On Inverse Utility and Third-Order Effects in the Economics of Uncertainty

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Abstract

We prove that the coefficient of absolute prudence is greater than k - times coefficient of absolute risk aversion for the utility function if and only if the coefficient of absolute prudence is (3-k) times the coefficient of absolute risk aversion for the inverse utility function. Moreover this is also equivalent to (k-2)-concavity of the first derivative of the inverse utility function.

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1 Introduction

Recent advances in the economics of uncertainty have emphasized the importance of various conditions involving the second and third derivatives of the von Neumann-Morgenstern utility function. Several of these conditions assume the following form for suitable values of k.

$$-u'''(x)u'(x) + ku''(x)^2 \leq 0 \tag{1}$$

It is often convenient to rewrite (**??**) in the equivalent form:

$$-\frac{u'''(x)}{u''(x)} \geq -k\frac{u''(x)}{u'(x)}, \tag{2}$$

which states that the coefficient of absolute prudence[1], given by the left-hand side of (**??**), as defined by Kimball (1990), is greater than k times the coefficient of absolute risk aversion.

In particular, the special cases $k = 1$ and $k = 2$ have been widely used in the microeconomics of uncertainty, (e.g. Pratt, 1964 and Gollier, 2000). Furthermore, Carroll and Kimball (1996) have shown that condition (**??**) is equivalent to the $(k-1)$-concavity (Caplin and Nalebuff, 1991) of the reciprocal of the first derivative of the utility function.

In this note we point out an alternative interpretation of condition (**??**) using the first three derivatives of the inverse utility function and relate it to the notion of k-concavity, thereby providing another perspective on this condition. These two alternative interpretations will be presented in two propositions. The discussion of some specific economic applications pertaining to the cases $k = 1, k = 2$ and $k = 3$ will follow.

2 Conditions Involving Inverse Utility

We consider a utility function u satisfying the standard properties $u' > 0$ and $u'' < 0$. Define the inverse utility function $v = u^{-1}$. The following observation is one of the main results of this note.

Proposition 1 *The following two conditions are equivalent:*

$(a) - u'''(x)u'(x) + ku''(x)^2 \leq 0$ *for all* $x \geq 0$.

[1]Prudence implies that an agent increases the precautionary savings if faced with additional second period background risk, (Kimball, 1990) or that a risk-averse agent prefers a mean-preserving spread being attached to the better outcomes of the lottery if forced to accept one (Eeckhoudt et al., 1995).

$(b) v'''(y)v'(y) + (k-3)v''(y)^2 \leq 0$ *for all* $y \geq 0$.

Proof. Since $v = u^{-1}$, we have $u(x) = y$ if and only if $v(y) = x$. By differentiating the identity $u(v(y)) = y$ three times, we get the closed form expressions for first three derivatives of the utility function:

$$u'(x) = \frac{1}{v'(y)},\ u''(x) = -\frac{v''(y)}{v'(y)^3},\ u'''(x) = \frac{3v''(y)^2}{v'(y)^5} - \frac{v'''(y)}{v'(y)^4}$$

Evaluating the expression $-u'''(x)u'(x) + ku''(x)^2$ using the above three identities yields the result. ■

The condition stated in Proposition 1(b) is equivalent to

$$-\frac{v'''(x)}{v''(x)} \geq -(3-k)\frac{v''(x)}{v'(x)} \tag{3}$$

Using conditions (**??**) and (**??**), we can reinterpret Proposition 1. The coefficient of absolute prudence being greater than k times the coefficient of absolute risk aversion for the utility function is equivalent to the coefficient of absolute prudence being greater than $(3-k)$ the coefficient of absolute risk aversion for the inverse utility function. We will also express condition (**??**) using ρ-concavity of the first derivative of the inverse utility function.

We first review the notion of ρ-concavity, as in Caplin and Nalebuff (1991).

Definition 1 *A nonnegative function $f(x)$ with convex support is called ρ - concave, $\rho \neq 0$, if for all x_1, x_2 and all $\alpha \in [0,1]$ such that $f(x_1)f(x_2) \neq 0$, we have*

$$f(\alpha x_1 + (1-\alpha)x_2) \geq [\alpha f(x_1)^\rho + (1-\alpha)f(x_2)^\rho]^{\frac{1}{\rho}} \tag{4}$$

Thus ρ-concavity amounts to f^ρ being concave for positive ρ and $-f^\rho$ being concave for negative ρ. The definition of ρ-concavity can be extended, by continuity arguments to include cases where ρ equals $+\infty, 0$ or $-\infty$. To this end, we can calculate the limits of the right hand side of equation (**??**) and replace (**??**) by

$$f(\alpha x_1 + (1-\alpha)x_2) \geq \max\{f(x_1), f(x_2)\}, \text{ for } \rho = \infty \tag{5}$$

$$\ln f(\alpha x_1 + (1-\alpha)x_2) \geq \alpha \ln f(x_1) + (1-\alpha)\ln f(x_2), \text{ for } \rho = 0, \tag{6}$$

$$f(\alpha x_1 + (1-\alpha)x_2) \geq \min\{f(x_1), f(x_2)\}, \text{ for } \rho = -\infty \tag{7}$$

Here, (**??**) is satisfied only by the constant function, (**??**) amounts to log-concavity and (**??**) to quasi-concavity.

We now restate the result of Carroll and Kimball (1996)[2] using the introduced notion of ρ-concavity.

Proposition 2 *For all $k \in (-\infty, \infty)$, we have the equivalence*

$$v'(y) \text{ is } (k-2)\text{-concave} \Longleftrightarrow -u'''(x)u'(x) + ku''(x)^2 \leq 0 \tag{8}$$

Proof. The result is obtained by direct calculation of the second derivative of the function $[v'(y)]^{(k-2)}$ for $k \neq 2$ and the second derivative of the function $\ln v'(y)$ for $k = 2$. Proposition 2 follows by direct application of Proposition 1 and the definition of ρ-concavity. ■

Remark 1 *By condition* 5*, ∞-concavity of the function $v\prime(y) = [u'(x)]^{-1}$ implies that marginal utility is constant, which is fulfilled only by the linear utility function, i.e. in the case of risk neutrality.*

We will now discuss the economic interpretations of condition (1) and its alternative representations for the cases $k = 1, 2,$ and 3, which have appeared in applied work using these advanced notions of uncertainty.

1. Decreasing Absolute Risk Aversion ($k = 1$)

   It is well known that the condition:

   $$-u'''(x)u'(x) + u''(x)^2 \leq 0 \tag{9}$$

   is equivalent to decreasing absolute risk aversion (or DARA) of the investor (Pratt, 1964). Thus a DARA utility function can be interpreted as the utility function with (-1)-concave first derivative of inverse utility function or equivalently with convex logarithm of the utility first derivative function. Invoking Proposition 1, we can say that DARA is equivalent to the condition that the coefficient of absolute prudence is greater than twice the coefficient of absolute risk aversion for the inverse utility function.

2. Complementarity of risky assets ($k = 2$)

   Using Proposition 2, we can say that the condition:

   $$-u'''(x)u'(x) + 2u''(x)^2 \leq 0 \tag{10}$$

[2]More precisely, in their paper, the equivalent condition is expressed as convexity of $\frac{[u'(x)]^{1-k}}{1-k}$ for $k \neq 1$ and convexity of $\log u'(x)$ for $k = 1$.

is equivalent to 0-concavity of the first derivative of the inverse utility function. Moreover, using Proposition 1, we can say that the condition that the coefficient of absolute prudence being 2 times greater than the coefficient of absolute risk aversion for the utility function is equivalent to the condition that the coefficient of absolute prudence is greater than the coefficient of absolute risk aversion for the inverse utility function. Condition (**??**) was used by Gollier and Kimball (Gollier, 2000). They prove that decreasing absolute risk aversion, decreasing absolute prudence together with condition (**??**) constitute sufficient conditions for two assets with i.i.d. returns to be substitutes. Moreover they show that these sufficient conditions imply the convexity of the coefficient of absolute risk aversion. To show this, they notice that decreasing absolute prudence is equivalent to the condition

$$A''(x) \geq -A'(x)\left[P(x) - 2A(x)\right] \tag{11}$$

We know that under DARA $A'(x) \leq 0$. Using condition (**??**) we can conclude that the condition $P(x) \geq 2A(x)$ implies that $A"(x) \geq 0$.

3. Prudence of the inverse utility function ($k = 3$)

   Consider the condition:

$$-u'''(x)u'(x) + 3u''(x)^2 \leq 0 \tag{12}$$

   By Proposition 1, condition (**??**) is equivalent to the concavity of the first derivative of the inverse utility function (i.e. $v'''(y) \leq 0$). Because the inverse utility function satisfies $v''(y) > 0$, we can conclude that $v'''(y) \leq 0$ implies that the coefficient of absolute prudence is nonnegative. This condition emerged in Fagart and Sinclair-Desgagne (2003) as the sufficient condition for the optimal auditing policy in a principal - agent model to be relatively more upper - tailed.

Inspection of Proposition 1 indicates that only in case $k = 3$ does one get such a simple interpretation in terms of the inverse utility function. Note that the same interpretation of (**??**) can be obtained via Proposition 2, as 1-concavity is equivalent to the standard notion of concavity.

References

[1] Caplin, A. and B. Nalebuff, 1991, Aggregation and Social Choice: A Mean Voter Theorem, Econometrica 59, 1-23

[2] Carroll, C.D. and M. S. Kimball, 1996, On the Concavity of the Consumption Function, Econometrica 64, 981-992

[3] Eeckhoudt L.,Ch. Gollier and T. Schneider, 1995, Risk-aversion, prudence and temperance: A unified approach, Economics letters 48, 331-336

[4] Fagart, M-C. and B. Sinclair-Desgagne, 2003, Auditing Policies and Information, working paper

[5] Gollier, C., 2000, The Economics of Risk and Time (MIT Press: Cambridge, MA)

[6] Kimball, M.S., 1990, Precautionary Saving in the Small and in the Large, Econometrica 58, 53-73

[7] Pratt J. W., 1964, Risk Aversion in the Small and in the Large, Econometrica 32, 122-136