

Exponential moments for disk counting statistics of random normal matrices in the critical regime

Christophe Charlier* and Jonatan Lenells†

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Abstract

We obtain large n asymptotics for the m -point moment generating function of the disk counting statistics of the Mittag-Leffler ensemble, where n is the number of points of the process and m is arbitrary but fixed. We focus on the critical regime where all disk boundaries are merging at speed $n^{-\frac{1}{2}}$, either in the bulk or at the edge. As corollaries, we obtain two central limit theorems and precise large n asymptotics of all joint cumulants (such as the covariance) of the disk counting function. Our results can also be seen as large n asymptotics for $n \times n$ determinants with merging planar discontinuities.

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1 Introduction and statement of results

In recent years, there has been a growing interest in both the physics and mathematics literature in understanding the counting statistics of various two-dimensional point processes, see e.g. [37, 11, 35, 36, 26, 28, 47, 12, 48, 9, 1]. Most of the focus, so far, has been on the one-point counting statistics (see however [28, 12]). In this work we study the m -point counting statistics for general $m \in \mathbb{N}_{>0}$ of the Mittag-Leffler ensemble in the critical regime where all disk boundaries are merging.

The Mittag-Leffler ensemble with parameters $b > 0$ and $\alpha > -1$ is the following probability density function for n points in the complex plane

$$\frac{1}{n!Z_n} \prod_{1 \leq j < k \leq n} |z_k - z_j|^2 \prod_{j=1}^n |z_j|^{2\alpha} e^{-n|z_j|^{2b}}, \quad z_1, \dots, z_n \in \mathbb{C}, \quad (1.1)$$

where Z_n is the normalization constant. This is a two-dimensional determinantal point process arising in the random normal matrix model [43] and generalizing the complex Ginibre process (which corresponds to $(b, \alpha) = (1, 0)$ [32]). As $n \rightarrow +\infty$, the empirical measures $\frac{1}{n} \sum_{j=1}^n \delta_{z_j}$ converge weakly to the equilibrium measure $\mu(d^2z) = \frac{b^2}{\pi} |z|^{2b-2} d^2z$ with high probability, where the support of μ is the disk centered at 0 of radius $b^{-\frac{1}{2b}}$ [46]. The Mittag-Leffler ensemble has been widely studied over the years, see e.g. [17, 3, 29, 4, 5, 10] for various universality and finite- n results. We also refer the interested reader to [33] for more background on two-dimensional point processes.

*Centre for Mathematical Sciences, Lund University, 22100 Lund, Sweden. e-mail: christophe.charlier@math.lu.se

†Department of Mathematics, KTH Royal Institute of Technology, 10044 Stockholm, Sweden. e-mail: jlenells@kth.se

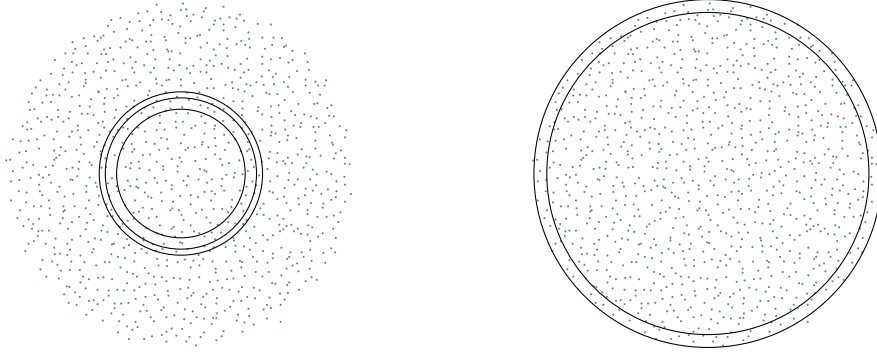


Figure 1: Left: three merging disks in the bulk; this case is covered by Theorem 1.2 with $m = 3$. Right: two merging disks at the edge; this case is covered by Theorem 1.3 with $m = 2$. For both pictures, $b = 1$ and $\alpha = 0$.

For $y > 0$, we let $N(y) := \#\{z_j : |z_j| < y\}$, i.e. $N(y)$ is the random variable that counts the number of points in the disk centered at 0 of radius y . In this paper we study the joint statistics of $N(r_1), \dots, N(r_m)$, where $m \in \mathbb{N}_{>0}$ is arbitrary but fixed, in the critical situation where $n \rightarrow +\infty$ and all radii are merging near a certain value $r \in (0, b^{-\frac{1}{2b}}]$:

$$0 < r_1 < \dots < r_m, \quad r_\ell = r \left(1 + \frac{\sqrt{2} \mathfrak{s}_\ell}{r^b \sqrt{n}} \right)^{\frac{1}{2b}}, \quad \mathfrak{s}_\ell \in \mathbb{R}, \quad (1.2)$$

see also Figure 1. The case $r \in (0, b^{-\frac{1}{2b}})$ corresponds to “the bulk regime” and $r = b^{-\frac{1}{2b}}$ to “the edge regime”. Our main results can be summarized as follows:

- Theorems 1.2 and 1.3 give precise large n asymptotics for the moment generating function $\mathbb{E}[\prod_{j=1}^m e^{u_j N(r_j)}]$, up to and including the fourth term of order $n^{-\frac{1}{2}}$. Theorem 1.2 deals with the bulk regime and Theorem 1.3 deals with the edge regime.
- Corollary 1.5 (a) establishes precise large n asymptotics for all joint cumulants of $N(r_1), \dots, N(r_m)$ in the bulk regime. The analogue of that for the edge regime is given in Corollary 1.5 (c). We also obtain several central limit theorems for the joint fluctuations of $N(r_1), \dots, N(r_m)$; Corollary 1.5 (b) concerns the bulk regime and Corollary 1.5 (d) the edge regime.

The problem of determining the large n asymptotics for the one-point generating function, i.e. the case $m = 1$ in our setting, was already considered in [11, 36, 28]. The work [11] considers counting statistics of Ginibre-type ensembles in a more general geometric setting, and second order asymptotics for the generating function of counting statistics of general domains (not just centered disks) are obtained in [11, Proposition 8.1]. The work [36] focuses on disk counting statistics of rotation-invariant ensembles with a general potential, and second order asymptotics for the generating function are given in [36, eq (34)]. More precise asymptotics, including the third term, were obtained in [28, Proposition 2.3] for the Ginibre ensemble (see also [28, Propositions 2.4 and 2.9] for analogous results on other rotation-invariant processes). Large n asymptotics for $\mathbb{E}[\prod_{j=1}^m e^{u_j N(r_j)}]$, including the fourth term and for general $m \in \mathbb{N}_{>0}$, were then obtained in [12] for Mittag-Leffler ensembles, in the case where $r_1, \dots, r_m$ are fixed (independent of n). This case contrasts with the regime (1.2) considered here. In the setting of [12], the general case $m \geq 2$ is actually not much different from the

case $m = 1$. Indeed, it follows from [12, Theorem 1.1] that for general $b > 0, \alpha > -1, u_1, \dots, u_m \in \mathbb{R}$ and $r_1 < \dots < r_m$ fixed, we have

$$\mathbb{E} \left[\prod_{j=1}^m e^{u_j N(r_j)} \right] = \prod_{j=1}^m \mathbb{E} \left[e^{u_j N(r_j)} \right] \times \left(1 + \mathcal{O} \left(\frac{(\ln n)^2}{n} \right) \right), \quad \text{as } n \rightarrow +\infty. \quad (1.3)$$

In fact, by straightforward modifications of the proof of [12], the error term in (1.3) can be shown to be exponentially small, see also [12, Remark 1.5]. In other words, when $r_1, \dots, r_m$ are fixed, the m -point generating function can be expressed asymptotically as the product of m one-point generating functions (up to an exponentially small error term). This, in turn, implies that all cumulants involving two random variables or more among $N(r_1), \dots, N(r_m)$ are exponentially small — for the covariance, this fact (namely that $\text{Cov}(N(r_1), N(r_2)) = \mathcal{O}(e^{-cn})$ as $n \rightarrow +\infty$) was already noticed in [45, Theorem 1.7] for $(b, \alpha) = (1, 0)$, and for the joint fluctuations of $N(r_1), \dots, N(r_m)$, this fact (namely that they become independent Gaussian random variables in the large n limit) was already proved in [28, Proposition 1.3] for Ginibre-type ensembles. (We mention en passant that the decoupling (1.3) is a particular feature of two-dimensional point processes such as (1.1). Indeed, the analogues of (1.3) for the one-dimensional sine, Airy, Bessel and Pearcey point processes involve explicit constant pre-factors of order 1, and the associated covariances are not small but of order 1, see e.g. [14, 16].) In the regime considered here, namely (1.2), the m -point generating function does not decouple as in (1.3), and all joint cumulants of $N(r_1), \dots, N(r_m)$ have non-trivial asymptotics as $n \rightarrow +\infty$.

Let us introduce the following functions:

$$\mathcal{H}_1(t; \vec{u}, \vec{s}) := 1 + \sum_{\ell=1}^m \frac{e^{u_\ell} - 1}{2} \exp \left[\sum_{j=\ell+1}^m u_j \right] \text{erfc}(t - \mathfrak{s}_\ell), \quad (1.4)$$

$$\mathcal{H}_2(t; \vec{u}, \vec{s}) := 1 + \sum_{\ell=1}^m \frac{e^{-u_\ell} - 1}{2} \exp \left[- \sum_{j=1}^{\ell-1} u_j \right] \text{erfc}(t + \mathfrak{s}_\ell), \quad (1.5)$$

$$\mathcal{G}_1(t; \vec{u}, \vec{s}) := \frac{1}{\mathcal{H}_1(t; \vec{u}, \vec{s})} \sum_{\ell=1}^m (e^{u_\ell} - 1) \exp \left[\sum_{j=\ell+1}^m u_j \right] \frac{e^{-(t-\mathfrak{s}_\ell)^2}}{\sqrt{2\pi}} \frac{1 - 2\mathfrak{s}_\ell^2 + t\mathfrak{s}_\ell - 5t^2}{3}, \quad (1.6)$$

$$\begin{aligned} \mathcal{G}_2(t; \vec{u}, \vec{s}) := & \frac{1}{\mathcal{H}_1(t; \vec{u}, \vec{s})} \sum_{\ell=1}^m (e^{u_\ell} - 1) \exp \left[\sum_{j=\ell+1}^m u_j \right] \frac{e^{-(t-\mathfrak{s}_\ell)^2}}{18\sqrt{2\pi}} \left(50t^5 - 70t^4\mathfrak{s}_\ell - t^3(73 - 62\mathfrak{s}_\ell^2) \right. \\ & \left. + t^2\mathfrak{s}_\ell(33 - 50\mathfrak{s}_\ell^2) - t(3 + 18\mathfrak{s}_\ell^2 - 16\mathfrak{s}_\ell^4) - \mathfrak{s}_\ell(3 - 22\mathfrak{s}_\ell^2 + 8\mathfrak{s}_\ell^4) \right), \end{aligned} \quad (1.7)$$

where $t \in \mathbb{R}$, $\vec{u} = (u_1, \dots, u_m) \in \mathbb{C}^m$, $\vec{s} = (\mathfrak{s}_1, \dots, \mathfrak{s}_m) \in \mathbb{R}^m$, and erfc is the complementary error function

$$\text{erfc}(t) = \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-x^2} dx. \quad (1.8)$$

The functions $\mathcal{H}_j$, $j = 1, 2$, appear in the denominators of (1.6)–(1.7) and inside logarithms in the statements of our main theorems. The next lemma ensures that $\{\mathcal{G}_j, \ln \mathcal{H}_j\}_{j=1}^2$ are well-defined and real-valued for $t \in \mathbb{R}$, $\vec{u} = (u_1, \dots, u_m) \in \mathbb{R}^m$, $\mathfrak{s}_1 < \dots < \mathfrak{s}_m$. Here and below, $\ln$ always denotes the principal branch of the logarithm.

Lemma 1.1. $\mathcal{H}_j(t; \vec{u}, \vec{s}) > 0$ for $j = 1, 2$ and for all $t \in \mathbb{R}$, $\vec{u} = (u_1, \dots, u_m) \in \mathbb{R}^m$, $s_1 < \dots < s_m$.

Proof. In view of the identity $\mathcal{H}_1(t; \vec{u}, \vec{s}) = e^{u_1 + \dots + u_m} \mathcal{H}_2(-t; \vec{u}, \vec{s})$, it is enough to consider $\mathcal{H}_1$. Since

$$\partial_{u_1} \mathcal{H}_1(t; \vec{u}, \vec{s}) = \frac{e^{u_1 + \dots + u_m}}{2} \operatorname{erfc}(t - s_1) > 0,$$

we only have to check that $\mathcal{H}_1|_{u_1=-\infty} \geq 0$. It is easy to verify that

$$\mathcal{H}_1(t; \vec{u}, \vec{s})|_{u_1=-\infty} = \frac{1}{2} [2 - \operatorname{erfc}(t - s_m)] + \sum_{\ell=2}^m \frac{e^{u_\ell + \dots + u_m}}{2} [\operatorname{erfc}(t - s_\ell) - \operatorname{erfc}(t - s_{\ell-1})].$$

Since $\mathbb{R} \ni x \mapsto \operatorname{erfc}(x)$ is decreasing from 2 to 0 and $s_1 < \dots < s_m$, each of the m terms in the above right-hand side is > 0 , which implies $\mathcal{H}_1|_{u_1=-\infty} > 0$. $\square$

The following two theorems are our main results.

Theorem 1.2. (*Merging radii in the bulk*)

Let $m \in \mathbb{N}_{>0}$, $r \in (0, b^{-\frac{1}{2b}})$, $s_1, \dots, s_m \in \mathbb{R}$, $\alpha > -1$ and $b > 0$ be fixed parameters such that $s_1 < \dots < s_m$, and for $n \in \mathbb{N}_{>0}$, define

$$r_\ell = r \left(1 + \frac{\sqrt{2} s_\ell}{r^b \sqrt{n}} \right)^{\frac{1}{2b}}, \quad \ell = 1, \dots, m.$$

For any fixed $x_1, \dots, x_m \in \mathbb{R}$, there exists $\delta > 0$ such that

$$\mathbb{E} \left[\prod_{j=1}^m e^{u_j N(r_j)} \right] = \exp \left(C_1 n + C_2 \sqrt{n} + C_3 + \frac{C_4}{\sqrt{n}} + \mathcal{O} \left(\frac{(\ln n)^2}{n} \right) \right), \quad \text{as } n \rightarrow +\infty \quad (1.9)$$

uniformly for $u_1 \in \{z \in \mathbb{C} : |z - x_1| \leq \delta\}, \dots, u_m \in \{z \in \mathbb{C} : |z - x_m| \leq \delta\}$, where

$$\begin{aligned} C_1 &= br^{2b} \sum_{j=1}^m u_j, \\ C_2 &= \sqrt{2} br^b \int_0^{+\infty} \left(\ln \mathcal{H}_1(t; \vec{u}, \vec{s}) + \ln \mathcal{H}_2(t; \vec{u}, \vec{s}) \right) dt, \\ C_3 &= - \left(\frac{1}{2} + \alpha \right) \sum_{j=1}^m u_j + 4b \int_0^{+\infty} t \left(\ln \mathcal{H}_1(t; \vec{u}, \vec{s}) - \ln \mathcal{H}_2(t; \vec{u}, \vec{s}) \right) dt + \sqrt{2} b \int_{-\infty}^{+\infty} \mathcal{G}_1(t; \vec{u}, \vec{s}) dt, \\ C_4 &= \frac{6\sqrt{2}b}{r^b} \int_0^{+\infty} t^2 \left(\ln \mathcal{H}_1(t; \vec{u}, \vec{s}) + \ln \mathcal{H}_2(t; \vec{u}, \vec{s}) \right) dt \\ &\quad + \frac{b}{r^b} \int_{-\infty}^{+\infty} \left(4t \mathcal{G}_1(t; \vec{u}, \vec{s}) - \frac{\mathcal{G}_1(t; \vec{u}, \vec{s})^2}{\sqrt{2}} + \mathcal{G}_2(t; \vec{u}, \vec{s}) \right) dt. \end{aligned}$$

In particular, since $\mathbb{E} \left[\prod_{j=1}^m e^{u_j N(r_j)} \right]$ depends analytically on $u_1, \dots, u_m \in \mathbb{C}$ and is positive for $u_1, \dots, u_m \in \mathbb{R}$, the asymptotic formula (1.9) together with Cauchy's formula shows that

$$\partial_{u_1}^{k_1} \dots \partial_{u_m}^{k_m} \left\{ \ln \mathbb{E} \left[\prod_{j=1}^m e^{u_j N(r_j)} \right] - \left(C_1 n + C_2 \sqrt{n} + C_3 + \frac{C_4}{\sqrt{n}} \right) \right\} = \mathcal{O} \left(\frac{(\ln n)^2}{n} \right), \quad \text{as } n \rightarrow +\infty, \quad (1.10)$$

for any $k_1, \dots, k_m \in \mathbb{N}$, and $u_1, \dots, u_m \in \mathbb{R}$.

Theorem 1.3. (*Merging radii at the edge*)

Let $m \in \mathbb{N}_{>0}$, $\mathfrak{s}_1, \dots, \mathfrak{s}_m \in \mathbb{R}$, $\alpha > -1$ and $b > 0$ be fixed parameters such that $\mathfrak{s}_1 < \dots < \mathfrak{s}_m$, and for $n \in \mathbb{N}_{>0}$, define

$$r_\ell = b^{-\frac{1}{2b}} \left(1 + \sqrt{2b} \frac{\mathfrak{s}_\ell}{\sqrt{n}} \right)^{\frac{1}{2b}}, \quad \ell = 1, \dots, m.$$

For any fixed $x_1, \dots, x_m \in \mathbb{R}$, there exists $\delta > 0$ such that

$$\mathbb{E} \left[\prod_{j=1}^m e^{u_j N(r_j)} \right] = \exp \left(C_1 n + C_2 \sqrt{n} + C_3 + \frac{C_4}{\sqrt{n}} + \mathcal{O} \left(\frac{(\ln n)^2}{n} \right) \right), \quad \text{as } n \rightarrow +\infty \quad (1.11)$$

uniformly for $u_1 \in \{z \in \mathbb{C} : |z - x_1| \leq \delta\}, \dots, u_m \in \{z \in \mathbb{C} : |z - x_m| \leq \delta\}$, where

$$\begin{aligned} C_1 &= \sum_{j=1}^m u_j, \\ C_2 &= \sqrt{2b} \int_0^{+\infty} \ln \mathcal{H}_2(t; \vec{u}, \vec{\mathfrak{s}}) dt, \\ C_3 &= \left(\frac{1}{2} + \alpha \right) \ln \mathcal{H}_2(0; \vec{u}, \vec{\mathfrak{s}}) - 4b \int_0^{+\infty} t \ln \mathcal{H}_2(t; \vec{u}, \vec{\mathfrak{s}}) dt + \sqrt{2} b \int_{-\infty}^0 \mathcal{G}_1(t; \vec{u}, \vec{\mathfrak{s}}) dt, \\ C_4 &= 6\sqrt{2} b^{\frac{3}{2}} \int_0^{+\infty} t^2 \ln \mathcal{H}_2(t; \vec{u}, \vec{\mathfrak{s}}) dt + b^{3/2} \int_{-\infty}^0 \left(4t \mathcal{G}_1(t; \vec{u}, \vec{\mathfrak{s}}) - \frac{\mathcal{G}_1(t; \vec{u}, \vec{\mathfrak{s}})^2}{\sqrt{2}} + \mathcal{G}_2(t; \vec{u}, \vec{\mathfrak{s}}) \right) dt \\ &\quad - \frac{1 + 6\alpha + 6\alpha^2}{12\sqrt{2b}} \frac{\mathcal{H}'_2(0; \vec{u}, \vec{\mathfrak{s}})}{\mathcal{H}_2(0; \vec{u}, \vec{\mathfrak{s}})} + \left(\frac{1}{2} + \alpha \right) \sqrt{b} \mathcal{G}_1(0; \vec{u}, \vec{\mathfrak{s}}). \end{aligned}$$

In particular, since $\mathbb{E} \left[\prod_{j=1}^m e^{u_j N(r_j)} \right]$ depends analytically on $u_1, \dots, u_m \in \mathbb{C}$ and is positive for $u_1, \dots, u_m \in \mathbb{R}$, the asymptotic formula (1.9) together with Cauchy's formula shows that

$$\partial_{u_1}^{k_1} \dots \partial_{u_m}^{k_m} \left\{ \ln \mathbb{E} \left[\prod_{j=1}^m e^{u_j N(r_j)} \right] - \left(C_1 n + C_2 \sqrt{n} + C_3 + \frac{C_4}{\sqrt{n}} \right) \right\} = \mathcal{O} \left(\frac{(\ln n)^2}{n} \right), \quad \text{as } n \rightarrow +\infty, \quad (1.12)$$

for any $k_1, \dots, k_m \in \mathbb{N}$, and $u_1, \dots, u_m \in \mathbb{R}$.

Remark 1.4. We believe that $\mathcal{O} \left(\frac{(\ln n)^2}{n} \right)$ in (1.9) and (1.10) is not optimal and can be improved to $\mathcal{O}(n^{-1})$.

Let $(\mathbb{N}^m)_{>0} := \{\vec{j} = (j_1, \dots, j_m) \in \mathbb{N} : j_1 + \dots + j_m \geq 1\}$.

Recall that the joint cumulants $\{\kappa_{\vec{j}} = \kappa_{\vec{j}}(r_1, \dots, r_m; n, b, \alpha)\}_{\vec{j} \in (\mathbb{N}^m)_{>0}}$ of $N(r_1), \dots, N(r_m)$ are defined by

$$\kappa_{\vec{j}} = \kappa_{j_1, \dots, j_m} := \partial_{\vec{u}}^{\vec{j}} \ln \mathbb{E} [e^{u_1 N(r_1) + \dots + u_m N(r_m)}] \Big|_{\vec{u}=\vec{0}}, \quad \vec{j} \in (\mathbb{N}^m)_{>0}, \quad (1.13)$$

where $\partial_{\vec{u}}^{\vec{j}} := \partial_{u_1}^{j_1} \dots \partial_{u_m}^{j_m}$ and $\vec{0} := (0, \dots, 0)$. In particular, we have

$$\mathbb{E}[N(r)] = \kappa_1(r), \quad \text{Var}[N(r)] = \kappa_2(r), \quad \text{Cov}[N(r_1), N(r_2)] = \kappa_{(1,1)}(r_1, r_2).$$

Corollary 1.5 below follows from Theorems 1.2 and 1.3. As already mentioned, it contains the large n asymptotics of all joint cumulants of $N(r_1), \dots, N(r_m)$ when the radii are merging, either in the bulk or at the edge, and also contains several central limit theorems for the joint fluctuations of $N(r_1), \dots, N(r_m)$.

Corollary 1.5. (a) (bulk regime) Let $m \in \mathbb{N}_{>0}$, $\vec{j} \in (\mathbb{N}^m)_{>0}$, $\alpha > -1$, $b > 0$, $r \in (0, b^{-\frac{1}{2b}})$ and $-\infty < \mathfrak{s}_1 < \dots < \mathfrak{s}_m < +\infty$ be fixed, and for $n \in \mathbb{N}_{>0}$, define

$$r_\ell = r \left(1 + \frac{\sqrt{2} \mathfrak{s}_\ell}{r^b \sqrt{n}} \right)^{\frac{1}{2b}}, \quad \ell = 1, \dots, m.$$

As $n \rightarrow +\infty$, we have

$$\kappa_{\vec{j}} = \partial_{\vec{u}}^{\vec{j}} C_1|_{\vec{u}=\vec{0}} n + \partial_{\vec{u}}^{\vec{j}} C_2|_{\vec{u}=\vec{0}} \sqrt{n} + \partial_{\vec{u}}^{\vec{j}} C_3|_{\vec{u}=\vec{0}} + \frac{\partial_{\vec{u}}^{\vec{j}} C_4|_{\vec{u}=\vec{0}}}{\sqrt{n}} + \mathcal{O}\left(\frac{(\ln n)^2}{n}\right), \quad (1.14)$$

where $C_1, \dots, C_4$ are as in Theorem 1.2. In particular, for any $1 \leq \ell < k \leq m$, as $n \rightarrow +\infty$ we have

$$\mathbb{E}[N(r_\ell)] = br^{2b}n + \sqrt{2}br^b \mathfrak{s}_\ell \sqrt{n} + \frac{b-1-2\alpha}{2} + \mathcal{O}\left(\frac{(\ln n)^2}{n}\right), \quad (1.15)$$

$$\text{Var}[N(r_\ell)] = \frac{br^b}{\sqrt{\pi}} \sqrt{n} + \frac{b \mathfrak{s}_\ell}{\sqrt{2\pi}} - \frac{b(1+4\mathfrak{s}_\ell^2)}{16\sqrt{\pi}r^b} \frac{1}{\sqrt{n}} + \mathcal{O}\left(\frac{(\ln n)^2}{n}\right), \quad (1.16)$$

$$\text{Cov}(N(r_\ell), N(r_k)) = c_{(1,1)}(\mathfrak{s}_\ell, \mathfrak{s}_k) \sqrt{n} + d_{(1,1)}(\mathfrak{s}_\ell, \mathfrak{s}_k) + e_{(1,1)}(\mathfrak{s}_\ell, \mathfrak{s}_k) n^{-\frac{1}{2}} + \mathcal{O}\left(\frac{(\ln n)^2}{n}\right),$$

where

$$c_{(1,1)}(\mathfrak{s}_\ell, \mathfrak{s}_k) = \frac{br^b}{\sqrt{2}} \int_0^{+\infty} \left\{ \text{erfc}(t - \mathfrak{s}_\ell) \left(1 - \frac{1}{2} \text{erfc}(t - \mathfrak{s}_k)\right) + \text{erfc}(t + \mathfrak{s}_k) \left(1 - \frac{1}{2} \text{erfc}(t + \mathfrak{s}_\ell)\right) \right\} dt, \quad (1.17)$$

$$\begin{aligned} d_{(1,1)}(\mathfrak{s}_\ell, \mathfrak{s}_k) &= b \int_0^{+\infty} t \left\{ \text{erfc}(t - \mathfrak{s}_\ell) (2 - \text{erfc}(t - \mathfrak{s}_k)) - \text{erfc}(t + \mathfrak{s}_k) (2 - \text{erfc}(t + \mathfrak{s}_\ell)) \right\} dt \\ &+ b \int_{-\infty}^{+\infty} \left\{ (2 - \text{erfc}(t - \mathfrak{s}_k)) \frac{e^{-(t-\mathfrak{s}_\ell)^2}}{2\sqrt{\pi}} \frac{1 - 5t^2 + t\mathfrak{s}_\ell - 2\mathfrak{s}_\ell^2}{3} \right. \\ &\quad \left. - \text{erfc}(t - \mathfrak{s}_\ell) \frac{e^{-(t-\mathfrak{s}_k)^2}}{2\sqrt{\pi}} \frac{1 - 5t^2 + t\mathfrak{s}_k - 2\mathfrak{s}_k^2}{3} \right\} dt, \\ e_{(1,1)}(\mathfrak{s}_\ell, \mathfrak{s}_k) &= -\frac{br^{-b} e^{-\frac{(\mathfrak{s}_\ell - \mathfrak{s}_k)^2}{2}}}{288\sqrt{\pi}} \left(51 + 55\mathfrak{s}_\ell^4 + 55\mathfrak{s}_k^4 + 96\mathfrak{s}_\ell^2 + 96\mathfrak{s}_k^2 + 128\mathfrak{s}_\ell^3 \mathfrak{s}_k + 128\mathfrak{s}_\ell \mathfrak{s}_k^3 + 180\mathfrak{s}_\ell \mathfrak{s}_k \right. \\ &\quad \left. + 210\mathfrak{s}_\ell^2 \mathfrak{s}_k^2 \right) + \frac{3br^{-b}}{\sqrt{2}} \int_0^{+\infty} t^2 \left\{ \text{erfc}(t - \mathfrak{s}_\ell) (2 - \text{erfc}(t - \mathfrak{s}_k)) + (2 - \text{erfc}(t + \mathfrak{s}_\ell)) \text{erfc}(t + \mathfrak{s}_k) \right\} dt \\ &+ \frac{br^{-b}}{36\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left[(2 - \text{erfc}(t - \mathfrak{s}_k)) e^{-(t-\mathfrak{s}_\ell)^2} \mathfrak{p}(t, \mathfrak{s}_\ell) - \text{erfc}(t - \mathfrak{s}_\ell) e^{-(t-\mathfrak{s}_k)^2} \mathfrak{p}(t, \mathfrak{s}_k) \right] dt, \\ \mathfrak{p}(t, \mathfrak{s}) &= -3\mathfrak{s} + 22\mathfrak{s}^3 - 8\mathfrak{s}^5 + t(21 - 66\mathfrak{s}^2 + 16\mathfrak{s}^4) + t^2(57\mathfrak{s} - 50\mathfrak{s}^3) + t^3(-193 + 62\mathfrak{s}^2) - 70t^4\mathfrak{s} + 50t^5. \end{aligned}$$

(b) (joint fluctuations in the bulk) Let $\alpha > -1$, $b > 0$, $r \in (0, b^{-\frac{1}{2b}})$, $m \in \mathbb{N}_{>0}$ and $\mathfrak{s}_1, \dots, \mathfrak{s}_m \in \mathbb{R}$ be fixed, and for $n \in \mathbb{N}_{>0}$, define

$$r_\ell = r \left(1 + \frac{\sqrt{2} \mathfrak{s}_\ell}{r^b \sqrt{n}} \right)^{\frac{1}{2b}}, \quad \ell = 1, \dots, m.$$

Consider the random variables

$$\mathcal{N}_\ell := \pi^{1/4} \frac{N(r_\ell) - (br^{2b}n + \sqrt{2}br^b \mathfrak{s}_\ell \sqrt{n})}{\sqrt{br^b} n^{1/4}}, \quad \ell = 1, \dots, m. \quad (1.18)$$

As $n \rightarrow +\infty$, $(\mathcal{N}_1, \dots, \mathcal{N}_m)$ convergences in distribution to a multivariate normal random variable of mean $(0, \dots, 0)$ and whose covariance matrix Σ is given by

$$\Sigma_{\ell, \ell} = 1, \quad \Sigma_{\ell, k} = \Sigma_{k, \ell} = \frac{c_{(1,1)}(\mathfrak{s}_\ell, \mathfrak{s}_k)}{br^b/\sqrt{\pi}}, \quad 1 \leq \ell < k \leq m,$$

where $c_{(1,1)}(\mathfrak{s}_\ell, \mathfrak{s}_k)$ is given by (1.17).

(c) (edge regime) Let $m \in \mathbb{N}_{>0}$, $\vec{j} \in (\mathbb{N}^m)_{>0}$, $\alpha > -1$, $b > 0$ and $-\infty < \mathfrak{s}_1 < \dots < \mathfrak{s}_m < +\infty$ be fixed, and for $n \in \mathbb{N}_{>0}$, define

$$r_\ell = b^{-\frac{1}{2b}} \left(1 + \frac{\sqrt{2b}\mathfrak{s}_\ell}{\sqrt{n}} \right)^{\frac{1}{2b}}, \quad \ell = 1, \dots, m.$$

As $n \rightarrow +\infty$, we have

$$\kappa_{\vec{j}} = \partial_{\vec{u}}^{\vec{j}} C_1|_{\vec{u}=\vec{0}} n + \partial_{\vec{u}}^{\vec{j}} C_2|_{\vec{u}=\vec{0}} \sqrt{n} + \partial_{\vec{u}}^{\vec{j}} C_3|_{\vec{u}=\vec{0}} + \frac{\partial_{\vec{u}}^{\vec{j}} C_4|_{\vec{u}=\vec{0}}}{\sqrt{n}} + \mathcal{O}\left(\frac{(\ln n)^2}{n}\right), \quad (1.19)$$

where $C_1, \dots, C_4$ are as in Theorem 1.3. In particular, for any $1 \leq \ell < k \leq m$, as $n \rightarrow +\infty$ we have

$$\begin{aligned} \mathbb{E}[N(r_\ell)] &= n + c_1(\mathfrak{s}_\ell)\sqrt{n} + d_1(\mathfrak{s}_\ell) + e_1(\mathfrak{s}_\ell)n^{-\frac{1}{2}} + \mathcal{O}\left(\frac{(\ln n)^2}{n}\right), \\ \text{Var}[N(r_\ell)] &= c_2(\mathfrak{s}_\ell)\sqrt{n} + d_2(\mathfrak{s}_\ell) + e_2(\mathfrak{s}_\ell)n^{-\frac{1}{2}} + \mathcal{O}\left(\frac{(\ln n)^2}{n}\right), \\ \text{Cov}(N(r_\ell), N(r_k)) &= c_{(1,1)}(\mathfrak{s}_\ell, \mathfrak{s}_k)\sqrt{n} + d_{(1,1)}(\mathfrak{s}_\ell, \mathfrak{s}_k) + e_{(1,1)}(\mathfrak{s}_\ell, \mathfrak{s}_k)n^{-\frac{1}{2}} + \mathcal{O}\left(\frac{(\ln n)^2}{n}\right), \end{aligned}$$

where

$$c_1(\mathfrak{s}) = \frac{\sqrt{b}\mathfrak{s}}{\sqrt{2}} \text{erfc}(\mathfrak{s}) - \frac{\sqrt{b}}{\sqrt{2\pi}} e^{-\mathfrak{s}^2}, \quad (1.20)$$

$$\begin{aligned} d_1(\mathfrak{s}) &= -\frac{1}{2} \left(\frac{1}{2} + \alpha - \frac{b}{2} \right) \text{erfc}(\mathfrak{s}) - \frac{b\mathfrak{s}}{3\sqrt{\pi}} e^{-\mathfrak{s}^2}, \\ e_1(\mathfrak{s}) &= \frac{e^{-\mathfrak{s}^2}}{\sqrt{2\pi}} \left(\frac{b(2+4\alpha)-1-6\alpha-6\alpha^2}{12\sqrt{b}} + \frac{(3b-2-4\alpha)\mathfrak{s}^2}{6} \sqrt{b} - \frac{2\mathfrak{s}^4}{9} b^{3/2} \right), \\ c_2(\mathfrak{s}) &= \frac{\sqrt{b}}{2\sqrt{\pi}} \text{erfc}(\sqrt{2}\mathfrak{s}) + \sqrt{b} \frac{e^{-\mathfrak{s}^2}}{\sqrt{2\pi}} (1 - \text{erfc}(\mathfrak{s})) + \frac{\sqrt{b}\mathfrak{s}}{\sqrt{2}} \text{erfc}(\mathfrak{s}) \left(\frac{1}{2} \text{erfc}(\mathfrak{s}) - 1 \right), \end{aligned} \quad (1.21)$$

$$\begin{aligned} d_2(\mathfrak{s}) &= -\frac{b}{12\pi} e^{-2\mathfrak{s}^2} + \frac{b\mathfrak{s}}{2\sqrt{2\pi}} \text{erfc}(\sqrt{2}\mathfrak{s}) + \frac{b\mathfrak{s}}{3\sqrt{\pi}} e^{-\mathfrak{s}^2} (1 - \text{erfc}(\mathfrak{s})) \\ &\quad + \frac{b-1-2\alpha}{4} \text{erfc}(\mathfrak{s}) \left(\frac{1}{2} \text{erfc}(\mathfrak{s}) - 1 \right), \\ e_2(\mathfrak{s}) &= \frac{e^{-\mathfrak{s}^2}}{12\sqrt{2\pi}b} \left(1 - 2b + 6\alpha - 4b\alpha + 6\alpha^2 + 2(2-3b+4\alpha)b\mathfrak{s}^2 + \frac{8b^2}{3}\mathfrak{s}^4 \right) (1 - \text{erfc}(\mathfrak{s})) \\ &\quad - \frac{b^{3/2}\mathfrak{s}}{72\sqrt{2}\pi} e^{-2\mathfrak{s}^2} - \frac{b^{3/2}(1+4\mathfrak{s}^2)}{32\sqrt{\pi}} \text{erfc}(\sqrt{2}\mathfrak{s}), \\ c_{(1,1)}(\mathfrak{s}_\ell, \mathfrak{s}_k) &= \frac{\sqrt{b}}{2\sqrt{2}} \int_0^{+\infty} \text{erfc}(t + \mathfrak{s}_k) (2 - \text{erfc}(t + \mathfrak{s}_\ell)) dt, \end{aligned} \quad (1.22)$$

$$\begin{aligned}
d_{(1,1)}(\mathfrak{s}_\ell, \mathfrak{s}_k) &= \frac{1+2\alpha}{8} (2 - \operatorname{erfc}(\mathfrak{s}_\ell)) \operatorname{erfc}(\mathfrak{s}_k) - b \int_0^{+\infty} t \operatorname{erfc}(t + \mathfrak{s}_k) (2 - \operatorname{erfc}(t + \mathfrak{s}_\ell)) dt \\
&+ b \int_{-\infty}^0 \left\{ (2 - \operatorname{erfc}(t - \mathfrak{s}_k)) \frac{e^{-(t-\mathfrak{s}_\ell)^2}}{2\sqrt{\pi}} \frac{1 - 5t^2 + t\mathfrak{s}_\ell - 2\mathfrak{s}_\ell^2}{3} \right. \\
&\quad \left. - \operatorname{erfc}(t - \mathfrak{s}_\ell) \frac{e^{-(t-\mathfrak{s}_k)^2}}{2\sqrt{\pi}} \frac{1 - 5t^2 + t\mathfrak{s}_k - 2\mathfrak{s}_k^2}{3} \right\} dt, \\
e_{(1,1)}(\mathfrak{s}_\ell, \mathfrak{s}_k) &= (2 - \operatorname{erfc}(\mathfrak{s}_\ell)) \frac{e^{-\mathfrak{s}_k^2}}{\sqrt{2\pi}} \frac{1 + 6\alpha + 6\alpha^2 + 2b(1 + 2\alpha)(2\mathfrak{s}_k^2 - 1)}{24\sqrt{b}} \\
&\quad - \operatorname{erfc}(\mathfrak{s}_k) \frac{e^{-\mathfrak{s}_\ell^2}}{\sqrt{2\pi}} \frac{1 + 6\alpha + 6\alpha^2 + 2b(1 + 2\alpha)(2\mathfrak{s}_\ell^2 - 1)}{24\sqrt{b}} \\
&- \frac{b^{\frac{3}{2}} e^{-\frac{(\mathfrak{s}_\ell - \mathfrak{s}_k)^2}{2}}}{288\sqrt{\pi}} \frac{1}{2} \operatorname{erfc}\left(\frac{\mathfrak{s}_\ell + \mathfrak{s}_k}{\sqrt{2}}\right) \left(51 + 55\mathfrak{s}_\ell^4 + 55\mathfrak{s}_k^4 + 96\mathfrak{s}_\ell^2 + 96\mathfrak{s}_k^2 + 128\mathfrak{s}_\ell^3\mathfrak{s}_k + 128\mathfrak{s}_\ell\mathfrak{s}_k^3 \right. \\
&\quad \left. + 180\mathfrak{s}_\ell\mathfrak{s}_k + 210\mathfrak{s}_\ell^2\mathfrak{s}_k^2 \right) + \frac{b^{\frac{3}{2}}}{144\sqrt{2}} \frac{e^{-\mathfrak{s}_\ell^2 - \mathfrak{s}_k^2}}{2\pi} \left(55(\mathfrak{s}_\ell^3 + \mathfrak{s}_k^3) + 73(\mathfrak{s}_\ell + \mathfrak{s}_k + \mathfrak{s}_\ell^2\mathfrak{s}_k + \mathfrak{s}_\ell\mathfrak{s}_k^2) \right) \\
&+ \frac{3b^{\frac{3}{2}}}{\sqrt{2}} \int_0^{+\infty} t^2 (2 - \operatorname{erfc}(t + \mathfrak{s}_\ell)) \operatorname{erfc}(t + \mathfrak{s}_k) dt \\
&+ \frac{b^{\frac{3}{2}}}{36\sqrt{2\pi}} \int_{-\infty}^0 \left[(2 - \operatorname{erfc}(t - \mathfrak{s}_k)) e^{-(t-\mathfrak{s}_\ell)^2} \mathfrak{p}(t, \mathfrak{s}_\ell) - \operatorname{erfc}(t - \mathfrak{s}_\ell) e^{-(t-\mathfrak{s}_k)^2} \mathfrak{p}(t, \mathfrak{s}_k) \right] dt, \\
\mathfrak{p}(t, \mathfrak{s}) &= -3\mathfrak{s} + 22\mathfrak{s}^3 - 8\mathfrak{s}^5 + t(21 - 66\mathfrak{s}^2 + 16\mathfrak{s}^4) + t^2(57\mathfrak{s} - 50\mathfrak{s}^3) + t^3(-193 + 62\mathfrak{s}^2) - 70t^4\mathfrak{s} + 50t^5.
\end{aligned}$$

(d) (joint fluctuations at the edge) Let $\alpha > -1$, $b > 0$, $m \in \mathbb{N}_{>0}$ and $\mathfrak{s}_1, \dots, \mathfrak{s}_m \in \mathbb{R}$ be fixed, and for $n \in \mathbb{N}_{>0}$, define

$$r_\ell = b^{-\frac{1}{2b}} \left(1 + \sqrt{2b} \frac{\mathfrak{s}_\ell}{\sqrt{n}} \right)^{\frac{1}{2b}}, \quad \ell = 1, \dots, m.$$

Consider the random variables

$$\mathcal{N}_\ell := \frac{N(r_\ell) - (n + c_1(\mathfrak{s}_\ell)\sqrt{n})}{\sqrt{c_2(\mathfrak{s}_\ell)} n^{1/4}}, \quad \ell = 1, \dots, m, \quad (1.23)$$

where c_1 is given by (1.20) and c_2 is given by (1.21). As $n \rightarrow +\infty$, $(\mathcal{N}_1, \dots, \mathcal{N}_m)$ converges in distribution to a multivariate normal random variable of mean $(0, \dots, 0)$ and whose covariance matrix Σ is given by

$$\Sigma_{\ell, \ell} = 1, \quad \Sigma_{\ell, k} = \Sigma_{k, \ell} = \frac{c_{(1,1)}(\mathfrak{s}_\ell, \mathfrak{s}_k)}{\sqrt{c_2(\mathfrak{s}_\ell)} \sqrt{c_2(\mathfrak{s}_k)}}, \quad 1 \leq \ell < k \leq m,$$

where $c_{(1,1)}(\mathfrak{s}_\ell, \mathfrak{s}_k)$ is given by (1.17).

Remark 1.6. Some of the results contained in Corollary 1.5 were already known:

- In [37, eq (70)], Lee and Riser obtained second-order asymptotics for the number of points lying outside the droplet of the Elliptic Ginibre ensemble (in particular, the coefficients $c_1(0)|_{(b, \alpha)=(1,0)}$ and $d_1(0)|_{(b, \alpha)=(1,0)}$ of part (c) above were contained in their results).
- Given a Borel set A , let $N_A := \#\{z_j : z_j \in A\}$. In [11], Charles and Estienne proved that if A is independent of n , has smooth boundary and lies strictly in the bulk of the Ginibre ensemble,

the cumulants $\{\kappa_j(A)\}_{j=1}^{+\infty}$ enjoy an all-order expansion of the form

$$\kappa_j(A) = \begin{cases} \alpha_{j,0}n + \sum_{k=1}^N \alpha_{j,k}n^{1-k} + \mathcal{O}(n^{-N}), & \text{if } j = 1, \\ \sum_{k=1}^N \alpha_{j,k}n^{1-k} + \mathcal{O}(n^{-N}), & \text{if } j \text{ is odd and } j \geq 3, \\ \beta_{j,0}n^{\frac{1}{2}} + \sum_{k=1}^N \beta_{j,k}n^{\frac{1}{2}-k} + \mathcal{O}(n^{-N-\frac{1}{2}}), & \text{if } j \text{ is even,} \end{cases} \quad (1.24)$$

where $N \in \mathbb{N}$ is arbitrary. Furthermore, the coefficients $\alpha_{j,0}$ and $\beta_{j,0}$ were computed explicitly. It can be verified (see [12, Corollary 1.4 (a)]) that (1.14) is consistent with (1.24): for $m = 1$ and $\mathfrak{s}_1 = 0$, we have $\partial_u^j C_1|_{u=0} = 0$ for $j \geq 2$, $\partial_u^j C_2|_{u=0} = 0 = \partial_u^j C_4|_{u=0} = 0$ for j odd, and $\partial_u^j C_3|_{u=0} = 0$ for j even.

- Second order asymptotics for the cumulants $\{\kappa_j\}_{j=1}^{+\infty}$ (i.e. the case $m = 1$) were obtained in [35, eqs (55)–(67)] for general $b > 0$ and $\alpha > -1$, both in the bulk and the edge regimes.
- Third order asymptotics for the cumulants $\{\kappa_j\}_{j=1}^{+\infty}$ (i.e. the case $m = 1$) were obtained in [28, Remark 4] for the Ginibre case, and the leading coefficient $c_{(1,1)}(\mathfrak{s}_\ell, \mathfrak{s}_k)|_{(b,\alpha)=(1,0)}$ (given in (1.17) for the bulk and in (1.22) for the edge) was obtained in [28, Proposition 2.3]. Parts (b) and (d) above, when specialized to $(b, \alpha) = (1, 0)$, were also already known from [28, Proposition 2.3].
- Part (a) with $m = 1$ and $\mathfrak{s}_1 = 0$ and part (c) with $m = 1$ and general $\mathfrak{s}_1 \in \mathbb{R}$ were already known from [12, Corollary 1.4].

Proof of Corollary 1.5. Proof of parts (a) and (c): The asymptotics (1.14) and (1.19) directly follow from (1.10), (1.12) and (1.13). The simplified asymptotics for $\mathbb{E}[\mathbf{N}(r_\ell)]$, $\text{Var}[\mathbf{N}(r_\ell)]$, and $\text{Cov}(\mathbf{N}(r_\ell), \mathbf{N}(r_k))$ are then obtained by performing a long but straightforward computation. Proof of part (d): Let $t_1, \dots, t_m \in \mathbb{R}$ be arbitrary but fixed. Note that $c_2(\mathfrak{s}) > 0$ for $\mathfrak{s} \in \mathbb{R}$, because $c'_2(\mathfrak{s}) = 2^{-3/2}\sqrt{b}(\text{erfc}(\mathfrak{s}) - 2)\text{erfc}(\mathfrak{s}) < 0$ and $\lim_{\mathfrak{s} \rightarrow +\infty} c_2(\mathfrak{s}) = 0$. By Theorem 1.3, we know that (1.11) holds uniformly for $u_1, \dots, u_m \in \{z \in \mathbb{C} : |z| \leq \delta\}$ for a certain $\delta > 0$. Hence, using Theorem 1.3 with

$$u_\ell = \frac{t_\ell}{\sqrt{c_2(\mathfrak{s}_\ell)} n^{1/4}}, \quad \ell = 1, \dots, m,$$

we obtain

$$\mathbb{E} \left[\prod_{\ell=1}^m e^{t_\ell \mathcal{N}_\ell} \right] = \exp \left(\frac{1}{2} \sum_{1 \leq \ell < k \leq m} \Sigma_{\ell,k} t_\ell t_k + \mathcal{O}(n^{-\frac{1}{4}}) \right), \quad \text{as } n \rightarrow +\infty.$$

Thus the above asymptotics imply pointwise convergence in $(t_1, \dots, t_m) \in \mathbb{R}^m$ of $\mathbb{E}[\prod_{\ell=1}^m e^{t_\ell \mathcal{N}_\ell}]$ to $\exp(\frac{1}{2} \sum_{1 \leq \ell < k \leq m} \Sigma_{\ell,k} t_\ell t_k)$ as $n \rightarrow +\infty$. This, in turn, implies by standard theorems that $(\mathcal{N}_1, \dots, \mathcal{N}_m)$ converges in distribution to a multivariate normal random variable of mean $\vec{0}$ and covariance matrix Σ , which finishes the proof of (d). The proof of (b) is similar. $\square$

Determinants. Here we express $\mathbb{E}[\prod_{\ell=1}^m e^{u_\ell \mathbf{N}(r_\ell)}]$ as a ratio of two determinants. Using that $\prod_{1 \leq j < k \leq n} |z_k - z_j|^2$ is the product of two Vandermonde determinants, we obtain after standard manipulations that

$$\begin{aligned} \mathbb{E} \left[\prod_{\ell=1}^m e^{u_\ell \mathbf{N}(r_\ell)} \right] &= \frac{1}{n! Z_n} \int_{\mathbb{C}} \cdots \int_{\mathbb{C}} \prod_{1 \leq j < k \leq n} |z_k - z_j|^2 \prod_{j=1}^n w(z_j) d^2 z_j \\ &= \frac{1}{Z_n} \det \left(\int_{\mathbb{C}} z^j \bar{z}^k w(z) d^2 z \right)_{j,k=0}^{n-1} \end{aligned} \quad (1.25)$$

$$= \frac{1}{Z_n} (2\pi)^n \prod_{j=0}^{n-1} \int_0^{+\infty} u^{2j+1} w(u) du, \quad (1.26)$$

where the weight w is defined by

$$w(z) := |z|^{2\alpha} e^{-n|z|^{2b}} \omega(|z|), \quad \omega(x) := \prod_{\ell=1}^m \begin{cases} e^{u_\ell}, & \text{if } x < r_\ell, \\ 1, & \text{if } x \geq r_\ell. \end{cases} \quad (1.27)$$

Formula (1.26) directly follows from (1.25) and the fact that w is rotation-invariant. Indeed, since $w(z) = w(|z|)$, the integral $\int_{\mathbb{C}} z^j \bar{z}^k w(z) d^2 z$ is 0 for $j \neq k$ and is $2\pi \int_0^{+\infty} u^{2j+1} w(u) du$ for $j = k$. So only the main diagonal contributes for the determinants in (1.25).

Related works. We note from (1.25) that the problem of determining the large n asymptotics of $\mathbb{E}[\prod_{j=1}^m e^{u_j N(r_j)}]$ can equivalently be seen as a problem of obtaining large n asymptotics for an $n \times n$ determinant whose weight is supported on $\mathbb{C}$, rotation-invariant, and with m merging discontinuities along circles. For Theorem 1.2, the discontinuities are merging in the bulk, while for Theorem 1.3 the discontinuities are merging at the edge.

The one-dimensional analogue of this merging of discontinuities has been studied by several authors in the context of Toeplitz, Hankel and Toeplitz+Hankel determinants. Large n asymptotics of $n \times n$ Toeplitz determinants with *two* merging discontinuities were first obtained in the important works [21, 22]. In both [21] and [22], the term of order 1 in the asymptotics is characterized in terms of the solution to a Painlevé V equation. The generalization of [22] to the case where an arbitrary number of discontinuities are merging is a challenging problem and only recently important progress has been made [27]. Toeplitz+Hankel determinants with merging singularities have also recently been studied in [30, 20], and some applications of these results are given in [19]. In the aforementioned works, the discontinuities are merging in the bulk. Hankel determinants with merging discontinuities at the edge are also related to the Painlevé theory, see [51, 41] for soft edges, [42] for a hard edge, and e.g. [18, 15, 14] for studies on related Fredholm determinants. It is interesting to note that, in contrast to these works, the asymptotics obtained in Theorems 1.2 and 1.3 for merging circular discontinuities do not involve transcendental functions.

Let us also discuss related results on determinants with singularities in a two-dimensional setting. The works [11, 36, 28, 12] were already mentioned at the beginning of the introduction and deal with determinants having (non-merging) discontinuities. Beyond determinants with discontinuities, determinants with root-type singularities and related planar orthogonal polynomials have also attracted considerable attention in recent years [6, 8, 38, 50, 7, 39, 40, 24]. The analogues of Theorems 1.2 and 1.3 for planar root-type singularities can be found in [24, Theorem 1.5] (two merging singularities in the bulk) and [24, Theorem 1.14] (an arbitrary number of merging singularities at the edge).

Finally, we mention that our approach is based on the asymptotic evaluation of sums of logarithms involving the incomplete gamma function; related asymptotic evaluations are present also in e.g. [31, 34, 2, 23, 25, 12, 13] although the problems treated there are different.

2 Proof of Theorem 1.2

Recall that ω was defined in (1.27). For convenience, let us rewrite it as

$$\omega(x) = \sum_{\ell=1}^{m+1} \omega_\ell \mathbf{1}_{[0, r_\ell)}(x) = \sum_{\ell=1}^{m+1} \Omega_\ell \mathbf{1}_{[r_{\ell-1}, r_\ell)}(x), \quad (2.1)$$

where $r_{m+1} := +\infty$ and

$$\omega_\ell := \begin{cases} e^{u_\ell + \dots + u_m} - e^{u_{\ell+1} + \dots + u_m}, & \text{if } \ell < m, \\ e^{u_m} - 1, & \text{if } \ell = m, \\ 1, & \text{if } \ell = m+1, \end{cases} \quad \Omega_\ell = \sum_{j=\ell}^{m+1} \omega_j = \begin{cases} e^{u_\ell + \dots + u_m}, & \text{if } \ell \leq m, \\ 1 & \text{if } \ell = m+1. \end{cases} \quad (2.2)$$

The starting point of our proof is the following formula:

$$\ln \mathcal{E}_n = \sum_{j=1}^n \ln \left(1 + \sum_{\ell=1}^m \omega_\ell \frac{\gamma(\frac{j+\alpha}{b}, nr_\ell^{2b})}{\Gamma(\frac{j+\alpha}{b})} \right), \quad (2.3)$$

where $\mathcal{E}_n := \mathbb{E}[\prod_{\ell=1}^m e^{u_\ell N(r_\ell)}]$ and $\gamma(a, z)$ is the incomplete gamma function

$$\gamma(a, z) = \int_0^z t^{a-1} e^{-t} dt.$$

The identity (2.3) can easily be derived from (1.26) and was also obtained in [12, eqs (1.23) and (1.26)]. We infer from (2.3) that the asymptotics of $\gamma(a, z)$ as $z \rightarrow +\infty$ uniformly for $a \in [\frac{1+\alpha}{b}, \frac{z}{br_1^{2b}} + \frac{\alpha}{b}]$ are needed to obtain large n asymptotics for $\mathcal{E}_n$ — we recall these asymptotics in Appendix A.

In (2.3) and below, $\ln$ always denotes the principal branch of the logarithm.

Our proof strategy follows [12]. Let us define

$$j_- := \lceil \frac{bnr^{2b}}{1+\epsilon} - \alpha \rceil, \quad j_+ := \lfloor \frac{bnr^{2b}}{1-\epsilon} - \alpha \rfloor,$$

where $\epsilon > 0$ is independent of n . We assume that ϵ is sufficiently small such that

$$\frac{br^{2b}}{1-\epsilon} < \frac{1}{1+\epsilon},$$

so that we can write

$$\ln \mathcal{E}_n = S_0 + S_1 + S_2 + S_3, \quad (2.4)$$

where

$$S_0 = \sum_{j=1}^{M'} \ln \left(1 + \sum_{\ell=1}^m \omega_\ell \frac{\gamma(\frac{j+\alpha}{b}, nr_\ell^{2b})}{\Gamma(\frac{j+\alpha}{b})} \right), \quad S_1 = \sum_{j=M'+1}^{j_- - 1} \ln \left(1 + \sum_{\ell=1}^m \omega_\ell \frac{\gamma(\frac{j+\alpha}{b}, nr_\ell^{2b})}{\Gamma(\frac{j+\alpha}{b})} \right), \quad (2.5)$$

$$S_2 = \sum_{j=j_-}^{j_+} \ln \left(1 + \sum_{\ell=1}^m \omega_\ell \frac{\gamma(\frac{j+\alpha}{b}, nr_\ell^{2b})}{\Gamma(\frac{j+\alpha}{b})} \right), \quad S_3 = \sum_{j=j_++1}^n \ln \left(1 + \sum_{\ell=1}^m \omega_\ell \frac{\gamma(\frac{j+\alpha}{b}, nr_\ell^{2b})}{\Gamma(\frac{j+\alpha}{b})} \right). \quad (2.6)$$

In the above, $M' > 0$ is an integer independent of n . For $j = 1, \dots, n$ and $k = 1, \dots, m$, we also define $a_j := \frac{j+\alpha}{b}$, $z_k := nr_k^{2b}$ and

$$\lambda_{j,k} := \frac{z_k}{a_j} = \frac{bnr_k^{2b}}{j+\alpha}, \quad \lambda_j := \frac{bnr^{2b}}{j+\alpha}, \quad \eta_{j,k} := (\lambda_{j,k} - 1) \sqrt{\frac{2(\lambda_{j,k} - 1 - \ln \lambda_{j,k})}{(\lambda_{j,k} - 1)^2}}. \quad (2.7)$$

Lemma 2.1. *For any $x_1, \dots, x_m \in \mathbb{R}$, there exists $\delta > 0$ such that*

$$S_0 = M' \ln \Omega_1 + \mathcal{O}(e^{-cn}), \quad \text{as } n \rightarrow +\infty, \quad (2.8)$$

uniformly for $u_1 \in \{z \in \mathbb{C} : |z - x_1| \leq \delta\}, \dots, u_m \in \{z \in \mathbb{C} : |z - x_m| \leq \delta\}$.

Proof. We infer from (2.5) and Lemma A.1 that

$$S_0 = \sum_{j=1}^{M'} \ln \left(\sum_{\ell=1}^{m+1} \omega_\ell [1 + \mathcal{O}(e^{-cn})] \right) = \sum_{j=1}^{M'} \ln \Omega_1 + \mathcal{O}(e^{-cn}), \quad \text{as } n \rightarrow +\infty.$$

In the above, the error terms before the second equality are independent of $u_1, \dots, u_m$, so the claim follows. $\square$

Lemma 2.2. *The constant M' can be chosen sufficiently large such that the following holds. For any $x_1, \dots, x_m \in \mathbb{R}$, there exists $\delta > 0$ such that*

$$S_1 = (j_- - M' - 1) \ln \Omega_1 + \mathcal{O}(e^{-cn}), \quad S_3 = \mathcal{O}(e^{-cn}),$$

as $n \rightarrow +\infty$ uniformly for $u_1 \in \{z \in \mathbb{C} : |z - x_1| \leq \delta\}, \dots, u_m \in \{z \in \mathbb{C} : |z - x_m| \leq \delta\}$.

Proof. The proof is identical to the proof of [12, Lemma 2.2], so we omit it. $\square$

We now focus on S_2 . Let $M := M' \sqrt{\ln n}$. For the analysis we need to split S_2 as follows

$$S_2 = S_2^{(1)} + S_2^{(2)} + S_2^{(3)}, \quad S_2^{(v)} := \sum_{j: \lambda_j \in I_v} \ln \left(1 + \sum_{\ell=1}^m \omega_\ell \frac{\gamma(\frac{j+\alpha}{b}, nr_\ell^{2b})}{\Gamma(\frac{j+\alpha}{b})} \right), \quad v = 1, 2, 3, \quad (2.9)$$

where

$$I_1 = [1 - \epsilon, 1 - \frac{M}{\sqrt{n}}), \quad I_2 = [1 - \frac{M}{\sqrt{n}}, 1 + \frac{M}{\sqrt{n}}], \quad I_3 = (1 + \frac{M}{\sqrt{n}}, 1 + \epsilon].$$

From (2.9), we see that the large n asymptotics of $\{S_2^{(v)}\}_{v=1,2,3}$ involve the asymptotics of $\gamma(a, z)$ when $a \rightarrow +\infty$, $z \rightarrow +\infty$ with $\lambda = \frac{z}{a} \in [1 - \epsilon, 1 + \epsilon]$. These sums can also be rewritten using

$$\sum_{j: \lambda_j \in I_3} = \sum_{j=j_-}^{g_- - 1}, \quad \sum_{j: \lambda_j \in I_2} = \sum_{j=g_-}^{g_+}, \quad \sum_{j: \lambda_j \in I_1} = \sum_{j=g_+ + 1}^{j_+}, \quad (2.10)$$

where $g_- := \lceil \frac{bnr^{2b}}{1 + \frac{M}{\sqrt{n}}} - \alpha \rceil$, $g_+ := \lfloor \frac{bnr^{2b}}{1 - \frac{M}{\sqrt{n}}} - \alpha \rfloor$.

It turns out that $S_2^{(1)}$, $S_2^{(2)}$ and $S_2^{(3)}$ have oscillatory asymptotics as $n \rightarrow +\infty$. To handle these oscillations, we follow [12] and introduce the following quantities:

$$\begin{aligned} \theta_-^{(n,M)} &:= g_- - \left(\frac{bnr^{2b}}{1 + \frac{M}{\sqrt{n}}} - \alpha \right) = \left\lceil \frac{bnr^{2b}}{1 + \frac{M}{\sqrt{n}}} - \alpha \right\rceil - \left(\frac{bnr^{2b}}{1 + \frac{M}{\sqrt{n}}} - \alpha \right), \\ \theta_+^{(n,M)} &:= \left(\frac{bnr^{2b}}{1 - \frac{M}{\sqrt{n}}} - \alpha \right) - g_+ = \left(\frac{bnr^{2b}}{1 - \frac{M}{\sqrt{n}}} - \alpha \right) - \left\lfloor \frac{bnr^{2b}}{1 - \frac{M}{\sqrt{n}}} - \alpha \right\rfloor. \end{aligned}$$

Note that $\theta_-^{(n,M)}, \theta_+^{(n,M)} \in [0, 1)$ are oscillatory but remain bounded as $n \rightarrow +\infty$.

Lemma 2.3. *The constant M' can be chosen sufficiently large such that the following holds. For any $x_1, \dots, x_m \in \mathbb{R}$, there exists $\delta > 0$ such that*

$$S_2^{(1)} = \mathcal{O}(n^{-10}),$$

as $n \rightarrow +\infty$ uniformly for $u_1 \in \{z \in \mathbb{C} : |z - x_1| \leq \delta\}, \dots, u_m \in \{z \in \mathbb{C} : |z - x_m| \leq \delta\}$.

Proof. Using (2.9) and Lemma A.2, we get

$$S_2^{(1)} = \sum_{j:\lambda_j \in I_1} \ln \left(1 + \sum_{\ell=1}^m \omega_\ell \left[\frac{1}{2} \operatorname{erfc} \left(-\eta_{j,\ell} \sqrt{a_j/2} \right) - R_{a_j}(\eta_{j,\ell}) \right] \right).$$

Furthermore, for all sufficiently large n we have

$$\eta_{j,\ell} = \lambda_{j,\ell} - 1 + \mathcal{O}((\lambda_{j,\ell} - 1)^2) \leq -\frac{M}{\sqrt{n}} + \mathcal{O}(\frac{1}{\sqrt{n}}), \quad -\eta_{j,\ell} \sqrt{a_j/2} \geq \frac{Mr^b}{\sqrt{2}} + \mathcal{O}(1), \quad (2.11)$$

uniformly for $j \in \{j : \lambda_j \in I_1\}$. Hence, for sufficiently large M' we have

$$R_{a_j}(\eta_{j,\ell}) = \mathcal{O}(e^{-\frac{r^{2b}M^2}{4}}) = \mathcal{O}(n^{-11}), \quad \frac{1}{2} \operatorname{erfc} \left(-\eta_{j,\ell} \sqrt{a_j/2} \right) = \mathcal{O}(e^{-\frac{r^{2b}M^2}{4}}) = \mathcal{O}(n^{-11}),$$

as $n \rightarrow +\infty$ uniformly for $j \in \{j : \lambda_j \in I_1\}$. Thus, by (2.10),

$$S_2^{(1)} = \mathcal{O}(n^{-10}), \quad \text{as } n \rightarrow +\infty. \quad (2.12)$$

The error terms in (2.11) are independent of $\omega_1, \dots, \omega_m$, and therefore the error term in (2.12) is uniform for $u_1 \in \{z \in \mathbb{C} : |z - x_1| \leq \delta\}, \dots, u_m \in \{z \in \mathbb{C} : |z - x_m| \leq \delta\}$. $\square$

Lemma 2.4. *The constant M' can be chosen sufficiently large such that the following holds. For any $x_1, \dots, x_m \in \mathbb{R}$, there exists $\delta > 0$ such that*

$$S_2^{(3)} = \left(br^{2b}n - j_- - bMr^{2b}\sqrt{n} + bM^2r^{2b} - \alpha + \theta_-^{(n,M)} - bM^3r^{2b}n^{-\frac{1}{2}} \right) \ln \Omega_1 + \mathcal{O}(M^4n^{-1}),$$

as $n \rightarrow +\infty$ uniformly for $u_1 \in \{z \in \mathbb{C} : |z - x_1| \leq \delta\}, \dots, u_m \in \{z \in \mathbb{C} : |z - x_m| \leq \delta\}$.

Proof. The claim can be proved in a similar way as Lemma 2.3. Using (2.9) and Lemma A.2, we obtain

$$S_2^{(3)} = \sum_{j:\lambda_j \in I_3} \ln \left(1 + \sum_{\ell=1}^m \omega_\ell \left[\frac{1}{2} \operatorname{erfc} \left(-\eta_{j,\ell} \sqrt{a_j/2} \right) - R_{a_j}(\eta_{j,\ell}) \right] \right).$$

Since for all sufficiently large n we have

$$\eta_{j,\ell} = \lambda_{j,\ell} - 1 + \mathcal{O}((\lambda_{j,\ell} - 1)^2) \geq \frac{M}{\sqrt{n}} + \mathcal{O}(\frac{1}{\sqrt{n}}), \quad -\eta_{j,\ell} \sqrt{a_j/2} \leq -\frac{Mr^b}{\sqrt{2}} + \mathcal{O}(1),$$

uniformly for $j \in \{j : \lambda_j \in I_3\}$, we can choose M' large enough such that

$$R_{a_j}(\eta_{j,\ell}) = \mathcal{O}(e^{-\frac{r^{2b}M^2}{4}}) = \mathcal{O}(n^{-10}), \quad \frac{1}{2} \operatorname{erfc} \left(-\eta_{j,\ell} \sqrt{a_j/2} \right) = 1 - \mathcal{O}(e^{-\frac{r^{2b}M^2}{4}}) = 1 - \mathcal{O}(n^{-10}),$$

as $n \rightarrow +\infty$ uniformly for $j \in \{j : \lambda_j \in I_3\}$, and thus, by (2.10),

$$S_2^{(3)} = \sum_{j=j_-}^{g_- - 1} \ln \Omega_1 + \mathcal{O}(n^{-9}) = (g_- - j_-) \ln \Omega_1 + \mathcal{O}(n^{-9}).$$

The claim now follows from

$$\begin{aligned} \sum_{j=j_-}^{g_- - 1} 1 &= g_- - j_- = \left(\frac{bnr^{2b}}{1 + \frac{M}{\sqrt{n}}} - \alpha \right) + \theta_-^{(n,M)} - j_- \\ &= br^{2b}n - j_- - bMr^{2b}\sqrt{n} + bM^2r^{2b} - \alpha + \theta_-^{(n,M)} - bM^3r^{2b}n^{-\frac{1}{2}} + \mathcal{O}(M^4n^{-1}), \quad \text{as } n \rightarrow +\infty. \end{aligned}$$

$\square$

We now turn to the asymptotic analysis of $S_2^{(2)}$. For all $k \in \{1, \dots, m\}$ and $j \in \{j : \lambda_j \in I_2\} = \{g_-, \dots, g_+\}$, define

$$M_{j,k} := \sqrt{n}(\lambda_{j,k} - 1), \quad M_j := \sqrt{n}(\lambda_j - 1).$$

Note that $M_{j,k}$ and M_j decrease as j increases. Since $I_2 = [1 - \frac{M}{\sqrt{n}}, 1 + \frac{M}{\sqrt{n}}]$, as $n \rightarrow +\infty$ the points $M_{g_-,k}, \dots, M_{g_+,k}$ run over the interval

$$[\sqrt{n}((\frac{r_k}{r})^{2b}(1 - \frac{M}{\sqrt{n}}) - 1), \sqrt{n}((\frac{r_k}{r})^{2b}(1 + \frac{M}{\sqrt{n}}) - 1)] \approx [-M + \sqrt{2}r^{-b}\mathfrak{s}_k, M + \sqrt{2}r^{-b}\mathfrak{s}_k]$$

for each $k \in \{1, \dots, m\}$, and the points $M_{g_-}, \dots, M_{g_+}$ run over the interval $[-M, M]$. For the large n asymptotics of $S_2^{(2)}$ we will need the following lemma.

Lemma 2.5. (Taken from [12, Lemma 2.7]) Let $f \in C^3(\mathbb{R})$ be a function such that $|f|, |f'|, |f''|, |f'''|$ are bounded. As $n \rightarrow +\infty$ we have

$$\begin{aligned} \sum_{j=g_-}^{g_+} f(M_j) &= br^{2b} \int_{-M}^M f(t) dt \sqrt{n} - 2br^{2b} \int_{-M}^M t f(t) dt + \left(\frac{1}{2} - \theta_-^{(n,M)}\right) f(M) + \left(\frac{1}{2} - \theta_+^{(n,M)}\right) f(-M) \\ &+ \frac{1}{\sqrt{n}} \left[3br^{2b} \int_{-M}^M t^2 f(t) dt + \left(\frac{1}{12} + \frac{\theta_-^{(n,M)}(\theta_-^{(n,M)} - 1)}{2}\right) \frac{f'(M)}{br^{2b}} - \left(\frac{1}{12} + \frac{\theta_+^{(n,M)}(\theta_+^{(n,M)} - 1)}{2}\right) \frac{f'(-M)}{br^{2b}} \right] \\ &+ \mathcal{O}(M^4 n^{-1}). \end{aligned} \quad (2.13)$$

Lemma 2.6. For any $x_1, \dots, x_m \in \mathbb{R}$, there exists $\delta > 0$ such that

$$\begin{aligned} S_2^{(2)} &= \tilde{C}_2^{(M)} \sqrt{n} + \tilde{C}_3^{(n,M)} + \frac{1}{\sqrt{n}} \tilde{C}_4^{(n,M)} + \mathcal{O}(M^4 n^{-1}), \\ \tilde{C}_2^{(M)} &= br^{2b} \int_{-M}^M f_1(t) dt, \\ \tilde{C}_3^{(n,M)} &= br^{2b} \int_{-M}^M \left(-2t f_1(t) + f_2(t) \right) dt + \left(\frac{1}{2} - \theta_-^{(n,M)}\right) f_1(M) + \left(\frac{1}{2} - \theta_+^{(n,M)}\right) f_1(-M), \\ \tilde{C}_4^{(n,M)} &= br^{2b} \int_{-M}^M \left(3t^2 f_1(t) - 2t f_2(t) + f_3(t) \right) dt + \left(\frac{1}{2} - \theta_-^{(n,M)}\right) f_2(M) + \left(\frac{1}{2} - \theta_+^{(n,M)}\right) f_2(-M) \\ &+ \left(\frac{1}{12} + \frac{\theta_-^{(n,M)}(\theta_-^{(n,M)} - 1)}{2}\right) \frac{f_1'(M)}{br^{2b}} - \left(\frac{1}{12} + \frac{\theta_+^{(n,M)}(\theta_+^{(n,M)} - 1)}{2}\right) \frac{f_1'(-M)}{br^{2b}}, \end{aligned}$$

as $n \rightarrow +\infty$ uniformly for $u_1 \in \{z \in \mathbb{C} : |z - x_1| \leq \delta\}, \dots, u_m \in \{z \in \mathbb{C} : |z - x_m| \leq \delta\}$, where

$$\begin{aligned} g(x) &= 1 + \sum_{\ell=1}^m \frac{\omega_\ell}{2} \operatorname{erfc} \left(-\frac{r^b}{\sqrt{2}} (x + \sqrt{2} r^{-b} \mathfrak{s}_\ell) \right), \quad f_1(x) = \ln(g(x)), \\ f_2(x) &= \frac{1}{g(x)} \sum_{\ell=1}^m \frac{-\omega_\ell e^{-\frac{(x + \sqrt{2} r^{-b} \mathfrak{s}_\ell)^2 r^{2b}}{2}}}{6r^b \sqrt{2\pi}} \left(5r^{2b} x^2 + \sqrt{2} r^b x \mathfrak{s}_\ell + 4\mathfrak{s}_\ell^2 - 2 \right), \\ f_3(x) &= -\frac{f_2(x)^2}{2} + \frac{1}{g(x)} \sum_{\ell=1}^m \frac{\omega_\ell e^{-\frac{(x + \sqrt{2} r^{-b} \mathfrak{s}_\ell)^2 r^{2b}}{2}}}{72r^{2b} \sqrt{2\pi}} \left\{ -25r^{5b} x^5 - 35\sqrt{2} r^{4b} x^4 \mathfrak{s}_\ell + r^{3b} x^3 (73 - 62\mathfrak{s}_\ell^2) \right. \\ &\quad \left. + \sqrt{2} r^{2b} x^2 \mathfrak{s}_\ell (33 - 50\mathfrak{s}_\ell^2) + 2r^b x (3 + 18\mathfrak{s}_\ell^2 - 16\mathfrak{s}_\ell^4) - 2\sqrt{2} \mathfrak{s}_\ell (3 - 22\mathfrak{s}_\ell^2 + 8\mathfrak{s}_\ell^4) \right\}. \end{aligned}$$

Proof. Using (2.9) and Lemma A.2, we obtain

$$S_2^{(2)} = \sum_{j: \lambda_j \in I_2} \ln \left(1 + \sum_{\ell=1}^m \omega_\ell \left[\frac{1}{2} \operatorname{erfc} \left(-\eta_{j,\ell} \sqrt{a_j/2} \right) - R_{a_j}(\eta_{j,\ell}) \right] \right). \quad (2.14)$$

For $j \in \{j : \lambda_j \in I_2\}$, we have $1 - \frac{M}{\sqrt{n}} \leq \lambda_j = \frac{bnr^{2b}}{j+\alpha} \leq 1 + \frac{M}{\sqrt{n}}$, $-M \leq M_j \leq M$, and

$$M_{j,k} = M_j + \sqrt{2} r^{-b} \mathfrak{s}_k + \frac{\sqrt{2} r^{-b} \mathfrak{s}_k M_j}{\sqrt{n}}, \quad k = 1, \dots, m.$$

Furthermore, as $n \rightarrow +\infty$ we have

$$\begin{aligned} \eta_{j,\ell} &= (\lambda_{j,\ell} - 1) \left(1 - \frac{\lambda_{j,\ell} - 1}{3} + \frac{7}{36} (\lambda_{j,\ell} - 1)^2 + \mathcal{O}((\lambda_{j,\ell} - 1)^3) \right) = \frac{M_{j,\ell}}{\sqrt{n}} - \frac{M_{j,\ell}^2}{3n} + \frac{7M_{j,\ell}^3}{36n^{3/2}} + \mathcal{O}\left(\frac{M^4}{n^2}\right), \\ &= \frac{M_j + \sqrt{2} r^{-b} \mathfrak{s}_\ell}{\sqrt{n}} + \frac{1}{3n} \left(\sqrt{2} r^{-b} \mathfrak{s}_\ell M_j - M_j^2 - 2r^{-2b} \mathfrak{s}_\ell^2 \right) \\ &\quad + \frac{1}{3n^{3/2}} (M_j + \sqrt{2} r^{-b} \mathfrak{s}_\ell) \left(\frac{7}{12} (M_j + \sqrt{2} r^{-b} \mathfrak{s}_\ell)^2 - 2\sqrt{2} r^{-b} \mathfrak{s}_\ell M_j \right) + \mathcal{O}\left(\frac{M^4}{n^2}\right) \end{aligned} \quad (2.15)$$

and

$$\begin{aligned} -\eta_{j,\ell} \sqrt{a_j/2} &= -\frac{M_{j,\ell} r_\ell^b}{\sqrt{2}} + \frac{5M_{j,\ell}^2 r_\ell^b}{6\sqrt{2}\sqrt{n}} - \frac{53M_{j,\ell}^3 r_\ell^b}{72\sqrt{2}n} + \mathcal{O}(M^4 n^{-\frac{3}{2}}) \\ &= -\frac{(M_j + \sqrt{2} r^{-b} \mathfrak{s}_\ell) r^b}{\sqrt{2}} + \frac{r^b}{12\sqrt{n}} \left(5\sqrt{2} M_j^2 + 2r^{-b} M_j \mathfrak{s}_\ell + 4\sqrt{2} r^{-2b} \mathfrak{s}_\ell^2 \right) \\ &\quad - \frac{r^b}{144n} \left(53\sqrt{2} M_j^3 + 18r^{-b} M_j^2 \mathfrak{s}_\ell + 12\sqrt{2} r^{-2b} M_j \mathfrak{s}_\ell^2 + 56r^{-3b} \mathfrak{s}_\ell^3 \right) + \mathcal{O}(M^4 n^{-3/2}), \end{aligned} \quad (2.16)$$

uniformly for $j \in \{j : \lambda_j \in I_2\}$. Hence, by (A.2), as $n \rightarrow +\infty$ we have

$$\begin{aligned} R_{a_j}(\eta_{j,\ell}) &= \frac{e^{-\frac{(M_j + \sqrt{2} r^{-b} \mathfrak{s}_\ell)^2 r^{2b}}{2}}}{\sqrt{2\pi}} \left(\frac{-1}{3r^b \sqrt{n}} \right. \\ &\quad \left. - \frac{10M_j^3 r^{3b} + 12\sqrt{2} M_j^2 r^{2b} \mathfrak{s}_\ell + 12M_j r^b \mathfrak{s}_\ell^2 + 8\sqrt{2} \mathfrak{s}_\ell^3 + 3M_j r^b - 3\sqrt{2} \mathfrak{s}_\ell}{36r^{2b} n} + \mathcal{O}((1 + M_j^6) n^{-\frac{3}{2}}) \right) \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2} \operatorname{erfc} \left(-\eta_{j,\ell} \sqrt{a_j/2} \right) &= \frac{1}{2} \operatorname{erfc} \left(-\frac{r^b}{\sqrt{2}} (M_j + \sqrt{2} r^{-b} \mathfrak{s}_\ell) \right) \\ &\quad - \frac{e^{-\frac{(M_j + \sqrt{2} r^{-b} \mathfrak{s}_\ell)^2 r^{2b}}{2}}}{12\sqrt{\pi} r^b \sqrt{n}} \left(5\sqrt{2} r^{2b} M_j^2 + 2r^b M_j \mathfrak{s}_\ell + 4\sqrt{2} \mathfrak{s}_\ell^2 \right) \\ &\quad + \frac{e^{-\frac{(M_j + \sqrt{2} r^{-b} \mathfrak{s}_\ell)^2 r^{2b}}{2}}}{144\sqrt{\pi} r^{2b} n} \left\{ 53\sqrt{2} r^{3b} M_j^3 + 18r^{2b} M_j^2 \mathfrak{s}_\ell + 12\sqrt{2} r^b M_j \mathfrak{s}_\ell^2 + 56\mathfrak{s}_\ell^3 \right. \\ &\quad \left. - \sqrt{2} (r^b M_j + \sqrt{2} \mathfrak{s}_\ell) \left(5r^{2b} M_j^2 + \sqrt{2} r^b M_j \mathfrak{s}_\ell + 4\mathfrak{s}_\ell^2 \right)^2 \right\} + \mathcal{O} \left(e^{-\frac{(M_j + \sqrt{2} r^{-b} \mathfrak{s}_\ell)^2 r^{2b}}{2}} (1 + M_j^8) n^{-\frac{3}{2}} \right), \end{aligned}$$

uniformly for $j \in \{j : \lambda_j \in I_2\}$. The identity $g(-\sqrt{2}r^{-b}t) = \mathcal{H}_1(t; \vec{u}, \vec{s})$ in combination with Lemma 1.1 shows that $g(x) > 0$ for all $x \in \mathbb{R}$; in particular, the functions $f_1(x), f_2(x), f_3(x)$ are well-defined and real-valued for $x \in \mathbb{R}$. Substituting the above asymptotics into (2.14) and using that the error terms are suppressed by exponentials of the form $e^{-cM_j^2}$, we obtain

$$S_2^{(2)} = \Sigma_1^{(n)} + \frac{1}{\sqrt{n}}\Sigma_2^{(n)} + \frac{1}{n}\Sigma_3^{(n)} + \mathcal{O}(n^{-1}), \quad \text{as } n \rightarrow +\infty, \quad (2.17)$$

where

$$\Sigma_1^{(n)} = \sum_{j=g_-}^{g_+} f_1(M_j), \quad \Sigma_2^{(n)} = \sum_{j=g_-}^{g_+} f_2(M_j), \quad \Sigma_3^{(n)} = \sum_{j=g_-}^{g_+} f_3(M_j).$$

The functions f_j , $j = 1, 2, 3$, satisfy the assumptions of Lemma 2.5. Moreover, $f_2(x), f_3(x)$, and their derivatives have exponential decay as $x \rightarrow \pm\infty$. Hence, by (2.13), we have

$$\begin{aligned} \Sigma_1^{(n)} &= \Sigma_{1,2}\sqrt{n} + \Sigma_{1,3} + \frac{1}{\sqrt{n}}\Sigma_{1,4} + \mathcal{O}(M^4n^{-1}), \\ \frac{1}{\sqrt{n}}\Sigma_2^{(n)} &= \Sigma_{2,3} + \frac{1}{\sqrt{n}}\Sigma_{2,4} + \mathcal{O}(n^{-1}), \quad \frac{1}{n}\Sigma_3^{(n)} = \frac{1}{\sqrt{n}}\Sigma_{3,4} + \mathcal{O}(n^{-1}), \end{aligned}$$

as $n \rightarrow +\infty$, for some explicit $\Sigma_{1,2}, \Sigma_{1,3}, \Sigma_{1,4}, \Sigma_{2,3}, \Sigma_{2,4}, \Sigma_{3,4}$. A computation gives

$$\Sigma_{1,2} = \tilde{C}_2^{(M)}, \quad \Sigma_{1,3} + \Sigma_{2,3} = \tilde{C}_3^{(n,M)}, \quad \Sigma_{1,4} + \Sigma_{2,4} + \Sigma_{3,4} = \tilde{C}_4^{(n,M)},$$

which is the claim. $\square$

We are now ready to derive the asymptotics of S_2 as $n \rightarrow +\infty$.

Lemma 2.7. *The constant M' can be chosen sufficiently large such that the following holds. For any $x_1, \dots, x_m \in \mathbb{R}$, there exists $\delta > 0$ such that*

$$S_2 = \left(br^{2b}n - j_-\right) \ln \Omega_1 + C_2\sqrt{n} + \tilde{C}_3 + \frac{1}{\sqrt{n}}C_4 + \mathcal{O}(M^4n^{-1}),$$

as $n \rightarrow +\infty$ uniformly for $u_1 \in \{z \in \mathbb{C} : |z - x_1| \leq \delta\}, \dots, u_m \in \{z \in \mathbb{C} : |z - x_m| \leq \delta\}$, where

$$\begin{aligned} C_2 &= br^{2b} \left[\int_{-\infty}^0 f_1(t) dt + \int_0^{+\infty} (f_1(t) - \ln \Omega_1) dt \right], \\ \tilde{C}_3 &= \left(\frac{1}{2} - \alpha \right) \ln \Omega_1 + br^{2b} \left[\int_{-\infty}^0 (-2tf_1(t) + f_2(t)) dt + \int_0^{+\infty} (-2t[f_1(t) - \ln \Omega_1] + f_2(t)) dt \right], \\ C_4 &= br^{2b} \left[\int_{-\infty}^0 (3t^2 f_1(t) - 2tf_2(t) + f_3(t)) dt + \int_0^{+\infty} (3t^2(f_1(t) - \ln(\Omega_1)) - 2tf_2(t) + f_3(t)) dt \right]. \end{aligned}$$

Proof. It follows from Lemmas 2.3, 2.4 and 2.6 that

$$S_2 = \left(br^{2b}n - j_-\right) \ln \Omega_1 + \hat{C}_2^{(M)}\sqrt{n} + \hat{C}_3^{(n,M)} + \frac{1}{\sqrt{n}}\hat{C}_4^{(n,M)} + \mathcal{O}(M^4n^{-1}),$$

as $n \rightarrow +\infty$ uniformly for $u_1 \in \{z \in \mathbb{C} : |z - x_1| \leq \delta\}, \dots, u_m \in \{z \in \mathbb{C} : |z - x_m| \leq \delta\}$, where

$$\hat{C}_2^{(M)} := \tilde{C}_2^{(M)} - bMr^{2b} \ln \Omega_1,$$

$$\begin{aligned}\widehat{C}_3^{(n,M)} &:= \widetilde{C}_3^{(n,M)} + \left(bM^2r^{2b} - \alpha + \theta_-^{(n,M)}\right) \ln \Omega_1, \\ \widehat{C}_4^{(n,M)} &:= \widetilde{C}_4^{(n,M)} - bM^3r^{2b} \ln \Omega_1.\end{aligned}$$

Provided M' is chosen sufficiently large, as $n \rightarrow +\infty$ we get

$$\widehat{C}_2^{(M)} = C_2 + \mathcal{O}(n^{-10}), \quad \widehat{C}_3^{(n,M)} = \widetilde{C}_3 + \mathcal{O}(n^{-10}), \quad \widehat{C}_4^{(n,M)} = C_4 + \mathcal{O}(n^{-10}),$$

and the claim follows. $\square$

End of the proof of Theorem 1.2. Let $M' > 0$ be sufficiently large such that Lemmas 2.2 and 2.7 hold. Using (2.4) and Lemmas 2.1, 2.2 and 2.7, we conclude that for any $x_1, \dots, x_m \in \mathbb{R}$, there exists $\delta > 0$ such that

$$\begin{aligned}\ln \mathcal{E}_n &= S_0 + S_1 + S_2 + S_3 \\ &= M' \ln \Omega_1 + (j_- - M' - 1) \ln \Omega_1 + \left(br^{2b}n - j_-\right) \ln \Omega_1 + C_2\sqrt{n} + \widetilde{C}_3 + \frac{1}{\sqrt{n}}C_4 + \mathcal{O}(M^4n^{-1}) \\ &= \left(br^{2b} \ln \Omega_1\right)n + C_2\sqrt{n} + C_3 + \frac{1}{\sqrt{n}}C_4 + \mathcal{O}(M^4n^{-1}),\end{aligned}$$

as $n \rightarrow +\infty$ uniformly for $u_1 \in \{z \in \mathbb{C} : |z - x_1| \leq \delta\}, \dots, u_m \in \{z \in \mathbb{C} : |z - x_m| \leq \delta\}$, where $C_3 = \widetilde{C}_3 - \ln \Omega_1$. Using (1.4)–(1.7), (2.1) and (2.2), the constants C_2 , C_3 and C_4 can be rewritten as in (1.9) after a change of variables. $\square$

3 Proof of Theorem 1.3

As in the proof of Theorem 1.2, our starting point is formula (2.3).

Let

$$j_- := \lceil \frac{n}{1+\epsilon} - \alpha \rceil,$$

where $\epsilon > 0$ is a small constant independent of n . Using (2.3), we write $\ln \mathcal{E}_n$ in 3 parts

$$\ln \mathcal{E}_n = S_0 + S_1 + S_2, \tag{3.1}$$

with

$$S_0 = \sum_{j=1}^{M'} \ln \left(1 + \sum_{\ell=1}^m \omega_\ell \frac{\gamma(\frac{j+\alpha}{b}, nr_\ell^{2b})}{\Gamma(\frac{j+\alpha}{b})} \right), \quad S_1 = \sum_{j=M'+1}^{j_- - 1} \ln \left(1 + \sum_{\ell=1}^m \omega_\ell \frac{\gamma(\frac{j+\alpha}{b}, nr_\ell^{2b})}{\Gamma(\frac{j+\alpha}{b})} \right), \tag{3.2}$$

$$S_2 = \sum_{j=j_-}^n \ln \left(1 + \sum_{\ell=1}^m \omega_\ell \frac{\gamma(\frac{j+\alpha}{b}, nr_\ell^{2b})}{\Gamma(\frac{j+\alpha}{b})} \right), \tag{3.3}$$

where $M' > 0$ is a large integer independent of n . Recall the definition of Ω_ℓ in (2.2).

Lemma 3.1. *For any $x_1, \dots, x_m \in \mathbb{R}$, there exists $\delta > 0$ such that*

$$S_0 = M' \ln \Omega_1 + \mathcal{O}(e^{-cn}), \quad \text{as } n \rightarrow +\infty, \tag{3.4}$$

uniformly for $u_1 \in \{z \in \mathbb{C} : |z - x_1| \leq \delta\}, \dots, u_m \in \{z \in \mathbb{C} : |z - x_m| \leq \delta\}$.

Proof. The proof is identical to the proof of Lemma 2.1. $\square$

Lemma 3.2. *The constant M' can be chosen sufficiently large such that the following holds. For any $x_1, \dots, x_m \in \mathbb{R}$, there exists $\delta > 0$ such that*

$$S_1 = (j_- - M' - 1) \ln \Omega_1 + \mathcal{O}(e^{-cn}),$$

as $n \rightarrow +\infty$ uniformly for $u_1 \in \{z \in \mathbb{C} : |z - x_1| \leq \delta\}, \dots, u_m \in \{z \in \mathbb{C} : |z - x_m| \leq \delta\}$.

Proof. The claim follows as in Lemma 2.2. $\square$

We now focus on S_2 . For $j = 1, \dots, n$ and $k = 1, \dots, m$, define a_j , z_k , $\lambda_{j,k}$, λ_j and $\eta_{j,k}$ as in (2.7) with r replaced by $b^{-\frac{1}{2b}}$, i.e.

$$\lambda_{j,k} := \frac{z_k}{a_j} = \frac{bnr_k^{2b}}{j + \alpha}, \quad \lambda_j := \frac{n}{j + \alpha}, \quad \eta_{j,k} := (\lambda_{j,k} - 1) \sqrt{\frac{2(\lambda_{j,k} - 1 - \ln \lambda_{j,k})}{(\lambda_{j,k} - 1)^2}},$$

with $a_j := \frac{j+\alpha}{b}$, $z_k := nr_k^{2b}$ and $r_1, \dots, r_m$ as in the statement of Theorem 1.3. Let $M := M' \sqrt{\ln n}$. We split S_2 in two pieces as follows

$$S_2 = S_2^{(2)} + S_2^{(3)},$$

where

$$S_2^{(v)} = \sum_{j: \lambda_j \in I_v} \ln \left(1 + \sum_{\ell=1}^m \omega_\ell \frac{\gamma(\frac{j+\alpha}{b}, nr_\ell^{2b})}{\Gamma(\frac{j+\alpha}{b})} \right), \quad v = 2, 3, \quad (3.5)$$

and

$$I_2 = [\frac{1}{1+\frac{\alpha}{n}}, 1 + \frac{M}{\sqrt{n}}], \quad I_3 = (1 + \frac{M}{\sqrt{n}}, 1 + \epsilon].$$

The sums $S_2^{(2)}$ and $S_2^{(3)}$ can also be rewritten using

$$\sum_{j: \lambda_j \in I_3} = \sum_{j=j_-}^{g_- - 1}, \quad \sum_{j: \lambda_j \in I_2} = \sum_{j=g_-}^n, \quad (3.6)$$

where $g_- := \lceil \frac{n}{1+\frac{M}{\sqrt{n}}} - \alpha \rceil$. Let us also define

$$\theta_-^{(n,M)} := g_- - \left(\frac{n}{1+\frac{M}{\sqrt{n}}} - \alpha \right) = \left\lceil \frac{n}{1+\frac{M}{\sqrt{n}}} - \alpha \right\rceil - \left(\frac{n}{1+\frac{M}{\sqrt{n}}} - \alpha \right).$$

Clearly, $\theta_-^{(n,M)} \in [0, 1)$.

Lemma 3.3. *The constant M' can be chosen sufficiently large such that the following holds. For any $x_1, \dots, x_m \in \mathbb{R}$, there exists $\delta > 0$ such that*

$$S_2^{(3)} = \left(n - j_- - M\sqrt{n} + M^2 - \alpha + \theta_-^{(n,M)} - M^3 n^{-\frac{1}{2}} \right) \ln \Omega_1 + \mathcal{O}(M^4 n^{-1}),$$

as $n \rightarrow +\infty$ uniformly for $u_1 \in \{z \in \mathbb{C} : |z - x_1| \leq \delta\}, \dots, u_m \in \{z \in \mathbb{C} : |z - x_m| \leq \delta\}$.

Proof. The proof follows the proof of Lemma 2.4, except that at the last step we need to use

$$\begin{aligned} \sum_{j=j_-}^{g_- - 1} 1 &= g_- - j_- = \left(\frac{n}{1+\frac{M}{\sqrt{n}}} - \alpha \right) + \theta_-^{(n,M)} - j_- \\ &= n - j_- - M\sqrt{n} + M^2 - \alpha + \theta_-^{(n,M)} - M^3 n^{-\frac{1}{2}} + \mathcal{O}(M^4 n^{-1}), \quad \text{as } n \rightarrow +\infty. \end{aligned} \quad (3.7)$$

$\square$

For $k \in \{1, \dots, m\}$ and $j \in \{j : \lambda_j \in I_2\} = \{g_-, \dots, n\}$, we define $M_{j,k} := \sqrt{n}(\lambda_{j,k} - 1)$ and $M_j := \sqrt{n}(\lambda_j - 1)$.

Lemma 3.4. (Taken from [12, Lemma 2.7]) Let $f \in C^3(\mathbb{R})$ be a function such that $|f|, |f'|, |f''|, |f'''|$ are bounded. As $n \rightarrow +\infty$ we have

$$\begin{aligned} \sum_{j=g_-}^n f(M_j) &= \int_0^M f(t) dt \sqrt{n} - 2 \int_0^M t f(t) dt + \left(\frac{1}{2} - \theta_-^{(n,M)}\right) f(M) + \left(\frac{1}{2} + \alpha\right) f(0) \\ &+ \frac{1}{\sqrt{n}} \left[3 \int_0^M t^2 f(t) dt + \left(\frac{1}{12} + \frac{\theta_-^{(n,M)}(\theta_-^{(n,M)} - 1)}{2}\right) f'(M) - \frac{1 + 6\alpha + 6\alpha^2}{12} f'(0) \right] + \mathcal{O}(M^4 n^{-1}). \end{aligned} \quad (3.8)$$

Lemma 3.5. For any $x_1, \dots, x_m \in \mathbb{R}$, there exists $\delta > 0$ such that

$$\begin{aligned} S_2^{(2)} &= \tilde{C}_2^{(M)} \sqrt{n} + \tilde{C}_3^{(n,M)} + \frac{1}{\sqrt{n}} \tilde{C}_4^{(n,M)} + \mathcal{O}(M^4 n^{-1}), \\ \tilde{C}_2^{(M)} &= \int_0^M f_1(t) dt, \\ \tilde{C}_3^{(n,M)} &= \int_0^M \left(-2t f_1(t) + f_2(t) \right) dt + \left(\frac{1}{2} - \theta_-^{(n,M)}\right) f_1(M) + \left(\frac{1}{2} + \alpha\right) f_1(0), \\ \tilde{C}_4^{(n,M)} &= \int_0^M \left(3t^2 f_1(t) - 2t f_2(t) + f_3(t) \right) dt + \left(\frac{1}{2} - \theta_-^{(n,M)}\right) f_2(M) + \left(\frac{1}{2} + \alpha\right) f_2(0) \\ &+ \left(\frac{1}{12} + \frac{\theta_-^{(n,M)}(\theta_-^{(n,M)} - 1)}{2}\right) f_1'(M) - \frac{1 + 6\alpha + 6\alpha^2}{12} f_1'(0), \end{aligned}$$

as $n \rightarrow +\infty$ uniformly for $u_1 \in \{z \in \mathbb{C} : |z - x_1| \leq \delta\}, \dots, u_m \in \{z \in \mathbb{C} : |z - x_m| \leq \delta\}$, where g, f_1, f_2 and f_3 are as in the statement of Lemma 2.6 with r replaced by $b^{-\frac{1}{2b}}$.

Proof. The first part of the proof is identical to the beginning of the proof of Lemma 2.6, except that one needs to replace r and g_+ by $b^{-\frac{1}{2b}}$ and n , respectively. In particular, we find

$$S_2^{(2)} = \Sigma_1^{(n)} + \frac{1}{\sqrt{n}} \Sigma_2^{(n)} + \frac{1}{n} \Sigma_3^{(n)} + \mathcal{O}(n^{-1}), \quad \text{as } n \rightarrow +\infty, \quad (3.9)$$

where $\Sigma_1^{(n)}, \Sigma_2^{(n)}$ and $\Sigma_3^{(n)}$ are in the proof of Lemma 2.6 with r and g_+ replaced by $b^{-\frac{1}{2b}}$ and n , respectively. The asymptotics of these sums can then be obtained using Lemma 3.4. After a computation, we then find the claim. $\square$

Lemma 3.6. The constant M' can be chosen sufficiently large such that the following holds. For any $x_1, \dots, x_m \in \mathbb{R}$, there exists $\delta > 0$ such that

$$S_2 = (n - j_-) \ln \Omega_1 + C_2 \sqrt{n} + \tilde{C}_3 + \frac{1}{\sqrt{n}} C_4 + \mathcal{O}(M^4 n^{-1}),$$

as $n \rightarrow +\infty$ uniformly for $u_1 \in \{z \in \mathbb{C} : |z - x_1| \leq \delta\}, \dots, u_m \in \{z \in \mathbb{C} : |z - x_m| \leq \delta\}$, where

$$\begin{aligned} C_2 &= \int_0^{+\infty} (f_1(t) - \ln \Omega_1) dt, \\ \tilde{C}_3 &= \left(\frac{1}{2} - \alpha\right) \ln \Omega_1 + \int_0^{+\infty} (-2t[f_1(t) - \ln \Omega_1] + f_2(t)) dt + \left(\frac{1}{2} + \alpha\right) f_1(0), \\ C_4 &= \int_0^{+\infty} (3t^2(f_1(t) - \ln(\Omega_1)) - 2t f_2(t) + f_3(t)) dt - \frac{1 + 6\alpha + 6\alpha^2}{12} f_1'(0) + \left(\frac{1}{2} + \alpha\right) f_2(0). \end{aligned}$$

Proof. By combining Lemmas 3.3 and 3.5, we have

$$S_2 = (n - j_-) \ln \Omega_1 + \widehat{C}_2^{(M)} \sqrt{n} + \widehat{C}_3^{(n,M)} + \frac{1}{\sqrt{n}} \widehat{C}_4^{(n,M)} + \mathcal{O}(M^4 n^{-1}),$$

as $n \rightarrow +\infty$ uniformly for $u_1 \in \{z \in \mathbb{C} : |z - x_1| \leq \delta\}, \dots, u_m \in \{z \in \mathbb{C} : |z - x_m| \leq \delta\}$, where

$$\begin{aligned} \widehat{C}_2^{(M)} &:= \widetilde{C}_2^{(M)} - M \ln \Omega_1, \\ \widehat{C}_3^{(n,M)} &:= \widetilde{C}_3^{(n,M)} + \left(M^2 - \alpha + \theta_-^{(n,M)}\right) \ln \Omega_1, \\ \widehat{C}_4^{(n,M)} &:= \widetilde{C}_4^{(n,M)} - M^3 \ln \Omega_1. \end{aligned}$$

Provided M' is chosen sufficiently large, as $n \rightarrow +\infty$ we get

$$\widehat{C}_2^{(M)} = C_2 + \mathcal{O}(n^{-10}), \quad \widehat{C}_3^{(n,M)} = \widetilde{C}_3 + \mathcal{O}(n^{-10}), \quad \widehat{C}_4^{(n,M)} = C_4 + \mathcal{O}(n^{-10}),$$

and the claim follows. $\square$

End of the proof of Theorem 1.3. Let $M' > 0$ be sufficiently large such that Lemmas 3.2 and 3.6 hold. Using (3.1) and Lemmas 3.1, 3.2 and 3.6, we conclude that for any $x_1, \dots, x_m \in \mathbb{R}$, there exists $\delta > 0$ such that

$$\begin{aligned} \ln \mathcal{E}_n &= S_0 + S_1 + S_2 \\ &= M' \ln \Omega_1 + (j_- - M' - 1) \ln \Omega_1 + (n - j_-) \ln \Omega_1 + C_2 \sqrt{n} + \widetilde{C}_3 + \frac{1}{\sqrt{n}} C_4 + \mathcal{O}(M^4 n^{-1}) \\ &= n \ln \Omega_1 + C_2 \sqrt{n} + C_3 + \frac{1}{\sqrt{n}} C_4 + \mathcal{O}(M^4 n^{-1}), \end{aligned}$$

as $n \rightarrow +\infty$ uniformly for $u_1 \in \{z \in \mathbb{C} : |z - x_1| \leq \delta\}, \dots, u_m \in \{z \in \mathbb{C} : |z - x_m| \leq \delta\}$, where $C_3 = \widetilde{C}_3 - \ln \Omega_1$. Using (1.4)–(1.7), (2.1) and (2.2), the constants C_2, C_3 and C_4 can be rewritten as in (1.11) after a simple change of variables. This concludes the proof of Theorem 1.3. $\square$

A Uniform asymptotics of the incomplete gamma function

Lemma A.1. (From [44, formula 8.11.2]). Let $a > 0$ be fixed. As $z \rightarrow +\infty$,

$$\gamma(a, z) = \Gamma(a) + \mathcal{O}(e^{-\frac{z}{2}}).$$

Lemma A.2. (From [49, Section 11.2.4]). We have

$$\frac{\gamma(a, z)}{\Gamma(a)} = \frac{1}{2} \operatorname{erfc}(-\eta \sqrt{a/2}) - R_a(\eta), \quad R_a(\eta) = \frac{e^{-\frac{1}{2}a\eta^2}}{2\pi i} \int_{-\infty}^{\infty} e^{-\frac{1}{2}au^2} g(u) du,$$

where erfc is defined in (1.8),

$$\lambda = \frac{z}{a}, \quad \eta = (\lambda - 1) \sqrt{\frac{2(\lambda - 1 - \ln \lambda)}{(\lambda - 1)^2}}, \quad g(u) := \frac{dt}{du} \frac{1}{\lambda - t} + \frac{1}{u + i\eta}, \quad (\text{A.1})$$

with t and u being related by the bijection $t \mapsto u$ from $\mathcal{L} := \{\frac{\theta}{\sin \theta} e^{i\theta} : -\pi < \theta < \pi\}$ to $\mathbb{R}$ given by

$$u = -i(t - 1) \sqrt{\frac{2(t - 1 - \ln t)}{(t - 1)^2}}, \quad t \in \mathcal{L},$$

and the principal branch is used for the roots. Furthermore, as $a \rightarrow +\infty$, uniformly for $z \in [0, \infty)$,

$$R_a(\eta) \sim \frac{e^{-\frac{1}{2}a\eta^2}}{\sqrt{2\pi a}} \sum_{j=0}^{\infty} \frac{c_j(\eta)}{a^j}, \quad (\text{A.2})$$

where all coefficients $c_j(\eta)$ are bounded functions of $\eta \in \mathbb{R}$ (i.e. bounded for $\lambda \in [0, +\infty)$). The first two coefficients are given by (see [49, p. 312])

$$c_0(\eta) = \frac{1}{\lambda-1} - \frac{1}{\eta}, \quad c_1(\eta) = \frac{1}{\eta^3} - \frac{1}{(\lambda-1)^3} - \frac{1}{(\lambda-1)^2} - \frac{1}{12(\lambda-1)}.$$

In particular, the following hold:

(i) Let $z = \lambda a$ and let $\delta > 0$ be fixed. As $a \rightarrow +\infty$, uniformly for $\lambda \geq 1 + \delta$,

$$\gamma(a, z) = \Gamma(a)(1 + \mathcal{O}(e^{-\frac{a\eta^2}{2}})).$$

(ii) Let $z = \lambda a$. As $a \rightarrow +\infty$, uniformly for λ in compact subsets of $(0, 1)$,

$$\gamma(a, z) = \Gamma(a)\mathcal{O}(e^{-\frac{a\eta^2}{2}}).$$

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