BOUSSINESQ'S EQUATION FOR WATER WAVES: ASYMPTOTICS IN SECTOR I

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Abstract. In a recent paper, we showed that the large (x,t) behavior of a class of physically relevant solutions of Boussinesq's equation for water waves is described by ten main asymptotic sectors. In the sector adjacent to the positive x-axis, referred to as Sector I, we stated without proof an exact expression for the leading asymptotic term together with an error estimate. Here we provide a proof of this asymptotic formula.

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1. Introduction

In a recent paper [10], we developed an inverse scattering approach to the Boussinesq [4] equation

$$u_{tt} = u_{xx} + (u^2)_{xx} + u_{xxxx}. \tag{1.1}$$

This approach yields an expression for the solution $u(x,t)$ in terms of the solution $n(x,t,k)$ of a 1×3-row-vector Riemann–Hilbert (RH) problem, and by performing a Deift–Zhou steepest descent analysis of this RH problem, it is possible to derive asymptotic formulas for the large (x,t) behavior of $u(x,t)$. In [10], we identified in this way ten main sectors in the half-plane $t \geq 0$ describing the asymptotics of $u(x,t)$ for a class of physically relevant initial data (see Figure 1), and in each sector we provided an asymptotic formula for the solution. For conciseness, the formula in the sector adjacent to the positive x-axis, referred to as Sector I in [10], was stated without proof. Here we provide a proof of this formula.

Equation (1.1) first appeared in a paper by Boussinesq from 1872 [4], where it was derived as a model for long small-amplitude waves in a channel. About a century later, Hirota discovered that (1.1) supports soliton solutions [23], and soon afterwards Zakharov [38] constructed a Lax pair for (1.1), thus demonstrating its formal integrability. In the early 1980's, Deift, Tomei, and Trubowitz [16] developed an inverse scattering formalism for an equation closely related to the Boussinesq equation (more precisely, for the equation obtained from (1.1) by deleting the u_{xx}-term). However, the problem of developing an inverse scattering transform formalism for (1.1) is substantially more complicated, because (1.1) involves a spectral problem with coefficients that do not decay to zero at infinity. This introduces new jumps on the unit circle in the RH problem, and it turns out that all the main asymptotic contributions to $u(x,t)$ as $(x,t) \to \infty$ originate from these new jumps. In particular, as shown below, the main contributions to the asymptotic behavior of $u(x,t)$ in Sector I stem from a global parametrix which is used to cancel jumps on certain parts of the unit circle, as well as from six saddle points on the unit circle.

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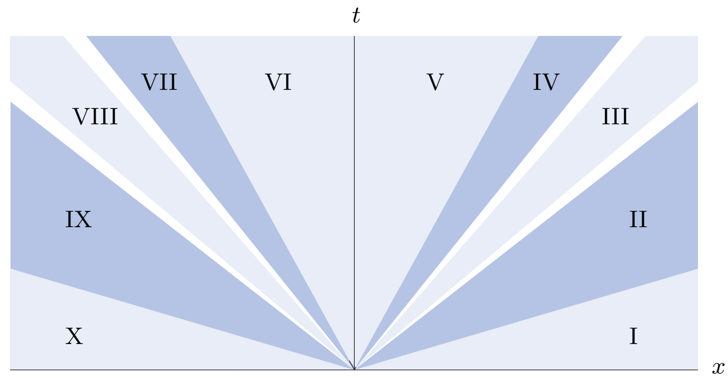

Figure 1. *The ten asymptotic sectors for (1.1) in the xt-plane.*

To describe the large (x,t) asymptotics of $u(x,t)$, it is sufficient to consider Sectors I–V because Sectors IV–X are related to Sectors I–V by symmetry. The derivation of the asymptotic formula in Sector I presented here features a number of differences compared to the derivations in Sectors II–V. For example, in Sector I, x plays the role of the large parameter instead of t, because t is not uniformly large as $(x,t) \to \infty$ in Sector I. The analysis needed for Sector I is also complicated by the fact that as (x,t) approaches the x-axis, the six saddle points relevant for the analysis merge with the six points $\{ie^{\pi ij/3}\}_{j=0}^{5}$, which are all intersection points for the jump contour of the RH problem. This makes the local analysis near the saddle points more involved. It also forces us to perform an additional transformation of the RH problem (see Section 5) and for this purpose additional analytic approximations of the reflection coefficients r_1 and r_2 are required on the six rays that start at the origin and pass through $\{ie^{\pi ij/3}\}_{j=0}^{5}$. Since another set of analytic approximations of r_1 and r_2 is needed on the unit circle, this leads to the curious situation that we have two different approximations of the same functions on overlapping domains. Nevertheless, thanks to certain nonlinear relations obeyed by r_1 and r_2 (see (4.4) and (4.5)), the relevant new jumps can still be estimated (see Lemma 7.3).

Obtaining long-time asymptotics for integrable equations is an important problem with a long history, see e.g. [2, 5–8, 14, 15, 17–21, 24, 25, 29, 31–33, 35].

Equation (1.1) is sometimes referred to as the "bad" Boussinesq equation. The so-called "good" Boussinesq equation has the same form as (1.1) except that the term u_{xxxx} has the opposite sign. Works on the "good" Boussinesq equation or related Boussinesq-type equations include [1, 3, 9, 13, 22, 26–28, 30, 34, 36, 37, 39].

2. Main result

Let $u_0(x)$ and $u_1(x)$ be real-valued functions in the Schwartz class $\mathcal{S}(\mathbb{R})$ and consider the initial value problem for (1.1) on the line with initial conditions

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x) \qquad \text{for } x \in \mathbb{R}. \tag{2.1}$$

2.1. **Assumptions.** Our results will be valid under the same asumptions on u_0 and u_1 as in [10, Theorem 2.14]. To formulate these assumptions, we define two scattering matrices $s(k)$ and $s^A(k)$ by

$$s(k) = I - \int_{\mathbb{R}} e^{-x\mathcal{L}(k)}(\mathsf{U}X)(x,k)e^{x\mathcal{L}(k)}dx, \quad s^A(k) = I + \int_{\mathbb{R}} e^{x\mathcal{L}(k)}(\mathsf{U}^T X^A)(x,k)e^{-x\mathcal{L}(k)}dx,$$

where the following notation is used:

- $\mathcal{L} = \operatorname{diag}(l_1, l_2, l_3)$ where the functions $\{l_j(k)\}_{j=1}^3$ are defined by
$$l_j(k) = i\frac{\omega^j k + (\omega^j k)^{-1}}{2\sqrt{3}}, \qquad k \in \mathbb{C} \setminus \{0\}, \tag{2.2}$$
with $\omega := e^{\frac{2\pi i}{3}}$.
- $\mathsf{U}(x,k)$ is given by
$$\mathsf{U}(x,k) = P(k)^{-1} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{u_{0x}}{4} - \frac{iv_0}{4\sqrt{3}} & -\frac{u_0}{2} & 0 \end{pmatrix} P(k),$$
where $v_0(x) := \int_{-\infty}^x u_1(x')dx'$ and
$$P(k) := \begin{pmatrix} 1 & 1 & 1 \\ l_1(k) & l_2(k) & l_3(k) \\ l_1(k)^2 & l_2(k)^2 & l_3(k)^2 \end{pmatrix}.$$
- $X(x,k)$ and $X^A(x,k)$ are the unique solutions of the Volterra equations
$$X(x,k) = I - \int_x^\infty e^{(x-x')\mathcal{L}(k)}(\mathsf{U}X)(x',k)e^{-(x-x')\mathcal{L}(k)}dx',$$
$$X^A(x,k) = I + \int_x^\infty e^{-(x-x')\mathcal{L}(k)}(\mathsf{U}^T X^A)(x',k)e^{(x-x')\mathcal{L}(k)}dx'.$$

Let $\Gamma = \cup_{j=1}^9 \Gamma_j$ be the contour shown and oriented as in Figure 2. We define two spectral functions $r_1(k)$ and $r_2(k)$ by
$$\begin{cases} r_1(k) = \frac{(s(k))_{12}}{(s(k))_{11}}, & k \in \hat{\Gamma}_1 \setminus \hat{\mathcal{Q}}, \\ r_2(k) = \frac{(s^A(k))_{12}}{(s^A(k))_{11}}, & k \in \hat{\Gamma}_4 \setminus \hat{\mathcal{Q}}, \end{cases} \tag{2.3}$$
where $\hat{\Gamma}_j = \Gamma_j \cup \partial\mathbb{D}$ denotes the union of Γ_j and the unit circle $\partial\mathbb{D}$, and $\hat{\mathcal{Q}} = \{\kappa_j\}_{j=1}^6 \cup \{0\}$ with $\kappa_j = e^{\frac{\pi i(j-1)}{3}}$, $j = 1, \dots, 6$, the sixth roots of unity.

Our assumptions on $\{u_0, u_1\}$ can now be formulated as follows:

(i) Mass conservation: we suppose that $\int_{\mathbb{R}} u_1(x)dx = 0$.

(ii) Absence of solitons: we assume that $s(k)_{11} \neq 0$ for $k \in (\bar{D}_2 \cup \partial\mathbb{D}) \setminus \hat{\mathcal{Q}}$, where $\{D_n\}_1^6$ are the open subsets of the complex plane shown in Figure 2, and $\bar{D}_2$ denotes the closure of D_2.

(iii) Generic behavior of s and s^A near $k = 1$ and $k = -1$: for $k_\star = \pm 1$, we assume that
$$\lim_{k\to k_\star} (k - k_\star)s(k)_{11} \neq 0, \quad \lim_{k\to k_\star} (k - k_\star)s(k)_{13} \neq 0, \quad \lim_{k\to k_\star} s(k)_{31} \neq 0, \quad \lim_{k\to k_\star} s(k)_{33} \neq 0,$$
$$\lim_{k\to k_\star} (k - k_\star)s^A(k)_{11} \neq 0, \quad \lim_{k\to k_\star} (k - k_\star)s^A(k)_{31} \neq 0, \quad \lim_{k\to k_\star} s^A(k)_{13} \neq 0, \quad \lim_{k\to k_\star} s^A(k)_{33} \neq 0.$$

(iv) Existence of a global solution of the initial value problem: we suppose that $r_1(k) = 0$ for all $k \in [0, i]$, where $[0, i]$ is the vertical segment from 0 to i.

2.2. **Statement of the main result.** It is shown in [10] that the solution $u(x,t)$ to the initial value problem for (1.1) with initial data u_0, u_1 satisfying the assumptions (i)–(iv) can be expressed in terms of the solution n of a row-vector RH problem with jump contour Γ. In particular, the assumptions (i)–(iv) imply that the solution $u(x,t)$ exists globally. Our main theorem provides the asymptotics of $u(x,t)$ in Sector I given by $\tau := t/x \in [0, \tau_{\max}]$, where $\tau_{\max} \in (0,1)$ is a constant. We define $\tilde{\Phi}_{ij}(\tau,k)$ for $1 \le j < i \le 3$ by
$$\tilde{\Phi}_{ij}(\tau,k) = (l_i(k) - l_j(k)) + (z_i(k) - z_j(k))\tau, \tag{2.4}$$

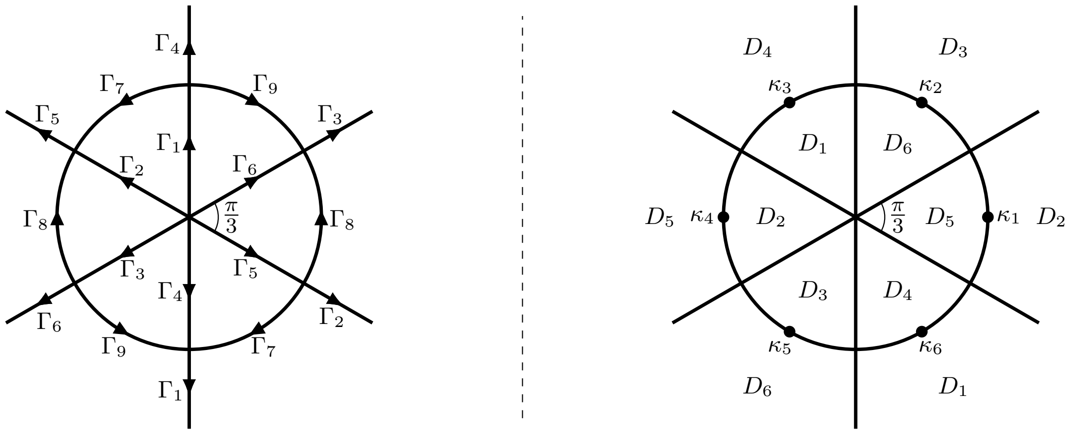

Figure 2. *The contour $\Gamma = \cup_{j=1}^{9} \Gamma_j$ in the complex k-plane (left) and the open sets D_n, $n = 1, \dots, 6$, together with the sixth roots of unity κ_j, $j = 1, \dots, 6$ (right).*

where $l_j(k)$ are as in (2.2) and the functions $z_j(k)$, $j = 1, 2, 3$, are defined similarly:

$$z_j(k) = i\frac{(\omega^j k)^2 + (\omega^j k)^{-2}}{4\sqrt{3}}, \qquad k \in \mathbb{C} \setminus \{0\}. \tag{2.5}$$

Finally, we introduce the saddle points $\{k_j = k_j(\tau)\}_{j=1}^{4}$ of $\tilde{\Phi}_{21}$ which are given by

$$k_1 = \frac{1}{4\tau}\left(1 - \sqrt{8\tau^2+1} + i\sqrt{2}\sqrt{4\tau^2 - 1 + \sqrt{8\tau^2+1}}\right), \qquad k_2 = \bar{k}_1, \tag{2.6a}$$

$$k_3 = \frac{1}{4\tau}\left(1 + \sqrt{8\tau^2+1} + \sqrt{2}\sqrt{-4\tau^2 + 1 + \sqrt{8\tau^2+1}}\right), \qquad k_4 = k_3^{-1}, \tag{2.6b}$$

and satisfy $|k_1| = 1$, $k_3 \in (1, +\infty)$, and $\arg k_1 \in (\frac{\pi}{2}, \frac{2\pi}{3})$. The following is our main result.

Theorem 2.1 (Asymptotics in Sector I)**.** *Let $u_0, u_1 \in \mathcal{S}(\mathbb{R})$ be real-valued functions such that assumptions (i)–(iv) are fulfilled. For any integer $N \geq 1$, there exists a constant $\tau_{\max} \in (0,1)$ such that the global solution $u(x,t)$ of the initial value problem for (1.1) with initial data u_0, u_1 satisfies*

$$u(x,t) = \frac{A(\tau)}{\sqrt{x}} \cos\alpha(x,\tau) + O\left(\frac{1}{x^N} + \frac{C_N(\tau)\ln x}{x}\right) \qquad \text{as } x \to +\infty, \tag{2.7}$$

uniformly for $\tau = t/x \in [0, \tau_{\max}]$, where $C_N(\tau) \geq 0$ is a smooth function of τ that vanishes to all orders at $\tau = 0$, and $A(\tau)$ and $\alpha(x,\tau)$ are defined by

$$A(\tau) := 2\sqrt{3}\frac{\sqrt{-\nu}\sqrt{-1-2\cos(2\arg k_1)}}{-ik_1 z_\star}\operatorname{Im} k_1,$$

$$\alpha(x,\tau) := \frac{3\pi}{4} + \arg r_2(k_1) + \arg\Gamma(i\nu) + \arg d_0 + x \operatorname{Im}\tilde{\Phi}_{21}(\tau, k_1).$$

Here $\Gamma(k)$ is the Gamma function, $z_\star$ is given by

$$z_\star = z_\star(\tau) := \sqrt{2}e^{\frac{\pi i}{4}}\sqrt{\frac{4\tau - 3k_1 - k_1^3}{4k_1^4}}, \tag{2.8}$$

where the branch of the square root is such that $-ik_1 z_\star > 0$, and

$$\nu = \nu(k_1) := -\frac{1}{2\pi}\ln(1 + r_1(k_1) r_2(k_1)) \leq 0,$$

$$\arg d_0 = \arg d_0(x,\tau) := \nu \ln\left|\frac{(\frac{1}{\omega^2 k_1} - k_1)(\frac{1}{\omega k_1} - k_1)}{3(\frac{1}{k_1} - k_1)^2 z_\star^2}\right| - \nu \ln x$$
$$+ \frac{1}{2\pi}\int_i^{k_1} \ln\left|\frac{(k_1 - s)^2(\frac{1}{\omega^2 k_1} - s)(\frac{1}{\omega k_1} - s)}{(\frac{1}{k_1} - s)^2(\omega k_1 - s)(\omega^2 k_1 - s)}\right| d\ln(1 + r_1(s) r_2(s)), \tag{2.9}$$

where the path of the integral $\int_i^{k_1}$ starts at i, follows the unit circle in the counterclockwise direction, and ends at k_1.

Remark 2.2. *Since k_1 tends to i as $\tau \to 0$ and $r_1(k)$ vanishes to all orders at $k = i$, it follows that ν, and hence also $A(\tau)$, vanishes to all orders as $\tau \to 0$. Consequently, (2.7) implies that*

$$u(x,t) = O\bigg(\frac{1}{x^N} + \frac{C_N(\tau)}{\sqrt{x}}\bigg) \qquad \text{as } x \to +\infty$$

uniformly for $\tau \in [0, \tau_{\max}]$. In particular, for any fixed $t \ge 0$, the formula (2.7) reduces to $u(x,t) = O(x^{-N})$ as $x \to +\infty$, in consistency with the fact that $u(x,t)$ is a Schwartz class solution.

Remark 2.3. *By expressing the asymptotic formula in Theorem 2.1 in terms of $\zeta = x/t$ instead of $\tau = t/x$, we see that it is equivalent to the formula stated in [10] for Sector I (note that $z_\star$ here should be identified with $z_\star/\sqrt{\zeta}$ in [10]). We will use the variable τ in this paper, because it allows us to evaluate certain expressions at $t = 0$ and $\tau = 0$ without taking limits.*

Remark 2.4. *By combining the result of Theorem 2.1 with the asymptotic formula for $u(x,t)$ in Sector II (see [10]), it can be seen that the conclusion of Theorem 2.1 in fact holds for any choice of $\tau_{\max} \in (0,1)$.*

The proof of Theorem 2.1 is given in Sections 4–12. It is based on a steepest descent analysis of a RH problem derived in [10] for a 1×3-row-vector valued solution n, which we next recall.

3. The RH problem for n

For $j = 1, \dots, 6$, let $\Gamma_{j'} = \Gamma_j \setminus \mathbb{D}$ be the part of Γ_j outside the open unit disk $\mathbb{D} = \{k \in \mathbb{C} \,|\, |k| < 1\}$ and let $\Gamma_{j''} := \Gamma_j \setminus \Gamma_{j'}$ be the part inside $\mathbb{D}$, where Γ_j is as in Figure 2. Let $\theta_{ij}(x,t,k) = x\,\tilde{\Phi}_{ij}(\tau,k)$ and denote by $\Gamma_\star = \{i\kappa_j\}_{j=1}^6 \cup \{0\}$ the set of intersection points of Γ. Define the function f on the unit circle by

$$f(k) := 1 + r_1(k) r_2(k) + r_1(\tfrac{1}{\omega^2 k}) r_2(\tfrac{1}{\omega^2 k}), \qquad k \in \partial\mathbb{D}, \tag{3.1}$$

and define the jump matrix $v(x,t,k)$ for $k \in \Gamma$ by

$$v_{1'} = \begin{pmatrix} 1 & -r_1(k)e^{-\theta_{21}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; v_{1''} = \begin{pmatrix} 1 & 0 & 0 \\ r_1(\frac{1}{k})e^{\theta_{21}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; v_{2'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -r_2(\frac{1}{\omega k})e^{-\theta_{32}} \\ 0 & 0 & 1 \end{pmatrix},$$

$$v_{2''} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & r_2(\omega k)e^{\theta_{32}} & 1 \end{pmatrix}, \; v_{3'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -r_1(\omega^2 k)e^{\theta_{31}} & 0 & 1 \end{pmatrix}, \; v_{3''} = \begin{pmatrix} 1 & 0 & r_1(\frac{1}{\omega^2 k})e^{-\theta_{31}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$v_{4'} = \begin{pmatrix} 1 & -r_2(\frac{1}{k})e^{-\theta_{21}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; v_{4''} = \begin{pmatrix} 1 & 0 & 0 \\ r_2(k)e^{\theta_{21}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; v_{5'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -r_1(\omega k)e^{-\theta_{32}} \\ 0 & 0 & 1 \end{pmatrix},$$

$$v_{5''} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & r_1(\frac{1}{\omega k})e^{\theta_{32}} & 1 \end{pmatrix}, \; v_{6'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -r_2(\frac{1}{\omega^2 k})e^{\theta_{31}} & 0 & 1 \end{pmatrix}, \; v_{6''} = \begin{pmatrix} 1 & 0 & r_2(\omega^2 k)e^{-\theta_{31}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$v_7 = \begin{pmatrix} 1 & -r_1(k)e^{-\theta_{21}} & r_2(\omega^2 k)e^{-\theta_{31}} \\ -r_2(k)e^{\theta_{21}} & 1 + r_1(k)r_2(k) & \big(r_2(\frac{1}{\omega k}) - r_2(k)r_2(\omega^2 k)\big)e^{-\theta_{32}} \\ r_1(\omega^2 k)e^{\theta_{31}} & \big(r_1(\frac{1}{\omega k}) - r_1(k)r_1(\omega^2 k)\big)e^{\theta_{32}} & f(\omega^2 k) \end{pmatrix},$$

$$v_8 = \begin{pmatrix} f(k) & r_1(k)e^{-\theta_{21}} & \big(r_1(\frac{1}{\omega^2 k}) - r_1(k)r_1(\omega k)\big)e^{-\theta_{31}} \\ r_2(k)e^{\theta_{21}} & 1 & -r_1(\omega k)e^{-\theta_{32}} \\ \big(r_2(\frac{1}{\omega^2 k}) - r_2(\omega k)r_2(k)\big)e^{\theta_{31}} & -r_2(\omega k)e^{\theta_{32}} & 1 + r_1(\omega k)r_2(\omega k) \end{pmatrix},$$

$$v_9 = \begin{pmatrix} 1 + r_1(\omega^2 k)r_2(\omega^2 k) & \big(r_2(\frac{1}{k}) - r_2(\omega k)r_2(\omega^2 k)\big)e^{-\theta_{21}} & -r_2(\omega^2 k)e^{-\theta_{31}} \\ \big(r_1(\frac{1}{k}) - r_1(\omega k)r_1(\omega^2 k)\big)e^{\theta_{21}} & f(\omega k) & r_1(\omega k)e^{-\theta_{32}} \\ -r_1(\omega^2 k)e^{\theta_{31}} & r_2(\omega k)e^{\theta_{32}} & 1 \end{pmatrix}, \tag{3.2}$$

where $v_j, v_{j'}, v_{j''}$ are the restrictions of v to Γ_j, $\Gamma_{j'}$, and $\Gamma_{j''}$, respectively. The jump matrix v obeys the symmetries

$$v(x,t,k) = \mathcal{A}v(x,t,\omega k)\mathcal{A}^{-1} = \mathcal{B}v(x,t,k^{-1})^{-1}\mathcal{B}, \qquad k \in \Gamma, \tag{3.3}$$

where

$$\mathcal{A} := \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \mathcal{B} := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.4}$$

The following RH problem was derived in [10].

RH problem 3.1 (RH problem for n)**.** *Find a 1×3-row-vector valued function $n(x,t,k)$ with the following properties:*

(*a*) $n(x,t,\cdot) : \mathbb{C} \setminus \Gamma \to \mathbb{C}^{1\times 3}$ *is analytic.*

(*b*) *The limits of $n(x,t,k)$ as k approaches $\Gamma \setminus \Gamma_\star$ from the left and right exist, are continuous on $\Gamma \setminus \Gamma_\star$, and are denoted by n_+ and n_-, respectively. Moreover,*

$$n_+(x,t,k) = n_-(x,t,k)v(x,t,k) \qquad \textit{for} \quad k \in \Gamma \setminus \Gamma_\star. \tag{3.5}$$

(*c*) $n(x,t,k) = O(1)$ *as* $k \to k_\star \in \Gamma_\star$.

(*d*) *For $k \in \mathbb{C} \setminus \Gamma$, n obeys the symmetries*

$$n(x,t,k) = n(x,t,\omega k)\mathcal{A}^{-1} = n(x,t,k^{-1})\mathcal{B}. \tag{3.6}$$

(*e*) $n(x,t,k) = (1,1,1) + O(k^{-1})$ *as* $k \to \infty$.

4. Brief overview of the proof

Let $n_3(x,t,k)$ denote the third component of the row-vector $n(x,t,k)$. Since u_0 and u_1 satisfy assumptions (i)–(iv), [10, Theorems 2.6 and 2.12] imply that RH problem 3.1 has a unique solution $n(x,t,k)$ for each $(x,t) \in \mathbb{R} \times [0,\infty)$, that

$$n_3^{(1)}(x,t) := \lim_{k\to\infty} k(n_3(x,t,k) - 1)$$

is well-defined and smooth for $(x,t) \in \mathbb{R} \times [0,\infty)$, and that

$$u(x,t) = -i\sqrt{3}\frac{\partial}{\partial x}n_3^{(1)}(x,t) \tag{4.1}$$

is a real-valued Schwartz class solution of (1.1) on $\mathbb{R}\times[0,\infty)$ with initial data $u(x,0) = u_0(x)$ and $u_t(x,0) = u_1(x)$. We can therefore obtain the asymptotics of $u(x,t)$ by analyzing the RH problem 3.1 with the help of Deift–Zhou [17] steepest descent techniques. In this analysis the saddle points of the three phase functions $\tilde{\Phi}_{21}$, $\tilde{\Phi}_{31}$, and $\tilde{\Phi}_{32}$ play a central role. The saddle points of $\tilde{\Phi}_{21}$, i.e., the solutions of $\partial_k\tilde{\Phi}_{21}(\tau,k)=0$, were denoted by $\{k_j\}_{j=1}^4$ in Section 2. Since $\tilde{\Phi}_{31}(\tau,k)=-\tilde{\Phi}_{21}(\tau,\omega^2 k)$ and $\tilde{\Phi}_{32}(\tau,k)=\tilde{\Phi}_{21}(\tau,\omega k)$, it follows that $\{\omega k_j\}_{j=1}^4$ are the saddle points of $\tilde{\Phi}_{31}$ and that $\{\omega^2 k_j\}_{j=1}^4$ are the saddle points of $\tilde{\Phi}_{32}$. For Sector I only the six saddle points $\{\omega^j k_1,\omega^j k_2\}_{j=0}^2$ play a role in the asymptotic analysis. The signature tables for $\tilde{\Phi}_{21}$, $\tilde{\Phi}_{31}$, and $\tilde{\Phi}_{32}$ are shown in Figure 3.

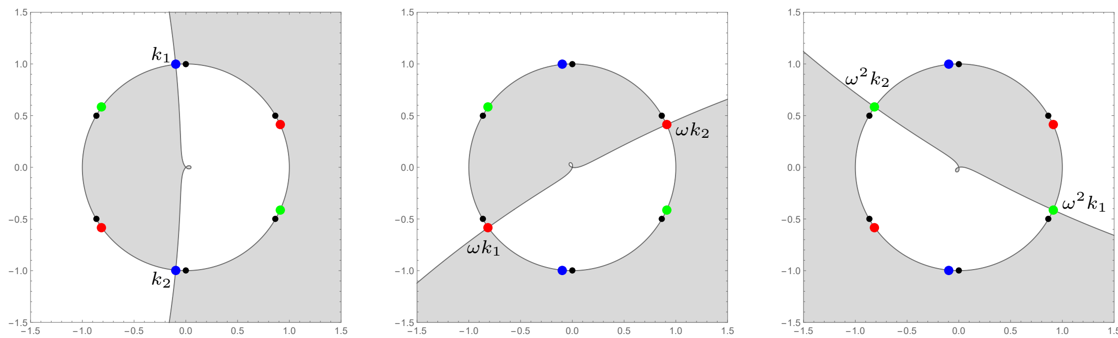

Figure 3. *From left to right: The signature tables for $\tilde{\Phi}_{21}$, $\tilde{\Phi}_{31}$, and $\tilde{\Phi}_{32}$ for $\tau=\frac{1}{10}$. The grey regions correspond to $\{k\,|\,\mathrm{Re}\,\tilde{\Phi}_{ij}>0\}$ and the white regions to $\{k\,|\,\mathrm{Re}\,\tilde{\Phi}_{ij}<0\}$. The saddle points k_1,k_2 of $\tilde{\Phi}_{21}$ are shown blue, the saddle points $\omega k_1,\omega k_2$ of $\tilde{\Phi}_{31}$ red, and the saddle points $\omega^2 k_1,\omega^2 k_2$ of $\tilde{\Phi}_{32}$ green. The black dots are the points $i\kappa_j$, $j=1,\dots,6$.*

The steepest descent analysis proceeds via several transformations $n\mapsto n^{(1)}\mapsto n^{(2)}\mapsto n^{(3)}\mapsto n^{(4)}\mapsto\hat{n}$, such that the RH problems satisfied by $n^{(1)},n^{(2)},n^{(3)},n^{(4)}$, and $\hat{n}$ are equivalent to RH problem 3.1 satisfied by n. We will let $\Gamma^{(j)},\hat{\Gamma}$ and $v^{(j)},\hat{v}$ denote the contours and jump matrices of the RH problems for $n^{(j)},\hat{n}$, respectively. The symmetries (3.3) and (3.6) will be preserved by each transformation, so that, for $j=1,2,3,4$,

$$v^{(j)}(x,t,k)=\mathcal{A}v^{(j)}(x,t,\omega k)\mathcal{A}^{-1}=\mathcal{B}v^{(j)}(x,t,k^{-1})^{-1}\mathcal{B},\qquad k\in\Gamma^{(j)},\tag{4.2}$$

$$n^{(j)}(x,t,k)=n^{(j)}(x,t,\omega k)\mathcal{A}^{-1}=n^{(j)}(x,t,k^{-1})\mathcal{B},\qquad k\in\mathbb{C}\setminus\Gamma^{(j)},\tag{4.3}$$

and similarly for $\hat{n}$ and $\hat{v}$. We will see that six local parametrices near the saddle points $\{k_j,\omega k_j,\omega^2 k_j\}_{j=1}^2$ as well as a global parametrix Δ^{-1} contribute to the asymptotics of $u(x,t)$. However, due to the symmetries (4.2)–(4.3), we will only need to construct the local parametrix near k_1 explicitly, and we will only need to define the transformations $n^{(j)}\mapsto n^{(j+1)}$ and $n^{(4)}\mapsto\hat{n}$ in the sector $\mathsf{S}:=\{k\in\mathbb{C}|\arg k\in[\frac{\pi}{3},\frac{2\pi}{3}]\}$. In the end, we will find that $\hat{n}$ satisfies a small-norm RH problem whose solution can be expressed as an integral over the contour $\hat{\Gamma}$. This leads to an asymptotic formula for $\hat{n}^{(1)}(x,t):=\lim_{k\to\infty}k(\hat{n}(x,t,k)-(1,1,1))$, and hence, by inverting the above transformations, an asymptotic formula also for $n_3^{(1)}(x,t)$. By substituting this formula into (4.1), we obtain the formula of Theorem 2.1.

The following properties of the functions r_1 and r_2 established in [10, Theorem 2.3] will be used repeatedly throughout the proof: $r_1\in C^\infty(\hat{\Gamma}_1)$, $r_2\in C^\infty(\hat{\Gamma}_4\setminus\{\pm\omega^2\})$, $r_1(\kappa_j)\neq 0$ for $j=1,\dots,6$, $r_2(k)$ has simple zeros at $k=\pm\omega$ and simple poles at $k=\pm\omega^2$, and r_1,r_2 have rapid decay as $|k|\to\infty$. Moreover,

$$r_1(\tfrac{1}{\omega k})+r_2(\omega k)+r_1(\omega^2 k)r_2(\tfrac{1}{k})=0,\qquad k\in\partial\mathbb{D}\setminus\{\pm\omega\},\tag{4.4}$$

$$r_2(k)=\tilde{r}(k)\overline{r_1(\bar{k}^{-1})},\qquad \tilde{r}(k):=\frac{\omega^2-k^2}{1-\omega^2k^2},\qquad k\in\hat{\Gamma}_4\setminus\{0,\pm\omega^2\},\tag{4.5}$$

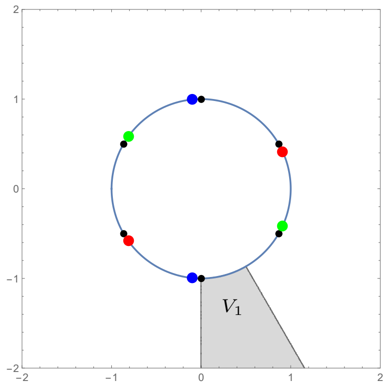

Figure 4. *The open subset V_1 of the complex k-plane.*

$$r_1(1) = r_1(-1) = 1, \qquad r_2(1) = r_2(-1) = -1. \tag{4.6}$$

5. The $n \to n^{(1)}$ transformation

Let $\tau = t/x$ and note that $x\tilde{\Phi}_{ij}(\tau, k) = \theta_{ij}$. We will use the notation $\mathcal{I} := [0, \tau_{\max}]$, where $\tau_{\max} \in (0,1)$ is a constant, so that Sector I corresponds to $\tau \in \mathcal{I}$. The jump matrices $v_{1''}$ and $v_{4'}$ involve the off-diagonal entries $r_1(\frac{1}{k})e^{x\tilde{\Phi}_{21}(\tau,k)}$ and $-r_2(\frac{1}{k})e^{-x\tilde{\Phi}_{21}(\tau,k)}$, respectively. As long as $\tau > 0$ stays away from 0, the exponentials $e^{x\tilde{\Phi}_{21}(\tau,k)}$ and $e^{-x\tilde{\Phi}_{21}(\tau,k)}$ are exponentially small as $x \to \infty$ for $k \in (0, i)$ and $k \in (i, i\infty)$, respectively. However,

$$\tilde{\Phi}_{21}(\tau, k) = \frac{k^2 - 1}{2k} - \frac{k^4 - 1}{4k^2}\tau,$$

so that $\tilde{\Phi}_{21}(0, i\mathrm{Im}\, k) = i\frac{(\mathrm{Im}\, k)^2+1}{2\mathrm{Im}\, k}$ is pure imaginary for all $k \in i\mathbb{R}$, meaning that the above exponentials lose their decay as $\tau \to 0$. The goal of the first transformation $n \to n^{(1)}$ is to deform the jumps $v_{1''}$ and $v_{4'}$ into regions where the exponentials retain their decay as $x \to \infty$ uniformly for small τ. For this purpose, we need to introduce analytic approximations of the functions r_1 and r_2. Let $V_1 \subset \mathbb{C}$ be the open set (see Figure 4)

$$V_1 = \{k \in \mathbb{C} \,|\, \arg k \in (-\tfrac{\pi}{2}, -\tfrac{\pi}{3}),\ |k| > 1\}$$

and let $\bar{V}_1$ denote the closure of V_1.

Lemma 5.1 (Decomposition lemma I)**.** *For every integer $N \geq 1$, there exists a decomposition*

$$r_1(k) = \tilde{r}_{1,a}(x,k) + \tilde{r}_{1,r}(x,k), \qquad k \in \partial V_1 \cap i\mathbb{R} = (-i\infty, -i], \tag{5.1}$$

such that the functions $\tilde{r}_{1,a}, \tilde{r}_{1,r}$ have the following properties:

(a) *For each $x \geq 1$, $\tilde{r}_{1,a}(x,k)$ is defined and continuous for $k \in \bar{V}_1$ and analytic for $k \in V_1$.*

(b) *For $x \geq 1$ and $\tau \in \mathcal{I}$, the function $\tilde{r}_{1,a}$ satisfies*

$$\Big|\tilde{r}_{1,a}(x,k) - \sum_{j=0}^{N} \frac{r_1^{(j)}(k_\star)}{j!}(k - k_\star)^j\Big| \leq C|k - k_\star|^{N+1} e^{\frac{x}{4}|\mathrm{Re}\,\tilde{\Phi}_{21}(\tau,k)|}, \qquad k \in \bar{V}_1, \tag{5.2}$$

$$|\tilde{r}_{1,a}(x,k)| \leq \frac{C}{1 + |k|^{N+1}} e^{\frac{x}{4}|\mathrm{Re}\,\tilde{\Phi}_{21}(\tau,k)|}, \qquad k \in \bar{V}_1, \tag{5.3}$$

where $k_\star = -i$ and the constant C is independent of x, τ, k.

(c) *For each $1 \leq p \leq \infty$, the L^p-norm of $\tilde{r}_{1,r}(x,\cdot)$ on $\partial V_1 \cap i\mathbb{R}$ is $O(x^{-N})$ as $x \to \infty$.*

Remark 5.2. *We will later need to introduce yet another decomposition of* r_1*, see Lemma 6.1. We use tildes on the functions* $\tilde{r}_{1,a}$ *and* $\tilde{r}_{1,r}$ *in Lemma 5.1 to distinguish them from the functions* $r_{1,a}$ *and* $r_{1,r}$ *in Lemma 6.1. The functions* $\tilde{r}_{1,a}$ *and* $r_{1,a}$ *have partially overlapping domains of definition, but are in general not equal on the intersection of these domains. Since the derivations of Lemma 5.1 and Lemma 6.1 use different phase functions* ($-i\tilde{\Phi}(0,k)$ *and* $-i\tilde{\Phi}(\tau,k)$*, respectively), it seems difficult to construct* $\tilde{r}_{1,a}$ *in such a way that it coincides with* $r_{1,a}$ *on this intersection.*

Proof of Lemma 5.1. The proof follows the idea of [17] (see [17, Proposition 1.92]) but is nonstandard because the real part of $\tilde{\Phi}_{21}(\tau,k)$ is nonzero for $k \in (-i\infty,-i)$ whenever $\tau > 0$. We therefore present a proof, whose main idea is to use $-i\tilde{\Phi}_{21}(0,k)$ instead of $-i\tilde{\Phi}_{21}(\tau,k)$ as the phase function.

Let $M \geq N+1$ be a large integer. There exists a rational function $f_0(k)$ with no poles in $\bar{V}_1$ such that r_1 and f_0 coincide to order $4M$ at $k=-i$ and such that $f_0(k) = O(k^{-4M})$ as $(-i\infty,-i] \ni k \to \infty$. Since $r_1 \in C^\infty((-i\infty,-i])$ has rapid decay as k tends to ∞, r_1 and f_0 coincide to order $4M$ also at $k=\infty$. Let $f_1(k) = r_1(k) - f_0(k)$. The map $k \mapsto \phi = \phi(k)$ where

$$\phi = -i\tilde{\Phi}_{21}(0,k) = -\frac{i}{2}\left(k - \frac{1}{k}\right) = \frac{1+(\operatorname{Im} k)^2}{2\operatorname{Im} k} \tag{5.4}$$

is a bijection $(-i\infty,-i] \to (-\infty,-1]$, so we may define

$$F(\phi) := \begin{cases} \frac{k^{2M}}{(k-k_\star)^M} f_1(k), & \phi \leq -1, \\ 0, & \phi > -1, \end{cases} \tag{5.5}$$

with $k_\star := -i$. The function $F(\phi)$ is smooth for $\phi \in \mathbb{R} \setminus \{-1\}$. Also, for $n \geq 1$,

$$F^{(n)}(\phi) = \left(\frac{2ik^2}{(k-k_\star)(k-i)}\frac{d}{dk}\right)^n \left[\frac{k^{2M}}{(k-k_\star)^M} f_1(k)\right], \qquad \phi \leq -1, \tag{5.6}$$

so by choosing M large enough, we can achieve that F belongs to the Sobolev space $H^{N+1}(\mathbb{R})$. We conclude that

$$\hat{F}(s) = \frac{1}{2\pi}\int_{\mathbb{R}} F(\phi) e^{-i\phi s} d\phi, \qquad F(\phi) = \int_{\mathbb{R}} \hat{F}(s) e^{i\phi s} ds \tag{5.7}$$

and, by the Plancherel theorem, $\|s^{N+1}\hat{F}(s)\|_{L^2(\mathbb{R})} = \|\frac{F^{(N+1)}(\phi)}{\sqrt{2\pi}}\|_{L^2(\mathbb{R})} < \infty$. By (5.5) and (5.7),

$$\frac{(k-k_\star)^M}{k^{2M}} \int_{\mathbb{R}} \hat{F}(s) e^{s\tilde{\Phi}_{21}(0,k)} ds = f_1(k)$$

for $k \in (-i\infty,-i]$. In particular, $f_1(k) = f_a(x,k) + f_r(x,k)$ for $x \geq 1$ and $k \in (-i\infty,-i]$, where

$$f_a(x,k) := \frac{(k-k_\star)^M}{k^{2M}} \int_{-\infty}^{\frac{x}{4}} \hat{F}(s) e^{s\tilde{\Phi}_{21}(0,k)} ds, \qquad k \in \bar{V}_1,$$
$$f_r(x,k) := \frac{(k-k_\star)^M}{k^{2M}} \int_{\frac{x}{4}}^{+\infty} \hat{F}(s) e^{s\tilde{\Phi}_{21}(0,k)} ds, \qquad k \in (-i\infty,-i].$$

Since $\operatorname{Re}\tilde{\Phi}_{21}(0,k) = 0$ for $k \in (-i\infty,-i]$, we have

$$|f_r(x,k)| \leq \frac{(k-k_\star)^M}{k^{2M}} \int_{\frac{x}{4}}^{+\infty} s^{N+1}|\hat{F}(s)| s^{-N-1} ds$$
$$\leq \frac{C}{1+|k|^M} \|s^{N+1}\hat{F}(s)\|_{L^2(\mathbb{R})} \sqrt{\int_{\frac{x}{4}}^{+\infty} s^{-2N-2} ds} \leq \frac{C}{1+|k|^M} x^{-N-\frac{1}{2}} \tag{5.8}$$

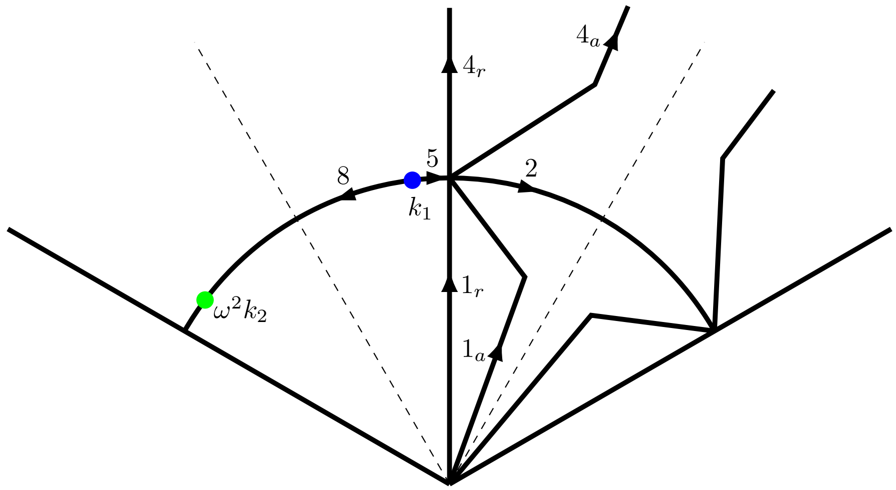

Figure 5. *The contour $\Gamma^{(1)}$ (solid), the boundary of S (dashed), and the saddle points k_1 (blue) and $\omega^2 k_2$ (green).*

for $x \geq 1$ and $k \in (-i\infty, -i]$. Hence the L^1 and L^∞ norms of $f_r(x,\cdot)$ on $(-i\infty,-i]$ are $O(x^{-N-\frac{1}{2}})$. On the other hand, $f_a(x,\cdot)$ is clearly continuous on $\bar{V}_1$ and analytic in V_1. Since $\operatorname{Re}\tilde{\Phi}_{21}(\tau,k) \geq \operatorname{Re}\tilde{\Phi}_{21}(0,k) \geq 0$ for $k \in \bar{V}_1$ and $\tau \in \mathcal{I}$ (cf. Figure 3), it also holds that

$$\begin{aligned}|f_a(x,k)| &\leq \frac{(k-k_\star)^M}{k^{2M}} \|\hat{F}\|_{L^1(\mathbb{R})} \sup_{s \leq \frac{x}{4}} e^{s \operatorname{Re}\tilde{\Phi}_{21}(0,k)} \leq C \frac{(k-k_\star)^M}{k^{2M}} e^{\frac{x}{4}\operatorname{Re}\tilde{\Phi}_{21}(0,k)} \\ &\leq C(k-k_\star)^M e^{\frac{x}{4}\operatorname{Re}\tilde{\Phi}_{21}(\tau,k)} \end{aligned} \tag{5.9}$$

for $x \geq 1$, $\tau \in \mathcal{I}$, and $k \in \bar{V}_1$. Since $M \geq N+1$, it follows that the assignments

$$\begin{aligned} &\tilde{r}_{1,a}(x,k) := f_0(k) + f_a(x,k) && \text{for} \quad k \in \bar{V}_1, \\ &\tilde{r}_{1,r}(x,k) := f_r(x,k) && \text{for} \quad k \in (-i\infty,-i], \end{aligned}$$

yield a decomposition of r_1 with the desired properties. ☐

Lemma 5.1 establishes a decomposition of r_1. The symmetry (4.5) then yields an analogous decomposition $r_2 = \tilde{r}_{2,a} + \tilde{r}_{2,r}$ of $\tilde{r}_2$ as follows:

$$\tilde{r}_{2,a}(x,k) := \tilde{r}(k)\overline{\tilde{r}_{1,a}(x,\bar{k}^{-1})}, \quad k \in V_2; \quad \tilde{r}_{2,r}(x,k) := \tilde{r}(k)\overline{\tilde{r}_{1,r}(x,\bar{k}^{-1})}, \quad k \in (-i,0),$$

where $V_2 := \{k | \bar{k}^{-1} \in V_1\}$.

We are now in a position to define the first transformation $n \mapsto n^{(1)}$. As explained in Section 4, we will focus our attention on the sector S. Let $\Gamma^{(1)}$ be as in Figure 5. Note that

$$\Gamma_8^{(1)} := \{e^{i\theta} \,|\, \theta \in (\arg k_1, \tfrac{2\pi}{3})\}, \qquad \Gamma_2^{(1)} := \{e^{i\theta} \,|\, \theta \in (\tfrac{\pi}{3}, \tfrac{\pi}{2})\}.$$

The matrices $v_{1''}$ and $v_{4'}$ admit the factorizations

$$v_{1''} = v_{1_a}^{(1)} v_{1_r}^{(1)}, \qquad v_{4'} = v_{4_a}^{(1)} v_{4_r}^{(1)},$$

where

$$v_{1_a}^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ \tilde{r}_{1,a}(\frac{1}{k})e^{\theta_{21}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad v_{1_r}^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ \tilde{r}_{1,r}(\frac{1}{k})e^{\theta_{21}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$v_{4_a}^{(1)} = \begin{pmatrix} 1 & -\tilde{r}_{2,a}(\frac{1}{k})e^{-\theta_{21}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad v_{4_r}^{(1)} = \begin{pmatrix} 1 & -\tilde{r}_{2,r}(\frac{1}{k})e^{-\theta_{21}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.10}$$

The function $n^{(1)}$ is defined by

$$n^{(1)}(x,t,k) = n(x,t,k)G^{(1)}(x,t,k), \qquad k \in \mathbb{C} \setminus \Gamma^{(1)}, \tag{5.11}$$

where the function $G^{(1)}$ is analytic in $\mathbb{C} \setminus \Gamma^{(1)}$. It is given for $k \in \mathsf{S}$ by

$$G^{(1)} = \begin{cases} v_{1_a}^{(1)}, & k \text{ on the } - \text{ side of } \Gamma_{1_r}^{(1)}, \\ v_{4_a}^{(1)}, & k \text{ on the } - \text{ side of } \Gamma_{4_r}^{(1)}, \\ I, & \text{otherwise}, \end{cases} \tag{5.12}$$

and extended to $\mathbb{C} \setminus \Gamma^{(1)}$ by means of the symmetry

$$G^{(1)}(x,t,k) = \mathcal{A}G^{(1)}(x,t,\omega k)\mathcal{A}^{-1} = \mathcal{B}G^{(1)}(x,t,k^{-1})\mathcal{B}. \tag{5.13}$$

The next lemma follows from the signature tables of Figure 3 and Lemma 5.1.

Lemma 5.3. *$G^{(1)}(x,t,k)$ and $G^{(1)}(x,t,k)^{-1}$ are uniformly bounded for $k \in \mathbb{C} \setminus \Gamma^{(1)}$, $x \geq 1$, and $\tau \in \mathcal{I}$.*

Using (5.11), we infer that $n_+^{(1)} = n_-^{(1)} v_j^{(1)}$ on $\Gamma_j^{(1)}$, where $v_j^{(1)}$ is given by (5.10) for $j = 1_a, 1_r, 4_a, 4_r$ and for $j = 2, 5, 8$ by

$$v_2^{(1)} = v_9, \qquad v_5^{(1)} = v_7^{-1}, \qquad v_8^{(1)} = v_7.$$

The L^1 and L^∞ norms of $v^{(1)} - I$ on $\Gamma_{1_r}^{(1)} \cup \Gamma_{4_r}^{(1)}$ are uniformly of order $O(x^{-N})$ in Sector I as a consequence of Lemma 5.1. The estimate (5.3) of $\tilde{r}_{1,a}$ ensures that $n^{(1)}$ has the same behavior as n as $k \to \infty$ up to terms of $O(k^{-N})$.

6. The $n^{(1)} \to n^{(2)}$ transformation

The purpose of our next transformation, $n^{(1)} \to n^{(2)}$, is to open lenses. To implement this transformation, we need another set of decompositions of the functions r_1, r_2. We also need decompositions of the functions $\hat{r}_1, \hat{r}_2$ defined for all $k \in \partial\mathbb{D}$ with $\arg k \in [\frac{\pi}{2}, \arg k_1]$ by

$$\hat{r}_j(k) := \frac{r_j(k)}{1 + r_1(k)r_2(k)} = \frac{r_j(k)}{1 + \tilde{r}(k)|r_1(k)|^2}, \qquad j = 1, 2, \tag{6.1}$$

where the second equality follows from (4.5). Since $\arg k_1 \in [\frac{\pi}{2}, \frac{2\pi}{3})$ for $\tau \in [0,1)$, and since $\tilde{r}(e^{i\theta}) > 0$ for $\theta \in [\frac{\pi}{2}, \frac{2\pi}{3})$, we have $1 + r_1(k)r_2(k) = 1 + \tilde{r}(k)|r_1(k)|^2 > 1$ for $|k| = 1, \arg k \in [\frac{\pi}{2}, \arg k_1]$. Thus $\hat{r}_1$, $\hat{r}_2$ are well-defined. Given $K > 1$, we define the open sets $U_1 = U_1(\tau, K)$ and $\hat{U}_1 = \hat{U}_1(\tau, K) \subset \mathbb{C}$ by (see Figure 6)

$$\begin{aligned} U_1 = \big(&\{k \,|\, \arg k \in [-\pi, -\tfrac{5\pi}{6}) \cup (\arg k_1, \pi],\ K^{-1} < |k| < 1\} \\ &\cup \{k \,|\, \arg k \in (-\tfrac{\pi}{2}, \tfrac{\pi}{3}),\ 1 < |k| < K\}\big) \cap \{k \,|\, \operatorname{Re} \Phi_{21} > 0\}, \\ \hat{U}_1 = \{&k \,|\, \arg k \in (\tfrac{\pi}{4}, \arg k_1),\ 1 < |k| < K\} \cap \{k \,|\, \operatorname{Re} \Phi_{21} > 0\}. \end{aligned}$$

Lemma 6.1 (Decomposition lemma II)**.** *For every integer $N \geq 1$, there exist $K > 1$ and decompositions*

$$\begin{aligned} r_1(k) &= r_{1,a}(x,t,k) + r_{1,r}(x,t,k), && k \in \partial U_1 \cap \partial\mathbb{D}, \\ \hat{r}_1(k) &= \hat{r}_{1,a}(x,t,k) + \hat{r}_{1,r}(x,t,k), && k \in \partial \hat{U}_1 \cap \partial\mathbb{D}, \end{aligned}$$

such that $r_{1,a}, r_{1,r}, \hat{r}_{1,a}, \hat{r}_{1,r}$ satisfy the following properties:

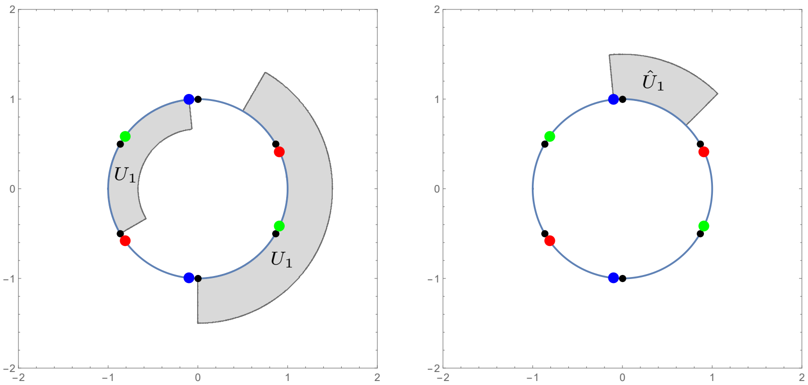

Figure 6. *The open subsets U_1 and $\hat{U}_1$ of the complex k-plane.*

(a) *For each $x \geq 1$ and $\tau \in \mathcal{I}$, $r_{1,a}(x,t,k)$ is defined and continuous for $k \in \bar{U}_1$ and analytic for $k \in U_1$, while $\hat{r}_{1,a}(x,t,k)$ is defined and continuous for $k \in \bar{\hat{U}}_1$ and analytic for $k \in \hat{U}_1$.*

(b) *For $x \geq 1$ and $\tau \in \mathcal{I}$, the functions $r_{1,a}$ and $\hat{r}_{1,a}$ obey*

$$\bigg| r_{1,a}(x,t,k) - \sum_{j=0}^{N} \frac{r_1^{(j)}(k_\star)}{j!}(k-k_\star)^j \bigg| \leq C|k-k_\star|^{N+1} e^{\frac{x}{4}|\mathrm{Re}\,\tilde{\Phi}_{21}(\tau,k)|}, \quad k \in \bar{U}_1,\ k_\star \in \mathcal{R}_1,$$

$$\bigg| \hat{r}_{1,a}(x,t,k) - \sum_{j=0}^{N} \frac{\hat{r}_1^{(j)}(k_\star)}{j!}(k-k_\star)^j \bigg| \leq C|k-k_\star|^{N+1} e^{\frac{x}{4}|\mathrm{Re}\,\tilde{\Phi}_{21}(\tau,k)|}, \quad k \in \bar{\hat{U}}_1,\ k_\star \in \hat{\mathcal{R}}_1,$$

where $\mathcal{R}_1 = \{k_1, \pm\omega, \omega^2 k_2, \pm e^{\frac{\pi i}{6}}, \pm e^{-\frac{\pi i}{6}}, \pm 1, -i, \omega^2 k_1, \omega k_2\}$, $\hat{\mathcal{R}}_1 = \{i, k_1\}$, and $C > 0$ is independent of x, τ, k. In fact, for $k_\star = k_1$, the following stronger estimates hold:

$$\bigg| r_{1,a}(x,t,k) - \sum_{j=0}^{N} \frac{r_1^{(j)}(k_\star)}{j!}(k-k_\star)^j \bigg| \leq C_N(\tau)|k-k_\star|^{N+1} e^{\frac{x}{4}|\mathrm{Re}\,\tilde{\Phi}_{21}(\tau,k)|}, \qquad k \in \bar{U}_1,$$

$$\bigg| \hat{r}_{1,a}(x,t,k) - \sum_{j=0}^{N} \frac{\hat{r}_1^{(j)}(k_\star)}{j!}(k-k_\star)^j \bigg| \leq C_N(\tau)|k-k_\star|^{N+1} e^{\frac{x}{4}|\mathrm{Re}\,\tilde{\Phi}_{21}(\tau,k)|}, \qquad k \in \bar{\hat{U}}_1,$$

where $C_N(\tau) \geq 0$ is a smooth function of τ which is independent of x, k and which vanishes to all orders at $\tau = 0$.

(c) *For each $1 \leq p \leq \infty$, the L^p-norm of $r_{1,r}(x,t,\cdot)$ on $\partial U_1 \cap \partial\mathbb{D}$ is $O(x^{-N})$ and the L^p-norm of $\hat{r}_{1,r}(x,t,\cdot)$ on $\partial\hat{U}_1 \cap \partial\mathbb{D}$ is $O(x^{-N})$ as $x \to \infty$ uniformly for $\tau \in \mathcal{I}$.*

Proof. On each arc of $\partial U_1 \cap \partial\mathbb{D}$ and $\partial\hat{U}_1 \cap \partial\mathbb{D}$, the function $\theta \mapsto -i\tilde{\Phi}_{21}(\tau, e^{i\theta}) = (1 - \tau\cos\theta)\sin\theta$ is real-valued and monotone. The statement therefore follows using the method introduced in [17] (see Case $1.25_>$ beginning on p. 313 of [17]); see also the proof of Lemma 5.1. The stronger estimates in assertion (b) valid in the case of $k_\star = k_1$ follow because $r_1(k)$ and $\hat{r}_1(k)$ vanish to all orders at $k = i$, and k_1 tends to i as $\tau \to 0$. ☐

Lemma 6.1 provides decompositions of r_1 and $\hat{r}_1$; using the symmetry (4.5), we now deduce decompositions of r_2 and $\hat{r}_2$. Let $U_2 := \{k | \bar{k}^{-1} \in U_1\}$ and $\hat{U}_2 := \{k | \bar{k}^{-1} \in \hat{U}_1\}$. We define decompositions $r_2 = r_{2,a} + r_{2,r}$ and $\hat{r}_2 = \hat{r}_{2,a} + \hat{r}_{2,r}$ by

$$r_{2,a}(k) := \tilde{r}(k)\overline{r_{1,a}(\bar{k}^{-1})}, \quad k \in U_2, \qquad r_{2,r}(k) := \tilde{r}(k)\overline{r_{1,r}(\bar{k}^{-1})}, \quad k \in \partial U_2 \cap \partial\mathbb{D},$$

$$\hat{r}_{2,a}(k) := \tilde{r}(k)\overline{\hat{r}_{1,a}(\bar{k}^{-1})}, \quad k \in \hat{U}_2, \qquad \hat{r}_{2,r}(k) := \tilde{r}(k)\overline{\hat{r}_{1,r}(\bar{k}^{-1})}, \quad k \in \partial\hat{U}_2 \cap \partial\mathbb{D},$$

where we have omitted the (x,t)-dependence for conciseness.

On $\Gamma_2^{(2)}$, we factorize $v_2^{(1)}$ as follows:

$$v_2^{(1)} = v_3^{(2)} v_2^{(2)} v_1^{(2)},$$

where

$$v_3^{(2)} = \begin{pmatrix} 1 & 0 & -r_{2,a}(\omega^2 k)e^{-\theta_{31}} \\ r_{1,a}(\frac{1}{k})e^{\theta_{21}} & 1 & r_{1,a}(\omega k)e^{-\theta_{32}} \\ 0 & 0 & 1 \end{pmatrix}, \; v_1^{(2)} = \begin{pmatrix} 1 & r_{2,a}(\frac{1}{k})e^{-\theta_{21}} & 0 \\ 0 & 1 & 0 \\ -r_{1,a}(\omega^2 k)e^{\theta_{31}} & r_{2,a}(\omega k)e^{\theta_{32}} & 1 \end{pmatrix},$$

$$v_2^{(2)} = I + v_{2,r}^{(2)}, \qquad v_{2,r}^{(2)} = \begin{pmatrix} r_{1,r}(\omega^2 k)r_{2,r}(\omega^2 k) & g_2(\omega k)e^{-\theta_{21}} & -r_{2,r}(\omega^2 k)e^{-\theta_{31}} \\ g_1(\omega k)e^{\theta_{21}} & g(\omega k) & h_1(\omega k)e^{-\theta_{32}} \\ -r_{1,r}(\omega^2 k)e^{\theta_{31}} & h_2(\omega k)e^{\theta_{32}} & 0 \end{pmatrix}, \tag{6.2}$$

and

$$\begin{aligned} h_1(k) &= r_{1,r}(k) + r_{1,a}(\tfrac{1}{\omega^2 k})r_{2,r}(\omega k), \qquad h_2(k) = r_{2,r}(k) + r_{2,a}(\tfrac{1}{\omega^2 k})r_{1,r}(\omega k), \\ g_1(k) &= r_{1,r}(\tfrac{1}{\omega^2 k}) - r_{1,r}(\omega k)\big(r_{1,r}(k) + r_{1,a}(\tfrac{1}{\omega^2 k})r_{2,r}(\omega k)\big), \\ g_2(k) &= r_{2,r}(\tfrac{1}{\omega^2 k}) - r_{2,r}(\omega k)\big(r_{2,r}(k) + r_{2,a}(\tfrac{1}{\omega^2 k})r_{1,r}(\omega k)\big), \\ g(k) &= r_{1,r}(k)\big(r_{1,r}(\omega k)r_{2,a}(\tfrac{1}{\omega^2 k}) + r_{2,r}(k)\big) \\ &\quad + r_{1,a}(\tfrac{1}{\omega^2 k})r_{2,r}(\omega k)\big(r_{1,r}(\omega k)r_{2,a}(\tfrac{1}{\omega^2 k}) + r_{2,r}(k)\big) + r_{1,r}(\tfrac{1}{\omega^2 k})r_{2,r}(\tfrac{1}{\omega^2 k}). \end{aligned}$$

On $\Gamma_5^{(2)}$, we factorize $v_5^{(1)}$ as follows:

$$v_5^{(1)} = v_6^{(2)} v_5^{(2)} v_4^{(2)}, \qquad v_5^{(2)} = \begin{pmatrix} 1 + r_1(k)r_2(k) & 0 & 0 \\ 0 & \frac{1}{1+r_1(k)r_2(k)} & 0 \\ 0 & 0 & 1 \end{pmatrix} + v_{5,r}^{(2)},$$

$$v_6^{(2)} = \begin{pmatrix} 1 & 0 & r_{1,a}(\frac{1}{\omega^2 k})e^{-\theta_{31}} \\ \hat{r}_{2,a}(k)e^{\theta_{21}} & 1 & -r_{2,a}(\frac{1}{\omega k})e^{-\theta_{32}} \\ 0 & 0 & 1 \end{pmatrix}, \; v_4^{(2)} = \begin{pmatrix} 1 & \hat{r}_{1,a}(k)e^{-\theta_{21}} & 0 \\ 0 & 1 & 0 \\ r_{2,a}(\frac{1}{\omega^2 k})e^{\theta_{31}} & -r_{1,a}(\frac{1}{\omega k})e^{\theta_{32}} & 1 \end{pmatrix}, \tag{6.3}$$

where we have used (4.4). Finally, on $\Gamma_8^{(2)}$, we factorize $v_8^{(1)}$ as follows:

$$v_8^{(1)} = v_7^{(2)} v_8^{(2)} v_9^{(2)}, \quad v_8^{(2)} = I + v_{8,r}^{(2)}, \quad v_{8,r}^{(2)} = \begin{pmatrix} 0 & -r_{1,r}(k)e^{-\theta_{21}} & h_2(\omega^2 k)e^{-\theta_{31}} \\ -r_{2,r}(k)e^{\theta_{21}} & r_{1,r}(k)r_{2,r}(k) & g_2(\omega^2 k)e^{-\theta_{32}} \\ h_1(\omega^2 k)e^{\theta_{31}} & g_1(\omega^2 k)e^{\theta_{32}} & g(\omega^2 k) \end{pmatrix}$$

$$v_7^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ -r_{2,a}(k)e^{\theta_{21}} & 1 & 0 \\ r_{1,a}(\omega^2 k)e^{\theta_{31}} & r_{1,a}(\frac{1}{\omega k})e^{\theta_{32}} & 1 \end{pmatrix}, \qquad v_9^{(2)} = \begin{pmatrix} 1 & -r_{1,a}(k)e^{-\theta_{21}} & r_{2,a}(\omega^2 k)e^{-\theta_{31}} \\ 0 & 1 & r_{2,a}(\frac{1}{\omega k})e^{-\theta_{32}} \\ 0 & 0 & 1 \end{pmatrix}. \tag{6.4}$$

We do not write down the long expression for $v_{5,r}^{(2)}$, but note that Lemma 6.1 yields[1]

$$\|v_{j,r}^{(2)}\|_{(L^1 \cap L^\infty)(\Gamma_j^{(2)})} = O(x^{-N}) \qquad \text{as } x \to \infty, \; j = 2, 5, 8.$$

Let $\Gamma^{(2)}$ be the contour shown in Figure 7. The function $n^{(2)}$ is defined by

$$n^{(2)}(x,t,k) = n^{(1)}(x,t,k)G^{(2)}(x,t,k), \qquad k \in \mathbb{C} \setminus \Gamma^{(2)}, \tag{6.5}$$

[1] For a matrix-valued function A defined on a piecewise smooth contour $\gamma \subset \mathbb{C}$, we define $\|A\|_{L^p(\gamma)} := \||A|\|_{L^p(\gamma)}$, $1 \le p \le \infty$, where $|A| := \sqrt{\Sigma_{i,j}|A_{ij}|^2}$.

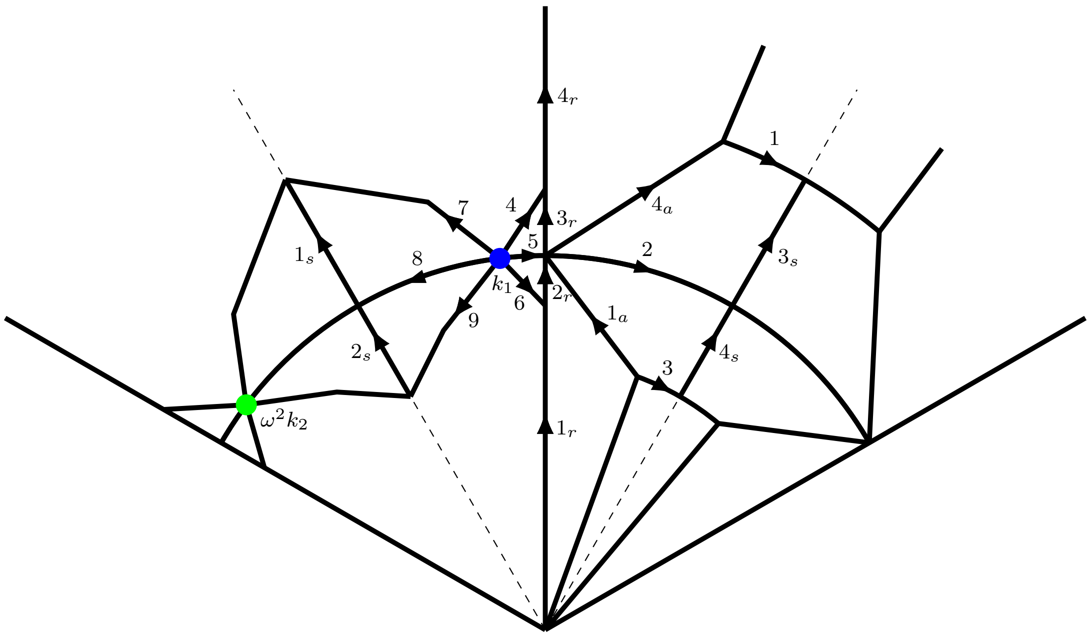

Figure 7. *The contour $\Gamma^{(2)}$ (solid), the boundary of S (dashed), and the saddle points k_1 (blue) and $\omega^2 k_2$ (green).*

where $G^{(2)}$ is analytic in $\mathbb{C} \setminus \Gamma^{(2)}$. It is defined for $k \in \mathsf{S}$ by

$$G^{(2)} = \begin{cases} v_3^{(2)}, & k \text{ on the } - \text{ side of } \Gamma_2^{(2)}, \\ (v_1^{(2)})^{-1}, & k \text{ on the } + \text{ side of } \Gamma_2^{(2)}, \\ v_6^{(2)}, & k \text{ on the } - \text{ side of } \Gamma_5^{(2)}, \\ (v_4^{(2)})^{-1}, & k \text{ on the } + \text{ side of } \Gamma_5^{(2)}, \end{cases} \quad G^{(2)} = \begin{cases} v_7^{(2)}, & k \text{ on the } - \text{ side of } \Gamma_8^{(2)}, \\ (v_9^{(2)})^{-1}, & k \text{ on the } + \text{ side of } \Gamma_8^{(2)}, \\ I, & \text{otherwise}, \end{cases} \tag{6.6}$$

and extended to $\mathbb{C} \setminus \Gamma^{(2)}$ by means of the symmetries

$$G^{(2)}(x,t,k) = \mathcal{A} G^{(2)}(x,t,\omega k) \mathcal{A}^{-1} = \mathcal{B} G^{(2)}(x,t,k^{-1}) \mathcal{B}. \tag{6.7}$$

The next lemma follows from Lemma 6.1 and the signature tables of Figure 3.

Lemma 6.2. *$G^{(2)}(x,t,k)$ and $G^{(2)}(x,t,k)^{-1}$ are uniformly bounded for $k \in \mathbb{C} \setminus \Gamma^{(2)}$, $x \geq 1$, and $\tau \in \mathcal{I}$. Furthermore, $G^{(2)}(x,t,k) = I$ for all large enough $|k|$.*

Using (6.5), we obtain $n_+^{(2)} = n_-^{(2)} v_j^{(2)}$ on $\Gamma_j^{(2)}$, where $v_j^{(2)}$ is given by (6.2) for $j = 1,2,3$, by (6.3) for $j = 4,5,6$, by (6.4) for $j = 7,8,9$, and

$$v_{1_r}^{(2)} = v_{1_r}^{(1)}, \qquad v_{2_r}^{(2)} = v_{1_r}^{(1)} v_6^{(2)}, \qquad v_{3_r}^{(2)} = v_{4_r}^{(1)} (v_4^{(2)})^{-1}, \qquad v_{4_r}^{(2)} = v_{4_r}^{(1)},$$
$$v_{1_a}^{(2)} = (v_3^{(2)})^{-1} v_{1_a}^{(1)}, \qquad v_{4_a}^{(2)} = v_1^{(2)} v_{4_a}^{(1)},$$
$$v_{1_s}^{(2)} = G_-^{(2)}(k)^{-1} G_+^{(2)}(k) = v_7^{(2)}(k)^{-1} \mathcal{A}\mathcal{B} v_9^{(2)}(\tfrac{1}{\omega k})^{-1} \mathcal{B}\mathcal{A}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ (v_{1_s}^{(2)})_{31} & 0 & 1 \end{pmatrix},$$
$$(v_{1_s}^{(2)})_{31} := -\big(r_{1,a}(\omega^2 k) + r_{2,a}(k) r_{1,a}(\tfrac{1}{\omega k}) + r_{2,a}(\tfrac{1}{\omega^2 k})\big) e^{\theta_{31}}.$$

We omit the explicit expressions for $v_{2_s}, v_{3_s}, v_{4_s}$ which are similar to the expression for v_{1_s}. The jumps on the parts of $\Gamma^{(2)} \cap \mathsf{S}$ that are unlabeled in Figure 7 are small as $x \to \infty$, so we do not write them down.

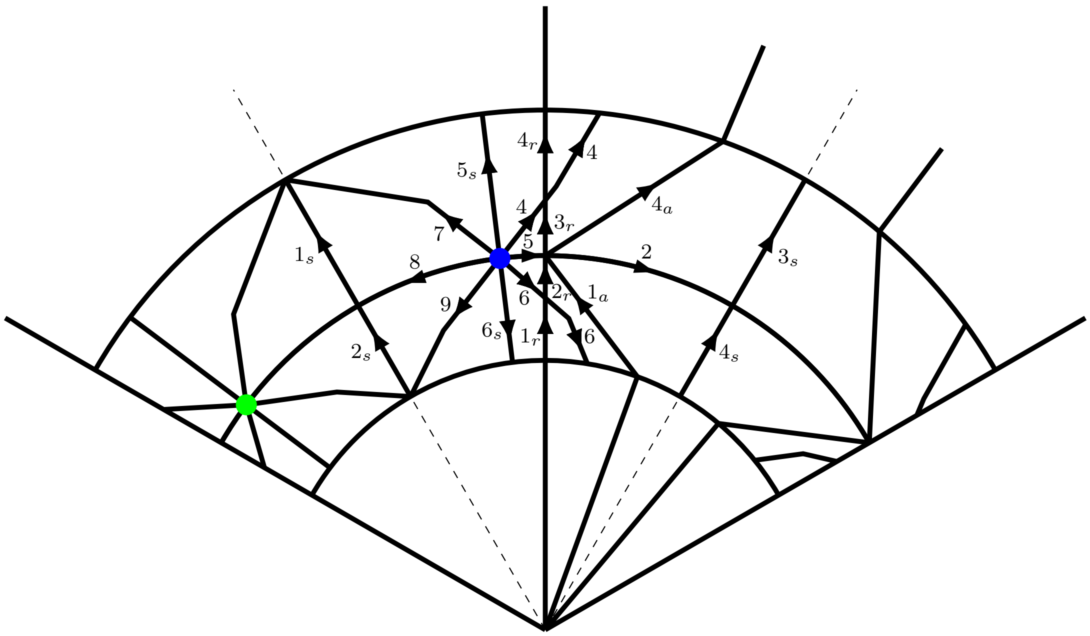

Figure 8. *The contour $\Gamma^{(3)}$ (solid), the boundary of S (dashed), and the saddle points k_1 (blue) and $\omega^2 k_2$ (green).*

7. The $n^{(2)} \to n^{(3)}$ transformation

For the third transformation, which is a second opening of lenses, we will deform contours up and down (see (7.3) below) using the following factorizations:

$$v_4^{(2)} = v_4^{(3)} v_{4u}^{(3)}, \quad v_4^{(3)} = \begin{pmatrix} 1 & \hat{r}_{1,a}(k)e^{-\theta_{21}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad v_{4u}^{(3)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ r_{2,a}(\frac{1}{\omega^2 k})e^{\theta_{31}} & -r_{1,a}(\frac{1}{\omega k})e^{\theta_{32}} & 1 \end{pmatrix},$$

$$v_6^{(2)} = v_{6d}^{(3)} v_6^{(3)}, \quad v_{6d}^{(3)} = \begin{pmatrix} 1 & 0 & r_{1,a}(\frac{1}{\omega^2 k})e^{-\theta_{31}} \\ 0 & 1 & -r_{2,a}(\frac{1}{\omega k})e^{-\theta_{32}} \\ 0 & 0 & 1 \end{pmatrix}, \quad v_6^{(3)} = \begin{pmatrix} 1 & 0 & 0 \\ \hat{r}_{2,a}(k)e^{\theta_{21}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$v_7^{(2)} = v_{7u}^{(3)} v_7^{(3)}, \quad v_7^{(3)} = \begin{pmatrix} 1 & 0 & 0 \\ -r_{2,a}(k)e^{\theta_{21}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad v_9^{(2)} = v_9^{(3)} v_{9d}^{(3)}, \quad v_9^{(3)} = \begin{pmatrix} 1 & -r_{1,a}(k)e^{-\theta_{21}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$v_{7u}^{(3)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ (r_{1,a}(\omega^2 k) + r_{1,a}(\frac{1}{\omega k})r_{2,a}(k))e^{\theta_{31}} & r_{1,a}(\frac{1}{\omega k})e^{\theta_{32}} & 1 \end{pmatrix},$$

$$v_{9d}^{(3)} = \begin{pmatrix} 1 & 0 & (r_{2,a}(\omega^2 k) + r_{1,a}(k)r_{2,a}(\frac{1}{\omega k}))e^{-\theta_{31}} \\ 0 & 1 & r_{2,a}(\frac{1}{\omega k})e^{-\theta_{32}} \\ 0 & 0 & 1 \end{pmatrix}. \tag{7.1}$$

Let $\Gamma^{(3)}$ be the contour shown in Figure 8. We define $n^{(3)}$ by

$$n^{(3)}(x,t,k) = n^{(2)}(x,t,k)G^{(3)}(x,t,k), \qquad k \in \mathbb{C} \setminus \Gamma^{(3)}, \tag{7.2}$$

where $G^{(3)}$ is given for $k \in \mathsf{S}$ by

$$G^{(3)}(x,t,k) = \begin{cases} (v_4^{(2)})^{-1}, & k \text{ on the } + \text{ side of } \Gamma_{4_a}^{(3)}, \\ v_6^{(2)}, & k \text{ on the } + \text{ side of } \Gamma_{1_a}^{(3)}, \\ (v_{4u}^{(3)})^{-1}, & k \text{ on the } + \text{ side of } \Gamma_4^{(3)}, \\ v_{6d}^{(3)}, & k \text{ on the } - \text{ side of } \Gamma_6^{(3)}, \\ v_{7u}^{(3)}, & k \text{ on the } - \text{ side of } \Gamma_7^{(3)}, \\ (v_{9d}^{(3)})^{-1}, & k \text{ on the } + \text{ side of } \Gamma_9^{(3)}, \\ I, & \text{otherwise}, \end{cases} \tag{7.3}$$

and $G^{(3)}$ is defined to $\mathbb{C} \setminus \Gamma^{(3)}$ using the $\mathcal{A}$- and $\mathcal{B}$-symmetries (as in (5.13)). The following lemma can be deduced from Lemma 6.1 and Figure 3.

Lemma 7.1. *$G^{(3)}(x,t,k)$ and $G^{(3)}(x,t,k)^{-1}$ are uniformly bounded for $k \in \mathbb{C} \setminus \Gamma^{(3)}$, $\tau \in \mathcal{I}$, and $x \geq 1$. Furthermore, $G^{(3)}(x,t,k) = I$ for all large enough $|k|$.*

The jump matrices $v_j^{(3)}$ are given for $j = 4, 6, 7, 9$ by (7.1) and by

$$\begin{aligned} & v_j^{(3)} := v_j^{(2)}, \ j = 2, 5, 8, 1_s, 2_s, 3_s, 4_s, \qquad v_{5_s}^{(3)} := v_{4u}^{(3)} v_{7u}^{(3)}, \qquad v_{6_s}^{(3)} := v_{9d}^{(3)} v_{6d}^{(3)}, \\ & v_{1_r}^{(3)} = (v_{6d}^{(3)})^{-1} v_{1_r}^{(2)} v_{6d}^{(3)}, \quad v_{2_r}^{(3)} = (v_6^{(2)})^{-1} v_{2_r}^{(2)}, \quad v_{3_r}^{(3)} = v_4^{(2)} v_{3_r}^{(2)}, \quad v_{4_r}^{(3)} = v_{4u}^{(3)} v_{4_r}^{(2)} (v_{4u}^{(3)})^{-1}, \\ & v_{1_a}^{(3)} = v_{1_a}^{(2)} v_6^{(2)}, \qquad v_{4_a}^{(3)} = v_{4_a}^{(2)} (v_4^{(2)})^{-1}. \end{aligned} \tag{7.4}$$

On the subcontours of $\Gamma^{(3)} \cap \mathsf{S}$ that are unlabeled in Figure 8, the matrix $v^{(3)}$ is close to I as $x \to \infty$, so we omit its explicit expression. On $\Gamma^{(3)} \setminus \mathsf{S}$, $v^{(3)}$ can be obtained using (4.2).

Our next lemma shows that the jumps on $\Gamma_j^{(1)}$, $j = 1_r, \ldots, 4_r$, are uniformly small for large x.

Lemma 7.2. *For any $1 \leq p \leq \infty$ and $j = 1_r, 2_r, 3_r, 4_r$, the L^p-norm of $v^{(3)} - I$ on $\Gamma_j^{(3)}$ is $O(x^{-N})$ as $x \to \infty$ uniformly for $\tau \in \mathcal{I}$.*

Proof. We have

$$v_{1_r}^{(3)} - I = (v_{6d}^{(3)})^{-1} (v_{1_r}^{(1)} - I) v_{6d}^{(3)}, \qquad v_{2_r}^{(3)} - I = (v_6^{(2)})^{-1} (v_{1_r}^{(1)} - I) v_6^{(2)}.$$

The matrices $(v_{6d}^{(3)})^{\pm 1}$ and $(v_6^{(2)})^{\pm 1}$ are uniformly bounded on $\Gamma_{1_r}^{(3)}$ and $\Gamma_{2_r}^{(3)}$, respectively. Indeed, consider for example the (13)-entry of $v_{6d}^{(3)}$ given by $(v_{6d}^{(3)})_{13} = r_{1,a}(\frac{1}{\omega^2 k}) e^{-\theta_{31}}$. Using Lemma 6.1 and the fact that $\tilde{\Phi}_{21}(\tau, \frac{1}{\omega^2 k}) = \tilde{\Phi}_{31}(\tau, k)$ has nonnegative real part on $k \in \Gamma_{1_r}^{(3)}$, we obtain

$$|(v_{6d}^{(3)}(x,t,k))_{13}| \leq C e^{\frac{x}{4} |\text{Re}\, \tilde{\Phi}_{21}(\tau, \frac{1}{\omega^2 k})|} e^{-x \text{Re}\, \tilde{\Phi}_{31}(\tau, k)} \leq C e^{-\frac{3x}{4} |\text{Re}\, \tilde{\Phi}_{31}(\tau, k)|} \leq C$$

uniformly for $k \in \Gamma_{1_r}^{(3)}$ and $\tau \in \mathcal{I}$, as desired. On the other hand, for any $1 \leq p \leq \infty$, the L^p-norm of the matrix $v_{1_r}^{(1)} - I$ on $[0, i]$ is of order $O(x^{-N})$ uniformly for $\tau \in \mathcal{I}$ by Lemma 5.1(c). This proves the desired statement $j = 1_r, 2_r$; the proof for $j = 3_r, 4_r$ is analogous. ∎

Lemma 7.3. *For any $1 \leq p \leq \infty$, the L^p-norm of $v^{(3)} - I$ on $\Gamma_{1_a}^{(3)} \cup \Gamma_{4_a}^{(3)}$ is $O(x^{-N})$ as $x \to \infty$ uniformly for $\tau \in \mathcal{I}$.*

Proof. We give a proof for $v_{1_a}^{(3)}$; the proof for $v_{4_a}^{(3)}$ is analogous. We have

$$v_{1_a}^{(3)} - I = \begin{pmatrix} 0 & 0 & \big(r_{1,a}(\frac{1}{k\omega^2}) + r_{2,a}(k\omega^2)\big)e^{-\theta_{31}} \\ \big(\tilde{r}_{1,a}(\frac{1}{k}) - r_{1,a}(\frac{1}{k}) + \hat{r}_{2,a}(k)\big)e^{\theta_{21}} & 0 & (v_{1_a}^{(3)})_{23} \\ 0 & 0 & 0 \end{pmatrix},$$

$$(v_{1_a}^{(3)})_{23} := \Big(\big(\tilde{r}_{1,a}(\tfrac{1}{k}) - r_{1,a}(\tfrac{1}{k})\big)r_{1,a}(\tfrac{1}{\omega^2 k}) - \big(r_{1,a}(\omega k) + r_{2,a}(\tfrac{1}{\omega k}) + r_{1,a}(\tfrac{1}{k})r_{2,a}(\omega^2 k)\big)\Big)e^{-\theta_{32}}.$$

Let us first estimate the (21)-entry of $v_{1_a}^{(3)}$. Lemma 5.1 (with N replaced by $2N$) implies that

$$\left|\tilde{r}_{1,a}(\tfrac{1}{k}) - \sum_{j=0}^{2N} \frac{\partial_k^j\big(r_1(\frac{1}{k})\big)\big|_{k=i}}{j!}(k-i)^j\right| \leq C|k-i|^{2N+1}e^{\frac{x}{4}|\operatorname{Re}\tilde{\Phi}_{21}(\tau,\frac{1}{k})|}, \qquad k \in \Gamma_{1_a}^{(3)},$$

and, similarly, Lemma 6.1 implies that

$$\left|r_{1,a}(\tfrac{1}{k}) - \sum_{j=0}^{2N} \frac{\partial_k^j\big(r_1(\frac{1}{k})\big)\big|_{k=i}}{j!}(k-i)^j\right| \leq C|k-i|^{2N+1}e^{\frac{x}{4}|\operatorname{Re}\tilde{\Phi}_{21}(\tau,\frac{1}{k})|}, \qquad k \in \Gamma_{1_a}^{(3)}. \tag{7.5}$$

Hence, applying the triangle inequality and the identity $\operatorname{Re}\tilde{\Phi}_{21}(\tau,\frac{1}{k}) = -\operatorname{Re}\tilde{\Phi}_{21}(\tau,k)$,

$$\big|\tilde{r}_{1,a}(\tfrac{1}{k}) - r_{1,a}(\tfrac{1}{k})\big| \leq 2C|k-i|^{2N+1}e^{\frac{x}{4}|\operatorname{Re}\tilde{\Phi}_{21}(\tau,k)|}. \tag{7.6}$$

On the other hand, the function $\hat{r}_2(k)$ vanishes to all orders at $k = i$ and there exists a $c > 0$ such that $\operatorname{Re}\tilde{\Phi}_{21}(\tau,k) \leq -c|k-i|^2$ uniformly for $k \in \Gamma_{1_a}^{(3)}$ and $\tau \in \mathcal{I}$. Consequently, using (7.6) and the estimate for $\hat{r}_{2,a}(k)$ that follows from Lemma 6.1, we arrive at

$$|(v_{1_a}^{(3)}(x,t,k))_{21}| \leq C|k-i|^{2N+1}e^{-\frac{3}{4}x|\operatorname{Re}\tilde{\Phi}_{21}(\tau,k)|} \leq C|k-i|^{2N+1}e^{-cx|k-i|^2} \leq Cx^{-N}$$

for $k \in \Gamma_{1_a}^{(3)}$ and $\tau \in \mathcal{I}$, which gives the desired estimate of the (21)-entry of $v_{1_a}^{(3)} - I$.

Let us next consider the (23)-entry of $v_{1_a}^{(3)}$. We have $(v_{1_a}^{(3)})_{23} = (g_a(k) - h_a(k))e^{-x\tilde{\Phi}_{32}(\tau,k)}$, where

$$g_a(k) := \big(\tilde{r}_{1,a}(\tfrac{1}{k}) - r_{1,a}(\tfrac{1}{k})\big)r_{1,a}(\tfrac{1}{\omega^2 k}), \quad h_a(k) := r_{1,a}(\omega k) + r_{2,a}(\tfrac{1}{\omega k}) + r_{1,a}(\tfrac{1}{k})r_{2,a}(\omega^2 k).$$

Recalling (7.6) and estimating $r_{1,a}(\frac{1}{\omega^2 k})$ by means of Lemma 6.1, we get the bound

$$\big|g_a(k)e^{-x\tilde{\Phi}_{32}(\tau,k)}\big| \leq C|k-i|^{2N+1}e^{\frac{x}{4}|\operatorname{Re}\tilde{\Phi}_{21}(\tau,k)|}e^{\frac{x}{4}|\operatorname{Re}\tilde{\Phi}_{21}(\tau,\frac{1}{\omega^2 k})|}e^{-x|\tilde{\Phi}_{32}(\tau,k)|}.$$

Since $-\tilde{\Phi}_{21}(\tau,k) + \tilde{\Phi}_{21}(\tau,\frac{1}{\omega^2 k}) = \tilde{\Phi}_{21}(\tau,\omega k) = \tilde{\Phi}_{32}(\tau,k)$, the above can be rewritten as

$$\begin{aligned}\big|g_a(k)e^{-x\tilde{\Phi}_{32}(\tau,k)}\big| &\leq C|k-i|^{2N+1}e^{-x(\operatorname{Re}\tilde{\Phi}_{32}(\tau,k) - \frac{1}{4}|\operatorname{Re}\tilde{\Phi}_{21}(\tau,\omega k)|)} \\ &\leq C|k-i|^{2N+1}e^{-\frac{3x}{4}\operatorname{Re}\tilde{\Phi}_{32}(\tau,k)}.\end{aligned} \tag{7.7}$$

On the other hand, considering the relation (4.4) with k replaced by $\frac{1}{\omega^2 k}$, we infer that the function

$$h(k) := r_1(\omega k) + r_2(\tfrac{1}{\omega k}) + r_1(\tfrac{1}{k})r_2(\omega^2 k)$$

vanishes to all orders at $k = i$. Hence, since $\operatorname{Re}\tilde{\Phi}_{32}(\tau,k) \geq c|k-i|$ for $k \in \Gamma_{1_a}^{(3)}$ (cf. Figure 3),

$$\begin{aligned}\big|h_a(k)e^{-x\tilde{\Phi}_{32}(\tau,k)}\big| &\leq |h_a(k) - h(k)|e^{-x\operatorname{Re}\tilde{\Phi}_{32}(\tau,k)} + |h(k)|e^{-x\operatorname{Re}\tilde{\Phi}_{32}(\tau,k)} \\ &\leq |h_a(k) - h(k)|e^{-x\operatorname{Re}\tilde{\Phi}_{32}(\tau,k)} + Cx^{-N}.\end{aligned} \tag{7.8}$$

To estimate $|h_a(k) - h(k)|$, we note that Lemma 6.1 yields

$$\Big|r_{1,a}(\omega k) - \sum_{j=0}^{2N} \frac{\partial_k^j\big(r_1(\omega k)\big)\big|_{k=i}}{j!}(k-i)^j\Big| \leq C|k-i|^{2N+1}e^{\frac{x}{4}|\mathrm{Re}\,\tilde{\Phi}_{21}(\tau,\omega k)|}, \qquad k \in \Gamma_{1_a}^{(3)},$$

$$\Big|r_{2,a}(\omega^2 k) - \sum_{j=0}^{2N} \frac{\partial_k^j\big(r_2(\omega^2 k)\big)\big|_{k=i}}{j!}(k-i)^j\Big| \leq C|k-i|^{2N+1}e^{\frac{x}{4}|\mathrm{Re}\,\tilde{\Phi}_{21}(\tau,\omega^2 k)|}, \qquad k \in \Gamma_{1_a}^{(3)},$$

$$\Big|r_{2,a}(\tfrac{1}{\omega k}) - \sum_{j=0}^{2N} \frac{\partial_k^j\big(r_2(\frac{1}{\omega k})\big)\big|_{k=i}}{j!}(k-i)^j\Big| \leq C|k-i|^{2N+1}e^{\frac{x}{4}|\mathrm{Re}\,\tilde{\Phi}_{21}(\tau,\frac{1}{\omega k})|}, \qquad k \in \Gamma_{1_a}^{(3)},$$

uniformly for $\tau \in \mathcal{I}$. The identities

$$\tilde{\Phi}_{21}(\tau,\omega k) = -\tilde{\Phi}_{21}(\tau,\tfrac{1}{\omega k}) = \tilde{\Phi}_{21}(\tau,\tfrac{1}{k}) - \tilde{\Phi}_{21}(\tau,\omega^2 k), \qquad k \in \mathbb{C},$$

together with Figure 3, imply that

$$|\mathrm{Re}\,\tilde{\Phi}_{21}(\tau,\omega k)| = |\mathrm{Re}\,\tilde{\Phi}_{21}(\tau,\tfrac{1}{\omega k})| = |\mathrm{Re}\,\tilde{\Phi}_{21}(\tau,\tfrac{1}{k})| + |\mathrm{Re}\,\tilde{\Phi}_{21}(\tau,\omega^2 k)|, \qquad k \in \Gamma_{1_a}^{(3)},$$

and hence, using also (7.5) and the relation $\tilde{\Phi}_{21}(\tau,\omega k) = \tilde{\Phi}_{32}(\tau,k)$,

$$\begin{aligned} |h_a(k) - h(k)|e^{-x\mathrm{Re}\,\tilde{\Phi}_{32}(\tau,k)} &\leq C|k-i|^{2N+1}e^{-x(|\mathrm{Re}\,\tilde{\Phi}_{32}(\tau,k)|-\frac{1}{4}|\mathrm{Re}\,\tilde{\Phi}_{21}(\tau,\omega k)|)} \\ &\leq C|k-i|^{2N+1}e^{-\frac{3x}{4}|\mathrm{Re}\,\tilde{\Phi}_{32}(\tau,k)|} \end{aligned} \tag{7.9}$$

for $k \in \Gamma_{1_a}^{(3)}$ and $\tau \in \mathcal{I}$. Shrinking the size of the lenses if necessary, there exists a $c > 0$ such that

$$\mathrm{Re}\,\tilde{\Phi}_{32}(\tau,k) \geq c|k-i| \tag{7.10}$$

uniformly for $k \in \Gamma_{1_a}^{(3)}$ and $\tau \in \mathcal{I}$. Combining the inequalities (7.7), (7.8), (7.9), and (7.10), it transpires that

$$|(v_{1_a}^{(3)}(x,t,k))_{23}| \leq C|k-i|^{2N+1}e^{-c|k-i|} + Cx^{-N} \leq Cx^{-N},$$

which proves the desired estimate for the (23)-entry of $v_{1_a}^{(3)} - I$; the (13)-entry can be handled similarly since $r_1(\frac{1}{k\omega^2}) + r_2(k\omega^2)$ vanishes to all orders at $k = i$ (this follows from (4.4) with k replaced by ωk and the assumption that $r_1(k)$ vanishes to all orders at $k = i$). ☐

The next lemma is proved in the same way as [12, Lemma 6.3].

Lemma 7.4. *It is possible to choose the analytic approximations of Lemma 6.1 so that the L^∞-norm of $v_j^{(3)} - I$ on $\Gamma_j^{(3)}$, $j = 1_s, \dots, 6_s$, is $O(x^{-N})$ as $x \to \infty$ uniformly for $\tau \in \mathcal{I}$.*

Using Lemmas 7.2, 7.3, and 7.4, we conclude that $v^{(3)} - I = O(x^{-N})$ as $x \to \infty$, uniformly for $k \in \Gamma^{(3)} \setminus \big(\cup_{j=0}^2 \cup_{\ell=1}^2 D_\epsilon(\omega^j k_\ell) \cup \partial\mathbb{D}\big)$.

8. Global parametrix

For each $\tau \in \mathcal{I}$, let $\delta(\tau,\cdot) : \mathbb{C} \setminus \Gamma_5^{(3)} \to \mathbb{C}$ be a function obeying the jump condition

$$\delta_+(\tau,k) = \delta_-(\tau,k)(1 + r_1(k)r_2(k)), \qquad k \in \Gamma_5^{(3)},$$

and the normalization condition $\delta(\tau,k) = 1 + O(k^{-1})$ as $k \to \infty$. As mentioned earlier (see the text below (6.1)), $1 + r_1(k)r_2(k) > 1$ for all $k \in \Gamma_5^{(3)}$. Therefore, by taking the logarithm and using Plemelj's formula, we obtain

$$\delta(\tau,k) = \exp\left\{\frac{-1}{2\pi i}\int_i^{k_1} \frac{\ln(1 + r_1(s)r_2(s))}{s-k}ds\right\}, \qquad k \in \mathbb{C} \setminus \Gamma_5^{(3)}, \tag{8.1}$$

where the path of integration follows the unit circle in the counterclockwise direction, and the principal branch is used for the logarithm.

Lemma 8.1. *The function $\delta(\tau,k)$ has the following properties:*

(*a*) *$\delta(\tau,k)$ admits the representation*

$$\delta(\tau,k) = e^{-i\nu \ln_{k_1}(k-k_1)} e^{-\chi(\tau,k)}, \tag{8.2}$$

$$\chi(\tau,k) = \frac{-1}{2\pi i}\int_i^{k_1} \ln_s(k-s) d\ln(1+r_1(s)r_2(s)), \tag{8.3}$$

where the path of integration follows $\partial\mathbb{D}$ in the counterclockwise direction and $\nu \leq 0$ is defined in (2.9).

For $s \in \bar{\Gamma}_5^{(3)}$, $k \mapsto \ln_s(k-s) = \ln|k-s| + i\arg_s(k-s)$ has a cut along $\{e^{i\theta} : \theta \in [\frac{\pi}{2}, \arg s],\ \arg s \in [\frac{\pi}{2}, \frac{2\pi}{3}]\} \cup (i, i\infty)$ and $\arg_s(1) = 2\pi$.

(*b*) *For each $\tau \in \mathcal{I}$, $\delta(\tau,k)$ and $\delta(\tau,k)^{-1}$ are analytic functions of $k \in \mathbb{C}\setminus\Gamma_5^{(3)}$. Furthermore,*

$$\sup_{\tau\in\mathcal{I}} \sup_{k\in\mathbb{C}\setminus\Gamma_5^{(3)}} |\delta(\tau,k)^{\pm 1}| < \infty.$$

(*c*) *For $k \in \partial\mathbb{D}\setminus\Gamma_5^{(3)}$,*

$$\operatorname{Re}\chi(\tau,k) = \frac{\pi + \arg_i k}{2}\nu + \frac{1}{2\pi}\int_{\Gamma_5^{(3)}} \frac{\arg s}{2} d\ln(1+r_1(s)r_2(s)), \tag{8.4}$$

where $\arg_i k \in (\frac{\pi}{2}, \frac{5\pi}{2})$ and $\arg s \in (-\pi,\pi)$. In particular, for $k \in \partial\mathbb{D}\setminus\Gamma_5^{(3)}$, $|\delta(\tau,k)|$ is constant and given by

$$|\delta(\tau,k)| = \exp\left(\nu\frac{\arg k_1}{2} - \frac{1}{2\pi}\int_{\Gamma_5^{(3)}} \frac{\arg s}{2} d\ln(1+r_1(s)r_2(s))\right), \tag{8.5}$$

where $\arg k_1 \in [\frac{\pi}{2}, \frac{2\pi}{3}]$.

(*d*) *As $k \to k_1$ along a path which is nontangential to $\Gamma_5^{(3)}$, we have*

$$|\chi(\tau,k) - \chi(\tau,k_1)| \leq C|k-k_1|(1+|\ln|k-k_1||), \tag{8.6}$$

where C is independent of $\tau \in \mathcal{I}$.

(*e*) *As $k \to \infty$,*

$$\delta(\tau,k) = e^{\frac{1}{2\pi i k}\int_i^{k_1} \ln(1+r_1(s)r_2(s))ds + O(k^{-2})}. \tag{8.7}$$

Proof. To prove part (*a*), it suffices to perform an integration by parts in (8.1), recalling that $r_1(i) = 0$ due to assumption (*iv*) in Section 2.1. Part (*b*) is then a consequence of (8.2) and the fact that $\operatorname{Im}\nu = 0$. By (8.3) we have, for $k \in \partial\mathbb{D}\setminus\Gamma_5^{(3)}$,

$$\begin{aligned}\operatorname{Re}\chi(\tau,k) &= \frac{1}{2\pi}\int_{\Gamma_5^{(3)}} \arg_s(k-s) d\ln(1+r_1(s)r_2(s)) \\ &= \frac{1}{2\pi}\int_{\Gamma_5^{(3)}} \frac{\pi + \arg_i k + \arg s}{2} d\ln(1+r_1(s)r_2(s)),\end{aligned}$$

and (8.4) then follows from the assumption that $r_1 = 0$ on $[0,i]$ and the definition (2.9) of ν. By (8.2),

$$|\delta(\tau,k)| = e^{\nu\arg_{k_1}(k-k_1)} e^{-\operatorname{Re}\chi(\tau,k)}.$$

We obtain (8.5) after substituting (8.4) and $\arg_{k_1}(k-k_1) = \frac{\pi+\arg_i k+\arg k_1}{2}$ into the above equation. This finishes the proof of part (*c*). We obtain part (*d*) using (8.3) and standard estimates and part (*e*) is immediate from (8.1). ☐

For $\tau \in \mathcal{I}$ and $k \in \mathbb{C} \setminus \partial\mathbb{D}$, we let $\{\Delta_{jj}(\tau,k)\}_{j=1}^3$ be given by

$$\Delta_{33}(\tau,k) = \frac{\delta(\tau,\omega k)}{\delta(\tau,\omega^2 k)} \frac{\delta(\tau,\frac{1}{\omega^2 k})}{\delta(\tau,\frac{1}{\omega k})}, \quad \Delta_{11}(\tau,k) = \Delta_{33}(\tau,\omega k), \quad \Delta_{22}(\tau,k) = \Delta_{33}(\tau,\omega^2 k).$$

We have $\Delta_{33}(\tau,\frac{1}{k}) = \Delta_{33}(\tau,k)$ and, using (8.7),

$$\begin{aligned}\Delta_{33}(\tau,k) &= e^{\frac{1}{2\pi i}(\frac{1}{\omega k}-\frac{1}{\omega^2 k})\int_i^{k_1}\ln(1+r_1(s)r_2(s))(1+s^{-2})ds+O(k^{-2})} \\ &= 1 - \frac{\sqrt{3}}{2\pi k}\int_i^{k_1}\ln(1+r_1(s)r_2(s))(1+s^{-2})ds + O(k^{-2}), \qquad k\to\infty. \end{aligned} \tag{8.8}$$

We define the inverse of the global parametrix by

$$\Delta(\tau,k) = \begin{pmatrix} \Delta_{11}(\tau,k) & 0 & 0 \\ 0 & \Delta_{22}(\tau,k) & 0 \\ 0 & 0 & \Delta_{33}(\tau,k)\end{pmatrix}, \quad \tau\in\mathcal{I},\ k\in\mathbb{C}\setminus\partial\mathbb{D}. \tag{8.9}$$

It satisfies the symmetries

$$\Delta(\tau,k) = \mathcal{A}\Delta(\tau,\omega k)\mathcal{A}^{-1} = \mathcal{B}\Delta(\tau,\tfrac{1}{k})\mathcal{B},$$

and obeys

$$\Delta_+^{-1}(\tau,k) = \Delta_-^{-1}(\tau,k)\begin{pmatrix} 1+r_1(k)r_2(k) & 0 & 0 \\ 0 & \frac{1}{1+r_1(k)r_2(k)} & 0 \\ 0 & 0 & 1\end{pmatrix}, \qquad k\in\Gamma_5^{(3)}, \tag{8.10a}$$

$$\Delta_+^{-1}(\tau,k) = \Delta_-^{-1}(\tau,k), \qquad k\in\Gamma_2^{(3)}\cup\Gamma_8^{(3)}. \tag{8.10b}$$

Using (8.9) and Lemma 8.1, we deduce the following.

Lemma 8.2. *$\Delta(\tau,k)$ and $\Delta(\tau,k)^{-1}$ are uniformly bounded for $k\in\mathbb{C}\setminus\partial\mathbb{D}$ and $\tau\in\mathcal{I}$. Furthermore, $\Delta(\tau,k) = I + O(k^{-1})$ as $k\to\infty$.*

9. The $n^{(3)} \to n^{(4)}$ transformation

The purpose of the fourth transformation $n^{(3)}\to n^{(4)}$ is to make $v^{(4)} - I$ uniformly small as $x\to\infty$ for $|k|=1$. Let $n^{(4)}$ be given by

$$n^{(4)}(x,t,k) = n^{(3)}(x,t,k)\Delta(\tau,k), \qquad k\in\mathbb{C}\setminus\Gamma^{(4)}, \tag{9.1}$$

where $\Gamma^{(4)} = \Gamma^{(3)}$. This function satisfies $n_+^{(4)} = n_-^{(4)}v^{(4)}$ on $\Gamma^{(4)}$, where $v^{(4)} = \Delta_-^{-1}v^{(3)}\Delta_+$. We define a small cross centered at k_1 by $\mathcal{X}^\epsilon := \cup_{j=1}^4\mathcal{X}_j^\epsilon$, where

$$\mathcal{X}_1^\epsilon = \Gamma_4^{(4)}\cap D_\epsilon(k_1), \quad \mathcal{X}_2^\epsilon = \Gamma_7^{(4)}\cap D_\epsilon(k_1), \quad \mathcal{X}_3^\epsilon = \Gamma_9^{(4)}\cap D_\epsilon(k_1), \quad \mathcal{X}_4^\epsilon = \Gamma_6^{(4)}\cap D_\epsilon(k_1),$$

are oriented outwards from k_1. We also define the union of six small crosses centered at $\{\omega^j k_1, \omega^j k_2\}_{j=0}^2$ by $\hat{\mathcal{X}}^\epsilon = \mathcal{X}^\epsilon\cup\omega\mathcal{X}^\epsilon\cup\omega^2\mathcal{X}^\epsilon\cup(\mathcal{X}^\epsilon)^{-1}\cup(\omega\mathcal{X}^\epsilon)^{-1}\cup(\omega^2\mathcal{X}^\epsilon)^{-1}$.

Lemma 9.1. *$v^{(4)}$ converges to the identity matrix I as $x\to\infty$ uniformly for $\tau\in\mathcal{I}$ and $k\in\Gamma^{(4)}\setminus\hat{\mathcal{X}}^\epsilon$. More precisely, for $\tau\in\mathcal{I}$,*

$$\|v^{(4)} - I\|_{(L^1\cap L^\infty)(\Gamma^{(4)}\setminus\hat{\mathcal{X}}^\epsilon)} \le Cx^{-N}. \tag{9.2}$$

Proof. Using (6.2), (6.3), (6.4), and (7.4) together with the jumps (8.10) of Δ, we obtain

$$v_j^{(4)} = \Delta_-^{-1}v_j^{(3)}\Delta_+ = I + \Delta_-^{-1}v_{j,r}^{(3)}\Delta_+, \quad j=2,5,8.$$

Using Lemmas 6.1 and 8.2, we infer that the matrices $\Delta_-^{-1}v_{j,r}^{(3)}\Delta_+$, $j=2,5,8$, are $O(x^{-N})$ as $x\to\infty$. Using also Lemmas 7.2, 7.3, and 7.4, we conclude that the L^1 and L^∞ norms of $v^{(4)} - I$ are $O(x^{-N})$ as $x \to \infty$, uniformly for $\tau \in \mathcal{I}$ and $k \in \mathsf{S} \cap \Gamma^{(4)} \setminus \hat{\mathcal{X}}^\epsilon$. Then (9.2) directly follows from (4.2). ☐

The matrix $v^{(4)} - I$ is not uniformly small on $\hat{\mathcal{X}}^\epsilon$. The goal of the next section is to build a local parametrix that approximates $n^{(4)}$ in a neighborhood of k_1. The local parametrices near $\{k_2, \omega k_1, \omega k_2, \omega^2 k_1, \omega^2 k_2\}$ will then be obtained using the $\mathcal{A}$- and $\mathcal{B}$-symmetries.

10. Local parametrix near k_1

As $k \to k_1$, we have

$$\tilde{\Phi}_{21}(\tau, k) - \tilde{\Phi}_{21}(\tau, k_1) = \tilde{\Phi}_{21,k_1}(k - k_1)^2 + O((k-k_1)^3), \qquad \tilde{\Phi}_{21,k_1} = \frac{4\tau - 3k_1 - k_1^3}{4k_1^4}.$$

Let $z = z(x, \tau, k)$ be given by

$$z = z_\star \sqrt{x}(k - k_1)\hat{z}, \qquad \hat{z} = \sqrt{\frac{2i(\tilde{\Phi}_{21}(\tau, k) - \tilde{\Phi}_{21}(\tau, k_1))}{z_\star^2 (k - k_1)^2}},$$

where the principal branch is taken for $\hat{z} = \hat{z}(\tau, k)$, and

$$z_\star = \sqrt{2} e^{\frac{\pi i}{4}} \sqrt{\tilde{\Phi}_{21,k_1}}, \qquad -ik_1 z_\star > 0.$$

We have $\hat{z}(\tau, k_1) = 1$, and $x(\tilde{\Phi}_{21}(\tau, k) - \tilde{\Phi}_{21}(\tau, k_1)) = -\frac{iz^2}{2}$. We fix $\epsilon > 0$ and choose it so small that the function z is conformal from $D_\epsilon(k_1)$ to a neighborhood of 0. As $k \to k_1$,

$$z = z_\star \sqrt{x}(k - k_1)(1 + O(k - k_1)),$$

and for all $k \in D_\epsilon(k_1)$, we have

$$\ln_{k_1}(k - k_1) = \ln_0[z_\star(k - k_1)\hat{z}] - \ln \hat{z} - \ln z_\star$$

where $\ln_0(k) := \ln|k| + i \arg_0 k$, $\arg_0(k) \in (0, 2\pi)$, and $\ln$ is the principal logarithm. Reducing $\epsilon > 0$ if needed, $k \mapsto \ln \hat{z}$ is analytic in $D_\epsilon(k_1)$, $\ln \hat{z} = O(k - k_1)$ as $k \to k_1$, and

$$\ln z_\star = \ln|z_\star| + i \arg z_\star = \ln|z_\star| + i\big(\tfrac{\pi}{2} - \arg k_1\big),$$

where $\arg k_1 \in (\frac{\pi}{2}, \frac{2\pi}{3})$. By (8.2), we have

$$\delta(\tau, k) = e^{-i\nu \ln_{k_1}(k - k_1)} e^{-\chi(\tau, k)} = e^{-i\nu \ln_0 z} x^{\frac{i\nu}{2}} e^{i\nu \ln \hat{z}} e^{i\nu \ln z_\star} e^{-\chi(\tau, k)}.$$

From now on, for brevity we will use the notation $z_{(0)}^{-i\nu} := e^{-i\nu \ln_0 z}$, $\hat{z}^{i\nu} := e^{i\nu \ln \hat{z}}$, and $z_\star^{i\nu} := e^{i\nu \ln z_\star}$, so that

$$\delta(\tau, k) = z_{(0)}^{-i\nu} x^{\frac{i\nu}{2}} \hat{z}^{i\nu} z_\star^{i\nu} e^{-\chi(\tau, k)}.$$

Using this, we get

$$\frac{\Delta_{11}(\tau, k)}{\Delta_{22}(\tau, k)} = \frac{\delta(\frac{1}{k})^2 \delta(\omega k)\delta(\omega^2 k)}{\delta(k)^2 \delta(\frac{1}{\omega^2 k})\delta(\frac{1}{\omega k})} = z_{(0)}^{2i\nu} d_0(x, \tau) d_1(\tau, k), \tag{10.1}$$

$$d_0(x, \tau) = e^{2\chi(\tau, k_1)} x^{-i\nu} z_\star^{-2i\nu} \frac{\delta(\frac{1}{k_1})^2 \delta(\omega k_1)\delta(\omega^2 k_1)}{\delta(\frac{1}{\omega^2 k_1})\delta(\frac{1}{\omega k_1})}, \tag{10.2}$$

$$d_1(\tau, k) = e^{2(\chi(\tau, k) - \chi(\tau, k_1))} \hat{z}^{-2i\nu} \frac{\delta(\frac{1}{k})^2 \delta(\omega k)\delta(\omega^2 k)}{\delta(\frac{1}{\omega^2 k})\delta(\frac{1}{\omega k})} \bigg(\frac{\delta(\frac{1}{k_1})^2 \delta(\omega k_1)\delta(\omega^2 k_1)}{\delta(\frac{1}{\omega^2 k_1})\delta(\frac{1}{\omega k_1})}\bigg)^{-1}.$$

Define

$$Y(x, \tau) = d_0(x, \tau)^{-\frac{\tilde{\sigma}_3}{2}} \tilde{r}(k_1)^{-\frac{1}{4}\tilde{\sigma}_3} e^{-\frac{x}{2}\tilde{\Phi}_{21}(\tau, k_1)\tilde{\sigma}_3},$$

where $\tilde{\sigma}_3 = \mathrm{diag}\,(1,-1,0)$ and $\tilde{r}(k_1)^{\pm\frac{1}{4}} > 0$. For $k \in D_\epsilon(k_1)$, we define

$$\tilde{n}(x,t,k) = n^{(4)}(x,t,k)Y(x,\tau). \tag{10.3}$$

Let $\tilde{v}$ be the jump matrix of $\tilde{n}$, and let $\tilde{v}_j$ be the restriction of $\tilde{v}$ to $\Gamma_j^{(4)} \cap D_\epsilon(k_1)$. By (7.1), (9.1), and (10.1), we have

$$\tilde{v}_4 = \begin{pmatrix} 1 & \tilde{r}(k_1)^{\frac{1}{2}}\hat{r}_{1,a}(k)d_1^{-1}z_{(0)}^{-2i\nu}e^{\frac{iz^2}{2}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \tilde{v}_7 = \begin{pmatrix} 1 & 0 & 0 \\ -\tilde{r}(k_1)^{-\frac{1}{2}}r_{2,a}(k)d_1z_{(0)}^{2i\nu}e^{-\frac{iz^2}{2}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\tilde{v}_9 = \begin{pmatrix} 1 & -\tilde{r}(k_1)^{\frac{1}{2}}r_{1,a}(k)d_1^{-1}z_{(0)}^{-2i\nu}e^{\frac{iz^2}{2}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{v}_6 = \begin{pmatrix} 1 & 0 & 0 \\ \tilde{r}(k_1)^{-\frac{1}{2}}\hat{r}_{2,a}(k)d_1z_{(0)}^{2i\nu}e^{-\frac{iz^2}{2}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Define $q := -\tilde{r}(k_1)^{-\frac{1}{2}}r_2(k_1)$, and note that $-\tilde{r}(k_1)^{\frac{1}{2}}r_1(k_1) = \bar{q}$, $1 + r_1(k_1)r_2(k_1) = 1 + |q|^2$ and $\nu = -\frac{1}{2\pi}\ln(1+|q|^2)$. We also have $d_1(\tau,k_1) = 1$. Hence, we expect that $\tilde{n}$ should be close to $(1,1,1)Y\tilde{m}^{k_1}$, where $\tilde{m}^{k_1}(x,t,k)$ is analytic for $k \in D_\epsilon(k_1) \setminus \mathcal{X}^\epsilon$, satisfies $\tilde{m}_+^{k_1} = \tilde{m}_-^{k_1}\tilde{v}_{\mathcal{X}^\epsilon}^X$ on $\mathcal{X}^\epsilon$ with

$$\tilde{v}_{\mathcal{X}_1^\epsilon}^X = \begin{pmatrix} 1 & \frac{\tilde{r}(k_1)^{\frac{1}{2}}r_1(k_1)}{1+r_1(k_1)r_2(k_1)}z_{(0)}^{-2i\nu}e^{\frac{iz^2}{2}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{-\bar{q}}{1+|q|^2}z_{(0)}^{-2i\nu}e^{\frac{iz^2}{2}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\tilde{v}_{\mathcal{X}_2^\epsilon}^X = \begin{pmatrix} 1 & 0 & 0 \\ -\tilde{r}(k_1)^{-\frac{1}{2}}r_2(k_1)z_{(0)}^{2i\nu}e^{-\frac{iz^2}{2}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ qz_{(0)}^{2i\nu}e^{-\frac{iz^2}{2}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\tilde{v}_{\mathcal{X}_3^\epsilon}^X = \begin{pmatrix} 1 & -\tilde{r}(k_1)^{\frac{1}{2}}r_1(k_1)z_{(0)}^{-2i\nu}e^{\frac{iz^2}{2}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \bar{q}z_{(0)}^{-2i\nu}e^{\frac{iz^2}{2}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\tilde{v}_{\mathcal{X}_4^\epsilon}^X = \begin{pmatrix} 1 & 0 & 0 \\ \frac{\tilde{r}(k_1)^{-\frac{1}{2}}r_2(k_1)}{1+r_1(k_1)r_2(k_1)}z_{(0)}^{2i\nu}e^{-\frac{iz^2}{2}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{-q}{1+|q|^2}z_{(0)}^{2i\nu}e^{-\frac{iz^2}{2}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and is such that $\tilde{m}^{k_1}(x,t,k) \to I$ as $t \to \infty$ uniformly for $k \in \partial D_\epsilon(k_1)$. Let m^{k_1} be given by

$$m^{k_1}(x,t,k) = Y(x,\tau)m^X(q,z(x,\tau,k))Y(x,\tau)^{-1}, \qquad k \in D_\epsilon(k_1), \tag{10.4}$$

where m^X is the unique solution to the model RH problem of Lemma A.1.

Lemma 10.1. *The function $Y(x,\tau)$ is uniformly bounded:*

$$\sup_{\tau\in\mathcal{I}}\sup_{x\geq 2}|Y(x,\tau)^{\pm1}| \leq C, \tag{10.5}$$

where $|Y(x,\tau)^{\pm1}| := \sqrt{\Sigma_{i,j=1}^3|(Y(x,\tau)^{\pm1})_{ij}|^2}$. Furthermore, $d_0(x,\tau)$ and $d_1(\tau,k)$ satisfy

$$|d_0(x,\tau)| = e^{2\pi\nu}, \qquad x \geq 2,\ \tau \in \mathcal{I}, \tag{10.6}$$

$$|d_1(\tau,k) - 1| \leq C|k-k_1|(1+|\ln|k-k_1||), \qquad \tau \in \mathcal{I},\ k \in \mathcal{X}^\epsilon. \tag{10.7}$$

Proof. The inequalities (10.5) and (10.7) are direct consequences of Lemma 8.1. We now turn to the proof of (10.6). Recall that $\nu \in \mathbb{R}$. Hence, by (10.2),

$$|d_0(x,\tau)| = e^{2\operatorname{Re}\chi(\tau,k_1)} e^{2\nu \arg z_\star} \left| \frac{\delta(\frac{1}{k_1})^2 \delta(\omega k_1)\delta(\omega^2 k_1)}{\delta(\frac{1}{\omega^2 k_1})\delta(\frac{1}{\omega k_1})} \right|. \tag{10.8}$$

Recall also that $\arg z_\star = \frac{\pi}{2} - \arg k_1$. Using (8.5), we get

$$\left| \frac{\delta(\frac{1}{k_1})^2 \delta(\omega k_1)\delta(\omega^2 k_1)}{\delta(\frac{1}{\omega^2 k_1})\delta(\frac{1}{\omega k_1})} \right| = \exp\left(\nu \arg k_1 - \frac{1}{2\pi} \int_{\Gamma_5^{(3)}} \arg s \, d\ln(1 + r_1(s) r_2(s)) \right). \tag{10.9}$$

We find (10.6) after substituting (8.4) (with k replaced by k_1) and (10.9) into (10.8). $\square$

In the statement of the next lemma, and in what follows, $C_N(\tau)$ denotes a generic nonnegative smooth function of τ that vanishes to all orders at $\tau = 0$.

Lemma 10.2. *For each $x \geq 2$ and $\tau \in \mathcal{I}$, $m^{k_1}(x,t,k)$ defined in (10.4) is analytic for $k \in D_\epsilon(k_1) \setminus \mathcal{X}^\epsilon$. Moreover, $m^{k_1}(x,t,k)$ is uniformly bounded for $x \geq 2$, $\tau \in \mathcal{I}$ and $k \in D_\epsilon(k_1) \setminus \mathcal{X}^\epsilon$. On $\mathcal{X}^\epsilon$, m^{k_1} satisfies $m^{k_1}_+ = m^{k_1}_- v^{k_1}$, where the jump matrix v^{k_1} obeys*

$$\begin{cases} \|v^{(4)} - v^{k_1}\|_{L^1(\mathcal{X}^\epsilon)} \leq C_N(\tau) x^{-1} \ln x, \\ \|v^{(4)} - v^{k_1}\|_{L^\infty(\mathcal{X}^\epsilon)} \leq C_N(\tau) x^{-1/2} \ln x, \end{cases} \qquad x \geq 2, \ \tau \in \mathcal{I}. \tag{10.10}$$

Furthermore, as $x \to \infty$,

$$\|m^{k_1}(x,t,\cdot) - I\|_{L^\infty(\partial D_\epsilon(k_1))} \leq C_N(\tau) x^{-1/2}, \tag{10.11}$$

$$m^{k_1}(x,t,k) - I = \frac{Y(x,\tau) m_1^X(q) Y(x,\tau)^{-1}}{z_\star \sqrt{x}(k - k_1)\hat{z}(\tau,k)} + O(C_N(\tau) x^{-1}) \tag{10.12}$$

uniformly for $\tau \in \mathcal{I}$, where $m_1^X(q)$ is given by (A.3) with $q = -\tilde{r}(k_1)^{-\frac{1}{2}} r_2(k_1)$.

Proof. By (10.3) and (10.4), we have $v^{(4)} - v^{k_1} = Y(\tilde{v} - v^X)Y^{-1}$. Thus, in view of Lemma 10.1, the estimates in (10.10) will follow if we can show that

$$\|\tilde{v}(x,t,\cdot) - v^X(x,t,z(x,\tau,\cdot))\|_{L^1(\mathcal{X}_j^\epsilon)} \leq C_N(\tau) x^{-1} \ln x, \tag{10.13a}$$

$$\|\tilde{v}(x,t,\cdot) - v^X(x,t,z(x,\tau,\cdot))\|_{L^\infty(\mathcal{X}_j^\epsilon)} \leq C_N(\tau) x^{-1/2} \ln x, \tag{10.13b}$$

for $j = 1, \ldots, 4$. We show (10.13) for $j = 4$; similar arguments apply when $j = 1, 2, 3$.

For $k \in \mathcal{X}_4^\epsilon$, only the (21)-entry of the matrix $\tilde{v} - v^X$ is nonzero. Note from Lemma 6.1 that $|\hat{r}_{2,a}(k)| \leq C_N(\tau) e^{\frac{x}{4}|\operatorname{Re}\tilde{\Phi}_{21}(\tau,k)|}$ for all $k \in \mathcal{X}_4^\epsilon$ and $\tau \in \mathcal{I}$. Using also the identities

$$\frac{-q}{1+|q|^2} = \tilde{r}(k_1)^{-\frac{1}{2}} \hat{r}_2(k_1), \qquad \left|e^{-\frac{iz_1^2}{2}}\right| = e^{-\operatorname{Re}(\frac{iz_1^2}{2})} = e^{-\frac{|z_1|^2}{2}},$$

and Lemma 6.1, we find, for $k \in \mathcal{X}_4^\epsilon$,

$$\begin{aligned} |(\tilde{v} - v^X)_{21}| &= \left| \tilde{r}(k_1)^{-\frac{1}{2}} \hat{r}_{2,a}(k) d_1 z_{(0)}^{2i\nu} e^{-\frac{iz^2}{2}} - \frac{-q}{1+|q|^2} z_{(0)}^{2i\nu} e^{-\frac{iz^2}{2}} \right| \\ &= \left|z_{(0)}^{2i\nu}\right| \left| (d_1 - 1)\tilde{r}(k_1)^{-\frac{1}{2}} \hat{r}_{2,a}(k) + \tilde{r}(k_1)^{-\frac{1}{2}} \big(\hat{r}_{2,a}(k) - \hat{r}_2(k_1)\big) \right| \left|e^{-\frac{iz^2}{2}}\right| \\ &\leq C\bigg(|d_1 - 1||\hat{r}_{2,a}(k)| + |\hat{r}_{2,a}(k) - \hat{r}_2(k_1)| \bigg) e^{-\frac{|z|^2}{2}} \\ &\leq C_N(\tau)\bigg(|d_1 - 1| + |k - k_1| \bigg) e^{\frac{x}{4}|\operatorname{Re}\tilde{\Phi}_{21}(\tau,k)|} e^{-\frac{|z|^2}{2}} \\ &\leq C_N(\tau)\bigg(|d_1 - 1| + |k - k_1| \bigg) e^{-cx|k-k_1|^2}. \end{aligned}$$

Using (10.7), this gives

$$|(\tilde{v} - v^X)_{21}| \leq C_N(\tau)|k - k_1|(1 + |\ln|k - k_1||)e^{-cx|k-k_1|^2}, \qquad k \in \mathcal{X}_4^\epsilon.$$

Hence

$$\|\tilde{v} - v^X\|_{L^1(\mathcal{X}_4^\epsilon)} \leq C_N(\tau) \int_0^\infty u(1 + |\ln u|)e^{-cxu^2} du \leq C_N(\tau)x^{-1} \ln x,$$
$$\|\tilde{v} - v^X\|_{L^\infty(\mathcal{X}_4^\epsilon)} \leq C_N(\tau) \sup_{u \geq 0} u(1 + |\ln u|)e^{-cxu^2} \leq C_N(\tau)x^{-1/2} \ln x,$$

which completes the proof of (10.13), and hence also of (10.10).

For $k \in \partial D_\epsilon(k_1)$, we have $|z| = |z_\star|\sqrt{x}|k - k_1||\hat{z}| \geq c\sqrt{x}$, so the variable z tends to infinity as $x \to \infty$. Thus (A.3) can be used to infer that

$$m^X(q, z(x, \tau, k)) = I + \frac{m_1^X(q)}{z_\star \sqrt{x}(k - k_1)\hat{z}(\tau, k)} + O(qx^{-1}), \qquad x \to \infty,$$

uniformly for $k \in \partial D_\epsilon(k_1)$ and $\tau \in \mathcal{I}$. Recalling the definition (10.4) of m^{k_1}, this gives

$$m^{k_1} - I = \frac{Y(x, \tau)m_1^X(q)Y(x, \tau)^{-1}}{z_\star \sqrt{x}(k - k_1)\hat{z}(\tau, k)} + O(qx^{-1}), \qquad x \to \infty, \tag{10.14}$$

uniformly for $k \in \partial D_\epsilon(\omega k_4)$ and $\tau \in \mathcal{I}$. Since $r_2(k)$ vanishes to all orders at $k = i$ and $q = -\tilde{r}(k_1)^{-\frac{1}{2}} r_2(k_1)$, we see that $|q| = O(C_N(\tau))$. Since $m_1^X(q) = O(q)$ as $q \to 0$, the estimates (10.11) and (10.12) are a consequence of (10.14). □

11. The $n^{(4)} \to \hat{n}$ transformation

The symmetries

$$m^{k_1}(x, t, k) = \mathcal{A} m^{k_1}(x, t, \omega k)\mathcal{A}^{-1} = \mathcal{B} m^{k_1}(x, t, k^{-1})\mathcal{B}$$

allow us to extend the domain of definition of m^{k_1} from $D_\epsilon(k_1)$ to $\mathcal{D}$, where $\mathcal{D} := D_\epsilon(k_1) \cup \omega D_\epsilon(k_1) \cup \omega^2 D_\epsilon(k_1) \cup D_\epsilon(k_1^{-1}) \cup \omega D_\epsilon(k_1^{-1}) \cup \omega^2 D_\epsilon(k_1^{-1})$. Shrinking $\tau_{\max}$ if necessary, we may assume that i belongs to $D_\epsilon(k_1)$ for all $\tau \in \mathcal{I}$. Our next goal is to prove that

$$\hat{n} := \begin{cases} n^{(4)}(m^{k_1})^{-1}, & k \in \mathcal{D}, \\ n^{(4)}, & \text{elsewhere}, \end{cases} \tag{11.1}$$

is close to I for large x and $\tau \in \mathcal{I}$. Let $\hat{\Gamma} = \Gamma^{(4)} \cup \partial\mathcal{D}$ be the contour shown in Figure 9, and let us orient each circle that is part of $\partial\mathcal{D}$ in the clockwise direction. Define

$$\hat{v} = \begin{cases} v^{(4)}, & k \in \hat{\Gamma} \setminus \bar{\mathcal{D}}, \\ m^{k_1}, & k \in \partial\mathcal{D}, \\ m_-^{k_1} v^{(4)} (m_+^{k_1})^{-1}, & k \in \hat{\Gamma} \cap \mathcal{D}. \end{cases} \tag{11.2}$$

We denote the set of self-intersection points of $\hat{\Gamma}$ by $\hat{\Gamma}_\star$. The function $\hat{n}$ satisfies the following RH problem.

RH problem 11.1 (RH problem for $\hat{n}$)**.** *Find a 1×3-row-vector valued function $\hat{n}(x, t, k)$ with the following properties:*

(a) $\hat{n}(x, t, \cdot) : \mathbb{C} \setminus \hat{\Gamma} \to \mathbb{C}^{1\times 3}$ *is analytic.*

(b) *The limits of $\hat{n}(x, t, k)$ as k approaches $\hat{\Gamma} \setminus \hat{\Gamma}_\star$ from the left and right exist, are continuous on $\hat{\Gamma} \setminus \hat{\Gamma}_\star$, and are denoted by $\hat{n}_+$ and $\hat{n}_-$, respectively. Moreover,*

$$\hat{n}_+(x, t, k) = \hat{n}_-(x, t, k)\hat{v}(x, t, k) \qquad \textit{for} \quad k \in \hat{\Gamma} \setminus \hat{\Gamma}_\star.$$

(c) $\hat{n}(x, t, k) = (1, 1, 1) + O(k^{-1})$ *as* $k \to \infty$.

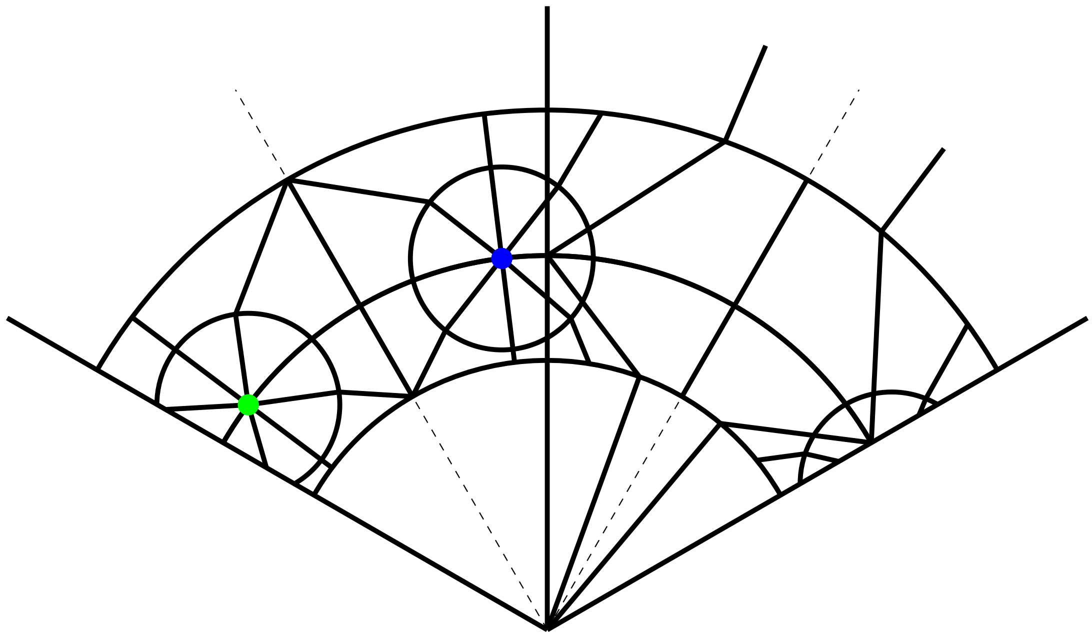

Figure 9. *The contour* $\hat{\Gamma} = \Gamma^{(4)} \cup \partial\mathcal{D}$ *for* $\arg k \in [\frac{\pi}{6}, \frac{5\pi}{6}]$ *(solid), the boundary of* S *(dashed), and the saddle points* k_1 *(blue) and* $\omega^2 k_2$ *(green).*

(*d*) *For* $k \in \mathbb{C} \setminus \hat{\Gamma}$, $\hat{n}$ *obeys the symmetries*

$$\hat{n}(x,t,k) = \hat{n}(x,t,\omega k)\mathcal{A}^{-1} = \hat{n}(x,t,k^{-1})\mathcal{B}.$$

(*e*) $\hat{n}(x,t,k) = O(1)$ *as* $k \to k_\star \in \hat{\Gamma}_\star$.

Lemma 11.2. *Let* $\hat{w} = \hat{v} - I$. *The following inequalities hold uniformly for* $x \geq 2$ *and* $\tau \in \mathcal{I}$:

$$\|\hat{w}\|_{(L^1 \cap L^\infty)(\hat{\Gamma} \setminus (\partial\mathcal{D} \cup \hat{\mathcal{X}}^\epsilon))} \leq C x^{-N}, \tag{11.3a}$$

$$\|\hat{w}\|_{(L^1 \cap L^\infty)(\partial\mathcal{D})} \leq C_N(\tau) x^{-1/2}, \tag{11.3b}$$

$$\|\hat{w}\|_{L^1(\hat{\mathcal{X}}^\epsilon)} \leq C_N(\tau) x^{-1} \ln x, \tag{11.3c}$$

$$\|\hat{w}\|_{L^\infty(\hat{\mathcal{X}}^\epsilon)} \leq C_N(\tau) x^{-1/2} \ln x. \tag{11.3d}$$

Proof. Note that $\hat{\Gamma} \setminus (\partial\mathcal{D} \cup \hat{\mathcal{X}}^\epsilon) = \Gamma^{(4)} \setminus \hat{\mathcal{X}}^\epsilon$. Thus (11.3a) is a consequence of (11.2), Lemma 9.1, and the uniform boundedness of m^{k_1}. The estimate (11.3b) follows from (11.2) and (10.11). The inequalities (11.3c) and (11.3d) are direct consequences of (11.2), (10.10), and the uniform boundedness of m^{k_1}. $\square$

The Cauchy transform $\hat{\mathcal{C}}h$ of a function h defined on $\hat{\Gamma}$ is defined by

$$(\hat{\mathcal{C}}h)(z) = \frac{1}{2\pi i} \int_{\hat{\Gamma}} \frac{h(z')dz'}{z' - z}, \qquad z \in \mathbb{C} \setminus \hat{\Gamma}.$$

The inequalities in Lemma 11.2 imply that

$$\begin{cases} \|\hat{w}\|_{L^1(\hat{\Gamma})} \leq C x^{-N} + C_N(\tau) x^{-1/2}, \\ \|\hat{w}\|_{L^2(\hat{\Gamma})} \leq C x^{-N} + C_N(\tau) x^{-1/2} (\ln x)^{1/2}, \\ \|\hat{w}\|_{L^\infty(\hat{\Gamma})} \leq C x^{-N} + C_N(\tau) x^{-1/2} \ln x, \end{cases} \qquad x \geq 2, \ \tau \in \mathcal{I}. \tag{11.4}$$

In particular, we have $\hat{w} \in L^2(\hat{\Gamma}) \cap L^\infty(\hat{\Gamma})$, so that the operator $\hat{\mathcal{C}}_{\hat{w}} = \hat{\mathcal{C}}_{\hat{w}(x,t,\cdot)} : L^2(\hat{\Gamma}) + L^\infty(\hat{\Gamma}) \to L^2(\hat{\Gamma})$, $h \mapsto \hat{\mathcal{C}}_{\hat{w}} h := \hat{\mathcal{C}}_-(h\hat{w})$ is well-defined. We denote the space of bounded linear operators on $L^2(\hat{\Gamma})$ by $\mathcal{B}(L^2(\hat{\Gamma}))$. Using (11.4) we infer that there exists an $X > 0$ such that $I - \hat{\mathcal{C}}_{\hat{w}(x,t,\cdot)} \in \mathcal{B}(L^2(\hat{\Gamma}))$ is invertible for all $x \geq X$ and $\tau \in \mathcal{I}$. Hence $\hat{n}$ satisfies a small-norm RH problem and can be expressed as

$$\hat{n}(x,t,k) = (1,1,1) + \hat{\mathcal{C}}(\hat{\mu}\hat{w}) = (1,1,1) + \frac{1}{2\pi i}\int_{\hat{\Gamma}} \hat{\mu}(x,t,s)\hat{w}(x,t,s)\frac{ds}{s-k} \tag{11.5}$$

for $x \geq X$ and $\tau \in \mathcal{I}$, where

$$\hat{\mu} = (1,1,1) + (I - \hat{\mathcal{C}}_{\hat{w}})^{-1}\hat{\mathcal{C}}_{\hat{w}}(1,1,1) \in (1,1,1) + L^2(\hat{\Gamma}). \tag{11.6}$$

Furthermore, using (11.4), (11.6), and the fact that $\|\hat{\mathcal{C}}_-\|_{\mathcal{B}(L^2(\hat{\Gamma}))} < \infty$, we find

$$\|\hat{\mu} - (1,1,1)\|_{L^2(\hat{\Gamma})} \leq \frac{C\|\hat{w}\|_{L^2(\hat{\Gamma})}}{1 - \|\hat{\mathcal{C}}_{\hat{w}}\|_{\mathcal{B}(L^2(\hat{\Gamma}))}} \leq Cx^{-N} + C_N(\tau)x^{-1/2}(\ln x)^{1/2} \tag{11.7}$$

for all $\tau \in \mathcal{I}$ and all large enough x. To determine the long-time asymptotics of $\hat{n}$, define $\hat{n}^{(1)}$ as the nontangential limit

$$\hat{n}^{(1)}(x,t) := \lim_{k\to\infty}^{\angle} k(\hat{n}(x,t,k) - (1,1,1)) = -\frac{1}{2\pi i}\int_{\hat{\Gamma}} \hat{\mu}(x,t,k)\hat{w}(x,t,k)dk.$$

Lemma 11.3. *As $x \to \infty$,*

$$\hat{n}^{(1)}(x,t) = -\frac{(1,1,1)}{2\pi i}\int_{\partial\mathcal{D}} \hat{w}(x,t,k)dk + O\big(x^{-N} + C_N(\tau)x^{-1}\ln x\big). \tag{11.8}$$

Proof. It suffices to write

$$\begin{aligned}\hat{n}^{(1)}(x,t) = &-\frac{(1,1,1)}{2\pi i}\int_{\partial\mathcal{D}} \hat{w}(x,t,k)dk - \frac{(1,1,1)}{2\pi i}\int_{\hat{\Gamma}\setminus\partial\mathcal{D}} \hat{w}(x,t,k)dk \\ &- \frac{1}{2\pi i}\int_{\hat{\Gamma}}(\hat{\mu}(x,t,k) - (1,1,1))\hat{w}(x,t,k)dk,\end{aligned}$$

and then to use (11.7) and Lemma 11.2. ◻

Let $\{F^{(l)}\}_{l\in\mathbb{Z}}$ be given by

$$F^{(l)}(x,\tau) = -\frac{1}{2\pi i}\int_{\partial D_\epsilon(k_1)} k^{l-1}\hat{w}(x,t,k)dk = -\frac{1}{2\pi i}\int_{\partial D_\epsilon(k_1)} k^{l-1}(m^{k_1} - I)dk.$$

Using (10.12) and $\hat{z}(\tau,k_1) = 1$, and recalling that $\partial D_\epsilon(k_1)$ is oriented clockwise, we get

$$F^{(l)}(x,\tau) = k_1^{l-1}\frac{Y(x,\tau)m_1^X(q)Y(x,\tau)^{-1}}{z_\star\sqrt{x}} + O(C_N(\tau)x^{-1}) = -ik_1^l Z(x,\tau) + O(C_N(\tau)x^{-1}) \tag{11.9}$$

as $x \to \infty$ uniformly for $\tau \in \mathcal{I}$, where

$$Z(x,\tau) = \frac{Y(x,\tau)m_1^X Y(x,\tau)^{-1}}{-ik_1 z_\star\sqrt{x}} = \frac{1}{-ik_1 z_\star\sqrt{x}}\begin{pmatrix} 0 & \frac{\beta_{12}}{d_0 e^{x\tilde{\Phi}_{21}(\tau,k_1)}\tilde{r}(k_1)^{\frac{1}{2}}} & 0 \\ d_0 e^{x\tilde{\Phi}_{21}(\tau,k_1)}\tilde{r}(k_1)^{\frac{1}{2}}\beta_{21} & 0 & 0 \\ 0 & 0 & 0\end{pmatrix}.$$

Lemma 11.4. *For $l \in \mathbb{Z}$ and $j = 0,1,2$, we have*

$$-\frac{1}{2\pi i}\int_{\omega^j\partial D_\epsilon(k_1)} k^l\hat{w}(x,t,k)dk = \omega^{j(l+1)}\mathcal{A}^{-j}F^{(l+1)}(x,\tau)\mathcal{A}^j, \tag{11.10}$$

$$-\frac{1}{2\pi i}\int_{\omega^j\partial D_\epsilon(k_1^{-1})} k^l \hat{w}(x,t,k)dk = -\omega^{j(l+1)}\mathcal{A}^{-j}\mathcal{B}F^{(-l-1)}(x,\tau)\mathcal{B}\mathcal{A}^j. \tag{11.11}$$

Proof. These identities follow from $\hat{w}(x,t,k) = \mathcal{A}\hat{w}(x,t,\omega k)\mathcal{A}^{-1} = \mathcal{B}\hat{w}(x,t,k^{-1})\mathcal{B}$ which hold for $k \in \partial\mathcal{D}$. A more detailed argument in a similar situation can be found in [12, Proof of Lemma 12.4]. □

It follows from Lemma 11.4 that

$$\frac{-1}{2\pi i}\int_{\partial\mathcal{D}} \hat{w}(x,t,k)dk = \sum_{j=0}^{2}\frac{-1}{2\pi i}\int_{\omega^j\partial D_\epsilon(k_1)}\hat{w}(x,t,k)dk - \sum_{j=0}^{2}\frac{1}{2\pi i}\int_{\omega^j\partial D_\epsilon(k_1^{-1})}\hat{w}(x,t,k)dk$$
$$= \sum_{j=0}^{2}\omega^j\mathcal{A}^{-j}F^{(1)}(x,\tau)\mathcal{A}^j - \sum_{j=0}^{2}\omega^j\mathcal{A}^{-j}\mathcal{B}F^{(-1)}(x,\tau)\mathcal{B}\mathcal{A}^j.$$

Hence, (11.8) and (11.9) show that $\hat{n}^{(1)} = (1,1,1)\hat{m}^{(1)}$, where

$$\begin{aligned}\hat{m}^{(1)}(x,t) &= \sum_{j=0}^{2}\omega^j\mathcal{A}^{-j}F^{(1)}(x,\tau)\mathcal{A}^j - \sum_{j=0}^{2}\omega^j\mathcal{A}^{-j}\mathcal{B}F^{(-1)}(x,\tau)\mathcal{B}\mathcal{A}^j + O\big(x^{-N} + C_N(\tau)x^{-1}\ln x\big)\\ &= -ik_1\sum_{j=0}^{2}\omega^j\mathcal{A}^{-j}Z(x,\tau)\mathcal{A}^j + ik_1^{-1}\sum_{j=0}^{2}\omega^j\mathcal{A}^{-j}\mathcal{B}Z(x,\tau)\mathcal{B}\mathcal{A}^j + O\big(x^{-N} + C_N(\tau)x^{-1}\ln x\big)\end{aligned} \tag{11.12}$$

as $x \to \infty$, uniformly for $\tau \in \mathcal{I}$.

12. Asymptotics of u

By (4.1), we have $u(x,t) = -i\sqrt{3}\frac{\partial}{\partial x}n_3^{(1)}(x,t)$, where $n_3(x,t,k) = 1 + n_3^{(1)}(x,t)k^{-1} + O(k^{-2})$ as $k \to \infty$. By inverting the transformations $n \mapsto n^{(1)} \mapsto n^{(2)} \mapsto n^{(3)} \mapsto n^{(4)} \mapsto \hat{n}$ using (5.11), (6.5), (7.2), (9.1), and (11.1), we obtain

$$n = \hat{n}\Delta^{-1}(G^{(3)})^{-1}(G^{(2)})^{-1}(G^{(1)})^{-1}, \qquad k \in \mathbb{C}\setminus(\hat{\Gamma}\cup\bar{\mathcal{D}}),$$

where $G^{(1)}$, $G^{(2)}$, $G^{(3)}$, Δ are given by (5.12), (6.6), (7.3), and (8.9), respectively. Thus

$$u(x,t) = -i\sqrt{3}\frac{\partial}{\partial x}\bigg(\hat{n}_3^{(1)}(x,t) + \overset{\angle}{\lim_{k\to\infty}} k(\Delta_{33}(\tau,k)^{-1} - 1)\bigg),$$

where $\hat{n}_3^{(1)}$ is defined via the expansion $\hat{n}_3(x,t,k) = 1 + \hat{n}_3^{(1)}(x,t)k^{-1} + O(k^{-2})$ as $k \to \infty$. By virtue of (8.8) and the change of variables formula

$$\frac{\partial}{\partial x}\bigg|_{t\ \mathrm{const}} = \frac{\partial}{\partial x}\bigg|_{\tau\ \mathrm{const}} - \frac{\tau}{x}\frac{\partial}{\partial\tau}\bigg|_{x\ \mathrm{const}}, \tag{12.1}$$

we obtain

$$\frac{\partial}{\partial x}\overset{\angle}{\lim_{k\to\infty}} k(\Delta_{33}(\tau,k)^{-1} - 1) = \frac{\tau}{x}\frac{\sqrt{3}}{2\pi}\frac{d}{d\tau}\int_i^{k_1(\tau)}\ln(1 + r_1(s)r_2(s))(1 + s^{-2})ds. \tag{12.2}$$

Since $r_1(s)r_2(s)$ vanishes to all orders at $s = i$ and $k_1 = ie^{i\tau + O(\tau^3)}$ as $\tau \to 0$, the right-hand side of (12.2) is $O(C_N(\tau)x^{-1})$. Hence

$$u(x,t) = -i\sqrt{3}\frac{\partial}{\partial x}\hat{n}_3^{(1)}(x,t) + O(C_N(\tau)x^{-1}), \qquad x \to \infty, \tag{12.3}$$

uniformly for $\tau \in \mathcal{I}$. By (11.12), we have

$$\hat{n}_3^{(1)} = Z_{12}\big(\omega i k_1^{-1} - \omega^2 i k_1\big) + Z_{21}\big(\omega^2 i k_1^{-1} - \omega i k_1\big) + O\big(x^{-N} + C_N(\tau)x^{-1}\ln x\big)$$

$$= \frac{\beta_{12}(\omega i k_1^{-1} - \omega^2 i k_1)}{-ik_1 z_\star \sqrt{x} d_0 e^{x\tilde{\Phi}_{21}(\tau,k_1)} \tilde{r}(k_1)^{\frac{1}{2}}} + \frac{d_0 e^{x\tilde{\Phi}_{21}(\tau,k_1)} \tilde{r}(k_1)^{\frac{1}{2}} \beta_{21}(\omega^2 i k_1^{-1} - \omega i k_1)}{-ik_1 z_\star \sqrt{x}}$$
$$+ O\big(x^{-N} + C_N(\tau) x^{-1} \ln x\big)$$

as $x \to \infty$ uniformly for $\tau \in \mathcal{I}$. The identity $|d_0(x,\tau)| = e^{2\pi\nu}$ established in Lemma 10.1 implies that $\frac{\beta_{12}}{d_0} = -\bar{d}_0 \bar{\beta}_{21}$. Therefore, using also that $(\omega i k_1^{-1} - \omega^2 i k_1)/\tilde{r}(k_1)^{\frac{1}{2}} = \tilde{r}(k_1)^{\frac{1}{2}}(\omega^2 i k_1^{-1} - \omega i k_1) \in \mathbb{R}$ and $-ik_1 z_\star > 0$,

$$\hat{n}_3^{(1)} = 2i \operatorname{Im} \frac{\beta_{12}(\omega i k_1^{-1} - \omega^2 i k_1)}{-ik_1 z_\star \sqrt{x} d_0 e^{x\tilde{\Phi}_{21}(\tau,k_1)} \tilde{r}(k_1)^{\frac{1}{2}}} + O\big(x^{-N} + C_N(\tau) x^{-1} \ln x\big) \qquad \text{as } x \to \infty.$$

Since

$$\beta_{12} = \frac{\sqrt{2\pi} e^{\frac{\pi i}{4}} e^{\frac{3\pi\nu}{2}}}{q\Gamma(i\nu)}, \qquad d_0(x,\tau) = e^{2\pi\nu} e^{i \arg d_0(x,\tau)}, \qquad |q| = \sqrt{e^{-2\pi\nu} - 1},$$
$$|\Gamma(i\nu)| = \frac{\sqrt{2\pi}}{\sqrt{-\nu}\sqrt{e^{-\pi\nu} - e^{\pi\nu}}} = \frac{\sqrt{2\pi}}{\sqrt{-\nu} e^{\frac{\pi\nu}{2}} |q|}, \qquad \frac{\omega i k_1^{-1} - \omega^2 i k_1}{\tilde{r}(k_1)^{\frac{1}{2}}} = -\sqrt{-1 - 2\cos(2\arg k_1)},$$

we find

$$\hat{n}_3^{(1)} = -2i \frac{\sqrt{-\nu}\sqrt{-1 - 2\cos(2\arg k_1)}}{-ik_1 z_\star \sqrt{x}} \sin\Big(\frac{\pi}{4} - \arg q - \arg\Gamma(i\nu) - \arg d_0 - x \operatorname{Im} \tilde{\Phi}_{21}(\tau,k_1)\Big)$$
$$+ O\big(x^{-N} + C_N(\tau) x^{-1} \ln x\big) \qquad \text{as } x \to \infty,$$

uniformly for $\tau \in \mathcal{I}$. As in [13], we can prove that the above asymptotics can be differentiated with respect to x without increasing the error term. Hence, by substituting the above formula into (12.3) and using (12.1) and the relation $\operatorname{Im}\big(\tilde{\Phi}_{21}(\tau,k_1) - \tau \frac{d}{d\tau}\tilde{\Phi}_{21}(\tau,k_1(\tau))\big) = \operatorname{Im} k_1$, we arrive at

$$u = -2\sqrt{3}\Big(\frac{\partial}{\partial x} - \frac{\tau}{x}\frac{\partial}{\partial \tau}\Big)\Bigg(\sin\Big(\frac{\pi}{4} - \arg q - \arg\Gamma(i\nu) - \arg d_0 - x \operatorname{Im} \tilde{\Phi}_{21}(\tau,k_1)\Big)$$
$$\times \frac{\sqrt{-\nu}\sqrt{-1 - 2\cos(2\arg k_1)}}{-ik_1 z_\star \sqrt{x}}\Bigg) + O\big(x^{-N} + C_N(\tau) x^{-1} \ln x\big)$$
$$= \frac{A(\tau)}{\sqrt{x}} \cos\Big(\frac{\pi}{4} - \arg q - \arg\Gamma(i\nu) - \arg d_0 - x \operatorname{Im} \tilde{\Phi}_{21}(\tau,k_1)\Big) + O\big(x^{-N} + C_N(\tau) x^{-1} \ln x\big)$$

as $x \to \infty$ uniformly for $\tau \in \mathcal{I}$, where $A(\tau)$ is as in the statement of Theorem 2.1. Since $\arg q = \pi + \arg r_2(k_1) \pmod{2\pi}$ and $\cos(-x) = \cos(x)$, we can write this as

$$u(x,t) = \frac{A(\tau)}{\sqrt{x}} \cos\Big(\frac{3\pi}{4} + \arg r_2(k_1) + \arg\Gamma(i\nu) + \arg d_0 + x \operatorname{Im} \tilde{\Phi}_{21}(\tau,k_1)\Big)$$
$$+ O\big(x^{-N} + C_N(\tau) x^{-1} \ln x\big)$$

as $x \to \infty$ uniformly for $\tau \in \mathcal{I}$. The expression (2.9) for $\arg d_0(x,\tau)$ follows from a long but direct computation using (10.2), (8.2), and (8.3). This finishes the proof of Theorem 2.1.

Appendix A. Model RH problem

The lemma presented in this appendix is needed for the local parametrix near k_1. The proof involves parabolic cylinder functions as in [24], and we refer to [11, Appendix A] for a constructive proof of a similar lemma.

Let $X = X_1 \cup \cdots \cup X_4 \subset \mathbb{C}$ be the cross consisting of the four rays

$$\begin{aligned} X_1 &= \left\{se^{\frac{i\pi}{4}} \,\middle|\, 0 \le s < \infty\right\}, & X_2 &= \left\{se^{\frac{3i\pi}{4}} \,\middle|\, 0 \le s < \infty\right\}, \\ X_3 &= \left\{se^{-\frac{3i\pi}{4}} \,\middle|\, 0 \le s < \infty\right\}, & X_4 &= \left\{se^{-\frac{i\pi}{4}} \,\middle|\, 0 \le s < \infty\right\}, \end{aligned} \tag{A.1}$$

all oriented away from the origin.

Lemma A.1 (Model RH problem needed near $k = k_1$)**.** *Let $q \in \mathbb{C}$, and define*

$$\nu = -\tfrac{1}{2\pi} \ln(1 + |q|^2) \le 0.$$

Define the jump matrix $v^X(z)$ for $z \in X$ by

$$\begin{aligned} &\begin{pmatrix} 1 & \frac{-\bar{q}}{1+|q|^2} z_{(0)}^{-2i\nu} e^{\frac{iz^2}{2}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ if } z \in X_1, && \begin{pmatrix} 1 & 0 & 0 \\ q z_{(0)}^{2i\nu} e^{-\frac{iz^2}{2}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ if } z \in X_2, \\ &\begin{pmatrix} 1 & \bar{q} z_{(0)}^{-2i\nu} e^{\frac{iz^2}{2}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ if } z \in X_3, && \begin{pmatrix} 1 & 0 & 0 \\ \frac{-q}{1+|q|^2} z_{(0)}^{2i\nu} e^{-\frac{iz^2}{2}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ if } z \in X_4, \end{aligned} \tag{A.2}$$

where $z_{(0)}^{i\nu}$ has a branch cut along $[0, +\infty)$, such that $z_{(0)}^{i\nu} = |z|^{i\nu} e^{-\nu \arg_0(z)}$, $\arg_0(z) \in (0, 2\pi)$. Then the RH problem

(a) $m^X(q, \cdot) = m^X(q, \cdot) : \mathbb{C} \setminus X \to \mathbb{C}^{3\times 3}$ is analytic;
(b) on $X \setminus \{0\}$, the boundary values of $m^X(q, \cdot)$ exist, are continuous, and satisfy $m_+^X = m_-^X v^X$;
(c) $m^X(q, z) = I + O(z^{-1})$ as $z \to \infty$, and $m^X(q, z) = O(1)$ as $z \to 0$;

has a unique solution $m^X(q, z)$. This solution satisfies

$$m^X(q, z) = I + \frac{m_1^X(q)}{z} + O\left(\frac{q}{z^2}\right), \quad z \to \infty, \quad m_1^X(q) := \begin{pmatrix} 0 & \beta_{12} & 0 \\ \beta_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{A.3}$$

where the error term is uniform for $\arg z \in [-\pi, \pi]$ and q in compact subsets of $\mathbb{C}$, and

$$\beta_{12} := \frac{\sqrt{2\pi} e^{\frac{\pi i}{4}} e^{\frac{3\pi\nu}{2}}}{q \Gamma(i\nu)}, \qquad \beta_{21} := \frac{\sqrt{2\pi} e^{-\frac{\pi i}{4}} e^{-\frac{5\pi\nu}{2}}}{-\bar{q} \Gamma(-i\nu)}. \tag{A.4}$$

Remark A.2. *Since $|\Gamma(i\nu)| = \frac{\sqrt{2\pi}}{\sqrt{-\nu}\sqrt{e^{-\pi\nu} - e^{\pi\nu}}} = \frac{\sqrt{2\pi}}{\sqrt{-\nu} e^{\frac{\pi\nu}{2}} |q|}$, it follows that $\beta_{12}\beta_{21} = \nu$ and that $m_1^X(q) = O(q)$ as $q \to 0$.*

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