s}^{[j]}) + \mathcal{O}(r_j - r_{j-1}),$$

for any $\ell \in \{1, \dots, k\} \setminus \{j, j-1\}$. Let $j \in \{2, \dots, k\}$. From (4.6.3), (4.6.11) and (4.7.1), we obtain

$$b_{j-1}(\sqrt{x}; \vec{r}, \vec{s}) = \frac{s_j - s_{j-1}}{s_{j+1} - s_{j-1}} b_{j-1}(\sqrt{x}; \vec{r}^{[j]}, \vec{s}^{[j]}) + \mathcal{O}(r_j - r_{j-1}),$$

$$b_j(\sqrt{x}; \vec{r}, \vec{s}) = \frac{s_{j+1} - s_j}{s_{j+1} - s_{j-1}} b_{j-1}(\sqrt{x}; \vec{r}^{[j]}, \vec{s}^{[j]}) + \mathcal{O}(r_j - r_{j-1}).$$

This proves part 2 of Theorem 4.1.6. If $j = 1$, the computations are very similar to (4.6.12). From (4.6.4), (4.6.11) and (4.7.1), we obtain

$$b_1(\sqrt{x}; \vec{r}, \vec{s}) = i\sqrt{x} \frac{s_2 - s_1}{2\pi i} (r_1 x)^\alpha (\Phi_{0,0}^*(-r_1 x))_{11}^2 (1 + \mathcal{O}(r_1)) = \mathcal{O}(r_1^\alpha),$$

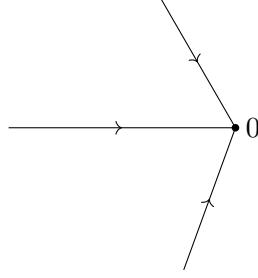
which is part 3 of Theorem 4.1.6.

4.8 Bessel model RH problem

In this final section we present the well-known Bessel RH problem, whose solution is denoted P_{Be} and depends on a parameter $\alpha > -1$. At the end of the section, we also define $\tilde{P}_{\text{Be}}$, which is obtained from P_{Be} by a simple transformation and satisfies a modified version of the Bessel model RH problem.

RH problem for $P_{\text{Be}}(z) = P_{\text{Be}}(z; \alpha)$

- (a) $P_{\text{Be}} : \mathbb{C} \setminus \Sigma_B \rightarrow \mathbb{C}^{2 \times 2}$ is analytic, where Σ_B is shown in Figure 4.2.

Figure 4.2: The jump contour Σ_B for $P_{\text{Be}}(z)$.

(b) P_{Be} satisfies the jump conditions

$$\begin{aligned} P_{\text{Be},+}(z) &= P_{\text{Be},-}(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in \mathbb{R}^-, \\ P_{\text{Be},+}(z) &= P_{\text{Be},-}(z) \begin{pmatrix} 1 & 0 \\ e^{\pi i \alpha} & 1 \end{pmatrix}, & z \in e^{\frac{2\pi i}{3}} \mathbb{R}^+, \\ P_{\text{Be},+}(z) &= P_{\text{Be},-}(z) \begin{pmatrix} 1 & 0 \\ e^{-\pi i \alpha} & 1 \end{pmatrix}, & z \in e^{-\frac{2\pi i}{3}} \mathbb{R}^+. \end{aligned} \quad (4.8.1)$$

(c) As $z \rightarrow \infty$, $z \notin \Sigma_B$, we have

$$P_{\text{Be}}(z) = (I + \mathcal{O}(z^{-1})) z^{\frac{-\sigma_3}{4}} N e^{z^{\frac{1}{2}} \sigma_3}. \quad (4.8.2)$$

(d) As z tends to 0, the behaviour of $P_{\text{Be}}(z)$ is

$$\begin{aligned} P_{\text{Be}}(z) &= \begin{cases} \mathcal{O}(1) z^{\frac{\alpha}{2} \sigma_3}, & |\arg z| < \frac{2\pi}{3}, \\ \mathcal{O}(z^{-\frac{\alpha}{2}}), & \frac{2\pi}{3} < |\arg z| < \pi, \end{cases} \quad \text{if } \alpha > 0, \\ P_{\text{Be}}(z) &= \mathcal{O}(\log z), & \text{if } \alpha = 0, \\ P_{\text{Be}}(z) &= \mathcal{O}(z^{\frac{\alpha}{2}}), & \text{if } \alpha < 0. \end{aligned} \quad (4.8.3)$$

Note that by deleting the jumps of P_{Be} near the origin, we obtain the relation

$$P_{\text{Be}}(z) = P_{\text{Be},0}(z) z^{\frac{\alpha}{2} \sigma_3} \begin{pmatrix} 1 & h(z) \\ 0 & 1 \end{pmatrix} H_0(z), \quad (4.8.4)$$

where $P_{\text{Be},0}$ is an entire function, h is defined in (4.3.11) and H_0 is defined in (4.3.1). It was shown in [KMAV04] that the unique solution

to this RH problem is given by

$$P_{\text{Be}}(z) = \begin{pmatrix} 1 & 0 \\ \frac{i}{8}(4\alpha^2 + 3) & 1 \end{pmatrix} \pi^{\frac{\sigma_3}{2}} \begin{pmatrix} I_\alpha(z^{\frac{1}{2}}) & \frac{i}{\pi} K_\alpha(z^{\frac{1}{2}}) \\ \pi i z^{\frac{1}{2}} I'_\alpha(z^{\frac{1}{2}}) & -z^{\frac{1}{2}} K'_\alpha(z^{\frac{1}{2}}) \end{pmatrix} H_0(z). \quad (4.8.5)$$

where I_α and K_α are the modified Bessel functions of the first and second kind.

Note that $\begin{pmatrix} I_\alpha(z^{\frac{1}{2}}) & \frac{i}{\pi} K_\alpha(z^{\frac{1}{2}}) \\ \pi i z^{\frac{1}{2}} I'_\alpha(z^{\frac{1}{2}}) & -z^{\frac{1}{2}} K'_\alpha(z^{\frac{1}{2}}) \end{pmatrix} H_0(z)$ is equal to

$$\begin{cases} \begin{pmatrix} I_\alpha(z^{\frac{1}{2}}) & \frac{i}{\pi} K_\alpha(z^{\frac{1}{2}}) \\ \pi i z^{\frac{1}{2}} I'_\alpha(z^{\frac{1}{2}}) & -z^{\frac{1}{2}} K'_\alpha(z^{\frac{1}{2}}) \end{pmatrix}, & \text{if } |\arg z| < \frac{2\pi}{3}, \\ \begin{pmatrix} \frac{1}{2} H_\alpha^{(1)}((-z)^{\frac{1}{2}}) & \frac{1}{2} H_\alpha^{(2)}((-z)^{\frac{1}{2}}) \\ \frac{1}{2} \pi z^{\frac{1}{2}} (H_\alpha^{(1)})'((-z)^{\frac{1}{2}}) & \frac{1}{2} \pi z^{\frac{1}{2}} (H_\alpha^{(2)})'((-z)^{\frac{1}{2}}) \end{pmatrix} e^{\frac{\pi i \alpha}{2} \sigma_3}, & \text{if } \frac{2\pi}{3} < \arg z < \pi, \\ \begin{pmatrix} \frac{1}{2} H_\alpha^{(2)}((-z)^{\frac{1}{2}}) & -\frac{1}{2} H_\alpha^{(1)}((-z)^{\frac{1}{2}}) \\ -\frac{1}{2} \pi z^{\frac{1}{2}} (H_\alpha^{(2)})'((-z)^{\frac{1}{2}}) & \frac{1}{2} \pi z^{\frac{1}{2}} (H_\alpha^{(1)})'((-z)^{\frac{1}{2}}) \end{pmatrix} e^{-\frac{\pi i \alpha}{2} \sigma_3}, & \text{if } -\pi < \arg z < -\frac{2\pi}{3}, \end{cases} \quad (4.8.6)$$

where $H_\alpha^{(1)}$ and $H_\alpha^{(2)}$ are the Hankel functions of the first and second kind.

We will also use a modified version of the above RH problem. We define

$$\tilde{P}_{\text{Be}}(z) = e^{-\frac{\pi i}{4} \sigma_3} \pi^{\frac{\sigma_3}{2}} \begin{pmatrix} I_\alpha(z^{\frac{1}{2}}) & \frac{i}{\pi} K_\alpha(z^{\frac{1}{2}}) \\ \pi i z^{\frac{1}{2}} I'_\alpha(z^{\frac{1}{2}}) & -z^{\frac{1}{2}} K'_\alpha(z^{\frac{1}{2}}) \end{pmatrix} H_{-x_k}(z). \quad (4.8.7)$$

From the jumps for P_{Be} , we have that $\tilde{P}_{\text{Be}}$ has exactly the same jumps as P_{Be} on $\Sigma_1 \cup \Sigma_2 \cup (-\infty, -x_k)$, Σ_1 and Σ_2 being depicted on Figure 4.1. We can compute the jumps of $\tilde{P}_{\text{Be}}$ on $(-x_k, 0)$ either from the properties of the Bessel functions, or from the jumps of P_{Be} : it is given

by the product $\begin{pmatrix} 1 & 0 \\ e^{-\pi i \alpha} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{\pi i \alpha} & 1 \end{pmatrix}$, i.e. we have

$$\tilde{P}_{\text{Be},+}(z) = \tilde{P}_{\text{Be},-}(z) \begin{pmatrix} e^{\pi i \alpha} & 1 \\ 0 & e^{-\pi i \alpha} \end{pmatrix}, \quad z \in (-x_k, 0). \quad (4.8.8)$$

Also, from (4.8.2), as $z \rightarrow \infty$, $z \notin \Sigma_B$, we have

$$\tilde{P}_{\text{Be}}(z) = e^{-\frac{\pi i}{4} \sigma_3} \begin{pmatrix} 1 & 0 \\ \frac{-i}{8}(4\alpha^2 + 3) & 1 \end{pmatrix} (I + \mathcal{O}(z^{-1})) z^{\frac{-\sigma_3}{4}} N e^{z^{\frac{1}{2}} \sigma_3}. \quad (4.8.9)$$

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