

SDEs with uniform distributions: Peacocks, Conic martingales and mean reverting uniform diffusions

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Abstract

Peacocks are increasing processes for the convex order. To any peacock, one can associate martingales with the same marginal laws. We are interested in finding the *diffusion* associated to the *uniform peacock*, i.e., to the peacock with uniform law at all times on a time-varying support $[a(t), b(t)]$. Following an idea from Dupire [9], Madan and Yor [20] propose a construction to find a diffusion martingale associated to a peacock, under the assumption of existence of a solution to a particular stochastic differential equation (SDE). In this paper we derive the SDE associated to the uniform peacock and give sufficient conditions on the (conic) boundary to have a unique strong or weak solution and analyse the local time at the boundary. Eventually, we focus on the *constant support* case. Given that the only uniform martingale with time-independent support proves to be a constant, we consider more general (mean reverting) diffusions. We prove existence of a solution to the related SDE and derive the moments of transition densities. Limit-laws and ergodic results show that the transition law tends to a uniform distribution.

Keywords: Uniformly distributed Stochastic Differential Equation, Conic Martingales, Peacock Process, Uniformly distributed Diffusion, Mean Reverting Uniform SDE.

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1 Introduction

A *peacock* is an integrable process that is increasing in the convex order. In other words, a peacock is a process with (i) constant expected value and (ii) whose transform via any

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positive and convex function Ψ has an increasing expectation; see [13, Definition 1.3]. More precisely, a process X with a constant expectation is a peacock if the map $t \mapsto \mathbb{E}(\Psi(X_t))$ is increasing for any convex function Ψ such that $\mathbb{E}(|\Psi(X_t)|) < \infty$ for all t . From this equivalent representation, it is trivial to show via the law of iterated expectations and Jensen's inequality that any martingale is a peacock. Reciprocally, it is known from Kellerer [17] that for any peacock there exist martingales (called *associated martingales*) with the same marginal laws. As Kellerer's proof is not constructive, specifying explicitly the martingales associated to a given peacock is not trivial. Madan and Yor tackled this problem by providing three different construction schemes: *Skorohod embedding*, *inhomogeneous independent increments* and *continuous martingales* [20]. The first two methods provide martingales taking the form of time-changed Brownian motions. The last method consists of inverting Fokker-Plank's formula (also known as the Forward Kolmogorov equation). Those methods require various assumptions. For instance, in the last approach, the authors of [20] mention that the solution exists if the diffusion coefficient is Lipschitz.

Peacock's theory is of practical interest. Numerous examples from financial mathematics are provided in the introduction of [13]. Actually, the main purpose of the whole book is to find various martingales associated to a given peacock, stressing the relevance of this mathematical problem. The first goal of our paper is to contribute to the peacock literature by studying the specific case of the *uniform peacock*. The associated diffusion martingale, called uniform diffusion martingale (UDM), is found explicitly. Diffusions with uniform marginal distributions have important applications. Indeed, any stochastic process $(X_t)_{t \geq 0}$ taking the form of conditional expectations $X_t := \mathbb{E}[Y | \mathcal{F}_t]$ for some random variable Y with values in a bounded interval is obviously a martingale. When no information is provided about Y on top of its bounds, it makes sense to work under the maximum-entropy principle, and postulate uniform martingale dynamics for X . Such a framework covers for example the case where Y represents the loss given default (or recovery rate) of a firm. In this context, X_t represents the "best guess" (in the mean-square sense) of the loss given default (or recovery rate) of a firm based on the information available at time t . Moreover, diffusions are specifically appealing when one wishes to introduce correlation between processes. For example, suppose that in a mathematical finance context one has designed a modeling suite that is based on diffusion processes technology. This is common for volatility smile modeling in several asset classes for example, or for interest rate and credit modeling in particular. One now wishes to add a model for a dynamics of the loss given default of a portfolio. Without any prior knowledge, it is decided that this loss given default will have a maximum entropy uniform distribution. Now using one of our uniform diffusions for this would have the advantage of aligning the random shock structure with the rest of the modeling suite. Indeed, we would be able to simulate the loss given default in time using the same Gaussian shocks that we use for all other modeling processes. This also allows one to correlate the loss given default to other processes in an easy way, be they loss given default of other portfolios or different variables. This can be done by simply correlating the Brownian motions driving the relevant processes. Also, one may have control of the volatility of the loss given default through the diffusion parameters. A similar reasoning could be applied to maximum entropy random correlations or even random

default probabilities. This altogether stresses the relevance of studying UDM. While the latter appears to be essentially unique, we show that there are infinitely many associated uniform diffusions (UD, not necessarily UDM), whose drift term can take very different forms. Analyzing the latter is the second goal of the paper; we give two motivations for this. First, bounded stochastic processes received surprisingly little attention in the literature (see e.g. the Jacobi process or the Φ -martingale, [8, 12, 14]). UD could be used to model random probabilities or stochastic correlations in a maximum-entropy framework. Second, UD are of particular interest due to the probability integral transform. Indeed, let $(U_t)_{t \geq 0}$ be a UD with support $[0, 1]$ and $G(\cdot; t)$ be the quantile function associated to a given distribution function $F(\cdot; t)$, in the sense that $G(F(x; t); t) = x$. Assume further that F is $\mathcal{C}^2$ in the first argument and $\mathcal{C}^1$ in the second. Then, $X_t := G(U_t; t)$ has distribution function $F(\cdot; t)$ and (from Ito's lemma) $(X_t)_{t \geq 0}$ is a diffusion. Hence, one can generate a diffusion with the required marginal laws starting with a UD.

Starting from an idea of [9] further used in [3] and [20], we will show that the SDE

$$dX_t = \left(\mathbb{I}_{\{X_t \in [-b(t), b(t)]\}} \frac{\dot{b}(t)}{b(t)} (b(t)^2 - X_t^2) \right)^{1/2} dW_t ,$$

has a solution which is a martingale, whose marginals X_t have a uniform density on $[-b(t), b(t)]$ for any $t > 0$, under some conditions on the increasing function b . In particular, we show strong existence and uniqueness for the case $b(t) = t^\alpha$, with $\alpha \geq 1$.

The case $b(t) = \sqrt{t}$ has to be dealt with using different techniques. For the case $0 < \alpha < 1$ case, we use the approach in [13, p. 253–260], and we obtain a weak solution and uniqueness in law.

We further show that the solution process spends zero time at the boundaries. In a second part, we extend our analysis to deal with uniform diffusions that may not be martingales. One reason for this is that martingales constant in time, i.e. $X_t = X_0$ where X_0 can be a random variable, seem to be the only UDMs when dealing with the time-independent support case ($b(t) = c$). Hence, UD with constant support are also analyzed.

The paper is structured as follows. In Section 2 we formulate the problem consisting of finding a UDM evolving in $[-b(t), b(t)]$ where b is regular enough, and we give the form of the diffusion coefficient. Because the latter fails to be Lipschitz, we study the solution rigorously in Section 3, and prove that the related SDE admits a unique strong solution. Being bounded on a finite horizon, the solution is a genuine martingale and, from Fokker-Planck, it is the diffusion martingale associated to the uniform peacock, i.e. a diffusion with uniform marginal law at all times, evolving in the desired conic boundary (Theorem 1 as well as the broader Theorem 2). In Section 4 we relax the martingale condition and study mean reverting UD obtained by re-scaling the conic UDM, where now the uniform law has a constant support that is no longer expanding. A special case of interest is uniform in $[-1, 1]$; the other cases can be obtained by scaling and shifting transforms. In Section 5 we prove limit-law results for the mean reverting uniform $[-1, 1]$ re-scaled process. In doing so, we derive the exact transition density of the SDE solution.

While we know that the solution margins are uniform by construction, this will not hold for the transition densities in general. We characterize the limit laws via their moments (Theorem 4). We show that, under some conditions, any initial condition $U_s = z$ at a given time s in the transition density is “forgotten” after a long enough time, and the limit tends again to a uniform. We further show in Section 6 that the rescaled processes have zero local time at the boundaries (Theorem 5). In Section 7 we revisit the two previously known cases and hint at new choices for the time-dependent boundaries. In the linear case we study the process’ pathwise activity, finding that the pathwise activity of the mean reverting diffusion vanishes asymptotically. The behavior of the process is illustrated based on numerical simulations that confirm our earlier characterization of the SDE having the desired marginal distribution and our limit-law type results.

2 Conic diffusion martingales with uniform distribution

We set out to construct a martingale diffusion process X , i.e. a diffusion process driven by a Brownian motion W that is a martingale, with marginal at time t having a uniform distribution in an interval $[a(t), b(t)]$ with $a(t) < b(t)$ for all $t > 0$. The martingale condition implies that $\mathbb{E}(X_t) = \mathbb{E}(X_0)$ for all $t \geq 0$, whereas the uniform distribution requirement implies that $\mathbb{E}(X_t) = [a(t) + b(t)]/2$ for all $t \geq 0$. Thus we have $a(t) + b(t) = a(0) + b(0)$ for all $t \geq 0$. We will assume $a(0) = b(0) = 0$, taking the initial condition X_0 to be deterministic and with value zero (Dirac delta law in 0). Hence $b(t) = -a(t)$ for all $t \geq 0$. With such preliminaries in mind, we state the following

Problem 1 (Designing conic martingale diffusions with given uniform law). *Consider the diffusion process*

$$dX_t = \sigma(X_t, t)dW_t, \quad X_0 = 0. \quad (1)$$

Find a diffusion coefficient $\sigma(x, t)$ such that

1. *The SDE (1) has a unique strong solution;*
2. *For any $t > 0$, the solution of (1) at time t , i.e., the random variable X_t , is uniformly distributed in $[-b(t), b(t)]$ for a non-negative strictly increasing continuous function $t \mapsto b(t)$ with $b(0) = 0$.*

In other terms, our aim is to build a diffusion martingale X as in (1) such that the process X has a density at time $t > 0$ at the point x given by the uniform density

$$\rho(x, t) := \mathbb{I}_{\{x \in [-b(t), b(t)]\}} / (2 b(t)). \quad (2)$$

Following [14], we call such martingales “conic” to stress that their support opens up in time.

In Problem 1, function b is restricted in a first step to be increasing in time. The reason is that the tight upper (resp. lower) bound of any bounded martingale must be

a non-decreasing (resp. non-increasing) function [25]. Hence, X is a *conic martingale*; it is a martingale that exhibits a conic behavior. We will need strict monotonicity in the following derivation, so we assumed b to be strictly increasing in Problem 1.

Following [20, eq. (8)], the martingale solution of the SDE $dX_t = \sigma(X_t, t)dW_t$ will have marginals with density $g(x, t)$ if the diffusion coefficient is $\sigma^2(x, t) = \frac{2C_t}{C_{xx}}$, where $C(t, x) = \int_x^\infty (y - x)g(y, t)dy$. In the case $g(x, t) = \rho(x, t)$, where ρ is defined in (2) one obtains

$$\sigma^2(x, t) = \mathbb{I}_{\{x \in [-b(t), b(t)]\}} \frac{\dot{b}(t)}{b(t)} (b(t)^2 - x^2). \quad (3)$$

For the ease of the reader, a derivation of this formula is given in the Appendix A the way it had been originally derived in the preprint [3], see also [5].

3 Analysis of the SDE: solutions and distributions

The diffusion coefficient (3) does not satisfy the Lipschitz condition (mentioned in [20]) at the frontier. Therefore, the SDE (1) is not guaranteed to admit a solution. In order to deal rigorously with existence and uniqueness issues, we now give suitable regularity conditions on the function $t \mapsto b(t)$ under which eq. (1) with diffusion coefficient (3) admits a unique strong solution that has indeed a uniform law at all times. In the more general case where regularity of the boundary is relaxed we shall prove that the weak solution is unique in law.

Theorem 1 (Existence and uniqueness of solution for candidate SDE solving Problem 1). *Let $T > 0$ and b be a strictly increasing function defined on $[0, T]$, continuous in $[0, T]$ and continuously differentiable in $(0, T]$ and satisfying $b(0) = 0$. Assume $\dot{b}$ to be bounded in $(0, T]$. The stochastic differential equation*

$$dX_t = \mathbb{I}_{\{X_t \in [-b(t), b(t)]\}} \left(\frac{\dot{b}(t)}{b(t)} (b(t)^2 - X_t^2) \right)^{1/2} dW_t, \quad X_0 = 0, \quad (4)$$

whose diffusion coefficient is extended to $t = 0$ by continuity via

$$\sigma(x, 0) := 0 \quad \text{for all } x,$$

admits a unique strong solution which is uniformly distributed in $[-b(t), b(t)]$ at every point in time t . We thus have a conic diffusion martingale with the cone expansion controlled by the time function b .

Proof. Observe first that the solution X to (4), if it exists, necessarily belongs to $[-b(t), b(t)]$ almost surely in order for the square root to be well defined. Moreover, this diffusion coefficient has been guessed from Fokker-Plank's inversion, showing that the solution, if it exists, will be uniformly distributed in the $[-b(t), b(t)]$ interval at any time t . This uniform

behavior can be double checked by analyzing the moments of the resulting solution. This is done in Appendix B for the sake of completeness.

It remains to prove that the solution X to (4) exists and is unique. To that end, it is enough to show that $\sigma(x, t)$ satisfies the linear growth bound and is Holder-1/2 for all $t \in [0, T]$ [18, p.135]. Clearly, $\sigma(x, t)$ in (3) satisfies the linear growth bound since it is uniformly bounded on $[0, T]$. To see this, notice that

$$0 \leq \sigma^2(x, t) = \mathbb{I}_{\{x \in [-b(t), b(t)]\}} (\dot{b}(t)/b(t))(b^2(t) - x^2) \leq \dot{b}(t)b(t) \quad \text{for all } x,$$

and that $\dot{b}(t)b(t)$ is bounded on $(0, T]$ by assumption, with zero limit when $t \downarrow 0$. This allows us to conclude that

$$\lim_{t \downarrow 0} \sigma^2(x, t) = 0 \quad \text{for all } x.$$

Since $\sigma^2(x, t)$ is continuous and bounded on $(0, T]$ with the above limit, it admits a continuous extension at $t = 0$ taking value zero. The extended $\sigma(x, t)$ is unique and uniformly bounded on $[0, T]$.

We now proceed with the Holder continuity of σ , for which the boundedness of $\dot{b}$ is needed. Of course, $f(x) = \sqrt{|x|}$ is Holder-1/2 on $\mathbb{R}$ since $|\sqrt{|x|} - \sqrt{|y|}| \leq \sqrt{|x - y|}$ for all x, y . We now check that $\sigma(x, t)$ is Holder-1/2 uniformly in $t > 0$ ($t = 0$ is not a problem given the above extension by continuity). Define $I(t) := [-b(t), b(t)]$. We check the possible cases.

1. If $x, y \notin I(t)$, the diffusion coefficient vanishes and one gets $|\sigma(x, t) - \sigma(y, t)| = 0$
2. If $x, y \in I(t)$, using the Holder-1/2 continuity of $\sqrt{|x|}$:

$$\begin{aligned} |\sigma(x, t) - \sigma(y, t)| &= \sqrt{\frac{\dot{b}(t)}{b(t)}} |\sqrt{b^2(t) - x^2} - \sqrt{b^2(t) - y^2}| \\ &\leq \sqrt{\frac{\dot{b}(t)}{b(t)}} \sqrt{|(b^2(t) - x^2) - (b^2(t) - y^2)|} \\ &= \sqrt{\frac{\dot{b}(t)}{b(t)}} \sqrt{|y^2 - x^2|} \leq \sqrt{\frac{\dot{b}(t)}{b(t)}} \sqrt{|y + x|} \sqrt{|y - x|} \leq \sqrt{\frac{\dot{b}(t)}{b(t)}} \sqrt{2b(t)} \sqrt{|x - y|} \\ &= \sqrt{2\dot{b}(t)} \sqrt{|x - y|} \end{aligned}$$

and we are done since $\dot{b}$ is assumed to be bounded in $(0, T]$.

3. If $x \in I(t)$, $y > b(t)$:

$$|\sigma(x, t) - \sigma(y, t)| = |\sigma(x, t)| = \sqrt{\frac{\dot{b}(t)}{b(t)}} \sqrt{b^2(t) - x^2} = \sqrt{\frac{\dot{b}(t)}{b(t)}} \sqrt{b(t) + x} \sqrt{b(t) - x} \leq$$

$$\leq \sqrt{\frac{\dot{b}(t)}{b(t)}} \sqrt{2b(t)} \sqrt{b(t) - x} \leq \sqrt{2\dot{b}(t)} \sqrt{|x - y|}$$

and again we are done since $\dot{b}$ is bounded in t .

4. If $x \in I(t)$, $y < -b(t)$ (so that $-y > b(t)$) :

$$\begin{aligned} |\sigma(x, t) - \sigma(y, t)| &= |\sigma(x, t)| \leq \sqrt{\frac{\dot{b}(t)}{b(t)}} \sqrt{b(t) - x} \sqrt{b(t) + x} \leq \\ &\leq \sqrt{\frac{\dot{b}(t)}{b(t)}} \sqrt{2b(t)} \sqrt{x + b(t)} \leq \sqrt{2\dot{b}(t)} \sqrt{x - y} \end{aligned}$$

5. The case $x \notin I(t)$, $y \in I(t)$ is similar to steps 3 and 4.

Hence, the solution X to (4) exists, is unique and has the desired uniform behavior in the conic $[-b(t), b(t)]$ boundary. □

Remark 1 (Indicator function in the diffusion coefficient). The solution of the SDE without the indicator function exists and is equal to the solution with the indicator function in the diffusion coefficient. Therefore, one could omit the indicator whenever the coefficient is featured inside a SDE.

The special case $b(t) = kt$ gives us a conic martingale with uniform distribution where the boundaries grow symmetrically and linearly in time. This example was considered originally in [3] and also in [13] (see for instance ex. 6.5 p.253 with $\varphi(x) = x$ and $f(z) = 1/2 \mathbb{I}_{\{z \in [-1, 1]\}}$). More generally, Theorem 1 covers the case $b(t) = t^\alpha$, for $\alpha \geq 1$. Staying in the class of boundaries t^α , we see that the case $\alpha < 1$ violates our assumptions, since in that case $\dot{b}$ is not bounded in 0, and has to be dealt with differently. For $0 < \alpha < 1$, and with the square root case in mind in particular, we now introduce a different approach to prove existence of the SDE solution as well as uniqueness (in law), as done in the peacock processes literature [13].

Theorem 2 (Existence of solution for SDE solving Problem 1 under milder conditions on the boundary). *Let b a continuous strictly increasing function defined on $[0, T]$ and of class C^1 in $(0, T]$, with $b(0) = 0$ and T a positive real number. Assume $b\dot{b}$ to be integrable in $(0, T]$. The stochastic differential equation (4), namely*

$$dX_t = \mathbb{I}_{\{X_t \in [-b(t), b(t)]\}} \left(\frac{\dot{b}(t)}{b(t)} (b^2(t) - X_t^2) \right)^{1/2} dW_t, \quad t > 0, \quad X_0 = 0,$$

admits a weak solution that is unique in law and its solution X is distributed at every point in time t as a uniform distribution concentrated in $[-b(t), b(t)]$. We thus have a conic diffusion martingale with the cone expansion controlled by the time function b .

Proof. From the same argument as in Theorem 1, the solution X to the SDE (4), if it exists, is uniformly distributed in $[-b(t), b(t)]$. The solution of (4) has to be understood in a first step as a process satisfying, for any $t \geq \epsilon > 0$

$$X_t = X_\epsilon + \int_\epsilon^t \mathbb{I}_{\{X_s \in [-b(s), b(s)]\}} \left(\frac{\dot{b}(s)}{b(s)} (b^2(s) - X_s^2) \right)^{1/2} dW_s$$

where X_ϵ has a uniform law in $(-b(\epsilon), b(\epsilon))$. The value of X at time 0 is defined by continuity when ϵ goes to zero (we will prove that the limit exists), and (4) can be written $X_t = \int_0^t \mathbb{I}_{\{X_s \in [-b(s), b(s)]\}} \left(\frac{\dot{b}(s)}{b(s)} (b^2(s) - X_s^2) \right)^{1/2} dW_s$ which has a meaning even if $\sigma(0, x)$ is not well defined.

It remains to prove that a solution X to (4) exists. We apply the methodology introduced in [13, Lemma 6.8] in the particular case where, using the notation of [13, Lemma 6.8], h is the density of a uniform law on $[-1, 1]$, with $a_h(y)$ as defined in (6.49) therein set to $1 - y^2/2$. In that work, the authors introduce a process $Y = (Y_t)_{t \in \mathbb{R}}$ such that, for all $t \geq s$

$$Y_t = Y_s - \frac{1}{2} \int_s^t Y_u du + \frac{1}{\sqrt{2}} \int_s^t \sqrt{1 - Y_u^2} dB_u$$

with marginals having uniform distribution on $[-1, 1]$, where B is a Brownian motion on $\mathbb{R}$ (not merely $\mathbb{R}^+$), meaning that it is a process with continuous paths and stationary independent increments. Then, setting

$$X_t := b(t)Z_t, \quad Z_t := Y_{\gamma(t)} \quad (5)$$

for $t > 0$, where γ is an increasing differentiable function, leads to a process with uniform marginals on $[-b(t), b(t)]$ (since by construction Z_t has a uniform law). It remains to find γ such that X is a martingale with the prescribed dynamics. Using [23, lemma 5.1.3.], and defining $\beta(y) := \frac{1}{\sqrt{2}} \sqrt{1 - y^2}$ and U as $U_t := \int_s^{\gamma(t)} \beta(Y_u) dB_u$, there exists an $\mathbb{F} = (\mathcal{F}_{\gamma(t)})_{t \geq 0}$ Brownian motion W such that

$$dU_t = \beta(Z_t) \sqrt{\dot{\gamma}(t)} dW_t.$$

It follows that

$$dZ_t = -\frac{1}{2} Z_t \dot{\gamma}(t) dt + \beta(Z_t) \sqrt{\dot{\gamma}(t)} dW_t \quad (6)$$

and by integration by parts

$$dX_t = b(t) \beta(Z_t) \sqrt{\dot{\gamma}(t)} dW_t \quad (7)$$

and the process X is a local martingale. Equating the diffusion coefficient of (4) to that of (7) yields to identifying $\dot{\gamma}(t) = 2 \frac{\dot{b}(t)}{b(t)}$ so that a valid choice for our time-change process is $\gamma(t) = 2 \ln b(t)$. The process X is a true martingale on $[s, T]$ for any $s > 0$: indeed by assumption on the integrability of $b\dot{b}$

$$\sigma^2(x, t) = \mathbb{I}_{\{x \in [-b(t), b(t)]\}} (\dot{b}(t)/b(t)) (b^2(t) - x^2) \leq \dot{b}(t)b(t)$$

and hence

$$\mathbb{E} \left[\left(\int_s^t \sigma(u, X_u) dW_u \right)^2 \right] = \mathbb{E} \left[\int_s^t \sigma^2(u, X_u) du \right] \leq \int_s^T \dot{b}(t)b(t) dt \leq C.$$

It remains to prove that $X_t = b(t)Y_{2\ln b(t)}$ goes to 0 a.s. when t goes to 0. This proof is similar to the proof given in [13, p. 260-261], where it is shown that X converges to some X_0 when t goes to 0. This is done by giving a bound for the number of downcrossings of X on the interval $[a, b]$ for the time interval $]0, t]$, using Dubins' inequality and showing that X_u converges in probability to 0 since $\mathbb{E}(|X_u| \geq \epsilon)$ goes to 0 when u goes to 0. This identifies $X_0 = 0$. Again in [13, p. 261] it is shown that one has uniqueness in law and the argument can be straightforwardly repeated for our process here. $\square$

4 Mean reverting uniform diffusions with constant boundaries

Notice that the martingale that is uniform in the expanding boundary $t \mapsto [-b(t), b(t)]$ is unique, in the sense that there is only one diffusion coefficient that will make the diffusion martingale attain a uniform law in $[-b(t), b(t)]$. One can check this informally by inspecting the “inverse Fokker-Planck” equation approach detailed in Appendix A. This is not surprising since it can be shown, using a generalized form of the backward Kolmogorov equation, that continuous and strong Markov martingales are uniquely determined by their marginal distributions, see e.g. [21, Section 2].¹ However, there are infinitely many diffusions with uniform margins in general, because one can play with the drift/diffusion coefficient pair. For example, there are many different $b(t)$ that would lead to a uniform $Z = X/b$ in $[-1, 1]$ via re-scaling, as we show below. Before looking at that, we discuss briefly mean reversion, since this property will be discussed below in relation with Z .

In this paper we define mean reversion as follows. A real-valued squared-integrable Markov process $(\xi_t(\omega))_{t \geq 0}$ mean reverts towards a long term mean $\bar{\theta} \in \mathbb{R}$ if the following holds: for all s in the time domain of the process and all possible values $\bar{\xi}$ for the process at time s , one has

$$\lim_{t \uparrow \infty} \mathbb{E}[\xi_t | \xi_s = \bar{\xi}] = \bar{\theta},$$

where $\bar{\theta}$ is a deterministic constant. This condition implies that wherever the process state is found at a given future time, the long term mean from that time onward is a constant deterministic value that does not depend on the chosen time and state. We also require $\lim_{t \uparrow \infty} \text{Var}(\xi_t)$ to exist and be finite.

Mean reversion is an important property that tells us that the process expectation tends to forget a specific initial condition in the long run from any past time. However, it

¹We are grateful to an anonymous referee for providing this reference.

is a special case of a more general property. If we assume that the process has a density with respect to the Lebesgue measure at all times $t > 0$, denote by

$$p_{\xi_t|\xi_s}(x; y) dx = \mathbb{P}\{\xi_t \in dx | \xi_s = y\}$$

the conditional density of ξ_t at dx given $\xi_s = y$, with $s < t$. We have that the whole law forgets earlier conditions if

$$\lim_{t \uparrow +\infty} p_{\xi_t|\xi_s}(x; y) \text{ exists, is a density in } x \text{ and depends neither on } s \text{ nor on } y.$$

We now focus on mean reversion and will get to the general law later in Section 5.

Take $t_0 > 0$ and consider the solution of the SDE (4) for $t \geq t_0$. If one starts from X solving eq. (4), one immediate way to obtain a diffusion $(Z_t)_{t \geq t_0}$ with a standard uniform distribution at all times by taking $Z_t = X_t/b(t)$ and a random initial condition ζ independent from W . We will see that this leads in particular to a simple mean reverting linear drift. This does not mean however that this is the only way to obtain a mean reverting uniform diffusion, there are many others.

Theorem 3. *Assumptions on b as in Theorem 1 but extended to all T : let b be a strictly increasing function defined on $[0, +\infty)$, continuous and continuously differentiable in $(0, +\infty)$. Assume $b(0) = 0$. Assume $\dot{b}$ to be bounded in $(0, T]$ for all $T > 0$. Assume further that $\lim_{t \uparrow +\infty} b(t) = +\infty$. Consider, for $t \geq t_0$, the SDE*

$$d\tilde{Z}_t = -\frac{\dot{b}(t)}{b(t)}\tilde{Z}_t dt + \mathbb{1}_{\{\tilde{Z}_t \in [-1, 1]\}} \left(\frac{\dot{b}(t)}{b(t)}(1 - \tilde{Z}_t^2) \right)^{1/2} dW_t, \quad \tilde{Z}_{t_0} = \zeta \sim U([-1, 1]) \quad (8)$$

with ζ independent of W . The unique solution to this equation mean reverts to 0 with reversion speed (defined as minus the drift rate) $\dot{b}/b$ and is distributed at any point in time as a uniform- $[-1, 1]$ random variable.

Proof. The proof is immediate. Indeed, if we consider the SDE for Z_t derived by Leibnitz rule, for $t > 0$, we see that the coefficients of the SDE of Z are the same as the coefficients for the SDE $\tilde{Z}$ above. Since X is distributed as a uniform in $[-b, b]$, $Z := X/b$ will be distributed as a uniform in $[-1, 1]$. For the mean reverting behavior, we first note that $\lim_{t \uparrow +\infty} \mathbb{E}(\tilde{Z}_t) = 0$, and $\lim_{t \uparrow +\infty} \text{Var}(\tilde{Z}_t) = 1/3$. Actually, we are in a special case where mean and variance are constant. Furthermore, whenever $\tilde{Z}_t$ is above the long term mean 0, the drift is negative, pointing back to 0, while the variance remains bounded. A similar symmetric pattern is observed when $\tilde{Z}_t$ is below zero. To address mean reversion more generally, we may analyze Z . From the martingale property of X , for $t > s > 0$,

$$\mathbb{E}(Z_t | \mathcal{F}_s) = \frac{b(s)}{b(t)} Z_s$$

hence

$$\mathbb{E}(Z_t | Z_s = z) = \frac{b(s)}{b(t)} z$$

and it follows that, for $t > s > t_0$,

$$\lim_{t \uparrow +\infty} \mathbb{E} \left(\tilde{Z}_t | \tilde{Z}_s = z \right) = \lim_{t \uparrow +\infty} \frac{b(s)}{b(t)} z = 0 .$$

We thus see that after a sufficiently long time the value z at time s is forgotten by the mean. $\square$

Remark 2. Note that mean reversion holds also under the weaker assumptions of Theorem 2 similarly extended to $(0, +\infty)$, provided that again $\lim_{t \uparrow +\infty} b(t) = +\infty$. This is the case for example with t^α with $\alpha \in [1/2, 1)$. More generally, we can find uniform diffusions whose drift and diffusion coefficients take a completely different form with respect to the “ $\dot{b}/b$ ” proportional drift we studied in this section. A specific example is provided in Appendix C, obtained by transforming a Brownian motion using its own distribution function.

5 Exact transition moments, limit laws and ergodic properties

We have shown above that mean reversion holds. In fact, we can say more than this, and we now analyze the limit behaviour of the process law and its exact moments.

5.1 The special case $b(t) = b_0 \exp(kt)$

In the special case $b(t) = b_0 \exp(kt)$ with $b_0 > 0$ we need X starting with $X_{t_0} \sim U([-b_0 \exp(kt_0), b_0 \exp(kt_0)])$. In this case we could also take $t_0 = 0$ since there is no singularity at time 0. The setting is slightly different than our earlier setting because even with $t_0 = 0$ the cone would not start with a point but rather with the interval $[-b_0, b_0]$. In particular, the initial condition for X would not be $X_0 = 0$; instead, X_0 would be requested to be a random variable with uniform law in $[-b_0, b_0]$. In this case we have the special property that

$$\dot{b}(t)/b(t) = k$$

is constant and the general SDE

$$d\tilde{Z}_t = -\frac{\dot{b}(t)}{b(t)} \tilde{Z}_t dt + \mathbb{I}_{\{\tilde{Z}_t \in [-1, 1]\}} \left(\frac{\dot{b}(t)}{b(t)} (1 - \tilde{Z}_t^2) \right)^{1/2} dW_t, \quad \tilde{Z}_{t_0} = \zeta \sim U([-1, 1]) \quad (9)$$

is in fact a *time homogeneous diffusion*

$$d\tilde{Z}_t = -k \tilde{Z}_t dt + \mathbb{I}_{\{\tilde{Z}_t \in [-1, 1]\}} \left(k(1 - \tilde{Z}_t^2) \right)^{1/2} dW_t, \quad \tilde{Z}_{t_0} = \zeta \sim U([-1, 1]) \quad (10)$$

to which we can apply standard boundary and ergodic theory techniques for time homogeneous one-dimensional diffusions, see for example [16].

Let's analyze eq. (10) using the standard theory. First of all in this case we already know from our previous analysis of X that, if $\bar{p}$ is the density of a $U([-1, 1])$ random variable then $\bar{p}$ satisfies the Fokker Planck equation for the marginal density of the diffusion (10) so that

$$\mathcal{L}^* \bar{p} = 0 ,$$

where $\mathcal{L}^*$ is the forward diffusion operator of the Fokker Planck equation. This means that $\bar{p}$ is the invariant measure for the diffusion (10). This can be further confirmed by standard calculations: given a diffusion process with drift μ and diffusion coefficient σ , under suitable conditions (see for example [19]) the invariant measure is proportional to

$$\frac{2}{\sigma^2(x) \exp \left(-2 \int_{x_0}^x \frac{\mu(u)}{\sigma^2(u)} du \right)}$$

which, with our $\mu(x) = -kx$ and $\sigma(x) = (k(1-x^2))^{1/2}$ results immediately in a uniform density. Hence we have that the uniform is the invariant measure of our diffusion and that our diffusion is ergodic. We also have

$$\lim_{t \uparrow +\infty} p_{Z_{t+s}|Z_s}(y; x) = \lim_{t \uparrow +\infty} p_{Z_t|Z_0}(y; x) = \bar{p}(y) \text{ for all } s > 0, \ x \in [-1, 1] .$$

5.2 The general case with curved boundary

Now we move to the case of the general boundary $b(t)$ in eq. (9). We already know that the density $\bar{p}$ satisfies the Fokker Planck equation for the marginal density of (9). Given that $\partial \bar{p} / \partial t = 0$ and that the Fokker Planck equation reads $\partial p_t / \partial t = \mathcal{L}_t^* p_t$ we deduce that

$$\mathcal{L}_t^* \bar{p} = 0$$

for the operator $\mathcal{L}^*$ of (9). Hence $\bar{p}$ is also the invariant measure for the more general case (9). It's not clear beforehand however that the diffusion (9) has a limit transition law. To check this, we first derive the exact moments of the transition laws. We have the following, proven in Appendix D.

Theorem 4 (Transition moments for the time-inhomogeneous mean reverting uniform diffusion (9)). *Let $\xi_n := \text{mod}(n, 2)$ stand for the odd indicator and $\mu_n := (1 - \xi_n)/(n + 1)$ denote the n -th moment of a random variable uniformly distributed in $[-1, 1]$. Then, the conditional moments $M_n(s, t; z) := \mathbb{E}(\tilde{Z}_t^n | \tilde{Z}_s = z), t \geq s$ are given by*

$$M_n(s, t; z) = \mu_n + \sum_{k=1}^{\frac{n+\xi_n}{2}} (-1)^k (z^{2k-\xi_n} - \mu_{2k-\xi_n}) \sum_{j=k}^{\frac{n+\xi_n}{2}} \alpha_{j,k}[n] (-1)^j \left(\frac{b(s)}{b(t)} \right)^{j(2(j-\xi_n)+1)}$$

with $\alpha[n]$ being $\frac{n+\xi_n}{2}$ -by- $\frac{n+\xi_n}{2}$ lower triangular matrices (i.e. $\alpha_{j,k}[n] = 0$ for all $k > j$) whose lower entries are defined as

$$\alpha_{j,k}[n] = \begin{cases} 1 & \text{if } j = k = \frac{n+\xi_n}{2} \\ -(-1)^{\frac{n+\xi_n}{2}} \sum_{i=k}^{\frac{n+\xi_n}{2}-1} \alpha_{i,k}[n](-1)^i & \text{if } j = \frac{n+\xi_n}{2}, k < j \\ \frac{\alpha_{j,k}[n-2]n(n-1)}{n(n+1)-2j(2(j-\xi_n)+1)} & \text{otherwise} . \end{cases} \quad (11)$$

Corollary 1 (Limit law for the transition densities of (9)). When b is grounded and non-decreasing, the solution of the SDE (9) conditional on $\tilde{Z}_s = z \in [-1, 1]$, $s \geq 0$ admits a stationary law in the sense that each conditional moment of the solution tends to a constant. If, moreover, $\lim_{t \rightarrow \infty} 1/b(t) = 0$ then the stationary law is $U([-1, 1])$.

Proof. The only t -dependency in the $M_n(s, t; z)$ expression given in Theorem 4 appears in the ratio $b(s)/b(t)$. From the positivity and non-decreasingness assumptions about b , one clearly has $0 \leq \lim_{t \rightarrow \infty} b(s)/b(t) \leq 1$. Hence, $M_n(s, t; z)$ is the sum of two terms: the n -th uniform moment μ_n and a weighted sum of positive powers of terms in $[0, 1]$ (which obviously converges since it is finite). Moreover, in the special case where $\lim_{t \rightarrow \infty} 1/b(t) \rightarrow 0$, the finite sum collapses to 0, leading to $\lim_{t \rightarrow \infty} M_n(s, t; z) = \mu_n$. Because the uniform distribution is uniquely determined by its moments, the asymptotic law is uniform, and does no longer depend on s , asymptotically. $\square$

The above theorem is illustrated on Figure 1. This result suggests that one could connect the general case to the special time-homogeneous case discussed in Section 5.1. This intuition can be confirmed by introducing a deterministic time change. This is performed in Appendix E.

6 Local time at the boundaries and potential applications

We now discuss the behavior of the solution $\tilde{Z}$ of (8) at the boundaries -1 and 1, and thus the behaviour of the original X_t , solution of (4), at the boundaries $-b(t)$ and $b(t)$.

Theorem 5. *Given a strictly increasing function b defined in $[0, T]$, continuous, and differentiable in $(0, T]$, assume $b(0) = 0$ and $\dot{b}$ to be bounded in $(0, T]$. The local time for the process $b(t) - X_t$ (resp. $X_t + b(t)$) at level 0 is zero.*

Proof. Let us introduce $U_t = b(t) - X_t$. Then

$$d\langle U \rangle_t = \mathbb{1}_{\{0 \leq U_t \leq 2b(t)\}} \left(\frac{\dot{b}(t)}{b(t)} \right) U_t (2b(t) - U_t) dt .$$

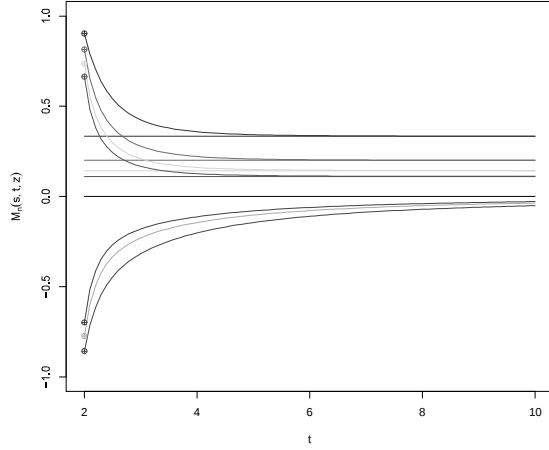
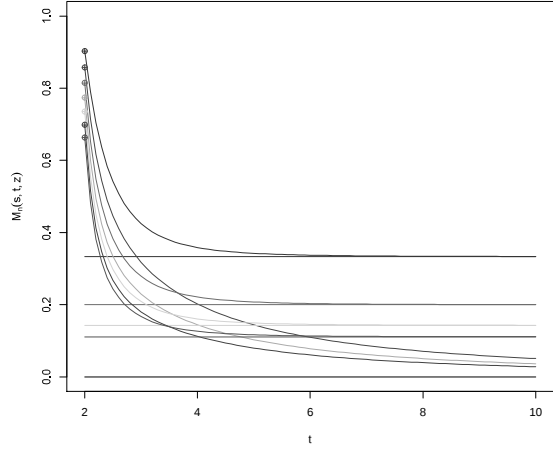
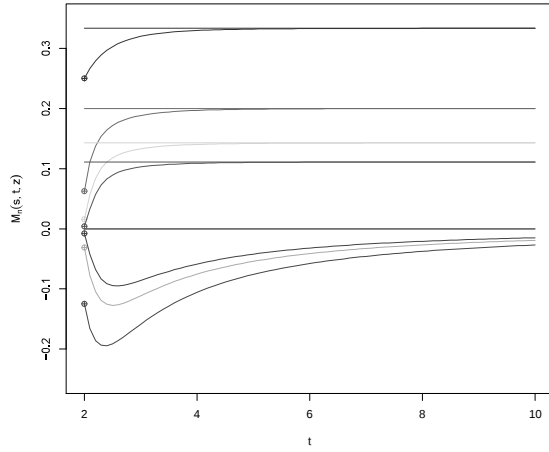
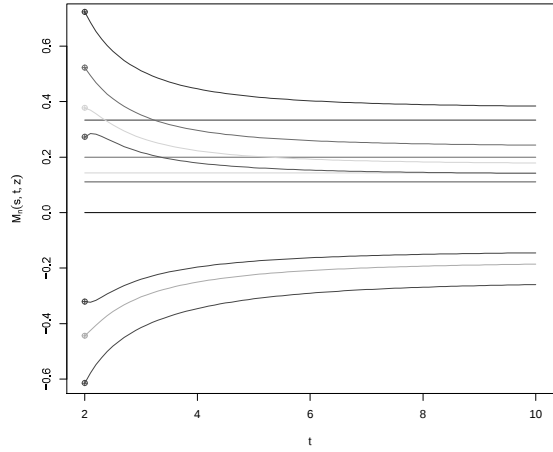
(a) $z = -0.95, b(t) = 2t^{3/2}$ (b) $z = 0.95, b(t) = 2t^{3/2}$ (c) $z = -0.5, b(t) = 2t^{3/2}$ (d) $z = -0.85, b(t) = 1 - e^{-t/3}$

Figure 1: Evolution of the 2, ..., 8-th conditional moments (solid) and those of $U([-1, 1])$ (dashed) for $s = 2$ up to $T = 10$ in the following order: blue, brown, red, orange, green, dark green, purple. The conditional moments converge to those of the uniform in panels (a), (b) and (c) since $\lim_{t \rightarrow \infty} 1/b(t) = 0$, in contrast to panel (d) where $\lim_{t \rightarrow \infty} 1/b(t) = 1$.

Then, denoting by L^a the local time of U at level a

$$\begin{aligned} t &\geq \int_0^t \mathbb{I}_{\{0 \leq U_s \leq 2b(s)\}} ds = \int_0^t \mathbb{I}_{\{0 \leq U_s \leq 2b(s)\}} \frac{b(s)}{\dot{b}(s)} \frac{1}{(2b(s) - U_s)U_s} d\langle U \rangle_s \\ &= \int_0^\infty da \int_0^t \mathbb{I}_{\{0 \leq a \leq 2b(s)\}} \frac{b(s)}{\dot{b}(s)} \frac{1}{(2b(s) - a)a} d_s L_s^a, \end{aligned}$$

where the last equality comes from an extension of the occupation time formula ([24], Chapter VI, Section 1, Corollary 1.6) as in [10].

We note that $b(s)/\dot{b}(s)$ is bounded from below by a positive constant C for all $s \geq \delta$. We can easily see that this is indeed the case since $\dot{b}(s)b(s)$ is bounded by above in $[0, T]$ by assumption, say by a constant $K > 0$, so that $\dot{b}(s)/b(s) = \dot{b}(s)b(s)/b(s)^2 \leq K/b(\delta)^2 =: C$. This implies that $b(s)/\dot{b}(s) \geq C$ for all $t \geq \delta$. We obtain

$$t \geq C \int_0^\infty da \int_\delta^t \mathbb{I}_{\{0 \leq a \leq 2b(\delta)\}} \frac{dL_s^a}{(2b(\delta) - a)a} \geq C \int_0^{2b(\delta)} \frac{L_t^a - L_\delta^a}{(2b(\delta) - a)a} da,$$

which implies that $L_t^0 - L_\delta^0 = 0$. By continuity, L_δ^0 goes to 0 when δ goes to 0. $\square$

As a corollary, the solution to (4) spends zero time at the boundaries $-b$ and b whenever $\dot{b}b$ is bounded.

We conclude this section with a hint at potential applications of our processes and with two remarks. $\tilde{Z}$ can be used as a model for stochastic correlation whereas $\tilde{Y} := (1 + \tilde{Z})/2$ can be used for example to model the dynamics of recovery rates or probabilities in the case of no information (maximum entropy).

Remark 3. The above construction for $\tilde{Z}$, mean reverting uniform diffusions with fixed boundaries based on rescaling the process X of Theorem 1, has the drawback of starting time at $t_0 > 0$, without defining the dynamics in $[0, t_0)$. This is done to avoid singularities in $t = 0$ with the rescaling. On the other hand, it has the advantage that the solution is unique in the strong sense. An alternative for obtaining a similar process, especially for cases like $b(t) = \sqrt{t}$, is to start from X constructed as in Theorem 2, requiring assumptions on b that are weaker than in Theorem 1. If we do so, and recalling Y in the proof of Theorem 2 and eq. (5) in particular, we obviously could have $Z_t = Y_{\gamma(t)}$ where $\gamma(t) = 2 \ln b(t)$, or even $Z_t = Y_t$. Notice however that to get a diffusion with uniform law in $[-1, 1]$ we could directly define a process $\hat{Z}$ as $\hat{Z}_t := Y_{\alpha(t)}$ for any time change function α provided that it is increasing. Indeed, this would not affect the marginals of $\hat{Z}$ as Y is a diffusion with uniform marginals in $[-1, 1]$ at all times.

Remark 4. The above rescaling approach yields a diffusion associated to the uniform peacock with constant boundaries $-1, 1$. It is also obvious from eq. (6) that defining Z as $Z_t = Y_{\gamma(t)}$ will lead to a mean reverting diffusion. However, this is a mean reverting diffusion process and not a diffusion martingale. Still, we know since [17] that there is a martingale associated to any peacock. Hence a natural question is: what is the diffusion martingale

associated with this peacock ? Looking at the Fokker-Planck equation, the answer turns out to be: only the trivial martingale diffusions with zero drift and zero diffusion coefficients. Indeed, forcing $\varphi(x, t)$ to be the density of a uniform with fixed boundaries at all time implies that the left hand side of (17) vanishes, leading to $\sigma(x, t) = 0$ for all x . In other words, the only diffusion martingale associated to this peacock is the trivial martingale $Z_t = \zeta$ for all t , where $\zeta \sim U([-1, 1])$. This is in line with the more general result in [21] cited above.

7 Specific choices of the boundary $b(t)$ and links with peacocks

In this section we present a number of qualitatively different choices for $b(t)$. With a slight abuse of notation, we will denote $\tilde{Z}$ by Z in the rest of the paper.

7.1 The square-root case $b(t) = \sqrt{t}$

As we pointed out earlier, the case $b(t) = \sqrt{t}$ for (4), which leads to

$$dX_t = \frac{1}{\sqrt{2}} \sqrt{1 - \frac{X_t^2}{t}} \mathbb{I}_{\{X_t \in [-\sqrt{t}, \sqrt{t}]\}} dW_t, \quad X_0 = 0$$

corresponds to the solution presented in [13].

7.2 The linear case $b(t) = kt$: numerical examples and activity

The case $b(t) = kt$ fits the assumption of Theorem 1 since $\dot{b}(t) = k$ is bounded on $[0, T]$ for any $T \in \mathbb{R}^+$. Notice also that $\dot{b}(t)b(t) = k^2t$ vanishes for $t \downarrow 0$. Our previous SDEs for X (4) and $\tilde{Z}$ (8) specialize to

$$dX_t = \mathbb{I}_{\{X_t \in [-kt, kt]\}} \frac{1}{\sqrt{t}} \sqrt{(kt)^2 - X_t^2} dW_t, \quad X_0 = 0, \quad X_t \sim U([-kt, kt]) \quad \text{for all } t > 0 \quad (12)$$

and

$$dZ_t = -\frac{1}{t} Z_t dt + \mathbb{I}_{\{Z_t \in [-1, 1]\}} \frac{1}{\sqrt{t}} \sqrt{1 - Z_t^2} dW_t, \quad Z_{t_0} = \zeta \sim U([-1, 1]) \quad \text{for all } t \geq t_0. \quad (13)$$

As a numerical example we implement the Euler scheme for X . We know from [11] that under our assumptions the Euler scheme converges in probability. We thus implement a Euler scheme for the SDE for X and show in Figure 2 a few sample paths.

We may also apply Theorem 5 to this particular case, to see that $Z_t = X_t/(kt)$ for $t \geq t_0 > 0$ spends zero time at the boundaries -1 and 1 . As a consequence, X_t spends zero time at the boundaries $-kt$ and kt .

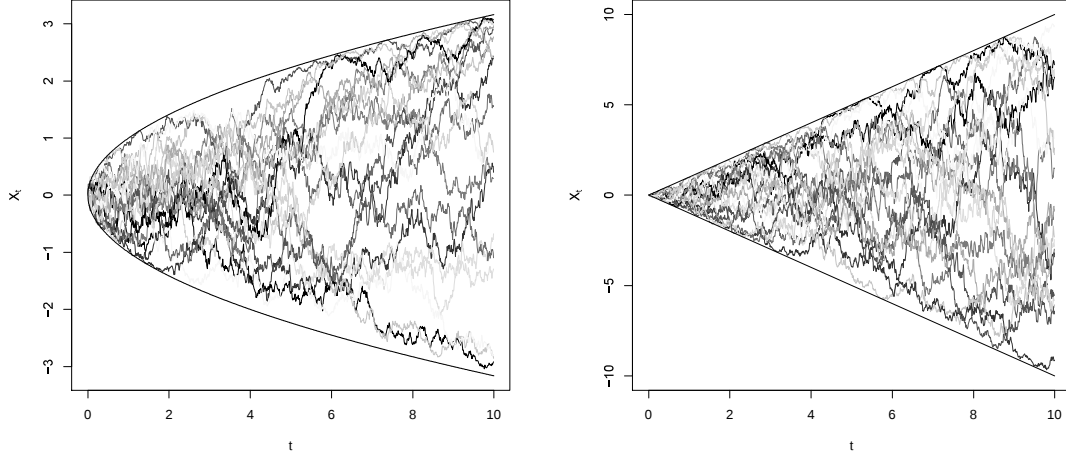


Figure 2: 20 paths of the SDE (12) at time 1y with $b(t) = \sqrt{t}$ (left) and $b(t) = t$ (right). Time step is 0.01 years. Euler Scheme.

More qualitatively, we observe that Z in (13) mean-reverts to 0 with speed $1/t$. The speed will be very large for small time but will become almost zero when time is large. The diffusion coefficient, similarly, is divided by $\sqrt{t}$, so it will tend to vanish for large t . This is confirmed by the following activity calculation. We may conclude that the process will not be absorbed in the boundary and will tend to “slow down” in time, while maintaining a uniform distribution.

We show that the pathwise activity of the uniform $[-1, 1]$ process Z is vanishing for large t in the sense that the deviation of $Z_{t+\delta}(\omega)$ from $Z_t(\omega)$ collapses to zero for all $\delta > 0$, all $\omega \in \Omega$ (the sample space) as $t \rightarrow \infty$.

Lemma 1.

$$\forall \delta > 0, \quad \mathbb{V}\text{ar}(Z_{t+\delta} - Z_t) \rightarrow 0 \text{ as } t \rightarrow \infty .$$

Proof. Notice that for all $t > 0$, $\mathbb{E}(Z_t) = 0$ so that $v := \mathbb{V}\text{ar}(Z_t) = \mathbb{E}(Z_t^2) = 1/3$ is the variance of a zero-mean uniform random variable distributed on $[-1, 1]$. Then,

$$\mathbb{V}\text{ar}(Z_{t+\delta} - Z_t) = \mathbb{V}\text{ar}(Z_{t+\delta}^2) + \mathbb{V}\text{ar}(Z_t^2) - 2\text{Cov}(Z_t, Z_{t+\delta}) = 2(v - \mathbb{E}(Z_t Z_{t+\delta})) .$$

Since Z is bounded, one can rely on Fubini’s theorem for all $t > 0$ and exchange time-integration and expectation,

$$\begin{aligned}
\mathbb{E}(Z_t Z_{t+\delta}) &= \mathbb{E} \left(Z_t \left(Z_t - \int_t^{t+\delta} \frac{Z_s}{s} ds + \int_t^{t+\delta} \sigma(s, Z_s) dW_s \right) \right) \\
&= \mathbb{E}(Z_t^2) - \mathbb{E} \left(Z_t \int_t^{t+\delta} \frac{Z_s}{s} ds \right) + \mathbb{E} \left(\int_t^{t+\delta} Z_t \sigma(s, Z_s) dW_s \right) \\
&= v - \int_t^{t+\delta} \frac{\mathbb{E}(Z_t Z_s)}{s} ds
\end{aligned}$$

where we have used the fact that $Z_s \sigma(s, Z_s)$ is bounded on the interval $[t, t + \delta]$.

Hence, $\mathbb{V}\text{ar}(Z_{t+\delta} - Z_t) = 2(v - f(t, t + \delta))$ where $f(t, s) := \mathbb{E}(Z_t Z_s)$ solves the ODE

$$\frac{\partial f(t, s)}{\partial s} = -\frac{f(t, s)}{s}.$$

Using the initial condition $f(t, t) = v$, the solution is $f(t, s) = vt/s$. Finally, $\lim_{t \rightarrow \infty} f(t, t + \delta) = \lim_{t \rightarrow \infty} vt/(t + \delta) = v$ showing that $\lim_{t \rightarrow 0} \mathbb{V}\text{ar}(Z_{t+\delta} - Z_t) = 0$.

□

The activity result can be generalized to the following lemma.

Lemma 2. *Let $X_t = x_0 + \int_0^t \theta_s dW_s$ and suppose $X = (X_t)_{t \geq 0}$ is a bounded martingale such that for all $t \geq 0$, $a \leq X_t \leq b$ and $\mathbb{P}(\theta_t = 0) < 1$. Then, the path activity of X is collapsing to zero as time passes.*

Proof. Since martingales have uncorrelated increments, the variance of increments is the increment of the variances:

$$\mathbb{V}\text{ar}(X_{t+\delta} - X_t) = \mathbb{V}\text{ar}(X_{t+\delta}) - \mathbb{V}\text{ar}(X_t).$$

Because the diffusion coefficient θ_s is not identically null, the variance of X_t (equal to $\int_0^t \mathbb{E}(\theta_s^2) ds$) is monotonically increasing. Since X is bounded, $\mathbb{V}\text{ar}(X_t)$ is bounded as well and thus converges. This proves that for all $\epsilon > 0$ there exists t^* such that $\mathbb{V}\text{ar}(X_{t+\delta} - X_t) < \epsilon$ for all $t > t^*$. □

We now illustrate the limiting distribution results with a numerical simulation. We simulate the same process as before but conditional on an initial condition at a given time. In particular, we plot in Figure 3 the histograms of the transition densities $p_{X_{100y}|X_{90y}}(\cdot; 0)$ and $p_{X_{400y}|X_{90y}}(\cdot; 0)$.

Our simulation confirms effectively our earlier results. For $p_{X_{100y}|X_{90y}}(\cdot; 0)$ we condition on time in 90 years, very far away in the future. Given the slowing activity of the SDE solution process, the process will move very slowly after 90 years. Indeed, in the time it takes to get 10 years further it shows a conditional density for the next ten years, at 100 years, that seems qualitatively Gaussian. This is compatible with the process being so slow as to behave not too differently from an arithmetic Brownian motion qualitatively. Still, our limit-law results tell us that in the very long run the conditional density should go back to uniform. Indeed, this is illustrated in the simulated density $p_{X_{400y}|X_{90y}}(\cdot; 0)$. We see that if we wait long enough, 310 years in this case, the density goes back to uniform.

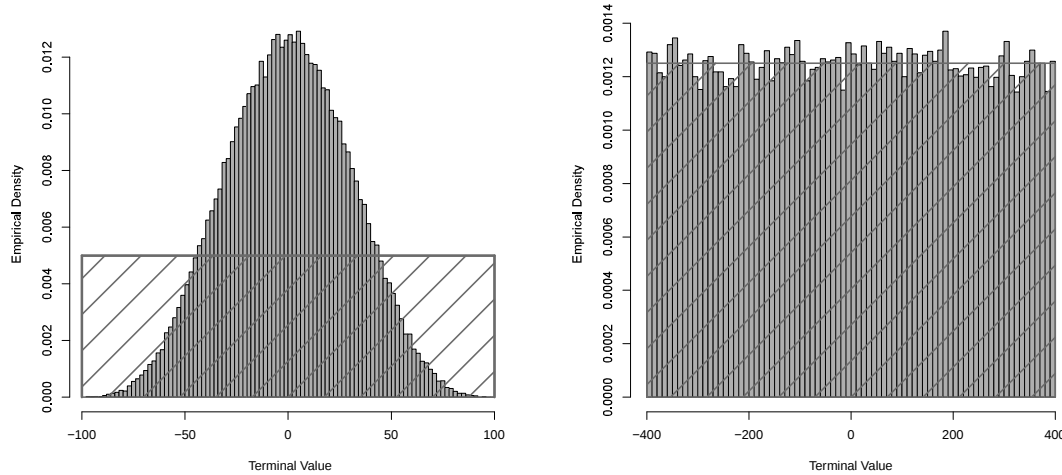


Figure 3: Histograms for the transition densities of the solution X of SDE (12) conditional on $X_s = 0$ at time $s = 90$ years. We take $k = 1$ and use the Euler scheme. Left hand side: density $p_{X_{100y}|X_{90y}}(\cdot; 0)$ at time $t = 100$ years; right hand side: density $p_{X_{400y}|X_{90y}}(\cdot; 0)$ at time $t = 400$ years.

Remark 5 (Other boundaries). One could choose time-boundaries that are concave and converge asymptotically to a constant value B , e.g. $b(t) = Bt/(t + \beta)$ or $b(t) = B(1 - e^{-\beta t})$ where $B > 0, \beta > 0$. It is also possible to use convex boundaries, like e.g. $b(t) = k(e^{\beta t} - 1)$, $k > 0, \beta > 0$. Finally, as mentioned earlier, we could study boundaries of the form $b(t) = kt^\alpha$, $\alpha > 1/2$, $k > 0$, since in this case too existence and uniqueness of the SDE strong solution is guaranteed.

8 Conclusions and further research

We introduced a way to design Stochastic Differential Equations of diffusion type admitting a unique strong solution distributed as a uniform law with conic time-boundaries. Our work has deep connections with peacocks' theory. For instance, existence results associated to linear and square-root boundaries had been dealt with previously in the related literature. In this paper however, we significantly extend the later by dealing with much more general boundary shapes. Moreover, we also provide conditions to have pathwise uniqueness of solutions. We then introduced mean reverting diffusion processes having constant uniform margins at all times and gave limit-law theorems proving that the moments of the transition densities also tend to uniform distributions after a long time. In doing so we derived the exact moments of the transition densities of the mean reverting uniform-margins diffusions, and by re-scaling, the exact transition densities of the uniform peacock SDEs we derived initially. Our results may be used to model random probabilities, random recovery rates

or random correlations.

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Appendix

A Deriving the candidate SDE for a uniformly distributed martingale

Let us guess a candidate solution σ for Problem 1. Here we follow the derivation in the preprint [3], although this guess for the diffusion coefficient can be obtained immediately from the published reference [20], as done in the main text above.

The Fokker-Planck eq. for (1) with ρ plugged in as a solution reads

$$\frac{\partial \rho(x, t)}{\partial t} = \frac{1}{2} \frac{\partial^2}{\partial x^2} (\sigma^2(x, t) \rho(x, t)) , \quad \rho(x, 0) = \delta_0(x) , \quad (14)$$

where the index in δ denotes the point where the Dirac delta distribution is centered. Now we integrate twice both sides of (14) with respect to x and assume we can switch integration with respect to x and differentiation with respect to t (one can check *a posteriori* that the solution we find has a continuous partial derivative with respect to t so that Leibniz's rule can be used). We obtain

$$\frac{\partial}{\partial t} \left(\int_{-\infty}^x \left(\int_{-\infty}^y \rho(z, t) dz \right) dy \right) = \frac{1}{2} \sigma^2(x, t) \rho(x, t) , \quad (15)$$

assuming the relevant first and second derivatives with respect to x on the right hand side vanish fast enough at minus infinity. Compute for $t > 0$, substituting from (2),

$$\varphi(x, t) := \int_{-\infty}^x \left(\int_{-\infty}^y \rho(z, t) dz \right) dy = \begin{cases} 0, & \text{if } x < -b(t) \\ \frac{(x+b(t))^2}{4b(t)}, & \text{if } x \in [-b(t), b(t)] \\ x, & \text{if } x > b(t) \end{cases}$$

and note that φ is continuous in x . Equivalently,

$$\varphi(x, t) = \frac{(x + b(t))^2}{4b(t)} \mathbb{I}_{\{x \in [-b(t), b(t)]\}} + x \mathbb{I}_{\{x > b(t)\}} . \quad (16)$$

Thus, rewriting (15) as

$$\frac{\partial \varphi(x, t)}{\partial t} = \frac{1}{2} \sigma^2(x, t) \rho(x, t) , \quad (17)$$

and substituting (16) we are done. To do this, we need to differentiate φ with respect to time. The calculations are all standard but one has to pay attention when differentiating terms in (16) such as

$$\mathbb{I}_{\{x \in [-b(t), b(t)]\}} = \mathbb{I}_{\{x \geq -b(t)\}} - \mathbb{I}_{\{x > b(t)\}}$$

which can be differentiated in the sense of distributions,

$$\frac{d}{dt} \mathbb{I}_{\{x > b(t)\}} = \frac{d}{dt} \mathbb{I}_{\{t < b^{-1}(x)\}} = -\delta_{b^{-1}(x)}(t)$$

One can check that all terms involving δ 's either offset each other or are multiplied by a function that vanishes at the point of evaluation.

Assuming b is differentiable, omitting time arguments and denoting differentiation with respect to time with a dot one gets:

$$\frac{\partial \varphi(x, t)}{\partial t} = \frac{x + b}{4b^2} \left(2b\dot{b} - \dot{b}(x + b) \right) \mathbb{I}_{\{x \in [-b, b]\}}.$$

We notice that $\dot{b}$ appears only in ratios $\dot{b}/b$, so that this quantity may be extended to time $t = 0$ by continuity if needed provided that the limit exists.

The above quantity is the left hand side of (17). We can substitute ρ on the right hand side and we have that

$$\frac{x + b}{4b^2} \left(2b\dot{b} - \dot{b}(x + b) \right) \mathbb{I}_{\{x \in [-b, b]\}} = \frac{1}{2} \frac{\sigma^2(x, t)}{2b} \mathbb{I}_{\{x \in [-b, b]\}}.$$

After some algebra, one obtains

$$\sigma^2(x, t) = \mathbb{I}_{\{x \in [-b(t), b(t)]\}} \frac{\dot{b}(t)}{b(t)} (b(t)^2 - x^2).$$

From the above development, we expect the diffusion coefficient $\sigma(x, t)$ defined as

$$\sigma(x, t) := \mathbb{I}_{\{x \in [-b(t), b(t)]\}} \sqrt{\frac{\dot{b}(t)}{b(t)} (b(t)^2 - x^2)} \quad (18)$$

to be a valid candidate for the solution X of (1) to be a martingale with marginals having a uniform law in $[-b, b]$.

B Proof that the solution of the peacock SDE (4) has uniform law

Theorem 6. *The solution X to the SDE (4) is a uniform martingale on $[-b(t), b(t)]$ in the sense that for all $t > 0$, X_t is uniformly distributed on $[-b(t), b(t)]$.*

Proof. We start by noting that from Carleman's theorem, it is well known that the continuous uniform distribution on an interval $[a, b]$ with finite $a, b \in \mathbb{R}$ having uniformly bounded moments, it is determined by its moments. See for example Chapter 30 of [1]. Hence, it is enough to show that all moments of the random variable X_t ($t > 0$) associated to (4) coincide with those of the density $\mathbb{I}_{\{-b(t) \leq x \leq b(t)\}} \frac{1}{2b(t)}$.

Let X be a random variable uniformly distributed on $[a, b]$. Then,

$$\mathbb{E}(X^n) = \frac{1}{n+1} \sum_{i=0}^n (a)^i (b)^{n-i}.$$

In the special case where $a = -b$, this expression reduces to

$$\mathbb{E}(X^n) = \begin{cases} \frac{b^n}{n+1} & \text{if } n \text{ is odd} \\ 0 & \text{otherwise .} \end{cases}$$

Let us now compute the moments of

$$X_t = \int_0^t \sigma(X_s, s) dW_s, \quad \sigma(t, x) = \mathbb{I}_{\{-b(t) \leq x \leq b(t)\}} \sqrt{\frac{\dot{b}(t)}{b(t)}} \sqrt{b^2(t) - x^2}$$

solving eq. (4). By Itô's lemma:

$$X_t^n = n \int_0^t X_s^{n-1} dX_s + \frac{1}{2} n(n-1) \int_0^t X_s^{n-2} \sigma^2(X_s, s) ds$$

and we can compute the expression for the n -th moment, $n \geq 2$ using a recursion. Using the property that Itô's integrals have zero expectation and exchanging integration and expectation operators, which is possible since $X_s^{n-2} \sigma^2(X_s, s)$ is bounded for all s and $n \geq 2$, we obtain

$$\begin{aligned} \mathbb{E}(X_t^n) &= n \mathbb{E} \left(\int_0^t X_s^{n-1} dX_s \right) + \frac{1}{2} n(n-1) \mathbb{E} \left(\int_0^t X_s^{n-2} \sigma^2(X_s, s) ds \right) \\ &= \frac{n(n-1)}{2} \left(\int_0^t b(s) \dot{b}(s) \mathbb{E}(X_s^{n-2}) ds - \int_0^t \frac{\dot{b}(s)}{b(s)} \mathbb{E}(X_s^n) ds \right). \end{aligned} \quad (19)$$

Notice that we have postulated in the last equality that the indicator $\mathbb{I}_{\{-b(s) \leq X_s \leq b(s)\}}$ in $\sigma(t, x)$ is always 1. This is a natural assumption: it says that X cannot stay on a boundary with a strict positive probability for a given period of time. This happens because in case X reaches $\pm b(t)$ at some time t , the process is locally frozen ($\sigma(t, x) = 0$) but the boundary $b(t)$ keeps on growing.

Obviously, $\mathbb{E}(X_t) = X_0 = 0$ since X is a martingale and one concludes from eq. (19) that the n -th moment of X_t is zero when n odd. For n even, eq. (19) can be written as

$$M_n(t) = \frac{n(n-1)}{2} \left(\int_0^t b(s) \dot{b}(s) M_{n-2}(s) ds - \int_0^t \frac{\dot{b}(s)}{b(s)} M_n(s) ds \right)$$

with $M_n(t) := \mathbb{E}(X_t^n)$. This can be written as a recursive differential equation

$$\frac{\partial M_n(t)}{\partial t} = \dot{b}(t) \frac{n(n-1)}{2} \left(b(t) M_{n-2}(t) - \frac{1}{b(t)} M_n(t) \right)$$

with the constraint that $f(t, 0) = \mathbb{E}(X_t^0) = 1$. The solution to this equation is $M_n(t) = b^n(t)/(n+1)$. One concludes that X_t is uniform on $[-b(t), b(t)]$ since all the odd moments are zero and all the even moments are given by

$$\mathbb{E}(X_t^n) = \frac{b^n(t)}{n+1}$$

and agree with those of a random variable uniformly distributed on $[-b(t), b(t)]$. $\square$

C Other uniform diffusions

It is easy to get uniform diffusion with non-linear drift. The idea consists of mapping a diffusion in a time-dependent way, using the corresponding distribution function. We provide here a specific example where the diffusion is a Brownian motion W , and analyze its asymptotic conditional distribution.

Proposition 1. *Let W be a standard Brownian motion. Define $Z_t := 2\Phi\left(\frac{W_t}{\sqrt{t}}\right) - 1$. Z is a stochastic process with uniform distribution in $(-1, 1)$ at all times (possibly with random initialization $Z_{t_0} = \zeta$ at $t_0 > 0$). The dynamics of Z are given by*

$$dZ_t = \underbrace{-\frac{2}{t}\Phi^{-1}\left(\frac{1+Z_t}{2}\right)\varphi\left(\Phi^{-1}\left(\frac{1+Z_t}{2}\right)\right)}_{\mu(t,x)}dt + \underbrace{\frac{2}{\sqrt{t}}\varphi\left(\Phi^{-1}\left(\frac{1+Z_t}{2}\right)\right)}_{\sigma(t,x)}dW_t.$$

It can be shown that this satisfies the Fokker-Planck equation with

$p(x, t) = (1/2) \mathbb{I}_{\{-1 \leq x \leq 1\}}$ as

$$\mu_x(t, x) = (\sigma_x(t, x))^2 + \sigma(t, x)\sigma_{xx}(t, x) = \left(\left(\Phi^{-1}\left(\frac{1+Z_t}{2}\right)\right)^2 - 1\right)/t.$$

Moreover, the law of $Z_t|Z_s$ tends to that of a uniform in $(-1, 1)$ as $t \rightarrow \infty$.

Proof. Conditioning upon Z_s is equivalent to conditioning w.r.t. W_s as $Z_s = 2\Phi\left(\frac{W_s}{\sqrt{s}}\right) - 1$.

But $\frac{W_t}{\sqrt{t}}\Big|_{W_s} \sim \frac{W_s + \sqrt{t-s}Z}{\sqrt{t}}$ where $Z \sim \mathcal{N}(0, 1)$. Hence,

$$\mathbb{P}(Z_t \leq x|Z_s) = \mathbb{P}\left(\Phi\left(\frac{W_t}{\sqrt{t}}\right) \leq \frac{1+x}{2}\Big|W_s\right) = \mathbb{P}\left(\Phi\left(\frac{W_s + \sqrt{t-s}Z}{\sqrt{t}}\right) \leq \frac{1+x}{2}\Big|W_s\right).$$

Inverting the standard Normal CDF,

$$\mathbb{P}(Z_t \leq x|Z_s) = \mathbb{P}\left(Z \leq \frac{\sqrt{t}\Phi^{-1}\left(\frac{1+x}{2}\right) - W_s}{\sqrt{t-s}}\right) = \Phi\left(\frac{\sqrt{t}\Phi^{-1}\left(\frac{1+x}{2}\right) - W_s}{\sqrt{t-s}}\right).$$

So $\mathbb{P}(Z_t \leq x|Z_s) \rightarrow (1+x)/2$ as $t \rightarrow \infty$ for all (s, Z_s) where $Z_s \in (-1, 1)$. $\square$

D Proof of Theorem 4 and first conditional moments

Let us prove Theorem 4.

Proof. The dynamics of powers of $\tilde{Z}$ solving (9) are easily found from Ito's formula. This yields the ODE governing the conditional expectations for all n . For $n = 0$ one trivially has $M_0(s, t; z) = 1$. Now, set $h(t) := \dot{b}(t)/b(t)$ satisfying

$$\exp \left\{ - \int_s^t h(u) du \right\} = \exp \left\{ \int_t^s d \ln u \right\} = b(s)/b(t) .$$

Hence,

$$\dot{M}_1(s, t; z) := \frac{\partial M_1(s, t; z)}{\partial t} = -h(t) M_1(s, t; z) \text{ s.t. } M_1(s, s; z) = z$$

which leads to $M_1(s, t; z) = z \frac{b(s)}{b(t)}$.

For $n \geq 2$, one gets a recursive first order inhomogeneous ODE

$$\begin{aligned} \dot{M}_n(s, t; z) &= -\frac{n(n+1)}{2} \frac{\dot{b}(t)}{b(t)} M_n(s, t; z) + \frac{n(n-1)}{2} \frac{\dot{b}(t)}{b(t)} M_{n-2}(s, t; z) \\ &= \frac{n(n+1)}{2} h(t) \left(\frac{n-1}{n+1} M_{n-2}(s, t; z) - M_n(s, t; z) \right) , \end{aligned} \quad (20)$$

whose solution is

$$M_n(s, t; z) = z^n \left(\frac{b(s)}{b(t)} \right)^{n(n+1)/2} + \frac{n(n-1)}{2} \int_s^t \left(\frac{b(u)}{b(t)} \right)^{n(n+1)/2} h(u) M_{n-2}(s, u; z) du . \quad (21)$$

Notice that the expression above satisfies the initial conditions $M_n(s, s; z) = z^n$ for all n in $2, 3, \dots$. This is also the case for the expression stated in the theorem as a result of the relationship between the entries of the $\alpha[n]$ matrices: as $b(s)/b(t) = 1$ when $s = t$, the double sum collapses to the single $j = k = (n + \xi_n)/2$ term. This concludes the check of the initial conditions.

Replacing n by $n + 2$ in the $M_n(s, t; z)$ expression given in the theorem yields

$$\begin{aligned}
M_{n+2}(s, t; z) &= \mu_{n+2} + \sum_{k=1}^{\frac{n+2+\xi_n}{2}} (-1)^k (z^{2k-\xi_n} - \mu_{2k-\xi_n}) \sum_{j=k}^{\frac{n+2+\xi_n}{2}} \alpha_{j,k}[n+2] (-1)^j \left(\frac{b(s)}{b(t)} \right)^{j(2(j-\xi_n)+1)} \\
&= \mu_{n+2} + \sum_{k=1}^{\frac{n+\xi_n}{2}} (-1)^k (z^{2k-\xi_n} - \mu_{2k-\xi_n}) \sum_{j=k}^{\frac{n+2+\xi_n}{2}} \alpha_{j,k}[n+2] (-1)^j \left(\frac{b(s)}{b(t)} \right)^{j(2(j-\xi_n)+1)} \\
&\quad + (-1)^{\frac{n+2+\xi_n}{2}} (z^{n+2} - \mu_{n+2}) \alpha_{\frac{n+2+\xi_n}{2}, \frac{n+2+\xi_n}{2}}[n+2] (-1)^{\frac{n+2+\xi_n}{2}} \left(\frac{b(s)}{b(t)} \right)^{\frac{n+2+\xi_n}{2}(n+3-\xi_n)} \\
&= \mu_{n+2} + (z^{n+2} - \mu_{n+2}) \left(\frac{b(s)}{b(t)} \right)^{\frac{n+2}{2}(n+3)} + I_1 + I_2 \\
I_1 &:= \sum_{k=1}^{\frac{n+\xi_n}{2}} (-1)^k (z^{2k-\xi_n} - \mu_{2k-\xi_n}) \sum_{j=k}^{\frac{n+2+\xi_n}{2}} \alpha_{j,k}[n+2] (-1)^j \left(\frac{b(s)}{b(t)} \right)^{j(2(j-\xi_n)+1)} \\
I_2 &:= (-1)^{\frac{n+2+\xi_n}{2}} \left(\frac{b(s)}{b(t)} \right)^{\frac{n+2}{2}(n+3)} \sum_{k=1}^{\frac{n+\xi_n}{2}} (-1)^k (z^{2k-\xi_n} - \mu_{2k-\xi_n}) \alpha_{\frac{n+2+\xi_n}{2}, k}[n+2]
\end{aligned}$$

where we have used $\xi_n = \xi_n^2$ and $\alpha_{\frac{n+2+\xi_n}{2}, \frac{n+2+\xi_n}{2}}[n+2] = 1$ from (11) with $n \leftarrow n+2$.

It remains to check that this expression agrees with the solution (21) when setting $n \leftarrow n+2$. The constant term trivially reads

$$z^{n+2} \left(\frac{b(s)}{b(t)} \right)^{\frac{n+2}{2}(n+3)}.$$

The integral can be split in two parts with respect to M_n . The first part of M_n is μ_n and the second is the double sum. The first part is

$$\begin{aligned}
\frac{(n+2)(n+1)}{2} \int_s^t \left(\frac{b(u)}{b(t)} \right)^{(n+2)(n+3)/2} h(u) \mu_n du &= \mu_n \frac{n+1}{n+3} \left(1 - \left(\frac{b(s)}{b(t)} \right)^{(n+2)(n+3)/2} \right) \\
&= \mu_{n+2} \left(1 - \left(\frac{b(s)}{b(t)} \right)^{(n+2)(n+3)/2} \right).
\end{aligned}$$

It remains to show that the remaining integral agrees with $I_1 + I_2$ defined above. It comes

$$\alpha_{jk}[n] \frac{(n+2)(n+1)}{2} \int_s^t \left(\frac{b(u)}{b(t)} \right)^{(n+2)(n+3)/2} h(u) \left(\frac{b(s)}{b(u)} \right)^{j(2(j-\xi_n)+1)} du = J_1(j, k, n) + J_2(j, k, n)$$

where

$$\begin{aligned}
J_1(j, k, n) &:= \alpha_{j,k}[n+2] \left(\frac{b(s)}{b(t)} \right)^{j(2(j-\xi_n)-1)}, \\
J_2(j, k, n) &:= -\alpha_{j,k}[n+2] \left(\frac{b(s)}{b(t)} \right)^{(n+2)(n+3)/2}.
\end{aligned}$$

It is easy to see that

$$\sum_{k=1}^{\frac{n+\xi_n}{2}} (-1)^k (z^{2k-\xi_n} - \mu_{2k-\xi_n}) \sum_{j=k}^{\frac{n+\xi_n}{2}} (-1)^j J_1(j, k, n) = I_1.$$

On the other hand,

$$\begin{aligned}
& \sum_{k=1}^{\frac{n+\xi_n}{2}} (-1)^k (z^{2k-\xi_n} - \mu_{2k-\xi_n}) \sum_{j=k}^{\frac{n+\xi_n}{2}} (-1)^j J_2(j, k, n) \\
&= - \left(\frac{b(s)}{b(t)} \right)^{(n+2)(n+3)/2} \sum_{k=1}^{\frac{n+\xi_n}{2}} (-1)^k (z^{2k-\xi_n} - \mu_{2k-\xi_n}) \sum_{j=k}^{\frac{n+\xi_n}{2}} (-1)^j \alpha_{j,k}[n+2] \\
&= (-1)^{\frac{n+2+\xi_n}{2}} \left(\frac{b(s)}{b(t)} \right)^{(n+2)(n+3)/2} \sum_{k=1}^{\frac{n+\xi_n}{2}} (-1)^k (z^{2k-\xi_n} - \mu_{2k-\xi_n}) \alpha_{\frac{n+2+\xi_n}{2},k}[n+2]
\end{aligned}$$

where the last inequality results from (11) with $n \leftarrow n+2$; this is nothing but I_2 . This completes the proof. $\square$

E General UD with time-changed homogeneous SDE

We confirm the intuition given the above moments result, showing that we can connect the general case to the special time-homogeneous case discussed in Section 5.1. To do this, it will be enough to introduce a deterministic time change. The following proposition is essentially equivalent to the methodology in [13] that we already used in the proof of Theorem 2, but given the different context we state and prove the proposition explicitly for convenience.

Proposition 2. *[General mean reverting SDE as a time-changed time homogeneous SDE] Consider the general SDE (9) for $\tilde{Z}_t$ with $t \geq t_0$. There exists a Brownian motion B such that*

$$\tilde{Z}_t = \xi_{\tau(t)} \quad \text{for the deterministic time change } \tau(t) = \ln(b(t)),$$

where ξ is the solution of the following SDE driven by B :

$$d\xi_t = -\xi_t dt + (1 - \xi_t^2)^{1/2} \mathbb{I}_{\{\xi_t \in [-1,1]\}} dB_t,$$

provided that

$$t_0 = b^{-1}(1), \quad \xi_{\tau(t_0)} = \xi_0 = \tilde{Z}_{t_0}$$

and that the initial condition is assumed to be a random variable ξ_0 with uniform law in $[-1, 1]$ and independent of B .

Proof. Consider the SDE (9) for $\tilde{Z}$. This is driven by the continuous martingale

$$M_t = \int_0^t \left(\frac{\dot{b}(s)}{b(s)} \right)^{1/2} dW_s$$

in that it can be written as

$$d\tilde{Z}_t = -\frac{\dot{b}(t)}{b(t)} \tilde{Z}_t dt + \left(1 - \tilde{Z}_t^2\right)^{1/2} \mathbb{I}_{\{\tilde{Z}_t \in [-1, 1]\}} dM_t .$$

Note that the quadratic variation of M is given by $\langle M \rangle_t = \tau(t)$. From the Dambis, Dubins–Schwarz (DDS) theorem we know that there exists a Brownian motion B such that

$$M_t = B_{\langle M \rangle_t} = B_{\tau(t)} .$$

If we further notice that $d\tau(t) = (\dot{b}(t)/b(t))dt$ we can write SDE (9) as

$$d\tilde{Z}_t = -\tilde{Z}_t d\tau(t) + \left(1 - \tilde{Z}_t^2\right)^{1/2} \mathbb{I}_{\{\tilde{Z}_t \in [-1, 1]\}} dB_{\tau(t)}$$

so that if we set $\xi_{\tau(t)} := \tilde{Z}_t$ and substitute in the last SDE above we conclude. $\square$

The assumption that $t_0 = b^{-1}(1)$ (we could also take a larger t_0) is needed to avoid negative time in the ξ SDE, but this is not an issue since we are interested in the limiting behaviour of the solution for the SDE of $\tilde{Z}_t$ for large t .

Given our discussion in Section 5.1, we know that ξ is ergodic and has a uniform invariant measure as limit law. We can then confirm our earlier result on the limit law of $\tilde{Z}$: it will be a uniform law that forgets the initial condition at an earlier time, and the $\tilde{Z}$ process will be a deterministic time-change of an ergodic process.